14.126 GAME THEORY

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1. EXISTENCE AND CONTINUITY OF NASH EQUILIBRIA

Follow Muhamet’s slides. We need the following result for future reference.

**Theorem 1.** Suppose that each $S_i$ is a convex and compact subset of an Euclidean space and that each $u_i$ is continuous in $s$ and quasi-concave in $s_i$. Then there exists a pure strategy Nash equilibrium.

2. BAYESIAN GAMES

When some players are uncertain about the characteristics or types of others, the game is said to have incomplete information. Most often a player’s type is simply defined by his payoff function. More generally, types may embody any private information that is relevant to players’ decision making. This may include, in addition to the player’s payoff function, his beliefs about other players’ payoff functions, his beliefs about what other players believe his beliefs are, and so on. The idea that a situation in which players are unsure about each other’s payoffs and beliefs can be modeled as a Bayesian game, in which a player’s type encapsulates all his uncertainty, is due to Harsanyi (1967, 1968) and has been formalized by Mertens and Zamir (1985). For simplicity, we assume that a player’s type is his own payoff and the type captures all the private information.

A Bayesian game is a list $\mathcal{B} = (N, S, \Theta, u, p)$ with

- $N = \{1, 2, \ldots, n\}$ is a finite set of players
- $S_i$ is the set of pure strategies of player $i$; $S = S_1 \times \ldots \times S_n$
- $\Theta_i$ is the set of types of player $i$; $\Theta = \Theta_1 \times \ldots \times \Theta_n$

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• $u_i : \Theta \times S \to \mathbb{R}$ is the payoff function of player $i$; $u = (u_1, \ldots, u_n)$
• $p \in \Delta(\Theta)$ is a common prior (we can relax this assumption).

We often assume that $\Theta$ is finite and the marginal $p_i(\theta_i)$ is positive for each type $\theta_i$.

Example 1 (First Price Auction with I.I.D. Private Values). One object is up for sale. Suppose that the value $\theta_i$ of player $i \in N$ for the object is uniformly distributed in $\Theta_i = [0, 1]$ and that the values are independent across players. This means that if $\bar{\theta}_i \in [0, 1], \forall i$ then $p(\theta_i \leq \bar{\theta}_i, \forall i) = \prod_i \bar{\theta}_i$. Each player $i$ submits a bid $s_i \in S_i = [0, \infty)$. The player with the highest bid wins the object and pays his bid. Ties are broken randomly. Hence the payoffs are given by

$$u_i(\theta, s) = \begin{cases} \frac{\theta_i - s_i}{|\{j \in N | s_i = s_j\}|} & \text{if } s_i \geq s_j, \forall j \in N \\ 0 & \text{otherwise.} \end{cases}$$

Example 2 (An exchange game). Each player $i = 1, 2$ receives a ticket on which there is a number in some finite set $\Theta_i \subset [0, 1]$. The number on a player’s ticket represents the size of a prize he may receive. The two prizes are independently distributed, with the value on $i$’s ticket distributed according to $F_i$. Each player is asked independently and simultaneously whether he wants to exchange his prize for the other player’s prize, hence $S_i = \{\text{agree, disagree}\}$. If both players agree then the prizes are exchanged; otherwise each player receives his own prize. Thus the payoff of player $i$ is

$$u_i(\theta, s) = \begin{cases} \theta_{3-i} & \text{if } s_1 = s_2 = \text{agree} \\ \theta_i & \text{otherwise.} \end{cases}$$

In the normal form representation $G(\mathcal{B})$ of the Bayesian game $\mathcal{B}$ player $i$ has strategies $(s_i(\theta_i))_{\theta_i \in \Theta_i} \in \mathcal{S}_i^{\Theta_i}$ and utility function $U_i$ given by

$$U_i((s_i(\theta_i))_{\theta_i \in \Theta_i, i \in N}) = \mathbb{E}_p(u_i(\theta, s_1(\theta_1), \ldots, s_n(\theta_n))).$$

The agent-normal form representation $AG(\mathcal{B})$ of the Bayesian game $\mathcal{B}$ has player set $\cup_i \Theta_i$. The strategy space of each player $\theta_i$ is $S_i$. A strategy profile $(s_{\theta_i})_{\theta_i \in \Theta_i, i \in N}$ yields utility

$$U_{\theta_i}((s_{\theta_i})_{\theta_i \in \Theta_i, i \in N}) = \mathbb{E}_p(u_i(\theta, s_{\theta_1}, \ldots, s_{\theta_n})|\theta_i)$$

for player $\theta_i$. For the conditional expectation to be well-defined we need $p_i(\theta_i) > 0$. 
Definition 1. A Bayesian Nash equilibrium of $\mathcal{B}$ is a Nash equilibrium of $G(\mathcal{B})$.

Proposition 1. If $p_i(\theta_i) > 0$ for all $\theta_i \in \Theta_i, i \in N$, a strategy profile is a Nash equilibrium of $G(\mathcal{B})$ iff it is a Nash equilibrium of $AG(\mathcal{B})$ (strategies are mapped across the two games by $s_i(\theta_i) \to s_{\theta_i}$).

Theorem 2. Suppose that
- $N$ and $\Theta$ are finite
- each $S_i$ is a compact and convex subset of an Euclidean space
- each $u_i$ is continuous in $s$ and concave in $s_i$.

Then $\mathcal{B}$ has a pure strategy Bayesian Nash equilibrium.

Proof. By Proposition 1, it is sufficient to show that $AG(\mathcal{B})$ has a pure Nash equilibrium. The latter follows from Theorem 1. We use the concavity of $u_i$ in $s_i$ to show that the corresponding $U_{\theta_i}$ is quasi-concave in $s_{\theta_i}$. Quasi-concavity of $u_i$ in $s_i$ does not typically imply quasi-concavity of $U_{\theta_i}$ in $s_{\theta_i}$ because $U_{\theta_i}$ is an integral of $u_i$ over variables other than $s_{\theta_i}$.

We can show that the set of Bayesian Nash equilibria of $\mathcal{B}^\pi$ is upper-hemicontinuous with respect to $x$ when payoffs are given by $w^\pi$, assumed continuous in $x$ in a compact set $X$, if $S, \Theta$ are finite. Indeed, $BNE(\mathcal{B}^\pi) = NE(AG(\mathcal{B}^\pi))$. Furthermore, we have upper-hemicontinuity with respect to beliefs.

Theorem 3. Suppose that $N, S, \Theta$ are finite. Let $P \subset \Delta(\Theta)$ be such that for every $p \in P$ $p_i(\theta_i) > 0, \forall \theta_i \in \Theta_i, i \in N$. Then $BNE(\mathcal{B}^p)$ is upper-hemicontinuous in $p$ over $P$.

Proof. Since $BNE(\mathcal{B}^p) = NE(G(\mathcal{B}^p))$, it is sufficient to note that

$$U_i((s_i(\theta_i))_{\theta_i \in \Theta, i \in N}) = E_p(u_i(\theta, s_1(\theta_1), \ldots, s_n(\theta_n)))$$

(as defined for $G(\mathcal{B}^p)$) is continuous in $p$. 

\footnote{Sums of quasi-concave functions are not necessarily quasi-concave.}
3. Extensive Form Games

An extensive form game consists of

- a finite set of players $N = \{1, 2, \ldots, n\}$; nature is denoted as “player 0"
- the order of moves specified by a tree
- each player’s payoffs at the terminal nodes in the tree
- information partition
- the set of actions available at every information set and a description of how actions lead to progress in the tree
- moves by nature.

A tree is a directed graph $(X, >)$—there is a link from node $x$ to node $y$ if $x > y$, which we interpret as “$x$ precedes $y$.” We assume that $X$ is finite, there is an initial node $\phi \in X$, $>$ is transitive $(x > y, y > z \Rightarrow x > z)$ and asymmetric $(x > y \Rightarrow y \nless x)$. Hence the tree has no cycles. We also require that each node $x \neq \phi$ has exactly one immediate predecessor, i.e., $\exists x' > x$ such that $x'' > x, x'' \neq x'$ implies $x'' > x'$. A node is terminal if it does not precede any other node; this means that the set of terminal nodes is $Z = \{z | \not\exists x, z > x\}$. Each $z \in Z$ completely determines a path of moves though the tree (recall the finiteness assumption), with associated payoff $u_i(z)$ for player $i$.

An information partition is a partition of $X \setminus Z$. Node $x$ belongs to the information set $h(x)$. The same player, denoted $i(h) \in N \cup \{0\}$, moves at each node $x \in h$ (otherwise players might disagree on whose turn to move is). The interpretation is that $i(h)$ is uncertain whether he is at $x$ or some other $x' \in h(x)$. We abuse notation writing $i(x) = i(h(x))$.

The set of available actions at $x \in X \setminus Z$ for player $i(x)$ is $A(x)$. We assume that $A(x) = A(x') =: A(h), \forall x' \in h(x)$ (otherwise $i(h)$ might play an infeasible action). A function $l$ labels each node $x \neq \phi$ with the last action taken to reach it. We require that the restriction of $l$ to the immediate successors of $x$ be bijective on $A(x)$. Finally, a move by nature at some node $x$ corresponds to a probability distribution over $A(x)$.

Let $H_i = \{h | i(h) = i\}$. The set of pure strategies for player $i$ is $S_i = \prod_{h \in H_i} A(h)$. As usual, $S = \prod_{i \in N} S_i$. A strategy is a complete contingent plan specifying an action to be taken at each information set (if reached). We can define mixed strategies as probability distributions over pure strategies, $\sigma_i \in \Delta(S_i)$. Any mixed strategy profile $\sigma \in \prod_{i \in N} \Delta(S_i)$,
along with the distribution of the moves by nature and the labeling of nodes with actions, leads to a probability distribution $O(\sigma) \in \Delta(Z)$. We denote by $u_i(\sigma) = \mathbb{E}_{O(\sigma)}(u_i(z))$. The associated normal form game is $(N, S, u)$.

Two strategies $s_i, s_i' \in S_i$ are equivalent if $O(s_i, s_{-i}) = O(s_i', s_{-i}), \forall s_{-i}$, that is, they lead to the same distribution over outcomes regardless of how the opponents play. See figure 3.9 in FT p. 86. $S^R_i$ is a subset of $S_i$ that contains exactly one strategy from each equivalence class. The reduced normal form game is given by $(N, S^R, u)$.

A behavior strategy specifies a distribution over actions for each information set. Formally, a behavior strategy $b_i(h)$ for player $i(h)$ at information set $h$ is an element of $\Delta(A(h))$. We use the notation $b_i(a|h)$ for the probability of action $a$ at information set $h$. A behavior strategy $b_i$ for $i$ is an element of $\prod_{h \in H_i} \Delta(A(h))$. A profile $b$ of behavior strategies determines a distribution over $Z$ in the obvious way. Clearly, $b_i$ is equivalent to $\sigma_i$ with

$$\sigma_i(s_i) = \prod_{h \in H_i} b_i(s_i(h)|h),$$

where $s_i(h)$ denotes the projection of $s_i$ on $A(h)$.

To guarantee that every mixed strategy is equivalent to a behavior strategy we need to impose the additional requirement of perfect recall. Basically, prefect recall means that no player ever forgets any information he once had and all players know the actions they have chosen previously. See figure 3.5 in FT, p. 81. Formally, perfect recall stipulates that if $x'' \in h(x')$, $x$ is a predecessor of $x'$ and the same player $i$ moves at both $x$ and $x'$ (and thus at $x''$) then there is a node $\hat{x}$ in the same information set as $x$ (possibly $x$ itself) such that $\hat{x}$ is a predecessor of $x''$ and the action taken at $x$ along the path to $x'$ is the same as the action taken at $\hat{x}$ along the path to $x''$. Intuitively, the nodes $x'$ and $x''$ are distinguished by information $i$ does not have, so he cannot have had it at $h(x)$; $x'$ and $x''$ must be consistent with the same action at $h(x)$ since $i$ must remember his action there.

Let $R_i(h)$ be the set of pure strategies for player $i$ that do not preclude reaching the information set $h \in H_i$, i.e., $R_i(h) = \{ s_i | h \text{ is on the path of some } (s_i, s_{-i}) \}$. If the game has prefect recall, a mixed strategy $\sigma_i$ is equivalent to a behavior strategy $b_i$ defined by

$$b_i(a|h) = \frac{\sum_{s_i \in R_i(h)|s_i(h)=a} \sigma_i(s_i)}{\sum_{s_i \in R_i(h)} \sigma_i(s_i)},$$

when the denominator is positive and any distribution when it is zero.
Many different mixed strategies can generate the same behavior strategy. Consider the example from FT p. 88, figure 3.12. Player 2 has four pure strategies, \( s_2 = (A, C), s'_2 = (A, D), s''_2 = (B, C), s'''_2 = (B, D) \). Now consider two mixed strategies, \( \sigma_2 = (1/4, 1/4, 1/4, 1/4) \), which assigns probability 1/4 to each pure strategy, and \( \sigma_2 = (1/2, 0, 0, 1/2) \), which assigns probability 1/2 to each of \( s_2 \) and \( s'''_2 \). Both of these mixed strategies generate the behavior strategy \( b_2 \) with \( b_2(A|h) = b_2(B|h) = 1/2 \) and \( b_2(C|h') = b_2(D|h') = 1/2 \). Moreover, for any strategy \( \sigma_1 \) of player 1, all of \( \sigma_2, \sigma'_2, b_2 \) lead to the same probability distribution over terminal nodes. For example, the probability of reaching node \( z_1 \) equals the probability of player 1 playing \( U \) times 1/2.

The relationship between mixed and behavior strategies is different in the game illustrated in FT p. 89, figure 3.13, which is not a game of perfect recall (player 1 forgets what he did at the initial node; formally, there are two nodes in his second information set which obviously succeed the initial node, but are not reached by the same action at the initial node). Player 1 has four strategies in the strategic form, \( s_1 = (A, C), s'_1 = (A, D), s''_1 = (B, C), s'''_1 = (B, D) \). Now consider the mixed strategy \( \sigma_1 = (1/2, 0, 0, 1/2) \). As in the last example, this generates the behavior strategy \( b_1 = \{(1/2, 1/2), (1/2, 1/2)\} \), where player 1 mixes 50 – 50 at each information set. But \( b_1 \) is not equivalent to the \( \sigma_1 \) that generated it. Indeed \( (\sigma_1, L) \) generates a probability 1/2 for the terminal node corresponding to \( (A, L, C) \) and a 1/2 probability for \( (B, L, D) \). However, since behavior strategies describe independent randomizations at each information set, \( (b_1, L) \) assigns probability 1/4 to each of the four paths \( (A, L, C), (A, L, D), (B, L, C), (B, L, D) \). Since both \( A \) vs. \( B \) and \( C \) vs. \( D \) are choices made by player 1, the strategy \( \sigma_1 \) under which player 1 makes all his decisions at once allows choices at different information sets to be correlated. Behavior strategies cannot produce this correlation in the example, because when it comes time to choose between \( C \) and \( D \), player 1 has forgotten whether he chose \( A \) or \( B \).

**Theorem 4** (Kuhn 1953). Under perfect recall, mixed and behavioral strategies are equivalent.

Hereafter we restrict attention to games with perfect recall, and use mixed and behavior strategies interchangeably. Behavior strategies prove more convenient in many arguments.
and constructions. We drop the notation \( b \) for behavior strategies and instead use \( \sigma_i(a_i|h) \) to denote player \( i \)'s probability of playing action \( a_i \) at information set \( h \).

4. Backward Induction and Subgame Perfection

An extensive form game has **perfect information** if all information sets are singletons. Backward induction can be applied to any extensive form game of perfect information with finite \( X \) (which means that the number of “stages” and the number of actions feasible at any stage are finite). The idea of backward induction is formalized by Zermelo’s algorithm. Since the game is finite, it has a set of penultimate nodes, i.e., nodes whose (all) immediate successors are terminal nodes. Specify that the player who moves at each such node chooses the strategy leading to the terminal node with the highest payoff for him. In case of a tie, make an arbitrary selection. Next each player at nodes whose immediate successors are penultimate (or terminal) nodes chooses the action maximizing his payoff over the feasible successors, given that players at the penultimate nodes play as assumed. We can now roll back through the tree, specifying actions at each node. At the end of the process we have a pure strategy for each player. It is easy to check that the resulting strategies form a Nash equilibrium.

**Theorem 5** (Zermelo 1913; Kuhn 1953). A finite game of perfect information has a pure-strategy Nash equilibrium.

Moreover, the backward induction solution has the nice property that each player’s actions are optimal at every possible history if the play of the opponents is held fixed, which we call subgame perfection. More generally, subgame perfection extends the logic of backward induction to games with imperfect information. The idea is to replace the “smallest” proper subgame with one of its Nash equilibria and iterate this procedure on the reduced tree. In stages where the “smallest” subgame has multiple Nash equilibria, the procedure implicitly assumes that all players believe the same equilibrium will be played. To define subgame perfection formally we first need the definition of a proper subgame. Informally, a proper subgame is a portion of a game that can be analyzed as a game in its own right.

**Definition 2.** A **proper subgame** \( G \) of an extensive form game \( T \) consists of a single node \( x \) and all its successors in \( T \), with the property that if \( x' \in G \) and \( x'' \in h(x') \) then
The information sets and payoffs of the subgame are inherited from the original game. That is, two nodes are in the same information set in $G$ if and only if they are in the same information set in $T$, and the payoff function on the subgame is just the restriction of the original payoff function to the terminal nodes of $G$.

The requirements that all the successors of $x$ be in the subgame and that the subgame does not “chop up” any information set ensure that the subgame corresponds to a situation that could arise in the original game. In figure 3.16, p. 95 of FT, the game on the right is not a subgame of the game on the left, because on the right player 2 knows that player 1 has not played $L$, which he did not know in the original game.

Together, the requirements that the subgame begin with a single node $x$ and respect information sets imply that in the original game $x$ must be a singleton information set, i.e. $h(x) = \{x\}$. This ensures that the payoffs in the subgame, conditional on the subgame being reached, are well defined. In figure 3.17, p. 95 of FT, the “game” on the right has the problem that player 2’s optimal choice depends on the relative probabilities of nodes $x$ and $x'$, but the specification of the game does not provide these probabilities. In other words, the diagram on the right cannot be analyzed as a separate game; it makes sense only as a component of the game on the left, which provides the missing probabilities.

Since payoffs conditional on reaching a proper subgame are well-defined, we can test whether strategies yield a Nash equilibrium when restricted to the subgame.

**Definition 3.** A behavior strategy profile $\sigma$ of an extensive form game is a **subgame perfect equilibrium** if the restriction of $\sigma$ to $G$ is a Nash equilibrium of $G$ for every proper subgame $G$.

Because any game is a proper subgame of itself, a subgame perfect equilibrium profile is necessarily a Nash equilibrium. If the only proper subgame is the whole game, the sets of Nash and subgame perfect equilibria coincide. If there are other proper subgames, some Nash equilibria may fail to be subgame perfect.

It is easy to see that subgame perfection coincides with backward induction in finite games of perfect information. Consider the penultimate nodes of the tree, where the last choices are made. Each of these nodes begins a trivial one-player proper subgame, and Nash equilibrium in these subgames requires that the player make a choice that maximizes his payoff. Thus...
any subgame perfect equilibrium must coincide with a backward induction solution at every penultimate node, and we can continue up the tree by induction.

5. Important Examples of Extensive Form Games

5.1. Repeated games with observable actions.
- **time** \( t = 0, 1, 2, \ldots \) (usually infinite)
- **stage game** is a normal-form game \( G \)
- \( G \) is played every period \( t \)
- players **observe** the realized actions at the end of each period
- **payoffs** are the sum of discounted payoffs in the stage game.

Repeated games are a widely studied class of dynamic games. There is a lot of research dealing with various restrictions on the information about past play.

5.2. Multistage games with observable actions.
- **stages** \( k = 0, 1, 2, \ldots \)
  - at stage \( k \), after having **observed** a “non-terminal” history of play \( h = (a^0, \ldots, a^{k-1}) \), each player \( i \) simultaneously chooses an **action** \( a^k_i \in A_i(h) \)
- **payoffs** given by \( u(h) \) for each terminal history \( h \).

Often it is natural to identify the “stages” of the game with time periods, but this is not always the case. A game of perfect information can be viewed as a multistage game in which every stage corresponds to some node. At every stage all but one player (the one moving at the node corresponding to that stage) have singleton action sets (“do nothing”; can refer to these players as “inactive”). Repeated versions of perfect information extensive form games also lead to multistage games, e.g., the Rubinstein (1982) alternating bargaining game, which we discuss later.

6. Single (or One-Shot) Deviation Principle

Consider a multi-stage game with observed actions. We show that in order to verify that a strategy profile \( \sigma \) is subgame perfect, it suffices to check whether there are any histories \( h_t \) where some player \( i \) can gain by deviating from the actions prescribed by \( \sigma_i \) at \( h_t \) and conforming to \( \sigma_i \) elsewhere.
Definition 4. A strategy $\sigma_i$ is unimprovable given $\sigma_{-i}$ if $u_i(\sigma_i, \sigma_{-i} | h_t) \geq u_i(\sigma'_i, \sigma_{-i} | h_t)$ for every $t \geq 0$, $h_t \in H_i$ and $\sigma'_i \in \Delta(S_i)$ with $\sigma'_i(h'_t) = \sigma_i(h'_t)$ for all $h'_t \in H_i \setminus \{h_t\}$.

Hence a strategy $\sigma_i$ is unimprovable if after every history, no strategy that differs from it at only one information set can increase utility. Obviously, if $\sigma$ is a subgame perfect equilibrium then $\sigma_i$ is unimprovable given $\sigma_{-i}$. To establish the converse, we need an additional condition.

Definition 5. A game is continuous at infinity if for each player $i$ the utility function $u_i$ satisfies

$$\lim_{t \to \infty} \sup_{(h, \tilde{h}) \in H_i = \tilde{h}_t} |u_i(h) - u_i(\tilde{h})| = 0.$$ 

This condition requires that events in the distant future are relatively unimportant. It is satisfied if the overall payoffs are a discounted sum of per-period payoffs and the stage payoffs are uniformly bounded.

Theorem 6. Consider a (finite or infinite horizon) multi-stage game with observed actions\(^2\) that is continuous at infinity. If $\sigma_i$ is unimprovable given $\sigma_{-i}$ then $\sigma_i$ is a best response to $\sigma_{-i}$ conditional on any history $h_t$.

Proof. Suppose that $\sigma_i$ is unimprovable given $\sigma_{-i}$, but $\sigma_i$ is not a best response to $\sigma_{-i}$ following some history $h_t$. Let $\sigma_i^1$ be a strictly better response and define

$$\varepsilon = u_i(\sigma_i^1, \sigma_{-i} | h_t) - u_i(\sigma_i, \sigma_{-i} | h_t) > 0.$$  

Since the game is continuous at infinity, there exists $t' > t$ and $\sigma_i^2$ such that $\sigma_i^2$ is identical to $\sigma_i^1$ at all information sets up to (and including) stage $t'$, $\sigma_i^2$ coincides with $\sigma_i$ across all longer histories and

$$|u_i(\sigma_i^2, \sigma_{-i} | h_t) - u_i(\sigma_i^1, \sigma_{-i} | h_t)| < \varepsilon/2.$$ 

In particular, 6.1 and 6.2 imply that

$$u_i(\sigma_i^2, \sigma_{-i} | h_t) > u_i(\sigma_i, \sigma_{-i} | h_t).$$

\(^2\)We allow for the possibility that the action set be infinite at some stages.
Denote by $\sigma_i^3$ the strategy obtained from $\sigma_i^2$ by replacing the stage $t'$ actions following any history $h_{t'}$ with the corresponding actions under $\sigma_i$. Conditional on any history $h_{t'}$, the strategies $\sigma_i$ and $\sigma_i^3$ coincide, hence

\begin{equation}
(6.3) \quad u_i(\sigma_i^3, \sigma_{-i}|h_{t'}) = u_i(\sigma_i, \sigma_{-i}|h_{t'}).
\end{equation}

As $\sigma_i$ is unimprovable given $\sigma_{-i}$, and $\sigma_i$ and $\sigma_i^2$ only differ at stage $t'$ conditional on $h_{t'}$, we need

\begin{equation}
(6.4) \quad u_i(\sigma_i, \sigma_{-i}|h_{t'}) \geq u_i(\sigma_i^2, \sigma_{-i}|h_{t'}).
\end{equation}

Then 6.3 and 6.4 lead to

\[ u_i(\sigma_i^3, \sigma_{-i}|h_{t'}) \geq u_i(\sigma_i^2, \sigma_{-i}|h_{t'}) \]

for all histories $h_{t'}$ (consistent with $h_t$). Since $\sigma_i^2$ and $\sigma_i^3$ coincide before reaching stage $t'$, we obtain

\[ u_i(\sigma_i^3, \sigma_{-i}|h_t) \geq u_i(\sigma_i^2, \sigma_{-i}|h_t). \]

Similarly, we can construct $\sigma_i^4, \ldots, \sigma_i^{t'-t+3}$. The strategy $\sigma_i^{t'-t+3}$ is identical to $\sigma_i$ conditional on $h_t$ and

\[ u_i(\sigma_i, \sigma_{-i}|h_t) = u_i(\sigma_i^{t'-t+3}, \sigma_{-i}|h_t) \geq \ldots \geq u_i(\sigma_i^3, \sigma_{-i}|h_t) \geq u_i(\sigma_i^2, \sigma_{-i}|h_t) > u_i(\sigma_i, \sigma_{-i}|h_t), \]

a contradiction. \qed

7. Iterated Conditional Dominance

**Definition 6.** In a multistage game with observable actions, an action $a_i$ is conditionally dominated at stage $t$ given history $h_t$ if in the subgame starting at $h_t$ every strategy for player $i$ that assigns positive probability to $a_i$ is strictly dominated.

**Proposition 2.** In any perfect information game, every subgame perfect equilibrium survives iterated elimination of conditionally dominated strategies.
8. Bargaining with Alternating Offers

The set of players is $N = \{1, 2\}$. For $i = 1, 2$ we write $j = 3 - i$. The set of feasible utility pairs is $U \subset \mathbb{R}^2$, assumed to be compact and convex with $(0,0) \in U$.\(^3\) Time is discrete and infinite, $t = 0, 1, \ldots$ Each player $i$ discounts payoffs by $\delta_i$, so receiving $u_i$ at time $t$ is worth $\delta_t^i u_i$.

Rubinstein (1982) analyzes the following perfect information game. At every time $t = 0, 1, \ldots$, player $i(t)$ proposes an alternative $u = (u_1, u_2) \in U$ to player $j = 3 - i(t)$; the bargaining protocol specifies that $i(t) = 1$ for $t$ even and $i(t) = 2$ for $t$ odd. If $j$ accepts the offer, then the game ends yielding a payoff vector $(\delta_1^i u_1, \delta_2^i u_2)$. Otherwise, the game proceeds to period $t + 1$. If agreement is never reached, each player receives a 0 payoff.

For each player $i$, it is useful to define the function $g_i$ by setting

$$g_i(u_j) = \max \{ u_i | (u_1, u_2) \in U \}.$$  

Notice that the graphs of $g_1$ and $g_2$ coincide with the Pareto-frontier of $U$.

8.1. Stationary subgame perfect equilibrium. Let $(m_1, m_2)$ be the unique solution to the following system of equations

$$m_1 = \delta_1 g_1 (m_2)$$
$$m_2 = \delta_2 g_2 (m_1).$$

Note that $(m_1, m_2)$ is the intersection of the graphs of the functions $\delta_2 g_2$ and $(\delta_1 g_1)^{-1}$.

We are going to argue that the following “stationary” strategies constitute the unique subgame perfect equilibrium. In any period where player $i$ has to make an offer to $j$, he offers $u$ with $u_j = m_j$ and $j$ accepts only offers $u$ with $u_j \geq m_j$. We can use the single-deviation principle to check that this is a subgame perfect equilibrium.

8.2. Equilibrium uniqueness. We prove that the subgame perfect equilibrium is unique by arguing that it is essentially the only strategy profile that survives iterated conditional dominance.

\(^3\)The set of feasible utility outcomes $U$ can be generated from a set of contracts or decisions $X$ in a natural way. Define $U = \{(v_1(x), v_2(x)) | x \in X\}$ for a pair of utility functions $v_1$ and $v_2$ over $X$. With additional assumptions on $X, v_1, v_2$ we can ensure that the resulting $U$ is compact and convex.
Theorem 7. If a strategy profile survives iterative elimination of conditionally dominated strategies, then it is identical to the stationary subgame perfect equilibrium except for the nodes at which a player is indifferent between accepting and rejecting an offer in the subgame perfect equilibrium.

Proof. Since player $i$ can get 0 by never reaching an agreement, offering an alternative that gives him less than

$$m_i^0 = 0$$

or accepting such an offer at any history is conditionally dominated. All such offers are eliminated at the first stage of the iteration. Then $i$ should never expect to receive more than

$$M_i^0 = \delta_i g_i(0)$$

in any future period following a disagreement. Hence rejecting an offer $u$ with $u_i > M_i^0$ is conditionally dominated by accepting such an offer for $i$. Once we eliminate the latter strategies, $i$ always accepts offers $u$ with $u_i > M_i^0$ from $j$. Then making offers $u$ with $u_i > M_i^0$ is dominated for $j$ by offers $\bar{u} = \lambda u + (1 - \lambda) (M_i^0, g_j(M_i^0))$ for $\lambda \in (0, 1)$. We remove all the strategies involving such offers.

Under the surviving strategies, $j$ can reject an offer from $i$ and make an offer next period that leaves him with slightly less than $g_j(M_i^0)$, which $i$ accepts. Hence accepting any offer that gives him less than

$$m_j^1 = \delta_j g_j(M_i^0)$$

is dominated for $j$. Moreover, making such offers is dominated for $j$ because we argued above that offers with $u_i > M_i^0$ are dominated. After we eliminate such moves, $i$ cannot expect more than

$$M_i^1 = \delta_i g_i(m_j^1) = \delta_i g_i(\delta_j g_j(M_i^0))$$

in any future period following a disagreement.

We can recursively define the sequences

$$m_j^{k+1} = \delta_j g_j(M_i^k)$$

$$M_i^{k+1} = \delta_i g_i(m_j^{k+1})$$
for \( i = 1, 2 \) and \( k \geq 1 \). Since both \( g_1 \) and \( g_2 \) are decreasing functions, we can easily show that the sequence \((m^*_k)\) is increasing and \((M^*_k)\) is decreasing. By arguments similar to those above, we can prove by induction on \( k \) that, at some stage in the iteration, player \( i = 1, 2 \)

- never accepts or makes an offer with \( u_i < m^*_k \)
- always accepts offers with \( u_i > M^*_k \), but making such offers is dominated for \( j \).

The sequences \((m^*_k)\) and \((M^*_k)\) are monotonic and bounded, so they need to converge. The limits satisfy

\[
\begin{align*}
  m^*_j &= \delta_j g_j (\delta_i g_i (m^*_j)) \\
  M^*_i &= \delta_i g_i (m^*_j).
\end{align*}
\]

It follows that \((m^*_j, m^*_i)\) is the (unique) intersection point of the graphs of the functions \( \delta_2 g_2 \) and \( (\delta_1 g_1)^{-1} \). Moreover, \( M^*_i = \delta_i g_i (m^*_j) = m^*_i \). Therefore, no strategy for \( i \) that rejects \( u \) with \( u_i > M^*_i = m^*_i \) or accepts \( u \) with \( u_i < m^*_i = M^*_i \) survives iterated elimination of conditionally dominated strategies. Also, no strategy for \( i \) to offer \( u \) with \( u_i \neq M^*_i = m^*_i \) survives.

9. **Nash Bargaining**

Assume that \( U \) is such that \( g_2 \) is decreasing, strictly concave and continuously differentiable (derivative exists and is continuous). The **Nash (1950) bargaining solution** \( u^* \) is defined by \( \{u^*\} = \arg \max_{u \in U} u_1 u_2 = \arg \max_{u \in U} u_1 g_2 (u_1) \). It is the outcome \((u^*_1, g_2(u^*_1))\) uniquely pinned down by the first order condition \( g_2(u^*_1) + u^*_1 g'_2(u^*_1) = 0 \). Indeed, since \( g_2 \) is decreasing and strictly concave, the function \( f \), given by \( f(x) = g_2(x) + xg'_2(x) \), is strictly decreasing and continuous and changes sign on the relevant range.

**Theorem 8.** Suppose that \( \delta_1 = \delta_2 =: \delta \) in the alternating bargaining model. Then the unique subgame perfect equilibrium payoffs converge to the Nash bargaining solution as \( \delta \to 1 \).

**Proof.** Recall that the subgame perfect equilibrium payoffs are given by \((g_1(m_2), m_2)\) where \((m_1, m_2)\) satisfies

\[
\begin{align*}
  m_1 &= \delta g_1 (m_2) \\
  m_2 &= \delta g_2 (m_1).
\end{align*}
\]
It follows that \( g_1(m_2) = m_1/\delta \), hence \( m_2 = g_2(g_1(m_2)) = g_2(m_1/\delta) \). We rewrite the equations as follows

\[
\begin{align*}
  g_2(m_1/\delta) &= m_2 \\
  g_2(m_1) &= m_2/\delta.
\end{align*}
\]

By the mean value theorem, there exists \( \xi \in (m_1, m_1/\delta) \) such that \( g_2(m_1/\delta) - g_2(m_1) = (m_1/\delta - m_1)g_2'(\xi) \), hence \( m_2 - m_2/\delta = (m_1/\delta - m_1)g_2'(\xi) \) or, equivalently, \( m_2 + m_1g_2'(\xi) = 0 \). Substituting \( m_2 = \delta g_2(m_1) \) we obtain \( \delta g_2(m_1) + m_1g_2'(\xi) = 0 \).

Note that \((g_1(m_2), m_2)\) converges to \( u^* \) as \( \delta \to 1 \) if and only if \((m_1, m_2)\) does. In order to show that \((m_1, m_2)\) converges to \( u^* \) as \( \delta \to 1 \), it is sufficient to show that any limit point of \((m_1, m_2)\) as \( \delta \to 1 \) is \( u^* \). Let \((m_1^*, m_2^*)\) be such a limit point corresponding to a sequence \((\delta_k)_{k \geq 0} \to 1 \). Recognizing that \( m_1, m_2, \xi \) are functions of \( \delta \), we have

\[
(9.1) \quad \delta_k g_2(m_1(\delta_k)) + m_1(\delta_k)g_2'(\xi(\delta_k)) = 0.
\]

Since \( \xi(\delta_k) \in (m_1(\delta_k), m_1(\delta_k)/\delta_k) \) with \( m_1(\delta_k), m_1(\delta_k)/\delta_k \to m_1^* \) as \( k \to \infty \) and \( g_2' \) is continuous by assumption, in the limit \(9.1 \) becomes \( g_2(m_1^*) + m_1^*g_2'(m_1^*) = 0 \). Therefore, \( m_1^* = u_1^* \). \( \square \)