14.126 (Game Theory) Final Examination

Instructions: This is an open-book exam – you may consult written material but you may not consult other humans. There are four questions, weighted equally. You have 24 hours to complete the exam from the time you first open the envelope.

Please begin your answer to each question on a new page.

1. Two players bargain over time in order to take a joint decision $x = (x_1, x_2) \in [0,1]^2$. Player i's "ideal point" is $\bar{x}^i \in [0,1]^2$, his payoff from reaching the decision x at time $t = 1, 2, \ldots$ is:

   $$u_i(x,t) = \delta_i^{t-1}[1 - (x_1 - \bar{x}_1^i)^2 - \gamma(x_2 - \bar{x}_2^i)^2]$$

   and his payoff from disagreement is 0. Assume that $0 < \gamma < 1$, $0 < \delta_i < 1$, $\bar{x}^1 = (1,0)$, and $\bar{x}^2 = (0,1)$.

   The bargaining process is as follows. In odd periods, player 1 makes an offer $x_1 \in [0,1]$, player 2 may either reject the offer in which case we proceed to the next period, or he may accept and choose $x_2 \in [0,1]$ in which case the decision is (x_1, x_2) and the game is over. In even periods, player 2 makes an offer $x_1 \in [0,1]$, player 1 may either reject the offer in which case we proceed to the next period, or he may accept and choose $x_2 \in [0,1]$ in which case the decision is (x_1, x_2) and the game is over.

   1. Verify that in SPE, at a subgame where player i just offered x_1, if player j accepts then he chooses $x_2 = \bar{x}_2^j$. Draw the achievable utility pairs U^i if j accepts some offer $x_1 \in [0,1]$ of i:

      $$U^i = \left\{ \left(u_i(x_1, \bar{x}_2^j, 0), u_j(x_1, \bar{x}_2^j, 0)\right) \in \mathbb{R}^2 \ : \ x_1 \in [0,1] \right\}$$

      for $i = 1, 2$ and $j \neq i$.

2. Find an SPE and argue that it is unique. You do not need to solve for the equilibrium strategies explicitly, a graphical argument using U^1 and U^2 suffices.

3. What happens to the SPE payoffs as $(\delta_1, \delta_2) \to (1,1)$? Again a graphical argument suffices. How do you contrast this limiting result to the SPE payoffs of the standard alternating offers bargaining model as $(\delta_1, \delta_2) \to (1,1)$?

2. Consider a partnership game with two players, who invest in a public good project at each date $t \in T = \{0, 1, 2, \ldots\}$ without observing each other's previous investments. We assume that a strategy of a player i is any function $x_i : T \to [0,1]$, where $x_i(t)$ is the investment level of i at $t \in T$. The payoff of of a player i is

$$U_i(x_1, x_2) = \sum_{t \in T} \delta^t \left[A f(x_1(t), x_2(t)) - c_i(x_i(t), t) \right]$$

where $\delta \in (0,1)$, $A \in [0,1]$ is a productivity parameter, $f : [0,1]^2 \to \mathbb{R}$ is a supermodular, increasing, and continuous production function, and c_i is a time dependent cost function for player i. Everything is common knowledge.

1. Show that the above game have equilibria $\underline{x}$ and $\bar{x}$ such that for each equilibrium x of this game,

$$\underline{x}_i(t) \le x_i(t) \le \bar{x}_i(t) \qquad (\forall i, t).$$

2. Let X be the set of all equilibria of this game. Construct an incomplete-information model in which (i) it is common knowledge that each player is rational and (ii) a strategy profile x is played at some state ω if and only if $x \in X$.

3. Show that, if $A \ge A'$, then the extremal equilibria for these parameters satisfy

$$\underline{x}_i(t; A) \ge \underline{x}_i(t; A') \text{ and } \bar{x}_i(t; A) \ge \bar{x}_i(t; A') \quad (\forall i, t).$$

4. Consider a strategy x_i with $x_i(0) > \bar{x}_i(0)$. Can you construct an incomplete-information model such that (i) each player is rational at each state and (ii) x_i is played by player i at some state?

3. Consider a finitely-repeated game, where players observe all previous moves, do not discount the future payoffs, and the stage game is repeated T times.

   1. At each stage t, the following game is played

	a	b
a	θ_t, θ_t	$\theta_t - 1, 0$
b	$0, \theta_t - 1$	0,0

      where $\theta_t = 1/3$. Show that for any feasible payoff vector v with $v \gg (0,0)$ and any $\varepsilon > 0$, there exists $\bar{T}$ such that, for each $T > \bar{T}$, there exists an equilibrium in which the average payoff of each player i is in ε-neighborhood of v_i.

   2. Suppose that in the stage game above, θ_t is a random variable with uniform distribution on $[-\infty, \infty]$, and at each t, each player i observes a signal $x_{i,t} = \theta_t + \varepsilon_{i,t}$ where $\varepsilon_{i,t}$ is distributed with $N(0,1)$ and $\{\theta_t, \varepsilon_{i,t} | i, t\}$ are all stochastically independent. Moreover players tremble so that at each history each player plays each action with at least probability ϵ, where ϵ is a very small but positive number. Everything above is common knowledge. Compute all the rationalizable strategies in the repeated game.

4. Consider the Cournot duopoly where the market price is given by $P(q_1 + q_2) = \max\{0, \theta - (q_1 + q_2)\}$. Firms have zero marginal costs, so the profits are $\pi_i = P(q_1 + q_2)q_i$. The intercept θ takes the value $\underline{\theta}$ with probability 1/2 and $\bar{\theta}$ with probability 1/2, where $0 < \underline{\theta} < \bar{\theta}$ and $2\underline{\theta} > \bar{\theta}$. Each firm i may produce $q_i \in [0, \bar{\theta})$. There is an additional dimension of uncertainty, W, the weather in Boston, which is either s(unny), or n(ot sunny). Firms just know that W is correlated with θ but nothing more, i.e. their set of priors over the joint uncertainty is given by:

$$\mathcal{P} = \left\{ P \in \Delta\left(\{\underline{\theta}, \bar{\theta}\} \times \{s, n\}\right) \; : \; \begin{array}{c} P(\underline{\theta}, s) + P(\underline{\theta}, n) = P(\bar{\theta}, s) + P(\bar{\theta}, n) = 1/2 \\ \& \; P(\underline{\theta}, s) \neq P(\bar{\theta}, s) \end{array} \right\}.$$

Firms use the maxmin criterion to evaluate uncertain profits, and upon receiving information they update their set of priors by using the Full Bayesian criterion[1]. Neither of them observes θ. Before the two firms engage in

[1] Just as in Kajii and Ui (2004), the Full Bayesian criterion means that the updating rule yields the set of posteriors from *every* possible prior.

Cournot competition, firm 1 has the option to observe W privately[2] and costlessly. Firm 2 can not observe W.

1. A pure strategy profile is an SPE if each player, at each one of his information sets, chooses an optimal action given his set of beliefs at that information set, taking into account his and his opponent's *possibly time inconsistent* future behavior. Compute the unique SPE in pure strategies.
2. Suppose instead that the firms had a single prior P, for some $P \in \mathcal{P}$. Does firm 1 observe W in the unique SPE in pure strategies? Briefly contrast this with what you found in (a).
3. How do your answers to (a) and (b) change if the market demand is known to be independent of the Boston weather, i.e. if you replace $\mathcal{P}$ above by

$$\mathcal{P}' = \left\{ P \in \Delta\left(\{\underline{\theta}, \bar{\theta}\} \times \{s, n\}\right) \; : \; \begin{array}{c} P(\underline{\theta}, s) + P(\underline{\theta}, n) = P(\bar{\theta}, s) + P(\bar{\theta}, n) = 1/2 \\ \& \; P(\underline{\theta}, s) = P(\bar{\theta}, s) \end{array} \right\}?$$

[2] That is, firm 2 does not observe whether firm 1 observes W.

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