Understanding Resilience in Large Networks

by

Tuhin Sarkar

B.Tech., Indian Institute of Technology Bombay (2013)

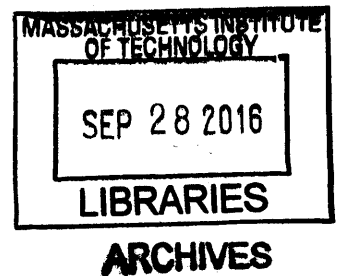Submitted to the Department of Electrical Engineering and Computer
Science
in partial fulfillment of the requirements for the degree of

Master of Science in Electrical Engineering

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

September 2016

Author . . . . **Signature redacted** . . . .

Department of Electrical Engineering and Computer Science

June 24, 2016

Certified by . . . . **Signature redacted** . . . .

Munther A. Dahleh

Professor of Electrical Engineering

Thesis Supervisor

Accepted by . . . . **Signature redacted** . . . .

Leslie A. Kolodziesjki

Professor of Electrical Engineering

Chair, Department Committee on Graduate Theses

Understanding Resilience in Large Networks

by

Tuhin Sarkar

Submitted to the Department of Electrical Engineering and Computer Science
on June 24, 2016, in partial fulfillment of the
requirements for the degree of
Master of Science in Electrical Engineering

Abstract

This thesis focuses on the analysis of robustness in large interconnected networks. Many real life systems in transportation, economics, finance and social sciences can be represented as networks. The individual constituents, or nodes, of the network may represent vehicles in the case of vehicular platoons, production sectors in the case of economic networks, banks in the case of financial sector, or people in the case of social networks. Due to interconnections between constituents in these networks, a disturbance to any one of the constituents of the network may propagate to other nodes of the network.

In any stable network, an incident noise, or disturbance, to any node of the network eventually fades away. However, in most real life situations, the object of interest is a finite time analysis of individual node behavior in response to input shocks, or noise, *i.e.*, *how* the effect of an incident disturbance fades away with time. Such transient behavior depends heavily on the interconnections between the nodes of the network.

In this thesis we build a framework to assess the transient behavior of large interconnected networks. Based on this formulation, we categorize each network into one of two broad classes - resilient or fragile. Intuitively, a network is resilient if the transient trajectory of every node of the network remains sufficiently close to the equilibrium, even as the network dimension grows. This is different from standard notion of stability wherein the trajectory excursion may grow arbitrarily with the network size. In order to quantify these transient excursions, we introduce a new notion of resilience that explicitly captures the effect of network interconnections on the resilience properties of the network. We further show that the framework presented here generalizes notions of robustness studied in many other applications, *e.g.*, economic input-output production networks, vehicular platoons and consensus networks. The main contribution of this thesis is that it builds a general framework to study resilience in arbitrary networks, thus aiding in more robust network design.

Thesis Supervisor: Munther A. Dahleh

3

Title: Professor of Electrical Engineering

4

Acknowledgments

I owe an enormous debt of gratitude to Prof. Munther Dahleh and Mardavij Roozbehani. They have been a great source of inspiration and ideas. Most of all, they have been very patient and encouraging even during times when I was not productive. I have really cherished their guidance, support and friendship during the two years of study at MIT. They have made me feel at home here and I look forward to continuing my PhD under their guidance.

I would like to thank my friends who have made my life here truly memorable; Nisha, who has been a great sounding board for any doubts or frustrations over the last year - you have been by my side whenever I needed you; Anuran, James, Nirav, Eren and David, who share my love for meat and travails in research; Akshay, who makes the best Indian food in Boston; Shelar, who introduced me to weightlifting and a healthier lifestyle; Zied, who introduced me to French cuisine, opera and cheese; Govind, Nicha, Tadayuki, and Hsin-Yu, who made settling in Cambridge easier. I cannot thank my office mates enough - Hajir, Ian, Sam, David, and Fang Chang, life would have been boring without you guys.

I will always be at a loss to express my gratitude to my family - Baba, Mumma, Chhoti and Thammi. Without your unconditional love, support and patience, I would not have been able to achieve what I have - I love you; you have given me the strength to move forward whenever I was faced with adversity.

Contents

List of Figures

List of Tables

Chapter 1

Introduction

Large interconnected networks arise commonly in economics, power systems, transportation systems and biology. Due to their ubiquity and high dimensionality, it is essential to study their robustness properties and further, analyze how the measures of robustness scale with the size of the networks.

In control systems theory, the robustness of systems with complex dynamics in response to external disturbances is well understood. Meanwhile, there exists a rich literature in algebraic graph theory that studies notions of criticality, centrality and robustness in graphs. Real life networks are characterized by their associated topology along with their complex interconnections and system dynamics. A simultaneous consideration of control and graph theoretic tools is therefore necessary for the design of robust networks. The complexity of network interconnections and its interplay with the dynamical behavior is responsible for interesting output sensitivity to input disturbances. An important question that arises in this regard is of predicting when small input disturbances can cause large deviations in the final system output.

1.1 Motivation

When does an input shock to a network have no effect? A network is a collection of individual entities, or nodes, with interconnection, or edges, between them. Each node has an associated state and the state dynamics depends on the set of inter-

connections of the node, and the external environment. Assume, now, that node i experiences some significant variation in the environment, then due to the interconnections in the network these variations experienced by node i can spread to other parts of the network, *for e.g.* that are not directly connected to node i. In fact, depending on the state dynamics of each node, there may be a positive feedback loop generated between i and rest of the network.

In this thesis, we focus on the resilience assessment of a large network with complex interconnections. Intuitively, a network is resilient to input shocks if there is no positive feedback loop between any two exclusive parts of the network. We specifically attempt to study how the complexity of interconnections affect the resiliency properties of a network. To better illustrate our motivation, we present an example that shows how variations in interconnections of a network result in substantial changes in its resilience properties.

1.1.1 An Interconnected Example

In this example we will present three networks - a completely disconnected network, a chain network and a chain network with self loops. Each node, i, has the following dynamics

$$x_i(k+1) = \sum_{j=1}^{n} a_{ij} x_j(k) + w_j(k) \tag{1.1}$$

where $a_{ij} \neq 0$ if and only if there exists a link from j to i and k is the time index. The a_{ij} values for each network are shown in Fig. (1.1). The dynamics proposed in Eqn. (1.1) reduces to the matrix formulation in Eqn. (1.2)

$$x(k+1) = Ax(k) + w(k) \tag{1.2}$$

Here, w is some noise. If $w(\cdot)$ is an input noise, then we are interested in the time evolution of the vector $x(\cdot)$. Such cases have been partly studied in [5] from a purely computational perspective. However, in a real network, the dynamics of $x(\cdot)$ may have physical ramifications that need to be understood. For example, in social networks, Eqn. (1.2) denotes a consensus system (See [16], [18]). In such a case it is important to understand:

12
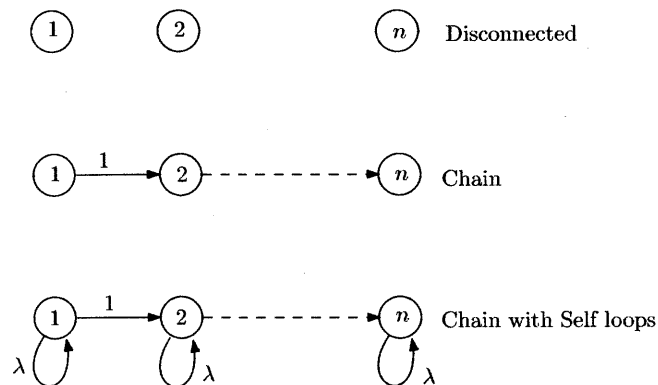
Figure 1.1: Different Network Topologies

- Does the network converge to a unique point ?

- If it converges, what is the speed of convergence ?

- How does a link $i \to j$, between nodes i and j,

    affect the speed of convergence ?

Now, if $\lambda < 1$ then the three networks in Fig. (1.1) are stable. From standard control theory, [11], we know that for each $k > 0$ and unit energy $w(\cdot)$, $||x(k)|| \leqslant C(n)||x(0)||$. Here $||a|| = \sqrt{\sum_{j=1}^{n} a_j^2}$, and $C(\cdot)$ is some function that depends on the number of nodes in the network. This implies that the trajectory of all the states of a stable network is bounded by some function of network dimension. In later chapters, we will show that this function is closely related to the effect of interconnections on the so-called resilience of the network. For now we will state, without proof, some properties of the $C(\cdot)$ for each of the networks shown in Fig. (1.1).

- For the disconnected graph, $C(n) = a$, for some constant $a > 0$. This happens because of the lack of links between the nodes, as a result any disturbance of nodal state values remains local to that node.

- For the chain network, $C(n) = bn$, for some constant $b > 0$. One might be tempted to attribute this to the addition of $n - 1$ links, however as we shall see in the case of chain network with self loops that it is, in general, not the case.

- $C(n) = \exp(cn)$ for some constant $c > 0$ for the chain network with self loops.

13

We will prove this rigorously in a later chapter.

When n is large, *for e.g.* $n > 1000$, we see that there will exist a time point, k, when some node, j, in the network observes a prohibitively large state value, *i.e.*, $x_j(k) > \exp(c1000)$. If each state variable corresponded to the requirement of physical resources, this would imply an acute nodal shortage. As a result, it becomes important to understand why such deviations occur in real networks and how they can be prevented.

1.1.2   Applications

In this section, we present three well studied problems spread across economics, transportation and social sciences that analyze resilience of networks occurring in that field. We find that the notions of fragility, or lack of resilience, in these areas are tied to poor dimensional scaling of some network property. The identification and study of these network properties will be a subject of interest in the chapters to follow.

Deviations from normal behavior in economic networks

GDP variations to input shocks in different network topologies have been studied in [2], [3] and [4]. In [2], it is shown how microeconomic shocks can lead to aggregate volatility (measured by the standard deviation of GDP), *i.e.*, the volatility of GDP does not diminish to zero as the network size increases, if some sectors are much larger than others.

In [4], it is shown that the interplay of idiosyncratic microeconomic shocks and sectoral heterogeneity results in significant departures of the distribution of economic downturns from the normal distribution. The factors behind "tail comovement" is also discussed there, whereby large recessions occur in a cascading fashion across a wide range of industries in the network. Further, in [4], domar weights in the economy, defined as sectoral sales divided by GDP, are shown to be the sufficient statistics of large economic downturns. The US GDP growth rate distribution QQ plot is shown in Fig. (1.2) (taken from [1]).
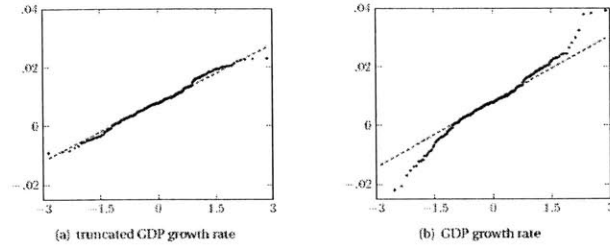
Figure 1.2: The QQ plots of postwar US GDP growth rate (1947 Q1 to 2013 Q3) vs standard normal distribution (dashed red line)

Vehicular Platoons

Platoons offer a promising solution to automated highway transport. In [13], a special form of string instability, disturbance amplification as an input shock propagates through the platoon, called harmonic instability is studied. The conditions under which an asymmetric control leads to an exponential, in the length of the platoon, magnification of input disturbance are studied there. In Fig. (1.3), $T_n(s)$ denotes the transfer function of the network, where the output, γ_n, is the distance, from the leader, of the last vehicle.
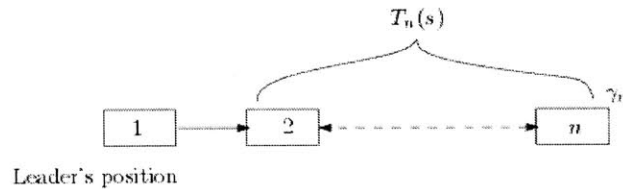
Figure 1.3: Dynamical model of a controlled vehicular platoon network

Consensus Systems

Consensus systems have been studied in [6], [16], [17]. Specifically, [16] builds a theoretical framework for the analysis of consensus algorithms for multi-agent networked systems and the dependence of consensus convergence times on network topology. Network dimension dependent bounds for convergence times in consensus networks are given in [6], [17] for a certain class of network topologies. Further, in [17], sufficient conditions for polynomial growth in convergence time, with re-

15

spect to network dimension, are studied.

In this thesis, we build a unifying framework to analyze the dependence of robustness or resiliency properties of a network on its topology. We introduce new measures of resiliency that are consistent with existing analyses of robustness in economic, transportation or social networks, and provide greater explanatory power.

1.2 Outline

In chapter 2, we formulate the model and introduce a new measure that captures the dependence of network topology and dimension on its resilience, or robustness. Based on this measure we create a hierarchy of resilience. Through examples, we will demonstrate how this hierarchy is consistent with existing literature on topology dependent robustness. In chapter 3, we show how this new notion of resilience is affected by the interconnections in the network and the network topology. In chapter 4, we return to the applications mentioned in this chapter. We show how the ideas introduced in chapter 2 can be used to study robustness in these, apparently unrelated, examples. In fact, our general framework gives us a broader set of tools for design and control of such networks. Finally, in chapter 5, we summarize our findings and provide possible future research directions.

Chapter 2

Resilience in Large Networks

In chapter 1, we motivated the need to study resilience of a network as a function of its dimension and topology. We will formulate the model and lay groundwork for results to follow in the following sections.

2.1   Network Model

Definition 1. — *A network is a graphical representation* $\mathcal{G} = (\mathcal{V}, \mathcal{E}_\mathcal{G})$, *where* $\mathcal{V} = \{v_1, v_2, \ldots, v_n\}$ *represents the set of nodes such that each node,* v_i, *has an associated dynamical behavior and* $\mathcal{E}_\mathcal{G} \subseteq \mathcal{V} \times \mathcal{V}$ *represents the set of edges or communication links. An edge or link from node* i *to node* j *is denoted by* $e[i,j] = (v_i, v_j) \in \mathcal{E}_\mathcal{G}$. *Further, the dynamics of each node* i *is given by*

$$x_i(k+1) = f_i(x_l(k), x_{l+1}(k), \ldots, x_m(k), w_i(k)) \tag{2.1}$$

Here $\{(l,i), (l+1, i), \ldots, (m,i)\} \subseteq \mathcal{E}_\mathcal{G}$ *and* $w_i(\cdot)$ *is the input at node* i.

Definition 2. — *A linear network is a network with the following dynamics*

$$x_i(k+1) = \sum_{j \in \mathcal{N}_i} a_{ij} x_j(k) + w_i(k) \tag{2.2}$$

Here $\{(j,i) | j \in \mathcal{N}_i\} \subseteq \mathcal{E}_\mathcal{G}$ *and we call* $\mathcal{N}_i$ *the incoming neighborhood of node* i. *In fact, Eqn. (2.2) can be compactly represented as Eqn. (1.2).*

Linear networks can be completely represented by A, from Eqn. (1.2), which we call the state transition matrix. Next, we show the different types of graphical networks that we encounter in our analysis.

- **Directed Networks**: Linear Networks where $a_{ij} \neq a_{ji}$ for at least one $(i,j) \in \mathcal{V} \times \mathcal{V}$ as in Fig. (2.1)
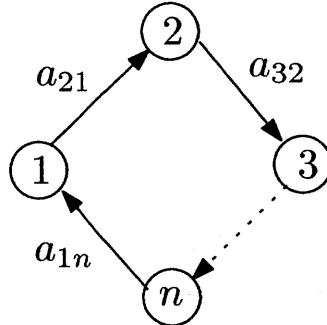
Figure 2.1: Directed Network

- **Undirected Networks**: Linear Networks where $a_{ij} = a_{ji} = a \, \forall \, i, j \in \mathcal{V}$ as in Fig. (2.2). Note that for an undirected network, it is not just sufficient to have a symmetric adjacency matrix.
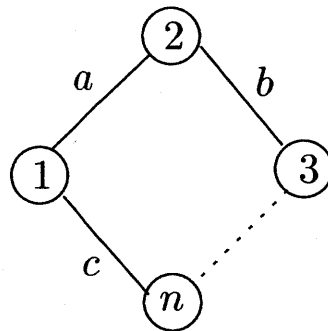
Figure 2.2: Undirected Network

In majority of this work we deal with linear networks. One might ask then,

Why do we work with linear networks ?

Linear models can appropriately model many real life scenarios as presented in chapter 1. Second, we are interested only in interactional effects on the robustness of a network, therefore assuming a nonlinear model for state dynamics obfuscates the effects solely due to links between the nodes of a network.

We impose the following (loose) assumptions on the state transition matrix, also called the network matrix,

Assumption 1. — *Any state transition matrix, A, is stable unless stated otherwise, i.e., $\rho(A) < 1$ where $\rho(\cdot)$ is the spectral radius function of matrix.*

The assumption of stability on the state transition matrix is reasonable, otherwise such a network would not be of any practical interest. There would always exist an input that would cause the system to misbehave.

Assumption 2. — *For any sequence of network matrices, $\{A_n\}_{n=1}^{\infty}$, where the $(i,j)^{th}$ entry of the matrix A_k is $a_{i,j}^{(k)}$, we have that $\limsup_{n \to \infty} \max_{i,j} |a_{ij}^{(n)}| < \infty$.*

Assumption (2) states that individual link weights should not grow with network dimension. This is reasonable because most real life networks are distributed, and individual nodes requirements are finite, and do not grow much with the scale of the network.

We would like to point out that we are interested in the resilience, or the lack thereof, of large networks, formally represented as a sequence of networks where the network dimension grows, *i.e.*, $|\mathcal{V}| \to \infty$. This representation is similar to the one studied in [4]. The key is to identify a limit of networks, that have similar structure as dimension grows.

What do we mean by a limiting sequence of networks?

A formal notion of a "limit" of a network sequence is necessary because we are interested in networks that are growing in dimension but have the same topology, *for e.g.*, we are interested in analyzing the behavior of a star network, so our network sequence comprises of a star of size $= 1$, a star of size $= 2$, ..., a star of size $= n$ and so on. This notion is formally characterized in Definition (3), and extended in Definition (14) in chapter 4, where we will need it to prove some equivalence results.

Definition 3. — *For each network matrix, A_n, in the network sequence, $\{A_k\}_{k=1}^{\infty}$, denote by $r_n(j)$ the jth row of the network matrix, A_n. Next, we define $\mathcal{F} = \{f : \mathbb{N} \to \mathbb{R} \mid \|f\|_{\infty} < \infty\}$. For each n, define $S_n = \{1, \ldots, n\}$, $r_n^A(j) = [r_n(j, l_1), \ldots r_n(j, l_m)]$ and $f^A = [f(l_1), \ldots f(l_m)]$, where $A = \{l_1, \ldots, l_m\} \subseteq S_n$ and $f \in \mathcal{F}$. Then, we say that the network sequence has a limit if $\sup_{A \subseteq S_n} \|r_n^A(j) - f_j^A\|_1 = o(1)$ for each*

19

$j \in S_n$, and for some $f_j(\cdot) \in \mathcal{F}$.

Assume that for every network matrix, A_n, in a sequence of networks we have the following dynamics

$$x_n(k+1) = A_n \, x_n(k) + \omega_n \, \delta(0,k), \quad k \in \{0,1,2,\ldots\} \tag{2.3}$$

Here $x_n(k)$ is the $n \times 1$ vector of state variables. A_n is the $n \times n$ state transition matrix. $\delta(0,k)$ is the Kronecker delta function, with $\delta(0,0) = 1$ and $\delta(0,k) = 0 \; \forall \, k \neq 0$ and ω_n is an $n \times 1$ input disturbance exogenous to the system.

What happens in Eqn. (2.3) is the following - at time $t = 0$, the system is given an input shock signal, ω_n, i.e., the state variable corresponding to each node, i, is perturbed. We are then interested in tracking the time evolution of the state variables of the network, to answer the following question -

How can we measure the effects of an impulse input to a network?
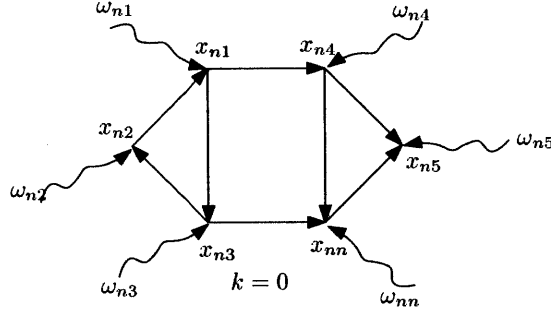
Figure 2.3: Network with noise on every node, $k = 0$

Definition 4. — *Given the network dynamics as in Eqn. (2.3), for each network matrix, A_n, in the sequence, $\{A_k\}_{k=1}^\infty$, and a deterministic input disturbance sequence, $\{\omega_k\}_{k=1}^\infty$, to the system, the max norm, $\mathcal{M}_n$, for every A_n, is given by:*

$$\mathcal{M}_n = \sup_{\|\omega_n\|_2 = 1} \sum_{k=0}^\infty x_n^T(k) x_n(k) \tag{2.4}$$

Definition 5. — *A sequence of vectors, $\{\omega_k\}_{k=1}^\infty$, is a white noise sequence when* $\mathbb{E}[\omega_n \omega_n^T] = I_{n \times n}$ *and* $\mathbb{E}[\omega_n] = 0$ *for each n.*

Definition 6. — *Given the network dynamics as in Eqn. (2.3), for each network matrix, A_n, in the sequence, $\{A_k\}_{k=1}^\infty$, and a white noise sequence, $\{\omega_k\}_{k=1}^\infty$, the av-*

erage norm, $\mathcal{E}_n$, for each A_n, is defined as the following:

$$\mathcal{E}_n = \mathbb{E}_{\omega_n}\left[\sum_{k=0}^{\infty} x_n^T(k)x_n(k) \right] \tag{2.5}$$

Proposition 1. — *The max norm, $\mathcal{M}_n$, is $\sigma_{max}(P_n)$, and the average norm, $\mathcal{E}_n$, is $tr(P_n)$, where*

$$A_n^T P_n A_n + I_n = P_n \tag{2.6}$$

Here $\sigma_{max}(P_n)$ is the largest singular value of P_n and $tr(P_n)$ is the trace of P_n.

Proof. Proof is in Appendix B.1. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Definition 7. — *Given a sequence of networks with network matrices $\{A_k\}_{k=1}^{\infty}$, the sequence is asymptotically robust, or resilient, if we have:*

- *Network matrix, A_k, is stable for each k*

- $\mathcal{E}_n = O(p(n))$

Here $p(\cdot) \in \mathcal{P}_d$ for some $d \in \mathbb{N}$, and $\mathcal{E}_n$ is the average norm of the matrix A_n in the sequence. Fragility is the lack of resilience in the sense of super-polynomial or exponential scaling of the average norm, $\mathcal{E}_n$, of the network matrix sequence $\{A_n\}_{n=1}^{\infty}$.

Remark 1. — *Since $\mathcal{M}_n \leqslant \mathcal{E}_n \leqslant n\mathcal{M}_n$, we note that a network sequence is resilient in $\mathcal{E}_n$-norm, if and only if it is resilient in $\mathcal{M}_n$-norm. However, the input (stochastic) to the system when evaluating the $\mathcal{E}_n$-norm is different from the input (deterministic) while evaluating the $\mathcal{M}_n$-norm. Therefore, the equivalence of the norms additionally tells us that the resilience property does not depend on the stochasticity of the input. It should be noted that the average norm of an LTI system is the same as the $\mathcal{H}_2$-norm when the input to the system is a white noise process as observed in [14].*

Definition (7) is general in the sense that resilience is a property of the sequence $\{P_k\}_{k=1}^{\infty}$, and hence $\{A_k\}_{k=1}^{\infty}$, and not specific to the $\mathcal{L}_p$ norm that we use. This follows from the following fact about vector norms, for $p > r > 0$:

$$||x||_p \leqslant ||x||_r \leqslant n^{1/r-1/p}||x||_p \tag{2.7}$$

Specifically, Eqn. (2.7) implies that if an induced norm of P_n is polynomial in dimension then so are all other induced norms of P_n. There is another intuition for

21

the demarcation between resilient and fragile network sequences that we will present in chapter 4.

Further, fixing an $\mathcal{L}_p$ norm gives us a hierarchy of resilience, in that norm, which we formally define below

Definition 8. — *For a given polynomial, $p(n)$, we call a network sequence, $\{A_k\}_{k=1}^{\infty}$, $\mathcal{L}_q - p(n)$ order if $||P_n||_q = \Theta(p(n))$ where P_n is generated as in Eqn. (2.6).*

It follows that, $\mathcal{M}_n = ||P_n||_2$, i.e., it is a measure of resilience in the $\mathcal{L}_2$ norm. In Table 2.1, we present the examples of resilience order that we will commonly visit in this work.

Resilient	Fragile
$\mathcal{M}_n = \Theta(1)$, $\mathcal{L}_2$ constant order	$\mathcal{M}_n = \text{Superpolynomial}(n)$
$\mathcal{M}_n = \Theta(n)$, $\mathcal{L}_2$ linear order	$\mathcal{M}_n = \Omega(\exp(an))$
$\mathcal{M}_n = $ Higher order polynomials in n	$\mathcal{M}_n = $ Higher order exponentials in n

Table 2.1: Hierarchy of Resilience under $\mathcal{L}_2$ norm

Proposition 2. — *Every undirected network sequence, $\{A_k\}_{k=1}^{\infty}$, is resilient. In fact, for each n, we have:*

$$P_n = (I - A_n^2)^{-1} \tag{2.8}$$

If further $\limsup_{n\to\infty} \rho(A_n) < 1$, then $\mathcal{M}_n = O(1)$.
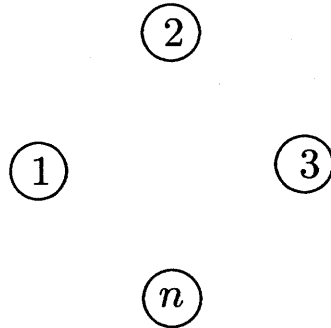
Proof. Proof is in Appendix B.2. □

Figure 2.4: Disconnected Network

Remark 2. — *We see that, in general, the average norm is affected both by the proximity of the spectral radius to unity and interconnection effects. In fact, the*

proximity of the spectral radius to unity is itself a result of interconnections in the network matrix. This is clearly visible when the network matrix is symmetric (as shown in the proof of Proposition 2), and holds true for general network matrices. In general, it is possible to have fragile behavior even when the spectral radii of the network matrices are uniformly bounded away from unity (we will show examples of such networks in chapter 4). Definition (7), therefore, attempts to encapsulate the dual effects of spectral radius proximity to unity and the influence of interconnections.

Remark 3. — *Suppose we had a disconnected network sequence, $\{A_k\}_{k=1}^{\infty}$, i.e., where each network matrix, A_n, looked of the form in Fig. (2.4), then it is easy to check that $\mathcal{M}_n = 1$. By Propostion (2), we see that for undirected network sequences, where each network matrix, A_n, is of the form in Fig. (2.2), we have that $\mathcal{M}_n = O(1)$. These observations imply that in any network the best, in terms of resilience, we can hope for is a symmetric flow of information.*

2.2   Resilience in arbitrary graphs

Are symmetric network matrices the only types of network that are resilient ?

We are interested in understanding and characterizing network properties that influence fragility in general networks. The following proposition gives us a sufficiency condition for asymptotic robustness, or resilience, of a general network sequence.

Proposition 3. — *A network matrix sequence, $\{A_n\}_{n=1}^{\infty}$, is asymptotically robust, or resilient, if there exist $k, d \in \mathbb{N}$ and a function $g(\cdot)$ such that*

$$\limsup_{n \to \infty} ||A_n^k||_p^{g(n)} < 1 \tag{2.9}$$

where $g(\cdot) \in \mathcal{P}_d$, and $|| \cdot ||_p$ is some $\mathcal{L}_p$ induced norm.

Proof. Proof is in Appendix B.3   □

Remark 4. — *An obvious implication of Proposition 2 is that all network sequences of contraction mappings are resilient. Proposition 2 further allows cases where $||A_n||_2 > 1$ but the sequence is still asymptotically robust. Also, if we are*

given an orthogonal projection, Π, *and a symmetric network matrix sequence* $\{A_k\}_{k=1}^{\infty}$ *then the sequence* $\{\|\Pi A_n\|\}_{n=1}^{\infty}$ *is also resilient. This shows us that asymmetry introduced in specific ways does not necessarily break resilience.*

2.2.1 Resilience in consensus graphs

So far we have dealt with stable networks only, however, in a large class of applications, *for e.g.*, consensus networks deal with network matrices that are strongly connected but have spectral radius at unity. As a result, we need a systematic way of extending our notion of resilience to such networks. We start by imposing Assumption (3) on the stochastic matrices that we study. This assumption ascertains that the network graph is strongly connected, for an analysis on consensus in graphs that are not strongly connected see [7].

Assumption 3. — *If A is a stochastic matrix, then we assume it is aperiodic and irreducible.*

Definition 9. — *A sequence of network matrix sequence,* $\{A_k\}_{k=1}^{\infty}$, *is a consensus network sequence when for every n, A_n is a stochastic matrix and for every network matrix, A_n, each node of the network has the dynamics given by Eqn. (2.10). Here $\delta(\cdot, \cdot)$ is the Kronecker delta function and ω_n is a $n \times 1$ input vector.*

$$x_n(k+1) = A_n x(k) + \omega_n \delta(0, k) \tag{2.10}$$

Proposition 4. — *For the consensus dynamics in Eqn. (2.10) we have that*

$$\lim_{k \to \infty} x_n(k) = c[1, 1, \ldots, 1]^T \tag{2.11}$$

Proof. Using Perron Frobenius as in [15]. $\qquad\square$

Next, we define the projection matrix, Π_n, that is perpendicular to the vector $[1, \ldots, 1]^T$ for every length n vector:

$$\Pi_n \triangleq I_n - 1_n 1_n^T / n \tag{2.12}$$

Here, I_n is the $n \times n$ identity matrix, and 1_n is the $n \times 1$ vector of all ones. Proposition (4) states that all the nodes approach the same state value eventually. As a result, we expect that in well behaved systems the trajectory in an orthogonal subspace be "well bounded". In light of this, we will study the Π_n projec-

24

tion of the state vector $x_n(k)$ at each time point k for every network, A_n, in the network sequence. Here $x_n(k)$ is generated as in Eqn. (2.10). Formally, we have $x_{n,\Pi}(k) = \Pi_n\, x_n(k)$, and the measures for the consensus network sequence are

$$\mathcal{M}_n^\Pi = \sup_{\|\omega_n\|_2 = 1} \left(\sum_{k=0}^{\infty} x_{n,\Pi}^T(k) x_{n,\Pi}(k) \right)$$

$$\mathcal{E}_n^\Pi = \mathbb{E}_{\omega_n}\left[\left(\sum_{k=0}^{\infty} x_{n,\Pi}^T(k) x_{n,\Pi}(k) \right) \right]$$

This approach is similar to the one studied in [14], [18].

Proposition 5. — *We have the following relations for every A_n in the consensus network sequence, $\{A_k\}_{k=1}^{\infty}$:*

$$\mathcal{E}_n^\Pi = tr(P_{n,\Pi})$$

$$\mathcal{M}_n^\Pi = \sigma_{max}(P_{n,\Pi}) \tag{2.13}$$

Here $P_{n,\Pi} = A_n^T P_{n,\Pi} A_n + \Pi_n$. Further, $P_{n,\Pi}$ can be represented by

$$P_{n,\Pi} = \sum_{k=1}^{\infty} (A_n^T)^k\, \Pi_n\, A_n^k + \Pi_n \tag{2.14}$$

Proof. Proof is in Appendix B.4. □

With the preceding framework we are now ready to define the resilience hierarchy for consensus network sequences as we did before

Definition 10. — *Given a sequence of consensus network matrices, $\{A_k\}_{k=1}^{\infty}$, we call the sequence asymptotically robust, or resilient, if:*

- *$\rho_2(A_n) < 1$ for each n*
- *$\mathcal{E}_n^\Pi = O(p(n))$*

Here $\rho_2(A_n)$ is the second largest eigenvalue (in absolute value) of the matrix A_n and $p(\cdot) \in \mathcal{P}_d$ for some $d \in \mathbb{N}$.

2.3 Examples

We conclude this chapter by presenting some network topologies that are commonly observed in real life, and computing their resilience using the tools defined previously. We further note that the reversed star and star network shown in Fig. (2.5) are behaviorally (in terms of resilience measures) same, as a result in Figs. (2.6), (2.7)

we refer them as simply a star network. The quantities E norm and M norm in Fig. (2.6) are $\mathcal{E}_n$ and $\mathcal{M}_n$ respectively.
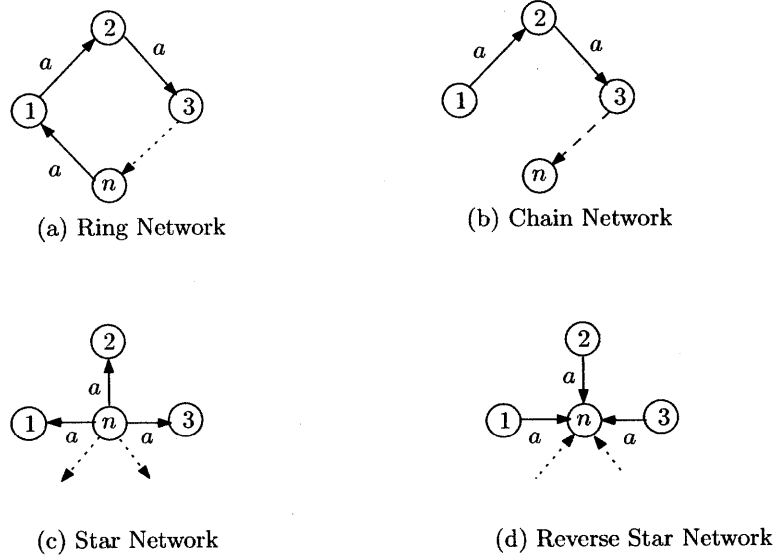

(a) Ring Network

(b) Chain Network

(c) Star Network

(d) Reverse Star Network

Figure 2.5: Different directed networks

Directed Graph type	$\mathcal{M}_n$	$\mathcal{E}_n$
Ring, $(a < 1)$	$\mathcal{M}_n = \Theta(1)$	$\mathcal{E}_n = \Theta(n)$
Chain, $(a < 1)$	$\mathcal{M}_n = \Theta(1)$	$\mathcal{E}_n = \Theta(n)$
Chain, $(a = 1)$	$\mathcal{M}_n = \Theta(n)$	$\mathcal{E}_n = \Theta(n^2)$
Star, $(a \leqslant 1)$	$\mathcal{M}_n = \Theta(n)$	$\mathcal{E}_n = \Theta(n)$

Table 2.2: Resilience of different network topologies

From Table 2.2, we observe that when $a < 1$, then $\mathcal{E}_n = O(n)$ and indeed we will formalize this in chapter 4.

The key takeaway here is that the resilience of a network sequence, $\{A_n\}_{n=1}^{\infty}$, can be completely determined by the grammian sequence, $\{P_n\}_{n=1}^{\infty}$, given by Eqn. (2.6) for each A_n. In the following chapters, we will show how measures of robustness across different disciplines are merely norms of the grammian sequence, and robust networks are equivalent to resilient networks in this framework.

The metric, P, or gramian, is the accumulated effect of a disturbance, at time $t = 0$, on a linear network. In fact, when any disturbance is input to the network, it spreads through to each node over time. Therefore, in a manner similar to [14], the
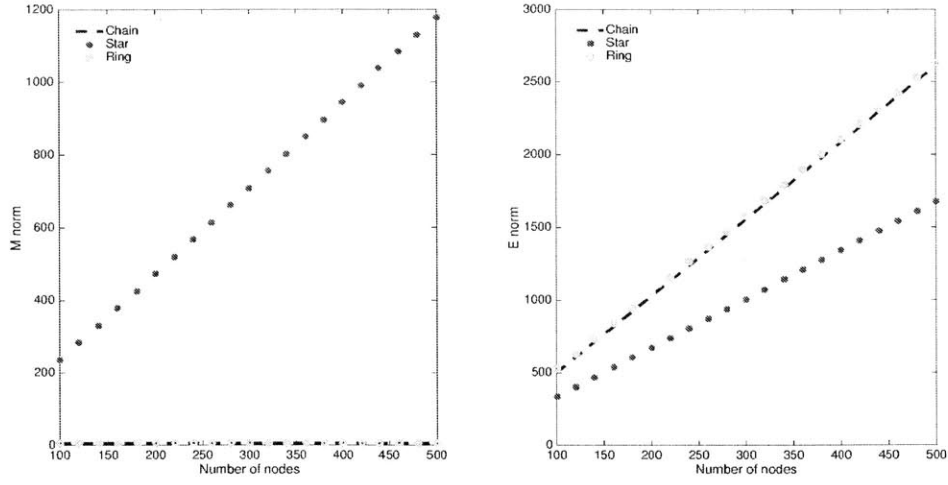
Figure 2.6: Resilience of different networks ($a < 1$)

singular values of the quantity $P_k = \sum_{l=0}^{k}(A^T)^l A^l$, measure the effect of an input disturbance at time, $t = k$, on every node of the network. In Fig. (2.8), we plot the energy evolution heat map for the chain network in Fig. (2.5) with $a = 1$. The diagonal elements are the singular values of P_k, where $t = k$.
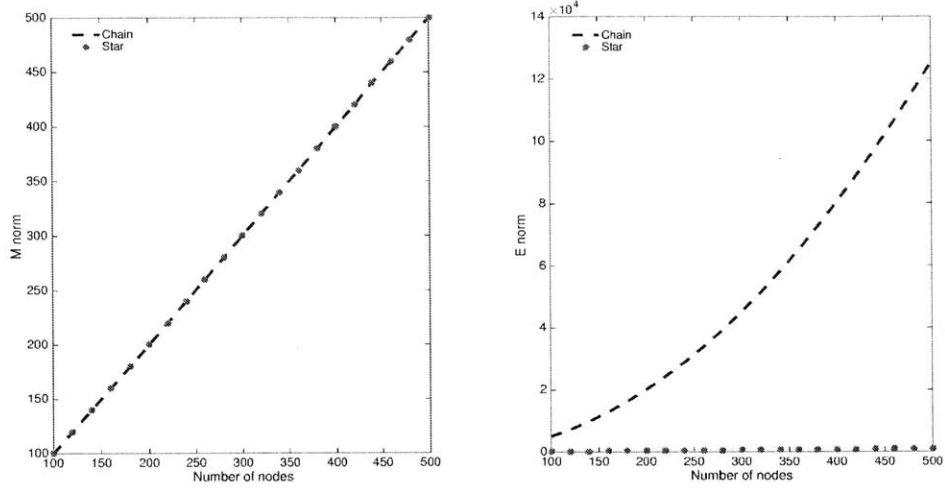
Figure 2.7: Resilience of different networks ($a = 1$)
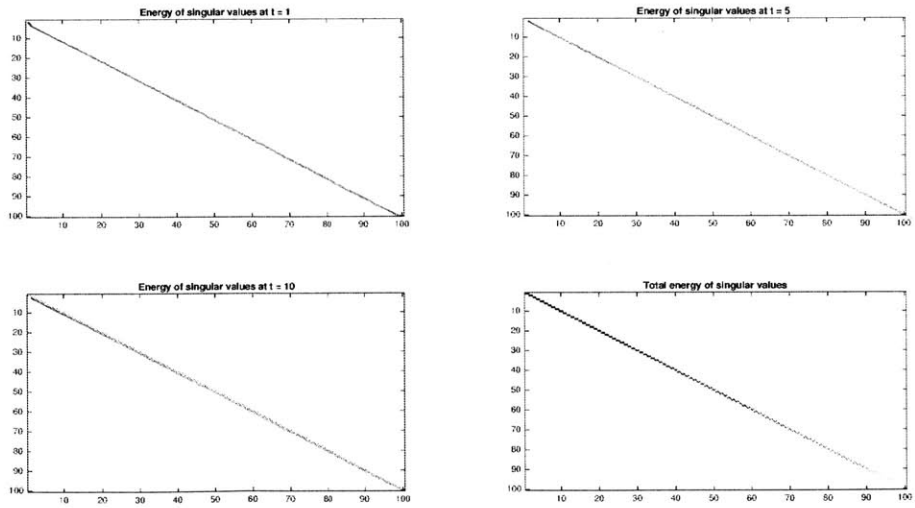
Figure 2.8: Disturbance energy evolution over time for the chain network with $a = 1$

Chapter 3

Edge link effects on resilience measures

In Chapters 1 and 2, we presented why quantifying resilience as a function of network dimension is important. We showed how different network topologies behave under an input shock. Specifically, we used the tools we developed in chapter 2 to analyze the resilience in these topologies. In this chapter we will demonstrate how individual edge links affect the resilience properties of a network topology.

3.1 Edge link sensitivity

The primary focus of this section is to determine if in a fragile network sequence, does their exist a set of bottleneck links, *i.e.*, those links that give substantial improvement when modified. Trivially, this set includes removing *all* the links of the network, however, the challenge is to find a nontrivial set, and formalize this notion of "substantial improvement".

When is an edge link fragile?

Definition 11. — *A link between nodes i, j in a network, A_n, in the network sequence, $\{A_k\}_{k=1}^{\infty}$ is fragile if*

$$\frac{\partial tr(P_n)}{\partial a_{ij}^n} = \Omega(SP(n))$$

29

Here $a_{ij}^n = [A_n]_{ij}$, P_n is defined in Eqn. (2.6) and $SP(\cdot)$ is some superpolynomial function in n.

If a network sequence is fragile, one might expect that there exists a critical link, or a set of critical links, in every network of the network sequence. Definition (11) attempts to capture this notion of criticality. Then, this leads to the question whether this definition of edge link fragility is consistent with the definition of network resilience, *i.e.*,

Do we find fragile links only when the network is part of a fragile sequence?

Lemma 1. — *For each network, A_n, in the network sequence, $\{A_k\}_{k=1}^{\infty}$, with $a_{ij}^n \geq 0$ we have:*

$$\frac{\partial tr(P_n)}{\partial a_{ij}^n} \geq 2[P_n A_n]_{ij} \tag{3.1}$$

Here $a_{ij}^n = [A_n]_{ij}$, P_n is defined in Eqn. (2.6).

Proof. Proof is in Appendix B.7 □

Theorem 1. — *There exists a fragile link in each network, A_n, of the network sequence, $\{A_k\}_{k=1}^{\infty}$ if and only if the network sequence is fragile.*

Proof. Proof is in Appendix B.8. □

As an example for Theorem (1), we show in Figs. (3.1), (3.2) the fragile links in the directed chain with self loops. In the extreme case when the link weight of all edges from $i \rightarrow i + 1$ are 0, this reduces to the disconnected network. However, such a reduction is in no way unique, since if we removed all the self loops we would obtain a simple chain network, that we already know is resilient. In the Fig. (3.2), the Link fragility is measured as $\dfrac{\partial tr(P_n)}{\partial a_{ij}^n}$.

3.2 Nodal Degree sensitivity

For the purpose of analyzing the effect of nodal degrees on the resilience of a network in this section, we will consider network sequences, $\{A_k\}_{k=1}^{\infty}$ where each network, A_n, is of the form

$$A_n = \gamma D_n^{-1} \mathbb{A}_n \tag{3.2}$$

30
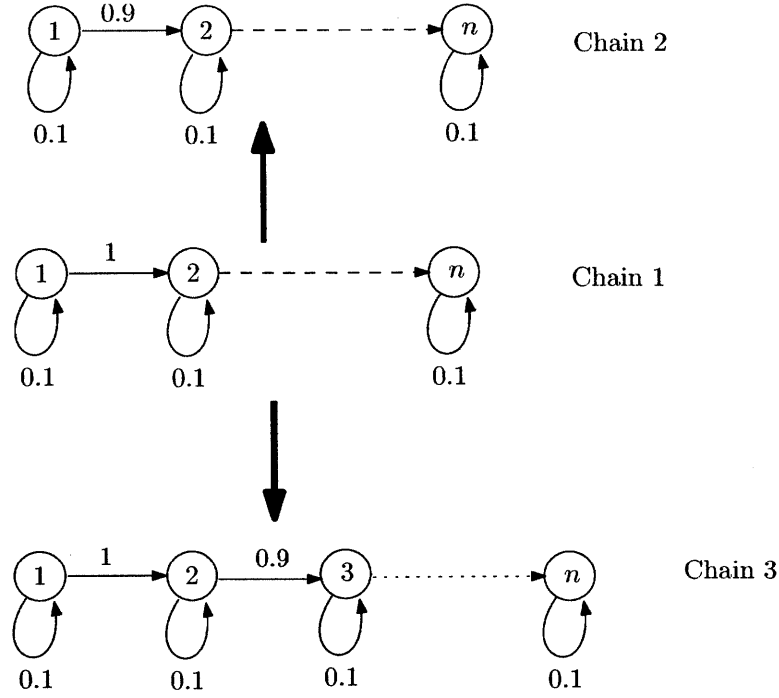
Figure 3.1: Different chain networks with resilience shown in Fig. (3.2)

Here D_n is nodal degree matrix of A_n, $\mathbb{A}_n$ is the incidence matrix corresponding to the network topology of A_n and $\gamma < 1$ for stability of the network A_n.

Definition 12. — *Network sequences, $\{A_k\}_{k=1}^{\infty}$, where each network, A_n, has the form given by Eqn. (3.2) are called degree network sequences. Further, we call networks with network matrix given by Eqn. (3.2) as degree networks.*

As an example, the regular degree network sequence, and the star degree network sequence in Fig. (3.3), have network matrices as given below

$$A_n^{\text{regular}} = \gamma \begin{bmatrix} 0 & 1/(n-1) & 1/(n-1) & \dots & 1/(n-1) \\ 1/(n-1) & 0 & 1/(n-1) & \dots & 1/(n-1) \\ 1/(n-1) & 1/(n-1) & 0 & \dots & 1/(n-1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1/(n-1) & 1/(n-1) & 1/(n-1) & \dots & 0 \end{bmatrix}$$

$$A_n^{\text{star}} = \gamma \begin{bmatrix} 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1/(n-1) & 1/(n-1) & 1/(n-1) & \dots & 0 \end{bmatrix}$$

31
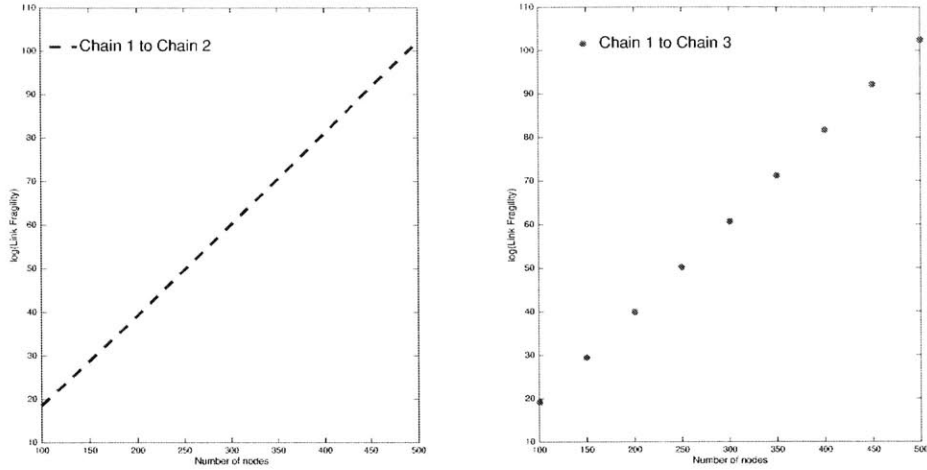
Figure 3.2: Substantial changes when a fragile link is changed

(a) Regular   (b) Star
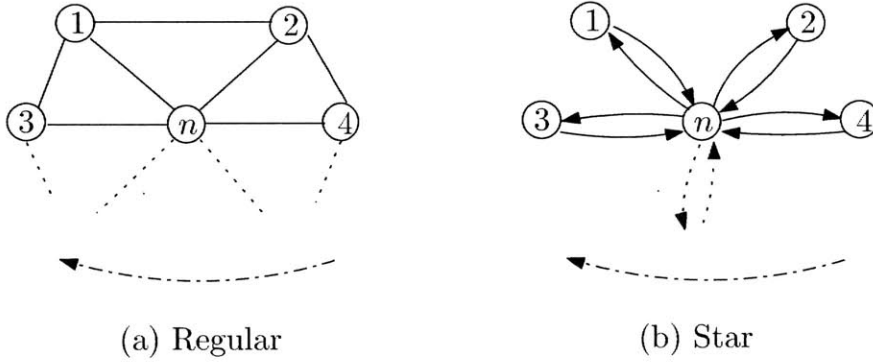
Figure 3.3: Network topologies as defined by Eqn. (3.2)

We observe that there is a strong relation between increasing network entropy, as discussed in [10] and reducing network fragility. As a future research direction, we would like to establish links between network entropy and network resilience. For the purpose of the following discussion, we restrict our attention to degree network sequences.

As an example, we start with a star network network sequence as shown in Fig. (3.3). It is known that $\mathcal{M}_n = \Theta(n)$ for the network sequence. Next, we build a network sequence, called improved star, as shown in Fig. (3.4).

As expected, we get substantial improvement in the resilience as shown in Fig. (3.5)

Degree Networks	$\mathcal{M}_n$	$\mathcal{E}_n$
Star	$\mathcal{M}_n = \Theta(n)$	$\mathcal{E}_n = \Theta(n)$
Regular	$\mathcal{M}_n = \Theta(1)$	$\mathcal{E}_n = \Theta(n)$

Table 3.1: Resilience of degree network topologies

(a) Star (b) Improved Star
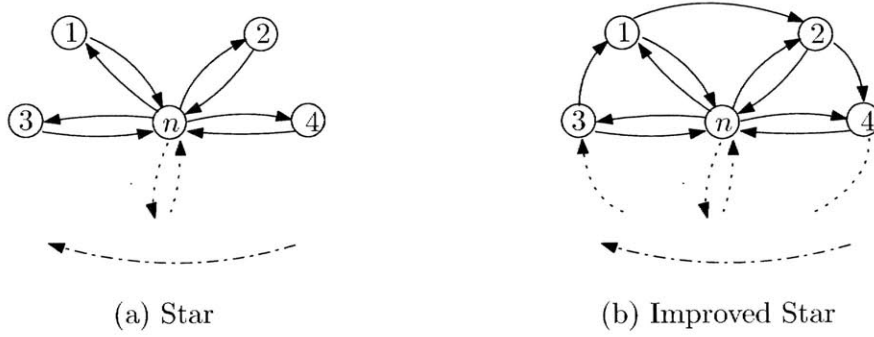
Figure 3.4: Improving the star topology
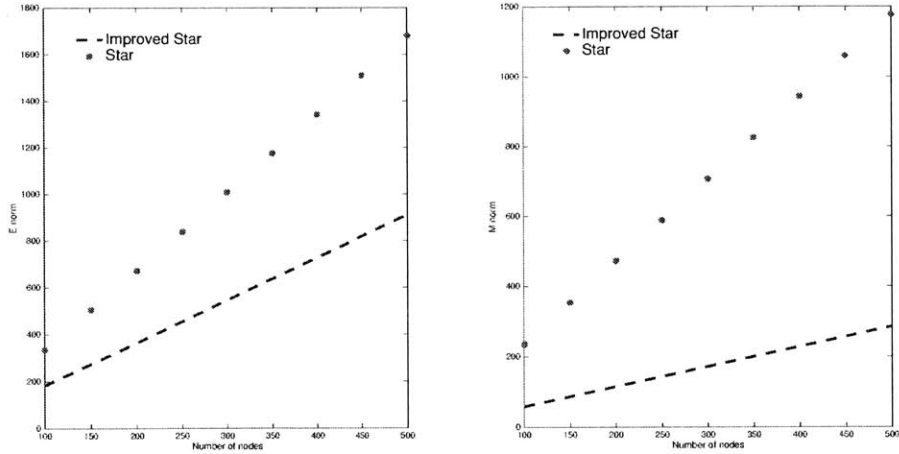
Figure 3.5: Resilience for improved star

33

Chapter 4

Applications in real life networks

We briefly discussed three problems in chapter 1 where robustness, or resilience, as a function of network dimension has been studied. In this chapter, we will revisit the three problems in greater detail and show that the robustness analysis there is essentially a form of resilience analysis under our unified framework. Further, our framework is not bound by the limitations imposed there, and as a result gives us more flexibility in analyses.

4.1 Deviations from normal distribution in economic networks

We study the model discussed in [2], [3], [4]. Consider a static economy consisting of n competitive sectors denoted by $\{1, 2, \ldots, n\}$, each producing a distinct product. Each sector corresponds to a node in the network graph. Firms in each sector employ Cobb-Douglas production technologies with constant returns to scale. Formally, for each sector, i, we have

$$x_i = \Sigma_i \eta_i l_i^{1-\mu} \left(\prod_{j=1}^{n} y_{ij}^{a_{ij}} \right)^{\mu} \tag{4.1}$$

x_i is the output of sector i, Σ_i is Hicks-neutral productivity shock, l_i is labor input to sector i, y_{ij} is amount of output of sector j used for the production of output of sector i, and $\eta_i > 0$ is some normalization constant. As [2] notes, a larger a_{ij} means that sector j is more important in the production of output of sector i. Constant

returns to scale implies $\sum_{j=1}^{n} a_{ij} = 1$ for all i, where $a_{ij} \geqslant 0$. From now, $A = [a_{ij}]$ will be referred as the economy's input-output matrix.

Definition 13. — *Each sector has input microeconomic shocks ϵ_i given by*

$$\epsilon_i = \log\left(\Sigma_i\right)$$

Assumption 4. — *Microeconomic shocks, ϵ_i, are i.i.d. across sectors, and symmetrically distributed around the origin with full support over $\mathbb{R}$.*

The economy has a representative household that supplies a unit of labor inelastically, and has the following preference function over each sector

$$u(c_1, \ldots, c_n) = \sum_{i=1}^{n} \beta_i \log\left(c_i\right) \qquad (4.2)$$

where c_i is amount of output of sector i consumed, given by

$$x_i = \sum_{j=1}^{n} y_{ij} + c_i$$

and $\beta_i > 0$ is i's share in the household's utility function, and $\sum_{i=1}^{n} \beta_i = 1$. In [2], it is noted that there are two forms of heterogeneity - primitive and network. Intuitively, primitive heterogeneity stems from the difference in preferences, *i.e.*, β_i, across different sectors, meanwhile the network heterogeneity is due to the interlinkages of the input-output matrix. Since, in this work we are concerned with the edge link dependencies only, we will impose the additional assumption.

Assumption 5. — *The utility function has no sectoral preferences, i.e., $\beta_i = 1/n$.*

Assumption 6. — *The economy has a competitive equilibrium as in [4]:*

- *Representative household maximizes its utility.*
- *Representative firm in each sector maximizes its profit, while prices and wages are known.*
- *All markets clear.*

Proposition 6 (See [4]). — *The aggregate output is of the economy is given by*

$$y = \log\left(GDP\right) = \sum_{i=1}^{n} v_i \epsilon_i$$

where

$$v_i = \frac{p_i x_i}{GDP} = \frac{1}{n} \sum_{j=1}^{n} l_{ji}$$

36

l_{ji} is (j,i) element of the economy's Leontief inverse $L = (I - \mu A)^{-1}$, and p_i is the price of output of sector i.

Proof. Proof can be found as Proposition 1 of [4]. $\qquad\square$

In this section, we will frequently deal with a sequence of economies, as mentioned in chapter 2, to model the effect of interconnections on the resilience of a network. Now we define what it means to be a limiting sequence of networks,

Definition 14. — *Consider the measurable space $\mathcal{M} = (\mathbb{N}, \mathcal{B})$, and for each $k \in \{1, 2, \ldots, N\}$ define a probability vector $p_N(j \mid k) \; \forall j \in \{1, 2, \ldots, N\}$ such that $\sum_{j=1}^{N} p_N(j \mid k) = 1$. Further, assume that $p_N(\cdot \mid k)$ strongly converges to $f_k(\cdot)$, a probability measure on $\mathcal{M}$, as $N \to \infty$. Then, a sequence of stochastic network sequences, $\{W_k\}_{k=1}^{\infty}$ is a limiting sequence of networks if for every $i = 1, 2, \ldots$, we have that $W_n[i]$ converges strongly to $f_i(\cdot)$, where $W_n[i]$ is the ith row of W_n. In other words, W_n must converge strongly to a markov kernel, $\mathcal{K}(\cdot, \cdot) : \mathbb{N} \times \mathcal{B} \to \mathbb{R}^+$.*

Assumption 7. — *All sequences of networks are a limiting sequence.*

For the purpose of this section, we impose Assumption (7) on the network sequences considered here.

Proposition 7. — *For any network matrix sequence, $\{W_k\}_{k=1}^{p}$, with the property that $||W_n||_p < 1$ for each n, and $p = \{1, \infty\}$ then we have that*

$$\mathcal{E}_n = O(n)$$

Proof. Proof is in Appendix B.10. $\qquad\square$

When $a < 1$, each network in Table 2.2 has $||W_n||_\infty < 1$ and as a result of Proposition (7) we have that $\mathcal{E}_n = O(n)$. A subset of such networks are studied in [4].

4.1.1 Tail Risks

In [4], conditions when idiosyncratic, microeconomic shocks lead to the emergence of large output deviations, or "macroeconomic tail risks", are studied. Ideally, when a network is hit by a microeconomic shock, the aggregate output, as given in Proposition (6), distribution should not deviate "too far" from the normal distribution for a resilient network. We have that,

37

Definition 15. — *For aggregate output, y_n, we have*

$$R_n(\tau_n) = \frac{\log\left(\mathbb{P}(y_n < -\tau_n \sigma_n)\right)}{\log\left(\Phi(-\tau_n)\right)}$$

*where $\Phi(\cdot)$ is the CDF of standard normal distribution, $\sigma_n = $ standard deviation(y_n) and $\tau_n > 0$. Aggregate output, y_n, exhibits tail risks (relative to the normal distribution) if $\lim_{n\to\infty} R_n(\sqrt{n}) = 0$*

Definition 16. — *A random variable, Z, has exponential tails if for some $\lambda > 0$ we have*

$$\mathbb{P}(Z > z) \leq \exp\left(-\lambda z\right)$$

Theorem 2. — *Suppose that microeconomic shocks have exponential tails, then a sequence of economies does not exhibit macroeconomic tail risks if and only if the sequence is $\mathcal{L}_1-$constant order resilient.*

Proof. Proof is in Appendix B.9. □
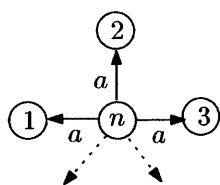
(a) Exhibits Macroeconomic risk   (b) No Macroeconomic risk
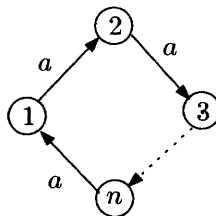
Figure 4.1: Networks with and without macroeconomic risks

As we had shown in Proposition (7), the sequences of economies with constant or diminishing returns to scale can have a worst case $\mathcal{E}_n = \Theta(n) \implies \mathcal{M}_n = O(n)$ for all such sequences. This is an important limitation in the analysis presented in [2], [3], [4]. On the contrary, our framework provides tools to analyze sequences that have increasing returns to scale, at the same time it is consistent with the analysis presented in past literature.

4.2 Vehicular Platoons

Asymmetric control has been well studied in the context of vehicular platoons (See [13], [19] and references therein). Specifically, in [13] a severe limitation of asymmetric control, harmonic instability, is analyzed.

38

Definition 17. — *In the Fig.* (1.3), γ_n *is the output of the platoon, and* $T_n(s)$ *is the transfer function from* $1 \to n$. *Then let* $\gamma_n = \sup_{\omega \in \mathbb{R}^+} |T_n(j\omega)|$, *where* $j = \sqrt{-1}$. *The platoon is harmonically stable if it is asymptotically stable, and* $\limsup_{n \to \infty} \gamma^{1/n} \leq 1$, *else it harmonically unstable.*

The dynamical model, as studied in [19], is given by

$$x_i(k+1) = A_i x_i(k) + B_i u_i(k)$$

$$y_i(k) = C_i x_i(k) \tag{4.3}$$

where $u_i(\cdot)$ is the external control input for the ith vehicle in the platoon.

Assumption 8. — *All vehicles in the platoon have identical behavior, i.e.,* $(A_i, B_i, C_i) = (A, B, C) \, \forall i$.

Now, from [13] and Eqn. (4.3), we have the following system dynamics,

$$x(k+1) = [I_n \otimes A - (I_n \otimes BC)(L \otimes I_N)]x(k) + (I_n \otimes B)r(k)$$

$$y(k+1) = (I_n \otimes C)x(k \dot{+} 1) \tag{4.4}$$

where n is the number of vehicles, N is the number of state variables per vehicle, and L is the graph laplacian of the form

$$L = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 0 \\ -\mu_2 & \mu_2(1+\epsilon_2) & -\mu_2\epsilon_2 & 0 & \dots & 0 \\ 0 & -\mu_3 & \mu_3(1+\epsilon_3) & -\mu_3\epsilon_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & -\mu_n & \mu_n \end{bmatrix}$$
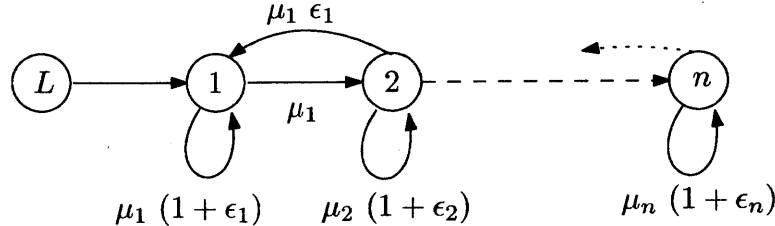
Further, r is the external noise.

Figure 4.2: Laplacian of the platoon

Proposition 8. — *If the leader following network sequence in Fig.* (1.3), *corresponding to the system dynamics in Eqn.* (4.4), *is resilient then the network se-*

quence is harmonically stable.

Proof. Proof is in Appendix B.11. □

There are a few key points to be noted here. First, Eqn. (2.6) changes to

$$P_n = A_{n,V}^T P_n A_{n,V} + C_{n,V}^T C_{n,V}$$

where $A_{n,V} = I_n \otimes \dot{A} - (I_n \otimes BC)(L \otimes I_N)$ and $C_{n,V} = I_n \otimes C$.

The model assumed in [13] is different from the disturbance model studied in this work.

- Noise incidence - In [13], the disturbance is assumed to be incident only on the leader vehicle, and the effect of the noise propagation is measured at the last vehicle. In general, a combination, possibly all, of nodes maybe incident by noise.

- The structure considered is strictly one dimensional, or string-like. However, our framework provides tools for "harmonic stability" of drones and higher dimensional vehicular formations.

The special case of Eqn. (4.3) when each vehicle contributes only one state variable, the problem reduces to finding the gramian matrix as in the familiar Eqn. (2.6).

$$x(k+1) = [a\,I_{n \times n} - bc\,L]x(k) + b\,r(k)$$

$$y(k+1) = c\,x(k+1) \tag{4.5}$$

where a, b, c are real numbers, with the laplacian shown in Fig. (4.3). The $\{\lambda_i, \mu_i \geq 1, \epsilon_i\}_{i=1}^n$ or in this case a, b, c are chosen so that the network is stable. Then,
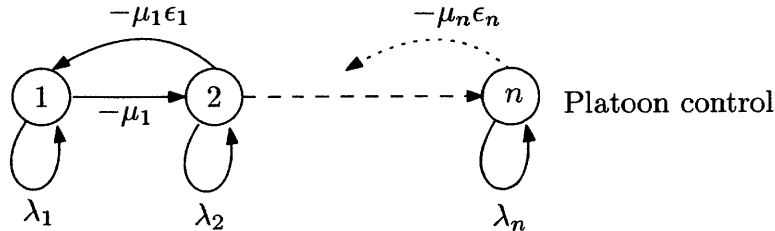

Figure 4.3: Platoon control network

Proposition 9. — *The predecessor following algorithm ($\epsilon_i = 0$, $\forall i$) is fragile, and the bidirectional symmetric control ($\epsilon_i = 1$, $\forall i$) is resilient.*

40

Proof. Proof is in Appendix B.12. □

In [13], it is shown that for $\max_i \epsilon_i < 1$ and $\min_i \epsilon_i > 0$, then the network is harmonically unstable, *i.e.*, fragile. As an example, we pick $\lambda_i = 0.04 \; \forall i$, $\mu_i = 0.8 \; \forall i$, and $\epsilon_i = 0.3 \; \forall i$, and show the $\mathcal{E}_n$, $\mathcal{M}_n$ norm variation with network size in Fig. (4.4),
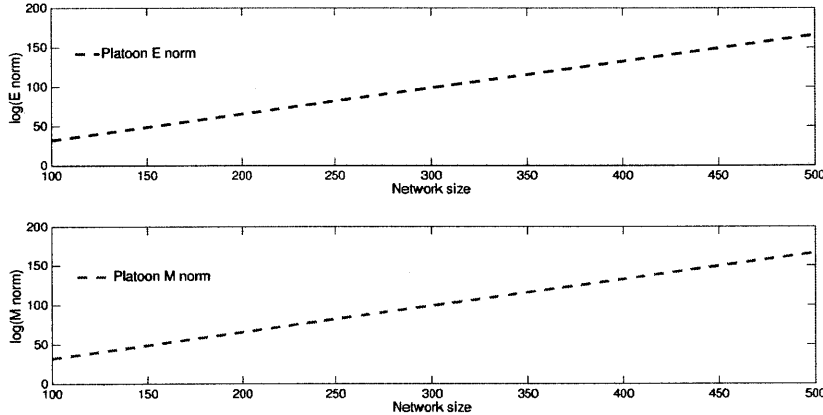
Figure 4.4: Norm variation in asymmetric platoon network

The network matrix, A_n, corresponding to Fig. (4.3) looks like

$$
A_n = \begin{bmatrix}
\lambda_1 & -\mu_1 \epsilon_1 & 0 & \dots & 0 \\
-\mu_1 & \lambda_2 & -\mu_2 \epsilon_2 & \dots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \dots & -\mu_n & \lambda_n
\end{bmatrix}
$$

4.3 Consensus Systems

A large body of work has been devoted to understanding convergence times of distributed consensus networks (See [6], [16], [17], [18]). In this section, we will show some fundamental connections between the convergence time and our metric of resilience. We will show that our measure is consistent with existing literature on convergence analysis for stochastic network sequences. For the purpose of analysis in this section, we will observe the assumptions in subsection 2.2.1, *i.e.*, assumptions of aperiodicity and irreducible.

41

In general, a stochastic network may only converge to the limiting distribution asymptotically. As a result, we need to define a time metric to study the time evolution of the closeness of a stochastic network to its limiting distribution. Further, assume $\Lambda = \lim_{k \to \infty} A^k = 1\pi^T$

Definition 18. — *Given any $\epsilon > 0$, $\epsilon-$convergence time to Λ, $t(\epsilon)$, for a stochastic matrix, A, is defined as*

$$t(\epsilon, \lambda) = \inf\{t \in \mathbb{N} | \|A^t - \Lambda\|_\infty < \epsilon\}$$

Definition 19. — *The sequence of stochastic, network matrices, $\{A_n\}_{n=1}^\infty$ converges in polynomial time if*

$$t_{A_n, \Lambda}(1/4) = O(p(n))$$

Here $t_{A_n, \Lambda}(\cdot)$ is the $1/4-$convergence time for the network matrix A_n to the limiting matrix Λ and $p(n) \in \mathcal{P}_d$ for some $d \in \mathbb{N}$.

As before, to analyze the resilience of stochastic matrix networks we define $A_\parallel^{(t)} = \Pi A^t$ as the projection of A on the subspace $\Pi = 11^T/n$ and $A_\perp^{(t)} = (I - \Pi)A^t = QA^t$ as the projection in the subspace perpendicular to $11^T/n$.

Proposition 10. — *For any stochastic matrix that is aperiodic and irreducible,*

$$\|A^t - 1\pi^T\|_\infty \leqslant \|A^{t-1} - 1\pi^T\|_\infty$$

Proof. Remember that $A1 = 1$, therefore we have $A^t - 1\pi^T = A(A^{t-1} - 1\pi^T)$. Thus, $\|A^t - 1\pi^T\| \leqslant \|A\|_\infty \|A^{t-1} - 1\pi^T\| = \|A^{t-1} - 1\pi^T\|$ □

Proposition 11. —

$$\|A_\parallel^{(2t)} - 1\pi^T\|_\infty \leqslant \|A_\parallel^{(t)} - 1\pi^T\|_\infty^2$$

$$\|A_\perp^{(2t)}\|_\infty \leqslant \|A_\perp^{(t)}\|_\infty^2$$

Proof. We have that $A_\perp^{2t} = QA^{2t} = QA^t(\Pi A^t + QA^t)$. Since A^t is stochastic we have that $A^t\Pi = \Pi$, then $QA^t\Pi A^t = Q\Pi A^t = 0$. This gives $QA^{2t} = QA^tQA^t$, and hence we have our inequality. The proof of the other part is similar. □

Next, we find a necessary and sufficient condition relating the convergence time of the consensus network and the norm decay of the probability transition matrix.

Proposition 12. — *A sequence of markov networks, $\{A_n\}_{n=1}^\infty$, converges in poly-*

nomial time if and only if $t_{A_n,\lambda}(1 - 1/(O(g(n)))) = O(p(n))$, *where* $g(\cdot), p(\cdot) \in \mathcal{P}_d$ *for some* $d \in \mathbb{N}$.

Proof. For any fixed $\epsilon < 1$, we have that

$$\epsilon = (1 - 1/O(g(n)))^m$$

$$\log{(\epsilon)} = m \log{(1 - 1/O(g(n)))}$$

We will show that m is at most some polynomial function in n. Assuming $O(g(n)) > 1$,

$$\log{(\epsilon)} \leqslant \frac{-m}{O(g(n))}$$

$$\frac{m}{O(g(n))} \leqslant -\log{(\epsilon)}$$

$$m \leqslant -O(g(n))\log{(\epsilon)}$$

Therefore, choosing $m = -O(g(n))\log{(\epsilon)}$ does the trick. Now, for any markov matrix we have the property in Proposition (10). Then it follows that if $t_{A_n,\lambda}(1 - 1/(O(g(n)))) = O(p(n))$, then so is $t_{A_n,\lambda}(\epsilon) \leqslant O(p(n)) - \log{(\epsilon)}O(g(n))$. The converse follows trivially from Proposition (10). $\qquad\square$

Based on Proposition (12), we have shown the $\mathcal{L}_\infty$ needs to fall below 1 in polynomial time for polynomial time convergence.

Lemma 2. — *For a stochastic matrix sequence,* $\{A_n\}_{n=1}^\infty$, *if* $A_{\perp,n}^{(t)}$ *has* $\epsilon-$*convergence time to* 0 *equal to* $p(n)$, *then* $A_{\parallel,n}^{(t)}$ *has convergence polynomial time to* $1\pi^T$ *equal to* $O(p(n))$.

Proof. Then, at time $t \geqslant p(n)$, we have the following, where $W_{\perp,n}$ is in the orthogonal subspace to $11^T/n$

$$A_n^t = 1\delta^T + \epsilon W_{\perp,n}$$

$$\implies A_n^{t+\tau} = A_n^\tau 1\delta^T + \epsilon A_n^\tau W_{\perp,n} \tag{4.6}$$

Now, we will show that δ and π does not vary to much in norm. From Eq. (4.6) we

see that

$$\lim_{\tau \to \infty} A_n^{t+\tau} = 1\delta^T + \epsilon 1\pi^T W_{\perp,n}$$

$$1\pi^T = 1\delta^T + \epsilon 1\pi^T W_{\perp,n}$$

$$||1\pi^T - 1\delta^T||_\infty \leqslant \epsilon \qquad\qquad (4.7)$$

Thus, $A_{\parallel,n}^t$ has ϵ-convergence time to $1\pi^T$ equal to $O(p(n))$. $\qquad\square$

The Lemma (2) paves way for the following equivalence result -

Theorem 3. — *A sequence of markov matrices, $\{A_n\}_{n=1}^\infty$ is resilient if and only if the sequence has polynomial convergence times. Further, if the ϵ-convergence time for A_n is $p(n)$, then the $\mathcal{L}_\infty$ resilience order for the stochastic network sequence is $\Theta(p(n))$.*

Proof. First from Lemma (2) observe that, if $A_{\perp,n}^{(t)}$ is at most $\epsilon = 1/4$ away after $p(n)$ time, then A^t is at most $2\epsilon = 1/2$ away from $1\pi^T$. By Proposition (10), we have that in $2p(n)$ time A^t will be at most $4\epsilon^2 = 1/4$ away from $1\pi^T$. Thus, the convergence time is $O(p(n))$. For the converse, let $||A_n^t - 1\pi^T||_\infty \leqslant 1/4$ after time $p(n)$, then $||A_n^t - 1\pi^T||_\infty \leqslant 1/16$ after $2p(n)$ time and $||Q(A_n^t - 1\pi^T)||_\infty \leqslant ||Q||_\infty ||A_n^t - 1\pi^T||_\infty \leqslant 1/8$ after $2p(n)$ time, thus the $||P_{n,\Pi}||_\infty = \Theta(p(n))$, where $P_{n,\Pi}$ is defined in Proposition (5). $\qquad\square$

A consequence of the results in [17] is that undirected flocking networks, *i.e.*, graphs that have network matrices of the form Eqn. (3.2) with symmetric adjacency matrices, have polynomial convergence times. Here we will provide an alternate proof for the same. Further, our setup allows for the convergence time analysis of any aperiodic and irreducible stochastic matrix, as opposed to [6], [17], where such an analysis is limited to flocking matrices.

Proposition 13. — *Any undirected, aperiodic and irreducible flocking network sequence, $\{A_k\}_{k=1}^\infty$, is resilient.*

Proof. An undirected flocking matrix, as in [17], has a symmetric adjacency matrix. Then, we know that $A_n = D_n^{-1}\mathbb{A}_n$ can be symmetrized to $S_n = D_n^{-1/2}\mathbb{A}_n D_n^{-1/2} =$

44

$D_n^{1/2} A_n D_n^{-1/2}$. Now, using the same notation as in Proposition (5), we have that,

$$P_{n,\Pi_n} = A_n^T P_{n,\Pi_n} A_n + \Pi_n$$

$$P_{n,\Pi_n} = A_n^T D_n^{1/2} D_n^{-1/2} P_{n,\Pi_n} D_n^{-1/2} A_n + \Pi_n$$

$$D_n^{-1/2} P_{n,\Pi_n} D_n^{-1/2} = S_n^T D_n^{-1/2} P_{n,\Pi_n} D_n^{-1/2} S_n + D_n^{-1/2} \Pi_n D_n^{-1/2}$$

$$P_{n,\Pi_n^D} = S_n^T P_{n,\Pi_n^D} S_n + \Pi_n^D$$

Here Π_n^D is orthogonal to $D^{1/2}[1,1,\ldots,1]^T$, which is the eigenvector of S_n corresponding to the eigenvalue 1. Since S_n is symmetric, the second greatest eigenvalue (in magnitude) will be in an orthogonal subspace, $i.e.$, in Π_n^D.

Since S_n is symmetric the singular values of S_n are the eigenvalues of S_n (albeit some permutation). From [6], we know that for a flocking matrix, $\lambda_2(S_n) \leqslant 1 - 1/n$, where $\lambda_2(A)$ is the second largest eigenvalue (in magnitude) of A. Then, we get that $\sigma_1(P_{n,\Pi_n^D}) = O(n^2)$, further $D_n^{1/2} P_{n,\Pi_n^D} D_n^{1/2} = P_{n,\Pi_n} \implies \sigma_1(P_{n,\Pi_n}) = O(n^3)$. $\qquad\square$

Chapter 5

Conclusions

5.1 Summary of results

In this thesis, we studied the dependence of transient dynamics of networks on their topology. Motivated by applications in economics, transportation systems and social networks, we provided a framework to assess resilience in each of these domains. We showed how existing analyses of resilience, or robustness as a function of network dimension are special cases of the unified framework we develop in this thesis. Based on this, we categorized networks that behaved poorly as their dimension grew, while keeping the topology constant, as fragile. We found that a network was fragile if and only if it had fragile links, which showed that the notion of resilience developed here consistently captures the effects of interconnections on the transient behavior of a networked system. We also provide foundations for analysis of networks that are marginally stable, specifically those where a notion of convergence exists. Additionally, it is shown here that resilience is equivalent to convergence times in such consensus systems.

5.2 Future Work

Although we show that our measures of resilience capture the effect of interconnections in a network, finding the critical links in a general network is a future direction of research. An important unexplored area is when the dynamics of the network are linear but time varying.

Appendix A

Notation

$f(n) = \Theta(g(n))$	$c_1 g(n) \leqslant f(n) \leqslant c_2 g(n)$ for some fixed constants $c_1 \neq c_2 > 0$ and all $n > n_0$
$f(n) = O(g(n))$	$f(n) \leqslant cg(n)$ for some fixed constant $c > 0$ and all $n > n_0$
$f(n) = o(g(n))$	$\lim_{n \to \infty} |f(n)/g(n)| = 0$
$f(n) = \Omega(g(n))$	$g(n) = O(f(n))$
$\rho(A)$	$\max_{1 \leqslant i \leqslant n} |\lambda_i(A)|$
$\|A\|_k$	$\sup_{\|v\|_k \leqslant 1} \|Av\|_k$ denotes the $\mathcal{L}_k$ norm of the matrix A
A_n	Denotes A, a matrix of size $n \times n$
v_n	Denotes v, a vector of size $n \times 1$
$\mathcal{P}_d$	Family of polynomials with maximum degree $d \in \mathbb{N}$

Appendix B

Additional Proofs

B.1 Proof of Proposition 1

Proof.

$$X = \sum_{k=0}^{\infty} x^T(k)x(k)$$

$$= \sum_{k=0}^{\infty} \omega^T (A^T)^k A^k \omega$$

$$= \omega^T \left(\sum_{k=0}^{\infty} (A^T)^k A^k \right) \omega$$

If A is stable, then $P = \left(\sum_{k=1}^{\infty} (A^T)^k A^k \right)$ is given by $P = A^T P A + I$ (this follows from [11]). Thus, we have $X = \omega^T P \omega$, now if ω is a deterministic signal with finite norm then we have

$$X^* = \sup_{\|\omega\|_2 \leqslant 1} \omega^T P \omega = \sigma_1(P)$$

Next if ω is white noise, then we have

$$X^\dagger = \text{tr}\big(\mathbb{E}_\omega[\omega^T P \omega]\big)$$

$$= \mathbb{E}_\omega[\text{tr}(\omega^T P \omega)]$$

$$= \text{tr}\big(\mathbb{E}_\omega[P \omega \omega^T]\big)$$

$$= \text{tr}(P)$$

$\square$

B.2 Proof of Proposition 2

Proof. Since A_n is symmetric, *i.e.*, $A_n = A_n^T$, it follows that $\rho(A_n) = |\lambda_1(A_n)| = \sigma_1(A_n)$. Further from [11] we have that as A_n is stable, the solution, P, to the matrix equation (2.6), with $A = A_n$, exists and is unique for all $n \in \mathbb{N}$.

Due to the stability of A_n for each n, we have from [11] that

$$A_n^T P A_n + I = P$$

$$P = I + \sum_{k=1}^{\infty}(A_n^T)^k A_n^k$$

$$P = I + \sum_{k=1}^{\infty}A_n^{2k} \tag{B.1}$$

$$P = (I - A_n^2)^{-1} \tag{B.2}$$

Equation (B.1) follows from the symmetric nature of A_n and Equation (B.2) follows from the fact that $\sigma_1(A_n^2) \leqslant (\sigma_1(A_n))^2 < 1$. From Equation (B.1) we have that

$$\|P\|_2 \leqslant \|I + \sum_{k=1}^{\infty}A_n^{2k}\|_2$$

$$\sigma_1(P) \leqslant 1 + \sum_{k=1}^{\infty}\sigma_1(A_n^{2k})$$

$$\leqslant 1 + \sum_{k=1}^{\infty}\sigma_1(A_n)^{2k}$$

$$\leqslant \frac{1}{1 - \sigma_1^2(A_n)} \leqslant \frac{1}{1 - \rho(A_n)}$$

The claim $\mathcal{E}_n = O(n)$ follows from the fact that $\mathrm{tr}(P) \leqslant n\sigma_1(P)$ and that $\mathrm{tr}(P) \geqslant$ n $\qquad\qquad\square$

B.3 Proof of Proposition 3

Proof. For the sake of this proof, we can assume $p = 2$ wlog. If equation (2.9) holds, then $||A_n^k||_2 = 1 - 1/O(g(n))$. Under Assumption (2), we have $||A_n||_2 = O(n)$. Next define $c(n) = \sum_{j=1}^k n^{2j}$, then we have from [11], due to the stability of A_n for every n, that

$$P = I + \sum_{m=1}^{\infty} \left(A_n^T \right)^m A_n^m$$

$$\sigma_1(P) \leqslant 1 + \sum_{l=1}^{k} O(n^{2l})$$

$$+ \sum_{l=1}^{\infty} (1 - 1/O(g(n)))^{2l} \left(\sum_{p=1}^{k} O(n^{2p}) \right)$$

$$= 1 + O(c(n))O(g(n))$$

$$= O(c(n)g(n))$$

Since $c(\cdot)$, $g(\cdot) \in \mathcal{P}_t$ for $t = \max\{d, 2k\}$, the sequence of network matrices is asymptotically robust. $\qquad\square$

B.4 Proof of Proposition 5

Proof. We first show that the sum in equation (2.14) is well defined. Since A is aperiodic and irreducible. We have $||A^k - vw^T|| \leq C\alpha^k$ for some $\alpha \in (0,1)$ as shown in [15] in Theorem 4.9 (the convergence is elementwise). Therefore there exists k_0 such that one can write for all $k > k_0$

$$A^k - vw^T \leq C\alpha^k 11^T$$

where $\leq$ denotes element wise inequality and v, w are left and right eigenvectors with eigenvalue 1 respectively. Notice, $\Pi v = 0$ as Π is orthogonal to $v = \gamma[1, 1, \ldots, 1]^T$.

$$\lim_{k \to \infty} ||\Pi A^k||^{1/k} = \lim_{k \to \infty} ||C\Pi \alpha^k B + \Pi vw^T||^{1/k}$$
$$= \lim_{k \to \infty} ||C\Pi \alpha^k B||^{1/k}$$
$$= \alpha \lim_{k \to \infty} ||\Pi B||^{1/k}$$
$$\leq \alpha < 1$$

where $B \leq 11^T$

Using the fact that $\Pi^p = \Pi$ for all $p \in \mathbb{N}$ and $\Pi^T = \Pi$, we have that infinite sum in equation (2.14) converges from [11].

For the next part, we have $x(k) = A^k \omega$ then $x_\Pi(k) = \Pi A^k \omega$. Then $x_\Pi^T(k) x_\Pi(k) = \omega^T(A^T)^k \Pi A^k \omega$ and we have

$$\mathcal{M} = \sup_{||\omega||=1} \sum_{k=0}^{\infty} x_\Pi^T(k) x_\Pi(k)$$
$$= \sup_{||\omega||=1} \sum_{k=0}^{\infty} \omega^T(A^T)^k \Pi^2 A^k \omega$$
$$= \sup_{||\omega||=1} \omega^T \sum_{k=0}^{\infty} (A^T)^k \Pi A^k \omega$$
$$= \sigma_{\max}(P_\Pi)$$

The other claim follows similarly. □

B.5 Another Measure for Gramian

Generally, the gramian does not have a closed form that is suitable for theoretical analysis. Here we will define a "resilience" proxy for the gramian matrix that will aid us in capturing many properties of the gramian matrix.

Lemma 3. — *A network sequence, $\{A_k\}_{k=1}^{\infty}$, with $a_{ij}^n \geq 0$ for each A_n and (i,j) pair, is resilient if and only if $||(I_{n\times n} - A_n)^{-1}|| = O(p(n))$*

Proof. Since, $\rho(A_n) < 1$, a series expansion exists. Note that (assuming $||A_n^k||_2 \geq 1$, otherwise the proof is trivial),

$$||(A_n^T)^k A_n^k||_2 \leq ||A_n^k||_2^2 \leq n||A_n^k||_1^2$$

$$||(A_n^T)^k A_n^k||_1 \geq ||A_n^k||_2$$

Now, assume $||(I_{n\times n} - A_n)||_1 = O(p(n))$, then $||(I_{n\times n} - A_n)||_1^2 = O(p^2(n))$

$$(1 + ||A_n||_1 + ||A_n^2||_1 + \ldots)^2 \geq (1 + \sum_{k=1}^{\infty} ||A_n^k||_1^2)$$

$$\implies O(p^2(n)) \geq (1 + \sum_{k=1}^{\infty} ||A_n^k||_1^2)$$

$$\implies O(p^2(n)) = 1 + \sum_{k=1}^{\infty} ||(A_n^T)^k A_n^k||_2$$

$$\implies O(p^2(n)) = ||I + \sum_{k=1}^{\infty} (A_n^T)^k A_n^k||_2$$

The proof of the converse now follows similarly. $\qquad \square$

B.6 Eigenvalue Perturbations

Our definition of resilience hinges on the fact that the network sequence is stable, *i.e.*, the spectral radius is less than one. Since the perturbation of an edge link may cause the network to become unstable, therefore to show that $\dfrac{\partial P_n}{\partial a_{ij}^n}$ exists we need to show that we can always perturb the edges of a gramian matrix without making the state transition matrix unstable.

Lemma 4. — *If a network sequence, $\{A_k\}_{k=1}^{\infty}$, is resilient, then we can perturb some element, a_{ij}^n, of a network, A_n, in the sequence, by $\epsilon = 1/\Omega(n^2 p(n))$ for some $p(\cdot) \in \mathcal{P}_d$ and for some $d \in \mathbb{N}$, without the network losing stability.*

Proof. From [12], we have that

$$|\lambda - \widehat{\lambda}| \leqslant \|I - A_n\| \, \|(I - A_n)^{-1}\| \epsilon$$

Now, we will show that if A_n is resilient, then $\|(I - A_n)^{-1}\| = O(p(n))$ for some $p(\cdot) \in \mathcal{P}_d$ and for some $d \in \mathbb{N}$. Then we have that $|\lambda - \widehat{\lambda}| = O(1/n)$ and hence our claim follows. $\qquad\square$

B.7 Proof of Lemma 1

Proof. From Lemma (4) we have that $\dfrac{\partial P_n}{\partial a_{ij}^n}$ exists.

We know that $P = I + A^T P A$. Define $P_\delta = I + (A + \Delta)^T P_\delta (A + \Delta)$, where $\Delta_{ij} = \delta$ and 0 otherwise. Then we have

$$P_\delta = I + A^T P_\delta \Delta + A^T P_\delta A$$

$$+ \Delta^T P_\delta \Delta + \Delta^T P_\delta A$$

$$P_\delta - P = A^T P_\delta \Delta + A^T (P_\delta - P) A$$

$$+ \Delta^T P_\delta \Delta + \Delta^T P_\delta A$$

$$\mathrm{tr}(P_\delta - P) = \mathrm{tr}((P_\delta - P)(AA^T)) +$$

$$\mathrm{tr}(P_\delta \Delta A^T + P_\delta \Delta \Delta^T + P_\delta A \Delta^T)$$

$$\mathrm{tr}((P_\delta - P)) \geqslant \mathrm{tr}(P_\delta \Delta A^T + P_\delta \Delta \Delta^T + P_\delta A \Delta^T)$$

$$\lim_{\delta \to 0} \mathrm{tr}\left(\frac{P_\delta - P}{\delta}\right) \geqslant 2[PA]_{ij}$$

and the claim follows. The second last inequality follows from the fact that $\mathrm{tr}((P_\delta - P)AA^T) \geqslant 0$, and from the fact that

$$\lim_{\delta \to 0} \frac{\mathrm{tr}(P_\delta \Delta \Delta^T)}{\delta} = 0$$

where $P_\delta \to P$ as $\delta \to 0$, due to continuity of Lyapunov equation. $\qquad \square$

B.8 Proof of Theorem 1

Proof. Let us assume that no fragile link exist, however the network sequence is fragile. Then there must exist some $[PA]_{ij}$ such that $[PA]_{ij} = \Omega(\text{Superpolynomial(n)}) > 0$. If it did not then it would mean that $P = A^T P A + I < I_{n \times n} O(\text{Polynomial}(n))$ which would lead to a contradiction because P is fragile $\implies P_{ii} = \Omega(\text{Superpolynomial(n)})$ for some i. But by Lemma (1) we have that then a fragile link exists, which contradicts our hypothesis. For the converse, we use the Lemma (3), if the network sequence is resilient then $\text{tr}((I - A)^{-1}) = O(p(n))$, where $p(\cdot)$ is some polynomial. We then need to see what happens to $(I - A)^{-1}$ when we perturb an edge. Since the edge perturbation is a rank 1 addition to the existing matrix we use Sherman-Morrison formula

$$(I - A + uv^T)^{-1} = (I - A)^{-1} - \frac{(I - A)^{-1} uv^T (I - A)^{-1}}{1 + v^T (I - A)^{-1} u}$$

Define $\Delta P = (I - A + uv^T)^{-1} - (I - A)^{-1}$, and a link perturbation can be represented as uv^T, where $u = [0, 0, \ldots, u_i = 1, 0, \ldots, 0]^T$, and $v = [0, 0, \ldots, v_j = \epsilon, 0, \ldots, 0]^T$. Then,

$$\lim_{\epsilon \to 0} \text{tr}(\Delta P / \epsilon) \leqslant \lim_{\epsilon \to 0} \frac{\text{tr}(uv^T (I - A)^{-2})}{\epsilon(1 + v^T (I - A)^{-1} u)}$$

$$\leqslant O(p^2(n))$$

where $p(\cdot)$ is a polynomial function, and we see that a link perturbation does not cause $\text{tr}((I - A + uv^T)^{-1})$ to be exponential, and hence no link is fragile. $\square$

58

B.9 Proof of Theorem 2

Proof. For a stochastic matrix, W, we have that $[W^T W]_{ij} = \sum_{k=1}^n w_{ki} w_{kj}$. Then, $r_j^{W^T W} = \sum_{i=1}^n \sum_{k=1}^n w_{ki} w_{kj} = r_j^W$ where r_j^A is the jth column sum of A. Now, assume $(I - \gamma W_n)^{-1} = c$, then

$$\|I + \sum_{k=1}^\infty \gamma^{2k} (W_n^T)^k W_n^k\|_1 = 1 + \sum_{k=1}^\infty \gamma^{2k} \|(W_n^T)^k W_n^k\|_1$$

$$= 1 + \sum_{k=1}^\infty \gamma^{2k} \|W_n^k\|_1$$

$$\leqslant 1 + \sum_{k=1}^\infty \gamma^k \|W_n^k\|_1$$

$$= \|(I - \gamma W_n)^{-1}\|_1$$

Now, for the converse, we first note that $\|(I - \gamma^2 W_n)^{-1}\|_1 = 1 + \sum_{k=1}^\infty \gamma^{2k} \|(W^T)^k W^k\|_1 \implies \|W_n\|_1 = O(1)$. Now, due to the sequence $\{W_k\}_{k=1}^\infty$ being a limiting sequence, we have that $\lim_{n\to\infty} W_n = \mathcal{K}$, where $\mathcal{K}$ is a markov kernel. Then $\mu \mathcal{K} = f$, where $0 < \mu < 1$.

Next, define the l_1 space and call it X, with the associated norm being $\|\cdot\|_1$. X is a Banach space, which is the space of sequence that are absolutely convergent. Next, define the linear operator $f : X \to X$, and with the property that $\|f\|_1, \|f\|_2 < \infty$, $\|f\|_\infty < 1$. Call $\mathcal{I}$ as the identity operator on the space $\mathcal{I} : X \to X$, then $(\mathcal{I} - f) : X \to X$ is a bounded, in l_1, linear operator. Second, $\rho(f) \leqslant \|f\|_\infty < 1$, as a result $(\mathcal{I} - f)^{-1}$ exists, and by bounded inverse theorem $(\mathcal{I} - f)^{-1}$ is bounded. Now, notice that the sequence $\{\sqrt{\gamma} W_n\}_{n=1}^\infty$ is a special case of the operators defined above, and as a result will be uniformly bounded. $\square$

B.10 Proof of Proposition 7

Proof. $||W_n||_\infty < 1 \implies ||W_n^k||_\infty < 1$ for all k. Further note that $\mathrm{tr}(WW^T) = \mathrm{tr}(W^TW)$, then $[WW^T]_{ij} \leqslant \sum_{k=1}^{\infty} |w_{ik}^n||w_{jk}^n| < 1 \implies \mathrm{tr}((W_n^k)^T W_n^k) = \mathrm{tr}((W_n^k)(W_n^k)^T) = O(n)$. Thus we have that, $\sup_{k>0} \mathrm{tr}((W_n^k)(W_n^k)^T) = O(n)$, and by definition of P_n, we have that $\mathrm{tr}(P_n) = O(n)$. $\qquad\square$

B.11   Proof of Proposition 8

Proof. The disturbance here is incident only on the leader vehicle. Define $||T_{\mathrm{VP}}(j\omega)||_\infty$ as the $\mathcal{H}_\infty$ norm of the vehicular platoon, $||T(j\omega)||_\infty$ as the $\mathcal{H}_\infty$ norm when disturbance can be incident on all the vehicles, then clearly we have

$$||T_{\mathrm{VP}}(j\omega)||_\infty \leqslant ||T(j\omega)||_\infty$$

From [9], for a LTI system we know that

$$\mathcal{H}_\infty \leqslant p(n)\mathcal{H}_2$$

where $p(n) \in \mathcal{P}_d$ for some $d \in \mathbb{N}$. Then our claim follows since the trace of the gramian is the $\mathcal{H}_2$ norm of a DT LTI system. $\square$

B.12 Proof of Proposition 9

Proof. We assume that the network matrix is A_n corresponding to Fig. (4.3). Note that $\mathcal{H}_\infty$ norm is $\max_\omega \sigma_1((e^{(j\omega)}I - A_n)^{-1}) \geqslant \max_\omega \sigma_1((I - A_n)^{-1})$. Notice that A_n is a tridiagonal matrix, then $|\det(A_n)| \leqslant p^n$, where $p < 1$. Then in [8] it has been shown that $|[T^{-1}]_{1,n}| = \Omega(\exp(cn))$ if $\mu_i \geqslant 1$, and as a result the network sequence is fragile. For the bidirectional case, the network looks like the following

$$A_s = \begin{bmatrix} \lambda_1 & -\mu_1\epsilon_1 & 0 & \dots & 0 \\ -\mu_1 & \lambda_2 & -\mu_2\epsilon_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -\mu_n & \lambda_n \end{bmatrix}$$

Now, A_s is symmetric when $\epsilon = 1$, and by Proposition 2, it is resilient. $\square$

Bibliography

[1] What really causes economic downturns? https://www.weforum.org/agenda/2015/03/what-really-causes-economic-downturns/. Published: 03/27/2015.

[2] Daron Acemoglu, Vasco Carvalho, Asuman Ozdaglar, and Alireza Tahbaz-Salehi. The network origins of aggregate fluctuations. *Econometrica*, 80(5):1977–2016, 2012.

[3] Daron Acemoglu, Asuman Ozdaglar, and Alireza Tahbaz-Salehi. Cascades in networks and aggregate volatility. Technical report, National Bureau of Economic Research, 2010.

[4] Daron Acemoglu, Asuman Ozdaglar, and Alireza Tahbaz-Salehi. Microeconomic origins of macroeconomic tail risks. Working Paper 20865, National Bureau of Economic Research, January 2015.

[5] N. Borovykh and M. N. Spijker. Resolvent conditions and bounds on the powers of matrices, with relevance to numerical stability of initial value problems. *Journal of Computational and Applied Mathematics*, 125:41–56, December 2000.

[6] Ming Cao, Daniel A Spielman, and A Stephen Morse. A lower bound on convergence of a distributed network consensus algorithm. In *Decision and Control, 2005 and 2005 European Control Conference. CDC-ECC'05. 44th IEEE Conference on*, pages 2356–2361. IEEE, 2005.

[7] Giacomo Como and Fabio Fagnani. From local averaging to emergent global behaviors: the fundamental role of network interconnections. *CoRR*, abs/1509.08572, 2015.

[8] CM Da Fonseca and J Petronilho. Explicit inverses of some tridiagonal matrices. *Linear Algebra and its Applications*, 325(1):7–21, 2001.

[9] F. De Bruye, B. D. O. Anderson, and M. Gevers. Relating h_2 and h_∞ bounds for finite-dimensional systems. *Syst. Control Lett.*, 24(3):173–181, February 1995.

[10] Lloyd Demetrius and Thomas Manke. Robustness and network evolution?an entropic principle. *Physica A: Statistical Mechanics and its Applications*, 346(3):682–696, 2005.

[11] Geir E Dullerud and Fernando G Paganini. *A course in robust control theory: a convex approach*, volume 36. Springer Verlag, 2000.

[12] Stanley C. Eisenstat and Ilse C. F. Ipsen. Three absolute perturbation bounds for matrix eigenvalues imply relative bounds. *SIAM Journal on Matrix Analysis and Applications*, 20(1):149–158, 1998.

[13] Ivo Herman, Dan Martinec, Zdenek Hurak, and Michael Sebek. Nonzero bound on fiedler eigenvalue causes exponential growth of h-infinity norm of vehicular platoon. *CoRR*, abs/1409.4605, 2014.

[14] Q. Huang, Y. Yuan, J. Gonçalves, and M. A. Dahleh. H2 norm based network volatility measures. In *2014 American Control Conference*, pages 3310–3315, June 2014.

[15] David Asher Levin, Yuval Peres, and Elizabeth Lee Wilmer. *Markov chains and mixing times*. American Mathematical Soc., 2009.

[16] R. Olfati-Saber, J. A. Fax, and R. M. Murray. Consensus and cooperation in networked multi-agent systems. *Proceedings of the IEEE*, 95(1):215–233, Jan 2007.

[17] Alex Olshevsky and John N Tsitsiklis. Degree fluctuations and the convergence time of consensus algorithms. *Automatic Control, IEEE Transactions on*, 58(10):2626–2631, 2013.

[18] George Forrest Young, Luca Scardovi, and Naomi E Leonard. Robustness of noisy consensus dynamics with directed communication. In *American Control Conference (ACC), 2010*, pages 6312–6317. IEEE, 2010.

[19] Y. Zhao, P. Minero, and V. Gupta. On disturbance propagation in vehicle platoon control systems. In *2012 American Control Conference (ACC)*, pages 6041–6046, June 2012.