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Markov Jump Dynamics with Additive Intensities in Continuum: State Evolution and Mesoscopic Scaling

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Abstract We investigate stochastic (conservative) non-equilibrium jump dynamics of interacting particles in continuum. The corresponding evolutions of correlation functions are constructed. The mesoscopic scaling (Vlasov scaling) of the dynamics is studied and the corresponding kinetic equations for the particle densities are derived.

Keywords Interacting particle system · Jump dynamics · Non-equilibrium evolution · Vlasov scaling · Kinetic equation

1 Introduction

Usually, an infinite group of identical particles with different locations in \mathbb{R}^d can be characterized by locally finite subset of \mathbb{R}^d , the points of which are

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positions of these particles. In such a case, the space of configurations of a continuous interacting particle system is given by

$$\Gamma \equiv \Gamma(\mathbb{R}^d) := \{ \gamma \subset \mathbb{R}^d \mid |\gamma \cap K| < \infty \text{ for any compact } K \subset \mathbb{R}^d \},$$

where $|\cdot|$ denotes the cardinality of a finite set. The elements of $\gamma = \{x_1, x_2, \dots\} \in \Gamma$ describe the locations of particles in \mathbb{R}^d .

In this paper, we discuss particular stochastic (conservative) jump dynamics in continuum. By general jump process in continuum we usually mean a process in which particles randomly hop over the space \mathbb{R}^d . More precisely, a large class of jump dynamics in continuum is given by the following mechanism: suppose that the system of identical particles in \mathbb{R}^d at a fixed moment of time is described by the configuration $\gamma \in \Gamma$, then each particle $x \in \gamma$ jumps during the infinitesimal time interval to a point $y \in \mathbb{R}^d$ according to a (probability) rate $c(x, y, \gamma)$. This leads to a change of the configuration γ to $(\gamma \setminus \{x\}) \cup \{y\}$. The transition rate $c(x, y, \gamma)$ contains all information about the evolution of the system. The concrete forms of these rates appear from modeling a particular system. In terms of observables, i.e., appropriate functions F on Γ , the general jump process is given heuristically by:

$$(LF)(\gamma) = \sum_{x \in \gamma} \int_{\mathbb{R}^d} dy c(x, y, \gamma) (F((\gamma \setminus \{x\}) \cup \{y\}) - F(\gamma)). \quad (1)$$

The question about construction of the Markov process with such generator seems to be the most natural for mathematicians. However, for the most cases appearing in applications, the description of the evolution model in terms of the classical Markov process seems to be too difficult. It is related with the complex structure of the space Γ which is an infinite dimensional space. On the other hand, the evolution of a state (i.e. probability measure on Γ) in the course of stochastic dynamics is an important question in its own right. The latter evolution does not give the description of the evolving system in terms of trajectories but still gives possibility to analyze system statistically. In the equilibrium case, where the initial state is invariant for the considered dynamics, the construction of the corresponding functional evolution (the time evolution of observables) can be realized for a quite general class of transition rates (via the Dirichlet form approach, see [28]). The construction of non-equilibrium dynamics is much more interesting (from the point of view of applications) and much more complex. Within the non-equilibrium framework it is possible to start the dynamics in a state which is "far away" from thermal equilibrium. The long time asymptotic behavior of such systems has a special significance. For a system of particles without interactions such questions were discussed in [30]. The evolution of states as well as mesoscopic scaling for the non-equilibrium Kawasaki dynamics in continuum has been thoroughly analyzed in [5] and [6].

The aim of the first part of the present paper is to construct and study the non-equilibrium evolution of states in terms of correlation functions ("statistical" process) corresponding to the generator (1) with transition rates of the following form:

$$c(x, y, \gamma) := a(x - y)(1 + \langle \gamma, c_{x,y} \rangle), \quad x, y \in \mathbb{R}^d, \gamma \in \Gamma, \quad (2)$$

where a is a non-negative even function in $L^1(dx)$ with $\|a\|_1 = 1$ and

$$\langle \gamma, c_{x,y} \rangle := \sum_{\tilde{x} \in \gamma} c_{x,y}(\tilde{x}).$$

Here, $c_{x,y}$ is a non-negative function in $L^1(dx)$ which depends on x and y . We will assume that $c_{x,y}$ is given by one of the following three expressions:

$$c_{x,y}(\tilde{x}) = \kappa(x - \tilde{x}), \quad (3)$$

$$c_{x,y}(\tilde{x}) = \kappa(y - \tilde{x}) \quad (4)$$

or

$$c_{x,y}(\tilde{x}) = \frac{1}{2}(\kappa(x - \tilde{x}) + \kappa(y - \tilde{x})), \quad (5)$$

where κ is a bounded, even, and non-negative function in $L^1(dx)$. The rates (2) are linear in γ and for $\kappa \equiv 0$ we obtain the free dynamics, i.e. dynamics of non-interacting particles. Such model was carefully analyzed, e.g., in [23]. In the case (5), the rate $c_{x,y}$ is symmetric in x and y . From now on, we will refer to this case as to the symmetric case. The free dynamics have a Poisson measure as equilibrium state. It turns out, that the symmetric case has this property as well. In this sense, the corresponding model describes particles with a "minimal" interaction.

The second part of this paper is devoted to a mesoscopic scaling (Vlasov scaling) for the considered microscopic models. The limit of the rescaled micro dynamics leads to a kinetic equation. It is worth noting that the Vlasov scaling coincides with the well-known Lebowitz-Penrose scaling (see e.g. [39]). This fact as well as the definitions of the corresponding scalings will be discussed in Section 8.

The complex evolution of a many-body system is often described (approximately) by kinetic equations, see e.g. [46], [47]. Besides the Boltzmann equation (which describes the evolution of the particle density of a dilute gas, see e.g. [33]), the Vlasov equation plays an important role in physics. This equation is a good approximation in situations where long range forces (i.e. forces caused by the collective effects of a large number of particles over relatively long distance) are present and short range forces (i.e. forces caused by collisions) are neglectable. Such circumstances are (approximately) valid in a plasma (due to long range Coulomb forces), see e.g. [20]. One can derive the Vlasov equation from the BBGKY-hierarchy by assuming that propagation of chaos holds, see e.g. [19]. In this situation the equation for the particle density is a closed equation since the second correlation function factorizes and this yields the Vlasov equation. In [8] the authors have shown that in the mean field scaling limit for Hamiltonian dynamics the empirical distribution of the particles has at any time $t > 0$ a Lebesgue density (if so, for $t = 0$) and this density satisfies a Vlasov-type equation. More general deterministic dynamical systems were considered in [9]. Note that the resulting Vlasov-type equations for particle densities are considered in the class of finite measures (in the weak form) or integrable functions (in the strong form). The latter implies, in fact, that we are restricted to the case of finite-volume systems or

systems with zero mean density in an infinite volume. A detailed analysis of Vlasov-type equations for integrable functions is presented in [32].

For the models considered in the present work, the approaches mentioned above are not applicable since a description in terms of proper stochastic evolution equations for particle motion is, generally speaking, absent. For that reason we have to follow in this work a general approach, proposed in [11], to study the Vlasov-type scaling for some classes of stochastic evolutions in continuum. The first step is to derive hierarchical equations for the evolution of correlation functions which generalizes the BBGKY-hierarchy from Hamiltonian to the dynamics considered here. Then, we perform the scaling, which, roughly speaking, ensures that on the one hand, the interaction gets weaker and on the other hand, the correlations between particles get stronger. The limiting hierarchy possesses a chaos preservation property. Namely, if we start with an initial correlation function which corresponds to a (non-homogeneous) Poisson state of the system, then this property will be preserved during the time evolution. This special property of the virtual Vlasov system allows us to derive a non-linear evolutionary equation for the evolving Poisson state which is the macroscopic Vlasov-type equation derived from the microscopic infinite-particle system. We remark, that we are working in an infinite volume with non-zero averaged density. The zero density case corresponds to a different physical situation of the underlying microscopic model, see e.g. [4].

2 General Facts and Notions

Let $\mathcal{B}(\mathbb{R}^d)$ be the family of all Borel sets in \mathbb{R}^d , $d \geq 1$; $\mathcal{B}_b(\mathbb{R}^d)$ denotes the system of all bounded sets from $\mathcal{B}(\mathbb{R}^d)$.

The space of n -point configurations in $Y \in \mathcal{B}(\mathbb{R}^d)$ is defined by

$$\Gamma_0^{(0)}(Y) := \{\emptyset\}, \quad \Gamma_0^{(n)}(Y) := \left\{ \eta \subset Y \mid |\eta| = n \right\}, \quad n \in \mathbb{N}.$$

Here $|\cdot|$ means the cardinality of a finite set. As a set, $\Gamma_0^{(n)}(Y)$ may be identified with the symmetrization of

$$\widetilde{Y}^n = \{(x_1, \dots, x_n) \in Y^n \mid x_k \neq x_l \text{ if } k \neq l\}.$$

Hence one can introduce the corresponding Borel σ -algebra, which we denote by $\mathcal{B}(\Gamma_0^{(n)}(Y))$. The space of finite configurations in $Y \in \mathcal{B}(\mathbb{R}^d)$ is defined as

$$\Gamma_0(Y) := \bigsqcup_{n \in \mathbb{N}_0} \Gamma_0^{(n)}(Y), \quad \mathbb{N}_0 := \mathbb{N} \cup \{0\}.$$

This space is equipped with the topology of the disjoint union. Let $\mathcal{B}(\Gamma_0(Y))$ denote the corresponding Borel σ -algebra. In the case of $Y = \mathbb{R}^d$ we will omit the index Y in the previously defined notation. Namely, $\Gamma_0 := \Gamma_0(\mathbb{R}^d)$, $\Gamma_0^{(n)} := \Gamma_0^{(n)}(\mathbb{R}^d)$.

The configuration space over space \mathbb{R}^d consists of all locally finite subsets (configurations) of \mathbb{R}^d . Namely,

$$\Gamma = \Gamma(\mathbb{R}^d) := \left\{ \gamma \subset \mathbb{R}^d \mid |\gamma_A| < \infty, \text{ for all } A \in \mathcal{B}_b(\mathbb{R}^d) \right\}, \quad (6)$$

where $\gamma_A := \gamma \cap A$. The space Γ is equipped with the vague topology, i.e. the smallest topology for which all mappings

$$\Gamma \ni \gamma \mapsto \langle \gamma, f \rangle := \sum_{x \in \gamma} f(x) \in \mathbb{R}$$

are continuous for any continuous function f on \mathbb{R}^d with compact support; note that the summation in $\sum_{x \in \gamma} f(x)$ is taken over finitely many points of γ that belongs to the support of f . In [24], it was shown that Γ with the vague topology is metrizable and becomes a Polish space (i.e. a complete separable metric space). Corresponding to this topology, the Borel σ -algebra $\mathcal{B}(\Gamma)$ is the smallest σ -algebra for which all mappings $\Gamma \ni \gamma \mapsto |\gamma_A| \in \mathbb{N}_0$ are measurable for any $A \in \mathcal{B}_b(\mathbb{R}^d)$.

The restriction of the Lebesgue product measure $(dx)^n$ to $(\Gamma_0^{(n)}, \mathcal{B}(\Gamma_0^{(n)}))$ we denote by $\sigma^{(n)}$. We set $\sigma^{(0)} := \delta_{\{\emptyset\}}$. The Lebesgue–Poisson measure λ_z on Γ_0 is defined by

$$\lambda_z := \sum_{n=0}^{\infty} \frac{z^n}{n!} \sigma^{(n)}, \quad z > 0. \quad (7)$$

For any $A \in \mathcal{B}_b(\mathbb{R}^d)$ the restriction of λ_z to $\Gamma(A) := \Gamma_0(A)$ will be also denoted by λ_z . Then $\lambda_z(\Gamma(A)) = e^{zm(A)}$, where $m(A)$ denotes the Lebesgue measure of $A \in \mathcal{B}_b(\mathbb{R}^d)$.

The space $(\Gamma, \mathcal{B}(\Gamma))$ is the projective limit of the family of spaces $\{(\Gamma(A), \mathcal{B}(\Gamma(A)))\}_{A \in \mathcal{B}_b(\mathbb{R}^d)}$. The Poisson measure π_z , $z > 0$ on $(\Gamma, \mathcal{B}(\Gamma))$ is given as the projective limit of the family of measures $\{\pi_z^A\}_{A \in \mathcal{B}_b(\mathbb{R}^d)}$, where $\pi_z^A := e^{-zm(A)} \lambda_z$ is the probability measure on $(\Gamma(A), \mathcal{B}(\Gamma(A)))$ (see e.g. [1] for details). If $z = 1$ it is customary to write λ and π rather than λ_1 and π_1 .

A set $M \in \mathcal{B}(\Gamma_0)$ is called bounded if there exists a $A \in \mathcal{B}_b(\mathbb{R}^d)$ and $N \in \mathbb{N}$ such that $M \subset \bigsqcup_{n=0}^N \Gamma_0^{(n)}(A)$. We denote the set of all bounded and measurable functions with bounded support by $B_{\text{bs}}(\Gamma_0)$, i.e. $G \in B_{\text{bs}}(\Gamma_0)$ if G is bounded and $G|_{\Gamma_0 \setminus M} \equiv 0$ for some bounded $M \in \mathcal{B}(\Gamma_0)$. Any $\mathcal{B}(\Gamma_0)$ -measurable function G on Γ_0 , in fact, is defined by a sequence of functions $\{G^{(n)}\}_{n \in \mathbb{N}_0}$, where $G^{(n)}$ is a $\mathcal{B}(\Gamma_0^{(n)})$ -measurable function on $\Gamma_0^{(n)}$. We also consider the set $\mathcal{F}_{\text{cyl}}(\Gamma)$ of all cylinder functions on Γ . Each $F \in \mathcal{F}_{\text{cyl}}(\Gamma)$ is characterized by the following property: $F(\gamma) = F(\gamma_A)$ for some $A \in \mathcal{B}_b(\mathbb{R}^d)$. Functions on Γ will be called *observables* whereas functions on Γ_0 will be called *quasi-observables*.

There exists mapping from $B_{\text{bs}}(\Gamma_0)$ into $\mathcal{F}_{\text{cyl}}(\Gamma)$, which plays the key role in our further considerations:

$$(KG)(\gamma) := \sum_{\eta \in \gamma} G(\eta), \quad \gamma \in \Gamma, \quad (8)$$

where $G \in \mathcal{B}_{\text{bs}}(\Gamma_0)$, see e.g. [22], [34], [35]. The summation in (8) is taken over all finite subconfigurations $\eta \in \Gamma_0$ of the (infinite) configuration $\gamma \in \Gamma$; we denote this by the symbol, $\eta \Subset \gamma$. The mapping K is linear, positivity preserving, and invertible, with

$$(K^{-1}F)(\eta) := \sum_{\xi \subset \eta} (-1)^{|\eta \setminus \xi|} F(\xi), \quad \eta \in \Gamma_0.$$

The so-called coherent state corresponding to a $\mathcal{B}(\mathbb{R}^d)$ -measurable function f is defined by

$$e_\lambda(f, \eta) := \prod_{x \in \eta} f(x), \quad \eta \in \Gamma_0 \setminus \{\emptyset\}, \quad e_\lambda(f, \emptyset) := 1. \quad (9)$$

Then

$$(Ke_\lambda(f))(\eta) = e_\lambda(f + 1, \eta), \quad \eta \in \Gamma_0 \quad (10)$$

and for any $f \in L^1(\mathbb{R}^d, dx)$

$$\int_{\Gamma_0} e_\lambda(f, \eta) d\lambda(\eta) = \exp\left\{\int_{\mathbb{R}^d} f(x) dx\right\}. \quad (11)$$

Let $\mathcal{M}_{\text{fm}}^1(\Gamma)$ be the set of all probability measures μ which have finite local moments of all orders, i.e. $\int_{\Gamma} |\gamma_A|^n \mu(d\gamma) < \infty$ for all $A \in \mathcal{B}_b(\mathbb{R}^d)$ and $n \in \mathbb{N}$. A measure ρ on Γ_0 is called locally finite iff $\rho(M) < \infty$ for all bounded sets $M \in \mathcal{B}(\Gamma_0)$. The set of such measures is denoted by $\mathcal{M}_{\text{lf}}(\Gamma_0)$. One can define a transform $K^* : \mathcal{M}_{\text{fm}}^1(\Gamma) \rightarrow \mathcal{M}_{\text{lf}}(\Gamma_0)$ which is dual to the K -transform, i.e. for every $\mu \in \mathcal{M}_{\text{fm}}^1(\Gamma)$, $G \in \mathcal{B}_{\text{bs}}(\Gamma_0)$ holds

$$\int_{\Gamma} KG(\gamma) \mu(d\gamma) = \int_{\Gamma_0} G(\eta) (K^* \mu)(d\eta).$$

The measure $\rho_\mu := K^* \mu$ is called the correlation measure of μ . If ρ_μ has a density with respect to (w.r.t. for short) the Lebesgue-Poisson measure λ i.e. $d\rho_\mu = k_\mu d\lambda$, the functions

$$k_\mu^{(n)} : (\mathbb{R}^d)^n \rightarrow \mathbb{R}_+, \quad n \in \mathbb{N},$$

$$k_\mu^{(n)}(x_1, \dots, x_n) := \begin{cases} k_\mu(\{x_1, \dots, x_n\}) & \text{if } (x_1, \dots, x_n) \in \widetilde{(\mathbb{R}^d)^n} \\ 0 & \text{otherwise.} \end{cases}$$

are the well-known correlation functions of statistical physics, see e.g. [42], [43]. As shown in [22], for $\mu \in \mathcal{M}_{\text{fm}}^1(\Gamma)$ and $G \in L^1(\Gamma_0, \rho_\mu)$, the series

$$KG(\eta) := \sum_{\eta \in \gamma} G(\eta)$$

is μ -a.s. absolutely convergent. Furthermore, $KG \in L^1(\Gamma, \mu)$ and

$$\int_{\Gamma} KG(\gamma) \mu(d\gamma) = \int_{\Gamma_0} G(\eta) (K^* \mu)(d\eta).$$

Thus, we can extend the K -transform to a mapping

$$K_\mu : L^1(\Gamma_0, d\rho_\mu) \rightarrow L^1(\Gamma, d\mu). \quad (12)$$

The following lemma will play a crucial role in many computations (cf. [31]):

Lemma 1 (Minlos lemma) *Let $n \in \mathbb{N}$, $n \geq 2$ be given. Then*

$$\begin{aligned} \int_{\Gamma_0} \cdots \int_{\Gamma_0} G(\eta_1 \cup \dots \cup \eta_n) H(\eta_1, \dots, \eta_n) \lambda(d\eta_1) \cdots \lambda(d\eta_n) \\ = \int_{\Gamma_0} G(\eta) \sum_{(\eta_1, \dots, \eta_n) \in \mathcal{P}_n^0(\eta)} H(\eta_1, \dots, \eta_n) \lambda(d\eta), \end{aligned} \quad (13)$$

for all positive measurable functions $G : \Gamma_0 \mapsto \mathbb{R}$ and $H : \Gamma_0 \times \dots \times \Gamma_0 \mapsto \mathbb{R}$. Here $\mathcal{P}_n^0(\eta)$ denotes the family of all ordered partitions of η into n parts, which may be empty.

3 Hierarchical equations

In this section we derive the hierarchical equations for the jump dynamics given by the transition rates (2). These equations are the analog of the BBGKY-hierarchy for Hamiltonian dynamics and describe the time evolution of correlation functions. According to (1), the dynamics, acting on observables, i.e., appropriate functions $F : \Gamma \mapsto \mathbb{R}$, is generated by the following heuristic ‘operator’

$$(LF)(\gamma) = \sum_{x \in \gamma} \int_{\mathbb{R}^d} dy a(x-y) (1 + \langle \gamma, c_{x,y} \rangle) (F((\gamma \setminus \{x\}) \cup \{y\}) - F(\gamma)), \quad (14)$$

where $c_{x,y}$ is given by (3), (4), or (5). By abuse of notation we continue to write x for $\{x\}$ when no confusion can arise. The mathematically rigorous scheme to derive the evolution equations for correlation functions determined by the mechanism (14) was proposed, e.g., in [12, 25]. It is not our purpose to repeat all steps of this scheme here. Below we introduce and study objects which are necessary for the construction of evolution of correlation function as well as for the mesoscopic description of the system under consideration.

The evolution of the initial state $\mu_0 \in \mathcal{M}^1(\Gamma)$ of the system is informally given by the solution to the following Cauchy problem

$$\frac{d}{dt} \int_{\Gamma} F(\gamma) d\mu_t(\gamma) = \int_{\Gamma} LF(\gamma) d\mu_t(\gamma), \quad t > 0, \quad (15)$$

$$\mu_t|_{t=0} = \mu_0.$$

As it was shown in [13], the corresponding evolution of the correlation functions is given by

$$\frac{\partial k_t}{\partial t} = L^\Delta k_t, \quad k_t|_{t=0} = k_0, \quad (16)$$

where k_0 is the correlation function of measure μ_0 and L^Δ is the dual operator to $\widehat{L} := K^{-1}LK$ with respect to the duality

$$\int_{\Gamma_0} \lambda(d\eta) (\widehat{L}G)(\eta)k(\eta) = \int_{\Gamma_0} \lambda(d\eta) G(\eta)(L^\Delta k)(\eta). \quad (17)$$

The hierarchical structure of (16) is described by the countable infinite system of equations

$$\frac{\partial}{\partial t} k_t^{(n)} = (L^\Delta k_t)^{(n)}, \quad k_t^{(n)} := k_t|_{\Gamma_0^{(n)}}, \quad (L^\Delta k_t)^{(n)} := (L^\Delta k_t)|_{\Gamma_0^{(n)}}, \quad n \in \mathbb{N}.$$

It was also shown in [13] that the operator \widehat{L} on $B_{\text{bs}}(\Gamma_0)$ is given by the following formula

$$\begin{aligned} (\widehat{L}G)(\eta) &= \sum_{x \in \eta} \int_{\mathbb{R}^d} dy a(x-y)(G((\eta \setminus x) \cup y) - G(\eta)) \\ &\quad + \sum_{x \in \eta} \sum_{\tilde{x} \in \eta \setminus x} \int_{\mathbb{R}^d} dy a(x-y) c_{x,y}(\tilde{x})(G((\eta \setminus \{x, \tilde{x}\}) \cup y) - G(\eta \setminus \tilde{x})) \\ &\quad + \sum_{x \in \eta} \sum_{\tilde{x} \in \eta \setminus x} \int_{\mathbb{R}^d} dy a(x-y) c_{x,y}(\tilde{x})(G((\eta \setminus x) \cup y) - G(\eta)). \end{aligned} \quad (18)$$

Moreover,

$$\begin{aligned} (L^\Delta k)(\eta) &= \sum_{y \in \eta} \int_{\mathbb{R}^d} d\tilde{x} \int_{\mathbb{R}^d} dx k(\tilde{x} \cup (\eta \setminus y) \cup x) a(x-y) c_{x,y}(\tilde{x}) \\ &\quad - \int_{\mathbb{R}^d} d\tilde{x} k(\eta \cup \tilde{x}) \sum_{x \in \eta} \int_{\mathbb{R}^d} dy a(x-y) c_{x,y}(\tilde{x}) \\ &\quad + \sum_{y \in \eta} \int_{\mathbb{R}^d} dx k((\eta \setminus y) \cup x) (a(x-y) + \sum_{\tilde{x} \in \eta \setminus y} a(x-y) c_{x,y}(\tilde{x})) \\ &\quad - k(\eta) \sum_{x \in \eta} \int_{\mathbb{R}^d} dy (a(x-y) + a(x-y) \sum_{\tilde{x} \in \eta \setminus x} c_{x,y}(\tilde{x})), \quad \eta \in \Gamma_0. \end{aligned} \quad (19)$$

Our next goal will be to construct the time evolution of correlation functions for the model defined by the mechanism (14). Namely, we have to solve the Cauchy problem (16) for the proper class of initial correlation functions. The admissible class of initial correlation functions will be introduced later. For these purposes we use the following strategy: the direct analysis of (16) meets some technical difficulties which will be explained in the next section. We overcome them by solving a pre-dual w.r.t. the duality (17) initial value problem

$$\begin{cases} \frac{\partial}{\partial t} G_t = \widehat{L}G_t \\ G_t|_{t=0} = G_0, \end{cases} \quad (20)$$

where \widehat{L} is given by (18). The evolution dual to the solution of (20) w.r.t. (17) will be a weak solution to (16).

4 Evolution of Quasi-Observables

The evolution equation (16) is an analogue to the BBGKY-hierarchy for Hamiltonian dynamics, see e.g. [3], [7]. As in the case of (infinite) Hamiltonian dynamics, the computation of the n -th correlation function requires the knowledge of the $(n + 1)$ -th correlation function. But the pre-dual evolution equation (given by the symbol \widehat{L}), which describes the evolution of quasi-observables, has an advantage that the computation of the n -th component of a quasi-observable requires the knowledge of the components of order less than n . This makes a recursive computation of the evolution of the components of quasi-observables possible. The duality between quasi-observables and correlation functions allows us to transfer this evolution to correlation functions.

Let us rewrite equation (20) in components. Each function G on Γ_0 one can associate to a sequence $(G^{(n)})_{n \in \mathbb{N}_0}$ of symmetric functions $G^{(n)}$ on $(\mathbb{R}^d)^n$ by defining

$$G^{(n)} := G|_{\Gamma_0^{(n)}}, \quad G^{(0)} := G(\emptyset).$$

We refer to the sequence $(G^{(n)})_{n \in \mathbb{N}_0}$ as to components or coordinates of the function G . Then, equation (20) is equivalent to

$$\begin{cases} \frac{\partial}{\partial t} G_t^{(n)} = (\widehat{L}G_t)^{(n)} \\ G_t^{(n)}|_{t=0} = G_0^{(n)}, \quad n \in \mathbb{N}_0. \end{cases} \quad (21)$$

Moreover, according to (18) we have

$$(\widehat{L}G)^{(0)} = 0, \quad (\widehat{L}G)^{(n)} = D^{(n)}G^{(n)} + R^{(n-1)}G^{(n-1)}, \quad n \in \mathbb{N} \quad (22)$$

with

$$\begin{aligned} (D^{(n)}G^{(n)})(\eta) &= \sum_{x \in \eta} \int_{\mathbb{R}^d} dy a(x-y) (G^{(n)}((\eta \setminus x) \cup y) - G^{(n)}(\eta)) \\ &+ \sum_{x \in \eta} \sum_{\tilde{x} \in \eta \setminus x} \int_{\mathbb{R}^d} dy a(x-y) c_{x,y}(\tilde{x}) (G^{(n)}((\eta \setminus x) \cup y) - G^{(n)}(\eta)) \end{aligned} \quad (23)$$

and

$$\begin{aligned} (R^{(n-1)}G^{(n-1)})(\eta) &= \sum_{x \in \eta} \sum_{\tilde{x} \in \eta \setminus x} \int_{\mathbb{R}^d} dy a(x-y) c_{x,y}(\tilde{x}) \\ &\times (G^{(n-1)}((\eta \setminus \{x, \tilde{x}\}) \cup y) - G^{(n-1)}(\eta \setminus \tilde{x})), \quad \eta \in \Gamma_0^{(n)}. \end{aligned}$$

Now equation (20) is read in components as follows

$$\frac{\partial}{\partial t} G_t^{(n)} = D^{(n)}G_t^{(n)} + R^{(n-1)}G_t^{(n-1)}, \quad G_t^{(n)}|_{t=0} = G_0^{(n)}, \quad n \in \mathbb{N}. \quad (24)$$

Due to this observation the following strategy for the construction of a solution to (20) should be reasonable:

Fix $n \in \mathbb{N}$ and assume that $D^{(n)}$ generates a semigroup (in some proper Banach space). If $G_t^{(n-1)}$ is already known, the informal solution of the system (24) is given by

$$G_t^{(n)} = e^{tD^{(n)}} G_0^{(n)} + \int_0^t e^{(t-s)D^{(n)}} R^{(n-1)} G_s^{(n-1)} ds, \quad t > 0, \quad (25)$$

where the above integral is interpreted in Bochner's sense. Hence, given G_0 , we can compute the components of the solution G_t of (20) successively. In the following we realize this approach.

To solve equation (20) we need some preparations and we have to introduce the spaces where the solution will be localized at each moment of time t . As it was mentioned in the introduction the most interesting situation for applications is non-zero density systems. Hence, it is natural to consider evolution of correlation functions in L^∞ -spaces. As a result, we have to search for a solution to (20) in a proper L^1 -space. Especially, we will need the operators $D^{(n)}$ induce contraction semigroups in such L^1 -spaces. This program will be realized in the lemmas below. We set $X_n := L^1(\Gamma_0^{(n)}, d\sigma^{(n)})$, for $n \in \mathbb{N}$.

Lemma 2 *Let $c_{x,y}(\tilde{x}) = \kappa(x - \tilde{x})$ and*

$$\int_{\mathbb{R}^d} a(x-y)\kappa(x-\tilde{x})dx \leq \kappa(y-\tilde{x}). \quad (26)$$

Then the operator $D^{(n)}$ generates a contraction semigroup in X_n .

Proof: It is easily seen by means of the Minlos lemma (Lemma 1), that the operator $D^{(n)}$ is bounded in X_n . Thus, $D^{(n)}$ induces a semigroup in X_n . In order to show that this semigroup is contractive, we use the Lumer-Phillips Theorem, see [38]. Thus, we have to show

- there exists some $\lambda > 0$ such that the range of $D^{(n)} - \lambda \mathbf{1}$ is the whole space X_n ;
- for all $\lambda > 0$ holds $\|(D^{(n)} - \lambda \mathbf{1})G^{(n)}\| \geq \lambda \|G^{(n)}\|$, where $\|\cdot\|$ is the norm in X_n .

To the first point: this is obvious since $D^{(n)}$ is bounded and therefore the spectrum $\sigma(D^{(n)}) \subset B_{\|D^{(n)}\|}(0)$. Now to the second point: let $\lambda > 0$. We write $D^{(n)} = D_1^{(n)} + D_2^{(n)}$ with

$$(D_1^{(n)} G^{(n)})(\eta) := - \left(|\eta| + E^\kappa(\eta) \right) G^{(n)}(\eta),$$

where

$$E^\kappa(\eta) := \sum_{x \in \eta} \sum_{\tilde{x} \in \eta \setminus x} \kappa(x - \tilde{x}),$$

and

$$D_2^{(n)} := D^{(n)} - D_1^{(n)}.$$

Then

$$\begin{aligned}
\|(D^{(n)} - \lambda \mathbf{1})G^{(n)}\| &= \|(D_1^{(n)} + D_2^{(n)} - \lambda \mathbf{1})G^{(n)}\| \\
&\geq \|(D_1^{(n)} - \lambda \mathbf{1})G^{(n)}\| - \|D_2^{(n)}G^{(n)}\| \\
&= \int_{\Gamma_0^{(n)}} \left(\lambda + |\eta| + E^\kappa(\eta) \right) |G^{(n)}(\eta)| \sigma^{(n)}(d\eta) \\
&\quad - \|D_2^{(n)}G^{(n)}\|
\end{aligned} \tag{27}$$

Now we write $D_2^{(n)}$ as $D_2^{(n)} := D_{2,1}^{(n)} + D_{2,2}^{(n)}$ with

$$(D_{2,1}^{(n)}G^{(n)})(\eta) := \sum_{x \in \eta} \int_{\mathbb{R}^d} dy a(x-y) G^{(n)}((\eta \setminus x) \cup y)$$

and

$$(D_{2,2}^{(n)}G^{(n)})(\eta) = \sum_{x \in \eta} \sum_{\tilde{x} \in \eta \setminus x} \kappa(x - \tilde{x}) \int_{\mathbb{R}^d} dy a(x-y) G^{(n)}((\eta \setminus x) \cup y).$$

Obviously,

$$\|D_{2,1}^{(n)}\| \leq \int_{\Gamma_0^{(n)}} \sigma^{(n)}(d\eta) |\eta| |G^{(n)}(\eta)|. \tag{28}$$

To estimate the norm of $D_{2,2}^{(n)}$ we use Minlos lemma and (26)

$$\begin{aligned}
\|D_{2,2}^{(n)}\| &\leq \int_{\Gamma_0^{(n)}} \sigma^{(n)}(d\eta) \sum_{x \in \eta} \sum_{\tilde{x} \in \eta \setminus x} \kappa(x - \tilde{x}) \int_{\mathbb{R}^d} dy a(x-y) |G^{(n)}((\eta \setminus x) \cup y)| \\
&= \int_{\Gamma_0^{(n)}} \sigma^{(n)}(d\eta) \sum_{y \in \eta} \sum_{\tilde{x} \in \eta \setminus y} \int_{\mathbb{R}^d} dx a(x-y) \kappa(x - \tilde{x}) |G^{(n)}(\eta)| \\
&\leq \int_{\Gamma_0^{(n)}} \sigma^{(n)}(d\eta) E^\kappa(\eta) |G^{(n)}(\eta)|.
\end{aligned} \tag{29}$$

Combining (27), (28) and (29) we get

$$\|(D^{(n)} - \lambda \mathbf{1})G^{(n)}\| \geq \lambda \|G^{(n)}\|. \quad \square$$

Lemma 3 Let $c_{x,y}(\tilde{x}) = \kappa(y - \tilde{x})$ and

$$\int_{\mathbb{R}^d} a(x-y) \kappa(y - \tilde{x}) dy \geq \kappa(x - \tilde{x}). \tag{30}$$

Then the operator $D^{(n)}$ generates a contraction semigroup in X_n .

Proof: The proof is analogous to Lemma 2. \square

Lemma 4 Let $c_{x,y}(\tilde{x}) = \frac{1}{2}(\kappa(x - \tilde{x}) + \kappa(y - \tilde{x}))$. Then $D^{(n)}$ generates a contraction semigroup in X_n .

Proof: The symmetry of $c_{x,y}(\tilde{x})$ in x and y and Minlos lemma yields

$$\begin{aligned} & \| (D^{(n)} - \lambda \mathbf{1}) G^{(n)} \| \\ & \geq \left| \int_{\Gamma_0^{(n)}} \sigma^{(n)}(d\eta) \left[\sum_{x \in \eta} \int_{\mathbb{R}^d} dy a(x-y) (G^{(n)}(\eta \setminus x \cup y) - G^{(n)}(\eta)) - \lambda G^{(n)}(\eta) \right] \right| \\ & = \lambda \| G^{(n)} \|. \end{aligned}$$

The rest of the proof is analogous to Lemma 2. \square

Next we will investigate the properties of the operators $R^{(n-1)}$, which have to be regarded as operators from the Banach space X_{n-1} to the Banach space X_n . To this end we define for a symmetric function $G^{(n)}$ on $(\mathbb{R}^d)^n$ a function $\widetilde{G^{(n)}}$ on Γ_0 by

$$\widetilde{G^{(n)}}(\eta) = \begin{cases} G^{(n)}(x_1, \dots, x_n), & \text{if } \eta = \{x_1, \dots, x_n\} \\ 0, & \text{otherwise.} \end{cases}$$

Then, we can write the L^1 -norm of X_n as

$$\| G^{(n)} \|_{X_n} = n! \int_{\Gamma_0} \lambda(d\eta) |\widetilde{G^{(n)}}(\eta)|.$$

We continue to write $\|\cdot\|_{X_n}$ for the norm of X_n to emphasize that it depends on n . The next lemma is valid for all three cases of $c_{x,y}(\tilde{x})$.

Lemma 5 *The operator $R^{(n-1)} : X_{n-1} \rightarrow X_n$ is a linear continuous operator. Moreover,*

$$\| R^{(n-1)} G^{(n-1)} \|_{X_n} \leq 2 \|\kappa\|_1 n(n-1) \| G^{(n-1)} \|_{X_{n-1}}$$

for all $G^{(n-1)} \in X_{n-1}$.

Proof: We divide operator $R^{(n-1)}$ into 2 parts. Namely, for $G^{(n-1)} \in X_{n-1}$ we have

$$R^{(n-1)} G^{(n-1)}(\eta) = R_1^{(n-1)} G^{(n-1)}(\eta) + R_2^{(n-1)} G^{(n-1)}(\eta), \quad \eta \in \Gamma_0^{(n)}$$

with

$$(R_1^{(n-1)} G^{(n-1)})(\eta) := \sum_{x \in \eta} \sum_{\tilde{x} \in \eta \setminus x} \int_{\mathbb{R}^d} dy a(x-y) c_{x,y}(\tilde{x}) G^{(n-1)}(\eta \setminus \{x, \tilde{x}\} \cup y)$$

and

$$(R_2^{(n-1)} G^{(n-1)})(\eta) := - \sum_{x \in \eta} \sum_{\tilde{x} \in \eta \setminus x} \int_{\mathbb{R}^d} dy a(x-y) c_{x,y}(\tilde{x}) G^{(n-1)}(\eta \setminus \tilde{x}).$$

Repeated application of Minlos lemma enables us to write

$$\begin{aligned}
\|R_1^{(n-1)}G^{(n-1)}\|_{X_n} &\leq n! \int_{\Gamma_0} \lambda(d\eta) \mathbf{1}_{\Gamma_0^{(n)}}(\eta) \sum_{x \in \eta} \sum_{\tilde{x} \in \eta \setminus x} \int_{\mathbb{R}^d} dy a(x-y) \\
&\quad \times c_{x,y}(\tilde{x}) |\widetilde{G^{(n-1)}}(\eta \setminus \{x, \tilde{x}\} \cup y)| \\
&= n! \int_{\Gamma_0} \lambda(d\eta) \int_{\mathbb{R}^d} dx \mathbf{1}_{\Gamma_0^{(n)}}(\eta \cup x) \sum_{\tilde{x} \in \eta} \int_{\mathbb{R}^d} dy a(x-y) \\
&\quad \times c_{x,y}(\tilde{x}) |\widetilde{G^{(n-1)}}(\eta \setminus \tilde{x} \cup y)| \\
&= n! \int_{\Gamma_0} \lambda(d\eta) \int_{\mathbb{R}^d} d\tilde{x} \int_{\mathbb{R}^d} dx \mathbf{1}_{\Gamma_0^{(n-1)}}(\eta \cup \tilde{x}) \int_{\mathbb{R}^d} dy a(x-y) \\
&\quad \times c_{x,y}(\tilde{x}) |\widetilde{G^{(n-1)}}(\eta \cup y)| \\
&= n! \int_{\Gamma_0} \lambda(d\eta) \sum_{y \in \eta} \mathbf{1}_{\Gamma_0^{(n-2)}}(\eta \setminus y) \int_{\mathbb{R}^d} dx a(x-y) \\
&\quad \times \int_{\mathbb{R}^d} d\tilde{x} \frac{1}{2} (\kappa(x-\tilde{x}) + \kappa(y-\tilde{x})) |\widetilde{G^{(n-1)}}(\eta)| \\
&= \|\kappa\|_1 n! \int_{\Gamma_0} \lambda(d\eta) \sum_{y \in \eta} \mathbf{1}_{\Gamma_0^{(n-1)}}(\eta) \int_{\mathbb{R}^d} dx a(x-y) |\widetilde{G^{(n-1)}}(\eta)| \\
&= \|\kappa\|_1 n! \int_{\Gamma_0} \lambda(d\eta) \sum_{y \in \eta} \mathbf{1}_{\Gamma_0^{(n-1)}}(\eta) |\widetilde{G^{(n-1)}}(\eta)| \\
&= \|\kappa\|_1 n! \int_{\Gamma_0} \lambda(d\eta) \mathbf{1}_{\Gamma_0^{(n-1)}}(\eta) |\eta| |\widetilde{G^{(n-1)}}(\eta)| \\
&= \|\kappa\|_1 (n-1)n(n-1)! \int_{\Gamma_0} \lambda(d\eta) |\widetilde{G^{(n-1)}}(\eta)| \\
&= \|\kappa\|_1 n(n-1) \|G^{(n-1)}\|_{X_{n-1}}.
\end{aligned}$$

As a result,

$$\|R_1^{(n-1)}G^{(n-1)}\|_{X_n} \leq \|\kappa\|_1 n(n-1) \|G^{(n-1)}\|_{X_{n-1}}. \quad (31)$$

We continue in this fashion obtaining the estimate for $\|R_2^{(n-1)}G^{(n-1)}\|_{X_n}$:

$$\begin{aligned}
&\|R_2^{(n-1)}G^{(n-1)}\|_{X_n} \\
&\leq n! \int_{\Gamma_0} \lambda(d\eta) \mathbf{1}_{\Gamma_0^{(n)}}(\eta) \sum_{x \in \eta} \sum_{\tilde{x} \in \eta \setminus x} \int_{\mathbb{R}^d} dy a(x-y) c_{x,y}(\tilde{x}) |\widetilde{G^{(n-1)}}(\eta \setminus \tilde{x})| \\
&= n! \int_{\Gamma_0} \lambda(d\eta) \int_{\mathbb{R}^d} dx \mathbf{1}_{\Gamma_0^{(n)}}(\eta \cup x) \sum_{\tilde{x} \in \eta} \int_{\mathbb{R}^d} dy a(x-y) \\
&\quad \times c_{x,y}(\tilde{x}) |\widetilde{G^{(n-1)}}(\eta \setminus \tilde{x} \cup x)|
\end{aligned}$$

$$\begin{aligned}
&= n! \int_{\Gamma_0} \lambda(d\eta) \int_{\mathbb{R}^d} dx \mathbf{1}_{\Gamma_0^{(n-1)}}(\eta) \sum_{\tilde{x} \in \eta} \int_{\mathbb{R}^d} dy a(x-y) \\
&\quad \times c_{x,y}(\tilde{x}) |\widetilde{G^{(n-1)}}(\eta \setminus \tilde{x} \cup x)| \\
&= n! \int_{\Gamma_0} \lambda(d\eta) \mathbf{1}_{\Gamma_0^{(n-2)}}(\eta) \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy a(x-y) |\widetilde{G^{(n-1)}}(\eta \cup x)| \\
&\quad \times \int_{\mathbb{R}^d} d\tilde{x} \frac{1}{2} (\kappa(x-\tilde{x}) + \kappa(y-\tilde{x})) \\
&= n! \|\kappa\|_1 \int_{\Gamma_0} \lambda(d\eta) \mathbf{1}_{\Gamma_0^{(n-2)}}(\eta) \int_{\mathbb{R}^d} dx |\widetilde{G^{(n-1)}}(\eta \cup x)| \\
&= n! \|\kappa\|_1 \int_{\Gamma_0} \lambda(d\eta) \mathbf{1}_{\Gamma_0^{(n-1)}}(\eta) \sum_{x \in \eta} |\widetilde{G^{(n-1)}}(\eta)| \\
&= n(n-1) \|\kappa\|_1 \|G^{(n-1)}\|_{X_{n-1}}
\end{aligned}$$

i.e.

$$\|R_2^{(n-1)} G^{(n-1)}\|_{X_n} \leq \|\kappa\|_1 n(n-1) \|G^{(n-1)}\|_{X_{n-1}} \quad (32)$$

Altogether, we derive by means of (31) and (32):

$$\|R^{(n-1)} G^{(n-1)}\|_{X_n} \leq 2 \|\kappa\|_1 n(n-1) \|G^{(n-1)}\|_{X_{n-1}}. \quad (33)$$

The proof for the cases (3) and (4) is completely analogous to the case considered above. This shows the assertions of Lemma 5. \square

We are now able to construct the evolution of quasi-observables. Define for $C > 0$ a Banach space \mathcal{I}_C consisting of all functions $G = (G^{(n)})_{n \in \mathbb{N}} \in \bigoplus_{n \in \mathbb{N}} X_n$ for which

$$\|G\|_{\mathcal{I}_C} := \sup_{n \in \mathbb{N}} \frac{\|G^{(n)}\|_{X_n} C^n}{n!} < \infty. \quad (34)$$

Remark 1 It is immediate that $B_{\text{bs}}(\Gamma_0) \subset \mathcal{I}_C$ for any $C > 0$.

The next theorem concerns the evolution of quasi-observables. Since we are primarily interested in the evolution of correlation function, we formulate the theorem below in a way suitable to transport the dynamics of quasi-observables to correlation functions.

Theorem 1 *Let $C > 0$ be arbitrary and fixed. Consider the evolution problem*

$$\begin{cases} \frac{\partial}{\partial t} G_t = \widehat{L} G_t \\ G_t|_{t=0} = G_0, \end{cases} \quad (35)$$

where \widehat{L} is given by (18) and $G_0 \in \mathcal{I}_C$. Suppose that $c_{x,y}$ is given by (3), (4), or (5). In the case of (3) (respectively (4)) we additionally assume that property (26) (respectively (30)) is satisfied. Then the initial value problem (35) has a unique solution $G_t \in \mathcal{I}_C$, where

$$C_t := \frac{C}{1 + 2t \|\kappa\|_1 C}, \quad t \geq 0. \quad (36)$$

Moreover,

$$\|G_t\|_{\mathcal{I}_{C_t}} \leq \|G_0\|_{\mathcal{I}_C} \quad (37)$$

for all $t \geq 0$.

Proof: Let us rewrite the initial value problem (35) in terms of components as an infinite system of differential equations

$$\frac{\partial}{\partial t} G_t^{(n)} = D^{(n)} G_t^{(n)} + R^{(n-1)} G_t^{(n-1)}, \quad G_t^{(n)}|_{t=0} = G_0^{(n)}, \quad n \in \mathbb{N}.$$

This system may be solved recurrently. Namely, for each $n \in \mathbb{N}$

$$G_t^{(n)} = e^{tD^{(n)}} G_0^{(n)} + \int_0^t e^{(t-s)D^{(n)}} R^{(n-1)} G_s^{(n-1)} ds, \quad t > 0.$$

Set $B := 2\|\kappa\|_1$. We proof that for all $n \in \mathbb{N}$ holds:

$$\|G_t^{(n)}\|_{X_n} \leq \|G_0\|_{\mathcal{I}_C} \frac{n!}{C^n} \left(1 + tBC\right)^n, \quad t \geq 0. \quad (38)$$

Clearly, from (38) follows (37). Since $G_0 \in \mathcal{I}_C$, it follows

$$\|G_0^{(n)}\|_{X_n} \leq \|G_0\|_{\mathcal{I}_C} \frac{n!}{C^n}. \quad (39)$$

By iterating the formula

$$G_t^{(n)} = e^{tD^{(n)}} G_0^{(n)} + \int_0^t e^{(t-s)D^{(n)}} R^{(n-1)} G_s^{(n-1)} ds,$$

we obtain

$$G_t^{(n)} = \sum_{k=0}^n A_{k,n}(t) G_0^{(n-k)}$$

where

$$A_{k,n}(t) := \int_0^t \int_0^{s_1} \dots \int_0^{s_{k-1}} e^{(t-s_1)D^{(n)}} R^{(n-1)} e^{(s_1-s_2)D^{(n-1)}} R^{(n-2)} \dots \\ \times R^{(n-k)} e^{s_k D^{(n-k)}} ds_k \dots ds_1,$$

$$A_{0,n}(t) := e^{tD^{(n)}}.$$

By means of the contraction property of $D^{(n)}$ (Lemma 2 - Lemma 4) and Lemma 5 we derive:

$$\begin{aligned} \|A_{k,n}(t) G_0^{(n-k)}\|_{X_n} &\leq \frac{t^k}{k!} B^k n(n-1)(n-1)(n-2) \dots \\ &\quad \times (n-k+1)(n-k) \|G_0^{(n-k)}\|_{X_{n-k}} \\ &= \frac{t^k}{k!} B^k \frac{n!}{(n-k)! (n-k-1)!} \|G_0^{(n-k)}\|_{X_{n-k}}. \end{aligned}$$

Thus, we obtain

$$\begin{aligned} \|G_t^{(n)}\|_{X_n} &\leq \sum_{k=0}^n \frac{t^k B^k}{k!} \frac{n!}{(n-k)!} \frac{(n-1)!}{(n-k-1)!} \|G_0^{(n-k)}\|_{X_{n-k}} \\ &= \sum_{k=0}^n (tB)^{n-k} \frac{n!(n-1)!}{(n-k)!k!(k-1)!} \|G_0^{(k)}\|_{X_k}. \end{aligned}$$

Now we use (39) and get

$$\begin{aligned} \|G_t^{(n)}\|_{X_n} &\leq \|G_0\|_{\mathcal{I}_C} n! \sum_{k=0}^n (tB)^{n-k} \binom{n}{k} \frac{k}{n} C^{-k} \\ &\leq \|G_0\|_{\mathcal{I}_C} \frac{n!}{C^n} \sum_{k=0}^n (tBC)^{n-k} \binom{n}{k} \\ &= \|G_0\|_{\mathcal{I}_C} \frac{n!}{C^n} (1 + tBC)^n. \end{aligned}$$

This shows (38). \square

Remark 2 We can define a propagator \widehat{P}_t by

$$\widehat{P}_t : \mathcal{I}_C \rightarrow \mathcal{I}_{C_t}, \quad \widehat{P}_t G := G_t$$

where G_t is the solution of (35) with initial data G . This propagator describes the time evolution of quasi-observables. According to Theorem 1, the propagator \widehat{P}_t is a linear contraction operator from \mathcal{I}_C to \mathcal{I}_{C_t} for any $t > 0$.

5 Evolution of Correlation Functions

In this section we construct the time evolution for correlation functions. Since quasi-observables and correlation functions are dual to each other (in the same manner as observables and states are dual to each other), we will construct this evolution as dual evolution to quasi-observables.

The natural space in which the evolution of correlation functions takes place is the space

$$\mathcal{K}_C := \left\{ k : \Gamma_0 \rightarrow \mathbb{R} \mid k \cdot C^{-|\cdot|} \in L^\infty(\Gamma_0, d\lambda) \right\}, \quad C > 0,$$

cf. [25]. The space \mathcal{K}_C is the dual space to

$$\mathcal{L}_C := L^1(\Gamma_0, C^{|\cdot|} d\lambda),$$

where the duality is given by the following expression:

$$\langle\langle k, G \rangle\rangle := \int_{\Gamma_0} k \cdot G \, d\lambda, \quad G \in \mathcal{L}_C.$$

It is clear that \mathcal{K}_C is a Banach space with the norm

$$\|k\|_C := \|k C^{-|\cdot|}\|_{L^\infty(\Gamma_0, d\lambda)}.$$

Note also, that $k \cdot C^{-|\cdot|} \in L^\infty(\Gamma_0, d\lambda)$ means that the function k satisfies the bound

$$|k(\eta)| \leq \text{const } C^{|\eta|}, \quad \lambda - \text{a.e.}$$

For $\alpha \in (0, 1)$ we prove the following inclusions

$$\mathcal{L}_C \subset \mathcal{I}_C \subset \mathcal{L}_{\alpha C}. \quad (40)$$

Indeed, it holds firstly:

$$\frac{(\alpha C)^n}{n!} \|G^{(n)}\|_{X_n} \leq \alpha^n \|G\|_{\mathcal{I}_C}, \quad n \in \mathbb{N}, \quad G \in \mathcal{I}_C,$$

which implies

$$\|G\|_{\mathcal{L}_{\alpha C}} \leq \frac{1}{1-\alpha} \|G\|_{\mathcal{I}_C} < \infty,$$

and hence

$$\mathcal{I}_C \subset \mathcal{L}_{\alpha C}.$$

Secondly, it holds for $G \in \mathcal{L}_C$

$$\|G\|_{\mathcal{I}_C} \leq \|G\|_{\mathcal{L}_C},$$

consequently

$$\mathcal{L}_C \subset \mathcal{I}_C.$$

As a result, we have (40). We also consider a functional space \mathcal{J}_C which consists of all functions $k = (k^{(n)})_{n \in \mathbb{N}} \in \bigoplus_{n \in \mathbb{N}} X_n^*$, $X_n^* := L^\infty(\Gamma_0^{(n)}, d\sigma^{(n)})$ for which

$$\|k\|_{\mathcal{J}_C} := \sum_{n=0}^{\infty} C^{-n} \|k^{(n)}\|_{X_n^*} < \infty$$

holds. Let $G \in \mathcal{I}_C$ and $k \in \mathcal{J}_C$. It follows

$$\begin{aligned} |\langle k, G \rangle| &\leq \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\Gamma_0^{(n)}} |k^{(n)}| |G^{(n)}| d\sigma^{(n)} \\ &\leq \sum_{n=0}^{\infty} \frac{1}{n!} \|G^{(n)}\|_{X_n} \|k^{(n)}\|_{X_n^*} \\ &= \sum_{n=0}^{\infty} \frac{C^n}{n!} \|G^{(n)}\|_{X_n} C^{-n} \|k^{(n)}\|_{X_n^*} \\ &\leq \left(\sup_{n \in \mathbb{N}} \left[\|G^{(n)}\|_{X_n} \frac{C^n}{n!} \right] \right) \left(\sum_{n=0}^{\infty} C^{-n} \|k^{(n)}\|_{X_n^*} \right) \\ &= \|G\|_{\mathcal{I}_C} \|k\|_{\mathcal{J}_C}. \end{aligned} \quad (41)$$

Hence, $G \mapsto \langle \langle k, G \rangle \rangle$ is a bounded linear functional on \mathcal{I}_C and therefore

$$\mathcal{J}_C \subset (\mathcal{I}_C)^*.$$

Next, we observe that for $k \in \mathcal{J}_C$ holds:

$$C^{-n} \|k^{(n)}\|_{X_n^*} \leq \|k\|_{\mathcal{J}_C}, \quad n \in \mathbb{N}.$$

It follows

$$\|k\|_{\mathcal{K}_C} \leq \|k\|_{\mathcal{J}_C}$$

which implies

$$\mathcal{J}_C \subset \mathcal{K}_C$$

Conversely, for $k \in \mathcal{K}_C$ holds

$$\|k^{(n)}\|_{X_n^*} \leq C^n \|k\|_{\mathcal{K}_C}, \quad n \in \mathbb{N}.$$

Using this inequality, for $\alpha \in (0, 1)$ we conclude

$$\|k\|_{\mathcal{J}_{\frac{\alpha}{C}}} = \sum_{n=0}^{\infty} \frac{\alpha^n}{C^n} \|k^{(n)}\|_{X_n^*} \leq \sum_{n=0}^{\infty} \frac{\alpha^n}{C^n} C^n \|k\|_{\mathcal{K}_C} = \frac{1}{1-\alpha} \|k\|_{\mathcal{K}_C}.$$

Therefore,

$$\mathcal{K}_C \subset \mathcal{J}_{\frac{\alpha}{C}} \subset \mathcal{K}_{\frac{\alpha}{C}}. \quad (42)$$

Further by means of (40), we get

$$\mathcal{K}_C \subset (\mathcal{I}_{\frac{\alpha}{C}})^* \subset \mathcal{K}_{\frac{\alpha}{C}}. \quad (43)$$

Summarizing, for $\alpha \in (0, 1)$

$$\mathcal{K}_C \subset \mathcal{J}_{\frac{\alpha}{C}} \subset (\mathcal{I}_{\frac{\alpha}{C}})^* \subset \mathcal{K}_{\frac{\alpha}{C}}.$$

Having disposed of these preliminary steps, we can now turn to the construction of the evolution of correlation functions:

Theorem 2 *Let $C > 0$, $\alpha \in (0, 1)$ be arbitrary and fixed. Define a time horizon $T > 0$ by*

$$T := \frac{\alpha}{2\|\kappa\|_1 C}.$$

Let $k_0 \in \mathcal{K}_C$. Then for all $t < T$ there exists unique $k_t \in (\mathcal{I}_{C_t^\alpha})^ \subset \mathcal{K}_{C_t^\alpha}$, with*

$$C_t^\alpha := \frac{C}{\alpha - 2t\|\kappa\|_1 C}, \quad (44)$$

such that for all $G \in \mathcal{I}_{C_t^\alpha}$ it holds

$$\langle \langle k_t, G \rangle \rangle = \langle \langle k_0, G_t \rangle \rangle,$$

where G_t is the solution $(G_s)_{s \geq 0}$ of (35) with initial data $G_0 = G$ evaluated at $s = t$.

Proof: Let $k_0 \in \mathcal{K}_C \subset (\mathcal{I}_{\underline{C}})^*$ and $t < T$. Since $G \in \mathcal{I}_{C_t^\alpha}$ it follows by Theorem 1 that G_t is an element of \mathcal{I}_{C_t} where

$$C_t = \frac{C_t^\alpha}{1 + 2t\|\kappa\|_1 C_t^\alpha}.$$

The definition of C_t^α together with a simple calculation shows that $C_t = C/\alpha$. Since $k_0 \in \mathcal{K}_C \subset \mathcal{J}_{\underline{C}}$ and $G_t \in \mathcal{I}_{\underline{C}}$, we obtain by means of (37) and (41):

$$|\langle\langle k_0, G_t \rangle\rangle| \leq \|k_0\|_{\mathcal{J}_{\underline{C}}} \|G_t\|_{\mathcal{I}_{\underline{C}}} \leq \|k_0\|_{\mathcal{J}_{\underline{C}}} \|G\|_{\mathcal{I}_{\alpha, C_t^\alpha}}.$$

Therefore, the mapping $G \mapsto \langle\langle k_0, G_t \rangle\rangle = \langle\langle k_0, \widehat{P}_t G \rangle\rangle$ is a linear continuous functional on the space $\mathcal{I}_{C_t^\alpha}$. Hence, there exists $k_t \in (\mathcal{I}_{C_t^\alpha})^*$ such that, for any $G \in \mathcal{I}_{C_t^\alpha}$

$$\langle\langle k_t, G \rangle\rangle = \langle\langle k_0, G_t \rangle\rangle = \langle\langle k_0, \widehat{P}_t G \rangle\rangle.$$

But since $(\mathcal{I}_{C_t^\alpha})^* \subset \mathcal{K}_{C_t^\alpha}$ (cf. (43)), we can regard $k_0 \mapsto k_t$ as a mapping

$$P_t^\Delta : \mathcal{K}_C \rightarrow \mathcal{K}_{C_t^\alpha}. \quad \square \quad (45)$$

Remark 3 The evolution $k_t := P_t^\Delta k_0$, $t \in [0, T]$ describes the time evolution of the initial correlation function k_0 . We can regard the mapping P_t^Δ as the dual propagator to \widehat{P}_t (cf. Remark 2), because it holds

$$\langle\langle P_t^\Delta k_0, G \rangle\rangle = \langle\langle k_0, G_t \rangle\rangle = \langle\langle k_0, \widehat{P}_t G \rangle\rangle.$$

6 Weak Solution

Now, let us consider our model of jumping particles and an initial function $k_0 \in \mathcal{K}_C$, $C > 0$.

Proposition 1 *Let $T = \frac{\alpha}{2\|\kappa\|_1 C}$, $\alpha \in (0, 1)$ and $(k_t)_{t \in [0, T]}$ be the evolution constructed in Theorem 2. Let $(G_s)_{s \geq 0}$ be the solution to (35) with initial data $G_0 = G \in B_{\text{bs}}(\Gamma_0)$. Then $\langle\langle k_t, G \rangle\rangle$ is differentiable on $(0, T)$ and*

$$\frac{d}{dt} \langle\langle k_t, G \rangle\rangle = \langle\langle k_t, \widehat{L}G \rangle\rangle.$$

In particular, it means that $(k_t)_{t \in (0, T)}$ is a weak solution to the following initial value problem

$$\begin{cases} \frac{\partial}{\partial t} k_t = L^\Delta k_t \\ k_t|_{t=0} = k_0 \in \mathcal{K}_C. \end{cases}$$

Remark 4 It is worth noting that $B_{\text{bs}}(\Gamma_0) \subset \mathcal{I}_{C_t^\alpha}$ (cf. Remark 1), where C_t^α is given by the formula (44).

Proof: Let us show that $\langle\langle k_0, G_t \rangle\rangle = \langle\langle k_t, G \rangle\rangle$ is differentiable on $(0, T)$. An easy computation shows that

$$\|D^{(n)}G^{(n)}\|_{X_n} \leq 2n((n-1)\|\kappa\|_\infty + 1)\|G^{(n)}\|_{X_n}, \quad n \in \mathbb{N},$$

where $\|\kappa\|_\infty$ stands for the L^∞ - norm in \mathbb{R}^d . We now apply Lemma 2 - Lemma 5 to estimate the following expression:

$$\begin{aligned} & \int_0^t \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} |k_0^{(n)}(x_1, \dots, x_n)| |(\widehat{L}G_s)^{(n)}(x_1, \dots, x_n)| dx_1 \dots dx_n ds \\ & \leq \|k_0\|_C \int_0^t \sum_{n=0}^{\infty} \frac{C^n}{n!} \|(\widehat{L}G_s)^{(n)}\|_{X_n} ds \\ & \leq \|k_0\|_C \int_0^t \sum_{n=0}^{\infty} \frac{C^n}{n!} (2n((n-1)\|\kappa\|_\infty + 1)\|G_s^{(n)}\|_{X_n} \\ & \quad + 2\|\kappa\|_1 n(n-1)\|G_s^{(n-1)}\|_{X_{n-1}}) ds. \end{aligned}$$

On account of Theorem 1 and Remark 4 we have $G_s \in \mathcal{I}_{C_s}$ with $C_s = \frac{C_t^\alpha}{1+2s\|\kappa\|_1 C_t^\alpha}$. Therefore

$$\|G_s^{(n)}\|_{X_n} \leq \|G_s\|_{\mathcal{I}_{C_s}} n! (C_s)^{-n} \leq \|G\|_{\mathcal{I}_{C_t^\alpha}} n! (C_s)^{-n}.$$

Using this bound, we conclude:

$$\begin{aligned} & \int_0^t \sum_{n=0}^{\infty} \frac{C^n}{n!} (2n((n-1)\|\kappa\|_\infty + 1)\|G_s^{(n)}\|_{X_n} + 2\|\kappa\|_1 n(n-1)\|G_s^{(n-1)}\|_{X_{n-1}}) ds \\ & \leq \|G\|_{\mathcal{I}_{C_t^\alpha}} \int_0^t \left(\sum_{n=0}^{\infty} 2n((n-1)\|\kappa\|_\infty + 1) \left(\frac{C}{C_s}\right)^n \right. \\ & \quad \left. + 2\|\kappa\|_1 C \sum_{n=0}^{\infty} (n-1) \left(\frac{C}{C_s}\right)^{n-1} \right) ds < \infty. \end{aligned}$$

In the last step we have used $\frac{C}{C_s} \leq \alpha < 1$. This is true since C_s is decreasing to $C_t = C/\alpha$ (cf. the proof of Theorem 2). Therefore, we obtain by means of Fubini's theorem:

$$\begin{aligned} \langle\langle k_t, G \rangle\rangle &= \langle\langle k_0, G_t \rangle\rangle \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} k_0^{(n)}(x_1, \dots, x_n) \left(G_0^{(n)}(x_1, \dots, x_n) \right. \\ & \quad \left. + \int_0^t (\widehat{L}G_s)^{(n)}(x_1, \dots, x_n) \right) dx_1 \dots dx_n ds \\ &= \langle\langle k_0, G \rangle\rangle + \int_0^t \langle\langle k_0, \widehat{L}G_s \rangle\rangle ds. \end{aligned} \tag{46}$$

It is easy to check that $\langle\langle k_0, \widehat{L}G_s \rangle\rangle$ as a series converges uniformly on $[0, t]$, which is due to the previous estimates. Hence, $\langle\langle k_0, \widehat{L}G_s \rangle\rangle$ is continuous in s since all components of the operator \widehat{L} are bounded operators. Finally $\langle\langle k_t, G \rangle\rangle$ is differentiable. Moreover,

$$\frac{d}{dt}\langle\langle k_t, G \rangle\rangle = \langle\langle k_0, \widehat{L}G_t \rangle\rangle, \quad G \in B_{\text{bs}}(\Gamma_0).$$

It is worth pointing out that if G belongs to the space $\mathcal{I}_{C_t^\alpha}$ and $G \notin B_{\text{bs}}(\Gamma_0)$ then G would depend implicitly on t and hence the relation (46) does not hold uniformly in t . What is left is to show that for $G \in B_{\text{bs}}(\Gamma_0)$

$$\widehat{L}G_t = \widehat{L}\widehat{P}_t G = \widehat{P}_t \widehat{L}G. \quad (47)$$

It is very well known that a bounded linear operator commutes with its semigroup. The proof of the relation (47) is based on the similar idea. One should modify it to the multicomponent case which is straightforward taking into account that initial function G is finitely based (it has finite number of non-zero components) and the operator \widehat{L} is bounded in components. \square .

7 Invariant Distribution

The arguments proposed in this section are only valid for the symmetric case, cf. (5). Let us start our symmetric jumping dynamics with a Poisson state $\mu_0(d\gamma) = \pi_z(d\gamma)$, $z > 0$. The initial correlation function $k_0 = k_{\pi_z}$ of π_z is given by

$$k_0(\eta) = z^{|\eta|}, \quad \eta \in \Gamma_0.$$

Let us fix $t \in [0, T]$ and $G \in \mathcal{I}_{C_t^\alpha}$ (the constant C in the definition of C_t^α is now equal to z). As in (46), we get:

$$\langle\langle k_t, G \rangle\rangle = \langle\langle k_0, G \rangle\rangle + \int_0^t \int_{\Gamma_0} \lambda(d\eta) z^{|\eta|} (\widehat{L}G_s)(\eta). \quad (48)$$

Using the symmetry of $c_{x,y}(\tilde{x})$ in x and y and Milnos Lemma, we conclude

$$\int_{\Gamma_0} \lambda(d\eta) z^{|\eta|} (\widehat{L}G_s)(\eta) = 0. \quad (49)$$

Combining (48) and (49) yields:

$$\langle\langle k_t, G \rangle\rangle = \langle\langle k_0, G \rangle\rangle, \quad G \in \mathcal{I}_{\alpha, C_t^*}.$$

Thus, $k_t = k_0$, $t \in [0, T]$. This shows that, the time evolution of the jump dynamics leaves the Poisson measure π_z invariant.

Commonly, the measure π_z refers to the free case, i.e. a dynamic without interaction. In a physical context, the measure π_z describes an ideal gas in equilibrium with activity $z > 0$ which depends on the chemical potential. Since the jump dynamics still have the state π_z as invariant measure, we have an interacting-particles system with a minimal interaction.

8 Vlasov Scaling

This section is devoted to the study of the Vlasov (mesoscopic) scaling. The main idea of this scaling is to make the particle system more dense whereas the interaction (repulsion) respectively weaker. This corresponds to the so called mean field approximation widely employed in theoretical physics.

In our particular case, we consider the model of jumping particles with rates (2), but we replace the function κ by $\epsilon\kappa$. Here, $\epsilon > 0$ is a scaling parameter which describes the strength of the interaction. We are interested in the behavior of the system when ϵ tends to 0. For small ϵ , the interaction becomes weaker. To compensate this, we perform an additional scaling, which ensures that the correlations between the particles become stronger.

A detailed description of the Vlasov scaling for stochastic dynamics of continuous system is given in [11]. For the convenience of the reader we propose below some informal arguments for a particular class of initial distributions to motivate the scaling, thus making our exposition self-contained: assume that we are interested in the time evolution of an initial Poisson state π_ϱ w.r.t. a (inhomogeneous) density ϱ , i.e. we have to study the evolution of the corresponding correlation function $k_{\pi_\varrho}(\eta) = e_\lambda(\varrho, \eta)$. Now, let L_ϵ^Δ be the operator given by (19) with κ replaced by $\epsilon\kappa$. Set $P_{t,\epsilon}^\Delta$ to be the corresponding evolution operator of correlation functions defined by (45). In order to make the correlations stronger, we replace the density ϱ by $\epsilon^{-1}\varrho$ (i.e. the system becomes more dense). Then $e_\lambda(\epsilon^{-1}\varrho, \eta) = (R_\epsilon e_\lambda(\varrho, \cdot))(\eta)$ where $(R_\epsilon k)(\eta) := \epsilon^{-|\eta|}k(\eta)$. Now, we let this dense system with weak interaction evolve, i.e. we consider $(e^{tL_\epsilon^\Delta} R_\epsilon)e_\lambda(\varrho, \cdot)$, where $e^{tL_\epsilon^\Delta}$ is a heuristic notation for $P_{t,\epsilon}^\Delta$. Afterwards we reverse the effect of increasing the density, i.e. we consider

$$(R_\epsilon^{-1} e^{tL_\epsilon^\Delta} R_\epsilon)e_\lambda(\varrho, \cdot) = (e^{tR_\epsilon^{-1}L_\epsilon^\Delta R_\epsilon})e_\lambda(\varrho, \cdot).$$

Motivated by these heuristic calculations, we introduce an operator

$$L_{\epsilon,\text{ren}}^\Delta := R_\epsilon^{-1}L_\epsilon^\Delta R_\epsilon. \quad (50)$$

It describes (for small ϵ) a dense and weakly interacting system. Clearly, on quasi-observables, we have to consider the operator

$$\hat{L}_{\epsilon,\text{ren}} := R_\epsilon \hat{L}_\epsilon R_\epsilon^{-1}.$$

Remark 5 As mentioned in the introduction, the Vlasov scaling limit coincides with the Lebowitz-Penrose scaling limit. Let \tilde{L}_ϵ be the evolutionary operator L^Δ with a and κ replaced by $\epsilon^d a(\epsilon \cdot)$ and $\epsilon^d \kappa(\epsilon \cdot)$, respectively. For a function k , we define an operator $S_\epsilon k$ by

$$(S_\epsilon k)^{(n)}(x_1, \dots, x_n) := k^{(n)}(\epsilon x_1, \dots, \epsilon x_n).$$

The rescaled dynamics according to the Lebowitz-Penrose scaling (cf. [39]) are described by the generator $L_{\epsilon,\text{LP}}$, which is defined by

$$L_{\epsilon,\text{LP}} = S_\epsilon^{-1} \tilde{L}_\epsilon S_\epsilon.$$

An easy computation shows that $L_{\epsilon,\text{LP}} = L_{\epsilon^d,\text{ren}}^\Delta$.

In the next section we analyze the Vlasov scaling limit on quasi-observables, later we transport this scaling to correlation functions.

8.1 Scaling of Quasi-Observables

We observe that the components of the operator $\widehat{L}_{\epsilon, \text{ren}}$ are given by

$$(\widehat{L}_{\epsilon, \text{ren}}G)^{(n)} = D_{\epsilon}^{(n)}G^{(n)} + R^{(n-1)}G^{(n-1)} \quad (51)$$

where $D_{\epsilon}^{(n)}$ is given by

$$\begin{aligned} & (D_{\epsilon}^{(n)}G^{(n)})(\eta) \\ &= \sum_{x \in \eta} \int_{\mathbb{R}^d} dy a(x-y)(G^{(n)}(\eta \setminus x \cup y) - G^{(n)}(\eta)) \\ &+ \epsilon \sum_{x \in \eta} \sum_{\tilde{x} \in \eta \setminus x} \int_{\mathbb{R}^d} dy a(x-y)c_{x,y}(\tilde{x})(G^{(n)}(\eta \setminus x \cup y) - G^{(n)}(\eta)), \end{aligned}$$

i.e. $D_{\epsilon}^{(n)}$ is given by (23) but κ is replaced by $\epsilon\kappa$. We also introduce an operator \widehat{L}_V by

$$\begin{aligned} (\widehat{L}_V G)(\eta) &= \sum_{x \in \eta} \int_{\mathbb{R}^d} dy a(x-y)(G(\eta \setminus x \cup y) - G(\eta)) \\ &+ \sum_{x \in \eta} \sum_{\tilde{x} \in \eta \setminus x} \int_{\mathbb{R}^d} dy a(x-y)c_{x,y}(\tilde{x})(G(\eta \setminus \{x, \tilde{x}\} \cup y) - G(\eta \setminus \tilde{x})). \end{aligned}$$

Clearly, the components of L_V are given by

$$(L_V G)^{(n)} = L_0^{(n)}G^{(n)} + R^{(n-1)}G^{(n-1)}$$

where L_0 is the generator of free jump dynamics w.r.t. the kernel a (cf. [23]). The operator \widehat{L}_V is the strong component-wise limit of $\widehat{L}_{\epsilon, \text{ren}}$ for $\epsilon \rightarrow 0$, i.e. the following holds: for any $G \in X_n$

$$(\widehat{L}_{\epsilon, \text{ren}}G)^{(n)} \rightarrow (\widehat{L}_V G)^{(n)} \quad \text{in } X_n \quad \text{for } \epsilon \rightarrow 0. \quad (52)$$

Since both $L_0^{(n)}$ and $D_{\epsilon}^{(n)}$ induce contraction semigroups in X_n , we can solve the following Cauchy problems for a proper class of initial functions

$$\begin{cases} \frac{\partial}{\partial t} G_{t,\epsilon} = \widehat{L}_{\epsilon, \text{ren}} G_{t,\epsilon} \\ G_{t,\epsilon}|_{t=0} = G_0. \end{cases} \quad (53)$$

respectively

$$\begin{cases} \frac{\partial}{\partial t} G_{t,V} = L_V G_{t,V} \\ G_{t,V}|_{t=0} = G_0. \end{cases} \quad (54)$$

To be more precise, the statements of Theorem 1 also hold for the above mentioned evolutionary problems.

Theorem 3 Let $\epsilon, C > 0$ be arbitrary and fixed. Consider the evolution problems (53) and (54) with $G_0 \in \mathcal{I}_C$. Suppose that $c_{x,y}$ is given by (3), (4), or (5). In the case of (3) (respectively (4)) we additionally assume that property (26) (respectively (30)) is satisfied. Then the initial value problems (53) and (54) have unique solutions $G_{t,\epsilon}, G_{t,V} \in \mathcal{I}_{C_t}$, where

$$C_t := \frac{C}{1 + 2t\|\kappa\|_1 C}, \quad t \geq 0. \quad (55)$$

Moreover,

$$\|G_{t,\epsilon}\|_{\mathcal{I}_{C_t}} \leq \|G_0\|_{\mathcal{I}_C}, \quad \|G_{t,V}\|_{\mathcal{I}_{C_t}} \leq \|G_0\|_{\mathcal{I}_C} \quad (56)$$

for all $t \geq 0$.

Now we can state the following

Theorem 4 Let $G_0 \in \mathcal{I}_C$ and consider the solutions $(G_{t,\epsilon})_{t>0}, (G_{t,V})_{t>0}$ of (53) resp. (54) with initial condition G_0 . Then, it holds for all $n \in \mathbb{N}$ and $t > 0$:

$$\left\| G_{t,\epsilon}^{(n)} - G_{t,V}^{(n)} \right\|_{X_n} \rightarrow 0, \quad \epsilon \rightarrow 0.$$

Proof: Using the representation of $G_{t,\epsilon}^{(n)}, G_{t,V}^{(n)}$, by the recurrent relation, we conclude

$$\begin{aligned} & \left\| G_{t,\epsilon}^{(n)} - G_{t,V}^{(n)} \right\|_{X_n} \\ & \leq \left\| e^{tD_\epsilon^{(n)}} G_0^{(n)} - e^{tL_0^{(n)}} G_0^{(n)} \right\|_{X_n} \\ & \quad + \left\| \int_0^t e^{(t-s)D_\epsilon^{(n)}} R^{(n-1)} G_{s,\epsilon}^{(n-1)} - e^{(t-s)L_0^{(n)}} R^{(n-1)} G_{s,V}^{(n-1)} ds \right\|_{X_n} \\ & \leq \left\| e^{tD_\epsilon^{(n)}} G_0^{(n)} - e^{tL_0^{(n)}} G_0^{(n)} \right\|_{X_n} \\ & \quad + \left\| \int_0^t e^{(t-s)D_\epsilon^{(n)}} R^{(n-1)} (G_{s,\epsilon}^{(n-1)} - G_{s,V}^{(n-1)}) ds \right\|_{X_n} \\ & \quad + \left\| \int_0^t e^{(t-s)D_\epsilon^{(n)}} R^{(n-1)} G_{s,V}^{(n-1)} - e^{(t-s)L_0^{(n)}} R^{(n-1)} G_{s,V}^{(n-1)} ds \right\|_{X_n}. \end{aligned}$$

Since $\lim_{\epsilon \rightarrow 0} D_\epsilon^{(n)} = L_0^{(n)}$ strongly in X_n and since $D_\epsilon^{(n)}, L_0^{(n)}$ are the generators of contraction semigroups in X_n , it follows (see [38])

$$\lim_{\epsilon \rightarrow 0} e^{tD_\epsilon^{(n)}} G^{(n)} = e^{tL_0^{(n)}} G^{(n)} \quad \text{in } X_n.$$

By the same arguments we get

$$\lim_{\epsilon \rightarrow 0} e^{(t-s)D_\epsilon^{(n)}} R^{(n-1)} G_{s,V}^{(n-1)} = e^{(t-s)L_0^{(n)}} R^{(n-1)} G_{s,V}^{(n-1)} \quad \text{in } X_n.$$

Further, using the bound

$$\begin{aligned}
& \|e^{(t-s)D_\epsilon^{(n)}} R^{(n-1)} G_{s,V}^{(n-1)} - e^{(t-s)L_0^{(n)}} R^{(n-1)} G_{s,V}^{(n-1)}\|_{X_n} \\
& \leq 2\|R^{(n-1)} G_{s,V}^{(n-1)}\|_{X_n} \\
& \leq 4\|\kappa\|_1 n(n-1) \|G_{s,V}^{(n-1)}\|_{X_{n-1}} \\
& \leq 4\|\kappa\|_1 n!(n-1) C_s^{-(n-1)} \|G_0\|_{\mathcal{I}_C} \in L^1([0, t], ds)
\end{aligned}$$

and dominated convergence theorem, we obtain

$$\lim_{\epsilon \rightarrow 0} \left\| \int_0^t e^{(t-s)D_\epsilon^{(n)}} R^{(n-1)} G_{s,V}^{(n-1)} - e^{(t-s)L_0^{(n)}} R^{(n-1)} G_{s,V}^{(n-1)} ds \right\|_{X_n} = 0.$$

Similar arguments yield

$$\lim_{\epsilon \rightarrow 0} \left\| \int_0^t e^{(t-s)D_\epsilon^{(n)}} R^{(n-1)} (G_{s,\epsilon}^{(n-1)} - G_{s,V}^{(n-1)}) ds \right\|_{X_n} = 0,$$

if we suppose that, for any $s \in [0, t]$

$$\lim_{\epsilon \rightarrow 0} \|G_{s,\epsilon}^{(n-1)} - G_{s,V}^{(n-1)}\|_{X_{n-1}} = 0$$

holds. Note that, for any $s \in [0, t]$ we have

$$|G_{s,\epsilon}^{(0)} - G_{s,V}^{(0)}| = |G_{0,\epsilon}^{(0)} - G_{0,V}^{(0)}| = 0.$$

Thus, the statement follows by induction. \square

8.2 Scaling Limit for Correlation Functions

We are now in a position to investigate the Vlasov scaling limit for correlation functions. We consider the operator $L_{\epsilon, \text{ren}}^\Delta$ (see (50)). The limiting operator $L_V^\Delta := \lim_{\epsilon \rightarrow 0} L_{\epsilon, \text{ren}}^\Delta$ is given by

$$\begin{aligned}
(L_V^\Delta k)(\eta) &= \sum_{x \in \eta} \int_{\mathbb{R}^d} d\tilde{x} \int_{\mathbb{R}^d} dy k((\eta \setminus x) \cup \{y, \tilde{x}\}) a(x-y) c_{x,y}(\tilde{x}) \\
&\quad - \sum_{x \in \eta} \int_{\mathbb{R}^d} d\tilde{x} k(\eta \cup \tilde{x}) \int_{\mathbb{R}^d} dy a(x-y) c_{x,y}(\tilde{x}) \\
&\quad + \sum_{x \in \eta} \int_{\mathbb{R}^d} dy a(x-y) (k((\eta \setminus x) \cup y) - k(\eta)).
\end{aligned}$$

We can now proceed analogously as in Theorem 2 to construct evolutions $k_{t,\epsilon}$ and r_t which are dual to the evolutions $G_{t,\epsilon}$ and $G_{t,V}$, respectively, for $t \in [0, T)$. We stress that the time interval is independent of ϵ .

Theorem 5 Let $C > 0$, $\alpha \in (0, 1)$ be arbitrary and fixed. Define a time horizon $T > 0$ by

$$T := \frac{\alpha}{2\|\kappa\|_1 C}.$$

Let $k_0 \in \mathcal{K}_C$. Then for all $t < T$ there exists $k_{t,\epsilon}$ (respectively r_t) $\in (\mathcal{I}_{C_t^\alpha})^* \subset \mathcal{K}_{C_t^\alpha}$, where

$$C_t^\alpha := \frac{C}{\alpha - 2t\|\kappa\|_1 C},$$

such that for all $G \in \mathcal{I}_{C_t^\alpha}$ it holds

$$\langle \langle k_{t,\epsilon}, G \rangle \rangle = \langle \langle k_0, G_{t,\epsilon} \rangle \rangle, \quad \langle \langle r_t, G \rangle \rangle = \langle \langle k_0, G_{t,V} \rangle \rangle,$$

where $G_{t,\epsilon}$ (respectively $G_{t,V}$) are the solution to (53) (respectively (54)) with initial data $G_0 = G$.

Moreover, the following convergence result holds.

Theorem 6 Let $k_0 \in \mathcal{K}_C$ for some $C > 0$. Then, for each $t \in [0, T)$, $k_{t,\epsilon}$ converges weakly to r_t as $\epsilon \rightarrow 0$, i.e.

$$\lim_{\epsilon \rightarrow 0} \langle \langle k_{t,\epsilon}, G \rangle \rangle = \langle \langle r_t, G \rangle \rangle$$

for all $G \in \mathcal{I}_{C_t^\alpha}$.

Proof: Let $G \in \mathcal{I}_{C_t^\alpha}$. Then, according to Theorem 1, $G_{t,\epsilon}, G_{t,V} \in \mathcal{I}_{C/\alpha}$ (cf. also the proof of Theorem 2). Hence

$$\begin{aligned} |\langle \langle k_{t,\epsilon}, G \rangle \rangle - \langle \langle r_t, G \rangle \rangle| &= |\langle \langle k_0, G_{t,\epsilon} \rangle \rangle - \langle \langle k_0, G_{t,V} \rangle \rangle| \\ &\leq \sum_{n=0}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}^d)^n} |k_0^{(n)}(x_1, \dots, x_n)| \\ &\quad \times |G_{t,\epsilon}^{(n)}(x_1, \dots, x_n) - G_{t,V}^{(n)}(x_1, \dots, x_n)| dx_1 \dots dx_n \\ &\leq \|k_0\|_C \sum_{n=0}^{\infty} \frac{C^n}{n!} \int_{(\mathbb{R}^d)^n} |G_{t,\epsilon}^{(n)}(x_1, \dots, x_n) - G_{t,V}^{(n)}(x_1, \dots, x_n)| dx_1 \dots dx_n \\ &= \|k_0\|_C \sum_{n=0}^{\infty} \frac{C^n}{n!} \|G_{t,\epsilon}^{(n)} - G_{t,V}^{(n)}\|_{X_n}. \end{aligned}$$

Using that $G_{t,\epsilon}, G_{t,V} \in \mathcal{I}_{C/\alpha}$ we obtain

$$\frac{C^n}{n!} \|G_{t,\epsilon}^{(n)} - G_{t,V}^{(n)}\|_{X_n} \leq 2\|G\|_{\mathcal{I}_{C_t^\alpha}} \alpha^n$$

Since $\alpha < 1$, the right side of the above estimate is summable. This completes the proof. \square

8.3 Vlasov Equation

Now we consider a coherent state $k_0 = e_\lambda(\varrho, \cdot)$ as initial correlation function. Here, ϱ is a (bounded) one particle density in \mathbb{R}^d . We choose $C > 0$, such that, $\varrho \leq C$ holds. According to Theorem 2, we can construct evolutions $k_{t,\epsilon}$ and r_t of $k_0 = e_\lambda(\varrho, \cdot)$ under the dynamics described by $\widehat{L}_{\epsilon, \text{ren}}$ and L_V , respectively. Moreover, according to Section 6, $r_t = \lim_{\epsilon \rightarrow 0} k_{t,\epsilon}$ is a weak solution of the equation

$$\begin{cases} \frac{\partial}{\partial t} r_t = L_V^\Delta r_t \\ r_t|_{t=0} = e_\lambda(\varrho, \cdot). \end{cases} \quad (57)$$

This Cauchy problem describes the time evolution of a virtual interacting particle system and has the following chaos preservation property: if ϱ_t is a solution to the following non-local nonlinear initial value problem

$$\begin{cases} \frac{\partial}{\partial t} \varrho_t = v(\varrho_t) \\ \varrho_t|_{t=0} = \varrho, \end{cases}$$

where, in the case of (3), $v(\varrho_t)$ is equal to

$$v(\varrho_t) = (\kappa * \varrho_t)(x)(a * \varrho_t)(x) - \langle a \rangle \varrho_t(x)(\kappa * \varrho_t)(x) + (\varrho_t * a)(x) - \langle a \rangle \varrho_t(x),$$

while in the case of (4), it is equal to

$$v(\varrho_t) = \left(((\kappa * \varrho_t)\varrho_t) * a \right)(x) - \varrho_t(x)(a * (\kappa * \varrho_t))(x) + (\varrho_t * a)(x) - \langle a \rangle \varrho_t(x)$$

and in the symmetric case (cf. (5)), $v(\varrho_t)$ is given by

$$\begin{aligned} v(\varrho_t) &= \frac{1}{2} \left(((\kappa * \varrho_t)\varrho_t) * a \right)(x) + \frac{1}{2} (\kappa * \varrho_t)(x)(a * \varrho_t)(x) + (\varrho_t * a)(x) \\ &\quad - \frac{1}{2} \varrho_t(x)(a * (\kappa * \varrho_t))(x) - \frac{1}{2} \langle a \rangle \varrho_t(x)(\kappa * \varrho_t)(x) - \langle a \rangle \varrho_t(x), \end{aligned}$$

then $r_t = e_\lambda(\varrho_t, \cdot)$ is a solution to (57). Here $\langle a \rangle = \|a\|_1 = 1$. This result can easily be obtained by using the formula

$$\frac{\partial}{\partial t} e_\lambda(\varrho_t, \eta) = \sum_{x \in \eta} e_\lambda(\varrho_t, \eta \setminus x) \frac{\partial}{\partial t} \varrho_t(x),$$

see also [11]. Thus, we have derived a mesoscopic (deterministic) kinetic equation from the microscopic (stochastic) particle evolution. The infinite linear chain of equations (57) reduces to one non-linear equation for ϱ_t .

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9 Compliance with Ethical Standards

Conflict of Interest: The authors declare that they have no conflict of interest.

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