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On Cohen–Macaulayness of Algebras Generated by Generalized Power Sums

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With an appendix by Misha Feigin

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To Sasha Veselov on his 60th birthday, with admiration.

Abstract: Generalized power sums are linear combinations of ith powers of coordinates. We consider subalgebras of the polynomial algebra generated by generalized power sums, and study when such algebras are Cohen–Macaulay. It turns out that the Cohen–Macaulay property of such algebras is rare, and tends to be related to quantum integrability and representation theory of Cherednik algebras. Using representation theoretic results and deformation theory, we establish Cohen–Macaulayness of the algebra of q,t-deformed power sums defined by Sergeev and Veselov, and of some generalizations of this algebra, proving a conjecture of Brookner, Corwin, Etingof, and Sam. We also apply representation-theoretic techniques to studying m-quasi-invariants of deformed Calogero–Moser systems. In an appendix to this paper, M. Feigin uses representation theory of Cherednik algebras to compute Hilbert series for such quasi-invariants, and show that in the case of one light particle, the ring of quasi-invariants is Gorenstein.

1. Introduction

The Cohen–Macaulay (shortly, CM) property is an important homological property of rings (see [Eis], Chapter 18). For the algebra of functions on an affine algebraic variety X, the CM property means, roughly, that the singularities of X are "not too wild". While this property has many powerful applications, it is rarely satisfied for singularities occurring in a "random" manner, and even when satisfied, it is often hard to prove. On the other hand, it was discovered by Feigin and Veselov that interesting examples of CM algebras arise in the theory of integrable systems, as algebras of quantum integrals of algebraically integrable quantum systems, or algebras of quasiinvariants [FV1]. In these cases, the CM property may be established by using the representation theory of rational Cherednik algebras [BEG]. These results were generalized in [EGL,BCES] using the theory of minimal support modules over rational Cherednik algebras developed

in [EGL]; namely, this theory allows one to prove the CM property of certain S_n-invariant subspace arrangements. Also, in [BCES] the CM property is studied for general S_n-invariant subspace arrangements in $\mathbb{C}^n$ defined by equalities between coordinates. In this paper, we continue this line of work.

Namely, let $a_{ij}, i \geq 1, 1 \leq j \leq N$, be nonzero complex numbers. Let

$$Q_i(x_1, \dots, x_N) = \sum_{j=1}^{N} a_{ij} x_j^i.$$

We call the functions Q_i generalized power sums. Let A be the algebra generated by $Q_i, i \geq 1$ inside $\mathbb{C}[x_1, \dots, x_N]$. For generic a_{ij}, this algebra is finitely generated. The main question studied in this paper is when the algebra A is CM. Specifically, following [BCES], for various collections of positive integers $(r_1, \dots, r_k)$ with $\sum r_i = N$, we study the CM property of algebras of generalized power sums with symmetry type $(r_1, \dots, r_k)$ (i.e., symmetric in the first r_1 variables, the next r_2 variables, etc.).

In Sect. 2, we study the simplest nontrivial case – type $(1, 1)$. In this case, by renormalizing Q_i, we can assume that $a_{i1} = a_i$ and $a_{i2} = 1$, so $Q_i = a_i y^i + z^i$. We show that if a_1, a_2, a_3 are generic, then A is CM if and only if $a_i = c^i \frac{q^i - 1}{1 - t^i}$ for some numbers $c, q, t \in \mathbb{C}$.

In Sect. 3, we extend this analysis to the case of type (r, s). Namely, we show that A is CM if $a_i = c^i \frac{q^i - 1}{1 - t^i}$, i.e., after rescaling the variables y_j,

$$Q_i = \frac{q^i - 1}{1 - t^i}\left(y_1^i + \cdots + y_r^i\right) + \left(z_1^i + \cdots + z_s^i\right)$$

In this case, the algebra A is the algebra of q, t-deformed Newton sums introduced by Sergeev and Veselov in [SV2]. If $t = q^{-n}$, where n is a positive integer, this is a subalgebra of the algebra of quantum integrals of the deformed Macdonald–Ruijsenaars system. Our proof of the CM property of this algebra is based on degeneration to the classical case,

$$Q_i = a\left(y_1^i + \cdots + y_r^i\right) + \left(z_1^i + \cdots + z_s^i\right)$$

(obtained by setting $q = t^{-a}$ and tending t to 1) where the CM property is established in [BCES] based on the methods of [EGL] (namely, the representation theory of rational Cherednik algebras with minimal support).

In Sect. 4, we study the case of type $(1, r, s)$. We show that in this case the CM property occurs generically for the generalized power sums

$$\frac{q^i - t^i}{1 - t^i} x^i + \frac{q^i - 1}{1 - t^i}\left(y_1^i + \cdots + y_r^i\right) + \left(z_1^i + \cdots + z_s^i\right).$$

and their degenerations

$$\left(a + 1\right)x^i + a\left(y_1^i + \cdots + y_r^i\right) + \left(z_1^i + \cdots + z_s^i\right)$$

(again obtained by setting $q = t^{-a}, t \to 1$). Namely, we prove this by reduction to type $(r + 1, s + 1)$. For $r = 1$, this confirms the first statement of Conjecture 7.4 in [BCES].

We also show that in all of the above cases, the CM algebras A can be defined by quasi-invariance conditions on hyperplanes. In the case $(1, r, s)$, these quasi-invariance conditions appear to be new.

In Sect. 5 we use a similar method to the one of Sect. 4 to prove that for any $m \geq 1$, $n \geq 3$, the union of S_{mn}-translates of the subspace

$$x_1 = \cdots = x_{2m},\ x_{2m+1} = \cdots = x_{3m},\ \ldots, x_{(n-1)m+1} = \cdots = x_{nm}$$

(i.e., one group of $2m$ equal coordinates and $n-2$ groups of m equal coordinates) is CM. This is done by reducing to the case of n m-tuples of equal coordinates, where the result is proved using representations of rational Cherednik algebras in [EGL].

In Sect. 6, we apply the theory of representations of the rational Cherednik algebra of minimal support to m-quasi-invariants considered in [FV1,FV2].

In the Appendix, Feigin uses this approach to prove a conjecture from [FV2] that the algebra of m-quasi-invariants in the case of one light particle ($s = 1$) is Gorenstein.

In particular, this paper explains all the instances of CM algebras found experimentally in [BCES], and confirms the philosophy (originating from [FV1] and developed further in [BEG]) that the CM property of algebras of this type should be rare, and, whenever it occurs, should be related to quasi-invariance conditions on hyperplanes, quantum integrable systems, and ultimately to representation theory. In fact, the proofs of the CM property in all the multivariate cases in this paper are ultimately based on the representation theory of Cherednik algebras.[1]

2. Type (1, 1)

2.1. The algebra $\Lambda_{\mathbf{a}}$. Let $\mathbf{a} = (a_1, a_2, ...)$ be a sequence of nonzero complex numbers. Let $\Lambda_{\mathbf{a}}$ be the subalgebra of $\mathbb{C}[y, z]$ generated by the polynomials Q_{i,a_i}, $i \geq 1$, where

$$Q_{i,a} := ay^i + z^i.$$

(When no confusion is possible, we will denote Q_{i,a_i} simply by Q_i.)

We will be interested in the question when the algebra $\Lambda_{\mathbf{a}}$ is CM. Note that by renormalizing y, we may replace a_i by a_i/a_1^i, and thus assume that $a_1 = 1$.

2.2. Commutative algebra preliminaries and auxiliary lemmas. Recall first that if $p_1, \ldots, p_n$ are homogeneous polynomials in $x_1, \ldots, x_n$ of positive degrees then $\mathbb{C}[p_1, \ldots, p_n]$ is module-finite over $\mathbb{C}[x_1, \ldots, x_n]$ if and only if the system of equations $p_1(\mathbf{x}) = 0, \ldots, p_n(\mathbf{x}) = 0$ has only the zero solution. Indeed, since p_i are homogeneous, by the Nakayama Lemma module finiteness is equivalent to the condition that the algebra $\mathbb{C}[x_1, \ldots, x_n]/(p_1, \ldots, p_n)$ is finite dimensional. This condition is equivalent to the zero set of $p_1, \ldots, p_n$ being finite. But this set is invariant under dilations, so it is finite iff it consists only of the origin.

We also recall basics on Cohen–Macaulay algebras, see [Eis], Chapter 18. Let R be a finitely generated $\mathbb{C}$-algebra. By the Noether normalization lemma, there exist algebraically independent $z_1, \ldots, z_n \in R$ such that R is module-finite over $\mathbb{C}[z_1, \ldots, z_n]$.

[1] We note, however, that we do not know a direct relation of the CM algebras of Sect. 4 to integrable systems or representation theory. It would be interesting to find such a relation.

The algebra R is called Cohen–Macaulay if R is a locally free module over $\mathbb{C}[z_1, \dots, z_n]$.[2] By Serre's theorem, if this property holds for some choice of $z_1, \dots, z_n$, then it holds for any such choice.

In addition, we need a few auxiliary lemmas.

Lemma 2.1. *If $a_2 + a_1^2 \neq 0$ then the algebra $\Lambda_{\mathbf{a}}$ is finitely generated as a module over the polynomial algebra $\mathbb{C}[Q_1, Q_2]$ (in particular, as an algebra).*

Proof. We may assume that $a_1 = 1$. It suffices to show that the equations $Q_1(y, z) = 0$, $Q_2(y, z) = 0$, i.e.

$$y + z = 0,\ a_2 y^2 + z^2 = 0$$

have only the zero solution (then the entire polynomial algebra $\mathbb{C}[y, z]$ is module-finite over $\mathbb{C}[Q_1, Q_2]$, so $\Lambda_{\mathbf{a}}$ is as well, by the Hilbert basis theorem). From the first equation we get $z = -y$, and substituting this into the second one, we get $(a_2 + 1)y^2 = 0$. Since $a_2 \neq -1$, we have $y = z = 0$. □

Lemma 2.2. *Let $a_2 \neq -a_1^2$, and $(a_2, a_3) \neq (a_1^2, a_1^3)$. Let $M \subset \Lambda_{\mathbf{a}}$ be the submodule over $\mathbb{C}[Q_1, Q_2]$ generated by 1 and Q_3. Then M is free of rank 2, so its Hilbert series is*

$$h(u) = \frac{1 + u^3}{(1 - u)(1 - u^2)}.$$

Proof. We may assume that $a_1 = 1$. First, we claim that $Q_3 \notin \mathbb{C}[Q_1, Q_2]$. Assume the contrary, that $Q_3 = \alpha Q_1^3 + \beta Q_1 Q_2$. Then from comparing coefficients we have

$$\alpha + \beta a_2 = a_3,\ 3\alpha + \beta a_2 = 0,\ \alpha + \beta = 1,\ 3\alpha + \beta = 0,$$

which implies that $(a_2, a_3) = (1, 1)$, a contradiction.

Since by Lemma 2.1, $\Lambda_{\mathbf{a}}$ is a finitely generated $\mathbb{C}[Q_1, Q_2]$-module, this implies that $Q_3 \notin \mathbb{C}(Q_1, Q_2)$ (as Q_3 is integral over $\mathbb{C}[Q_1, Q_2]$ and $\mathbb{C}[Q_1, Q_2]$ is integrally closed). Thus, the module M is indeed free with the stated Hilbert series. □

Now suppose that the assumptions of Lemma 2.2 are satisfied. The rank of $\Lambda_{\mathbf{a}}$ over $\mathbb{C}[Q_1, Q_2]$ is 2, since the system of equations $Q_1 = c_1$, $Q_2 = c_2$ has two solutions for generic $c_1, c_2 \in \mathbb{C}$. Thus, $\Lambda_{\mathbf{a}}$ is CM if and only if it coincides with M. Hence, since $\frac{1}{(1-u)^2} - h(u) = \frac{u}{1-u}$, we obtain the following lemma.

Lemma 2.3. *Under the assumptions of Lemma 2.2, the CM property of $\Lambda_{\mathbf{a}}$ is equivalent to saying that the codimension of $\Lambda_{\mathbf{a}}[i]$ in homogeneous polynomials of degree i is 1 for each $i \geq 1$.*

Note that this codimension is clearly at most 1.

[2] By the Quillen–Suslin theorem, any locally free finitely generated module over a polynomial algebra is free, but this is not important for us here.

2.3. The CM property of $\Lambda_{\mathbf{a}}$. Let q, t be not roots of unity, $t \neq q, q^{-1}$, and $c \neq 0$.

Theorem 2.4. (i) *If* $a_i = c^i \frac{q^i-1}{1-t^i}$ *for* $i \geq 1$ *then the algebra* $\Lambda_{\mathbf{a}}$ *is CM with Hilbert series* $h(u)$.

(ii) *Let* $\mathbf{a}$ *be any sequence of nonzero numbers, and* c, q, t *be such that* $a_i = c^i \frac{q^i-1}{1-t^i}$ *for* $i = 1, 2, 3$. *Assume that* q, t *are not roots of unity, and* $t \neq q, q^{-1}$. *If* $\Lambda_{\mathbf{a}}$ *is CM, then* $a_i = c^i \frac{q^i-1}{1-t^i}$ *for all* $i \geq 1$.

(iii) *Let* $a_i = c^i a$, *where* $a \neq \pm 1$. *Then the algebra* $\Lambda_{\mathbf{a}}$ *is CM with Hilbert series* $h(u)$.

(iv) *If* $a_i = c^i a$ *with* $a \neq \pm 1$ *for* $i = 1, 2, 3$, *and if* $\Lambda_{\mathbf{a}}$ *is CM, then* $a_i = c^i a$ *for all* $i \geq 1$.

Remark 2.5. It is easy to show that for generic a_1, a_2, a_3 the equations

$$a_i = c^i \frac{q^i - 1}{1 - t^i}, i = 1, 2, 3$$

lead to a quadratic equation, and thus have two solutions (c, q, t), related by the Galois symmetry $(c, q, t) \to (cqt^{-1}, q^{-1}, t^{-1})$. In particular, for generic a_1, a_2, a_3 a solution (c, q, t) exists, and Theorem 2.4(ii) applies.

Proof. By renormalizing y, we may assume without loss of generality that $c = 1$. Let us make this assumption.

(i) Any element $f \in \Lambda_{\mathbf{a}}$ satisfies the quasi-invariance condition

$$f(tx, qx) = f(x, x).$$

Indeed, this condition is satisfied for each generator Q_i, and if it is satisfied for f and g then it is satisfied for fg. This gives a codimension 1 subspace in $\mathbb{C}[y, z][i]$ for all $i \geq 1$ (since the function z^i does not satisfy this condition, as t is not a root of unity). So by Lemma 2.3, the result holds under the assumptions of Lemma 2.2. In terms of q and t, these assumptions turn into the conditions that $qt \neq 1$ and $q \neq t$, so they are satisfied.

(iii) is a limiting case of (i) (for $q = t^{-a}$ and $t \to 1$), so in this case $f \in \Lambda_{\mathbf{a}}$ satisfies the limiting quasi-invariance condition

$$((a\partial_2 - \partial_1) f)(x, x) = 0,$$

giving a codimension 1 condition in each positive degree. The assumptions of Lemma 2.2 in this case turn into the conditions $a \neq \pm 1$, so they are satisfied, and Lemma 2.3 implies the statement.

(ii) Let $i \geq 4$. By Lemma 2.2, homogeneous polynomials of degree i in Q_1, Q_2, Q_3 (linear in Q_3) span a subspace of codimension 1 in $\mathbb{C}[y, z][i]$ — the space of solutions of the quasi-invariance equation $f(tx, qx) = f(x, x)$. So Q_i must also satisfy this condition. Thus, $t^i a_i + q^i = a_i + 1$, i.e., $a_i = \frac{q^i-1}{1-t^i}$, as desired.

(iv) The proof is similar to (ii), except that we use the limiting quasi-invariance condition $((a\partial_2 - \partial_1) f)(x, x) = 0$. □

Remark 2.6. In spite of Theorem 2.4, there exist infinite-parameter families of sequences $\mathbf{a}$ for which $\Lambda_{\mathbf{a}}$ is CM. For instance, if q and t are primitive nth roots of unity with $t \neq q, q^{-1}$, then for any sequence with $a_i = c^i (q^i - 1)/(1 - t^i)$ when i is not a multiple of n, the corresponding $\Lambda_{\mathbf{a}}$ is CM. Indeed, in that case, the algebra generated by Q_1,

Q_2, Q_3 is determined by the same (codimension 1) quasi-invariance condition as in the generic case, but since both x^{mn} and y^{mn} are quasi-invariant, any linear combination of them is contained in the algebra. (Note that in this case $\Lambda_{\mathbf{a}}$ does not actually depend on $a_n, a_{2n}, \ldots$).

3. Type (r, s)

3.1. Finite generation. First let us prove a general result on finite generation (which is fairly standard, see e.g. [SV2], Theorem 5.1). Let $a_{ij}, i \geq 1, 1 \leq j \leq N$, be nonzero complex numbers. Let $Q_i(x_1, \ldots, x_N) = \sum_{j=1}^N a_{ij} x_j^i$. Let A be the algebra generated by $Q_i, i \geq 1$ inside $\mathbb{C}[x_1, \ldots, x_N]$.

Proposition 3.1. *A is finitely generated if and only if the system of equations*

$$Q_i(x_1, \ldots, x_N) = 0, \ i \geq 1 \tag{1}$$

has only the zero solution.

Proof. Suppose the system (1) has only the zero solution. By the Hilbert basis theorem, there is $k \geq 1$ such that this is true already for the first k equations. Then $\mathbb{C}[x_1, \ldots, x_N]$ is a finitely generated module over $\mathbb{C}[Q_1, \ldots, Q_k]$, and hence, by the Hilbert basis theorem, so is A. Thus, A is finitely generated as an algebra.

Conversely, suppose (1) has a nonzero solution $(x_1^*, \ldots, x_N^*)$. Without loss of generality we can assume that $x_1^* \neq 0$. Let $x_1 = y x_1^*$ and $x_i = z x_i^*$ for $i \geq 2$, where y, z are new variables. Then Q_i specialize to $Q_i^*(y, z) = a_{i1} x_1^{*i} (y^i - z^i)$. So we just need to show that the algebra generated by the polynomials $f_i(y, z) := y^i - z^i$ is not finitely generated. But this is easy and well known (see e.g. [BCES], Remark 2.7(3)). □

3.2. The algebra $\Lambda_{r,s,\mathbf{a}}$ and its CM properties. Now let $r, s \geq 1$ be integers, and

$$Q_{r,s,i,a}(y_1, \ldots, y_r, z_1, \ldots, z_s) := a(y_1^i + \cdots + y_r^i) + z_1^i + \cdots + z_s^i.$$

Let $\mathbf{a} = (a_1, a_2, \ldots)$ be a sequence of nonzero complex numbers, and $\Lambda_{r,s,\mathbf{a}}$ be the subalgebra of $\mathbb{C}[y_1, \ldots, y_r, z_1, \ldots, z_s]$ generated by Q_{r,s,i,a_i} for all $i \geq 1$. When no confusion is possible, we will denote Q_{r,s,i,a_i} simply by Q_i.

By Proposition 3.1, $\Lambda_{r,s,\mathbf{a}}$ is finitely generated if and only if the system

$$a_i(y_1^i + \cdots + y_r^i) + z_1^i + \cdots + z_s^i = 0, \quad i \geq 1 \tag{2}$$

has only the zero solution. If $r = 1$, this implies that the algebra is infinitely generated iff $a_i = -(\beta_1^i + \cdots + \beta_s^i)$ for some $\beta_1, \ldots, \beta_s \in \mathbb{C}$, and a similar statement holds for a_i^{-1} for $s = 1$. However, if $r, s \geq 2$, the set of sequences violating finite generation is infinite dimensional. For example, taking $y_1 = 1, y_2 = -1, z_1 = 1, z_2 = -1$, and the rest of y_j, z_l to be zero, we get that the sequence with $a_i = -1$ for even i and a_i arbitrary for odd i violates finite generation.

We would like to know when $\Lambda_{r,s,\mathbf{a}}$ is CM. Note that as before, we may assume that $a_1 = 1$ (or any other nonzero constant) by renormalizing y_i.

Our first result is the following theorem.

Let $c \neq 0, q, t$ be not roots of unity, and $a_i = c^i \frac{q^i - 1}{1 - t^i}$. For $c = t$, this is the algebra of q, t-deformed Newton sums (see [SV2], Sect. 5).

Theorem 3.2. (i) ([SV2], *Theorem* 5.1) $\Lambda_{r,s,\mathbf{a}}$ *is finitely generated if and only if* $q^m \neq t^n$ *for integers* $1 \leq m \leq r$, $1 \leq n \leq s$.

(ii) *If* q, t *are Weil generic (i.e., outside of a countable union of curves) then* $\Lambda_{r,s,\mathbf{a}}$ *is CM with Hilbert series*

$$h_{r,s}(u) = \frac{1}{(u,u)_r} \sum_{i=0}^{s} \frac{u^{i(r+1)}}{(u,u)_i}.$$

where $(u,u)_m := (1-u)\dots(1-u^m)$.

Note that Theorem 3.2 is a generalization (or, more precisely, a deformation) of the following theorem.

Let $\Lambda_{r,s,a}$ be the algebra corresponding to the sequence $a_i = a$.

Theorem 3.3. ([SV1], *Theorem* 5) $\Lambda_{r,s,a}$ *is finitely generated if and only if* $a \neq -n/m$ *for integers* $1 \leq m \leq r$, $1 \leq n \leq s$.

(ii)([BCES], *Theorem* 4.4) *For generic* a *the algebra* $\Lambda_{r,s,a}$ *is CM with Hilbert series* $h_{r,s}(u)$.

Remark 3.4. 1. By analogy with Conjecture 4.8 of [BCES], we expect that the exceptional set for Theorem 3.2(ii) is $q^m = t^{\pm n}$, where $1 \leq m \leq r$, $1 \leq n \leq s$ (assuming $q, t \neq 0$ and are not roots of unity).

2. The formula for the Hilbert series in Theorem 3.3(ii) is given in [SV1] and in the q,t-case in [SV2].

Proof. Without loss of generality, we may assume that $c = 1$.

(i) This is proved in [SV2], but we reproduce the proof for reader's convenience. Consider the system of equations $Q_i = 0$, $i \geq 1$. It can be written as

$$\sum_{j=1}^{r} (qy_j)^i + \sum_{l=1}^{s} z_l^i = \sum_{j=1}^{r} y_j^i + \sum_{l=1}^{s} (tz_l)^i. \tag{3}$$

Suppose that this system has a nontrivial solution. Let m be the number of nonzero coordinates y_j and n be the number of nonzero coordinates z_l in this solution. Since (3) holds for each i, each nonzero term on the LHS must equal some nonzero term on the RHS. By taking products, this implies that $q^m = t^n$. Note that $m, n > 0$ since $m+n > 0$ and q, t are not roots of 1. Conversely, suppose $q^m = t^n$. If $q = t = 0$, then (3) has a nonzero solution $y_1 = z_1 = 1$, $y_j = z_l = 0$ for $j, l \geq 2$. If q, t are not both zero, then taking $y_1 = t^n$, $y_2 = t^n q, \dots, y_n = t^n q^{m-1}$, $z_1 = q^m t^{n-1}, \dots, z_{n-1} = q^m t$, $z_n = q^m$, and the rest of y_j and z_l to be zero, we also obtain a nontrivial solution. Thus, the result follows from Proposition 3.1.

(ii) Let $q = t^{-a}$ and $t \to 1$. Then $\Lambda_{r,s,\mathbf{a}}$ degenerates to $\Lambda_{r,s,a}$. By Theorem 3.3(ii), for generic a, the algebra $\Lambda_{r,s,a}$ is a free module of finite rank over $\mathbb{C}[Q_1, \dots, Q_{r+s}]$, with Hilbert series $h_{r,s}(u)$. Thus, our job is to show that for Weil generic q, t, the Hilbert series of $\Lambda_{r,s,\mathbf{a}}$ is dominated by $h_{r,s}(u)$ coefficientwise (this will imply that it actually equals to $h_{r,s}(u)$).

This is proved in [SV2], Sect. 5, and we reproduce the proof for reader's convenience. Let Λ be the ring of symmetric functions, and define a surjective homomorphism $\phi : \Lambda \to \Lambda_{r,s,\mathbf{a}}$ given by the formula $\phi(p_i) = Q_i$, where p_i are the power sums. By Theorem 5.6 of [SV2], for generic q, t, $\mathrm{Ker}\phi$ has a basis consisting of Macdonald polynomials P_λ,

where λ is a Young diagram that does not fit into the fat (r,s)-hook (i.e., $\lambda_{r+1} > s$), while $\Lambda_{r,s,\mathbf{a}}$ has a basis formed by $P_\lambda(q,t)$ for λ fitting into the (r,s)-hook (i.e., $\lambda_{r+1} \le s$), with the Hilbert series $h_{r,s}(u)$. This means that the kernel does not shrink as we deform the limiting case to Weil generic q,t, as desired. □

If $a_i = \frac{q^i-1}{1-t^i}$, we will denote the algebra $\Lambda_{r,s,\mathbf{a}}$ by $\Lambda_{r,s,q,t}$.

3.3. The quasi-invariance conditions. Below we will use the following proposition, due to Sergeev and Veselov.

Proposition 3.5. (i) ([SV2]) *If $q,t \in \mathbb{C}^\times$ are not roots of unity, and $q^m \neq t^n$ for $n,m \ge 1$ then $\Lambda_{r,s,q,t}$ for $a_i = \frac{q^i-1}{1-t^i}$ is the algebra of symmetric polynomials in y_j and in z_l satisfying the quasi-invariance conditions*

$$f(y_1,\dots,ty_j,\dots,y_r,z_1,\dots,qz_l,\dots,z_s) = f(y_1,\dots,y_j,\dots,y_r,z_1,\dots,z_l,\dots,z_s), \tag{4}$$

when $y_j = z_l$ for all $j \in [1,r], l \in [1,s]$.

(ii) ([SV1]) *If a is generic then $\Lambda_{r,s,a}$ is the algebra of symmetric polynomials in y_j and in z_l satisfying the quasi-invariance conditions*

$$\left(\left(a\partial_{z_l} - \partial_{y_j}\right)f\right)(y_1,\dots,y_j,\dots,y_r,z_1,\dots,z_l,\dots,z_s) = 0, \tag{5}$$

when $y_j = z_l$ for all $j \in [1,r], l \in [1,s]$.

4. Type $(1, r, s)$

4.1. The result. As before, let $q,t \in \mathbb{C}^\times$ be not roots of unity such that $q \neq t$. Let r,s be positive integers. Consider the polynomials

$$P_{r,s,i,q,t} := \frac{q^i - t^i}{1-t^i}x^i + \frac{q^i-1}{1-t^i}\left(y_1^i + \cdots + y_r^i\right) + \left(z_1^i + \cdots + z_s^i\right).$$

Let $A_{r,s,q,t}$ be the algebra generated by the $P_{r,s,i,q,t}$, $i \ge 1$.

We will also be interested in the limiting case $q = t^{-a}$, $t \to 1$. In this limit, we get the polynomials

$$P_{r,s,i,a} := (a+1)x^i + a\left(y_1^i + \cdots + y_r^i\right) + \left(z_1^i + \cdots + z_s^i\right)$$

Let $A_{r,s,a}$ be the algebra generated by the $P_{r,s,i,a}$, $i \ge 1$.

In both cases, when no confusion is possible, we will denote the generating polynomials simply by P_i.

Note that if $a_i = \frac{q^i-1}{1-t^i}$ then the restriction of $Q_{r+1,s+1,i,a_i}$ to the hyperplane $y_{r+1} = z_{s+1}$ is $P_{r,s,i,q,t}$, where $x = y_{r+1} = z_{s+1}$. Similarly, the restriction of $Q_{r+1,s+1,i,a}$ is $P_{r,s,i,a}$. Thus, we have an epimorphism $\phi_{q,t} : \Lambda_{r+1,s+1,q,t} \to A_{r,s,q,t}$, which degenerates to an epimorphism $\phi_a : \Lambda_{r+1,s+1,a} \to A_{r,s,a}$.

Theorem 4.1. (i) *The algebra $A_{r,s,a}$ is CM for generic a. Moreover, the Hilbert series of this algebra is given by the formula*

$$h_{A_{r,s,a}}(u) = h_{\Lambda_{r+1,s+1,a}}(u) - \frac{u^{2(r+1)(s+1)}}{(u,u)_{r+1}(u,u)_{s+1}}.$$

(ii) *For Weil generic q, t, the algebra $A_{r,s,q,t}$ is CM with the same Hilbert series.*

In the special case $r = 1$, this confirms the first part of Conjecture 7.4 in [BCES]. A proof of Theorem 4.1 is given in the next subsection.

4.2. Proof of Theorem 4.1. We will need the following simple lemma from homological algebra.

Lemma 4.2. *Let C be a commutative algebra, I an ideal in C, and C' a subalgebra of C containing I. Let $B \subset C'$ be a subalgebra such that $C, C', C/I$ are all projective modules over B. Then so is C'/I.*

Proof. The short exact sequence

$$0 \to C' \to C \to C/C' \to 0$$

is a B-projective resolution of C/C', which therefore has homological dimension ≤ 1. Since C/I is B-projective, the short exact sequence

$$0 \to C'/I \to C/I \to C/C' \to 0$$

must also be a projective resolution, and thus C'/I is projective. □

We will apply Lemma 4.2 in the following situation:

$$C = \mathbb{C}\big[y_1, \dots, y_{r+1}, z_1, \dots, z_{s+1}\big]^{S_{r+1} \times S_{s+1}},\ C' = \Lambda_{r+1,s+1,a},\ I = \mathrm{Ker}\phi_a.$$

For this, we need to prove another auxiliary lemma.

Lemma 4.3. *I is an ideal in C. More precisely, I is the principal ideal generated by the polynomial*

$$D_{r+1,s+1}(\mathbf{y}, \mathbf{z}) := \prod_{j=1}^{r+1} \prod_{l=1}^{s+1} (y_j - z_l)^2,$$

and thus its Hilbert series is given by the formula

$$h_I(u) = \frac{u^{2(r+1)(s+1)}}{(u;u)_{r+1}(u;u)_{s+1}},$$

Proof. By Proposition 3.5(ii), C' is the algebra of polynomials symmetric in y_j and z_l and satisfying the quasi-invariance condition

$$\Big(\big(a\partial_{y_j} - \partial_{z_l}\big)f\Big)\big(y_1, \dots, y_j, \dots, y_{r+1}, z_1, \dots, z_l, \dots, z_{s+1}\big) = 0$$

when $y_j = z_l$ for all $j \in [1, r+1]$, $l \in [1, s+1]$. This implies that $D_{r+1,s+1}C \subset I \subset C'$ (as the restriction of $D_{r+1,s+1}$ to the hyperplane $y_{r+1} = z_{s+1}$ is zero, and any multiple of $D_{r+1,s+1}$ satisfies the quasi-invariance condition). Also, if $f \in I$ then its restriction to the hyperplane $y_j = z_l$ is zero and it satisfies the quasi-invariance condition, so must be divisible by $(y_j - z_l)^2$. Thus by symmetry f is divisible by $D_{r+1,s+1}$. Thus $f \in D_{r+1,s+1}C$ and $D_{r+1,s+1}C = I$. This implies all statements of the lemma. □

Now we prove part (i) of the theorem. To apply Lemma 4.2, we will now define $B := \mathbb{C}[Q_1, \dots, Q_{r+s+1}]$ (where $Q_i := Q_{r+1,s+1,i,a}$). Then C is clearly free over B (of infinite rank), as it is free of finite rank over $\mathbb{C}[Q_1, \dots, Q_{r+s+2}]$ by Serre's theorem (since C is a polynomial algebra). Also, C' is free over B (of infinite rank), as it is free of finite rank over $\mathbb{C}[Q_1, \dots, Q_{r+s+2}]$ by Theorem 3.3(ii) (since C' is a CM algebra). Finally, C/I is CM (as I is a principal ideal). So to show that C/I is free over B, it suffices to show that it is finitely generated as a module, i.e., the system of equations

$$Q_{r,s,i,a}(\mathbf{y}, \mathbf{z}) = 0,\ i = 1, \dots, r+s+1;\ D_{r+1,s+1}(\mathbf{y}, \mathbf{z}) = 0$$

has only the zero solution. By symmetry we may assume that $y_{r+1} = z_{s+1}$, so, substituting, we get

$$P_i(x, \mathbf{y}, \mathbf{z}) = 0,\ i = 1, \dots, r+s+1,$$

which we know has only the zero solution (see [BCES], proof of Proposition 2.6). Thus, by Lemma 4.2, $C'/I = A_{r,s,a}$ is a free module over B. It is also a finitely generated module. This implies that $A_{r,s,a}$ is a CM algebra with the claimed Hilbert series, as desired.

Let us now prove part (ii) of the theorem. Since the algebra $A_{r,s,q,t}$ is generated by polynomials which deform the polynomials generating $A_{r,s,a}$, it suffices to show that the Hilbert series $h_{A_{r,s,q,t}}(u)$ is dominated coefficientwise by the Hilbert series $h_{A_{r,s,a}}(u)$ (this will imply that these two series are actually the same). By Theorem 3.3(ii) and Theorem 3.2, The Hilbert series of $\Lambda_{r,s,q,t}$ and $\Lambda_{r,s,a}$ are the same, so it suffices to check that the Hilbert series of $\mathrm{Ker}\phi_{q,t}$ is dominated from below by the Hilbert series of $\mathrm{Ker}\phi_a$.

To this end, let

$$D_{r+1,s+1,b}(\mathbf{y}, \mathbf{z}) := \prod_{j=1}^{r+1}\prod_{l=1}^{s+1}\big(y_j - z_l\big)\big(y_j - bz_l\big).$$

Then any multiple of $D_{r+1,s+1,tq^{-1}}$ satisfies the quasi-invariance condition of Proposition 3.5(i), so $D_{r+1,s+1,tq^{-1}}C \subset \mathrm{Ker}\phi_{q,t}$, giving the desired lower bound for the Hilbert series.

4.3. The quasi-invariant description of $A_{r,s,a}$ and $A_{r,s,q,t}$. The construction of the algebras $A_{r,s,a}$ and $A_{r,s,q,t}$ implies that they can be described by quasi-invariance conditions on hyperplanes. Namely, we have the following result.

Proposition 4.4. (i) *For Weil generic q, t the algebra $A_{r,s,q,t}$ is the algebra of polynomials $f(x, \mathbf{y}, \mathbf{z})$ symmetric under $S_r \times S_s$ which satisfy the following quasi-invariance conditions:*

(a) $f(x; y_1, \dots, y_{r-1}, u; z_1, \dots, z_{s-1}, u) = f(u; y_1, \dots, y_{r-1}, x; z_1, \dots, z_{s-1}, x)$;

(b) $f(x; y_1, \dots, y_{r-1}, u; z_1, \dots, z_{s-1}, u) = f(x; y_1, \dots, y_{r-1}, tu; z_1, \dots, z_{s-1}, qu)$;

(c) $f(x; y_1, \dots, y_{r-1}, tq^{-1}x; z_1, \dots, z_s) = f(q^{-1}x; y_1, \dots, y_{r-1}, x; z_1, \dots, z_s)$;

(d) $f(x; y_1, \dots, y_r; z_1, \dots, z_{s-1}, qt^{-1}x) = f(xt^{-1}; y_1, \dots, y_r; z_1, \dots, z_{s-1}, x)$.

(ii) *For generic a the algebra $A_{r,s,a}$ is the algebra of polynomials $f(x, \mathbf{y}, \mathbf{z})$ symmetric under $S_r \times S_s$ which satisfy the following quasi-invariance conditions:*

(a) $f(x; y_1, \dots, y_{r-1}, u; z_1, \dots, z_{s-1}, u) = f(u; y_1, \dots, y_{r-1}, x; z_1, \dots, z_{s-1}, x)$;

(b) $((\partial_{y_r} - a\partial_{z_s})f)(x; y_1, \dots, y_{r-1}, u; z_1, \dots, z_{s-1}, u) = 0$;

(c) $(((a+1)\partial_{y_r} - a\partial_x)f)(x; y_1, \dots, y_{r-1}, x; z_1, \dots, z_s) = 0$;

(d) $(((a+1)\partial_{z_s} - \partial_x)f)(x; y_1, \dots, y_r; z_1, \dots, z_{s-1}, x) = 0$.

Proof. Let us prove (i). It is easy to check that conditions (a)–(d) (together with the $S_r \times S_s$-symmetry) are exactly the restriction of the quasi-invariance conditions of Proposition 3.5(i) for $\Lambda_{r+1,s+1,q,t}$ to the hyperplane $y_{r+1} = z_{s+1}$ (i.e., they define the equivalence relation on points induced by restricting the relation of Proposition 3.5(i) to this hyperplane). This implies the desired statement. The proof of (ii) is similar, using an infinitesimal version of this argument (as the equations (a)–(d) of (ii) are the infinitesimal versions of equations (a)–(d) of (i)). □

5. The CM Property of Subspace Arrangements of Type $(2m, m, \dots, m)$.

In this section we will use the same method as in the previous section to prove the following result about CM-ness of subspace arrangements, in the spirit of [BCES]. Namely, for a partition λ let X_λ be the union of subspaces in $\mathbb{C}^{|\lambda|}$ defined by the condition that some λ_1 coordinates are the same, some other λ_2 coordinates are the same, etc.

Theorem 5.1. *The variety $X_{(2m,m^{(r)})}$ is CM for any $r \geq 0$ and $m \geq 1$.*

Proof. Let $n = r+2$. Consider the variety $X_{m^{(n)}}$. Recall that $X_{m^{(n)}}$ is CM ([EGL], Proposition 3.11). The algebra $O(X_{m^{(n)}})$ can be viewed as a subalgebra of its normalization $O(\widetilde{X}_{m^{(n)}})$, a direct sum of polynomial rings. Let $I_{m^{(n)}}$ be the kernel of the morphism $O(X_{m^{(n)}}) \to O(X_{(2m,m^{(n-2)})})$, which we may again view as a module over $O(\widetilde{X}_{m^{(n)}})$.

Lemma 5.2. *$I_{m^{(n)}}$ is a principal ideal in $O(\widetilde{X}_{m^{(n)}})$.*

Proof. A point $\mathbf{x} = (x_1, x_2, \dots, x_{mn})$ with $x_1 = \dots = x_m, x_{m+1} = \dots = x_{2m}, \dots$ is in $X_{(2m,m^{(n-2)})}$ iff two of its m-blocks are equal, and thus a function in $I_{m^{(n)}}$ must be a multiple of the discriminant on each component in $\widetilde{X}_{m^{(n)}}$. Conversely, since the discriminant on one component vanishes on all other components, we find that any function on $\widetilde{X}_{m^{(n)}}$ which is a multiple of the discriminant in each summand is actually in $I_{m^{(n)}}$. It follows that the restriction of $I_{m^{(n)}}$ to each direct summand of $\widetilde{X}_{m^{(n)}}$ is the principal ideal generated by the discriminant, and thus $I_{m^{(n)}}$ is itself a principal ideal. □

Now, if we extend a generator of $I_{m^{(n)}}$ by a generic sequence of linear polynomials, the result will be a regular sequence, as it is regular in each direct summand of $O(\widetilde{X}_{m^{(n)}})$. Let B be the polynomial ring generated by the chosen sequence of linear polynomials. Then (since a generic sequence of linear polynomials is a regular sequence for $X_{m^{(n)}}$, and since the latter is CM) we have a chain of free B-modules:

$$I_{m^{(n)}} \subset O(X_{m^{(n)}}) \subset O(\widetilde{X}_{m^{(n)}}),$$

and thus a short exact sequence of B-modules of homological dimension 1:

$$0 \to O(X_{m^{(n)}})/I_{m^{(n)}} \to O(\widetilde{X}_{m^{(n)}})/I_{m^{(n)}} \to O(\widetilde{X}_{m^{(n)}})/O(X_{m^{(n)}}) \to 0$$

The middle term is free of finite rank since the algebra in the middle is CM (the function algebra on a disjoint union of hypersurfaces). Thus, so is $O(X_{m^{(n)}})/I_{m^{(n)}} = O(X_{(2m,m^{(r)})})$. Hence, $X_{(2m,m^{(r)})}$ is a CM variety, as desired. □

On the basis of the results of [BCES] and this paper, as well as computer calculations, we state the following conjecture.

Conjecture 5.3. X_λ is CM if and only if one of the following holds:
(1) $\lambda = (m^{(r)}, 1^{(s)})$ with $r \geq 1, m > s \geq 0$;
(2) $\lambda = (2^{(r)}, 1^{(s)})$ for $r \geq 1, s \geq 0$;
(3) $\lambda = (2m, m^{(s)})$, $m \geq 1$.

Note that the "if" part of the conjecture is known, and only the "only if" part is in question.

6. *m*-Quasi-Invariants

6.1. Rational m-quasi-invariants. Let $m \geq 1, r \geq 2, s \geq 1$ be integers. Following the paper [FV2] (which treats the case $s = 1$), define the algebra $\Lambda_{r,s}(m)$ to be the algebra of polynomials $P \in \mathbb{C}[y_1, \ldots, y_r, z_1, \ldots, z_s]$ which are symmetric in the z_l, satisfy the quasi-invariance conditions (5) for $a = m$, and also the m-quasi-invariance condition:

$$P(\ldots, y_j, \ldots, y_k, \ldots, z_1, \ldots, z_s) - P(\ldots, y_k, \ldots, y_j, \ldots, z_1, \ldots, z_s) \qquad (6)$$

is divisible by $(y_j - y_k)^{2m+1}$ for $1 \leq j < k \leq r$.

Theorem 6.1. *(M. Feigin) If $m > s$ then the algebra $\Lambda_{r,s}(m)$ is CM.*

Proof. Consider the algebra B generated by $\Lambda_{r,s,m}$ and the deformed Calogero–Moser operator L_2. We claim that B is the quotient of the type A spherical rational Cherednik algebra $\mathbf{e}H_{1/m}(mr + s)\mathbf{e}$ ([EGL], see also [F], and [BCES], Remark 4.10).

Indeed, consider the irreducible representation $L_{1/m}(\mathbb{C})$ of $H_{1/m}(n)$, $n = mr + s$. By [EGL], Proposition 3.8, $L_{1/m}(\mathbb{C})$ is the quotient of $\mathbb{C}[x_1, \ldots, x_n]$ by the ideal of functions vanishing on the set X of points having r groups of m coordinates in each such that the coordinates are equal inside every group. Thus, $\mathbf{e}L_{1/m}(\mathbb{C})$ is the ring of regular functions on X/S_n. This ring is generated by Newton (i.e., power) sums, which in terms of the coordinates y_i (repeated m times) and z_j (loose, i.e. not inside a group) have the form $m\sum y_i^k + \sum z_j^k$. These are deformed Newton sums, so $\mathbf{e}L_{1/m}(\mathbb{C}) = \Lambda_{r,s,m}$. Also, it is checked in [F] that the element $\mathcal{L}_2$ of the rational Cherednik algebra (the sum of

squares of the Dunkl operators) acts in terms of y_i and z_j as the deformed Calogero–Moser operator L_2. On the other hand, it is easy to see (by passing to underlying Poisson algebras) that $\mathbf{e}H_{1/m}(mr+s)\mathbf{e}$ is generated by $\mathcal{L}_2$ and symmetric functions in x_i. Thus, the algebra generated by L_2 and $\Lambda_{r,s,m}$, which acts on $\Lambda_{r,s,m}$, is the homomorphic image of $\mathbf{e}H_{1/m}(mr+s)\mathbf{e}$ in $\mathrm{End}(\mathbf{e}L_{1/m}(\mathbb{C}))$, as desired.

It is easy to see that L_2 preserves the space of polynomials satisfying (6) (a calculation in codimension 1 similar to the one in [FV1]). Thus, B acts naturally on $\Lambda_{r,s}(m)$. Hence, $\Lambda_{r,s}(m)$ is a module over the spherical Cherednik algebra $\mathbf{e}H_{1/m}(mr+s)\mathbf{e}$. Moreover, it is easy to see that it has minimal support. Therefore, by Theorem 1.2 of [EGL], $\Lambda_{r,s}(m)$ is a free module over $\mathbb{C}[Q_1,\dots,Q_{r+s}]$, hence it is a CM algebra, as claimed. □

Since characters of minimal support modules are explicitly known (see [EGL]), the method of proof of Theorem 6.1 can be used to derive explicit formulas for the Hilbert series of $\Lambda_{r,s}(m)$. In the appendix to this paper, M. Feigin uses these formulas to prove the conjecture from [FV2] that the algebra $\Lambda_{r,1}(m)$ is Gorenstein.

Remark 6.2. 1. For $s=1$, Theorem 6.1 is proved in [FV2].

2. Note that for $s=1$, Theorem 3.3 (i.e., Theorem 4.4 of [BCES]) follows from Theorem 6.1 (proved in this case in [FV2]) by interpolating with respect to m (using the fact that the homogeneous components of $\Lambda_{r,s}(m)$ stabilize as $m\to\infty$, and its structure constants depend rationally on m).

6.2. Trigonometric (non-homogeneous) quasi-invariants. Let $\Lambda^{\mathrm{trig}}_{r,s}(m)$ be the algebra of polynomials $P\in\mathbb{C}[y_1,\dots,y_r,z_1,\dots,z_s]$ which are symmetric in the z_l and satisfy the trigonometric (non-homogeneous) m-quasi-invariance conditions:

$$P(\dots,y_j+1,\dots,z_l-m,\dots)=P(\dots,y_j,\dots,z_l,\dots),\ y_j=z_l,$$

for $1\le j\le r$, $1\le l\le s$, and

$$P(\dots,y_j,\dots,y_k,\dots,z_1,\dots,z_s)-P(\dots,y_k,\dots,y_j,\dots,z_1,\dots,z_s) \tag{7}$$

is divisible by $\prod_{p=-m}^{m}(y_j-y_k-p)$ for $1\le j<k\le r$.

Note that the algebra $\Lambda^{\mathrm{trig}}_{r,s}(m)$ has a natural filtration by degree of polynomials.

Proposition 6.3. *If $m>s$, we have $\mathrm{gr}(\Lambda^{\mathrm{trig}}_{r,s}(m))=\Lambda_{r,s}(m)$. In particular, the algebra $\Lambda^{\mathrm{trig}}_{r,s}(m)$ is CM.*

Proof. Consider the completion of the type A trigonometric Cherednik algebra $\mathbf{e}H^{\mathrm{trig}}_{1/m}(mr+s)\mathbf{e}$ near the identity element of the torus $(\mathbb{C}^\times)^{mr+s}$. This algebra has a decreasing filtration with associated graded isomorphic to $\mathbf{e}H_{1/m}(mr+s)\mathbf{e}$ (in fact, this deformation is known to be trivial). One can check that the action of the algebra $\mathbf{e}H_{1/m}(mr+s)\mathbf{e}$ on $\Lambda_{r,s}(m)$ deforms to an action of $\mathbf{e}H^{\mathrm{trig}}_{1/m}(mr+s)\mathbf{e}$ on $\Lambda^{\mathrm{trig}}_{r,s}(m)$. Indeed, this amounts to checking that the rational deformed Macdonald–Ruijsenaars operator, i.e., the rational difference degeneration of the deformed Macdonald–Ruijsenaars operator (1) of [SV2] preserves the non-homogeneous m-quasi-invariance conditions, which is done by a straightforward computation similar to the one in [SV2]. Since the algebra $\Lambda^{\mathrm{trig}}_{r,s}(m)$ contains a principal ideal in $\mathbb{C}[y_1,\dots,y_r,z_1,\dots,z_s]^{S_s}$, the Hilbert series of the algebras $\mathrm{gr}(\Lambda^{\mathrm{trig}}_{r,s}(m))$ and $\Lambda_{r,s}(m)$ have the same asymptotics as $u\to 1$,

i.e., give the same value at 1 after multiplication by $(1-u)^{r+s}$ (namely, $1/s!$). Since $\Lambda_{r,s}(m)$ is a minimal support module over $\mathbf{e}H_{1/m}(mr+s)\mathbf{e}$, this implies that we must have $\mathrm{gr}(\Lambda^{\mathrm{trig}}_{r,s}(m)) = \Lambda_{r,s}(m)$ (as, because of equal growth, the quotient $\Lambda_{r,s}(m)/\mathrm{gr}(\Lambda^{\mathrm{trig}}_{r,s}(m))$ is a module over the rational Cherednik algebra with smaller support). □

Remark 6.4. Let $R \subset \mathfrak{h}$ be a root system with Weyl group W. For $\alpha \in R$ let s_α be the corresponding reflection. Let m a multiplicity function on roots (see [FV1]). In this case we can define the ring of quasi-invariants $\mathbf{Q}_m \subset \mathbb{C}[\mathfrak{h}]$, i.e. polynomials f on the reflection representation $\mathfrak{h}$ such that $f(x) - f(s_\alpha x)$ is divisible by $\alpha(x)^{2m_\alpha+1}$ for $\alpha \in R$, and the ring of trigonometric (non-homogeneous) quasi-invariants $\mathbf{Q}^{\mathrm{trig}}_m$, i.e. polynomials f on $\mathfrak{h}$ such that $f(x + \frac{1}{2}j\alpha^\vee) = f(x - \frac{1}{2}j\alpha^\vee)$ if $\alpha(x) = 0$ for $j = 1, \dots, m_\alpha$. Then one can use the same argument as in the proof of Proposition 6.3 (namely, the rational difference degeneration of [Cha], Proposition 2.1) to prove the following proposition:

Proposition 6.5. *One has* $\mathrm{gr}(\mathbf{Q}^{\mathrm{trig}}_m) = \mathbf{Q}_m$.

In particular, this implies that $\mathbf{Q}^{\mathrm{trig}}_m$ is CM and, moreover, Gorenstein (as by [EG, BEG], so is $\mathbf{Q}_m$).

Example 6.6. For the root system of type A_1 the rational Macdonald–Ruijsenaars operator has the form

$$(Mf)(x) = \frac{x-m}{x}\big(T-1\big) + \frac{x+m}{x}\big(T^{-1}-1\big),$$

where $(Tf)(x) = f(x+1)$. It is easy to see that this operator preserves the space $\mathbf{Q}^{\mathrm{trig}}_m$ of polynomials f such that $f(j) = f(-j)$ for $j = 1, 2, \dots, m$. The (completed) trigonometric Cherednik algebra acting on $\mathbf{Q}^{\mathrm{trig}}_m$ is generated by M and x^2. Note that M lives in filtration degrees $d \le -2$, and the degree -2 (leading) part of M equals $\partial^2 - \frac{2m}{x}\partial$, the rational Calogero–Moser operator for A_1.

Remark 6.7. Another proof of Proposition 6.5 can be obtained by using the rational difference degeneration $G^{\mathrm{trig}}_m : \mathbb{C}[\mathfrak{h}] \to \mathbb{C}[\mathfrak{h}]$ of Cherednik's shift operator ([Ch,Cha]). More precisely, Corollary 8.28 of [EG] proves that the image of the usual (differential) shift operator $G_m : \mathbb{C}[\mathfrak{h}] \to \mathbb{C}[\mathfrak{h}]$ is exactly $\mathbf{Q}_m$. Also, one can check that the image of G^{trig}_m is contained in $\mathbf{Q}^{\mathrm{trig}}_m$, which implies that $\mathbf{Q}^{\mathrm{trig}}_m$ is not smaller (i.e., the same size) as $\mathbf{Q}_m$, as desired.

7. Appendix. The Hilbert Series of $\Lambda_{r,s}(m)$.

By Misha Feigin

To Aleksandr Petrovich Veselov on the 60th birthday, with gratitude

In this Appendix we find the Hilbert series of the algebra $\Lambda_{r,s}(m)$ introduced in Sect. 6 assuming throughout that $m > s$. We also show that the algebra is Gorenstein if $s = 1$. The algebra $\Lambda_{r,1}(m)$ is isomorphic to the algebra of quasi-invariants for the configuration $\mathcal{A}_r(m)$ considered in [FV2,CFV]. The Gorenstein property of $\Lambda_{r,1}(m)$ was shown in [FV2] for $r = 2$ and it was conjectured to hold for any r.

Let $n = mr + s$. Let λ be a partition of n. Let $L_c(\lambda)$ be the corresponding irreducible module for the rational Cherednik algebra $H_c(S_n)$. Let $eL_c(\lambda)$ be the corresponding irreducible module for the spherical subalgebra, $e = \frac{1}{n!}\sum_{w\in S_n} w$. For a partition τ of r and a partition ν of s we denote by $m\tau + \nu$ the partition of n with terms $m\tau_i + \nu_i$. We will also denoted by τ the corresponding representation of S_r.

Theorem 7.1. *There is an isomorphism*

$$\Lambda_{r,s}(m) \cong \bigoplus_{\tau\vdash r} \tau \otimes eL_{1/m}(m\tau + s)$$

of $\mathbb{C}S_r \otimes eH_{1/m}(S_n)e$ *modules.*

Proof. It follows from the proof of Theorem 6.1 that $\Lambda_{r,s}(m)$ is a module over $eH_{1/m}(S_n)e$ of minimal support. It follows from [EGL] that as a module over $\mathbb{C}S_r \otimes eH_{1/m}(S_n)e$ it can be decomposed as

$$\Lambda_{r,s}(m) \cong \bigoplus_{\substack{\tau\vdash r\\ \nu\vdash s}} d_{\tau,\nu} \otimes eL_{1/m}(m\tau + \nu) \tag{8}$$

for some $\mathbb{C}S_r$ modules $d_{\tau,\nu}$. Let us consider the localised module $\Lambda_{r,s}(m)_{\mathrm{loc}}$, where localisation is at the powers of

$$\alpha(x) = \sum_{w\in S_n} w\left(\prod_{\substack{1\le i\le mr\\ mr+1\le j\le n}} (x_i - x_j)\right).$$

It is a module over the localised rational Cherednik algebra $eH_{1/m}(S_n, U)e$, where $U \subset \mathbb{C}^n$ is given by $\alpha(x) \neq 0$. Equivalently, we localise quasi-invariants $\Lambda_{r,s}(m) \subset \mathbb{C}[y_1, \dots, y_r, z_1, \dots, z_s]$ with respect to the powers of

$$\widehat{\alpha}(y, z) = \prod_{\substack{1\le i\le r\\ 1\le j\le s}} \big(y_i - z_j\big)^m.$$

Let $\Lambda'_{r,s}(m) \subset \mathbb{C}[y_1, \dots, y_r, z_1, \dots, z_s]$ consist of polynomials p which are symmetric in z-variables and satisfy quasi-invariant conditions (6). It is a module over the spherical rational Cherednik algebra

$$e' H_{m,1/m}(S_r \times S_s; \mathbb{C}^{r+s})e' \cong e_r H_m(S_r)e_r \otimes e_s H_{1/m}(S_s)e_s,$$

where $e_r = \frac{1}{r!}\sum_{w\in S_r} w$, $e_s = \frac{1}{s!}\sum_{w\in S_s} w$, $e' = \frac{1}{r!s!}\sum_{w\in S_r\times S_s} w$. This module has the form

$$\Lambda'_{r,s}(m) \cong Q_m \otimes e_s L_{1/m}(triv),$$

where Q_m are the ordinary S_r m-quasi-invariants as in [FV1]. The structure of $S_r \otimes e_r H_m(S_r)e_r$–module Q_m is obtained in [BEG], Proposition 6.6. It implies that $\Lambda'_{r,s}(m)$ as $\mathbb{C}S_r \otimes e' H_{m,1/m}(S_r \times S_s; \mathbb{C}^{r+s})e'$–module decomposes as

$$\Lambda'_{r,s}(m) = \bigoplus_{\tau\vdash r} \tau \otimes e_r L_m(\tau) \otimes e_s L_{1/m}(triv), \tag{9}$$

where $e_r = \frac{1}{r!}\sum_{w\in S_r} w$, $e_s = \frac{1}{s!}\sum_{w\in S_s} w$.

Consider localisation $\Lambda'_{r,s}(m)_{\mathrm{loc}}$ of the module $\Lambda'_{r,s}(m)$ at the powers of $\alpha'(y,z) = \prod_{\substack{1\le i\le r\\ 1\le j\le s}}(y_i - z_j)$, which is a module over the localised spherical Cherednik algebra $e'H_{m,1/m}(S_r \times S_s, U')e'$, where $U' \subset \mathbb{C}^{r+s}$ is given by $\alpha'|_{U'} \neq 0$. It is clear that $\Lambda_{r,s}(m)_{\mathrm{loc}} \subseteq \Lambda'_{r,s}(m)_{\mathrm{loc}}$. Since for any $p \in \Lambda'_{r,s}(m)$ we have $\alpha'^t p \in \Lambda_{r,s}(m)$ for any $t \geq 2$ the opposite inclusion follows so these spaces are equal.

Recall that it is established in [W], Theorem 4.4 that there is an equivalence of categories of $H_{m,1/m}(S_r \times S_s, \mathbb{C}^{r+s})$–modules and $H_{1/m}(S_n)$– modules with minimal support such that certain corresponding $\mathcal{D}$–modules and hence monodromy functors match. It follows from the proof of [W], Theorem 4.4 that under this equivalence an $H_{m,1/m}(S_r \times S_s, \mathbb{C}^{r+s})$–module M is mapped to an $H_{1/m}(S_n)$–module N such that $eN_{\mathrm{loc}} \cong \sigma^* e'M_{\mathrm{loc}}$, where $\sigma : eH_{1/m}(S_n, U)e \to e'H_{m,1/m}(S_r \times S_s, U')e'$ is the natural restriction homomorphism. And it follows then from [W], Theorem 1.8 that under the equivalence the irreducible $H_{1/m}(S_n)$–module $L_{1/m}(m\tau + \nu)$ is mapped to $L_m(\tau) \otimes L_{1/m}(\nu)$. So one has isomorphism

$$eL_{1/m}\big(m\tau + \nu\big)_{\mathrm{loc}} \cong \big(e_r L_m(\tau) \otimes e_s L_{1/m}(\nu)\big)_{\mathrm{loc}} \tag{10}$$

of $eH_{1/m}(S_n, U)e$ modules, where the action on the right-hand side is induced by homomorphism σ, and these modules are not isomorphic for different (τ, ν).

Since we localise at S_r-invariant elements $\widehat{\alpha}, \alpha'$ the equality $\Lambda_{r,s}(m)_{\mathrm{loc}} = \Lambda'_{r,s}(m)_{\mathrm{loc}}$ and decompositions (8)–(10) imply that $d_{\tau,\nu} = 0$ if ν has more than one part, and that $d_{\tau,s} \cong \tau$. □

Let s_λ be the Schur function corresponding to the partition λ. Define the coefficients $c^\nu_{\lambda;m}, b^\nu_{\lambda;m}$ by

$$s_\lambda\big(x_1^m, x_2^m, \dots\big) = \sum_\nu c^\nu_{\lambda;m} s_\nu\big(x_1, x_2, \dots\big), \tag{11}$$

$$s_\lambda\big(x_1^m, x_2^m, \dots\big) s_s\big(x_1, x_2, \dots\big) = \sum_\nu b^\nu_{\lambda,s;m} s_\nu\big(x_1, x_2, \dots\big). \tag{12}$$

Let λ be a partition of r. Define $\kappa(\lambda) = \sum_{1\le i<j\le r} s_{ij}|_\lambda$ the content of λ. Let $p_k(\lambda)$ be the multiplicity of the representation λ in the space of homogeneous polynomials of r variables of degree k. Define the Hilbert series

$$\chi_\lambda(t) = \sum_{k=0}^\infty p_k(\lambda).$$

It is known from [K] that

$$\chi_\lambda(t) = \prod_{\square\in\lambda} \frac{t^{l(\square)}}{1 - t^{h(\square)}}, \tag{13}$$

where $l(\square)$ is the leg length of a box, and $h(\square)$ is the hook length of a box.

Let $\Lambda^{(k)}_{r,s}(m) \subset \Lambda_{r,s}(m)$ be the subspace of homogeneous elements of degree k. Let

$$P_{r,s;m}(t) = \sum_{k=0}^\infty \dim \Lambda^{(k)}_{r,s}(m) t^k$$

be the Hilbert series of $\Lambda_{r,s}(m)$.

Theorem 7.2. *The Hilbert series of the algebra $\Lambda_{r,s}(m)$ has the form*

$$P_{r,s;m}(t) = \sum_{\lambda\vdash r} \dim \lambda \sum_{\nu\vdash n} b^{\nu}_{\lambda,s;m} t^{\frac{n(n-1)-2\kappa(\nu)}{2m}} \chi_\nu(t).$$

Proof. It is shown in [EGL], Theorem 1.4 that in the Grothendieck group

$$[L_{1/m}(m\lambda + s)] = \sum_{\nu\vdash n} b^{\nu}_{\lambda,s;m} [M_{1/m}(\nu)].$$

Therefore

$$[eL_{1/m}(m\lambda + s)] = \sum_{\nu\vdash n} b^{\nu}_{\lambda,s;m} [eM_{1/m}(\nu)],$$

and hence (cf. [EGL])

$$Tr_{eL_{1/m}(m\lambda+s)}(t^h) = \sum_{\nu\vdash n} b^{\nu}_{\lambda,s;m} t^{\frac{n}{2}-\frac{\kappa(\nu)}{m}} \chi_\nu(t),$$

where $h = \frac{1}{2}\sum_{i=1}^{n}(x_i\nabla_i + \nabla_i x_i)$ is the scaling element of the rational Cherednik algebra.

On the other hand the action of the operator h in the polynomial representation $\mathbb{C}[x_1, \ldots, x_n]$ is given by

$$h = \sum_{i=1}^{n} x_i \partial_{x_i} + \frac{n}{2} - \frac{1}{m}\sum_{i<j}^{n} s_{ij},$$

which reduces to $h^{res} = \sum_{i=1}^{n} x_i\partial_{x_i} + \frac{n}{2} - \frac{n(n-1)}{2m}$ on S_n-invariants. Its action in the representation $\Lambda_{r,s}(m) \subset \mathbb{C}[z_1, \ldots, z_r, y_1, \ldots, y_s]$ is by the same differential operator h^{res} where the Euler field component $\sum_{i=1}^{n} x_i\partial_{x_i}$ acts as its restriction to the Euler operator in y, z-space which is $\sum_{i=1}^{r} y_i\partial_{y_i} + \sum_{i=1}^{s} z_i\partial_{z_i}$. The statement now follows from Theorem 7.1. □

Let us now consider the case $s = 1$ so $n = mr + 1$. We will derive another formula for the Hilbert series of the quasi-invariants $\Lambda_r(m) := \Lambda_{r,1}(m)$. For the representations of rational Cherednik algebra $H_{1/m}(S_{mr})$ it is shown in [EGL], Theorem 1.4 that one has

$$\big[L_{1/m}(m\lambda)\big] = \sum_{\nu\vdash mr} c^{\nu}_{\lambda;m} \big[M_{1/m}(\nu)\big]. \tag{14}$$

Recall that there exists an exact functor F acting on category $\mathcal{O}$ objects

$$F : H_{1/m}(S_{mr}) - mod \to H_{1/m}(S_{mr+1}) - mod$$

which acts on standard modules as follows ([Sh], Section 4). Let ν be a partition of mr. Then in the Grothendieck groups

$$\big[FM_{1/m}(\nu)\big] = \bigoplus_{\widehat{\nu}\in B_\nu} \big[M_{1/m}(\widehat{\nu})\big], \tag{15}$$

where each diagram in the set B_ν is obtained from the diagram ν by adding a box with the content congruent to 0 modulo m.

It is shown in [EGL], Section 4.1 that

$$L_{1/m}(m\lambda+1) = FL_{1/m}(m\lambda).$$

Therefore presentation (14) implies that

$$\big[L_{1/m}(m\lambda+1)\big] = \sum_{\nu\vdash mr} c^{\nu}_{\lambda;m}\big[FM_{1/m}(\nu)\big], \tag{16}$$

and equality (15) allows to restate Theorem 7.2 in the following form.

Corollary 7.3. *The Hilbert series of the algebra $\Lambda_r(m)$ has the form*

$$P_{r;m}(t) = \sum_{\lambda\vdash r} \dim\lambda \sum_{\nu\vdash mr} \sum_{\widehat{\nu}\in B_\nu} c^{\nu}_{\lambda;m} t^{\frac{rn}{2}-\frac{\kappa(\widehat{\nu})}{m}} \chi_{\widehat{\nu}}(t). \tag{17}$$

Note that the right-hand side of the series (17) may contain fractional powers of t which would have to cancel.

It is established in [FV2] that the graded algebra $\Lambda_r(m)$ is Cohen–Macaulay. It is convenient to use the form (17) to show that the algebra $\Lambda_r(m)$ is Gorenstein.

Theorem 7.4. *The Hilbert series of the algebra of quasi-invariants $\Lambda_r(m)$ satisfies the symmetry property*

$$P_{r;m}(t^{-1}) = (-1)^{r+1}t^{n(1-r)}P_{r;m}(t).$$

Proof. Let us choose a term in the sum (17) corresponding to the diagrams $\lambda, \nu, \widehat{\nu}$. Notice that for the conjugate diagrams λ', ν' one can choose $\widehat{\nu'} = \widehat{\nu}'$. Indeed, if $\widehat{\nu}$ is obtained from ν by adding a box with the content k then the transposed partition $\widehat{\nu}'$ is obtained from ν' by adding a box with the content $-k$ so both contents are congruent to 0 modulo m and $\widehat{\nu}' \in B_{\nu'}$. Thus the series (17) decomposes as a sum of terms of the form

$$f(t) = (\dim\lambda)c^{\nu}_{\lambda;m}t^{\frac{rn}{2}-\frac{\kappa(\widehat{\nu})}{m}}\chi_{\widehat{\nu}}(t) + (\dim\lambda')c^{\nu'}_{\lambda';m}t^{\frac{rn}{2}-\frac{\kappa(\widehat{\nu}')}{m}}\chi_{\widehat{\nu}'}(t).$$

Recall that $\dim\lambda = \dim\lambda'$ and $c^{\nu}_{\lambda;m} = (-1)^{(m-1)r}c^{\nu'}_{\lambda';m}$ (see [EGL], Corollary 4.16). It is also easy to see from (13) that

$$\chi_{\widehat{\nu}}(t) = (-1)^n t^{-n}\chi_{\widehat{\nu}'}(t^{-1}).$$

Therefore

$$f(t) = (\dim\lambda)c^{\nu}_{\lambda;m}t^{\frac{rn}{2}}\left(t^{-\frac{\kappa(\widehat{\nu})}{m}}\chi_{\widehat{\nu}}(t) + (-1)^{(m-1)r}t^{\frac{\kappa(\widehat{\nu})}{m}}\chi_{\widehat{\nu}'}(t)\right),$$

and

$$\begin{aligned} f(t^{-1}) &= (\dim\lambda)c^{\nu}_{\lambda;m}t^{-\frac{rn}{2}}\left(t^{\frac{\kappa(\widehat{\nu})}{m}}(-1)^n t^n\chi_{\widehat{\nu}'}(t) + (-1)^{(m-1)r}t^{-\frac{\kappa(\widehat{\nu})}{m}}(-1)^n t^n\chi_{\widehat{\nu}}(t)\right) \\ &= (-1)^{r+1}t^{n(1-r)}f(t), \end{aligned}$$

so the statement follows. □

By Stanley criterion [S] we have the following

Corollary 7.5. *The algebra $\Lambda_r(m)$ is Gorenstein.*

We are going to obtain yet another form of the Hilbert series (17). Note that the coefficients $c^{\nu}_{\lambda;m}$ can be expressed in terms of characters of the symmetric group. Let μ be a partition of r and denote by C_μ the corresponding conjugacy class in S_r. Then

$$c^{\nu}_{\lambda;m} = \sum_{\mu\vdash r} \frac{|C_\mu|}{r!}\chi^{\lambda}(C_\mu)\chi^{\nu}(C_{m\mu}),$$

where χ^{λ}, χ^{ν} are characters of representations of S_r, S_{mr} corresponding to the partitions λ, ν (see e.g. [LZ]).

Let $\widehat{\chi}^{\lambda}$ be the character of the module U_λ which is induced from the trivial one for the parabolic subgroup corresponding to partition λ. Recall the Kostka matrix $K_{\mu\lambda}$ given by the relations $\widehat{\chi}^{\lambda} = \sum_\mu K_{\mu\lambda}\chi^{\mu}$. We will also need the inverse Kostka matrix K^{-1} satisfying $\chi^{\lambda} = \sum_\mu K^{-1}_{\mu\lambda}\widehat{\chi}^{\mu}$. Then we have

$$\sum_{\lambda\vdash r} \dim\lambda\, c^{\nu}_{\lambda;m} = \sum_{\substack{\lambda,\mu\vdash r\\ \tilde{\nu}\vdash mr}} \dim\lambda \frac{|C_\mu|}{r!}\chi^{\lambda}(C_\mu)K^{-1}_{\tilde{\nu}\nu}\widehat{\chi}^{\tilde{\nu}}(C_{m\mu}). \tag{18}$$

Note that $\widehat{\chi}^{\tilde{\nu}}(C_{m\mu})$ is non-zero only if the partition $\tilde{\nu}$ has the form $\tilde{\nu} = m\alpha$ for some $\alpha\vdash r$, in which case $\widehat{\chi}^{\tilde{\nu}}(C_{m\mu}) = \widehat{\chi}^{\alpha}(C_\mu)$. Taking into account orthogonality of characters, we continue (18) as

$$\begin{aligned}\sum_{\lambda,\mu,\alpha,\beta\vdash r} &\dim\lambda \frac{|C_\mu|}{r!}\chi^{\lambda}(C_\mu)K^{-1}_{m\alpha,\nu}K_{\beta\alpha}\chi^{\beta}(C_\mu)\\ &= \sum_{\alpha,\lambda\vdash r} \dim\lambda\, K^{-1}_{m\alpha,\nu}K_{\lambda\alpha}\\ &= \sum_{\alpha\vdash r}\dim U_\alpha\, K^{-1}_{m\alpha,\nu} = \sum_{\alpha\vdash r}\frac{r!}{\alpha!}K^{-1}_{m\alpha,\nu},\end{aligned}$$

where $\alpha! = \alpha_1!\alpha_2!\ldots$ Thus we get the following expression for Hilbert series (17):

$$P_{r;m}(t) = \sum_{\substack{\alpha\vdash r,\nu\vdash mr\\ \widehat{\nu}\in B_\nu}} \frac{r!}{\alpha!}K^{-1}_{m\alpha,\nu}t^{\frac{rm}{2}-\frac{\kappa(\widehat{\nu})}{m}}\chi_{\widehat{\nu}}(t). \tag{19}$$

It would be interesting to see if there is a simpler form of the Hilbert series $P_{r;m}(t)$.

Finally we note that the algebra $\Lambda_{r,s}(m)$ is not expected to be Gorenstein for $s > 1$ as the case $r = 1$ shows. Indeed, it is shown in [J] that for any non-zero m the Hilbert series $P_{1,s;m}$ is the same, which is known from [SV1] to be equal to $h = \frac{1-t+t^{s+1}}{(1-t)^2(1-t^2)\ldots(1-t^s)}$ so the algebra is not Gorenstein.

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