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MAXIMUM LIKELIHOOD ESTIMATION OF OBJECT SIZE AND ORIENTATION FROM PROJECTION DATA*

David J. Rossi
Schlumberger-Doll Research
P.O. Box 307
Ridgefield, CT 06877

Alan S. Willsky
Dept. of E.E.C.S., M.I.T.
Room 35-233
Cambridge, MA 02139

ABSTRACT

The problem of detecting, locating and characterizing objects in a 2D cross-section from noisy projection data has been considered recently [1-3], in which objects are characterized by a finite number of parameters, which are estimated directly from noisy projection measurements. In this paper, the problem of maximum likelihood (ML) estimation of those parameters characterizing the *geometry* of an object (e.g. size and orientation) is considered, and estimation performance is investigated.

Introduction

The problem of reconstructing a multi-dimensional function from its projections arises, typically in imaging applications, in a diversity of disciplines, including oceanography, medicine, and nondestructive evaluation. In the two-dimensional version of this problem, a 2D function $f(\mathbf{x})$ is observed via noisy samples of its Radon transform

$$g(t,\theta) = \int_{\mathbf{x}'\boldsymbol{\theta}=t} f(\mathbf{x})ds$$

where $\boldsymbol{\theta}$ is the unit vector $(\cos\theta \;\; \sin\theta)'$. The problem of locating and characterizing one or more objects in a cross-section from projection measurements has been considered recently [1-3], in which the 2D cross-section is represented as the superposition of a background field and N objects,

$$f(\mathbf{x}) = f_b(\mathbf{x}) + \sum_{k=1}^{N} d_k \, f(\mathbf{x}-\mathbf{c}_k;\boldsymbol{\gamma}_k)$$

Here, the k*th* object is located at the point $\mathbf{c}_k$ and has contrast or density d_k $(f(0;\boldsymbol{\gamma}_k)=1)$; $\boldsymbol{\gamma}_k$ is a finite-dimensional vector of parameters characterizing the density fluctuations of the k*th* object, it contains, for example, information about the object's size, shape and orientation. In this paper, we consider the special case where the background is known (and assumed to equal zero), and a single object (N=1) is situated at a known location; see [1-3] for a discussion of ML localization of an object from noisy projection measurements.

In the present analysis, a specific parameterization of object size and shape is chosen, and the performance of ML estimation of the geometry parameters in $\boldsymbol{\gamma}$ is evaluated. To begin, consider a circularly-symmetric normalized (i.e. unit-sized) object located at the origin, denoted $f_o(\mathbf{x})$ ($f_o(r)$ as a function of the radial polar coordinate r). Denote the object Radon transform as $g_o(t)$, which is independent of the projection angle θ, and its Radon transform energy as

$$\xi_o = \int_0^{\pi}\int_{-\infty}^{\infty} g_o^2(t)\,dt\,d\theta$$

The object whose projections are measured is modeled as arising from the object $f_o(\mathbf{x})$ by density scaling with a factor d, and by a series of coordinate transformations, specifically, isotropic *scaling* of the $\mathbf{x}$ coordinate system by a factor R, and/or *stretching* of the coordinate system by a factor λ, and possible *rotation* by an angle ϕ. Said another way, the object whose projections are measured is modeled as belonging to the class of objects $d\cdot f(\mathbf{x}) = d\cdot f_o(\tilde{\mathbf{x}})$ where

$$\tilde{\mathbf{x}} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \sqrt{\lambda} & 0 \\ 0 & 1/\sqrt{\lambda} \end{bmatrix} \begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix} \mathbf{x} \qquad (1)$$

$$0<R\leq\infty \qquad 0<\lambda\leq\infty \qquad -\frac{\pi}{2}\leq\phi<\frac{\pi}{2}$$

Throughout this discussion, the subscript "o" refers to the original unit-sized, circularly-symmetric object, and unsubscripted functions refer to the object after the transformation in (1) has been applied. The Radon transform of the object $d\cdot f(\mathbf{x};R,\lambda,\phi)$ resulting from this transformation will be denoted $d\cdot g(t,\theta;R,\lambda,\phi)$, and the Radon transform energy [1] is $\xi(d,R,\lambda) = d^2R^3q(\lambda)\xi_o$, where

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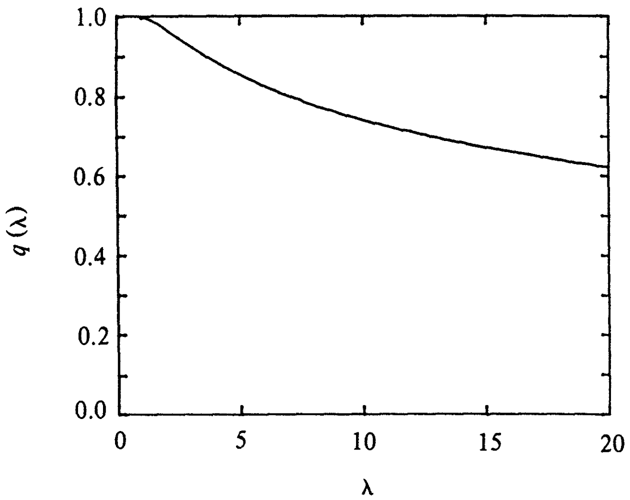

Figure 1. Radon Transform energy dependence on eccentricity λ.

$$q(\lambda) = \frac{2}{\pi} \int_0^{\pi/2} [\, \lambda \cos^2\psi + \lambda^{-1} \sin^2\psi]^{-1/2} \, d\psi$$

Note that $q(\lambda) = q(\lambda^{-1})$ and $q(1) = 1$; a plot of $q(\lambda)$ on the interval [1,20] is shown in Figure 1. Let the noisy projection measurements be given by

$$y(t, \theta) = d \cdot g(t, \theta; R, \lambda, \phi) + w(t, \theta) \qquad (2)$$

$$(t, \theta) \in S_Y \subset S = \{ (t, \theta) \colon -\infty < t < \infty \,,\ 0 \leq \theta < \pi \}$$

where $w(t, \theta)$ is a zero-mean white Gaussian noise process with spectral level $N_o/2$. In terms of the present notation, the problem of characterizing the object geometry may be stated as: *given* noisy incomplete measurements of the Radon transform as shown in (2), *estimate* the object density d, size R, eccentricity λ, and orientation ϕ. It should be noted that, with the exception of the density factor d, these parameters enter the problem nonlinearly, and lead to a nonlinear estimation problem of small dimensionality. This is in contrast to full image reconstruction, in which a linear estimation problem of high dimensionality is solved. In the present analysis, maximum likelihood (ML) estimation of object size R and orientation ϕ will be considered assuming that the object eccentricity λ is known; see [2] for a discussion of estimating object eccentricity.

ML Parameter Estimation

Let R_a, λ_a, ϕ_a, and $g(t, \theta; R_a, \lambda_a, \phi_a)$ be the actual object parameters and Radon transform, respectively, and consider the special case of a *full-view* set of measurements Y, that is, $S_Y = S$. ML estimates of object size and orientation correspond to those values of the parameters R and ϕ that maximize the log likelihood function [4]

$$L(R, \phi; Y) = \frac{2}{N_o} \int_0^{\pi} \int_{-\infty}^{\infty} y(t, \theta)\, g(t, \theta; R, \lambda_a, \phi)\, dt\, d\theta$$

$$- \frac{1}{N_o} \int_0^{\pi} \int_{-\infty}^{\infty} g^2(t, \theta; R, \lambda_a, \phi)\, dt\, d\theta$$

The *ambiguity function*, or expected value of the log likelihood function, is given by

$a(R, \phi; R_a, \lambda_a, \phi_a)$

$$= \frac{2}{N_o} \int_0^{\pi} \int_{-\infty}^{\infty} g(t, \theta; R_a, \lambda_a, \phi_a)\, g(t, \theta; R, \lambda_a, \phi)\, dt\, d\theta$$

$$- \frac{1}{N_o} \int_0^{\pi} \int_{-\infty}^{\infty} g^2(t, \theta; R, \lambda_a, \phi)\, dt\, d\theta$$

In the following two sections, expressions are presented for the ambiguity function and Cramer-Rao lower bound (CRLB) [4] for the individual problems of size and orientation estimation. For purposes of illustration, the ambiguity function and CRLB will also be evaluated in each section for the special case of the class of objects which arise when the coordinate transformation in (1) is applied to the circularly-symmetric *Gaussian* object $f_o(r) = exp(-r^2)$. This object has Hankel transform (central section of the 2D Fourier transform) $F_o(\rho) = \pi\, exp(-\pi^2 \rho^2)$, and Radon transform energy $\xi_o = \sqrt{\pi/2}\, \pi^2$.

Object Size Estimation

Consider first the problem of using noisy full-view projection measurements to estimate the size of an object that results from the isotropic coordinate scaling transformation in (1). In particular, begining with a unit-sized circularly-symmetric object $f_o(r)$ with Hankel transform $F_o(\rho)$ and Radon transform energy ξ_o, the size estimation ambiguity function is $a(R, R_a) = (\xi_a / N_o)\, a^*(R/R_a)$, where $\xi_a = d^2 R_a^3 \xi_o$ is the energy in the actual object Radon transform, and $a^*(\cdot)$ is the normalized ambiguity function

$a^*(R/R_a)$

$$= \frac{4\pi}{\xi_o} \left[R/R_a\right]^2 \int_0^{\infty} F_o(\rho)\, F_o(\rho R/R_a)\, d\rho - \left[R/R_a\right]^3 \qquad (3)$$

For the special case where the original object is the Gaussian object $f_o(r) = exp(-r^2)$, the normalized ambiguity function in (3) is plotted in Figure 2, along with the normalized ambiguity function for the *disk* object (indicator function on a disk of radius R_a). Qualitatively, since these ambiguity functions have their peak at $R = R_a$ and decrease rapidly as R moves away

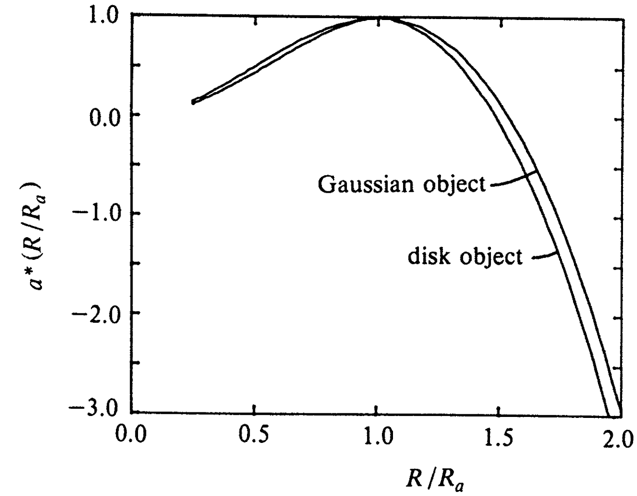

Figure 2. Normalized ambiguity functions for the estimation of Gaussian and disk object size.

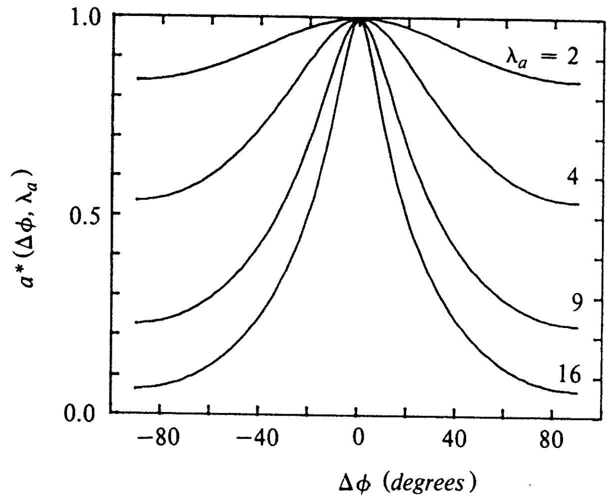

Figure 3. Orientation ambiguity function for a Gaussian object for several values of eccentricity.

from the true value R_a, it appears possible to estimate object size quite reliably from noisy projection measurements.

The CRLB for the size estimation problem may be obtained by evaluating the second partial derivative of the ambiguity function $a(R,R_a)$ with respect to the parameter R. If the first and second partial derivatives of $F_o(\rho)$ are denoted F_o' and F_o'', the CRLB on the size estimate error variance is [2,4]

$$\sigma_R^2 \geqslant \left[\left\{ \frac{\partial^2 a(R,R_a)}{\partial R^2} \right\}_{R=R_a} \right]^{-1} = \frac{N_o}{2d^2 R_a (3\xi_o - \zeta)} \qquad (4)$$

where

$$\zeta = 2\pi \int_0^\infty F_o(\rho) \Big[2F_o(\rho) + 4\rho F_o'(\rho) + \rho^2 F_o''(\rho) \Big] d\rho \qquad (5)$$

For the Gaussian object, ζ in (5) equals $(\pi/2)^{2.5}$, and the CRLB in (4) is $\sigma_R^2 \geqslant (2/\pi)^{2.5}[N_o/22d^2R_a]$. The CRLB is seen to depend inversely on the actual object size, indicating that the accuracy with which size may be estimated improves rapidly with increasing object size.

Object Orientation Estimation

Consider now the problem of using full-view projection measurements to estimate the orientation of an eccentric object. Begining with a circularly-symmetric object $f_o(r)$ with Hankel transform $F_o(\rho)$ and Radon transform energy ξ_o, an eccentric object is modeled as resulting from the application of known scaling and stretching transformations in (1) with $R = R_a$ and $\lambda = \lambda_a$. The orientation estimation ambiguity function, which depends on the modeled and actual orientations ϕ and ϕ_a only through their difference $\Delta\phi = \phi - \phi_a$, is given by $a(\Delta\phi, \lambda_a) = (\xi_a/N_o)\, a^*(\Delta\phi, \lambda_a)$. Here, $\xi_a = d^2R_a^3 q(\lambda_a)\xi_o$ is the energy in the actual object Radon transform, and $a^*(\cdot)$ is the normalized ambiguity function

$$a^*(\Delta\phi, \lambda_a) = \frac{2}{q(\lambda_a)\,\xi_o} \int_0^{2\pi}\int_0^\infty F_o\left(\rho\sqrt{\lambda_a \cos^2\psi + \lambda_a^{-1}\sin^2\psi}\right) \qquad (6)$$

$$\cdot F_o\left(\rho\sqrt{\lambda_a \cos^2(\psi+\Delta\phi) + \lambda_a^{-1}\sin^2(\psi+\Delta\phi)}\right) d\rho\, d\psi - 1$$

Here, $q(\cdot)$ is the function plotted in Figure 1. Note that a^* is symmetric in $\Delta\phi$ (because the eccentric object is centrally-symmetric or "balanced"), and $a^*(\Delta\phi, \lambda_a) = a^*(\Delta\phi, \lambda_a^{-1})$, since these are ambiguity functions for the same object (rotated by 90 degrees).

For the special case of a stretched Gaussian object, the ambiguity function in (6) is plotted in Figure 3 for several values of object eccentricity λ_a. The results presented in this figure agree with intuition -- narrow objects have a narrow orientation ambiguity function, implying that the orientation of eccentric objects can be estimated more reliably than can the orientation of nearly circular objects. This observation may be verified by calculating the CRLB for the orientation estimate error variance. Figure 4 is a plot of the normalized CRLB (the CRLB multiplied by $d^2R^3\xi_o/N_o$) for the special case of a stretched Gaussian object, presented as a function of the object eccentricity λ_a. This figure indicates the rapid rate of decrease in the bound with increasing λ_a, particularly for λ_a values close to unity; this indicates that more accurate orientation estimates are obtained with more eccentric objects.

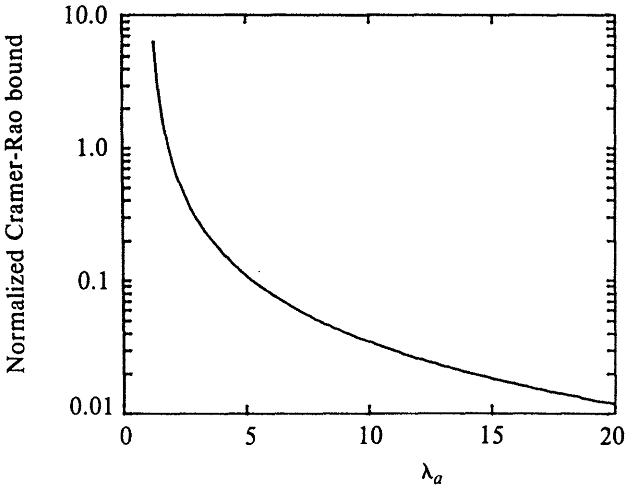

Figure 4. Normalized Cramer-Rao bound for the orientation estimation problem.

Conclusions

The problem has been considered of estimating the size and orientation of an object within a cross-section from noisy observations of the Radon transform (line integral or projection measurements). The object was assumed to belong to the class of objects that result from a circularly-symmetric object by the application of the coordinate transformation in (1), which is parameterized by three variables corresponding to final object size, eccentricity and orientation. ML estimation of the size and orientation parameters was investigated via evaluation of the ambiguity function and CRLB, and the results were illustrated for the class of Gaussian objects. It was shown that object size and orientation can be determined quite reliably, the latter particularly so for very eccentric objects.

Extensions of several of these results to non-circularly-symmetric objects may be found in [2], along with a discussion of object eccentrcity estimation. Also, several estimation robustness issues are addressed in [2], including robustness to a number of modeling errors. Generally, estimation of the geometry parameters in (1) appears to be very robust to a variety of modeling errors; this is further borne out in robustness simulations which were performed recently [5].

References

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