ON THE THEORY OF SMALL DEFORMATIONS OF CYLINDRICAL ELASTIC SHELLS

by

Millard W. Johnson, Jr.

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Signature of Author Signature Redacted
Department of Mathematics

Certified by Signature Redacted
Professor Eric Reissner
Thesis Supervisor

Accepted by Signature Redacted
Chairman, Departmental Committee
on Graduate Students

HRL

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Abstract

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A sequence of systems of equations for a small displacement theory of circular cylindrical shells is developed by starting with the linear three-dimensional equations of elasticity in appropriate non-dimensional form and expanding the dependent variables in terms of a small parameter. Each system can be integrated with respect to the radial coordinate and the results used together with a variational principle to obtain shell equations and appropriate boundary conditions. This is done for the first two systems.

For the axially symmetric problem conventional equations and boundary conditions are obtained as the first approximation. For the non-symmetric problem, Donnell's equations and a modification of Love's boundary conditions are obtained as the first approximation. Transverse shear and normal stress effects enter in a systematic manner into the higher order approximations.

Some explicit results are given for the axially symmetric problem. The theory is compared with other theories.

Thesis Supervisor: Eric Reissner

Title: Professor of Mathematics

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Introduction

In order to obtain equations for a theory of small deformations of elastic shells in a systematic manner, one ordinarily begins with the equations of linear elasticity in three dimensions and reduces them with the aid of appropriate assumptions to a two-dimensional system. For convenience, the two-dimensional coordinates are chosen on the middle surface of the shell. The three-dimensional coordinate system is formed by adding a coordinate giving the normal distance from the middle surface.

Thus, in order to obtain the system of equations which were first given by Love ([1], Chapter XXIV) and are known as Love's first approximation, the following assumptions are made.

(i) If ξ_1 and ξ_2 are middle surface coordinates, η the normal coordinate, we assume that the displacement components u_1, u_2 and w, corresponding to ξ_1, ξ_2, and η directions respectively, can be represented as a first approximation by

$$u_1(\xi_1, \xi_2, \eta) = U_1(\xi_1, \xi_2) + \eta U_1'(\xi_1, \xi_2)$$

$$u_2(\xi_1, \xi_2, \eta) = U_2(\xi_1, \xi_2) + \eta U_2'(\xi_1, \xi_2)$$

$$w(\xi_1, \xi_2, \eta) = W(\xi_1, \xi_2)$$

(ii) The third of the above expressions implies the assumption of vanishing normal strain, $\frac{\partial w}{\partial \eta}$.

(iii) Neglect η/R in comparision with unity where R is the radius of curvature.

(iv) Assume that the transverse shear strains vanish.

(v) Neglect the component of stress normal to the middle surface, which is small compared with the other normal components of stress, in the stress-strain relations.

A detailed description of Love's first approximation can be found in [2].

The system of shell equations derived on the basis of the above assumptions are usually regarded as satisfactory for thin shells except for special problems where transverse shear and normal effects may be important. Boundary conditions appropriate to the system of equations are given by Love ([1], page 536). They are four in number, being a statically equivalent combination of the five conditions one might expect on physical grounds. These five physically reasonable conditions can be obtained in theories which take transverse effects appropriately into account.

Another bothersome aspect of Love's first approximation is that assumptions (ii) and (v) taken together are not consistent as far as the three-dimensional equations are concerned.

Various attempts to improve on Love's first approximation by dropping some or all of the above assumptions are surveyed in reference [2]*. The theory given in [2] which is of most interest to us here is due to the authors themselves. In it the only assumption made is an extended form of (i). They add additional terms where appropriate involving higher powers of η, and by use of the principle of minimum potential energy a system of equations is obtained for the displacement functions as Euler differential equations together with appropriate boundary conditions. The five physically desirable conditions are obtained imbedded in a greater number appropriate to the order of the system**.

In reference [5] E. Reissner simplifies the above procedure through use of a general variational principle for both stresses and displacements formulated in [7]. The only assumptions made are of the form (i) for displacement components together with similar consistent assumptions for stress components. A system of equations and appropriate boundary conditions are obtained by use of the general variational principle. In [5], the theory

*For further discussion of these and other theories see [4] where also additional refernces are given.

**The reader is also referred to [3] where a systematic method is used to take transverse effects into account without the use of a variational principle. A system of equations is developed with the possibility of satisfying five boundary conditions.

is developed for the problem of axially symmetric deformations of shells of revolution while in [6] it is extended by P.M. Naghdi to the general deformation of arbitrary shells.

The above type of theory succeeds in the following ways as pointed out in [2] and [5].

(1) Physically desirable boundary conditions can be obtained.

(2) By adding terms in the expressions for the displacement and stress components involving higher powers of η one can obtain a sequence of more accurate theories.

(3) One can analyze effects associated not only with the bending boundary layer of thickness $\sqrt{Rh}$ but also with the boundary layer of thickness h, where R is the radius of curvature and h the thickness of the shell.

On the other hand, it appears to have the following drawbacks.

(1) In order to obtain more accurate theories, as explained above in (2), one must introduce higher moments of the stresses which have no physical significance in the conventional theory; and, in line with this,

(2) the systems of equations, associated with the more accurate theories, become of higher order and, hence, presumably more difficult to use.

(3) Additional boundary conditions become necessary in the more accurate theories which may not be ordinarily specified.

In this work we confine ourselves to a consideration of static small deformations of a semi-infinite circular cylindrical shell by stresses acting at the end. The method of attack is similar in some respects to that used by K.O. Friedrichs ([8],[9]) in analyzing the edge effect in the neighborhood of a free edge of a flat plate.

Friedrichs divides stresses and displacements into an interior system and an **excess** system. If ξ_1 and ξ_2 are middle surface coordinates, η the normal coordinate, and h the thickness of the plate, then with the interior system are associated variables ξ_1, ξ_2 and η/h while with the excess system variables ξ_1/h, ξ_2/h and η/h. Each system is expanded in terms of powers of h and substituted into the differential equations of linear three-dimensional elasticity, i.e. three equilibrium equations and six stress-strain relations, to obtain a succession of systems of equations, each involving quantities of higher accuracy. The interior system of lowest order corresponds to the conventional plate results satisfying Kirchhoff boundary conditions at the edge.

The interior stresses do not satisfy the exact free edge conditions of no stress. However, with the addition of the excess stresses, which vary rapidly within a boundary layer of width h at the edge, the exact free edge conditions are satisfied. Friedrichs proceeds in a step by step manner, using first interior stresses to determine proper boundary conditions for the excess stresses and then vice versa. The connection between the two systems comes through their boundary conditions, which must interact so as to satisfy the exact free edge conditions for each power of h.

Like Friedrichs we here start with no assumptions except that the displacements are small enough so that the linear equations of three dimensional elasticity can be used. These include linear equilibrium and strain displacement relations and the generalized Hooke's law. In the deformation of a cylindrical shell it is indicated by the results of conventional shell theory that due to the curvature of the middle surface there are boundary layer phenomena with a length scale $\sqrt{ah}$ in addition to phenomena with a length scale h, where a is the radius and h the thickness of the shell. The boundary layer width $\sqrt{ah}$ is associated with bending effects near the edge while the width h is associated with St. Venant effects. We associate an interior system of stresses and displacements with the length scale $\sqrt{ah}$

by non-dimensionalizing appropriately. The non-dimensional stresses and displacements are expanded in powers of $\lambda = h/2a$. When the expansions are substituted into the three-dimensional differential equations, a sequence of systems of equations is obtained. These systems can be integrated with respect to the normal coordinate to obtain expressions for displacements and stresses in form similar to assumption (i) above for each power of λ.

From here the development is similar in some respects to that in [5] and [6], use being made of E. Reissner's general variational theorem for stresses and displacements to obtain appropriate systems of equations for the displacement and stress functions together with appropriate boundary conditions. Systems of differential equations are also obtained by use of the boundary conditions on the surfaces $\eta = \pm \frac{1}{2} h$ and these are found to agree with those obtained by the variational method. The excess stress system is not here considered. Presumably, it could be used to obtain boundary conditions after the manner of Friedrichs.

The development is divided into two parts. In Chapter I we consider the axially symmetric problem in order to see where we are going before taking up in Chapter II the more difficult non-symmetric problem.

We assume a material with the normal to the middle surface a preferred elastic direction (but with elastic symmetry about the normal direction) in order to isolate in the formulae the transverse shear and normal effects.

Chapter I: Axially symmetric deformation

1. Non-dimensionalization. We consider axially symmetric deformation of a semi-infinite cylindrical shell of constant thickness. We start with the three-dimensional linearized equations for small deformations and with no body forces in cylindrical coordinates. With the usual notation, illustrated in figure 1, the equilibrium equations take the following form

$$\left.\begin{aligned} \frac{\partial}{\partial r}(r\sigma_r) + \frac{\partial}{\partial z}(r\tau_{rz}) - \sigma_\theta &= 0 \\ \frac{\partial}{\partial r}(r\tau_{rz}) + \frac{\partial}{\partial z}(r\sigma_z) &= 0 \end{aligned}\right\} \qquad (1.1)$$

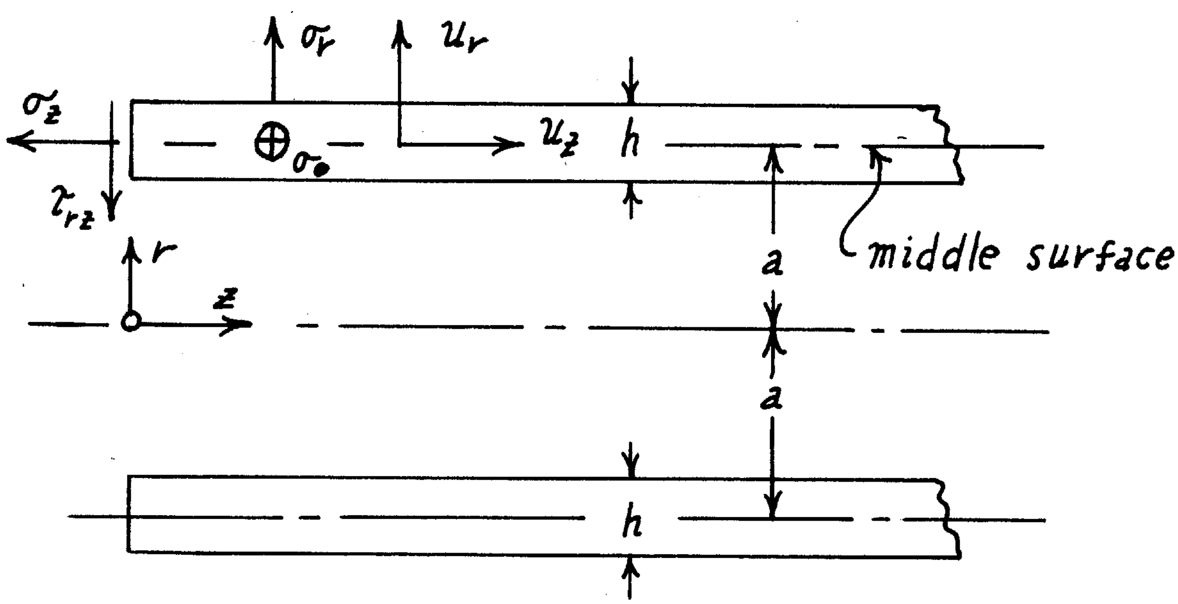

fig.1: Stress ($\sigma_z, \sigma_r, \sigma_\theta, \tau_{rz}$) and displacement ($u_r, u_z$) components and coordinates (r, z) for axially symmetric deformation.

If an orthotropic material is assumed with the direction normal to the middle surface, the r -direction, an axis of elastic symmetry, the stress-strain relations may be written as

$$\left.\begin{aligned} \frac{\partial u_r}{\partial r} &= \frac{\sigma_r}{E_t} - \frac{\nu_t}{E_t}(\sigma_z + \sigma_\theta) \\ \frac{\partial u_z}{\partial z} &= \frac{\sigma_z - \nu\sigma_\theta}{E} - \frac{\nu_t}{E_t}\sigma_r \\ \frac{u_r}{r} &= \frac{\sigma_\theta - \nu\sigma_z}{E} - \frac{\nu_t}{E_t}\sigma_r \\ \frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z} &= \frac{1}{G_t}\tau_{rz} \end{aligned}\right\} \qquad (1.2)$$

We assume that the deformation is due to a stress system applied at the end of the shell which we place at $z = 0$. Boundary conditions will be discussed later.

We introduce dimensionless coordinates ρ and ζ by

$$\rho = \frac{r-a}{\tfrac{1}{2}h}, \qquad \zeta = \frac{z}{b} \qquad (1.3)$$

and dimensionless parameters λ and μ by

$$\lambda = h/2a, \quad \mu = b/a \tag{1.4}$$

Dimensions a and h are the radius and thickness respectively (fig. 1) while b is a typical length in the z-direction to be determined presently. In fact, the proper determination of b is the crucial step of the non-dimensionalization. With (1.3) and (1.4) the equilibrium equations (1.1) become

$$\left.\begin{aligned} &\frac{1}{\lambda}\frac{\partial}{\partial\rho}\left[(1+\lambda\rho)\sigma_r\right] + \frac{1}{\mu}\frac{\partial}{\partial\zeta}\left[(1+\lambda\rho)\tau_{rz}\right] - \sigma_\theta = 0 \\ &\frac{1}{\lambda}\frac{\partial}{\partial\rho}\left[(1+\lambda\rho)\tau_{rz}\right] + \frac{1}{\mu}\frac{\partial}{\partial\zeta}\left[(1+\lambda\rho)\sigma_z\right] = 0 \end{aligned}\right\} \tag{1.5}$$

Of interest here is the solution away from the St. Venant boundary layer of width h at the edge of the shell. We shall call this the interior solution. In order to obtain equations appropriate for the interior solution, we non-dimensionalize so that all derivative terms in (1.5) are of the same order in λ and so that the curvature term σ_θ is also of the same order by setting

$$\left.\begin{aligned} \sigma_z &= \sigma S_z \,, & \sigma_\theta &= \sigma S_\theta \\ \sigma_r &= \lambda \sigma S_r \,, & \tau_{rz} &= \sqrt{\lambda}\, \sigma S_{rz} \end{aligned}\right\} \tag{1.6}$$

and by relating the two parameters (1.4) as follows.

$$\mu = \sqrt{\lambda} \tag{1.7}$$

We note that the stresses (1.6) agree insofar as the order in λ is concerned with the results of conventional shell theory. Introduction of (1.6) and (1.7) into (1.5) gives dimensionless equilibrium equations of the following form.

$$\left.\begin{aligned} \frac{\partial}{\partial \rho}\left[(1+\lambda\rho)S_r\right] + \frac{\partial}{\partial \xi}\left[(1+\lambda\rho)S_{rz}\right] - S_\theta &= 0 \\ \frac{\partial}{\partial \rho}\left[(1+\lambda\rho)S_{rz}\right] + \frac{\partial}{\partial \xi}\left[(1+\lambda\rho)S_z\right] &= 0 \end{aligned}\right\} \tag{1.8}$$

The quantity σ, which has the dimensions of stress, is to be determined from the boundary conditions so that the non-dimensional variables are $O(1)$. This will be done when boundary conditions are taken up.

Relation (1.7) determines the length scale in the z-direction as $b = \sqrt{\frac{1}{2} ah}$. This is the length over which significant changes of the interior stresses occur when the coordinate ξ undergoes a unit change. Thus, the interior solution contains the possibility of a boundary layer of thickness of order $\sqrt{\frac{1}{2} ah}$ near the end $\xi = 0$ and can give results corresponding to the conventional bending theory. However, we can not expect the interior solution to be capable of representing effects associated with a length scale h in the z-direction. In particular, we can not, in general, expect the interior solution to represent accurately the true stress distribution in the St. Venant boundary layer near the end of the shell.

Let us next non-dimensionalize the stress-strain relations (1.2). In order that the displacements agree in order of λ with conventional results, we set

$$\left.\begin{aligned} u_r &= (1-\nu^2)\,\sigma \frac{a}{E}\; V_r \\ u_z &= \sqrt{\lambda}\,(1-\nu^2)\,\sigma \frac{a}{E}\; V_z \end{aligned}\right\} \qquad (1.9)$$

which results in the following dimensionless stress-strain relations.

$$\left.\begin{aligned}
(1-\nu^2)\frac{\partial V_r}{\partial \rho} &= \lambda \frac{E}{E_t}\left[-\nu_t\,(S_z + S_\theta) + \lambda\, S_r\right] \\
(1-\nu^2)\frac{\partial V_z}{\partial \zeta} &= S_z - \nu S_\theta - \lambda\, \nu_n\, S_r \\
(1-\nu^2)\frac{V_r}{1+\lambda\rho} &= S_\theta - \nu S_z - \lambda\, \nu_n\, S_r \\
(1-\nu^2)\left[\frac{\partial V_z}{\partial \rho} + \frac{\partial V_r}{\partial \zeta}\right] &= 2\lambda\,(1+\nu)\, I_n\, S_{rz}
\end{aligned}\right\} \qquad (1.10)$$

In (1.10) we have set

$$\nu_n = \frac{\nu_t E}{E_t}, \qquad I_n = \frac{E}{2\,(1+\nu)\,G_t} \qquad (1.11)$$

Of particular interest are the following materials.

$I_n = 0$	no transverse shear strain
$\nu_n = E/E_t = 0$	no transverse normal strain
$I_n = E/E_t = 1,\ \nu_n = \nu$	isotropy

The factor $(1-\nu^2)$ is added in (1.9) in order to simplify some succeeding formulae.

We next arrange equations (1.8) and (1.10) in the order in which integration with respect to ρ will be possible.

$$\left.\begin{aligned}
\frac{\partial V_r}{\partial \rho} &= \frac{1}{1-\nu^2}\, \lambda\, \frac{E}{E_t}\left[-\nu_t\,(S_z + S_\theta) + \lambda\, S_r\right] \\
\frac{\partial V_z}{\partial \rho} &= -\frac{\partial V_r}{\partial \varsigma} + \lambda\, \frac{2\, I_n}{1-\nu}\; S_{rz} \\
S_\theta &= \frac{V_r}{1+\lambda\rho} + \nu\, \frac{\partial V_z}{\partial \varsigma} + \lambda\, \frac{\nu_n}{1-\nu}\, S_r \\
S_z &= \nu\, \frac{V_r}{1+\lambda\rho} + \frac{\partial V_z}{\partial \varsigma} + \lambda\, \frac{\nu_n}{1-\nu}\, S_r \\
\frac{\partial}{\partial \rho}\left[(1+\lambda\rho)\, S_{rz}\right] &= -\frac{\partial}{\partial \varsigma}\left[(1+\lambda\rho)\, S_z\right] \\
\frac{\partial}{\partial \rho}\left[(1+\lambda\rho)\, S_r\right] &= -\frac{\partial}{\partial \varsigma}\left[(1+\lambda\rho)\, S_{rz}\right] + S_\theta
\end{aligned}\right\} \quad (1.12)$$

2. <u>Expansion with respect to small parameter.</u> We assume that for sufficiently small λ all quantities in (1.12) can be expanded at each point (ρ, ς) in a power series in λ with

coefficients independent of λ.*

$$\left.\begin{aligned} S &= S_0(\rho,\varphi) + \lambda S_1(\rho,\varphi) + \ldots \\ V &= V_0(\rho,\varphi) + \lambda V_1(\rho,\varphi) + \ldots \end{aligned}\right\} \qquad (2.1)$$

We are interested in the solution for small λ and will study in detail only the first two systems obtained by substituting (2.1) into (1.12) and equating coefficients of λ^0 and λ^1 on both sides of the equations.

*Expansions (2.1) are not expected to be convergent. However, we expect that they are asymptotic in character. That is, if they are terminated at some power of λ, we expect them to approach the three-dimensional results as $\lambda \to 0$.

First system

$$
\left.
\begin{aligned}
\frac{\partial V_{r0}}{\partial \rho} &= 0 \\
\frac{\partial V_{z0}}{\partial \rho} &= -\frac{\partial V_{r0}}{\partial \zeta} \\
S_{\theta 0} &= V_{r0} + \nu \frac{\partial V_{z0}}{\partial \zeta} \\
S_{z0} &= \nu V_{r0} + \frac{\partial V_{z0}}{\partial \zeta} \\
\frac{\partial S_{rz0}}{\partial \rho} &= -\frac{\partial S_{z0}}{\partial \zeta} \\
\frac{\partial S_{r0}}{\partial \rho} &= -\frac{\partial S_{rz0}}{\partial \zeta} + S_{\theta 0}
\end{aligned}
\right\} \qquad (2.2)
$$

Second system

$$\left.\begin{aligned}
\frac{\partial V_{r1}}{\partial \rho} &= -\frac{\nu n}{1-\nu^2}\,(S_{z0} + S_{\theta 0}) \\
\frac{\partial V_{z1}}{\partial \rho} &= -\frac{\partial V_{r1}}{\partial \varsigma} + \frac{2 I n}{1-\nu}\, S_{rz0} \\
S_{\theta 1} &= V_{r1} - V_{r0}\,\rho + \nu \frac{\partial V_{z1}}{\partial \varsigma} + \frac{\nu n}{1-\nu}\, S_{r0} \\
S_{z1} &= \nu V_{r1} - \nu V_{r0}\,\rho + \frac{\partial V_{z1}}{\partial \varsigma} + \frac{\nu n}{1-\nu}\, S_{r0} \\
\frac{\partial}{\partial \rho}\left(S_{rz1} + \rho\, S_{rz0}\right) &= -\frac{\partial}{\partial \varsigma}\left(S_{z1} + \rho\, S_{z0}\right) \\
\frac{\partial}{\partial \rho}\left(S_{r1} + \rho\, S_{r0}\right) &= -\frac{\partial}{\partial \varsigma}\left(S_{rz1} + \rho\, S_{rz0}\right) + S_{\theta 1}
\end{aligned}\right\} \qquad (2.3)$$

Systems (2.2) and (2.3) can now be integrated with respect to ρ in a step by step fashion since the right hand sides of each equation are known functions of ρ at each step*. Introducing macroscopic displacement and stress functions $V_{ro}(\xi)$, etc., system (2.2) can be integrated to obtain

$$\left.\begin{aligned} V_{ro} &= V_{ro}(\xi) \\ V_{zo} &= V_{zo}(\xi) - V_{ro}'\rho \\ S_{zo} &= \nu V_{ro} + V_{zo}' - V_{ro}''\rho \\ S_{\theta o} &= V_{ro} + \nu V_{zo}' - \nu V_{ro}''\rho \end{aligned}\right\} \quad (2.4)$$

$$\left.\begin{aligned} S_{rzo} &= S_{rzo}(\xi) - (\nu V_{ro}' + V_{zo}'')\rho + \tfrac{1}{2} V_{ro}'''\rho^2 \\ S_{ro} &= S_{ro}(\xi) + (V_{ro} + \nu V_{zo}' - S_{rzo}')\rho \\ &\quad + \tfrac{1}{2} V_{zo}'''\rho^2 - \tfrac{1}{6} V_{ro}^{IV}\rho^3 \end{aligned}\right\} \quad (2.5)$$

*It is clear from the form of (1.12) that this integration can also be carried out in all higher order systems.

where primes indicate differentiation with respect to ξ . Expressions (2.4), when the proper equations and boundary conditions are given for the macroscopic displacements V_{ro} and V_{zo} , are those obtained in the usual theory of thin shells. Expressions (2.5) are not obtained in the usual theory which ignores σ_r and τ_{rz} .

Next, integration of (2.3) gives the following expressions.

$$\left.\begin{aligned}
V_{r1} &= V_{r1}(\xi) - \frac{\nu_n}{1-\nu}\left(V_{ro} + V_{zo}'\right)\rho + \frac{\nu_n}{2(1-\nu)} V_{ro}''\rho^2 \\
V_{z1} &= V_{z1}(\xi) - V_{r1}'\rho + \frac{2I_n}{1-\nu} S_{rzo}\rho \\
&\quad + \left[\frac{\nu_n - 2\nu I_n}{2(1-\nu)} V_{ro}' + \frac{\nu_n - 2I_n}{2(1-\nu)} V_{zo}''\right]\rho^2 + \frac{2I_n - \nu_n}{6(1-\nu)} V_{ro}'''\rho^3
\end{aligned}\right\} \quad (2.6)$$

$$
\left.\begin{aligned}
S_{z1} &= \nu V_{r1} + V'_{z1} - V''_{r1}\rho + \frac{\nu_n}{1-\nu} S_{r0} + (\nu_n - \nu) V_{r0}\rho \\
&\quad + \frac{2I_n - \nu_n}{1-\nu} S'_{rz0}\rho + \left[\frac{\nu_n(1+\nu) - 2\nu I_n}{2(1-\nu)} V''_{r0} - \frac{I_n - \nu_n}{1-\nu} V'''_{z0}\right]\rho^2 \\
&\quad + \frac{I_n - \nu_n}{3(1-\nu)} V^{IV}_{r0}\rho^3 \\[2ex]
S_{\theta 1} &= V_{r1} + \nu V'_{z1} - \nu V''_{r1}\rho + \frac{\nu_n}{1-\nu} S_{r0} + \Big[-\nu_n V'_{z0} - V_{r0} \\
&\quad + \frac{2\nu I_n - \nu_n}{1-\nu} S'_{rz0}\Big]\rho + \left[\frac{(1+\nu)\nu_n - 2\nu^2 I_n}{2(1-\nu)} V''_{r0} + \frac{(1+\nu)\nu_n - 2\nu I_n}{2(1-\nu)} V'''_{z0}\right]\rho^2 \\
&\quad + \frac{-(1+\nu)\nu_n + 2\nu I_n}{6(1-\nu)} V^{IV}_{r0}\rho^3 \\[2ex]
S_{rz1} &= S_{rz1}(\xi) - \left[\nu V'_{r1} + V''_{z1}\right]\rho + \frac{1}{2} V'''_{r1}\rho^2 - \left[S_{rz0} + \frac{\nu_n}{1-\nu} S'_{r0}\right]\rho \\
&\quad + \left[(2\nu - \nu_n) V'_{r0} + V''_{z0} + \frac{\nu_n - 2I_n}{1-\nu} S''_{rz0}\right]\frac{1}{2}\rho^2 \\
&\quad + \left[\frac{I_n - \nu_n}{3(1-\nu)} V^{IV}_{z0} - \frac{1}{6}\left(1 + \frac{(1+\nu)\nu_n - 2\nu I_n}{1-\nu}\right) V'''_{r0}\right]\rho^3 - \frac{I_n - \nu_n}{12(1-\nu)} V^{V}_{r0}\rho^4
\end{aligned}\right\} \quad (2.6)
$$

$$
\begin{aligned}
s_{r1} = {} & S_{r1}(\xi) + \left[V_{r1} + \nu V_{z1}' - S_{rz1}' + \left(-1 + \frac{\nu_n}{1-\nu}\right) S_{ro} \right] \rho \\
& + \left[V_{z1}''' - 3 V_{ro} - (2\nu + \nu_n) V_{zo}' + \left(2 + \frac{2\nu I_n - \nu_n}{1-\nu}\right) S_{rzo}' \right. \\
& \left. + \frac{\nu_n}{1-\nu} S_{ro}'' \right] \frac{1}{2} \rho^2 + \frac{1}{6} \left[-V_{r1}^{IV} + \left(-2 + \frac{(1+\nu)\nu_n - 2\nu I_n}{1-\nu}\right) V_{zo}''' \right. \\
& \left. + \frac{2\nu_n - 2\nu^2 I_n}{1-\nu} V_{ro}'' + \frac{2 I_n - \nu_n}{1-\nu} S_{rzo}''' \right] \rho^3 \\
& + \frac{1}{12} \left[V_{ro}^{IV} - \frac{I_n - \nu_n}{1-\nu} V_{zo}^{V} \right] \rho^4 + \frac{I_n - \nu_n}{60(1-\nu)} V_{ro}^{VI} \rho^5 \qquad (2.6)
\end{aligned}
$$

We next wish to obtain differential equations and boundary conditions for the macroscopic quantities V_{ro}, V_{zo}, etc. When these quantities have been obtained, (2.4), (2.5) and (2.6) enable us to calculate the microscopic stresses s and displacements v of the first two approximations. The differential equations can be obtained most easily as necessary conditions for the satisfaction of boundary conditions at $\rho = \pm 1$ and are discussed next. They are verified later by a variational procedure.

3. Equations for macroscopic quantities. Let us assume that the surfaces $r = a \pm h/2$ are stress free.*

$$r = a \pm h/2 : \quad \sigma_r = \tau_{rz} = 0 \tag{3.1}$$

In terms of dimensionless stresses (3.1) becomes

$$\rho = \pm 1 : \quad S_r = S_{rz} = 0 \tag{3.2}$$

Considering the expansion (2.1) in powers of λ, we replace (3.2) by the following conditions.

$$\rho = \pm 1 : \quad S_{rk} = S_{rzk} = 0, \quad k \geqslant 0 \tag{3.3}$$

In order that the stresses given by (2.5) and (2.6) satisfy (3.3) for $k = 0,1$, it is necessary (and sufficient) that the following conditions hold.

---

* We can consider infinitely differentiable surface stresses on $\rho = \pm 1$ within the framework of the present theory, but do not because of the increase in complexity of the resulting equations and because nothing is added concerning the nature of the boundary layer which is here of primary interest. The effects of surface stress can be accounted for by a Fourier integral solution. The residual end stresses can then be removed by superposition of a solution like the present with vanishing surface stresses. This method remains applicable even when the surface stresses are not infinitely differentiable.

First approximation

$$V_{z0}'' + \nu V_{r0}' = 0 \tag{3.4}$$

$$\frac{1}{3} V_{r0}^{IV} + V_{r0} + \nu V_{z0}' = 0 \tag{3.5}$$

$$S_{rz0} = -\frac{1}{2} V_{r0}''' \tag{3.6}$$

$$S_{r0} = -\frac{1}{2} V_{z0}''' \tag{3.7}$$

Second approximation

$$V_{z1}'' + \nu V_{r1}' + S_{rz0} + \frac{\nu_n}{1-\nu} S_{r0}' - \frac{I_n - \nu_n}{3(1-\nu)} V_{z0}^{IV}$$

$$+ \frac{1}{6}\left[1 + \frac{(1+\nu)\nu_n - 2\nu I_n}{1-\nu}\right] V_{r0}''' = 0 \tag{3.8}$$

$$V_{r1} + \nu V_{z1}' - S_{rz1}' - \frac{1}{6} V_{r1}^{IV} + \left(-1 + \frac{\nu_n}{1-\nu}\right) S_{r0}$$

$$+ \frac{I_n - \nu_n}{60(1-\nu)} V_{r0}^{VI} + \left[-\frac{1}{3} + \frac{(1+\nu)\nu_n - 2\nu I_n}{6(1-\nu)}\right] V_{z0}'''$$

$$+ \frac{\nu_n - \nu^2 I_n}{3(1-\nu)} V_{r0}'' + \frac{2I_n - \nu_n}{6(1-\nu)} S_{rz0}''' = 0 \tag{3.9}$$

$$S_{rz1} = -\frac{1}{2} V_{r1}''' + \left(\frac{1}{2} \nu_n - \nu\right) V_{r0}' - \frac{1}{2} V_{z0}''$$

$$+ \frac{2I_n - \nu_n}{2(1-\nu)} S_{rz0}'' + \frac{I_n - \nu_n}{12(1-\nu)} V_{r0}^{V} \tag{3.10}$$

$$S_{r1} = -\frac{1}{2} V_{z1}''' + \frac{3}{2} V_{r0} + \left(\nu + \frac{1}{2}\nu_n\right) V_{z0}' - \frac{\nu_n}{2(1-\nu)} S_{r0}''$$

$$-\left(1 + \frac{2\nu I_n - \nu_n}{2(1-\nu)}\right) S_{rz0}' - \frac{1}{12} V_{r0}^{IV} + \frac{I_n - \nu_n}{12(1-\nu)} V_{z0}^{V} \tag{3.11}$$

The above relations for the second approximation can be simplified by using the relations for the first approximation appropriately.

$$V_{z1}'' + \nu V_{r1}' = \left[\frac{1}{3} - \frac{\nu_n(1+2\nu)}{6(1-\nu)}\right] V_{r0}''' \tag{3.12}$$

$$\frac{1}{3} V_{r1}^{IV} + V_{r1} + \nu V_{z1}' = \left[-\frac{1}{3}\nu + \frac{4}{5}(1+\nu) I_n - \frac{\nu_n(4+25\nu-14\nu^2)}{30(1-\nu)}\right] V_{r0}'' \tag{3.13}$$

$$S_{rz1} = -\frac{1}{2} V_{r1}''' + \left[-\frac{1}{2}\nu - \frac{1}{2}\nu\nu_n + \frac{5}{4}(1+\nu) I_n\right] V_{r0}' \tag{3.14}$$

$$S_{r1} = -\frac{1}{2} V_{z1}''' + \frac{1}{4}\left[1 + \frac{(2\nu+3)\nu_n - 5\nu I_n}{1-\nu}\right] V_{r0}$$

$$+ \frac{1}{4}\left[-\nu + \frac{(2+\nu+2\nu^2)\nu_n - 5\nu^2 I_n}{1-\nu}\right] V_{z0}' \tag{3.15}$$

For most purposes expressions (3.12) through (3.15) would seem the most convenient form. It is possible to simplify expressions (2.5) and (2.6) in a similar manner but this will not be done here.

Expressions (3.4) and (3.5) are two ordinary differential equations with constant coefficients for V_{zo} and V_{ro}. Their solution can be found by elementary means (see section 8). In fact, these equations are equivalent to the conventional ones for symmetrical deformation of thin cylindrical shells. After solving these equations subject to suitable boundary conditions at $\xi = 0$ and $\xi = \infty$, we can find S_{rzo} and S_{ro} by (3.6) and (3.7), and then the microscopic displacements and stresses are given by (2.4) and (2.5).

Once the first approximation is known, we can calculate the right side of (3.12) and (3.13). The left sides are of the same form as (3.4) and (3.5). Therefore, these equations can be solved (for instance, by means of the method of variation of parameters) and the microscopic quantities of the second approximation determined by (2.6) with S_{rz1} and S_{r1} given by (3.14) and (3.15).

It is apparent from the nature of perturbation type solutions that at the k+1st approximation one must solve the system

$$\left.\begin{aligned} V_{zk}'' + \nu V_{rk}' &= G_k \\ \tfrac{1}{3} V_{rk}^{IV} + V_{rk} + \nu V_{zk}' &= H_k \end{aligned}\right\} \qquad (3.16)$$

where G_k and H_k are known from previous approximations. Thus, one can procede, in theory anyway, to obtain any desired degree of approximation, since we know that the solution of (3.16) can be obtained by elementary means.

We must still formulate suitable boundary conditions at $\zeta = 0$ and $\zeta = \infty$. Even though these conditions are fairly apparent for the axially symmetric problem, we choose to obtain them in a rigorous manner through use of a variational principle. This method is needed for the problem of non-symmetric deformation and we wish to see how it goes in a **simpler** problem as a guide. Also, it will at the same time reveal a few other interesting results. Before going to the variational principle we discuss the nature of the boundary conditions.

4. <u>End conditions</u>. We assume that the following boundary conditions are given at $z = 0$ and $z = \infty$,

$$z = 0 : \qquad \sigma_z = \overline{\sigma_z}(r) , \quad \tau_{rz} = \overline{\tau}_{rz}(r) \tag{4.1}$$

$$z = \infty : \qquad \sigma_z = \tau_{rz} = 0 \tag{4.2}$$

where $\overline{\sigma_z}$ and $\overline{\tau}_{rz}$ are given functions. We non-dimensionalize by setting

$$\overline{\sigma}_z = \sigma \overline{S}_z , \qquad \overline{\tau}_{rz} = \sqrt{\lambda}\, \sigma \overline{S}_{rz} \tag{4.3}$$

and by using (1.6). Then, non-dimensional relations corresponding to (4.1) and (4.2) are

$$\xi = 0: \qquad S_z = \bar{S}_z(\rho), \quad S_{rz} = \bar{S}_{rz}(\rho) \tag{4.4}$$

$$\xi = \infty: \qquad S_z = S_{rz} = 0 \tag{4.5}$$

Within the framework of the present theory we can replace $\bar{s}_z$ and $\bar{s}_{rz}$ in (4.4) by any statically equivalent system of loading as this will have the same effect upon the interior solution outside of the St. Venant boundary layer. This will be done in a way which seems natural and at the same time considerably simplifies the variational procedure to follow.

We first discuss the conventional stress resultants and couple associated with the z-direction which are defined as follows.

$$\left.\begin{aligned} N_z &= \int_{a-h/2}^{a+h/2} \sigma_z \frac{r}{a}\, dr \\ Q_z &= \int_{a-h/2}^{a+h/2} \tau_{rz} \frac{r}{a}\, dr \\ M_z &= \int_{a-h/2}^{a+h/2} \sigma_z \frac{r}{a}(r-a)\, dr \end{aligned}\right\} \tag{4.6}$$

Similar quantities, N_θ and M_θ, associated with the θ-direction are also defined but these are of no interest to us here. Expressions (4.6) are non-dimensionalized using (1.3) and (1.6) to obtain

$$\left.\begin{aligned} N_z &= \sigma a \lambda \left[N_{z0} + \lambda N_{z1} + \dots \right] \\ Q_z &= \sigma a \lambda^{3/2} \left[Q_{z0} + \lambda Q_{z1} + \dots \right] \\ M_z &= \sigma a^2 \lambda^2 \left[M_{z0} + \lambda M_{z1} + \dots \right] \end{aligned}\right\} \qquad (4.7)$$

where

$$\left.\begin{aligned} N_{z0} &= \int_{-1}^{1} s_{z0}\, d\rho, \quad N_{z1} = \int_{-1}^{1} (s_{z1} + \rho s_{z0})\, d\rho, \dots \\ Q_{z0} &= \int_{-1}^{1} s_{rz0}\, d\rho, \quad Q_{z1} = \int_{-1}^{1} (s_{rz1} + \rho s_{rz0})\, d\rho, \dots \\ M_{z0} &= \int_{-1}^{1} s_{z0} \rho\, d\rho, \quad M_{z1} = \int_{-1}^{1} (s_{z1} + \rho s_{z0}) \rho\, d\rho, \dots \end{aligned}\right\} \qquad (4.8)$$

Applied stress resultants and end moment are given by

$$\left.\begin{aligned}
\overline{N}_z &= \int_{a-h/2}^{a+h/2} \overline{\sigma}_z \frac{r}{a}\, dr = \sigma a \lambda \int_{-1}^{1} \overline{s}_z (1+\lambda\rho)\, d\rho \\
\overline{Q}_z &= \int_{a-h/2}^{a+h/2} \overline{\tau}_{rz} \frac{r}{a}\, dr = \sigma a \lambda^{3/2} \int_{-1}^{1} \overline{s}_{rz} (1+\lambda\rho)\, d\rho \\
\overline{M}_z &= \int_{a-h/2}^{a+h/2} \overline{\sigma}_z \frac{r}{a} (r-a)\, dr = \sigma a^2 \lambda^2 \int_{-1}^{1} \overline{s}_z (1+\lambda\rho)\rho\, d\rho
\end{aligned}\right\} \qquad (4.9)$$

Let us replace end conditions (4.4) by

$$\xi = 0: \quad s_z = \hat{s}_z(\rho), \quad s_{rz} = \hat{s}_{rz}(\rho) \qquad (4.10)$$

where

$$\left.\begin{aligned}\hat{s}_z &= \frac{1}{2}\,\frac{1}{1+\lambda\rho}\,\frac{\overline{N}_z}{\sigma a\lambda} + \frac{3}{2}\,\frac{\rho}{1+\lambda\rho}\,\frac{\overline{M}_z}{\sigma a^2\lambda^2}\\ \hat{s}_{rz} &= \frac{3}{4}\,\frac{1-\rho^2}{1+\lambda\rho}\,\frac{\overline{Q}_z}{\sigma a\lambda^{3/2}}\end{aligned}\right\} \tag{4.11}$$

Substitution of $\hat{s}_z$ and $\hat{s}_{rz}$ from (4.11) for $\bar{s}_z$ and $\bar{s}_{rz}$ in (4.9) shows that these two sets of end loadings are indeed statically equivalent.

We are now in a position to specify σ. With end stresses at $\xi = 0$ given by (4.11) and at $\xi = \infty$ by (4.5), any solution can be considered as the result of superposing solutions to three problems with conditions (4.5) at $\xi = \infty$ and with quantities in (4.9) taken as follows.

$$\left.\begin{aligned}&\text{(i) } \overline{N}_z \text{ given}, \quad \overline{M}_z = \overline{Q}_z = 0\\ &\text{(ii) } \overline{M}_z \text{ given}, \quad \overline{N}_z = \overline{Q}_z = 0\\ &\text{(iii) } \overline{Q}_z \text{ given}, \quad \overline{N}_z = \overline{M}_z = 0\end{aligned}\right\} \tag{4.12}$$

In order that $\hat{s}_z$ and $\hat{s}_{rz}$ be $O(1)$ we specify σ in the following manner.

$$\text{(i)}\ \sigma = \frac{\overline{N_z}}{a\lambda}\ , \quad \text{(ii)}\ \sigma = \frac{\overline{M_z}}{a^2\lambda^2}\ , \quad \text{(iii)}\ \sigma = \frac{\overline{Q_z}}{a\lambda^{3/2}} \tag{4.13}$$

With (4.13) we can write (4.11) as

$$\left.\begin{aligned} \hat{S}_z &= \frac{1}{2}\frac{1}{1+\lambda\rho}\ \overline{n}_z + \frac{3}{2}\frac{\rho}{1+\lambda\rho}\ \overline{m}_z \\ \hat{S}_{rz} &= \frac{3}{4}\frac{1-\rho^2}{1+\lambda\rho}\ \overline{q}_z \end{aligned}\right\} \tag{4.14}$$

where

$$\left.\begin{gathered} \text{(i)}\quad \overline{n}_z = 1\ ,\ \ \overline{m}_z = \overline{q}_z = 0\ ,\ \text{(ii)}\ \overline{m}_z = 1\ ,\ \overline{n}_z = \overline{q}_z = 0 \\ \text{(iii)}\ \overline{q}_z = 1\ ,\ \ \overline{n}_z = \overline{m}_z = 0 \end{gathered}\right\} \tag{4.15}$$

Let us assume the surfaces $\rho = \pm 1$ stress free and the end conditions at $\xi = \infty$ given by (4.5). For statical equilibrium of the entire shell we set $\overline{Nz} = 0$ and replace (4.14) by

$$\hat{S}_z = \frac{3}{2}\frac{\rho}{1+\lambda\rho}\ \overline{m}_z\ , \quad \hat{S}_{rz} = \frac{3}{4}\frac{1-\rho^2}{1+\lambda\rho}\ \overline{q}_z \tag{4.16}$$

where $\overline{m_z}$ and $\overline{q_z}$ are still given by (4.15).

In conventional shell theory microscopic end conditions (4.1) and (4.2) are replaced by

$$\left.\begin{aligned} \zeta = 0: \quad & N_z = 0, \; M_z = \overline{M}_z, \; Q_z = \overline{Q}_z \\ \zeta = \infty: \quad & N_z = M_z = Q_z = 0 \end{aligned}\right\} \qquad (4.17)$$

where $\overline{M_z}$ and $\overline{Q_z}$ are given by (4.9) and where there is no surface loading. In terms of quantities defined by (4.7), (4.8) and (4.15), (4.17) can be replaced by the following conditions on each approximation.

$$\left.\begin{aligned} \zeta = 0: \quad & N_{z0} = 0, \; M_{z0} = \overline{m}_z, \; Q_{z0} = \overline{q}_z \\ & N_{zk} = M_{zk} = Q_{zk} = 0, \; k \geq 1 \\ \zeta = \infty: \quad & N_{zk} = M_{zk} = Q_{zk} = 0, \; k \geq 0 \end{aligned}\right\} \qquad (4.18)$$

Let us now see how conditions (4.18) can be obtained from assumptions (4.5), (4.10), and (4.16) by the variational approach.

5. Application of variational principle. We make use of a variational theorem proven in reference [7]. For the present problem the expression to be varied becomes

$$I = \int_0^\infty \int_{a-h/2}^{a+h/2} \Big\{ \frac{\partial u_r}{\partial r}\sigma_r + \frac{1}{r}u_r\sigma_\theta + \frac{\partial u_z}{\partial z}\sigma_z$$

$$+\Big[\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\Big]\tau_{rz} - \Big[\frac{1}{2E_t}\sigma_r^2 + \frac{1}{2E}(\sigma_\theta^2 + \sigma_z^2)$$

$$-\frac{\nu}{E}\sigma_\theta\sigma_z - \frac{\nu_t}{E_t}\sigma_r(\sigma_\theta + \sigma_z) + \frac{1}{2G_t}\tau_{rz}^2\Big]\Big\} r\,dr\,dz$$

$$+\int_{a-h/2}^{a+h/2}\Big[\hat{\sigma}_z u_z + \hat{\tau}_{rz}u_r\Big]^{z=0} r\,dr \tag{5.1}$$

It can be shown by the usual methods of the variational calculus that for (5.1) to take on a stationary value it is necessary for equations (1.1) and (1.2) to be satisfied. In addition, if all variations are considered to be arbitrary on the boundaries, the following natural boundary conditions are necessarily satisfied.

$$\left.\begin{aligned} r = a \pm h/2 &: \quad \sigma_r = \tau_{rz} = 0 \\ z = 0 &: \quad \sigma_z = \hat{\sigma}_z(r), \quad \tau_{rz} = \hat{\tau}_{rz}(r) \\ z = \infty &: \quad \sigma_z = \tau_{rz} = 0 \end{aligned}\right\} \tag{5.2}$$

Note that the equation $\delta I = 0$ is equivalent to the complete system of differential equations and boundary conditions of the given problem.

We wish to see, by use of the variational method, in what sense the interior problem, in the form of a power series in λ, corresponds to the three dimensional problem given by (1.1), (1.2) and (5.2). First, let us non-dimensionalize (5.1) by making use of (1.3), (1.6) and (4.3). We further assume dimensionless edge stresses in the form (4.16). The result is

$$
\begin{aligned}
I = \lambda^{3/2} \frac{(1-\nu^2)\sigma^{-2}a^3}{E} \Bigg[\int_0^\infty \int_{-1}^1 \Bigg\{ & \frac{\partial v_r}{\partial \rho} s_r + \frac{v_r}{1+\lambda\rho} s_\theta + \frac{\partial v_z}{\partial \xi} s_z \\
& + \left[\frac{\partial v_z}{\partial \rho} + \frac{\partial v_r}{\partial \xi} \right] s_{rz} - \frac{1}{1-\nu^2} \Big[\frac{1}{2} \left(s_\theta^2 + s_z^2 \right) - \nu s_\theta s_z \\
& - \lambda \nu_n s_r (s_\theta + s_z) + \lambda (1+\nu) I_n s_{rz}^2 + \lambda^2 \frac{E}{E_t} s_r^2 \Big] \Bigg\} (1+\lambda\rho)\, d\rho\, d\xi \\
& + \int_{-1}^1 \left[\frac{3}{2} \rho\, \overline{m}_z v_z + \frac{3}{4} (1-\rho^2)\, \overline{q}_z v_r \right]^{\xi = 0} d\rho \Bigg]
\end{aligned}
\tag{5.3}
$$

Next, expand s and v in powers of λ as in (2.1) and write I as a power series in λ.

$$I = \lambda^{3/2} \frac{(1-\nu^2)\sigma^2 a^3}{E}\left[I_o + \lambda I_1 + \dots \right] \tag{5.4}$$

Expansion (5.4) must probably be considered valid in an asymptotic sense for small λ . The quantities I_o and I_1 are given by

$$\begin{aligned} I_o = \int_0^\infty \int_{-1}^1 \Big\{ & \frac{\partial V_{ro}}{\partial \rho} S_{ro} + V_{ro} S_{\theta o} + \frac{\partial V_{zo}}{\partial \varsigma} S_{zo} \\ & + \left[\frac{\partial V_{zo}}{\partial \rho} + \frac{\partial V_{ro}}{\partial \varsigma} \right] S_{rzo} - \frac{1}{1-\nu^2} \left[\frac{1}{2}(S_{\theta o}^2 + S_{zo}^2) - \nu S_{\theta o} S_{zo} \right] \Big\} d\rho\, d\varsigma \\ & + \int_{-1}^1 \left[\frac{3}{2} \overline{m}_z \rho V_{zo} + \frac{3}{4} \overline{q}_z (1-\rho^2) V_{ro} \right]^{\varsigma=0} d\rho \end{aligned} \tag{5.5}$$

$$\begin{aligned} I_1 = \int_0^\infty \int_{-1}^1 \Big\{ & \frac{\partial V_{ro}}{\partial \rho} (S_{r1} + \rho S_{ro}) + \frac{\partial V_{r1}}{\partial \rho} S_{ro} + V_{ro} S_{\theta 1} + V_{r1} S_{\theta o} \\ & + \frac{\partial V_{zo}}{\partial \varsigma} (S_{z1} + \rho S_{zo}) + \frac{\partial V_{z1}}{\partial \varsigma} S_{zo} + \left(\frac{\partial V_{z1}}{\partial \rho} + \frac{\partial V_{r1}}{\partial \varsigma} \right) S_{rzo} \\ & + \left(\frac{\partial V_{zo}}{\partial \rho} + \frac{\partial V_{ro}}{\partial \varsigma} \right)(S_{rz1} + \rho S_{rzo}) - \frac{1}{1-\nu^2} \Big[S_{\theta o} S_{\theta 1} + S_{zo} S_{z1} \\ & + \frac{1}{2}(S_{\theta o}^2 + S_{zo}^2)\rho - \nu (S_{\theta o} S_{z1} + S_{\theta 1} S_{zo}) - \nu S_{\theta o} S_{zo} \rho \\ & - \nu_n S_{ro}(S_{\theta o} + S_{zo}) + (1+\nu) I_n S_{rzo}^2 \Big] \Big\} d\rho\, d\varsigma \\ & + \int_{-1}^1 \left[\frac{3}{2} \overline{m}_z \rho V_{z1} + \frac{3}{4} \overline{q}_z (1-\rho^2) V_{r1} \right]^{\varsigma=0} d\rho \end{aligned} \tag{5.6}$$

As a first approximation we assume that

$$I = \lambda^{3/2} \frac{(1-\nu^2)\sigma^2 d^3}{E} I_o \; (s_o, v_o) \tag{5.7}$$

and that $\delta I = 0$ is approximated by $\delta I_o = 0$. We further assume that s_o and v_o are given by expressions (2.4) and (2.5). Then, since the differential equations (1.1) and (1.2) are satisfied identically to the order of the first approximation (i.e. equations (2.2) satisfied identically), $\delta I_o = 0$ should give results such that the boundary conditions (5.2) are approximated to the same order.* Making use of (2.2), I_o can be written more briefly as follows.

$$I_o = \frac{1}{2}\int_0^{\infty}\int_{-1}^{1}\left[V_{ro}^2 + 2\nu V_{ro}\frac{\partial V_{zo}}{\partial \xi} + \left(\frac{\partial V_{zo}}{\partial \xi}\right)^2\right] d\rho\, d\xi$$

$$+\int_{-1}^{1}\left[\frac{3}{2}\,\overline{m}_z\,\rho V_{zo} + \frac{3}{4}\,\overline{q}_z\,(1-\rho^2)\,V_{ro}\right]^{\xi=0} d\rho \tag{5.8}$$

Next, we substitute from (2.4) into (5.8) and perform the integrations with respect to ρ .

*Note that in assuming s_o and v_o given by (2.4)and (2.5) we do not identically satisfy boundary conditions on $\rho = \pm 1$.

$$I_o = \int_0^\infty \left[V_{ro}^2 + 2\nu V_{ro} V_{zo}' + (V_{zo}')^2 + \frac{1}{3}(V_{ro}'')^2 \right] d\rho$$

$$- \overline{m}_z V_{ro}'(0) + \overline{q}_z V_{ro}(0) \qquad (5.9)$$

After integratin by parts, the variation of I_o becomes

$$\delta I_o = \int_0^\infty \Big\{ 2\left[V_{ro} + \nu V_{zo}' + \frac{1}{3} V_{ro}^{IV} \right] \delta V_{ro}$$

$$- 2\left[V_{zo}'' + \nu V_{ro}' \right] \delta V_{zo} \Big\} d\rho + \Big\{ -\frac{2}{3} V_{ro}''' \delta V_{ro}$$

$$+ 2\left[V_{zo}' + \nu V_{ro} \right] \delta V_{zo} + \frac{2}{3} V_{ro}'' \delta V_{ro}' \Big\}_{\rho=0}^{\rho=\infty}$$

$$- \overline{m}_z \delta V_{ro}'(0) + \overline{q}_z \delta V_{ro}(0) \qquad (5.10)$$

As Euler differential equations we obtain (3.4) and (3.5) which were shown to be necessary for the satisfaction of boundary conditions at $\rho = \pm 1$. However, the variational method does not give expressions for S_{rzo} and S_{ro} as were previously given by (3.6) and (3.7). In this way in the first approximation the transverse stresses s_{rz} and s_r as given by (2.5) are incompletely determined. This agrees with the philosophy of conventional shell theory.

Natural boundary conditions are as follows.

$$\rho = \infty : \quad V_{zo}' + \nu V_{ro} = 0 \, , \quad V_{ro}'' = 0 \, , \quad V_{ro}''' = 0 \tag{5.11}$$

$$\rho = 0 : \quad V_{zo}' + \nu V_{ro} = 0, \quad -\frac{2}{3} V_{ro}''' = \overline{q}_z \, , \quad -\frac{2}{3} V_{ro}'' = \overline{m}_z \tag{5.12}$$

It will be shown later that conditions (5.12) agree with (4.18).

The first approximation as represented by differential equations (3.4) and (3.5), boundary conditions (5.11) and (5.12), and the relations (2.4) and (2.5) are equivalent to the conventional theory of axially symmetric deformation as presented in references [1], [11] or [12]. Note that the equations and boundary conditions of the first approximation do not involve the transverse elastic parameters I_n, ν_n and E/E_t, so that one can obtain the same results by assuming any values for these parameters. In particular, one can choose them so as to have no transverse shear or normal deformation as is done in the conventional theory. Transverse effects enter only into the higher approximations. It is important that these facts here are obtained as results rather than entering as basic assumptions as they do in conventional shell theory.

6. Second approximation. As a second approximation assume that

$$I = \lambda^{3/2} \frac{(1-\nu^2)\sigma^2 a^3}{E} \left[I_0 + \lambda I_1 \right] \tag{6.1}$$

Then, $\delta I = 0$ is approximated by

$$\delta \left[I_0 + \lambda I_1 \right] = 0 \tag{6.2}$$

We wish equation (6.2) to be true identically in λ which implies that δI_0 and δI_1 must vanish separately.

$$\delta I_0 (s_0, v_0) = 0 \tag{6.3}$$

$$\delta I_1 (s_0, v_0; s_1, v_1) = 0 \tag{6.4}$$

Quantities of both first and second order are to be varied independently in (6.3) and (6.4). The results of (6.3) have been investigated in the preceding section. The consequences of (6.4) must be compatible with those of (6.3) in order that the whole procedure make sense.

We assume s_0 and v_0 given by (2.4) and (2.5) and s_1 and v_1 given by (2.6) so that system (1.12) is satisfied identically up to the order of the second approximation. The systems (2.2) and (2.3) are satisfied identically and can be used to simplify expression (5.6) for I_1.

$$I_1 = \int_0^\infty \int_{-1}^{1} \Big[V_{r1} V_{r0} + \nu V_{r1} \frac{\partial V_{z0}}{\partial \xi} + \nu \frac{\partial V_{z1}}{\partial \xi} V_{r0} + \frac{\partial V_{z1}}{\partial \xi} \frac{\partial V_{z0}}{\partial \xi} + \frac{1}{2} \left(\frac{\partial V_{z0}}{\partial \xi}\right)^2 \rho - \frac{1}{2} V_{r0}^2 \rho + \frac{I_n}{1-\nu} S_{rz0}^2 \Big] d\rho \, d\xi + \int_{-1}^{1} \Big[\frac{3}{2} \overline{m}_z \rho V_{z1} + \frac{3}{4} \overline{q}_z (1-\rho^2) V_{r1} \Big]^{\xi=0} d\rho \qquad (6.5)$$

Now, substituting from (2.4), (2.5) and (2.6) and performing the integrations with respect to ρ gives, after some integrations by parts to shorten the result,

$$I_1 = \int_0^\infty \Big\{ 2 V_{r0} V_{r1} + 2\nu V_{r0} V_{z1}' + 2\nu V_{z0}' V_{r1} + 2 V_{z0}' V_{z1}' + \frac{2}{3} V_{r0}'' V_{r1}'' + \left[\frac{(1+4\nu)\nu_n - 8\nu I_n}{3(1-\nu)} - \frac{2}{3} \right] V_{z0}' V_{r0}'' + \frac{2 I_n}{1-\nu} S_{rz0}^2$$

$$+\frac{-(1+3\nu)\nu_n+4\nu^2 I_n}{3(1-\nu)}\left(V_{ro}'\right)^2+\frac{4I_n-\nu_n}{3(1-\nu)}\left(V_{zo}''\right)^2+\frac{7/2\, I_n-\nu_n}{15(1-\nu)}\left(V_{ro}'''\right)^2$$

$$-\frac{2I_n}{1-\nu}S_{rzo}'V_{ro}''\Big\}\,d\xi$$

$$+\Big\{\frac{(1+3\nu)\nu_n-2\nu^2 I_n}{3(1-\nu)}V_{ro}V_{ro}'+\frac{\nu\nu_n-2\nu I_n}{3(1-\nu)}V_{ro}V_{zo}''$$

$$+\frac{\nu_n-2I_n}{3(1-\nu)}V_{zo}'V_{zo}''+\frac{\nu_n-2I_n}{15(1-\nu)}V_{ro}''V_{ro}'''$$

$$+\frac{2I_n}{3(1-\nu)}S_{rzo}V_{ro}''+\frac{-\nu\nu_n+6\nu I_n}{3(1-\nu)}V_{ro}'V_{zo}'\Big\}_{\xi=0}^{\xi=\infty}$$

$$+\Big\{\bar{m}_z\Big[-V_{r1}'+\frac{2I_n}{1-\nu}S_{rzo}+\frac{-\nu_n+2I_n}{10(1-\nu)}V_{ro}'''\Big]$$

$$+\bar{q}_z\Big[V_{r1}+\frac{\nu_n}{10(1-\nu)}V_{ro}''\Big]\Big\}^{\xi=0} \qquad (6.6)$$

After integration by parts, the variation of I_1 becomes

$$\delta I_1 = \int_0^\infty \Big\{ 2\,\delta V_{r1}\Big[V_{r0} + \nu V_{z0}' + \tfrac{1}{3} V_{r0}^{IV}\Big] - 2\,\delta V_{z1}\Big[\nu V_{r0}' + V_{z0}''\Big]$$

$$+ \delta V_{r0}\Big[2V_{r1} + 2\nu V_{z1}' + \tfrac{2}{3} V_{r1}^{IV} + \frac{2(1+3\nu)\nu_n - 8\nu^2 I_n}{3(1-\nu)} V_{r0}''$$

$$- \frac{2I_n}{1-\nu} S_{rz0}''' + \frac{2\nu_n - 7I_n}{15(1-\nu)} V_{r0}^{VI} + \Big(\frac{(1+4\nu)\nu_n - 8\nu I_n}{3(1-\nu)} - \frac{2}{3}\Big) V_{z0}'''\Big]$$

$$- \delta V_{z0}\Big[2\nu V_{r1}' + 2V_{z1}'' + \Big(\frac{(1+4\nu)\nu_n - 8\nu I_n}{3(1-\nu)} - \frac{2}{3}\Big) V_{r0}''' + \frac{2\nu_n - 8I_n}{3(1-\nu)} V_{z0}^{IV}\Big]$$

$$+ \frac{4I_n}{1-\nu}\,\delta S_{rz0}\Big[S_{rz0} + \tfrac{1}{2} V_{r0}'''\Big]\Big\}\, d\rho$$

$$+ \Big\{ -\tfrac{2}{3}\,\delta V_{r1} V_{r0}''' + 2\,\delta V_{z1}\Big[\nu V_{r0} + V_{z0}'\Big] + \delta V_{r1}'\,\tfrac{2}{3} V_{r0}''$$

$$+ \delta V_{r0}\Big[-\tfrac{2}{3} V_{r1}''' + \frac{-(1+3\nu)\nu_n + 6\nu^2 I_n}{3(1-\nu)} V_{r0}' + \frac{2I_n}{1-\nu} S_{rz0}''$$

$$+ \Big(\frac{-(1+3\nu)\nu_n + 6\nu I_n}{3(1-\nu)} + \frac{2}{3}\Big) V_{z0}'' + \frac{-2\nu_n + 7I_n}{15(1-\nu)} V_{r0}^{V}\Big]$$

$$+ \delta V_{r0}'\Big[\tfrac{2}{3} V_{r1}'' + \Big(\frac{(1+3\nu)\nu_n - 2\nu I_n}{3(1-\nu)} - \frac{2}{3}\Big) V_{z0}' - \frac{2I_n}{1-\nu} S_{rz0}'$$

$$+ \frac{2\nu_n - 7I_n}{15(1-\nu)} V_{r0}^{IV} + \frac{(1+3\nu)\nu_n - 2\nu^2 I_n}{3(1-\nu)} V_{r0}\Big]$$

$$+ \delta V_{r0}''\Big[\frac{-\nu_n + 5I_n}{15(1-\nu)} V_{r0}''' + \frac{2I_n}{3(1-\nu)} S_{rz0}\Big] + \delta V_{r0}'''\,\frac{\nu_n - 2I_n}{15(1-\nu)} V_{r0}''$$

$$+ \delta V_{z0}\Big[2\nu V_{r1} + 2V_{z1}' + \Big(\frac{(1+4\nu)\nu_n - 8\nu I_n}{3(1-\nu)} - \frac{2}{3}\Big) V_{r0}'' + \frac{2\nu_n - 8I_n}{3(1-\nu)} V_{z0}'''\Big]$$

$$+\delta V_{z0}'\frac{-\nu_n+6I_n}{3(1-\nu)}\left[V_{z0}''+\nu V_{r0}'\right]+\delta V_{z0}''\frac{\nu_n-2I_n}{3(1-\nu)}\left[V_{z0}'+\nu V_{r0}\right]-\delta S_{rz0}\frac{4I_n}{3(1-\nu)}V_{r0}''\Big\}_{\xi=0}^{\xi=\infty}$$

$$+\Big\{\delta V_{r1}\bar{q}_z-\delta V_{r1}'\,\bar{m}_z+\delta V_{r0}''\frac{\nu_n}{10(1-\nu)}\,\bar{q}_z$$

$$+\delta V_{r0}'''\frac{-\nu_n+2I_n}{10(1-\nu)}\bar{m}_z+\delta S_{rz0}\frac{2I_n}{1-\nu}\bar{m}_z\Big\}^{\xi=0} \qquad (6.7)$$

The Euler equations are

$$\frac{1}{3}V_{r0}^{IV}+V_{r0}+\nu V_{z0}'=0 \qquad (6.8)$$

$$V_{z0}''+\nu V_{r0}'=0 \qquad (6.9)$$

$$S_{rz0}=-\frac{1}{2}V_{r0}''' \qquad (6.10)$$

$$\frac{1}{3}V_{r1}^{IV}+V_{r1}+\nu V_{z1}'+\frac{2\nu_n-7I_n}{30(1-\nu)}V_{r0}^{VI}-\frac{I_n}{1-\nu}S_{rz0}'''$$

$$+\frac{(1+3\nu)\nu_n-4\nu^2I_n}{3(1-\nu)}V_{r0}''+\left(\frac{(1+4\nu)\nu_n-8\nu I_n}{6(1-\nu)}-\frac{1}{3}\right)V_{z0}'''=0 \qquad (6.11)$$

$$V_{z1}''+\nu V_{r1}'+\left(\frac{(1+4\nu)\nu_n-8\nu I_n}{6(1-\nu)}-\frac{1}{3}\right)V_{r0}'''+\frac{\nu_n-4I_n}{3(1-\nu)}V_{z0}^{IV}=0 \qquad (6.12)$$

Equations (6.8) and (6.9) agree, as they must, with the Euler equations of the first approximation. In addition, the second approximation fully determines s_{rzo} (so that boundary condition $s_{rzo} = 0$ on $\rho \pm 1$ satisfied) by giving formula (6.10) for S_{rzo}. We note that (6.10) agrees with (3.6). Using (6.8), (6.9) and (6.10) equations (6.11) and (6.12) can be transformed to agree with (3.12) and (3.13). The variational method still says nothing up to this point about S_{ro} or about S_{rz1} and S_{r1}.

In order to obtain natural boundary conditions which approximate (5.2) to the degree of accuracy of the second approximation, we consider that all variations are arbitrary at the ends $\xi = 0, \infty$. The results are then as follows at $\xi = 0$.

$$\delta V_{r1}: \quad \frac{2}{3} V_{ro}''' + \bar{q}_z = 0 \tag{6.13}$$

$$\delta V_{z1}: \quad -2\left[\nu V_{ro} + V_{zo}'\right] = 0 \tag{6.14}$$

$$\delta V_{r1}': \quad -\frac{2}{3} V_{ro}'' - \bar{m}_z = 0 \tag{6.15}$$

$$\delta V_{ro}: -\frac{2}{3} V_{r1}''' + \frac{-(1+3\nu)\nu_n + 6\nu^2 I_n}{3(1-\nu)} V_{ro}' + \frac{2 I_n}{1-\nu} S_{rzo}''$$

$$+\left(\frac{-(1+3\nu)\nu_n + 6\nu I_n}{3(1-\nu)} + \frac{2}{3}\right) V_{zo}'' + \frac{-2\nu_n + 7 I_n}{15(1-\nu)} V_{ro}^{V} = 0 \tag{6.16}$$

$$\delta V_{z0}:\ 2\nu V_{r1} + 2V_{z1}' + \left(\frac{(1+4\nu)\nu_n - 8\nu I_n}{3(1-\nu)} - \frac{2}{3}\right) V_{r0}''$$

$$+ \frac{2\nu_n - 8I_n}{3(1-\nu)} V_{z0}''' = 0 \qquad (6.17)$$

$$\delta V_{r0}':\ \frac{2}{3} V_{r1}'' + \left(\frac{(1+3\nu)\nu_n - 2\nu I_n}{3(1-\nu)} - \frac{2}{3}\right) V_{z0}' - \frac{2I_n}{1-\nu} S_{rz0}'$$

$$+ \frac{2\nu_n - 7I_n}{15(1-\nu)} V_{r0}^{IV} + \frac{(1+3\nu)\nu_n - 2\nu^2 I_n}{3(1-\nu)} V_{r0} = 0 \qquad (6.18)$$

$$\delta V_{r0}'':\ \frac{\nu_n - 5I_n}{15(1-\nu)} V_{r0}''' - \frac{2I_n}{3(1-\nu)} S_{rz0} + \frac{\nu_n}{10(1-\nu)} \bar{q}_z = 0 \qquad (6.19)$$

$$\delta V_{r0}''':\ \frac{-\nu_n + 2I_n}{15(1-\nu)} V_{r0}'' + \frac{-\nu_n + 2I_n}{10(1-\nu)} \bar{m}_z = 0 \qquad (6.20)$$

$$\delta V_{z0}':\ \frac{-\nu_n + 6I_n}{3(1-\nu)} \left[V_{z0}'' + \nu V_{r0}'\right] = 0 \qquad (6.21)$$

$$\delta V_{z0}'':\ \frac{\nu_n - 2I_n}{3(1-\nu)} \left[V_{z0}' + \nu V_{r0}\right] = 0 \qquad (6.22)$$

$$\delta S_{rz0}:\ \frac{4I_n}{3(1-\nu)} V_{r0}'' + \frac{2I_n}{1-\nu} \bar{m}_z = 0 \qquad (6.23)$$

The end conditions at $\rho = \infty$ are the same as above with $\overline{m_z} = \overline{q_z} = 0$.

Conditions (6.13), (6.14) and (6.15) are the same as the conditions obtained in the first approximation, (5.12). Conditions (6.16), (6.17) and (6.18) are appropriate for the macroscopic displacements V_{r1} and V_{z1}. It would seem that we are left with five extra conditions, (6.19) through (6.23), except that these conditions add nothing new. With S_{rzo} given by (6.10), condition (6.19) is the same as (6.13). Condition (6.20) is the same as (6.15). Condition (6.21) is satisfied identically by solutions satisfying differential equation (6.9). Condition (6.22) is identical with (6.14), and, finally, (6.23) is the same as (6.15).

Conditions (6.16), (6.17) and (6.18) can be simplified by employing the equations and end conditions of the first approximation. The results are

$$\left.\begin{aligned} \rho = 0: \quad & \frac{2}{3} V_{r1}''' = \left[\frac{\nu_m}{15}(1-9\nu) + \frac{8}{5} I_m (1+\nu) - \frac{2}{3}\nu\right] V_{ro}' \\ & \frac{2}{3} V_{r1}'' = \left[\frac{\nu_m}{15}(1-9\nu) + \frac{8}{5} I_\eta (1+\nu) - \frac{2}{3}\nu\right] V_{ro} \\ & \nu V_{r1} + V_{z1}' = \left[-\frac{1}{2} + \frac{\nu_m(1+2\nu)}{4(1-\nu)}\right] \overline{m_z} \end{aligned}\right\} \qquad (6.24)$$

plus conditions at $\rho = \infty$ the same with $\overline{m_z} = 0$.

7. <u>End conditions in terms of stress resultants and couple.</u> We wish first to relate stress resultants and couples defined by (4.6), (4.7) and (4.8) to the macroscopic displacements V_{zo}, V_{ro}, etc. The results will be macroscopic stress-displacement relations which are of some interest in themselves. Substituting from (2.4), (2.5) and (2.6) into (4.8), performing the integrations with respect to ρ , and simplifying the results by use of (3.4) through (3.7) and (3.10), we obtain

$$\left.\begin{aligned} N_{z0} &= 2\left[\nu V_{r0} + V_{z0}'\right] \\ M_{z0} &= -\tfrac{2}{3} V_{r0}'' \quad , \qquad Q_{z0} = -\tfrac{2}{3} V_{r0}''' \end{aligned}\right\} \qquad (7.1)$$

$$\left.\begin{aligned} N_{z1} &= 2\left[\nu V_{r1} + V_{z1}'\right] + \left[\frac{\nu_n(1+2\nu)}{3(1-\nu)} - \frac{2}{3}\right] V_{r0}'' \\ M_{z1} &= -\frac{2}{3} V_{r1}'' + \frac{\nu_n(1-10\nu)+24 I_n}{15(1-\nu)} V_{r0} + \left[\frac{2}{3} + \frac{-3\nu\nu_n + 8\nu I_n}{5(1-\nu)}\right] V_{z0}' \\ Q_{z1} &= -\frac{2}{3} V_{r1}''' + \left[\frac{\nu_n}{15}(1-9\nu) + \frac{8}{5} I_\eta (1+\nu) - \frac{2}{3}\nu\right] V_{r0}' \end{aligned}\right\} \qquad (7.2)$$

Making use of (7.1) and (7.2) we can write the end conditions obtained by use of the variational method, (5.11), (5.12) and (6.24), in terms of stress resultants and couples as follows.

$$\left.\begin{aligned} \xi = 0: \quad & N_{z0} = 0, \quad M_{z0} = \overline{m}_z, \quad Q_{z0} = \overline{q}_z \\ & N_{z1} = 0, \quad M_{z1} = 0, \quad Q_{z1} = 0 \\ \xi = \infty: \quad & N_{z0} = M_{z0} = Q_{z0} = 0 \\ & N_{z1} = M_{z1} = Q_{z1} = 0 \end{aligned}\right\} \tag{7.3}$$

We note that (7.3) agrees with (4.18) which is as it should be. It seems reasonable to expect that the conditions given by (4.18) for higher approximations are also correct.

8. A solution. To illustrate use of the theory we solve the equations of the first and second approximation and obtain a few interesting formulae. For convenience we repeat the governing system of differential equations, (3.4), (3.5), (3.12) and (3.13).

$$V_{zo}'' + \nu V_{ro}' = 0 \tag{8.1}$$

$$\frac{1}{3} V_{ro}^{IV} + V_{ro} + \nu V_{zo}' = 0 \tag{8.2}$$

$$V_{z1}'' + \nu V_{r1}' = \left[\frac{1}{3} - \frac{(1+2\nu)\nu_n}{6(1-\nu)}\right] V_{ro}''' \tag{8.3}$$

$$\frac{1}{3} V_{r1}^{IV} + V_{r1} + \nu V_{z1}' = \left[-\frac{\nu}{3} + \frac{4}{5}(1+\nu)I_n - \frac{(4+25\nu-14\nu^2)\nu_n}{30(1-\nu)}\right] V_{ro}'' \tag{8.4}$$

Let us assume that end moment $\overline{M_z}$ is specified and that $\overline{Q_z} = 0$. This is problem (ii) of sectin 4, for which the edge conditions (5.11), (5.12) and (6.24) become

$$\rho = 0: \; V_{zo}' + \nu V_{ro} = 0 \tag{8.5}$$

$$-\frac{2}{3} V_{ro}'' = 1 \;, \quad V_{ro}''' = 0 \tag{8.6}$$

$$\nu V_{r1} + V_{z1}' = -\frac{1}{2} + \frac{(1+2\nu)\nu_n}{4(1-\nu)} \tag{8.7}$$

$$\left.\begin{aligned} \frac{2}{3} V_{r1}'' &= \left[\frac{\nu_n}{15}(1-9\nu) + \frac{8}{5} I_n (1+\nu) - \frac{2}{3}\nu\right] V_{ro} \\ \frac{2}{3} V_{r1}''' &= \left[\frac{\nu_n}{15}(1-9\nu) + \frac{8}{5} I_n (1+\nu) - \frac{2}{3}\nu\right] V_{ro}' \end{aligned}\right\} \tag{8.8}$$

$$\xi = \infty: \quad V_{zo}' + \nu V_{ro} = 0 \tag{8.9}$$

$$V_{ro}'' = 0 \,, \qquad V_{ro}''' = 0 \tag{8.10}$$

$$\nu V_{r1} + V_{z1}' = 0 \tag{8.11}$$

$$\left.\begin{aligned} \tfrac{2}{3} V_{r1}'' &= \left[\frac{\nu_n}{15}(1-9\nu) + \frac{8}{5} I_\eta (1+\nu) - \frac{2}{3}\nu\right] V_{ro} \\ \tfrac{2}{3} V_{r1}''' &= \left[\frac{\nu_n}{15}(1-9\nu) + \frac{8}{5} I_\eta (1+\nu) - \frac{2}{3}\nu\right] V_{ro}' \end{aligned}\right\} \tag{8.12}$$

Integrating equations (8.1) and using end conditions (8.5) and (8.9), we obtain

$$V_{zo}' = -\nu V_{ro} \tag{8.13}$$

which, upon substitution into (8.2), enables us to write the following differential equation for V_{ro}.*

*Equations (8.13) and (8.14) are the usual form of the governing equations of thin shell theory. (see [11], 391-392.)

$$V_{r0}^{IV} + 3(1-\nu^2)\, V_{r0} = 0 \tag{8.14}$$

The general solution of (8.14) is given by

$$V_{r0}(\varphi) = e^{-\kappa\varphi}\left[A \cos \kappa\varphi + B \sin \kappa\varphi\right] \tag{8.15}$$

where A and B are to be determined from the boundary conditions and where

$$\kappa^4 = \frac{3}{4}(1-\nu^2) \tag{8.16}$$

Satisfaction of (8.6) and (8.10) yields

$$V_{r0}(\varphi) = \frac{-\kappa^2}{1-\nu^2}\, e^{-\kappa\varphi}\left[\cos \kappa\varphi - \sin \kappa\varphi\right] \tag{8.17}$$

Integrating (8.3) and using conditions (8.10) and (8.11) to eliminate the constant of integration, we obtain

$$V_{z1}' = -\nu V_{r1} + \left[\frac{1}{3} - \frac{(1+2\nu)\nu_n}{6(1-\nu)}\right] V_{r0}'' \tag{8.18}$$

Note that condition (8.7) is satisfied by (8.18) if we make use of (8.6). We use (8.18) to eliminate V_{z1}' from (8.4).

$$\left.\begin{aligned} V_{r1}^{IV} + 3(1-\nu^2)V_{r1} &= \left[-2\nu + \tfrac{12}{5}(1+\nu)I_n - \tfrac{2}{5}(1+6\nu)\nu_n\right] V_{r0}'' \\ &= \left[3\nu - \tfrac{18}{5}(1+\nu)I_n + \tfrac{3}{5}(1+6\nu)\nu_n\right] e^{-\kappa\varphi}(\cos\kappa\varphi + \sin\kappa\varphi) \end{aligned}\right\} \tag{8.19}$$

The solution of the homogeneous part of (8.19) is of the same form as (8.15). A particular solution is obtained by the method of undetermined coefficients. Then, the complete solution satisfying conditions (8.8) and (8.12) is given by

$$V_{r1}(\xi) = e^{-\kappa\xi}\left[A\cos\kappa\xi + B\sin\kappa\xi\right] + Ce^{-\kappa\xi}\xi\sin\kappa\xi \qquad (8.20)$$

where

$$\left.\begin{aligned}(1-\nu^2)A &= \tfrac{1}{2}\nu + \tfrac{1}{4}(1+3\nu)\nu_n \\ (1-\nu^2)B &= \tfrac{3}{5}(1+\nu)I_n + \tfrac{3}{20}(1+\nu)\nu_n \\ (1-\nu^2)C &= \tfrac{\kappa}{10}\left[5\nu - 6(1+\nu)I_n + (1+6\nu)\nu_n\right]\end{aligned}\right\} \qquad (8.21)$$

We can obtain V_{zo} and V_{z1} by substituting (8.17) and (8.20) into (8.13) and (8.18) and integrating. The constants of integration, representing rigid body displacement, can be set equal to zero, and we get

$$V_{zo}(\xi) = \frac{\nu}{1-\nu^2}\,\kappa e^{-\kappa\xi}\sin\kappa\xi \qquad (8.22)$$

$$V_{z1}(\xi) = \frac{1}{2\kappa}e^{-\kappa\xi}\Big\{\nu A(\cos\kappa\xi - \sin\kappa\xi) + \nu B(\cos\kappa\xi + \sin\kappa\xi)$$

$$+\frac{\nu}{\kappa}C\left[\kappa\xi(\cos\kappa\xi + \sin\kappa\xi) + \cos\kappa\xi\right] + \left[1 - \frac{\nu_n(1+2\nu)}{2(1-\nu)}\right]\cos\kappa\xi\Big\} \qquad (8.23)$$

With expressions (8.17), (8.20), (8.22) and (8.23) for the macroscopic displacements, we can calculate, if we wish, dimensionless microscopic stresses and displacements with (2.4), (2.5) and (2.6) where S_{rzo}, etc., are given by formulae in section 3. True stresses and displacements can be obtained by use of (1.6), (1.9) and (2.1). We can also obtain stress resultants and moments with formulae (4.6) together with similar formulae for N_θ and M_θ. The quantity σ is given by formula (ii) of (4.13).

$$\sigma = \frac{\overline{M}_z}{a^2 \lambda^2} \tag{8.24}$$

As examples of the use of the solution presented in this section, we give the results of two simple calculations .

We may define an effective angle of rotation β of the end $z = 0$ by

$$-\frac{1}{2}(2\pi a \overline{M}_z)\beta = \text{strain energy of shell} \tag{8.25}$$

Upon carrying out the calculation of the strain energy up to contributions of order λ^1, we obtain

$$\beta = -\frac{2k^3\overline{M}_z}{a^2 E\lambda^{5/2}}\left\{1+\frac{\lambda}{2k^2}\left[-\frac{\nu}{2}+\frac{3E}{10G_t}-\frac{(1+6\nu)\nu E}{10E_t}\right]\right\} \tag{8.26}$$

As $\lambda \to 0$ (8.26) reduces to the conventional result for the end rotation.

For isotropy, (8.26) can be written in the following form.

$$\beta = -\frac{2k^3}{a^2 E\lambda^{5/2}}\left[1+\frac{2}{5}k^2\lambda\right]\overline{M}_z \tag{8.27}$$

We see that with the inclusion of effects of order λ^1, formula (8.27) predicts a larger angle of rotation than does the conventional result.

Another formula, giving a measure of the change of thickness of the shell at the end, is

$$\frac{u_r(a+\frac{1}{2}h,0)-u_r(a-\frac{1}{2}h,0)}{h} = \frac{\overline{M}_z}{a^2E\lambda^2}\,k^2\nu_n$$

$$= \sqrt{12(1-\nu^2)}\,\frac{\nu}{h^2E_t}\,\overline{M}_z \tag{8.28}$$

We note that transverse effects enter into (8.28) only through the parameter ν_n which seems right.

The results of this section include the contribution only of the interior solution. If, besides $\overline{M_z}$ and $\overline{N_z} = \overline{Q_z} = 0$, further information about the distribution of end stresses is known, it may be possible to calculate some of the excess solution (see introduction) which is important in the St. Venant boundary layer and to add its contribution.

The above results are given merely as illustrations of the use of the theory. More complete results would include the following calculations.

1. The solution of the problem with Q_z given at the end (problem (iii), section 4). By use of the principle of superposition relations (8.27) and (8.28) could then be replaced by relations of the form

$$\left.\begin{aligned} \beta &= c_{\beta M}\,\overline{M}_z + c_{\beta Q}\,\overline{Q}_z \\ \frac{u_r(a+\frac{1}{2}h,0) - u_r(a-\frac{1}{2}h,0)}{h} &= c_{uM}\,\overline{M}_z + c_{uQ}\,\overline{Q}_z \end{aligned}\right\} \quad (8.29)$$

The influence coefficients $c_{\beta m}$ and c_{um} are already given in (8.27) and (8.28).

2. An analysis of the effects of the excess solution.

3. A comparison of the results with those obtained from previous theories, such as are given in [5] and [6].

Chapter II: Non-symmetric deformation

1. Non-dimensionalization. Consider the deformation of a semi-infinite circular cylindrical shell of constant thickness by a non-symmetric loading at the end of the shell, $z = 0$. With the usual notation (Fig. 2) the three-dimensional equations of linear elasticity in cylindrical coordinates are as follows. The equilibrium equations are

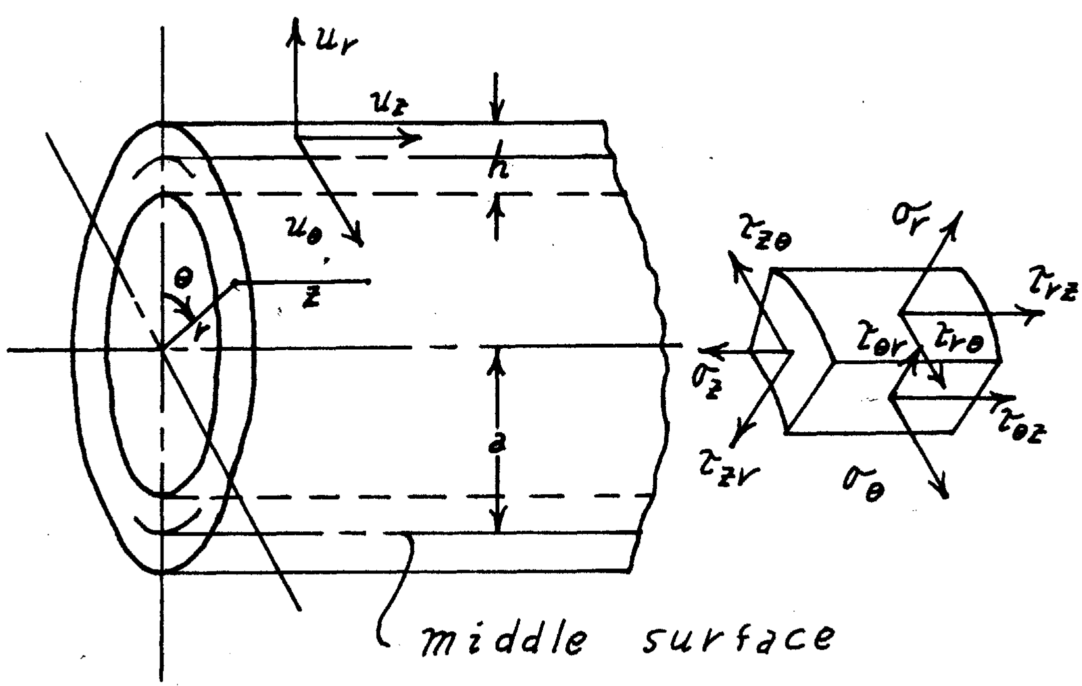

Fig. 2: Stress (σ_z, σ_r, σ_θ, τ_{rz}, $\tau_{r\theta}$, $\tau_{z\theta}$) and displacement (u_r, u_θ, u_z) components and coordinates (r, θ, z) for general deformation.

$$
\left.\begin{aligned}
\frac{\partial}{\partial r}(r\sigma_r) + \frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial (r\tau_{rz})}{\partial z} - \sigma_\theta &= 0 \\
\frac{\partial}{\partial r}(r\tau_{r\theta}) + \frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial (r\tau_{\theta z})}{\partial z} + \tau_{r\theta} &= 0 \\
\frac{\partial (r\tau_{rz})}{\partial r} + \frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial (r\sigma_z)}{\partial z} &= 0
\end{aligned}\right\} \qquad (1.1)
$$

Assuming an orthotropic material with the r -direction an axis of elastic symmetry, the stress-strain relations are found to be

$$
\left.\begin{aligned}
\frac{\partial u_r}{\partial r} &= \frac{1}{E_t}\sigma_r - \frac{\nu_t}{E_t}(\sigma_z + \sigma_\theta) \\
\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} &= \frac{1}{E}(\sigma_\theta - \nu\sigma_z) - \frac{\nu_t}{E_t}\sigma_r \\
\frac{\partial u_z}{\partial z} &= \frac{1}{E}(\sigma_z - \nu\sigma_\theta) - \frac{\nu_t}{E_t}\sigma_r \\
\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} &= \frac{1}{G_t}\tau_{r\theta} \\
\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z} &= \frac{1}{G_t}\tau_{rz} \\
\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta} &= \frac{2(1+\nu)}{E}\tau_{z\theta}
\end{aligned}\right\} \qquad (1.2)
$$

As in the axially symmetric problem we introduce dimensionless coordinates ρ and ς by

$$\rho = \frac{r-a}{1/2\, h}, \qquad \varsigma = \frac{z}{b} \tag{1.3}$$

and dimensionless parameters λ and μ by

$$\lambda = h/2a, \qquad \mu = b/a \tag{1.4}$$

We now also introduce a length c in the θ-direction, a new angular coordinate ϕ, and a new dimensionless parameter γ by

$$\theta = \gamma\phi, \qquad \gamma = c/a \tag{1.5}$$

The introduction of (1.3), (1.4) and (1.5) into the equilibrium equations (1.1) gives

$$\left.\begin{aligned}
&\frac{1}{\lambda}\frac{\partial}{\partial\rho}\left[(1+\lambda\rho)\sigma_r\right] + \frac{1}{\gamma}\frac{\partial\tau_{r\theta}}{\partial\phi} + \frac{1}{\mu}\frac{\partial}{\partial\varsigma}\left[(1+\lambda\rho)\tau_{rz}\right] - \sigma_\theta = 0 \\
&\frac{1}{\lambda}\frac{\partial}{\partial\rho}\left[(1+\lambda\rho)\tau_{r\theta}\right] + \frac{1}{\gamma}\frac{\partial\sigma_\theta}{\partial\phi} + \frac{1}{\mu}\frac{\partial}{\partial\varsigma}\left[(1+\lambda\rho)\tau_{\theta z}\right] + \tau_{r\theta} = 0 \\
&\frac{1}{\lambda}\frac{\partial}{\partial\rho}\left[(1+\lambda\rho)\tau_{rz}\right] + \frac{1}{\gamma}\frac{\partial\tau_{\theta z}}{\partial\phi} + \frac{1}{\mu}\frac{\partial}{\partial\varsigma}\left[(1+\lambda\rho)\sigma_z\right] = 0
\end{aligned}\right\} \tag{1.6}$$

In order to obtain equations appropriate to the interior solution, we identify the length b in the Z-direction with the boundary width $\sqrt{\frac{1}{2} ah}$ as was done in the axially symmetric problem. This is accomplished by non-dimensionalizing stresses so that all terms in (1.6) containing derivatives with respect to ρ and ς are of the same order in λ and so that the curvature term σ_θ is also of the same order in λ. This can be done by setting

$$\left.\begin{array}{ll} \sigma_z = \sigma S_z \,, & \sigma_\theta = \sigma S_\theta \\ \sigma_r = \lambda \sigma S_r \,, & \tau_{\theta z} = \dfrac{\sqrt{\lambda}}{\gamma} \sigma S_{\theta z} \\ \tau_{r\theta} = \dfrac{\lambda}{\gamma} \sigma S_{r\theta} \,, & \tau_{rz} = \sqrt{\lambda}\, \sigma S_{rz} \end{array}\right\} \qquad (1.7)$$

and by taking the parameter μ as

$$\mu = \sqrt{\lambda} \qquad (1.8)$$

The quantity σ will be discussed later.

With (1.7) and (1.8), (1.6) yields the following dimensionless equilibrium equations.

$$\left.\begin{aligned}
&\frac{\partial}{\partial\rho}\left[(1+\lambda\rho)S_r\right]+\frac{\lambda}{\gamma^2}\frac{\partial S_{r\theta}}{\partial\phi}+\frac{\partial}{\partial\zeta}\left[(1+\lambda\rho)S_{rz}\right]-S_\theta=0\\
&\frac{1}{\gamma}\left\{\frac{\partial}{\partial\rho}\left[(1+\lambda\rho)S_{r\theta}\right]+\frac{\partial S_\theta}{\partial\phi}+\frac{\partial}{\partial\zeta}\left[(1+\lambda\rho)S_{\theta z}\right]+\lambda S_{r\theta}\right\}=0\\
&\frac{\partial}{\partial\rho}\left[(1+\lambda\rho)S_{rz}\right]+\frac{\lambda}{\gamma^2}\frac{\partial S_{\theta z}}{\partial\phi}+\frac{\partial}{\partial\zeta}\left[(1+\lambda\rho)S_z\right]=0
\end{aligned}\right\}\quad(1.9)$$

The length c is to be identified with the length scale of the end loading in the θ-direction through the choice of the parameter γ. For instance, setting $\frac{1}{\gamma}=0$ reduces (1.7) and (1.9) to the proper forms for axially symmetric deformations as it should.

In order that the stresses (1.7) agree in order of λ with conventional results, we consider the theory which results from taking

$$\gamma=\sqrt{\lambda}\qquad(1.10)$$

With assumption (1.10), c is taken as $c=\sqrt{\frac{1}{2}ah}$, the same as the length scale in the z-direction. Thus, when the length scale of the end loading is different from $\sqrt{\frac{1}{2}ah}$, this theory gives results which may be innacurate due to the fact that $\partial/\partial\phi$ terms no longer enter the equations in the proper order in λ.

When the length scale of the end loading is shorter than $\sqrt{\frac{1}{2}ah}$, this would not seem important in the present theory, since the effect on the interior solution would be confined to a boundary layer near the end of thickness much smaller than $\sqrt{\frac{1}{2}ah}$ where we do not require the interior solution to represent accurately the true state of affairs anyway.

However, when the length scale of the end loading is greater than $\sqrt{\frac{1}{2}ah}$, the interior solution obtained by use of the present theory may be innaccurate for a region associated with the length scale $\sqrt{\frac{1}{2}ah}$. An investigation of the theory which results from taking $\gamma = 1$ and $c = a$ is therefore in order for its own sake and to determine the extent of applicability of the present theory. We do not pursue this question further here but go on to obtain the theory following from assumption (1.10).

Let us next non-dimensionalize the displacements so as to agree in order of λ with conventional results.

$$\left.\begin{aligned} u_r &= (1-\nu^2)\,\sigma\,\frac{a}{E}\,V_r \\ u_\theta &= \sqrt{\lambda}\,(1-\nu^2)\,\sigma\,\frac{a}{E}\,V_\theta\,, \qquad u_z = \sqrt{\lambda}\,(1-\nu^2)\,\sigma\,\frac{a}{E}\,V_z \end{aligned}\right\} \quad (1.11)$$

Introducing (1.11) together with (1.3), (1.4) and (1.7) into (1.2), we obtain the following non-dimensional stress-strain relations

$$\left.\begin{aligned}
(1-\nu^2)\frac{\partial V_r}{\partial \rho} &= \lambda \frac{E}{E_t}\left[-\nu_t\,(S_z+S_\theta)+\lambda\, S_r\right] \\
\frac{1-\nu^2}{1+\lambda\rho}\left[V_r+\frac{\partial V_\theta}{\partial \phi}\right] &= S_\theta-\nu S_z-\lambda\,\nu_n\, S_r \\
(1-\nu^2)\frac{\partial V_z}{\partial \varphi} &= S_z-\nu\, S_\theta-\lambda\,\nu_n\, S_r \\
(1-\nu^2)\left[\frac{1}{1+\lambda\rho}\frac{\partial V_r}{\partial \phi}+\frac{\partial V_\theta}{\partial \rho}-\frac{\lambda}{1+\lambda\rho}V_\theta\right] &= 2\lambda\,(1+\nu)\,I_n\, S_{r\theta} \\
(1-\nu^2)\left[\frac{\partial V_z}{\partial \rho}+\frac{\partial V_r}{\partial \varphi}\right] &= 2\lambda\,(1+\nu)\,I_n\, S_{rz} \\
(1-\nu^2)\left[\frac{\partial V_\theta}{\partial \varphi}+\frac{1}{1+\lambda\rho}\frac{\partial V_z}{\partial \phi}\right] &= 2\,(1+\nu)\, S_{z\theta}
\end{aligned}\right\} \qquad (1.12)$$

Here

$$\nu_n = \frac{\nu_t\, E}{E_t}, \qquad I_n = \frac{E}{2\,(1+\nu)\,G_t} \qquad (1.13)$$

Next, we arrange equations (1.9) and (1.12) in the order in which integration with respect to ρ will be possible.

$$
\left.
\begin{aligned}
\frac{\partial V_r}{\partial \rho} &= \frac{1}{1-\nu^2}\,\lambda\,\frac{E}{E_t}\left[-\nu_t\,(S_z + S_\theta) + \lambda\, S_r\right] \\
\frac{\partial V_z}{\partial \rho} &= -\frac{\partial V_r}{\partial \varsigma} + \lambda\,\frac{2\,I_n}{1-\nu}\,S_{rz} \\
\frac{\partial V_\theta}{\partial \rho} &= -\frac{1}{1+\lambda\rho}\,\frac{\partial V_r}{\partial \phi} + \frac{\lambda}{1+\lambda\rho}\,V_\theta + \lambda\,\frac{2\,I_n}{1-\nu}\,S_{r\theta} \\
S_\theta &= \frac{1}{1+\lambda\rho}\left[V_r + \frac{\partial V_\theta}{\partial \phi}\right] + \nu\,\frac{\partial V_z}{\partial \varsigma} + \lambda\,\frac{\nu_n}{1-\nu}\,S_r \\
S_z &= \frac{\nu}{1+\lambda\rho}\left[V_r + \frac{\partial V_\theta}{\partial \phi}\right] + \frac{\partial V_z}{\partial \varsigma} + \lambda\,\frac{\nu_n}{1-\nu}\,S_r \\
S_{\theta z} &= \frac{1}{2}\,(1-\nu)\left[\frac{\partial V_\theta}{\partial \varsigma} + \frac{1}{1+\lambda\rho}\,\frac{\partial V_z}{\partial \phi}\right] \\
\frac{\partial}{\partial \rho}\left[(1+\lambda\rho)\,S_{rz}\right] &= -\frac{\partial S_{\theta z}}{\partial \phi} - \frac{\partial}{\partial \varsigma}\left[(1+\lambda\rho)\,S_z\right] \\
\frac{\partial}{\partial \rho}\left[(1+\lambda\rho)\,S_{r\theta}\right] &= -\frac{\partial S_\theta}{\partial \phi} - \frac{\partial}{\partial \varsigma}\left[(1+\lambda\rho)\,S_{\theta z}\right] - \lambda\,S_{r\theta} \\
\frac{\partial}{\partial \rho}\left[(1+\lambda\rho)\,S_r\right] &= -\frac{\partial S_{r\theta}}{\partial \phi} - \frac{\partial}{\partial \varsigma}\left[(1+\lambda\rho)\,S_{rz}\right] + S_\theta
\end{aligned}
\right\}
\quad (1.14)
$$

2. Expansion with respect to small parameter. We assume that all quantities in (1.14) can be expanded at each point (ρ, ϕ, ς) in a power series in λ as follows.

$$\left.\begin{aligned} S &= S_0(\rho,\phi,\varsigma) + \lambda S_1(\rho,\phi,\varsigma) + \dots \\ V &= V_0(\rho,\phi,\varsigma) + \lambda V_1(\rho,\phi,\varsigma) + \dots \end{aligned}\right\} \quad (2.1)$$

We are interested in the solution for small λ and will be concerned only with the first two systems obtained by substituting (2.1) into (1.14) and equating coefficients of like powers of λ on both sides of the equations.

First system.

$$\left.\begin{aligned} \frac{\partial V_{r0}}{\partial \rho} &= 0 \\ \frac{\partial V_{z0}}{\partial \rho} &= -\frac{\partial V_{r0}}{\partial \varsigma} \\ \frac{\partial V_{\theta 0}}{\partial \rho} &= -\frac{\partial V_{r0}}{\partial \phi} \end{aligned}\right\} \quad (2.2)$$

$$\left.\begin{aligned}
S_{\theta o} &= \frac{\partial V_{\theta o}}{\partial \phi} + V_{ro} + \nu \frac{\partial V_{zo}}{\partial \varsigma} \\
S_{zo} &= \nu \left[\frac{\partial V_{\theta o}}{\partial \phi} + V_{ro} \right] + \frac{\partial V_{zo}}{\partial \varsigma} \\
S_{\theta z o} &= \frac{1}{2} (1-\nu) \left[\frac{\partial V_{\theta o}}{\partial \varsigma} + \frac{\partial V_{zo}}{\partial \phi} \right] \\
\frac{\partial S_{rzo}}{\partial \rho} &= - \frac{\partial S_{\theta z o}}{\partial \phi} - \frac{\partial S_{zo}}{\partial \varsigma} \\
\frac{\partial S_{r\theta o}}{\partial \rho} &= - \frac{\partial S_{\theta o}}{\partial \phi} - \frac{\partial S_{\theta z o}}{\partial \varsigma} \\
\frac{\partial S_{ro}}{\partial \rho} &= - \frac{\partial S_{r\theta o}}{\partial \phi} - \frac{\partial S_{rzo}}{\partial \varsigma} + S_{\theta o}
\end{aligned}\right\} \qquad (2.2)$$

Second system

$$
\left.
\begin{aligned}
\frac{\partial V_{r1}}{\partial \rho} &= -\frac{\nu_n}{1-\nu^2}\left(S_{z0}+S_{\theta 0}\right) \\
\frac{\partial V_{z1}}{\partial \rho} &= -\frac{\partial V_{r1}}{\partial \zeta} + \frac{2 I_n}{1-\nu} S_{rz0} \\
\frac{\partial V_{\theta 1}}{\partial \rho} &= -\frac{\partial V_{r1}}{\partial \phi} + \frac{\partial V_{r0}}{\partial \phi}\rho + V_{\theta 0} + \frac{2 I_n}{1-\nu} S_{r\theta 0} \\
S_{\theta 1} &= \frac{\partial V_{\theta 1}}{\partial \phi} + V_{r1} - \frac{\partial V_{\theta 0}}{\partial \phi}\rho - V_{r0}\rho + \nu \frac{\partial V_{z1}}{\partial \zeta} + \frac{\nu_n}{1-\nu} S_{r0} \\
S_{z1} &= \nu\left[\frac{\partial V_{\theta 1}}{\partial \phi} + V_{r1} - \frac{\partial V_{\theta 0}}{\partial \phi}\rho - V_{r0}\rho\right] + \frac{\partial V_{z1}}{\partial \zeta} + \frac{\nu_n}{1-\nu} S_{r0} \\
S_{\theta z1} &= \frac{1}{2}(1-\nu)\left[\frac{\partial V_{\theta 1}}{\partial \zeta} + \frac{\partial V_{z1}}{\partial \phi} - \frac{\partial V_{z0}}{\partial \phi}\rho\right] \\
\frac{\partial}{\partial \rho}\left(S_{rz1}+\rho S_{rz0}\right) &= -\frac{\partial S_{\theta z1}}{\partial \phi} - \frac{\partial}{\partial \zeta}\left(S_{z1}+\rho S_{z0}\right) \\
\frac{\partial}{\partial \rho}\left(S_{r\theta 1}+\rho S_{r\theta 0}\right) &= -\frac{\partial S_{\theta 1}}{\partial \phi} - \frac{\partial}{\partial \zeta}\left(S_{\theta z1}+\rho S_{\theta z0}\right) - S_{r\theta 0} \\
\frac{\partial}{\partial \rho}\left(S_{r1}+\rho S_{r0}\right) &= -\frac{\partial S_{r\theta 1}}{\partial \phi} - \frac{\partial}{\partial \zeta}\left(S_{rz1}+\rho S_{rz0}\right) + S_{\theta 1}
\end{aligned}
\right\} \quad (2.3)
$$

Systems (2.2) and (2.3) can now be integrated with respect to ρ in the same manner as was done in the axially symmetric problem*. To do this we introduce macroscopic functions $V_{ro}(\varphi, \phi)$, etc. and make use of a comma notation to indicate partial differentiation, i.e. $V_{ro,\phi^2} = \partial^2 V_{ro}(\varphi, \phi)/\partial \phi^2$. From the first system we obtain

$$
\left.
\begin{aligned}
V_{ro} &= V_{ro}(\varphi, \phi) \\
V_{zo} &= V_{zo}(\varphi, \phi) - \rho V_{ro,\varphi} \\
V_{\theta o} &= V_{\theta o}(\varphi, \phi) - \rho V_{ro,\phi} \\
S_{\theta o} &= V_{\theta o,\phi} + V_{ro} + \nu V_{zo,\varphi} - \left[V_{ro,\phi^2} + \nu V_{ro,\varphi^2}\right]\rho \\
S_{zo} &= \nu V_{\theta o,\phi} + \nu V_{ro} + V_{zo,\varphi} - \left[\nu V_{ro,\phi^2} + V_{ro,\varphi^2}\right]\rho \\
S_{\theta z o} &= \frac{1}{2}(1-\nu)\left[V_{\theta o,\varphi} + V_{zo,\phi} - 2 V_{ro,\varphi\phi}\rho\right]
\end{aligned}
\right\} \quad (2.4)
$$

*It is again clear from the form of (1.14) that the integration can be carried out in all higher systems.

$$\left.\begin{aligned}
S_{rz0} &= S_{rz0}(\xi,\phi) - \Big[\tfrac{1}{2}(1+\nu)V_{\theta 0,\xi\phi} + \tfrac{1}{2}(1-\nu)V_{z0,\phi^2} \\
&\quad + V_{z0,\xi^2} + \nu V_{r0,\xi}\Big]\rho + \tfrac{1}{2}\Delta V_{r0,\xi}\rho^2 \\
S_{r\theta 0} &= S_{r\theta 0}(\xi,\phi) - \Big[\tfrac{1}{2}(1+\nu)V_{z0,\xi\phi} + \tfrac{1}{2}(1-\nu)V_{\theta 0,\xi^2} \\
&\quad + V_{\theta 0,\phi^2} + V_{r0,\phi}\Big]\rho + \tfrac{1}{2}\Delta V_{r0,\phi}\rho^2 \\
S_{r0} &= S_{r0}(\xi,\phi) + \Big[-S_{r\theta 0,\phi} - S_{rz0,\xi} + V_{\theta 0,\phi} + V_{r0} + \nu V_{z0,\xi}\Big]\rho \\
&\quad + \Big[\Delta V_{\theta 0,\phi} + \Delta V_{z0,\xi}\Big]\tfrac{1}{2}\rho^2 - \tfrac{1}{6}\Delta^2 V_{r0}\rho^3
\end{aligned}\right\} \quad (2.5)$$

where

$$\Delta(\ ) = (\ )_{,\phi^2} + (\ )_{,\xi^2} \qquad (2.6)$$

The second system yields

$$V_{r1} = V_{r1}(\xi,\phi) - \frac{\nu_n}{1-\nu}\Big[V_{r0} + V_{\theta 0,\phi} + V_{z0,\xi}\Big]\rho + \frac{\nu_n}{2(1-\nu)}\Delta V_{r0}\rho^2 \qquad (2.7)$$

$$
\left.
\begin{aligned}
V_{z1} &= V_{z1}(\varphi,\phi) - V_{r1,\varphi}\rho + \frac{2I_n}{1-\nu} S_{rz0}\rho + \left[\frac{\nu_n - 2\nu I_n}{1-\nu} V_{r0,\varphi}\right. \\
&\quad \left. + \frac{\nu_n - (1+\nu)I_n}{1-\nu} V_{\theta 0,\varphi\phi} + \frac{\nu_n - 2I_n}{1-\nu} V_{z0,\varphi^2} - I_n V_{z0,\phi^2}\right]\frac{1}{2}\rho^2 \\
&\quad + \frac{-\nu_n + 2I_n}{6(1-\nu)} \Delta V_{r0,\varphi}\rho^3 \\
V_{\theta 1} &= V_{\theta 1}(\varphi,\phi) - V_{r1,\phi}\rho + \left[V_{\theta 0} + \frac{2I_n}{1-\nu} S_{r\theta 0}\right]\rho \\
&\quad + \left[\frac{\nu_n - 2I_n}{1-\nu} V_{r0,\phi} + \frac{\nu_n - 2I_n}{1-\nu} V_{\theta 0,\phi^2} + \frac{\nu_n - (1+\nu)I_n}{1-\nu} V_{z0,\varphi\phi}\right. \\
&\quad \left. - I_n V_{\theta 0,\varphi^2}\right]\frac{1}{2}\rho^2 + \frac{2I_n - \nu_n}{6(1-\nu)} \Delta V_{r0,\phi}\rho^3 \\
S_{\theta 1} &= V_{r1} + V_{\theta 1,\phi} + \nu V_{z1,\varphi} - \left[V_{r1,\phi^2} + \nu V_{r1,\varphi^2}\right]\rho + \frac{\nu_n}{1-\nu} S_{r0} \\
&\quad + \left[\frac{2I_n - \nu_n}{1-\nu} S_{r\theta 0,\phi} - \nu_n V_{z0,\varphi} - V_{r0} + \frac{2\nu I_n - \nu_n}{1-\nu} S_{rz0,\varphi}\right]\rho \\
&\quad + \left[\left(1 - \frac{I_n - \nu_n}{1-\nu}\right) V_{r0,\phi^2} + \frac{(1+\nu)\nu_n - 2\nu^2 I_n}{2(1-\nu)} V_{r0,\varphi^2} + \frac{\nu_n - I_n}{1-\nu} V_{\theta 0,\phi^3}\right. \\
&\quad + \frac{(1+\nu)\nu_n - (1+\nu^2)I_n}{2(1-\nu)} V_{\theta 0,\varphi^2\phi} + \frac{2\nu_n - (1+2\nu-\nu^2)I_n}{2(1-\nu)} V_{z0,\varphi\phi^2} \\
&\quad \left. + \frac{(1+\nu)\nu_n - 2\nu I_n}{2(1-\nu)} V_{z0,\varphi^3}\right]\rho^2 \\
&\quad + \left[\frac{I_n - \nu_n}{3(1-\nu)} \Delta V_{r0,\phi^2} + \frac{-(1+\nu)\nu_n + 2\nu I_n}{6(1-\nu)} \Delta V_{r0,\varphi^2}\right]\rho^3
\end{aligned}
\right\}
\quad (2.7)
$$

$$
\left.
\begin{aligned}
S_{z1} = {} & \nu V_{r1} + \nu V_{\theta 1,\phi} + V_{z1,\varphi} - \left[\nu V_{r1,\phi^2} + V_{r1,\varphi^2}\right]\rho + \frac{\nu_n}{1-\nu} S_{r0} \\
& + \left[\nu_n V_{\theta 0,\phi} + (\nu_n - \nu) V_{r0} + \frac{2\nu I_n - \nu_n}{1-\nu} S_{r\theta 0,\phi} + \frac{2 I_n - \nu_n}{1-\nu} S_{rz0,\varphi}\right]\rho \\
& + \left[\nu\left(1 - \frac{I_n - \nu_n}{1-\nu}\right) V_{r0,\phi^2} + \frac{(1+\nu)\nu_n - 2\nu I_n}{2(1-\nu)} V_{r0,\varphi^2} + \frac{(1+\nu)\nu_n - 2\nu I_n}{2(1-\nu)} V_{\theta 0,\phi^3}\right. \\
& \left. + \frac{2\nu_n - (1+2\nu-\nu^2) I_n}{2(1-\nu)} V_{\theta 0,\varphi^2\phi} + \frac{(1+\nu)\nu_n - (1+\nu^2) I_n}{2(1-\nu)} V_{z0,\varphi\phi^2} - \frac{I_n - \nu_n}{1-\nu} V_{z0,\varphi^3}\right]\rho^2 \\
& + \left[\frac{-(1+\nu)\nu_n + 2\nu I_n}{6(1-\nu)} \Delta V_{r0,\phi^2} + \frac{I_n - \nu_n}{3(1-\nu)} \Delta V_{r0,\varphi^2}\right]\rho^3 \\
\\
S_{\theta z1} = {} & \frac{1}{2}(1-\nu)\left[V_{\theta 1,\varphi} + V_{z1,\phi} - 2 V_{r1,\varphi\phi}\,\rho\right] \\
& + \left[\frac{1-\nu}{2}\left(V_{\theta 0,\varphi} - V_{z0,\phi}\right) + I_n\left(S_{r\theta 0,\varphi} + S_{rz0,\phi}\right)\right]\rho \\
& + \left[\left(1-\nu+\nu_n-(1+\nu)I_n\right) V_{r0,\varphi\phi} + \left(\nu_n - \frac{1}{2}(3+\nu) I_n\right) V_{\theta 0,\varphi\phi^2}\right. \\
& \left. + \left(\nu_n - \frac{1}{2}(3+\nu) I_n\right) V_{z0,\varphi^2\phi} - \frac{1}{2}(1-\nu) I_n \left(V_{\theta 0,\varphi^3} + V_{z0,\phi^3}\right)\right]\frac{1}{2}\rho^2 \\
& + \frac{1}{6}\left(-\nu_n + 2 I_n\right)\Delta V_{r0,\varphi\phi}\,\rho^3
\end{aligned}
\right\} \quad (2.7)
$$

$$
\left.
\begin{aligned}
s_{rz1} = {} & S_{rz1}(\varphi,\phi) - \left[\frac{1+\nu}{2} V_{\theta1,\varphi\phi} + \frac{1-\nu}{2} V_{z1,\phi^2} + V_{z1,\varphi^2} + \nu V_{r1,\varphi}\right]\rho \\
& + \frac{1}{2}\Delta V_{r1,\varphi}\rho^2 - \left[S_{rz0} + \frac{\nu_n}{1-\nu} S_{r0,\varphi}\right]\rho \\
& + \Big[\left(\frac{1+\nu}{2} - \nu_n\right) V_{\theta0,\varphi\phi} + \frac{3}{2}(1-\nu) V_{z0,\phi^2} + V_{z0,\varphi^2} + (2\nu - \nu_n) V_{r0,\varphi} \\
& \quad + \frac{\nu_n - 2I_n}{1-\nu} S_{rz0,\varphi^2} - I_n S_{rz0,\phi^2} + \frac{\nu_n - (1+\nu)I_n}{1-\nu} S_{r\theta0,\varphi\phi}\Big]\frac{1}{2}\rho^2 \\
& + \Big[\left(-2 + \frac{1}{2}\nu + \frac{-(1+\nu)\nu_n + (1+2\nu-\nu^2)I_n}{2(1-\nu)}\right) V_{r0,\varphi\phi^2} \\
& \quad + \left(-\frac{1}{2} + \frac{-(1+\nu)\nu_n + 2\nu I_n}{2(1-\nu)}\right) V_{r0,\varphi^3} + \frac{-4\nu_n + (3-\nu)(1+\nu)I_n}{4(1-\nu)} \Delta V_{\theta0,\varphi\phi} \\
& \quad + \frac{I_n - \nu_n}{1-\nu}\Delta V_{z0,\varphi^2} + \frac{1}{4}(1-\nu) I_n \Delta V_{z0,\phi^2}\Big]\frac{1}{3}\rho^3 \\
& - \frac{I_n - \nu_n}{12(1-\nu)} \Delta^2 V_{r0,\varphi}\rho^4 \\
s_{r\theta1} = {} & S_{r\theta1}(\varphi,\phi) - \left[\frac{1+\nu}{2} V_{z1,\varphi\phi} + \frac{1-\nu}{2} V_{\theta1,\varphi^2} + V_{\theta1,\phi^2} + V_{r1,\phi}\right]\rho \\
& + \frac{1}{2}\Delta V_{r1,\phi}\rho^2 - \left[\frac{\nu_n}{1-\nu} S_{r0,\phi} + 2 S_{r\theta0}\right]\rho \\
& + \Big[\frac{\nu_n - 2I_n}{1-\nu} S_{r\theta0,\phi^2} - I_n S_{r\theta0,\varphi^2} + \frac{\nu_n - (1+\nu)I_n}{1-\nu} S_{rz0,\varphi\phi} \\
& \quad + \left(\frac{3}{2}(1+\nu) + \nu_n\right) V_{z0,\varphi\phi} + 4 V_{r0,\phi} + \frac{1-\nu}{2} V_{\theta0,\varphi^2} + 3 V_{\theta0,\phi^2}\Big]\frac{1}{2}\rho^2
\end{aligned}
\right\} \quad (2.7)
$$

$$
\left.
\begin{aligned}
&+\Big[\Big(-3+\frac{I_n-\nu_n}{1-\nu}\Big)V_{r0,\phi^3}+\Big(-\frac{1}{2}(3+\nu)+\frac{-2\nu_n+(1+\nu^2)I_n}{2(1-\nu)}\Big)V_{r0,\xi^2\phi} \\
&\quad+\frac{1-\nu}{4}I_n\Delta V_{\theta 0,\xi^2}+\frac{I_n-\nu_n}{1-\nu}\Delta V_{\theta 0,\phi^2} \\
&\quad+\frac{-4\nu_n+(3-\nu)(1+\nu)I_n}{4(1-\nu)}\Delta V_{z0,\xi\phi}\Big]\frac{1}{3}\rho^3-\frac{I_n-\nu_n}{12(1-\nu)}\Delta^2 V_{r0,\phi}\rho^4 \\
s_{r1} &= S_{r1}(\xi,\phi)+\Big[-S_{r\theta 1,\phi}-S_{rz1,\xi}+V_{r1}+V_{\theta 1,\phi}+\nu V_{z1,\xi}\Big]\rho \\
&\quad+\Big[\Delta V_{z1,\xi}+\Delta V_{\theta 1,\phi}\Big]\frac{1}{2}\rho^2-\frac{1}{6}\Delta^2 V_{r1}\rho^3 \\
&\quad+\Big(-1+\frac{\nu_n}{1-\nu}\Big)S_{r0}\rho+\Big[\frac{\nu_n}{1-\nu}\Delta S_{r0}+\Big(4+\frac{2I_n-\nu_n}{1-\nu}\Big)S_{r\theta 0,\phi} \\
&\quad+\Big(2+\frac{2\nu I_n-\nu_n}{1-\nu}\Big)S_{rz0,\xi}-(\nu_n+2\nu)V_{z0,\xi}-3V_{r0}-2V_{\theta 0,\phi}\Big]\frac{1}{2}\rho^2 \\
&\quad+\Big[\frac{2I_n-\nu_n}{2(1-\nu)}\Big(\Delta S_{r\theta 0,\phi}+\Delta S_{rz0,\xi}\Big)+\Big(-1+\frac{(1+\nu)\nu_n-2\nu I_n}{2(1-\nu)}\Big)V_{z0,\xi^3} \\
&\quad+\Big(-\frac{5+\nu}{2}+\frac{(1+\nu)\nu_n-(1+2\nu-\nu^2)I_n}{2(1-\nu)}\Big)V_{z0,\xi\phi^2}-\Big(1+\frac{I_n-\nu_n}{1-\nu}\Big)V_{r0,\phi^2} \\
&\quad+\frac{\nu_n-\nu^2 I_n}{1-\nu}V_{r0,\xi^2}+\Big(\frac{-3+\nu}{2}+\frac{2\nu_n-(1+\nu^2)I_n}{2(1-\nu)}\Big)V_{\theta 0,\xi^2\phi}-\Big(3+\frac{I_n-\nu_n}{1-\nu}\Big)V_{\theta 0,\phi^3}\Big]\frac{1}{3}\rho^3 \\
&\quad+\Big[\Delta^2 V_{r0}+4\Delta V_{r0,\phi^2}-\frac{I_n-\nu_n}{1-\nu}\Big(\Delta^2 V_{\theta 0,\phi}+\Delta^2 V_{z0,\xi}\Big)\Big]\frac{1}{12}\rho^4 \\
&\quad+\frac{I_n-\nu_n}{60(1-\nu)}\Delta^3 V_{r0}\rho^5
\end{aligned}
\right\}\quad (2.7)
$$

We note that expressions (2.4), with the macroscopic quantities properly given, agree with what is obtained in usual shell theory. Expressions (2.5) and (2.7) are new.

3. Equations for macroscopic quantities. As in the axially symmetric problem we obtain equations for the macroscopic quantities by satisfying boundary conditions on $\rho = \pm 1$. They will later be found to agree with the results of the variational procedure.

We assume that the surfaces $r = a \pm h/2$ are free of stress,

$$r = a \pm h/2: \quad \sigma_r = \tau_{rz} = \tau_{r\theta} = 0 \tag{3.1}$$

or, in terms of dimensionless quantities,

$$\rho = \pm 1: \quad S_r = S_{rz} = S_{r\theta} = 0 \tag{3.2}$$

We replace conditions (3.2) by the following conditions on each approximation.

$$\rho = \pm 1: \; S_{rk} = S_{rzk} = S_{r\theta k} = 0, \quad k \geqslant 0 \tag{3.3}$$

In order that the stresses given by (2.5) satisfy (3.3) with k = 0 it is necessary (and sufficient) that the following relations hold.

$$\frac{1}{2}(1+\nu)V_{\theta o,\xi\phi}+\frac{1}{2}(1-\nu)V_{zo,\phi^2}+V_{zo,\xi^2}+\nu V_{ro,\xi} = 0 \tag{3.4}$$

$$\frac{1}{2}(1+\nu)V_{zo,\xi\phi}+\frac{1}{2}(1-\nu)V_{\theta o,\xi^2}+V_{\theta o,\phi^2}+V_{ro,\phi} = 0 \tag{3.5}$$

$$-S_{r\theta o,\phi}-S_{rzo,\xi}+V_{\theta o,\phi}+\nu V_{zo,\xi}+V_{ro}-\frac{1}{6}\Delta^2 V_{ro} = 0 \tag{3.6}$$

$$\left.\begin{aligned} S_{rzo} &= -\frac{1}{2}\Delta V_{ro,\xi} \\ S_{r\theta o} &= -\frac{1}{2}\Delta V_{ro,\phi} \\ S_{ro} &= -\frac{1}{2}\Delta\left[V_{\theta o,\phi}+V_{zo,\xi}\right] \end{aligned}\right\} \tag{3.7}$$

Using (3.7) we can eliminate $S_{r\theta o}$ and S_{rzo} from (3.6) to obtain

$$\frac{1}{3}\Delta^2 V_{ro}+V_{ro}+V_{\theta o,\phi}+\nu V_{zo,\xi} = 0 \tag{3.8}$$

Equations (3.4), (3.5) and (3.8) are a system of three partial differential equations with constant coefficients for V_{ro}, V_{zo} and $V_{\theta o}$. It will be shown in the next section that they are the non-dimensional versions of Donnell's equations [13] in the conventional theory of thin shells. Their solutions can be found by elementary means and have been discussed by several authors*. After a solution is found subject to suitable boundary conditions at $\rho = 0, \infty$, we can find S_{rzo}, $S_{r\theta o}$ and S_{ro} by (3.7) and then find the microscopic displacements and stresses, v_o and s_o, by use of (2.4) and (2.5).

Satisfaction of (3.3) with k=1 by (2.7) yields

$$\begin{aligned}
&\frac{1+\nu}{2} V_{\theta 1,\rho\phi} + \frac{1-\nu}{2} V_{z1,\phi^2} + V_{z1,\rho^2} + \nu V_{r1,\rho} \\
&\quad = -S_{rzo} - \frac{\nu_n}{1-\nu} S_{ro,\rho} + \frac{1}{3}\left[-2 + \frac{\nu}{2} + \frac{-(1+\nu)\nu_n + (1+2\nu-\nu^2)I_n}{2(1-\nu)}\right] V_{ro,\rho\phi^2} \\
&\qquad + \frac{1}{6}\left[-1 + \frac{-(1+\nu)\nu_n + 2\nu I_n}{1-\nu}\right] V_{ro,\rho^3} + \frac{-4\nu_n + (3-\nu)(1+\nu)I_n}{12(1-\nu)} \Delta V_{\theta o,\rho\phi} \\
&\qquad + \frac{I_n - \nu_n}{3(1-\nu)} \Delta V_{zo,\rho^2} + \frac{1}{12}(1-\nu) I_n \Delta V_{zo,\phi^2} \qquad (3.9)
\end{aligned}$$

---

*For references see [4] and [14].

$$\frac{1+\nu}{2} V_{z1,\xi\phi} + \frac{1-\nu}{2} V_{\theta 1,\xi^2} + V_{\theta 1,\phi^2} + V_{r1,\phi}$$

$$= -2 S_{r\theta 0} - \frac{\nu_n}{1-\nu} S_{r0,\phi} + \left[-1 + \frac{I_n - \nu_n}{3(1-\nu)}\right] V_{r0,\phi^3}$$

$$+ \frac{1}{6}\left[-(3+\nu) + \frac{-2\nu_n + (1+\nu^2) I_n}{1-\nu}\right] V_{r0,\xi^2\phi} + \frac{1}{12}(1-\nu) I_n \Delta V_{\theta 0,\xi^2}$$

$$+ \frac{I_n - \nu_n}{3(1-\nu)} \Delta V_{\theta 0,\phi^2} + \frac{-4\nu_n + (3-\nu)(1+\nu) I_n}{12(1-\nu)} \Delta V_{z0,\xi\phi} \qquad (3.10)$$

$$S_{r\theta 1,\phi} + S_{rz1,\xi} - V_{\theta 1,\phi} - \nu V_{z1,\xi} - V_{r1} + \frac{1}{6}\Delta^2 V_{r1}$$

$$= \left[-1 + \frac{\nu_n}{1-\nu}\right] S_{r0} + \frac{2I_n - \nu_n}{6(1-\nu)}\left[\Delta S_{r\theta 0,\phi} + \Delta S_{rz0,\xi}\right]$$

$$+ \frac{1}{6}\left[-(5+\nu) + \frac{(1+\nu)\nu_n - (1+2\nu-\nu^2) I_n}{1-\nu}\right] V_{z0,\xi\phi^2}$$

$$+ \frac{1}{3}\left[-1 + \frac{(1+\nu)\nu_n - 2\nu I_n}{2(1-\nu)}\right] V_{z0,\xi^3} - \frac{1}{3}\left[1 + \frac{I_n - \nu_n}{1-\nu}\right] V_{r0,\phi^2}$$

$$+ \frac{\nu_n - \nu^2 I_n}{3(1-\nu)} V_{r0,\xi^2} + \frac{1}{6}\left[-3 + \nu + \frac{2\nu_n - (1+\nu^2) I_n}{1-\nu}\right] V_{\theta 0,\xi^2\phi}$$

$$- \left[1 + \frac{I_n - \nu_n}{3(1-\nu)}\right] V_{\theta 0,\phi^3} + \frac{I_n - \nu_n}{60(1-\nu)} \Delta^3 V_{r0} \qquad (3.11)$$

$$S_{rz1} = -\frac{1}{2}\Delta V_{r1,\xi} - \frac{1}{2}\left[\frac{1+\nu}{2} - \nu_n\right] V_{\theta 0,\xi\phi} - \frac{3}{4}(1-\nu)\, V_{z0,\phi^2}$$

$$-\frac{1}{2} V_{z0,\xi^2} - \left(\nu - \frac{1}{2}\nu_n\right) V_{r0,\xi} + \frac{I_n - \nu_n}{12(1-\nu)}\,\Delta^2 V_{r0,\xi}$$

$$+\frac{2I_n - \nu_n}{2(1-\nu)} S_{rz0,\xi^2} + \frac{1}{2} I_n S_{rz0,\phi^2} + \frac{-\nu_n + (1+\nu) I_n}{2(1-\nu)} S_{r\theta 0,\xi\phi} \qquad (3.12)$$

$$S_{r\theta 1} = -\frac{1}{2}\Delta V_{r1,\phi} - \frac{1}{2}\left[\frac{3}{2}(1+\nu) + \nu_n\right] V_{z0,\xi\phi} - \frac{1-\nu}{4} V_{\theta 0,\xi^2}$$

$$-\frac{3}{2} V_{\theta 0,\phi^2} - 2 V_{r0,\phi} + \frac{I_n - \nu_n}{12(1-\nu)}\,\Delta^2 V_{r0,\phi}$$

$$+\frac{2I_n - \nu_n}{2(1-\nu)} S_{r\theta 0,\phi^2} + \frac{1}{2} I_n S_{r\theta 0,\xi^2} + \frac{-\nu_n + (1+\nu) I_n}{2(1-\nu)} S_{rz0,\xi\phi} \qquad (3.13)$$

$$S_{r1} = -\frac{1}{2}\Delta\left[V_{z1,\xi} + V_{\theta 1,\phi}\right] - \frac{\nu_n}{2(1-\nu)}\Delta S_{r0} - \left[2 + \frac{2I_n - \nu_n}{2(1-\nu)}\right] S_{r\theta 0,\phi}$$

$$-\left[1 + \frac{2\nu I_n - \nu_n}{2(1-\nu)}\right] S_{rz0,\xi} + \left(\nu + \frac{1}{2}\nu_n\right) V_{z0,\xi} + V_{\theta 0,\phi} + \frac{3}{2} V_{r0}$$

$$-\frac{1}{12}\Delta^2 V_{r0} - \frac{1}{3}\Delta V_{r0,\phi^2} + \frac{I_n - \nu_n}{12(1-\nu)}\,\Delta^2\left[V_{z0,\xi} + V_{\theta 0,\phi}\right] \qquad (3.14)$$

Equation (3.9) can be simplified by substituting S_{rzo} and S_{ro} from (3.7) and then using (3.4) to eliminate V_{zo,φ^2} from the result.

$$\frac{1+\nu}{2} V_{\theta 1,\varphi\phi} + \frac{1-\nu}{2} V_{z1,\phi^2} + V_{z1,\varphi^2} + \nu V_{r1,\varphi}$$

$$= \left[\frac{1}{3} - \frac{(1+2\nu)\nu_n}{6(1-\nu)}\right] \Delta V_{r0,\varphi} - \frac{1}{6}\left[3-\nu-(1+\nu)I_n\right] V_{r0,\varphi\phi^2}$$

$$+ \frac{1}{12}\left[\nu_n + (1+\nu)I_n\right] \Delta\left[V_{\theta 0,\varphi\phi} - V_{z0,\phi^2}\right] \qquad (3.15)$$

In a similar manner, using (3.7) and (3.5), equation (3.10) can be written as

$$\frac{1+\nu}{2} V_{z1,\varphi\phi} + \frac{1-\nu}{2} V_{\theta 1,\varphi^2} + V_{\theta 1,\phi^2} + V_{r1,\phi}$$

$$= -\frac{\nu_n}{2(1-\nu)} \Delta V_{r0,\phi} + \frac{1}{6}\left[3-\nu-(1+\nu)I_n\right] V_{r0,\varphi^2\phi}$$

$$+ \frac{1}{12}\left[\nu_n + (1+\nu)I_n\right] \Delta\left[V_{z0,\varphi\phi} - V_{\theta 0,\varphi^2}\right] \qquad (3.16)$$

Using expressions (3.7), (3.12) and (3.13) we can eliminate the macroscopic stress quantities from (3.11). If, furthermore, we use (3.8) to eliminate $\Delta^3 V_{ro}$ and (3.4) and (3.5) to eliminate V_{zo,ζ^3} and $V_{\theta o,\phi^3}$, we obtain the following more simple expression.

$$\frac{1}{3}\Delta^2 V_{r1} + V_{r1} + V_{\theta 1,\phi} + \nu V_{z1,\zeta}$$

$$= -\left[\frac{2}{3}+\frac{\nu_m}{2(1-\nu)}\right]V_{ro,\phi^2} + \left[-\frac{\nu}{3} - \frac{(4+25\nu-14\nu^2)\nu_m}{30(1-\nu)} + \frac{4}{5}(1+\nu)I_n\right]V_{ro,\zeta^2}$$

$$+\left[\frac{1}{3}(1-\nu)+\frac{1}{30}(8-7\nu)\nu_m+\frac{2}{5}(1+\nu)I_n\right]\left[V_{\theta o,\zeta^2\phi} - V_{zo,\zeta\phi^2}\right] \qquad (3.17)$$

System (3.15), (3.16) and (3.17) are three partial differential equations for the macroscopic displacements V_{z1}, V_{r1}, $V_{\theta 1}$. The right sides are given by the first approximation. Once the solution is found, we can determine S_{rz1}, $S_{r\theta 1}$ and S_{r1} with (3.12), (3.13) and (3.14) and then determine microscopic displacements and stresses with (2.7).

We note that the left sides of the equations of the second approximation are the same as in the first approximation. In fact, it is apparent that the k+1st system of equations for macroscopic displacements is of the form

$$
\left.\begin{aligned}
\frac{1+\nu}{2} V_{\theta k,\xi\phi} + \frac{1-\nu}{2} V_{zk,\phi^2} + V_{zk,\xi^2} + \nu V_{rk,\xi} &= G_k \\
\frac{1+\nu}{2} V_{zk,\xi\phi} + \frac{1-\nu}{2} V_{\theta k,\xi^2} + V_{\theta k,\phi^2} + V_{rk,\phi} &= H_k \\
\frac{1}{3}\Delta^2 V_{rk} + V_{rk} + V_{\theta k,\phi} + \nu V_{zk,\xi} &= J_k
\end{aligned}\right\} \quad (3.18)
$$

where G_k, H_k and J_k are given in terms of macroscopic displacement functions of preceding approximations. It is interesting that system (3.18) takes the form of dimensionless Donnell's equations with surface loading terms.

Before going to a variational formulation of the problem, we take up some other topics.

4. Donnell's equations. L.H. Donnell [13] has **derived** a system of equations to analyze the stability of cylindrical shells by an order of magnitude consideration of the theory given by Love [1]. The linearized Donnell system ([13], equations (8)) can be written as follows,

$$
\left.\begin{aligned}
&\frac{\partial^2 U_z}{\partial z^2} + \frac{1-\nu}{2a^2}\frac{\partial^2 U_z}{\partial \theta^2} + \frac{1+\nu}{2a}\frac{\partial^2 U_\theta}{\partial z \partial \theta} + \frac{\nu}{a}\frac{\partial U_r}{\partial z} = 0 \\
&\frac{1}{a^2}\frac{\partial^2 U_\theta}{\partial \theta^2} + \frac{1-\nu}{2}\frac{\partial^2 U_\theta}{\partial z^2} + \frac{1+\nu}{2a}\frac{\partial^2 U_z}{\partial z \partial \theta} + \frac{1}{a^2}\frac{\partial U_r}{\partial \theta} = 0 \\
&\frac{h^2}{12}\left(\frac{\partial^2}{\partial z^2} + \frac{1}{a^2}\frac{\partial^2}{\partial \theta^2}\right)^2 U_r + \frac{1}{a}\left(\frac{\partial U_\theta}{a\partial \theta} + \nu\frac{\partial U_z}{\partial z} + \frac{U_r}{a}\right) = 0
\end{aligned}\right\} \quad (4.1)
$$

where (U_z, U_θ, U_r) is the displacement vector of the middle surface and the other quantities are the same as here.

Non-dimensionalizing (4.1) in the manner of section 1 gives

$$\left.\begin{aligned} \frac{1+\nu}{2} V_{\theta,\varphi\phi} + \frac{1-\nu}{2} V_{z,\phi^2} + V_{z,\varphi^2} + \nu V_{r,\varphi} &= 0 \\ \frac{1+\nu}{2} V_{z,\varphi\phi} + \frac{1-\nu}{2} V_{\theta,\varphi^2} + V_{\theta,\phi^2} + V_{r,\phi} &= 0 \\ \frac{1}{3}\Delta^2 V_r + V_r + V_{\theta,\phi} + \nu V_{z,\varphi} &= 0 \end{aligned}\right\} \qquad (4.2)$$

With the dimensionless middle surface displacement (V_z, V_θ, V_r) identified with $(V_{zo}, V_{\theta o}, V_{ro})$, system (4.2) agrees with the system for the first approximation, (3.4), (3.5) and (3.8).

Therefore, it seems that, when stresses and displacements are taken with the order in λ given by (1.7) with $\gamma = \sqrt{\lambda}$ and (1.11) (which are the same orders as in conventional theory), the correct asymptotic limit for small λ of the three-dimensional equations is given by Donnell's equations.

In this regard we mention the equations developed by W. Flügge [10] on the basis of Love's theory [1] (see also [11] and [12]). These equations contain terms in addition to those present in Donnell's equations and are much more difficult to solve. On the basis of the present results it appears that these additional terms are of the same order as neglected terms arising from transverse effects and should not be included in a first approximation.

The superfluity of these terms has been demonstrated by a numerical comparison of results obtained by use of Flügge's and Donnell's equations by N.J. Höff [14]. Hoff shows that for all end loadings of interest here and for a practical range of values of λ the results of the two systems of equations agree satisfactorily for most engineering purposes. The agreement is satisfactory except in the case of pure bending which does not interest us here.

5. End conditions. Conventional stress resultants and couples are defined by

$$\left.\begin{aligned}
N_z &= \int_{a-h/2}^{a+h/2} \sigma_r \frac{r}{a}\, dr \\
M_z &= \int_{a-h/2}^{a+h/2} \sigma_z \frac{r}{a} (r-a)\, dr \\
Q_z &= \int_{a-h/2}^{a+h/2} \tau_{rz} \frac{r}{a}\, dr \\
M_{\theta z} &= \int_{a-h/2}^{a+h/2} \tau_{\theta z} \frac{r}{a} (r-a)\, dr \\
N_{\theta z} &= \int_{a-h/2}^{a+h/2} \tau_{\theta z} \frac{r}{a}\, dr
\end{aligned}\right\} \qquad (5.1)$$

Similar quantities associated with the Θ-direction are also defined, but these do not interest us here.

With the aid of dimensionless quantitites defined in section 1, we can write (5.1) in dimensionless form as follows.

$$\left.\begin{aligned} N_z &= \sigma a \lambda \left[N_{z0} + \lambda N_{z1} + \dots \right] \\ M_z &= \sigma a^2 \lambda^2 \left[M_{z0} + \lambda M_{z1} + \cdots \right] \\ Q_z &= \sigma a \lambda^{3/2} \left[Q_{z0} + \lambda Q_{z1} + \dots \right] \\ N_{\theta z} &= \sigma a \lambda \left[N_{\theta z0} + \lambda N_{\theta z1} + \dots \right] \\ M_{\theta z} &= \sigma a^2 \lambda^2 \left[M_{\theta z0} + \lambda M_{\theta z1} + \cdots \right] \end{aligned}\right\} \qquad (5.2)$$

Here

$$\left.\begin{aligned}
N_{z0} &= \int_{-1}^{1} s_{z0}\, d\rho, \quad N_{z1} = \int_{-1}^{1} (s_{z1} + \rho s_{z0})\, d\rho, \ \ldots \\
M_{z0} &= \int_{-1}^{1} s_{z0}\,\rho\, d\rho, \quad M_{z1} = \int_{-1}^{1} (s_{z1} + \rho s_{z0})\,\rho\, d\rho, \ \ldots \\
Q_{z0} &= \int_{-1}^{1} s_{rz0}\, d\rho, \quad Q_{z1} = \int_{-1}^{1} (s_{rz1} + \rho s_{rz0})\, d\rho, \ \ldots \\
N_{\theta z0} &= \int_{-1}^{1} s_{\theta z0}\, d\rho, \quad N_{\theta z1} = \int_{-1}^{1} (s_{\theta z1} + \rho s_{\theta z0})\, d\rho, \ \ldots \\
M_{\theta z0} &= \int_{-1}^{1} s_{\theta z0}\,\rho\, d\rho, \quad M_{\theta z1} = \int_{-1}^{1} (s_{\theta z1} + \rho s_{\theta z0})\,\rho\, d\rho, \ \ldots
\end{aligned}\right\} \quad (5.3)$$

Now, assume that the following boundary conditions are given at $z = 0$ and $z = \infty$.

$$z = 0: \quad \sigma_z = \bar{\sigma}_z(r,\theta), \quad \tau_{rz} = \bar{\tau}_{rz}(r,\theta), \quad \tau_{\theta z} = \bar{\tau}_{\theta z}(r,\theta) \qquad (5.4)$$

$$z = \infty: \quad \sigma_z = \tau_{rz} = \tau_{\theta z} = 0 \qquad (5.5)$$

Let us introduce dimensionless end stresses by setting

$$\bar{\sigma}_z = \sigma \bar{S}_z, \quad \bar{\tau}_{rz} = \sqrt{\lambda}\, \sigma \bar{S}_{rz}, \quad \bar{\tau}_{\theta z} = \sigma \bar{S}_{\theta z} \qquad (5.6)$$

Then, non-dimensional end conditions corresponding to (5.4) and (5.5) are

$$\zeta = 0 : \quad S_z = \bar{S}_z(\rho,\phi), \; S_{rz} = \bar{S}_{rz}(\rho,\phi), \; S_{\theta z} = \bar{S}_{\theta z}(\rho,\phi) \qquad (5.7)$$

$$\zeta = \infty : \quad S_z = S_{rz} = S_{\theta z} = 0 \qquad (5.8)$$

In addition, we assume that the surfaces $r = a \pm h/2$ are free of stress, as in section 3.

$$r = a \pm h/2 : \quad \sigma_r = \tau_{rz} = \tau_{r\theta} = 0 \qquad (5.9)$$

In non-dimensional form, (5.9) becomes

$$\rho = \pm 1 : \quad S_r = S_{rz} = S_{r\theta} = 0 \qquad (5.10)$$

Of course, conditions (5.4) must be so restricted that with (5.5) and (5.9) the entire shell is in static equilibrium.

Using (5.1) and (5.6) we write given stress resultants and moments at $z = 0$ as follows.

$$\left.\begin{aligned}
\overline{N}_z &= \int_{a-h/2}^{a+h/2} \overline{\sigma}_z \frac{r}{a}\, dr = \sigma a \lambda \int_{-1}^{1} \overline{s}_z (1+\lambda\rho)\, d\rho \\
\overline{M}_z &= \int_{a-h/2}^{a+h/2} \overline{\sigma}_z \frac{r}{a} (r-a)\, dr = \sigma a^2 \lambda^2 \int_{-1}^{1} \overline{s}_z (1+\lambda\rho)\rho\, d\rho \\
\overline{Q}_z &= \int_{a-h/2}^{a+h/2} \overline{\tau}_{rz} \frac{r}{a}\, dr = \sigma a \lambda^{3/2} \int_{-1}^{1} \overline{s}_{rz} (1+\lambda\rho)\, d\rho \\
\overline{N}_{\theta z} &= \int_{a-h/2}^{a+h/2} \overline{\tau}_{\theta z} \frac{r}{a}\, dr = \sigma a \lambda \int_{-1}^{1} \overline{s}_{\theta z} (1+\lambda\rho)\, d\rho \\
\overline{M}_{\theta z} &= \int_{a-h/2}^{a+h/2} \overline{\tau}_{\theta z} \frac{r}{a} (r-a)\, dr = \sigma a^2 \lambda^2 \int_{-1}^{1} \overline{s}_{\theta z} (1+\lambda\rho)\rho\, d\rho
\end{aligned}\right\} \quad (5.11)$$

We now replace $\overline{s_z}$, $\overline{s_{rz}}$ and $\overline{s_{\theta z}}$ by statically equivalent stresses in order to simplify the variational procedure to follow. As in the axially symmetric problem the replacement is justified on the grounds that all statically equivalent end loadings have the same effect on the interior solution outside of the St. Venant boundary layer. Thus, we replace conditions (5.7) at $\xi = 0$ by

$$\xi = 0: \quad s_z = \hat{s}_z, \quad s_{rz} = \hat{s}_{rz}, \quad s_{\theta z} = \hat{s}_{\theta z} \qquad (5.12)$$

where

$$\left.\begin{aligned}
\hat{S}_z &= \frac{1}{2}\,\frac{1}{1+\lambda\rho}\,\frac{\overline{N}_z(\theta)}{\sigma a\lambda} + \frac{3}{2}\,\frac{\rho}{1+\lambda\rho}\,\frac{\overline{M}_z(\theta)}{\sigma a^2\lambda^2} \\
\hat{S}_{rz} &= \frac{3}{4}\,\frac{1-\rho^2}{1+\lambda\rho}\,\frac{\overline{Q}_z(\theta)}{\sigma a\lambda^{3/2}} \\
\hat{S}_{\theta z} &= \frac{1}{2}\,\frac{1}{1+\lambda\rho}\,\frac{\overline{N}_{\theta z}(\theta)}{\sigma a\lambda} + \frac{3}{2}\,\frac{\rho}{1+\lambda\rho}\,\frac{\overline{M}_{\theta z}(\theta)}{\sigma a^2\lambda^2}
\end{aligned}\right\} \quad (5.13)$$

Substitution of $\hat{s}_z$ for $\overline{s_z}$, etc.,in (5.11) shows the two end loadings are indeed statically equivalent.

With end conditions given by (5.12) and (5.13) we are in a position to specify σ. Consider the five problems defined by taking quantities in (5.13) as follows.

$$\left.\begin{array}{lll}
\text{(i)} & \overline{N}_z \text{ given}, & \overline{M}_z = \overline{Q}_z = \overline{N}_{\theta z} = \overline{M}_{\theta z} = 0 \\
\text{(ii)} & \overline{M}_z \text{ given}, & \overline{N}_z = \overline{Q}_z = \overline{N}_{\theta z} = \overline{M}_{\theta z} = 0 \\
\text{(iii)} & \overline{Q}_z \text{ given}, & \overline{N}_z = \overline{M}_z = \overline{N}_{\theta z} = \overline{M}_{\theta z} = 0 \\
\text{(iv)} & \overline{N}_{\theta z} \text{ given}, & \overline{N}_z = \overline{M}_z = \overline{Q}_z = \overline{M}_{\theta z} = 0 \\
\text{(v)} & \overline{M}_{\theta z} \text{ given}, & \overline{N}_z = \overline{M}_z = \overline{Q}_z = \overline{N}_{\theta z} = 0
\end{array}\right\} \quad (5.14)$$

The solution for arbitrary $\overline{N_z}$, $\overline{M_z}$, etc., can be found by superposition of solutions to these five problems. In order that $\hat{s}_z$, $\hat{s}_{rz}$ and $\hat{s}_{\theta z}$ be O(1) we define σ for each of the problems as follows.

$$\left.\begin{aligned}
&(i)\ \sigma = \max\left|\frac{\overline{N}_z(\theta)}{a\lambda}\right|, \quad (ii)\ \sigma = \max\left|\frac{\overline{M}_z(\theta)}{a^2\lambda^2}\right| \\
&(iii)\ \sigma = \max\left|\frac{\overline{Q}_z(\theta)}{a\ \lambda^{3/2}}\right|, \quad (iv)\ \sigma = \max\left|\frac{\overline{N}_{\theta z}(\theta)}{a\lambda}\right| \\
&(v)\ \sigma = \max\left|\frac{\overline{M}_{\theta z}(\theta)}{a^2\lambda^2}\right|
\end{aligned}\right\} \quad (5.15)$$

Thus, (5.13) can be written as

$$\left.\begin{aligned}
\hat{S}_z &= \frac{1}{2}\,\frac{1}{1+\lambda\rho}\,\overline{n}_z + \frac{3}{2}\,\frac{\rho}{1+\lambda\rho}\,\overline{m}_z \\
\hat{S}_{rz} &= \frac{3}{4}\,\frac{1-\rho^2}{1+\lambda\rho}\,\overline{q}_z \\
\hat{S}_{\theta z} &= \frac{1}{2}\,\frac{1}{1+\lambda\rho}\,\overline{n}_{\theta z} + \frac{3}{2}\,\frac{\rho}{1+\lambda\rho}\,\overline{m}_{\theta z}
\end{aligned}\right\} \quad (5.16)$$

where

$$\left.\begin{aligned}
&(i)\ \bar{n}_z = \frac{\bar{N}_z(\theta)}{\max.|\bar{N}_z|}, \quad \bar{m}_z = \bar{q}_z = \bar{n}_{\theta z} = \bar{m}_{\theta z} = 0 \\
&(ii)\ \bar{m}_z = \frac{\bar{M}_z(\theta)}{\max.|\bar{M}_z|}, \quad \bar{n}_z = \bar{q}_z = \bar{n}_{\theta z} = \bar{m}_{\theta z} = 0 \\
&(iii)\ \bar{q}_z = \frac{\bar{Q}_z(\theta)}{\max.|\bar{Q}_z|}, \quad \bar{n}_z = \bar{m}_z = \bar{n}_{\theta z} = \bar{m}_{\theta z} = 0 \\
&(iv)\ \bar{n}_{\theta z} = \frac{\bar{N}_{\theta z}(\theta)}{\max.|\bar{N}_{\theta z}|}, \quad \bar{n}_z = \bar{m}_z = \bar{q}_z = \bar{m}_{\theta z} = 0 \\
&(v)\ \bar{m}_{\theta z} = \frac{\bar{M}_{\theta z}(\theta)}{\max.|\bar{M}_{\theta z}|}, \quad \bar{n}_z = \bar{m}_z = \bar{q}_z = \bar{n}_{\theta z} = 0
\end{aligned}\right\} \quad (5.17)$$

6. Love's boundary conditions. In conventional shell theory it is found that the solution can not, in general, take on the values of all five given stress resultants and couples defined by (5.11). The order of the system of differential equations allows one to specify only four conditions and these are usually taken as those given by Love ([1], page 537). In terms of the quantitites used in this paper Love's boundary conditions at $z = 0$ are

$$N_z = \overline{N}_z \ , \quad M_z = \overline{M}_z \tag{6.1}$$

$$N_{\theta z} + \frac{1}{a} M_{\theta z} = \overline{N}_{\theta z} + \frac{1}{a} \overline{M}_{\theta z} \tag{6.2}$$

$$Q_z + \frac{1}{a} \frac{\partial M_{\theta z}}{\partial \theta} = \overline{Q}_z + \frac{1}{a} \frac{d \overline{M}_{\theta z}}{d \theta} \tag{6.3}$$

In conditions (6.2) and (6.3) forces and moments are combined into statically equivalent forces along the edge. In particular, the combination in (6.3) corresponds to the resultant shearing force in the Kirchhoff conditions for a flat plate.

In view of the fact that the relevant macroscopic differential equations for each approximation are the conventional Donnell equations, we expect that appropriate boundary conditions for these equations can be obtained from Love's conditions by order of magnitude considerations. Non-dimensionalizing by use of (5.2) we obtain

$$\left.\begin{aligned}
\xi = 0:\quad & N_{z0} + \lambda N_{z1} + \dots && = \frac{\overline{N}_z}{\sigma a \lambda} \\
& M_{z0} + \lambda M_{z1} + \dots && = \frac{\overline{M}_z}{\sigma a^2 \lambda^2} \\
& N_{\theta z0} + \lambda \left[N_{\theta z1} + M_{\theta z0} \right] + \dots && = \frac{\overline{N}_{\theta z} + \frac{1}{a}\overline{M}_{\theta z}}{\sigma a \lambda} \\
& Q_{z0} + \frac{\partial}{\partial \phi} M_{\theta z0} + \lambda \left[Q_{z1} + \frac{\partial}{\partial \phi} M_{\theta z1} \right] + \dots && = \frac{\overline{Q}_z + \frac{1}{a}\frac{d\overline{M}_{\theta z}}{d\theta}}{\sigma a \lambda^{3/2}}
\end{aligned}\right\} \quad (6.4)$$

If one considers the problem as split into the five defined by (5.14) and assigns σ by (5.15), then it seems appropriate to assign boundary conditions to each approximation as follows.

$$\xi = 0:\quad k = 0:\quad \left.\begin{aligned}
& N_{z0} = \overline{n}_z ,\; M_{z0} = \overline{m}_z ,\; N_{\theta z0} = \overline{n}_{\theta z} \\
& Q_{z0} + \frac{\partial}{\partial \phi} M_{\theta z0} = \overline{q}_z + \frac{d}{d\phi} \overline{m}_{\theta z}
\end{aligned}\right\} \quad (6.5)$$

$$k = 1:\quad \left.\begin{aligned}
& N_{z1} = 0,\; M_{z1} = 0,\; N_{\theta z1} + M_{\theta z0} = \overline{m}_{\theta z} \\
& Q_{z1} + \frac{\partial}{\partial \phi} M_{\theta z1} = 0
\end{aligned}\right\} \quad (6.6)$$

$$\left.\begin{aligned} k>1:\quad & N_{zk} = 0,\; M_{zk} = 0,\; N_{\theta zk} + M_{\theta zk-1} = 0 \\ & Q_{zk} + \frac{\partial}{\partial \phi} M_{\theta zk} = 0 \end{aligned}\right\} \qquad (6.7)$$

Boundary conditions at $\phi = \infty$, if the shell is to be stress free there, are the same as (6.5), (6.6) and (6.7) with right sides zero.

Note that the third boundary condition of (6.5) for the first approximation is somewhat simpler than the corresponding condition given by Love, (6.2), in that the bending moment terms are omitted. We conclude that order of magnitude considerations allow us to replace (6.2) with

$$N_{\theta z} = \overline{N}_{\theta z} \qquad (6.2a)$$

for use with the conventional Donnell equations (4.1).

Let us now see by use of the variational principle how (6.5), (6.6) and (6.7) follow from the assumed conditions (5.8), (5.10) and (5.12).

7. Application of variational principle. We note that the boundary conditions (6.5)-(6.7), which we seek to verify,

do not explicitly involve the elastic parameters. This is fitting since they represent the conditions for static equilibrium of forces and moments at the edge. This leads us to assume that the end conditions to be derived now will not depend explicitly on the elastic parameters when they are written in terms of stress resultants and couples. A check of Chapter I shows that this assumption is indeed true in the axially symmetric problem.

Therefore, in the interest of simplifying the calculations to follow, let us find boundary conditions for a material with no transverse deformation and with $\nu = 0$.

$$I_n = \nu_n = \frac{E}{E_t} = \nu = 0 \tag{7.1}$$

Although the governing equations of section 3 can not with (7.1) be verified in their entirety, we expect to obtain boundary conditions in the form (6.5)-(6.7), which we then assume hold for arbitrary values of the elastic parameters.

With (7.1), the expression to be varied takes the following form.

$$I = \int_0^{2\pi}\int_0^{\infty}\int_{a-h/2}^{a+h/2}\Big\{\frac{\partial u_r}{\partial r}\sigma_r + \frac{1}{r}\Big[\frac{\partial u_\theta}{\partial \theta} + u_r\Big]\sigma_\theta + \frac{\partial u_z}{\partial z}\sigma_z$$

$$+\Big[\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\Big]\tau_{r\theta} + \Big[\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\Big]\tau_{rz}$$

$$+\Big[\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\Big]\tau_{\theta z} - \frac{1}{2E}\Big[\sigma_\theta^2 + \sigma_z^2 + 2\tau_{\theta z}^2\Big]\Big\} r\,dr\,dz\,d\theta$$

$$+\int_0^{2\pi}\int_{a-h/2}^{a+h/2}\Big[\hat{\sigma}_z u_z + \hat{\tau}_{zr} u_r + \hat{\tau}_{\theta z} u_\theta\Big]^{z=0} r\,dr\,d\theta \qquad (7.2)$$

It can be shown by the usual methods of the variational calculus that for (7.2) to take on a stationary value it is necessary for differential equations (1.1) and (1.2) to be satisfied. In addition, the following natural boundary conditions are obtained.

$$\left.\begin{aligned} r = a \pm h/2:&\quad \sigma_r = \tau_{rz} = \tau_{r\theta} = 0 \\ z = 0:&\quad \sigma_z = \hat{\sigma}_z\,,\ \tau_{rz} = \hat{\tau}_{rz}\,,\ \tau_{\theta z} = \hat{\tau}_{\theta z} \\ z = \infty:&\quad \sigma_z = \tau_{rz} = \tau_{\theta z} = 0 \end{aligned}\right\} \qquad (7.3)$$

Let us write (7.2) in non-dimensional form and assume end stresses in the form (5.17).

$$
\begin{aligned}
I = \frac{\lambda^2\sigma^2 d^3}{E}\int_0^{\frac{2\pi}{\sqrt{\lambda}}}\int_0^{\infty}\int_{-1}^{1}\Big\{ & \frac{\partial V_r}{\partial\rho} S_r + \frac{1}{1+\lambda\rho}\left[\frac{\partial V_\theta}{\partial\phi} + V_r\right] S_\theta \\
& + \frac{\partial V_z}{\partial\varsigma} S_z + \left[\frac{1}{1+\lambda\rho}\frac{\partial V_r}{\partial\phi} + \frac{\partial V_\theta}{\partial\rho} - \lambda\frac{V_\theta}{1+\lambda\rho}\right] S_{r\theta} \\
& + \left[\frac{\partial V_z}{\partial\rho} + \frac{\partial V_r}{\partial\varsigma}\right] S_{rz} + \left[\frac{\partial V_\theta}{\partial\varsigma} + \frac{1}{1+\lambda\rho}\frac{\partial V_z}{\partial\phi}\right] S_{\theta z} \\
& - \frac{1}{2}\left[S_\theta^2 + S_z^2 + 2S_{\theta z}^2\right]\Big\}(1+\lambda\rho)\, d\rho\, d\varsigma\, d\phi \\
+ \frac{\lambda^2\sigma^2 d^3}{E}\int_0^{\frac{2\pi}{\sqrt{\lambda}}}\int_{-1}^{1}\Big[& \left(\frac{1}{2}\bar{n}_z + \frac{3}{2}\rho\,\bar{m}_z\right)V_z + \frac{3}{4}(1-\rho^2)\bar{q}_z V_r \\
& + \left(\frac{1}{2}\bar{n}_{\theta z} + \frac{3}{2}\rho\,\bar{m}_{\theta z}\right)V_\theta\Big]^{\varsigma=0} d\rho\, d\phi \qquad (7.4)
\end{aligned}
$$

We next expand s and v in powers of λ as in (2.1) and write I as a power series in λ.

$$
I = \frac{\lambda^2\sigma^2 d^3}{E}\left\{ I_0 + \lambda I_1 + \dots \right\} \qquad (7.5)
$$

I_o and I_1 are given by

$$I_o = \int_0^{\frac{2\pi}{\sqrt{\lambda}}} \int_0^{\infty} \int_{-1}^{1} \left\{ \frac{\partial V_{r0}}{\partial \rho} S_{r0} + \left[\frac{\partial V_{\theta 0}}{\partial \phi} + V_{r0} \right] S_{\theta 0} + \frac{\partial V_{z0}}{\partial \zeta} S_{z0} \right.$$

$$+ \left[\frac{\partial V_{r0}}{\partial \phi} + \frac{\partial V_{\theta 0}}{\partial \rho} \right] S_{r\theta 0} + \left[\frac{\partial V_{z0}}{\partial \rho} + \frac{\partial V_{r0}}{\partial \zeta} \right] S_{rz0}$$

$$\left. + \left[\frac{\partial V_{\theta 0}}{\partial \zeta} + \frac{\partial V_{z0}}{\partial \phi} \right] S_{\theta z0} - \frac{1}{2} \left[S_{\theta 0}^2 + S_{z0}^2 + 2\, S_{\theta z0}^2 \right] \right\} d\rho\, d\zeta\, d\phi$$

$$+ \int_0^{\frac{2\pi}{\sqrt{\lambda}}} \int_{-1}^{1} \left[\left(\frac{1}{2} \overline{n}_z + \frac{3}{2} \rho\, \overline{m}_z \right) V_{z0} + \frac{3}{4} (1-\rho^2)\, \overline{q}_z V_{r0} \right.$$

$$\left. + \left(\frac{1}{2} \overline{n}_{\theta z} + \frac{3}{2} \rho\, \overline{m}_{\theta z} \right) V_{\theta 0} \right]^{\zeta = 0} d\rho\, d\phi \qquad (7.6)$$

$$
\begin{aligned}
I_1 = \int_0^{\frac{2\pi}{\sqrt{\lambda}}} \int_0^{\infty} \int_{-1}^{1} \Bigg\{ & \left[S_{r1} + \rho S_{r0} \right] \frac{\partial V_{r0}}{\partial \rho} + S_{r0} \frac{\partial V_{r1}}{\partial \rho} + \left[\frac{\partial V_{\theta 0}}{\partial \phi} + V_{r0} \right] S_{\theta 1} \\
& + \left[\frac{\partial V_{\theta 1}}{\partial \phi} + V_{r1} \right] S_{\theta 0} + \frac{\partial V_{z1}}{\partial \zeta} S_{z0} + \frac{\partial V_{z0}}{\partial \zeta} \left[S_{z1} + \rho S_{z0} \right] \\
& + \left[\frac{\partial V_{r1}}{\partial \phi} + \frac{\partial V_{\theta 1}}{\partial \rho} \right] S_{r\theta 0} + \left[\frac{\partial V_{r0}}{\partial \phi} + \frac{\partial V_{\theta 0}}{\partial \rho} \right] S_{r\theta 1} \\
& + \left[\frac{\partial V_{\theta 0}}{\partial \rho} \rho - V_{\theta 0} \right] S_{r\theta 0} + \left[\frac{\partial V_{z1}}{\partial \rho} + \frac{\partial V_{r1}}{\partial \zeta} \right] S_{rz0} \\
& + \left[\frac{\partial V_{z0}}{\partial \rho} + \frac{\partial V_{r0}}{\partial \zeta} \right] \left[S_{rz1} + \rho S_{rz0} \right] + \frac{\partial V_{\theta 1}}{\partial \zeta} S_{\theta z0} \\
& + \frac{\partial V_{\theta 0}}{\partial \zeta} \left[S_{\theta z1} + \rho S_{\theta z0} \right] + \frac{\partial V_{z1}}{\partial \phi} S_{\theta z0} + \frac{\partial V_{z0}}{\partial \phi} S_{\theta z1} \\
& - \frac{1}{2} \Big[2 S_{\theta 0} S_{\theta 1} + 2 S_{z0} S_{z1} + \left(S_{\theta 0}^2 + S_{z0}^2 \right) \rho \\
& \qquad + 4 S_{\theta z0} S_{\theta z1} + 2 S_{\theta z0}^2 \rho \Big] \Bigg\} \, d\rho \, d\zeta \, d\phi \\
+ \int_0^{\frac{2\pi}{\sqrt{\lambda}}} \int_{-1}^{1} \Bigg[& \left(\frac{1}{2} \overline{n}_z + \frac{3}{2} \rho \overline{m}_z \right) V_{z1} + \frac{3}{4} (1 - \rho^2) \overline{q}_z V_{r1} \\
& + \left(\frac{1}{2} \overline{n}_{\theta z} + \frac{3}{2} \rho \overline{m}_{\theta z} \right) V_{\theta 1} \Bigg]^{\zeta = 0} d\rho \, d\phi
\end{aligned}
\qquad (7.7)
$$

We assume that s_o, v_o, s_1 and v_1 in (7.6) and (7.7) are given by (2.4), (2.5) and (2.7) with elastic parameters taking on the values (7.1). We have not satisfied identically any boundary conditions, but differential equations (2.2) and (2.3) are satisfied identically. The variational method will be used to obtain boundary conditions approximating (7.3) to the same order in λ as systems (1.1) and (1.2) are approximated.

Making use of (2.4), (2.5) and (2.7) we can write (7.6) and (7.7) more simply as follows.

$$I_o = \frac{1}{2}\int_0^{\frac{2\pi}{\sqrt{\lambda}}}\int_0^{\infty}\int_{-1}^{1}\left\{\left(\frac{\partial V_{\theta o}}{\partial \phi} + V_{ro}\right)^2 + \left(\frac{\partial V_{zo}}{\partial \varphi}\right)^2 + \frac{1}{2}\left(\frac{\partial V_{\theta o}}{\partial \varphi} + \frac{\partial V_{zo}}{\partial \phi}\right)^2\right\} d\rho\, d\varphi\, d\phi$$

$$+\int_0^{\frac{2\pi}{\sqrt{\lambda}}}\int_{-1}^{1}\Big[\left(\frac{1}{2}\,\bar{n}_z + \frac{3}{2}\rho\,\bar{m}_z\right)V_{zo} + \frac{3}{4}\left(1-\rho^2\right)\bar{q}_z\, V_{ro}$$

$$+\left(\frac{1}{2}\,\bar{n}_{\theta z} + \frac{3}{2}\rho\,\bar{m}_{\theta z}\right)V_{\theta o}\Big]^{\varphi=0} d\rho\, d\phi \qquad (7.8)$$

$$I_1 = \int_0^{\frac{2\pi}{\sqrt{\lambda}}}\int_0^{\infty}\int_{-1}^{1}\left\{\left(\frac{\partial V_{\theta 1}}{\partial \phi} + V_{r1}\right)\left(\frac{\partial V_{\theta o}}{\partial \phi} + V_{ro}\right) + \frac{\partial V_{z1}}{\partial \varphi}\,\frac{\partial V_{zo}}{\partial \varphi}\right.$$

$$+\frac{1}{2}\left(\frac{\partial V_{\theta 1}}{\partial \varphi} + \frac{\partial V_{z1}}{\partial \phi}\right)\left(\frac{\partial V_{\theta o}}{\partial \varphi} + \frac{\partial V_{zo}}{\partial \phi}\right) + \frac{1}{2}\left(\frac{\partial V_{zo}}{\partial \varphi}\right)^2\rho$$

$$\left.-\frac{1}{2}\left(\frac{\partial V_{\theta o}}{\partial \phi} + V_{ro}\right)^2\rho + \frac{1}{4}\left(\frac{\partial V_{\theta o}}{\partial \varphi}\right)^2\rho - \frac{1}{4}\left(\frac{\partial V_{zo}}{\partial \phi}\right)^2\rho\right\} d\rho\, d\varphi\, d\phi$$

$$+\int_0^{\frac{2\pi}{\sqrt{\lambda}}}\int_{-1}^{1}\Big[\left(\tfrac{1}{2}\,\bar{n}_z+\tfrac{3}{2}\rho\,\bar{m}_z\right)V_{z1}+\tfrac{3}{4}(1-\rho^2)\bar{q}_z\,V_{r1}$$

$$+\left(\tfrac{1}{2}\,\bar{n}_{\theta z}+\tfrac{3}{2}\rho\,\bar{m}_{\theta z}\right)V_{\theta 1}\Big]^{\xi=0}d\rho\,d\phi \qquad (7.9)$$

Then, upon substitution from (2.4), (2.5) and (2.7) and performance of the integration with respect to ρ , (7.8) and (7.9) take the following form,where we use the comma notation to denote partial differentiation.

$$I_0=\int_0^{\frac{2\pi}{\sqrt{\lambda}}}\int_0^{\infty}\Big\{\left(V_{\theta 0,\phi}+V_{r0}\right)^2+\tfrac{1}{3}\left(V_{r0,\phi^2}\right)^2+\left(V_{z0,\xi}\right)^2$$

$$+\tfrac{1}{3}\left(V_{r0,\xi^2}\right)^2+\tfrac{1}{2}\left(V_{\theta 0,\xi}+V_{z0,\phi}\right)^2+\tfrac{2}{3}\left(V_{r0,\xi\phi}\right)^2\Big\}\,d\xi\,d\phi$$

$$+\int_0^{\frac{2\pi}{\sqrt{\lambda}}}\Big[\bar{n}_z V_{z0}-\bar{m}_z V_{r0,\xi}+\bar{q}_z V_{r0}+\bar{n}_{\theta z}V_{\theta 0}-\bar{m}_{\theta z}V_{r0,\phi}\Big]^{\xi=0}d\phi \quad (7.10)$$

$$
\begin{aligned}
I_1 = \int_0^{\frac{2\pi}{\sqrt{\lambda}}} \int_0^{\infty} \Big\{ & 2\left(V_{\theta 1,\phi} + V_{r1}\right)\left(V_{\theta 0,\phi} + V_{r0}\right) + \tfrac{2}{3} V_{r1,\phi^2} V_{r0,\phi^2} \\
& + \tfrac{4}{3} V_{r1,\psi\phi} V_{r0,\psi\phi} + \tfrac{2}{3} V_{r1,\psi^2} V_{r0,\psi^2} + 2 V_{z1,\psi} V_{z0,\psi} \\
& + \left(V_{\theta 1,\psi} + V_{z1,\phi}\right)\left(V_{\theta 0,\psi} + V_{z0,\phi}\right) - V_{r0,\psi\phi} V_{\theta 0,\psi} \\
& - \tfrac{2}{3} V_{z0,\psi} V_{r0,\psi^2} + \tfrac{2}{3} V_{r0,\phi^2} V_{r0} + \tfrac{1}{3} V_{z0,\phi} V_{r0,\psi\phi} \Big\} \, d\psi \, d\phi \\
+ \int_0^{\frac{2\pi}{\sqrt{\lambda}}} \Big[& \bar{n}_z V_{z1} - \bar{m}_z V_{r1,\psi} + \bar{q}_z V_{r1} + \bar{n}_{\theta z} V_{\theta 1} \\
& - \bar{m}_{\theta z}\left(V_{r1,\phi} - V_{\theta 0}\right) \Big]^{\psi = 0} d\phi
\end{aligned}
\qquad (7.11)
$$

8. First Approximation. As a first approximation to $\delta I = 0$ we take $\delta I_0 = 0$. The variation of (7.10) becomes upon integration by parts*

*Note that when integrating by parts with respect to ϕ, the integrated expression vanishes because of the periodicity of the solution in the angular direction .

$$\delta I_0 = \int_0^{\frac{2\pi}{\sqrt{\lambda}}} \int_0^{\infty} \Big\{ \Big[2\left(V_{\theta 0,\phi} + V_{r0}\right) + \tfrac{2}{3} \Delta^2 V_{r0} \Big] \delta V_{r0}$$

$$- \Big[2\left(V_{\theta 0,\phi^2} + V_{r0,\phi}\right) + V_{\theta 0,\varphi^2} + V_{z0,\varphi\phi} \Big] \delta V_{\theta 0}$$

$$- \Big[2 V_{z0,\varphi^2} + V_{\theta 0,\varphi\phi} + V_{z0,\phi^2} \Big] \delta V_{z0} \Big\} \, d\varphi \, d\phi$$

$$+ \int_0^{\frac{2\pi}{\sqrt{\lambda}}} \Big[-\left(\tfrac{2}{3} V_{r0,\varphi^3} + \tfrac{4}{3} V_{r0,\varphi\phi^2} \right) \delta V_{r0} + 2 V_{z0,\varphi} \, \delta V_{z0}$$

$$+ \left(V_{\theta 0,\varphi} + V_{z0,\phi} \right) \delta V_{\theta 0} + \tfrac{2}{3} V_{r0,\varphi^2} \, \delta V_{r0,\varphi} \Big]_{\varphi=0}^{\varphi=\infty} d\phi$$

$$+ \int_0^{\frac{2\pi}{\sqrt{\lambda}}} \Big[\left(\bar{q}_z + \frac{d \bar{m}_{\theta z}}{d\phi} \right) \delta V_{r0} + \bar{n}_z \, \delta V_{z0}$$

$$+ \bar{n}_{\theta z} \, \delta V_{\theta 0} - \bar{m}_z \, \delta V_{r0,\varphi} \Big]^{\varphi=0} d\phi \qquad (8.1)$$

As Euler equations we obtain

$$\left.\begin{aligned}
&\tfrac{1}{2} V_{\theta 0,\varphi\phi} + \tfrac{1}{2} V_{z0,\phi^2} + V_{z0,\varphi^2} = 0 \\
&\tfrac{1}{2} V_{z0,\varphi\phi} + \tfrac{1}{2} V_{\theta 0,\varphi^2} + V_{\theta 0,\phi^2} + V_{r0,\phi} = 0 \\
&\tfrac{1}{3} \Delta^2 V_{r0} + V_{r0} + V_{\theta 0,\phi} = 0
\end{aligned}\right\} \qquad (8.2)$$

which agree with equations (3.4), (3.5) and (3.8) when (7.1) is taken into account. Note that, similar to the axially symmetric problem, the variational method does not determine S_{rzo}, $S_{r\theta o}$ or S_{ro} and in this respect leaves s_{rzo}, $s_{r\theta o}$, and s_{ro} incompletely determined.

Natural boundary conditions are given by

$$\left.\begin{aligned}
\phi = 0:\quad & \delta V_{ro}: & -\tfrac{2}{3} V_{ro,\phi^3} - \tfrac{4}{3} V_{ro,\phi\phi^2} &= \bar{q}_z + \frac{d\bar{m}_{\theta z}}{d\phi} \\
& \delta V_{zo}: & 2V_{zo,\phi} &= \bar{n}_z \\
& \delta V_{\theta o}: & V_{\theta o,\phi} + V_{zo,\phi} &= \bar{n}_{\theta z} \\
& \delta V_{ro,\phi}: & -\tfrac{2}{3} V_{ro,\phi^2} &= \bar{m}_z
\end{aligned}\right\} \quad (8.3)$$

Boundary conditions at $\phi = \infty$ are the same as (8.3) with zero right sides.

9. Second approximation. We assume $\delta I = 0$ is approximated by

$$\delta(I_o + \lambda I_1) = 0 \qquad (9.1)$$

and are led by the same reasoning as in Chapter I to consider the variational equation

$$\delta I_1 = 0 \tag{9.2}$$

in which all quantities are to be varied independently. After integration by parts the variation of (7.11) becomes

$$\delta I_1 = \int_0^{\frac{2\pi}{\sqrt{\lambda}}}\int_0^{\infty}\Big\{2\Big[V_{\theta 0,\phi} + V_{r0} + \frac{1}{3}\Delta^2 V_{r0}\Big]\delta V_{r1}$$

$$-2\Big[V_{z0,\varphi^2} + \frac{1}{2}\left(V_{\theta 0,\varphi\phi} + V_{z0,\phi^2}\right)\Big]\delta V_{z1}$$

$$-2\Big[\frac{1}{2}\left(V_{\theta 0,\varphi^2} + V_{z0,\varphi\phi}\right) + V_{\theta 0,\phi^2} + V_{r0,\phi}\Big]\delta V_{\theta 1}$$

$$+2\Big[V_{\theta 1,\phi} + V_{r1} + \frac{1}{3}\Delta^2 V_{r1} + \frac{2}{3}V_{r0,\phi^2} - \frac{1}{2}V_{\theta 0,\varphi^2\phi}$$

$$-\frac{1}{3}V_{z0,\varphi^3} + \frac{1}{6}V_{z0,\varphi\phi^2}\Big]\delta V_{r0}$$

$$-2\Big[V_{z1,\varphi^2} + \frac{1}{2}\left(V_{\theta 1,\varphi\phi} + V_{z1,\phi^2}\right) - \frac{1}{3}V_{r0,\varphi^3} + \frac{1}{6}V_{r0,\varphi\phi^2}\Big]\delta V_{z0}$$

$$-2\left[V_{\theta 1,\phi^2}+V_{r1,\phi}+\frac{1}{2}\left(V_{\theta 1,\varphi^2}+V_{z1,\varphi\phi}\right)-\frac{1}{2}V_{r0,\varphi^2\phi}\right]\delta V_{\theta 0}\Big\}\,d\varphi\, d\phi$$

$$+\int_0^{\frac{2\pi}{\sqrt{\lambda}}}\Big[\left(\frac{2}{3}V_{r0,\varphi^3}+\frac{4}{3}V_{r0,\varphi\phi^2}+\bar{q}_z+\frac{d\bar{m}_{\theta z}}{d\phi}\right)\delta V_{r1}$$

$$-\left(2V_{z0,\varphi}-\bar{n}_z\right)\delta V_{z1}-\left(V_{\theta 0,\varphi}+V_{z0,\phi}-\bar{n}_{\theta z}\right)\delta V_{\theta 1}$$

$$-\left(\frac{2}{3}V_{r0,\varphi^2}+\bar{m}_z\right)\delta V_{r1,\varphi}-\Big(-\frac{2}{3}V_{r1,\varphi^3}-\frac{4}{3}V_{r1,\varphi\phi^2}$$

$$+V_{\theta 0,\varphi\phi}+\frac{2}{3}V_{z0,\varphi^2}-\frac{1}{3}V_{z0,\phi^2}\Big)\delta V_{r0}-\left(2V_{z1,\varphi}-\frac{2}{3}V_{r0,\varphi^2}\right)\delta V_{z0}$$

$$-\left(V_{\theta 1,\varphi}+V_{z1,\phi}-V_{r0,\varphi\phi}-\bar{m}_{\theta z}\right)\delta V_{\theta 0}-\left(\frac{2}{3}V_{r1,\varphi^2}-\frac{2}{3}V_{z0,\varphi}\right)\delta V_{r0,\varphi}\Big]^{\varphi=0} d\phi$$

$$+\int_0^{\frac{2\pi}{\sqrt{\lambda}}}\Big[---\Big]^{\varphi=\infty} d\phi \tag{9.3}$$

Three of the Euler equations indicated by (9.3) reproduce (8.2). Using the first equation of (8.2) to simplify one of the remaining three, we obtain

$$\left.\begin{aligned}
&\tfrac{1}{2} V_{\theta 1,\varphi\phi} + \tfrac{1}{2} V_{z1,\phi^2} + V_{z1,\varphi^2} = \tfrac{1}{3} V_{r0,\varphi^3} - \tfrac{1}{6} V_{r0,\varphi\phi^2} \\
&\tfrac{1}{2} V_{z1,\varphi\phi} + \tfrac{1}{2} V_{\theta 1,\varphi^2} + V_{\theta 1,\phi^2} + V_{r1,\phi} = \tfrac{1}{2} V_{r0,\varphi^2\phi} \\
&\tfrac{1}{3}\Delta^2 V_{r1} + V_{r1} + V_{\theta 1,\phi} = -\tfrac{2}{3} V_{r0,\phi^2} + \tfrac{1}{3} V_{\theta 0,\varphi^2\phi} - \tfrac{1}{3} V_{z0,\varphi\phi^2}
\end{aligned}\right\} \quad (9.4)$$

Equations (3.15), (3.16) and (3.17) reduce to (9.4) under assumption (7.1). The variational method still says nothing about the functions S_{rzo}, $S_{r\theta o}$, etc. However, this is now due to the fact that we neglect transverse shear and normal deformation. Presumably, if (7.1) were not assumed, we would obtain expressions in (3.7) for S_{rzo} and $S_{r\theta o}$ at this approximation, in analogy to the results of Chapter I.

Four of the natural boundary conditions of (9.3) duplicate (8.3). The other four at $\varphi = 0$ are as follows.

$$\left.\begin{aligned}
&\delta V_{r0}: -\tfrac{2}{3} V_{r1,\varphi^3} - \tfrac{4}{3} V_{r1,\varphi\phi^2} + V_{\theta 0,\varphi\phi} + \tfrac{2}{3} V_{z0,\varphi^2} - \tfrac{1}{3} V_{z0,\phi^2} = 0 \\
&\delta V_{z0}: \quad 2 V_{z1,\varphi} - \tfrac{2}{3} V_{r0,\varphi^2} = 0 \\
&\delta V_{\theta 0}: \quad V_{\theta 1,\varphi} + V_{z1,\phi} - V_{r0,\varphi\phi} = \overline{m}_{\theta z} \\
&\delta V_{r0,\varphi}: -\tfrac{2}{3} V_{r1,\varphi^2} + \tfrac{2}{3} V_{z0,\varphi} = 0
\end{aligned}\right\} \quad (9.5)$$

Conditions at $\varphi = \infty$ are the same as (9.5) with right sides all zero.

<u>10</u>. <u>End conditions in terms of stress resultants and couples</u>. We relate macroscopic displacement functions to stress resultants and couples by substituting from (2.4), (2.5) and (2.7) into expressions (5.3) and performing the indicated integrations.

$$\left.\begin{aligned}
N_{z0} &= 2\nu\left(V_{\theta 0,\phi} + V_{r0}\right) + 2\, V_{z0,\varphi} \\
M_{z0} &= -\tfrac{2}{3}\left(\nu V_{r0,\phi^2} + V_{r0,\varphi^2}\right) \\
Q_{z0} &= 2\, S_{rz0} + \tfrac{1}{3}\Delta\, V_{r0,\varphi} \\
N_{\theta z0} &= (1-\nu)\left(V_{\theta 0,\varphi} + V_{z0,\phi}\right) \\
M_{\theta z0} &= -\tfrac{2}{3}(1-\nu)\, V_{r0,\varphi\phi}
\end{aligned}\right\} \qquad (10.1)$$

$$
\left.
\begin{aligned}
N_{z1} &= 2\nu\left(V_{r1} + V_{\theta 1,\phi}\right) + 2V_{z1,\varphi} + \frac{2\nu_n}{1-\nu} S_{r0} \\
&\quad - \frac{2}{3}\nu \frac{I_n - \nu_n}{1-\nu} V_{r0,\phi^2} + \left[\frac{(1+\nu)\nu_n - 2\nu I_n}{3(1-\nu)} - \frac{2}{3}\right] V_{r0,\varphi^2} \\
&\quad + \frac{(1+\nu)\nu_n - 2\nu I_n}{3(1-\nu)} V_{\theta 0,\phi^3} + \frac{2\nu_n - (1+2\nu-\nu^2) I_n}{3(1-\nu)} V_{\theta 0,\varphi^2\phi} \\
&\quad + \frac{(1+\nu)\nu_n - (1+\nu^2) I_n}{3(1-\nu)} V_{z0,\varphi\phi^2} - \frac{2}{3}\frac{I_n - \nu_n}{1-\nu} V_{z0,\varphi^3} \\
M_{z1} &= -\frac{2}{3}\left(\nu V_{r1,\phi^2} + V_{r1,\varphi^2}\right) + \frac{2}{3}\frac{2\nu I_n - \nu_n}{1-\nu} S_{r\theta 0,\phi} \\
&\quad + \frac{2}{3}\frac{2I_n - \nu_n}{1-\nu} S_{rz0,\varphi} + \frac{2\nu I_n - (1+\nu)\nu_n}{15(1-\nu)} \Delta V_{r0,\phi^2} \\
&\quad + \frac{2}{15}\frac{I_n - \nu_n}{1-\nu} \Delta V_{r0,\varphi^2} + \frac{2}{3}(\nu + \nu_n) V_{\theta 0,\phi} + \frac{2}{3} V_{z0,\varphi} + \frac{2}{3}\nu_n V_{r0} \\
Q_{z1} &= 2S_{rz1} + \frac{1}{3}\Delta V_{r1,\varphi} + \frac{\nu_n - 2I_n}{3(1-\nu)} S_{rz0,\varphi^2} - \frac{1}{3} I_n S_{rz0,\phi^2} \\
&\quad + \frac{\nu_n - (1+\nu) I_n}{3(1-\nu)} S_{r\theta 0,\varphi\phi} - \frac{I_n - \nu_n}{30(1-\nu)} \Delta^2 V_{r0,\varphi} - \frac{1}{3}\nu_n V_{r0,\varphi} \\
&\quad - \frac{1}{3}\left(\frac{1+\nu}{2} + \nu_n\right) V_{\theta 0,\varphi\phi} + \frac{1}{6}(1-\nu) V_{z0,\phi^2} - \frac{1}{3} V_{z0,\varphi^2}
\end{aligned}
\right\} \quad (10.2)
$$

$$
\left.
\begin{aligned}
N_{\theta z1} &= (1-\nu)\left(V_{\theta 1,\varphi} + V_{z1,\phi}\right) + \tfrac{1}{3}\left[-(1-\nu)+\nu_n-(1+\nu)I_n\right]V_{r0,\varphi\phi} \\
&\quad + \tfrac{1}{3}\left[\nu_n - \tfrac{1}{2}(3+\nu)I_n\right]V_{\theta 0,\varphi\phi^2} - \tfrac{1}{6}(1-\nu)I_n V_{\theta 0,\varphi^3} \\
&\quad + \tfrac{1}{3}\left[\nu_n - \tfrac{1}{2}(3+\nu)I_n\right]V_{z0,\varphi^2\phi} - \tfrac{1}{6}(1-\nu)I_n V_{z0,\phi^3} \\
M_{\theta z1} &= -\tfrac{2}{3}(1-\nu)V_{r1,\varphi\phi} + \tfrac{2}{3}I_n\left(S_{r\theta 0,\varphi} + S_{rz0,\phi}\right) \\
&\quad + \tfrac{1}{15}(2I_n-\nu_n)\Delta V_{r0,\varphi\phi} + \tfrac{2}{3}(1-\nu)V_{\theta 0,\varphi}
\end{aligned}
\right\}
\qquad (10.2)
$$

Under assumption (7.1) formulas (10.1) and (10.2) become

$$
\left.
\begin{aligned}
N_{z0} &= 2V_{z0,\varphi}\,, \quad M_{z0} = -\tfrac{2}{3}V_{r0,\varphi^2} \\
Q_{z0} &= -\tfrac{2}{3}\Delta V_{r0,\varphi}\,, \quad N_{\theta z0} = V_{\theta 0,\varphi} + V_{z0,\phi} \\
M_{\theta z0} &= -\tfrac{2}{3}V_{r0,\varphi\phi}
\end{aligned}
\right\}
\qquad (10.3)
$$

$$\left.\begin{aligned}
N_{z1} &= 2V_{z1,\varphi} - \frac{2}{3} V_{r0,\varphi^2} \\
M_{z1} &= -\frac{2}{3} V_{r1,\varphi^2} + \frac{2}{3} V_{z0,\varphi} \\
Q_{z1} &= -\frac{2}{3}\Delta V_{r1,\varphi} - \frac{2}{3} V_{\theta0,\varphi\phi} - \frac{4}{3} V_{z0,\phi^2} - \frac{4}{3} V_{z0,\varphi^2} \\
N_{\theta z1} &= V_{\theta1,\varphi} + V_{z1,\phi} - \frac{1}{3} V_{r0,\varphi\phi} \\
M_{\theta z1} &= -\frac{2}{3} V_{r1,\varphi\phi} + \frac{2}{3} V_{\theta0,\varphi}
\end{aligned}\right\} \quad (10.4)$$

With (10.3) and (10.4) end conditions (8.3) and (9.5) at $\varphi = 0$ can be written as

$$\left.\begin{aligned}
&N_{z0} = \overline{n}_z \,, \qquad N_{\theta z0} = \overline{n}_{\theta z} \\
&M_{z0} = \overline{m}_z \,, \qquad Q_{z0} + M_{\theta z0,\phi} = \overline{q}_z + \frac{d\overline{m}_{\theta z}}{d\phi}
\end{aligned}\right\} \quad (10.5)$$

$$\left.\begin{aligned} N_{z1} &= 0\,, \qquad M_{z1} = 0 \\ N_{\theta z1} + M_{\theta z0} &= \overline{m}_{\theta z} \\ Q_{z1} + M_{\theta z1,\phi} &= 0 \end{aligned}\right\} \qquad (10.6)$$

Results (10.5) and (10.6) confirm end conditions (6.5) and (6.6) obtained by non-dimensionalizing Love's boundary conditions under assumption (7.1). As said before we assume that (10.5) and (10.6) also hold when the stress resultant and couple functions are given by (10.1) and (10.2)

It seems reasonable that the higher end conditions (6.7) are also perfectly valid.

11. Concluding remarks.

(a) The present approach of expansion in powers of λ offers a systematic means of taking transverse effects into account in the interior solution by a succession of approximations.

(b) The order of the systems of the higher approximations remains the same as that of the first approximation. In fact, the higher order systems are of the same form as the lowest order system with additional inhomogeneous terms.

(c) The boundary conditions for the interior solution involve only conventional stress resultant and couples. Higher moments do not arise. When only conventional end conditions are specified in a problem, this is a necessary requirement of any theory.

The above points show the advantage of the present theory over theories of the type given in [2], [5] and [6] when only the interior solution away from the St. Venant boundary layer is desired. On the other hand

(a) A theory of the type developed in [2], [5] and [6] can satisfy the five physically desirable end conditions and hence may furnish a more accurate solution at the edge. In problems where the exact edge stresses are known, such as at a free edge, the higher end moments of these theories are known so that systems of any degree of approximation can be used. In such problems the interior solution is to be corrected in the St. Venant layer by an excess solution. Which of these two approaches is best under what conditions is open to question.

(b) It remains to be seen if the approach of this paper or something like it can be developed with the generality present in the theories of [2], [5] and [6].

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Biographical Note

Millard W. Johnson, Jr. did his undergraduate work at Carleton College (1948), Northfield, Minnesota, and the University of Wisconsin (1948-1952), Madison, Wisconsin. He was honored in 1951 by election to the Phi Beta Kappa chapter in Madison, Wisconsin. He received the degree of Bachelor of Science (Applied Mathematics and Mechanics) in June, 1952. He continued his education in the graduate school at the University of Wisconsin, receiving a degree of Master of Science (Applied Mathematics and Mechanics) in June, 1953.

In the Fall of 1953 he accepted a research assistantship at the Massachusetts Institute of Technology supported by the Office of Naval Research and began work for the degree of Doctor of Philosophy under the supervision of Professor Eric Reissner.

He has published the following articles written in co-authorship with Professor Reissner.

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