NOTES ON LAYER STRIPPING SOLUTIONS OF HIGHER DIMENSIONAL INVERSE SEISMIC PROBLEMS

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This paper consists of some notes on inverse problems of higher dimensions, in which the quantities of interest (local density and wave speed) are functions of two or three spatial variables, e.g. $\rho(x,z)$ and $c(x,z)$, or $\rho(x,y,z)$ and $c(x,y,z)$. These notes are not designed to be complete, but to summarize the results obtained so far in applying layer stripping techniques to these problems.

I. Introduction

The subject of this paper is the inverse seismic problem in dimensions higher than one, in which local density and wave speed are functions of more than one spatial variable. To clarify matters, some terminology is introduced. The dimension of an inverse problem is defined as the number of spatial variables on which the quantities of interest (ρ and c) depend. Thus, the two-dimensional (2-D) problem is the inverse problem of determining $\rho(x,z)$ and $c(x,z)$ from surface measurements of the displacement $u(x, z=0, t)$, and the three-dimensional (3-D) problem is the inverse problem of determining $\rho(x,y,z)$ and $c(x,y,z)$ from surface measurements of the displacement $u(x, y, z=0, t)$.

Note that the dimension of a problem need not be the same as the dimension of the medium for which it is defined -- a problem of given dimension can be embedded in a medium of higher dimension. For example, the "offset problem" described in [1] is a 1-D problem embedded in a 2-D medium, while the point-source problem of that same paper is a 1-D problem embedded in a 3-D medium.

While a considerable amount of work has been done on the 1-D problem, much less has been done on the 2-D and 3-D problems.

Generalizing 1-D results and techniques to the 2-D and 3-D problems has proven to be very difficult, and applying other techniques to these higher-dimensional problems has been contingent on rather severe assumptions. The Born approximation (basically an assumption that medium parameters vary slowly with depth) was used by Cohen and Bleistein [2], and the WKBJ approximation (an assumption analogous to geometrical optics in which energy is assumed to propagate along rays) has been used by Clayton and Stolt [3]. Raz [4] has proposed a migration-like technique that involves a distorted-wave Born model. Various assumptions are made, including a straight-ray approximation between scattering and observation points. Results of a numerical 2-D inversion are presented, and a 3-D procedure proposed.

Newton [5] has described a general 3-D inverse scattering problem solution that reconstructs a Schrödinger potential from a scattering amplitude given as a function of energy and directions of incident and scattered particles. Solution of a generalized Marchenko integral equation is required. Morawetz and Kriegsmann [6] have proposed an iterative scheme in which an initial guess at a 2-D potential $V(x,y)$ is iteratively refined. In the numerical examples presented for a 1-D inverse potential problem, up to thirteen iterations were required, and also some smoothing to prevent numerical instability. The computations and memory required for 2-D inversion are admitted to be enormous.

The rest of this paper can be divided into three sections. First, a layer-stripping algorithm is given which reconstructs a 2-D density $\rho(x,z)$, with the assumption of constant wave speed c. This is given more for illustrating the application of layer stripping to higher-dimensional problems than as a useful algorithm. Next, the "offset problem" described in [1] is generalized to a 2-D problem embedded in a 3-D medium, and a layer-stripping solution specified. Finally, some thoughts on generalizing 1-D results to higher dimensions are given, and some difficulties in doing this are discussed.

II. Layer-Stripping Reconstruction of $\rho(x,z)$

In this section a recursive layer-stripping algorithm is derived for solving the 2-D problem with constant wave speed. In particular, the density $\rho(x,z)$ is reconstructed from surface observation of pressure $p(x, z=0, t)$ and medium acceleration $\vec{w}(x, z=0, t)$. The wave speed c is assumed to be constant throughout the medium, and will in fact be taken to be unity (this amounts to scaling depth z by c). Of course, this will not be a practical result; however, it will illustrate the application of the layer-stripping idea to a higher-dimensional problem. Assuming a constant wave speed removes the problem of defining the wavefront, which turns out to be the complicating factor in applying layer-stripping to higher-dimensional problems (see Section IV).

This problem was first formulated and solved using layer stripping ideas by Symes [7]. Symes's approach was to reconstruct the medium layer by layer by solving a Schrödinger equation in the lateral variable x to obtain the lateral dependence of density ρ at each depth. The layer stripping solution to this problem using first-order equations which can easily be adapted into a recursive algorithm is due to Levy [8].

The mathematical technique used to solve the partial differential equations in this algorithm consists of propagating the characteristic variables in depth z, lateral position x, and time t. This technique has been applied to the problem of the propagation of axial shear waves by Achenbach [9], and to the 1-D problem by Santosa and Schwetlick [10]; the form used here appears in Bruckstein et al. [11].

The acoustic and stress-strain equations for this problem are

$$\partial^2 p/\partial t^2 = -\rho(\partial w_x/\partial x + \partial w_z/\partial z) \tag{1}$$

$$\partial p/\partial x = -\rho w_x \tag{2a}$$

$$\partial p/\partial z = -\rho w_z \tag{2b}$$

where w_x and w_z are the horizontal and vertical, respectively, components of acceleration. Eliminating w_x by substituting (2a) in (1) yields

$$\partial w_z/\partial z = (\partial^2 p/\partial x^2 - \partial^2 p/\partial t^2)/\rho - (\partial\rho/\partial x)(\partial p/\partial x)/\rho^2 \tag{3}$$

It is assumed that the pressure p and vertical acceleration w_z (hereafter w_z will be replaced by w, for convenience) have the forms

$$w(x,z,t) = \delta(t-z) + w_0(x,z,t)\ 1(t-z) \tag{4}$$

$$p(x,z,t) = p_0(x,z,t)\ 1(t-z) \tag{5}$$

where $1(\cdot)$ is the unit step function. The impulse in (4) represents the source excitation; the step functions merely represent the causal natures of p and w.

Substituting (4) and (5) in (2b) and equating the two impulsive terms yields

$$\rho(x,z) = p_0(x,z,z) \quad . \tag{6}$$

Substituting (4) and (5) in (3) and equating the two $\dot{\delta}(\cdot)$ terms also yields (6), so the equations are consistent. In addition, equating the $\delta(\cdot)$ terms yields the added condition

$$\rho(x,z) = 2(\partial p_0/\partial t)/w_0\big|_{t=z}. \tag{7}$$

This additional condition ensures that the problem is not ill-posed and that a unique solution is forthcoming.

The algorithm in differential form consists of (2b), (3), and (6). In words it may be described as follows:

Assume all quantities known at depth z.
Update to depth $z + \Delta$:

(1) Update pressure p using (2b). Do for all x and t.
(2) Update acceleration w using (3). Do for all x and t.
(3) Obtain the updated density ρ from (6). Do for all x.

The derivatives in (3) could perhaps best be accomplished by using an FFT. The p and w updates are simple replacements, so the update can be done point by point.

This algorithm shows how layer-stripping is carried out in higher-dimensional problems -- the updates proceed along the entire wavefront, point by point. The wavefront itself describes the set of points currently being updated. This example was chosen to make the wavefront as simple as possible: a flat planar impulse moving vertically with unit velocity. When the wave speed c is also varying in space, the wavefront becomes distorted, and characterizing it becomes much more complicated.

III. The 2-D Offset Problem

In this section the 1-D offset problem of [1] is generalized to two dimensions. Recall that in the 1-D offset problem impulsive plane waves were incident upon a 2-D medium with 1-D material parameter variation, viz. $\rho(z)$ and $c(z)$. Running this experiment twice, at two different angles of incidence, allowed the recovery of $\rho(z)$ and $c(z)$ separately. A generalization of this experiment will now allow $\rho(x,z)$ and $c(x,z)$ to be recovered separately.

The problem set-up is as described in [1], only now $\rho(x,z)$ and $c(x,z)$ are functions of one lateral coordinate as well as depth, and the impulsive plane wave now has a normal lying in the y-z plane, where y is the other lateral coordinate. This may be visualized as a horizontal stack of identical inhomogeneous plates, with the normal to the impulsive plane wave having components in the direction of the stacking and in the direction of increasing depth.

The acoustic and stress-strain equations are now

$$\partial^2 p/\partial t^2 = -\rho c^2(\partial w_x/\partial x + \partial w_y/\partial y + \partial w_z/\partial z) \tag{8}$$

$$-\rho w_x = \partial p/\partial x \tag{9a}$$

$$-\rho w_y = \partial p/\partial y \tag{9b}$$

$$-\rho w_z = \partial p/\partial z \tag{9c}$$

where the quantities are the same as in (1) and (2) and w_y is the other acceleration component. Proceeding as in [1], the fact that $\rho(x,z)$ and $c(x,z)$ do not vary with y means that if the medium is subject to an impulsive plane wave whose Fourier transform for $z < 0$ (above the surface) is $e^{-j(k_x x + k_y y + k_z z)}$, then the wave number k_y will not vary with x and z either above the surface or below it. Hence the Fourier transform of the pressure takes the form

$$\hat{p}(x,y,z,\omega) = \hat{\psi}(x,z,\omega)\, e^{-jk_y y} = \hat{\psi}(x,z,\omega)\, e^{-j\omega y \sin\theta_i/c_0} \tag{10}$$

where θ_i is the angle of incidence for the plane wave and c_0 is the (homogeneous) wave speed for $z < 0$ (above the surface).

Taking Fourier transforms of (8) and (9), substituting (10), defining

$$\cos^2 \theta_i(x,z) = 1 - c(x,z)^2 \sin^2 \theta_i/c_0^2 , \tag{11}$$

and converting back to the time domain yields, in perfect analogy to [1],

$$(\partial^2 p/\partial t^2) \cos^2 \theta_i(x,z) = - \rho c^2 (\partial w_x/\partial x + \partial w_z/\partial z) . \tag{12}$$

Note that $\theta_i(x,z)$ can be interpreted as the angle between the tangent to the actual ray path at a point (x,y,z) and its projection on the x-z plane. Compare this to $\theta_i(z)$ in [1], which was the angle between the tangent to the ray path at depth z and the z-axis. Equation (12) shows that the problem has been reduced from a 2-D problem embedded in a 3-D medium to a 2-D problem embedded in a 2-D medium.

Since the partial derivatives in (9a), (9c), and (12) constitute a gradient and divergence, respectively, they must (taken collectively) be independent of the choice of coordinates. Thus we may change from x and z to the time-varying curvilinear coordinates s and e, where s is normal to the (2-D) wavefront and e is tangent to it (see Fig. 1). Note that s also represents arc length along the projection of a ray on the (x,z) plane, and e represents arc length along a projected wavefront. We have $s = 0$ along the surface $(z=0)$, and $e = 0$ along the ray passing through the origin $(x,z) = (0,0)$. This representation will be important in the next section.

Writing (9a), (9c), and (12) in terms of s and e yields

$$(\partial^2 p/\partial t^2) \cos^2 \theta_i(s,e) = - \rho c^2 (\partial w_s/\partial s + \partial w_e/\partial e) \tag{13}$$

$$- \rho w_s = \partial p/\partial s \tag{14a}$$

$$- \rho w_e = \partial p/\partial e$$

where w_s and w_e are the components of acceleration in the appropriate directions. Eliminating w_e gives

$$(\partial^2 p/\partial t^2) \cos^2 \theta_i(s,e) = - \rho c^2 (\partial w_s/\partial s) + c^2 \partial^2 p/\partial e^2 - (c^2/\rho) (\partial \rho/\partial e) (\partial p/\partial e) \tag{15}$$

and defining the travel times

$$d\tau/ds = 1/c\,(s,e) \tag{16}$$

$$d\tau_i/d\tau = \cos \theta_i(s,e) \quad , \quad i=1,2 \tag{17}$$

for two experiments with initial angles of incidence θ_1 and θ_2 allows the pressure and acceleration to be written in the forms

$$w_s^i(\tau e,t) = \delta(t-\tau_i) + w_{s0}^i(\tau,e,t)\ 1(t-\tau_i) \tag{18a}$$

$$p^i(\tau,e,t) = p_0^i(\tau,e,t)\ 1(t-\tau_i) \tag{18b}$$

where p^i is the pressure field resulting from the experiment at angle of incidence θ_i, and similarly for w_s^i.

Substituting (18) in (15) and (14a) yields

$$p_0^i(\tau,e,t = \tau_i) = \rho c(\tau,e)/\cos \theta_i(\tau,e) \tag{19}$$

which represents the information gained from the first reflection at τ for all e. From (19) (for <u>both</u> experiments) and (11) $c(\tau,e)$ may be found, and then $\rho(\tau,e)$ immediately follows.

Equations (14a), (15), (16), (17), and (19) taken together thus constitute a differential algorithm for computing $\rho(\tau,e)$ and $c(\tau,e)$, with the update taking place as an increment in the ray path travel time τ. The algorithm may be summarized as follows:

<u>Given:</u> $p^i(\tau,e,t)$, $w_s^i(\tau,e,t)$, $\rho(\tau,e)$, $c(\tau,e)$, $\cos\theta_i(\tau,e)$, $\tau_i(\tau,e)$, $i = 1,2$.
Update all quantities in τ.
Each step is done pointwise for all e and t.

(1) Update p^i : $\partial p^i/\partial\tau = -\rho c\, w_s^i$ (20)

(2) Update w_s^i : $\partial w_s^i/\partial\tau = -[(\partial^2 p^i/\partial t^2)\cos^2\theta_i(\tau,e) - c^2\partial^2 p^i/\partial e^2 + (c^2/\rho)(\partial\rho/\partial e)(\partial p^i/\partial e)]/(\rho c)$ (21)

(3) Update τ_i : $\partial\tau_i/\partial\tau = \cos\theta_i(\tau,e)$ (22)

(4) Compute U : $U(\tau^+,e) = (p^2(\tau^+,e,t=\tau_2^+)/p^1(\tau^+,e,t=\tau_1^+))^2$ (23)

(5) Compute c : $c(\tau^+,e) = c_0\,[(U-1)/(U\sin^2\theta_2 - \sin^2\theta_1)]^{1/2}$ (24)

(6) Compute $\cos\theta_i$: $\cos\theta_i(\tau^+,e) = [1 - c(\tau^+,e)^2/c_0^2\,\sin^2\theta_i]^{1/2}$ (25)

(7) Compute ρ : $\rho(\tau^+,e) = p^1(\tau^+,e,t=\tau_1^+)\cos\theta_1(\tau^+,e)/c(\tau^+,e)$. (26)

This algorithm bears a marked resemblance to the corresponding algorithm in [1], and it is not difficult to see why. In the 1-D offset problem algorithm updates similar to those above were carried out as the planar wavefront advanced from depth z to depth z + Δ. In the 2-D offset problem the wavefront is no longer a flat plane, but is described at time t by the equation $\tau(x,z) = t$. Hence the increment occurs in ray path travel time τ, which by definition is the same for all rays, i.e. all along the wavefront. When τ is incremented, the wavefront advances slightly, and information about the medium is obtained from the first reflection using (19), which applies all along the wavefront. Travel time τ is used instead of ray arc length s, since the latter changes by varying amounts along the wavefront in a given time increment. In the 1-D problem there was no variation of wave speed c along the wavefront.

Of course, it is still necessary to convert $\rho(\tau,e)$ and $c(\tau,e)$ back into the original (x,z) coordinates. This is done by a form of differential ray tracing or wave front tracing. Let $\phi(\tau,e)$ be the angle between a tangent to the wavefront at the point (τ,e) and the (horizontal) x-axis (see Fig. 1). Clearly the wavefront will advance locally in the direction ϕ - 90°.

Now, ϕ is of course a function of e, unless the medium is homogeneous. But ϕ changes with τ due to variation of the wave speed c along the wavefront -- without such variation, the wavefont would retain its shape. This allows the derivation of an update equation for ϕ. From Fig. 2, we have

$$\tan(\phi(\tau+\delta\tau,e) - \phi(\tau,e)) = (c(\tau,e+\delta e) - c(\tau,e))\delta\tau/\delta e \tag{27}$$

and letting $\delta\tau$ and δe go to zero yields

$$\partial\phi(\tau,e)/\partial\tau = \partial c(\tau,e)/\partial e \quad . \tag{28}$$

This equation is an update equation for ϕ, since $c(\tau,e)$ is assumed to be known at τ for all e, hence $\partial c/\partial e$ may be computed (although this computation is not very stable).

Now, suppose the coordinates (x,z) associated with the point (τ,e) are known for all e. When τ is incremented by Δ, these coordinates will change slightly, by amounts δx and δz. But clearly

$$\delta x(\tau,e) = c(\tau,e)\ \Delta \sin \phi(\tau,e) \tag{29a}$$

$$\delta z(\tau,e) = c(\tau,e)\ \Delta \cos \phi(\tau,e) \tag{29b}$$

This allows $\rho(x,z)$ and $c(x,z)$ to be computed recursively, as follows:

<u>Given</u>: $c(\tau,e)$, $\rho(\tau,e)$, $x(\tau,e)$, $z(\tau,e)$, $\cos \phi(\tau,e)$, $\sin \phi(\tau,e)$.

Update all quantities in τ. Each step is done for all e.

(1) Update $\cos \phi$ from $\partial \cos \phi/\partial\tau = -(\sin \phi)\ \partial c/\partial e$ (30)

(2) Update $\sin \phi$ from $\partial \sin \phi/\partial\tau = (\cos \phi)\ \partial c/\partial e$ (31)

(3) Update x and z from (29)

(4) Update $c(\tau,e)$ and $\rho(\tau,e)$ by the algorithm (18) - (24)

(5) Output $c(\tau+\Delta,e)$, $\rho(\tau+\Delta,e)$, $x(\tau+\Delta,e)$, $z(\tau+\Delta,e)$ as $c(x,z)$ and $\rho(x,z)$. This is quite suitable for plotting.

Note that (28) has been used in (30) and (31), and that $\phi(0,e)$ is initialized to zero.

IV. <u>Generalizations of 1-D Results to Higher Dimensions</u>

In this section some ways in which methods and results for the 1-D inverse seismic problem generalize to higher dimensions are discussed. Also, some difficulties in applying layer-stripping to higher-dimensional problems are noted.

It is known, e.g. [10], that for the 1-D inverse seismic problem in which an impulsive plane wave is normally incident on a 1-D medium, and the upgoing wave at the surface measured, then the only information about the medium that can be reconstructed exactly is the impedance as a function of travel time, viz. $\rho c(\tau)$. How might this result generalize to higher dimensions?

Writing the 2-D acoustic and stress-strain equations in the (s,e) coordinates defined in the last section gives

$$p = -\rho c^2 \ (\partial u_s/\partial s + \partial u_e/\partial e) \tag{32}$$

$$\partial p/\partial s = -\rho \, \partial^2 u_s/\partial t^2 \tag{33a}$$

$$\partial p/\partial e = -\rho \, \partial^2 u_e/\partial t^2 \tag{33b}$$

where u_s and u_e are components of displacement in the appropriate directions. Now, in the 1-D case changing variables from depth to travel time resulted in a set of equations entirely in terms of the impedance $\rho c(\tau)$, which allowed recovery of this quantity by layer-stripping. Unfortunately, this is <u>not</u> possible for (32) and (33), since e would also have to be differentially scaled by c, and this brings in other terms. And as long as ρ and c are present separately in these equations, there is no way they can be propagated from knowledge (from the first reflection) of their product ρc alone.

The solution here is to recognize an implicit feature of the 1-D inverse seismic problem: Since the problem takes place along a single vertical ray path, only acoustic (i.e. P-wave) wave propagation along this path need be considered. In the 2-D case, this is tantamount to considering only acoustic wave propagation along a ray path. From the nature of acoustic wave propagation, this means that u_e is negligible ([12], p. 95). (Note that this assumption would simplify the algorithms of the preceding sections.) With this assumption, (32) and (33) become

$$p = -\rho c^2 \, \partial u_s/\partial s \tag{34}$$

$$\partial p/\partial s = -\rho \partial^2 u_s/\partial t^2 \tag{35}$$

which have the same form as the basic 1-D equations. Defining outgoing and incoming waves as

$$O(s,e,t) = p/\sqrt{\rho c} + \sqrt{\rho c} \;\; \partial u_s/\partial t \tag{36a}$$

$$I(s,e,t) = p/\sqrt{\rho c} - \sqrt{\rho c} \;\; \partial u_s/\partial t \tag{36b}$$

and assuming an impulse present in the outgoing wave yields, as in [1], the fast Cholesky equations of the 1-D problem

$$(\partial/\partial\tau + \partial/\partial t) \; O(\tau,e,t) = -r(\tau,e) \; I(\tau,e,t) \tag{37a}$$

$$(\partial/\partial\tau - \partial/\partial t) \; I(\tau,e,t) = -r(\tau,e) \; O(\tau,e,t) \tag{37b}$$

$$r(\tau,e) = 2I(\tau,e,\tau) \tag{38}$$

now applied along each ray (i.e. for each e). Thus instead of reconstructing $\rho c(\tau)$, we now reconstruct $\rho c(\tau,e)$.

A variation on the 1-D problem provides for pure shear wave propagation, with $\rho c(\tau)$ again being reconstructed. For the 2-D problem, we simply neglect u_s instead of u_e. Since (32) and (33) are symmetric in u_s and u_e, the result is once again a fast Cholesky algorithm which reconstructs $\rho c(\tau,e)$.

As in the 1-D problem, some sort of offset experiment, involving the medium responses to impulsive plane waves at two different angles of incidence, is necessary in order to reconstruct ρ and c separately, and as functions of x and z. The 2-D offset problem where the normal to the plane wave lies in the (y,z) plane was solved in the previous section. More desirable would be a solution to the 2-D offset problem where the normal to the plane wave lies in the (x,z) plane (so that all of the action takes place in this plane), but there seems to be no way to relate the different wave front histories resulting from the two experiments to each other.

Why is a 2-D offset experiment necessary in order to reconstruct $\rho(x,z)$ and $c(x,z)$ separately? Considering the numbers of observed quantities, desired quantities, and dependent variables of both sheds some light on this. The following table summarizes the situation for each problem:

Problem	Observations	Total number of dependent variables	Output	Total number of dependent variables
1-D	upgoing wave $U(0,t)$	1	$\rho c(\tau)$	1
1-D offset	velocities $v^1(0,t)$, $v^2(0,t)$	2	$\rho(z)$, $c(z)$	2
2-D	upgoing wave $U(x,0,t)$	2	$\rho c(\tau,e)$	2
2-D offset	velocities $v^1(x,0,t)$, $v^2(x,0,t)$	4	$\rho(x,z)$, $c(x,z)$	4
3-D	upgoing wave $U(x,y,0,t)$	3	$\rho c(\tau,e_1,e_2)$	3
1-D elastic	upgoing P and S waves from P and S excitations	3	$\rho(z)$, $\lambda(z)$, $\mu(z)$	3
$\rho(x,y,z)$	acceleration $w(x,y,0,t)$	3	$\rho(x,y,z)$	3

For the elastic problem, the upgoing waves are constructed from different velocity components, and the converted reflection responses (P to S) and S to P) are the same. Hence there are only three measured quantities, instead of four.

V. References

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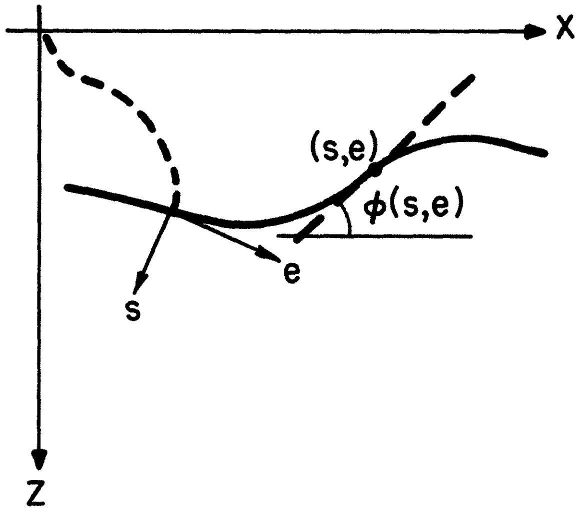

Fig. 1. Definitions of coordinates s and e.

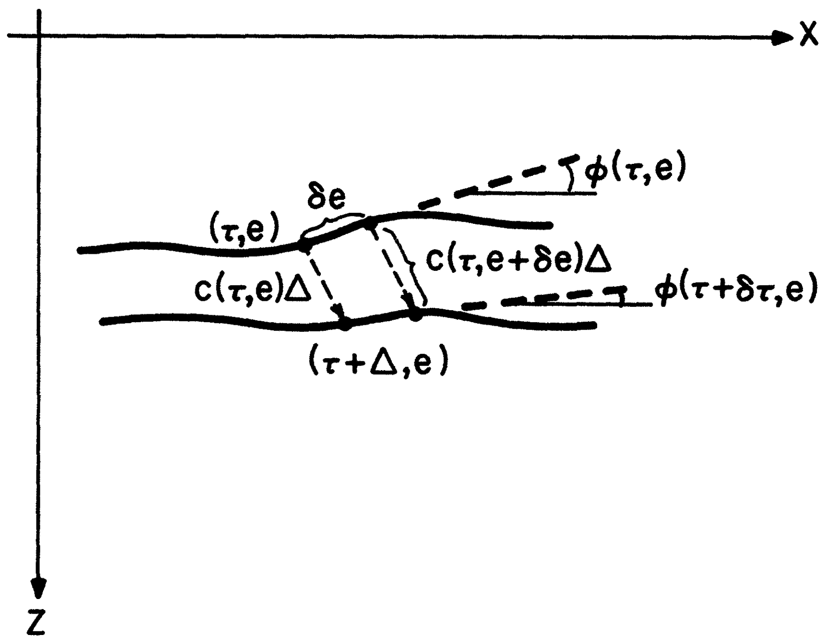

Fig. 2. Derivation of update equation for ϕ.