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#[superscript 1] -bundle bases and the prevalence of non-Higgsable structure in 4D F-theory models

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$\mathbb{P}^1$-bundle bases and the prevalence of non-Higgsable structure in 4D F-theory models

James Halverson[a] **and Washington Taylor**[b]

[a] *Kavli Institute for Theoretical Physics, University of California Santa Barbara,*
Kohn hall, Santa Barbara, CA 93106-4030, U.S.A.

[b] *Center for Theoretical Physics, Department of Physics, Massachusetts Institute of Technology,*
77 Massachusetts Avenue, Cambridge, MA 02139, U.S.A.

E-mail: jim@kitp.ucsb.edu, wati@mit.edu

ABSTRACT: We explore a large class of F-theory compactifications to four dimensions. We find evidence that gauge groups that cannot be Higgsed without breaking supersymmetry, often accompanied by associated matter fields, are a ubiquitous feature in the landscape of $\mathcal{N} = 1$ 4D F-theory constructions. In particular, we study 4D F-theory models that arise from compactification on threefold bases that are $\mathbb{P}^1$ bundles over certain toric surfaces. These bases are one natural analogue to the minimal models for base surfaces for 6D F-theory compactifications. Of the roughly 100,000 bases that we study, only 80 are weak Fano bases in which there are no automatic singularities on the associated elliptic Calabi-Yau fourfolds, and 98.3% of the bases have geometrically non-Higgsable gauge factors. The $\mathbb{P}^1$-bundle threefold bases we analyze contain a wide range of distinct surface topologies that support geometrically non-Higgsable clusters. Many of the bases that we consider contain $SU(3) \times SU(2)$ seven-brane clusters for generic values of deformation moduli; we analyze the relative frequency of this combination relative to the other four possible two-factor non-Higgsable product groups, as well as various other features such as geometrically non-Higgsable candidates for dark matter structure and phenomenological (SU(2)-charged) Higgs fields.

Contents

1 Introduction

F-theory [1–3] compactifications to four and six dimensions provide a broad perspective on the landscape of four- and six-dimensional string vacua. In six dimensions, F-theory may provide a construction for vacua in all branches of the space of consistent supersymmetric theories of gravity coupled to tensor fields, gauge fields, and matter [4–6]. In four dimensions, on the other hand, F-theory as currently formulated only captures one part of the apparently vast landscape of $\mathcal{N} = 1$ string vacua; for many heterotic vacua, infinite families of IIA flux vacua (see e.g. [7]), and large classes of G_2 M-theory compactifications (see e.g. [8, 9] for phenomenologically oriented work and [10] for recent progress on global compactifications), there is no known duality to F-theory, and in many cases it seems likely that such a duality cannot be found without a substantial extension of the framework of F-theory.

F-theory may nonetheless be the region of the $\mathcal{N} = 1$ landscape that is currently most amenable to study away from weakly coupled regimes, since supersymmetry and algebraic geometry together give strong control over nonperturbative aspects of the theory that are not accessible from other approaches. F-theory compactifications can be thought of as compactifications of type IIB string theory where the axiodilaton $\tau = C_0 + ie^{-\phi}$ varies over a compact space B, where B is a complex two-dimensional (three-dimensional) manifold B_2 (B_3) for an F-theory compactification to six (four) dimensions. Since e^{ϕ} determines the type IIB string coupling g_s, F-theory is more general than weakly coupled type IIB string theory in that it allows for the possibility of varying and/or strong coupling.

In recent years it has been found that nearly all six-dimensional supersymmetric F-theory compactifications give rise to non-Higgsable gauge groups and matter in the resulting low-energy 6D supergravity theory. These non-Higgsable structures arise from curved base geometries that force nonperturbative seven-brane configurations carrying nonabelian gauge groups to arise on certain divisors or combinations of divisors in the base. These gauge groups are not broken in any supersymmetric vacuum that can be reached without changing the topology of the base. A small number of basic irreducible "non-Higgsable clusters" [11] can be used to systematically study the space of elliptic Calabi-Yau threefolds and 6D F-theory models in their maximally Higgsed phase. In addition to single-factor non-Higgsable gauge groups, which can arise in F-theory compactifications with smooth heterotic duals, F-theory can also give rise to non-Higgsable product groups with matter charged under multiple gauge factors. Similar non-Higgsable clusters arise at the level of geometry in 4D F-theory models [12, 16].

The purpose of this paper is to initiate a systematic study of the kinds of non-Higgsable structures and bases that arise in concrete elliptic Calabi-Yau fourfolds and 4D F-theory constructions. In particular, we investigate here a specific class of compact toric threefold bases for 4D F-theory compactifications. The geometries we consider here are of interest for two reasons. First, they contain a broad class of topologically distinct complex surface types whose local geometry as divisors in the F-theory base gives rise to geometrically non-Higgsable gauge and matter content. Second, considering generic elliptic fibrations over the bases we construct here provides a view of a broad class of 4D F-theory vacua and gives some initial evidence that in 4D as well as in 6D, the vast majority of supersymmetric

F-theory vacua contain non-Higgsable gauge groups and matter. The construction used here gives rise to a class of $109,158$ threefold bases B, of which 98.3% (all but $1,824$) have non-Higgsable gauge groups at generic points in moduli space.

The structure of this paper is as follows: in section 2 we review some of the basics of F-theory constructions to six and four dimensions, and the notion of non-Higgsable clusters in F-theory geometry. In section 3, we describe the systematic classification of all threefold bases that can be constructed as $\mathbb{P}^1$ bundles over toric base surfaces S that themselves support elliptic Calabi-Yau threefolds. In section 4, we consider the Hodge structure of the generic elliptic fourfolds over the toric $\mathbb{P}^1$ bundle threefold bases and explore how this class of fourfolds fits into the known set of possible Hodge numbers for elliptic fourfolds. In section 5, we describe various aspects of the set of non-Higgsable clusters that arise for the bases we have constructed, and in section 6 we make some concluding remarks.

2 Review of F-theory, Weierstrass models and non-Higgsable clusters

2.1 Seven-branes and non-Higgsable clusters

In one description of F-theory, the IIB axiodilaton τ is conveniently encoded as the complex structure modulus of an elliptic curve (complex torus) that is fibered over the compact spatial dimensions. When the total space X of the resulting elliptic fibration $\pi : X \to B$ is Calabi-Yau, supersymmetry is preserved in the non-compact spacetime. One of the main features of F-theory is that the compactification manifold B can have curvature and a nontrivial canonical class K, unlike Calabi-Yau compactifications where the canonical class of the compactification manifold is trivial (up to torsion). When B has nontrivial (Ricci) curvature, for the total space X to be Calabi-Yau seven-branes must be added that wrap complex codimension one cycles in B. Specifically, seven-branes are located on loci in the base over which the elliptic fiber parameterized by τ becomes singular. For simple seven-brane configurations, the total space X of the elliptic fibration can still be smooth even when the fibration structure is singular. When multiple seven-branes come together over a common (complex) codimension one locus, however, the total space X of the fibration itself becomes singular. Such a situation is necessary for a nonabelian gauge group to arise in the low-energy supergravity theory of the F-theory compactification.

For some simple choices of the base manifold B, such as $\mathbb{P}^2$ or $\mathbb{P}^3$ for 6D or 4D F-theory compactifications, respectively, the generic elliptic fibration X over B is smooth, and the seven-branes can be separated so that there is no nonabelian gauge group in the low-energy theory. Nonabelian gauge groups can be "tuned" over such a base by piling up multiple seven-branes on a common codimension one locus, but in 6D, and in 4D in the absence of flux, the resulting nonabelian gauge groups can always be broken by varying the moduli of the elliptic fibration. In the low-energy theory this corresponds to a Higgsing transition where a vacuum expectation value is given to a matter field that is charged under the nonabelian gauge group. Such charged matter fields are encoded in the moduli of the holomorphic τ function describing the elliptic fibration.

For many choices of base geometry B, however, the curvature of the base is so strong that multiple seven-branes must pile up in certain places, giving nonabelian gauge groups

in the low-energy theory that cannot be removed by any simple deformation of the geometry that preserves the structure of the base as well as the Calabi-Yau property and supersymmetry. A nonabelian gauge group arising from such a seven-brane configuration in a compact space B is said to be *geometrically non-Higgsable*, meaning that there is no holomorphic deformation of the τ function for an elliptically fibered Calabi-Yau threefold X over the base B that breaks the gauge symmetry on the seven-branes. The geometrically non-Higgsable gauge symmetry can be enhanced on subloci in the moduli space, but it can never be broken by a geometric deformation. Whether or not this phenomenon occurs depends on the topology of B; it cannot occur for F-theory compactifications to eight dimensions, but it appears quite generally for F-theory compactifications to six dimensions, and there is growing evidence (to which this paper represents a contribution) that this is also the case in four dimensions.

One way to understand the appearance of non-Higgsable gauge groups and matter is from the local geometry of codimension one[1] algebraic cycles, or *divisors*, on the base B. When a divisor in B is embedded with a sufficiently negative normal bundle, this produces so much curvature in the local geometry that seven-branes must pile up on the divisor to maintain the Calabi-Yau nature of the total space X. A systematic analysis of circumstances under which a set of divisors in a threefold B must carry non-Higgsable gauge groups and matter is given in [16] in terms of the geometry of the divisors and their normal bundles. In particular, the normal bundle of a divisor is simply a complex line bundle over the divisor, and the presence of a non-Higgsable gauge group can be determined by the absence of sections for related line bundles over the divisor.

2.2 Weierstrass models and non-Higgsable clusters

To be more explicit and to discuss specific models, it is helpful to briefly review the relevant mathematics of Calabi-Yau elliptic fibrations. We start the discussion from the physical point of view, in terms of seven-branes of type IIB string theory. We consider a Calabi-Yau elliptic fibration $\pi : X \to B$ that has a global section and a representation as a Weierstrass model[2]

$$y^2 = x^3 + f\, x z^4 + g\, z^6 \tag{2.1}$$

where f and g are sections $f \in \Gamma(\mathcal{O}(-4K))$, $g \in \Gamma(\mathcal{O}(-6K))$, K is the canonical class of the base B, (x, y, z) are homogeneous coordinates in the weighted projective space $\mathbb{P}(2, 3, 1)$ and $z = 0$ is the section. One often passes to the patch where $z = 1$, giving the common form $y^2 = x^3 + f\, x + g$. For fixed f and g the Weierstrass equation represents a particular Calabi-Yau elliptic fibration X. More generally, a family $\mathcal{M}$ of Calabi-Yau elliptic fibrations is parameterized by a continuous set of choices for f and g. In the bulk of the moduli space $\mathcal{M}$, varying f and g varies the complex structure of a generic member of the family. We henceforth refer to $\mathcal{M}$ as the Weierstrass moduli space. Note

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[1]Note that in general when we refer to codimension k loci, we mean complex codimension k loci in the base manifold.

[2]By a theorem of Nakayama [17], any elliptic Calabi-Yau threefold with section can be described through a Weierstrass model. This is not proven for fourfolds, but holds at least for the class of examples we consider.

that $\mathcal{M}$ includes not only the complex structure moduli space for the Calabi-Yau threefold associated with the generic elliptic fibration over B, but also strata associated with the moduli spaces of other threefolds that arise when higher-order singularities, corresponding to enhanced gauge groups, are tuned on some divisors in the base.

The elliptic fiber becomes singular over the discriminant locus

$$\Delta \equiv 4f^3 + 27g^2 = 0, \tag{2.2}$$

which defines a divisor in B. An $SL(2, \mathbb{Z})$ monodromy is induced on τ by following any closed loop around the locus $\Delta = 0$. This implies in the physical language of F-theory that there are seven-branes on $\Delta = 0$ that source the axiodilaton. As f and g are varied the structure of seven-branes may change, giving rise to different low-energy physics. For example, obtaining non-abelian gauge symmetry along a seven-brane configuration on a locus described by $z = 0$ in a local coordinate system requires that Δ is of the form

$$\Delta = z^N \tilde{\Delta} \tag{2.3}$$

with $N \geq 2$.

Given this language for describing the elliptic fibration, a geometrically non-Higgsable seven-brane configuration arises when $\Delta = z^N \tilde{\Delta}$ with $N > 2$ for any allowed choice of f and g. In such a situation, there is no variation of f and g that removes this factorization, which means that there is no symmetry breaking flat direction in the Weierstrass moduli space. If B is toric, then f and g can be described as polynomials in homogeneous coordinates, and it is a straightforward exercise to classify all possible monomials that might appear in them. If all the monomials contain a nonzero power of some homogeneous coordinate z, then so do f and g, and therefore so does the discriminant. In this way, given a toric variety, it is straightforward to determine whether an F-theory compactification on a toric base B contains geometrically non-Higgsable seven-branes on toric divisors (see [18, 19] for explicit monomial analyses of toric base manifolds for F-theory fibrations to 6D and 4D respectively, in this context). This method is the one we use.

Let us be concrete about how the conditions that f, g vanish to given orders on a divisor D in a threefold base can be expressed particularly simply in terms of sections of line bundles on D. Though these can be computed easily in the toric case, we keep the discussion general since these methods apply even when B and D are not toric. If D is defined in terms of a local coordinate ζ via $\zeta = 0$, then we expand f and g

$$f = f_0 + f_1\zeta + \cdots \qquad g = g_0 + g_1\zeta + \cdots \tag{2.4}$$

in a power series around $\zeta = 0$. As described in [16, 19], the coefficient functions f_i, g_i are naturally described as global sections of line bundles over D

$$f_i \in \Gamma(\mathcal{O}(-4K_D) \otimes N_{D/B}^{4-i})$$
$$g_i \in \Gamma(\mathcal{O}(-6K_D) \otimes N_{D/B}^{6-i}).$$

When the divisors defining these line bundles are not effective (that is, when those bundles do not have global sections and accordingly some f_i or g_i do not exist), the corresponding

Type	ord (f)	ord (g)	ord (Δ)	singularity	nonabelian symmetry algebra
I_0	≥ 0	≥ 0	0	none	none
I_n	0	0	$n \geq 2$	A_{n-1}	$\mathfrak{su}(n)$ or $\mathfrak{sp}(\lfloor n/2 \rfloor)$
II	≥ 1	1	2	none	none
III	1	≥ 2	3	A_1	$\mathfrak{su}(2)$
IV	≥ 2	2	4	A_2	$\mathfrak{su}(3)$ or $\mathfrak{su}(2)$
I_0^*	≥ 2	≥ 3	6	D_4	$\mathfrak{so}(8)$ or $\mathfrak{so}(7)$ or $\mathfrak{g}_2$
I_n^*	2	3	$n \geq 7$	D_{n-2}	$\mathfrak{so}(2n-4)$ or $\mathfrak{so}(2n-5)$
IV^*	≥ 3	4	8	E_6	$\mathfrak{e}_6$ or $\mathfrak{f}_4$
III^*	3	≥ 5	9	E_7	$\mathfrak{e}_7$
II^*	≥ 4	5	10	E_8	$\mathfrak{e}_8$
non-min	≥ 4	≥ 6	≥ 12	does not appear for supersymmetric vacua	

Table 1. Table of codimension one singularity types for elliptic fibrations and associated non-abelian symmetry algebras. In cases where the algebra is not determined uniquely by the degrees of vanishing of f, g, the precise gauge algebra is fixed by monodromy conditions that can be identified from the form of the Weierstrass model.

terms in the expansion of f, g must vanish, giving rise to non-Higgsable structure when the resulting orders of vanishing are sufficiently large.

The types of singularities that can arise in an elliptic fibration, and the Dynkin diagrams associated with the corresponding low-energy gauge groups, are determined by the famous Kodaira classification; the singularity types and associated gauge group factors can be determined by the Tate algorithm [3] according to the orders of vanishing of f and g, augmented by more detailed structure associated with the effects of outer monodromy that can give rise to non-simply-laced groups in 6D and 4D. The standard table of singularity types is reproduced in table 1. When there is a singularity associated with a nonabelian gauge group, the Weierstrass model of the elliptically fibered Calabi-Yau X is singular, corresponding to a limit in which Kähler moduli associated with cycles on X that correspond to the nonabelian generators are taken to vanish; these moduli may be turned on in a related M-theory compactification to one lower space-time dimension. Alternatively, singularities can also be studied by deformation of a local geometry (which sometimes results from deformation of a global geometry); see [13–15] for recent work on the connection between deformation, singularities, and string junctions. When the nonabelian gauge content of the theory is enhanced by tuning a non-generic seven-brane configuration supporting additional gauge generators, the elliptic Calabi-Yau over B changes topology. Taken in the opposite direction, the gauge group can be broken by turning on moduli in the Weierstrass model. In the low-energy theory, this corresponds to a Higgsing of the additional nonabelian symmetry by giving vacuum expectation values to charged matter fields associated with the tuned Weierstrass moduli; see [14] for the explicit connection between such Higgsing processes and string junctions representing the massive W-bosons of the broken theory. When the singularity of the elliptic fibration becomes too strong, in particular when the order of vanishing of f, g on a codimension two locus reaches $(4, 6)$,

the theory becomes a superconformal field theory coupled to gravity [20]. At this point, it is often possible to turn on new moduli and enter a new branch of the F-theory moduli space. Geometrically, this corresponds to blowing up the codimension two locus on the base B, giving a new base variety B' [3, 21].

2.3 Non-Higgsable clusters for 6D F-theory vacua

A complete classification of non-Higgsable clusters for 6D F-theory vacua was given in [11], and derived from an alternative point of view in [16]. We review a few of the salient points here and describe some features of non-Higgsable clusters for 6D F-theory models that are relevant to the main points of this paper.

For 6D F-theory compactifications, we are interested in the case where the base variety used for the compactification is a complex surface $B_2 = S$. In this situation, the codimension one loci that support seven-branes are complex curves C. The only situation in which a curve $C \subset B$ can support a non-Higgsable cluster is when C has genus zero, i.e. is a rational curve with topology S^2; the complex structure in fact makes it $\mathbb{P}^1$. The normal bundle of such a curve $N = \mathcal{O}(n)$ is characterized by a single integer n, which also gives the self-intersection of the curve C within B, $C \cdot C = n$. Non-Higgsable clusters arise when there are effective divisors that are rational curves with sufficiently negative normal bundle that the self-intersection of the curve satisfies $n \leq -3$. Isolated curves of self-intersection $-3, -4, -5, -6, -7, -8, -12$ carry non-Higgsable gauge groups with Lie algebras $\mathfrak{su}_3, \mathfrak{so}_8, \mathfrak{f}_4, \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_7, \mathfrak{e}_8$. Curves of self-intersection $-9, -10, -11$ always contain points where f, g vanish to orders $(4, 6)$, so that F-theory models on bases containing such curves give superconformal field theories coupled to gravity. Blowing up these points can lead to well-behaved supergravity models with an additional tensor field. In addition to the non-Higgsable clusters associated with isolated curves of negative self-intersection, there are three configurations of multiple curves that give rise to non-Higgsable product groups. Chains of curves of self-intersection $(-3, -2), (-3, -2, -2)$ give rise to non-Higgsable gauge algebras $\mathfrak{g}_2 \oplus \mathfrak{su}_2$, and the chain $(-2, -3, -2)$ gives a non-Higgsable $\mathfrak{su}_2 \oplus \mathfrak{so}_7 \oplus \mathfrak{su}_2$.

An interesting feature of non-Higgsable clusters in 6D is that they can carry only very specific types of charged matter. The non-Higgsable cluster on a curve of self-intersection -7, for example, carries a gauge algebra $\mathfrak{e}_7$ and a half-hypermultiplet of matter in the real $\frac{1}{2}\mathbf{56}$ representation. That this matter cannot be given a vacuum expectation value to Higgs the $\mathfrak{e}_7$ can be seen from the fact that the D-term constraint cannot cancel for a single such field. Another way to understand this is from the fact that no gauge-invariant holomorphic function can be formed from a single such field [22].

In six dimensions, F-theory geometry is tightly coupled to the physics of the corresponding low-energy supergravity theory. All physical Weierstrass moduli in six-dimensional F-theory models are massless, and there is a precise correspondence between the moduli space in the low-energy theory and the complex structure moduli space associated with a Calabi-Yau threefold on each branch of the space of F-theory vacua. Thus, for six-dimensional F-theory models, gauge groups that are geometrically non-Higgsable are also physically non-Higgsable. Furthermore, all branches of the moduli space of F-theory

vacua are connected, suggesting that there is a unique consistent $\mathcal{N} = 1$ supersymmetric quantum theory of gravity in six dimensions.

The gauge groups associated with non-Higgsable clusters in 6D theory models are "generic" in two distinct senses. The first sense in which this type of structure is generic is the fact that when there is a non-Higgsable structure over a given base B, the resulting gauge group and matter are present in every 6D theory over that base, independent of the choice of Weierstrass moduli. While a larger gauge group can often be tuned at special subloci of the Weierstrass moduli space, the only way to reduce the size of the gauge group in this situation is through a tensionless string transition to a simpler base geometry, corresponding to blowing down one or more -1 curves in the base.

The second sense in which non-Higgsable clusters are generic in 6D F-theory models is that almost all base manifolds B that support elliptically fibered Calabi-Yau threefolds give rise to non-Higgsable clusters. While of course a measure on the space of vacua is difficult to define, the set of possible base manifolds B is a finite set, so a reasonable (if coarse) measure is to simply look at the fraction of the finite number of bases that support NHC's, or even more roughly to consider the fraction of the set of possible Hodge numbers for generic elliptic fibrations over B that are associated with bases forcing non-Higgsable structure. By either measure, virtually all allowed bases B have negative self-intersection curves that give non-Higgsable clusters. We summarize briefly the extent to which this is understood at a quantitative level. From the minimal model program for surfaces [23] and the work of Grassi [24], it is known that all twofold bases that support nontrivial elliptic fibrations can be formed by blowing up points on $\mathbb{P}^2$ or on the Hirzebruch surfaces $\mathbb{F}_m$. This approach has been used to systematically explore the space of possible bases B and to enumerate various classes of elliptically fibered Calabi-Yau threefolds. In [18, 25, 26], the set of all toric and "semi-toric" bases B that support an elliptically fibered Calabi-Yau threefold and the Hodge numbers of the corresponding generic elliptic fibrations were analyzed, and in [27] a systematic classification was given for all bases (including those without toric or semi-toric structure) that support elliptically fibered CY threefolds with small $h^{1,1}$ or large $h^{2,1}$. Rigorous upper bounds on the set of CY threefolds with large $h^{2,1}$ [25, 29], and the close correspondence between these analyses and the set of Calabi-Yau threefolds constructed through toric methods in [30],[3] indicate that the set of possible base surfaces B is not only finite but also fairly well understood, at least at a coarse level of detail. Of the over 60,000 toric bases constructed in [18], only 16 lack non-Higgsable clusters, and of the over 130,000 semi-toric bases found in [26] only 27 lack non-Higgsable clusters. Indeed, the only base surfaces that do not have at least one curve of self-intersection -3 or below are the generalized del Pezzo surfaces, which number on the order of several hundred. (Generalized del Pezzo surfaces can be defined as non-singular projective surfaces with the properties that the canonical class K satisfies $K \cdot K > 0$ and that the surface is *weak Fano*, i.e. $-K \cdot C \geq 0$ for every effective curve C in the surface.) For all these surfaces, the generic Calabi-Yau elliptic fibration has a set of Hodge numbers $(2 + T, 272 - 29T)$, where $T = 0, \ldots, 9$ corresponds to the number of tensor multiplets in the 6D theory resulting from

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[3]Data available online at [31].
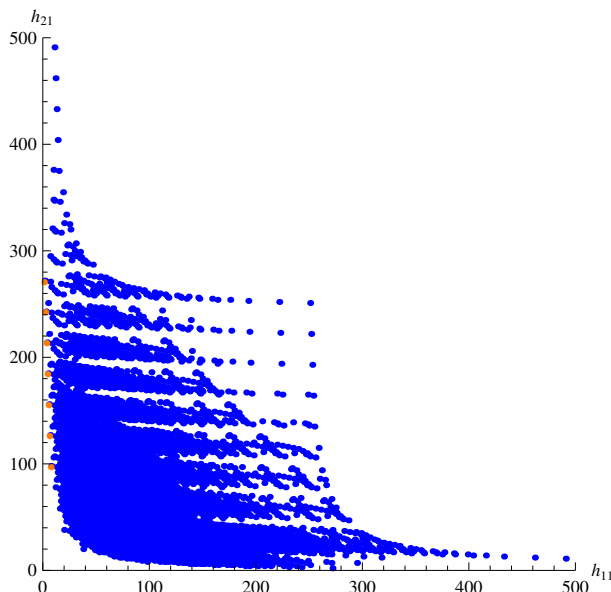
Figure 1. Hodge numbers for generic elliptic Calabi-Yau manifolds fibered over the 61,539 toric base surfaces that support elliptically fibered Calabi-Yau's [18, 25]. Only seven Hodge numbers (on the left in orange) correspond to elliptic fibrations over *weak Fano* bases that do not give rise to non-Higgsable clusters.

compactification on a generalized del Pezzo surface formed from blowing up $\mathbb{P}^2$ at T points. Thus, of all possible generic elliptic Calabi-Yau manifolds over all possible bases, there are only 10 possible Hodge number combinations associated with bases that do not admit non-Higgsable clusters. In figure 1, we have graphed the Hodge numbers associated with generic elliptic fibrations over all toric base surfaces; of these, only seven Hodge number pairs, on the left-hand side of the graph, correspond to F-theory compactifications that are free of non-Higgsable gauge groups.

2.4 Non-Higgsable clusters for 4D F-theory vacua

At the level of Calabi-Yau geometry, there is a close parallel between the structure of F-theory compactifications to 4D and 6D. For a 4D compactification, we have an elliptically fibered Calabi-Yau fourfold $X = X_4$ over a complex threefold base B. As in 6D, divisors $D \subset B$ with a "sufficiently negative" (*i.e.,* far from effective) normal bundle must support a geometrically non-Higgsable gauge group. A general approach to analyzing the geometry of non-Higgsable clusters for 4D F-theory models in terms of the geometry of surfaces and normal bundles was worked out in [16, 19]. While similar in general outline to the 6D story, the set of possible non-Higgsable clusters in 4D is rather more complicated than in 6D. In addition, for 4D F-theory compactifications there are complications in the relationship between the underlying geometry and the low-energy physics. For one thing, flux (G-flux) in F-theory compactifications can produce a superpotential that lifts some or all of the geometric moduli of the theory. This can in principle drive the model to special subloci of the moduli space with enhanced gauge symmetry. Furthermore, instantons and other

structure on the worldvolume of the seven-branes can break otherwise non-Higgsable gauge groups (see e.g. [19] for an explicit example), so that in 4D there is not necessarily a precise correspondence between non-Higgsable geometry and the gauge group and matter content in the low-energy theory. These are important issues, but a necessary prerequisite to a systematic study of the role of non-Higgsable clusters in low-energy theory must begin with the purely geometric aspect of the question. Thus, we focus here only on the geometric structure of non-Higgsable clusters, leaving further study of the role of fluxes and seven-brane worldvolume degrees of freedom to further work. While in principle all aspects of non-Higgsable clusters explored in the later part of this paper are purely based on geometry, and all such clusters should be referred to as "geometrically non-Higgsable" to be precise, we will often drop the adjective and leave it as understood in the remainder of this paper.

As found in [16], there are strong local constraints on the types of gauge groups that can appear in 4D (geometric) non-Higgsable clusters, and on the products of two gauge groups that can appear with jointly charged matter, either in an isolated cluster or as part of a larger cluster. The nonabelian gauge algebras that can appear in a 4D non-Higgsable cluster are [12, 16]

$$\mathfrak{su}_2, \mathfrak{su}_3, \mathfrak{g}_2, \mathfrak{so}_7, \mathfrak{so}_8, \mathfrak{f}_4, \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8 \, . \tag{2.5}$$

The only possible gauge algebra combinations that can appear in a product of two groups with jointly charged matter are

$$\begin{aligned} \mathfrak{su}_2 \oplus \mathfrak{su}_2, \qquad \mathfrak{su}_2 \oplus \mathfrak{su}_3, \qquad \mathfrak{su}_3 \oplus \mathfrak{su}_3, \\ \mathfrak{g}_2 \oplus \mathfrak{su}_2, \qquad \mathfrak{so}_7 \oplus \mathfrak{su}_2 \end{aligned} \tag{2.6}$$

Unlike in 6D, however, where only rational curves can support a non-Higgsable gauge group factor, the possible divisor geometries that can support non-Higgsable gauge group factors are quite varied in four dimensions. One of the goals of this work is to explore some of the range of possible divisor geometries that can support non-Higgsable gauge groups. Non-Higgsable clusters in four dimensions can also have much more complicated structure than for 6D theories. As shown in [16], non-Higgsable clusters can have gauge groups that are combined into "quiver" diagrams[4] that exhibit branching, loops, and long chains. We also explore some of this structure in the following sections.

3 Classification of $\mathbb{P}^1$ bundle bases

The threefold bases B that we consider here are defined as $\mathbb{P}^1$ bundles over the set of surfaces S classified in [18], which are those smooth toric surfaces that themselves support smooth elliptically fibered Calabi-Yau threefolds. Threefolds with $\mathbb{P}^1$ bundle structure have been considered previously in the context of heterotic/F-theory duality (see, e.g., [32]). In particular, in [19], a systematic analysis was given of all F-theory/heterotic dual constructions where the F-theory model is a $\mathbb{P}^1$ bundle and the dual Calabi-Yau on the heterotic side is

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[4]A quiver diagram for a gauge theory is a graph in which the nodes represent simple gauge group factors and edges represent bifundamental type matter between pairs of gauge factors.

smooth, which occurs when the surface S is a generalized del Pezzo surface. While in principle the F-theory models that result from the threefold bases we consider here also have heterotic duals, the compactification manifolds on the dual heterotic side are generally highly singular. The simplest classes of $\mathbb{P}^1$-bundle threefold bases, where the surface S is taken to be $\mathbb{P}^2$ or a Hirzebruch surface $\mathbb{F}_m$, were also studied in the context of F-theory in [33–36].

The bases in the class that we consider here are of interest for several reasons. Firstly, the $\mathbb{P}^1$ bundle threefold base over a surface S has a divisor (actually two divisors, corresponding to sections of the $\mathbb{P}^1$ bundle) with the geometry of S. Thus, this class of threefold bases provides a rich set of examples of distinct divisor topologies and normal bundles. In fact, the sections of each of the many threefold bases we consider here correspond to distinct divisor/normal bundle geometries for each distinct threefold base. The second reason that the bases in this class are of interest is because they are one natural analog of the Hirzebruch surfaces $\mathbb{F}_m$ (which are $\mathbb{P}^1$ bundles over $\mathbb{P}^1$), which form most of the "minimal model" bases for the set of 6D F-theory compactifications. Many of the threefold bases that we consider here are similarly, in a certain sense, minimal model bases for 4D F-theory compactifications. As we discuss further below, a more systematic and rigorous analysis along these lines would involve the framework of Mori theory [37], and will be left for further work. Finally, the geometries in the class we consider here are interesting because they give a rich sample of threefolds that can act as bases for elliptically fibered CY fourfolds, which goes well beyond the set of such bases that have been studied previously. This gives us a class of examples with which to explore what kinds of non-Higgsable structures may be typical in the 4D F-theory landscape. Because the class of bases explored here is rather specialized, the lessons taken in this latter context must be taken as only suggestive; nonetheless, some general conclusions that we can draw may be sufficiently robust that they will persist over larger distributions of threefold bases that can be studied using other methods.

3.1 Twists and allowed bundles

To define a threefold B that is a $\mathbb{P}^1$ bundle with section over a surface S we must choose a class $T \in H_2(S, \mathbb{Z})$. From one perspective, T can be thought of as a "twist" corresponding to the first Chern class of a line bundle $\mathcal{L}$ over S so that B is the projectivization $B = \mathbb{P}(\mathcal{L} \oplus \mathcal{O}_S)$. Alternatively, $\mathcal{O}_S(\pm T)$ are the normal bundles of the two sections. Note that choosing one line bundle $\mathcal{L}$ is sufficient for a general construction since $\mathbb{P}(\mathcal{L}_1 \oplus \mathcal{L}_2) = \mathbb{P}((\mathcal{L}_1 \otimes \mathcal{L}_2^*) \oplus \mathcal{O}_S) =: \mathbb{P}(\mathcal{L} \oplus \mathcal{O}_S)$.

In the language of toric geometry, if $v_i = (x_i, y_i), i = 1, \ldots, N$ give the rays that define the toric fan for S, then the fan for B can be defined using the set of rays [19]

$$w_i = (x_i, y_i, t_i), \ 1 \le i \le N \tag{3.1}$$

$$w_{N+1,N+2} = s_\pm = (0, 0, \mp 1), \tag{3.2}$$

where the coordinates t_i determine the twist T by $T = \sum_i t_i d_i$ with $d_i \in H_2(S, \mathbb{Z})$ the toric divisors in S associated to the vertices v_i. There are two linear relations amongst the d_i, so that the number of independent parameters in the twist T is $N - 2 = h^{1,1}(S)$.

As discussed above, an F-theory model can be defined over the base B when there exists an elliptic fibration (with section) over B that has a Calabi-Yau resolution, and a

Weierstrass model[5]

$$y^2 = x^3 + fx + g \,. \tag{3.3}$$

For B to admit an elliptic fibration with a Calabi-Yau resolution, there cannot be any divisors on B where the order of vanishing of f, g is generically $(4, 6)$. We also assume that we are in a supergravity phase of the theory that does not include superconformal sectors, so that there are no curves with $(4, 6)$ vanishing. These conditions impose stringent conditions on the geometry of B, and in particular place constraints on the set of allowed twists T for any given base surface S. Note that some weaker singularities may arise that cannot be resolved to a total space that is a Calabi-Yau fourfold, where consistent F-theory supergravity models may still be possible. For example, such singularities may include codimension 3 loci where f, g vanish to orders $(4, 6)$ [38], or certain other mild local singularity types that are weaker than $(4, 6)$ vanishing and yet cannot be resolved to smooth Calabi-Yau spaces. We allow for such milder singularity types in our analysis, leaving a more detailed study of the relevant physics to future work.

In [19], the F-theory conditions on the structure of a $\mathbb{P}^1$ bundle were shown to correspond to constraints on the components $\tilde{t}_i = T \cdot d_i$ that limit the set of possible T for a given surface S to a finite set. When the self-intersection of the curve d_i is $d_i \cdot d_i = -n$, then the constraints on the corresponding twist component are

$$n = 0 : \quad |T \cdot d_i| \leq 12 \tag{3.4}$$

$$n = 1 : \quad |T \cdot d_i| \leq 6 \tag{3.5}$$

$$6 \geq n \geq 2 : \quad |T \cdot d_i| \leq 1 \,. \tag{3.6}$$

$$n \geq 7 : \quad T \cdot d_i = 0 \,. \tag{3.7}$$

While these constraints in principle allow only a finite number of possible twists T for any given surface S, these constraints are only necessary and not sufficient. For any T satisfying these constraints, a more detailed check must be carried out to determine if the resulting base B is free of $(4, 6)$ divisors or curves. A general framework for carrying out such an analysis for an arbitrary base B was developed in [16]. For toric bases such as those considered here, the order of vanishing of f, g on each of the toric divisors and curves can be directly computed using toric methods, or can be determined using the more general methods of [16].

As the complexity of the surface S grows, the number of solutions T to the constraints (3.4)–(3.7) can grow exponentially, but most such solutions will have problems such as $(4, 6)$ singularities on curves. To reduce the combinatorial problem of identifying all allowed twists T over a given toric base surface S to a tractable computation, stronger constraints are needed. We have used methods related to those of [16, 19] to develop a set of stronger constraints that enable a complete classification of the twists T for any given surface S in a relatively modest computational time. The details of these stronger constraints are described in appendix A.1 and appendix A.2.

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[5]As mentioned above, unlike the case of threefolds, it has not been proven that every such fourfold admits a Weierstrass model, but one exists for all the elliptic fibrations over toric bases B that we consider here.

3.2 Classification of bases

We have used the constraints described in the appendices to carry out a systematic analysis and enumeration of all bases B that have the form of a $\mathbb{P}^1$ bundle over any of the 61,539 toric base surfaces S identified in [18]. This gives a total of 109,158 threefolds B that can support elliptic Calabi-Yau fourfolds, and thus can act as compactification spaces for F-theory to four dimensions. This set of threefolds includes the 4,962 bases B that were constructed in [19] as $\mathbb{P}^1$ bundles over the subset of 16 surfaces S that are generalized del Pezzo surfaces, corresponding to heterotic duals on elliptic Calabi-Yau manifolds with generically smooth Weierstrass models. Note that we have only counted as distinct bases B that have distinct topology, independent of the $\mathbb{P}^1$ bundle structure. So, for example, the $\mathbb{P}^1$ bundle with twist $(t_1, t_2, t_3, t_4) = (0, 0, 0, 1)$ over the base $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ is considered to be the same threefold base B as $\mathbb{P}^1 \times \mathbb{F}_1$.

Note that in this classification we have not included any bases that have curves on which f, g are forced to vanish to orders $(4, 6)$. In particular, we do not consider base threefolds that contain a divisor D that supports a non-Higgsable E_8 if g vanishes to order 6 on any curve $C \subset D$; such a curve can arise from a II^*-I_1 collision where the E_8 seven-brane intersects the residual discriminant. These would be the analogue of bases containing $-9, -10$, and -11 curves for 6D F-theory models. While in principle these base threefolds describe superconformal field theories coupled to gravity, and could give (generally non-toric) acceptable bases for 4D supergravity models without $(4, 6)$ curves after blowing up the offending curves C, this analysis is more complicated than the corresponding story in six dimensions, and we do not explore these bases further in this paper. The technical details of the procedure used to rule out these bases are described in appendix B.

In the remainder of this paper we provide some details of the structure of the 109,158 threefolds that we have constructed.

3.3 Distribution of $\mathbb{P}^1$ bundle bases

We begin by characterizing the range of bundles that are allowed for the different base surfaces S.

The distribution according to $h^{1,1}(S)$ of toric base surfaces S that support elliptically fibered CY threefolds is described in [18]. The maximum number of distinct such bases peaks around $h^{1,1}(S) \cong 25$ and then drops rapidly, with a relatively small number of bases having $h^{1,1}(S) > 80$, and the maximum value being $h^{1,1}(S) = T + 1 = 194$.

Only a subset of these surfaces admit $\mathbb{P}^1$ bundles over them that form valid bases B that can directly support elliptic CY fourfolds. In particular, if the surface S contains a -12 curve, then there is no good corresponding B. This can be seen as follows. From the analysis of twists in section 3.1, for a -12 curve d_i, we must have $T \cdot d_i = 0$. This means that in the local geometry of any $\mathbb{P}^1$ bundle B over S, the $\mathbb{P}^1$ bundle over the curve d_i is a divisor D in B with the geometry of $\mathbb{F}_0$, and normal bundle $N_{D/B} = \mathcal{O}_D(-12X)$, where X, Y are the standard basis for the cone of effective curves on D with $X^2 = Y^2 = 0$ and $X \cdot_D Y = 1$. There is a type II^* (E_8) singularity on D and, using the language of [16] we
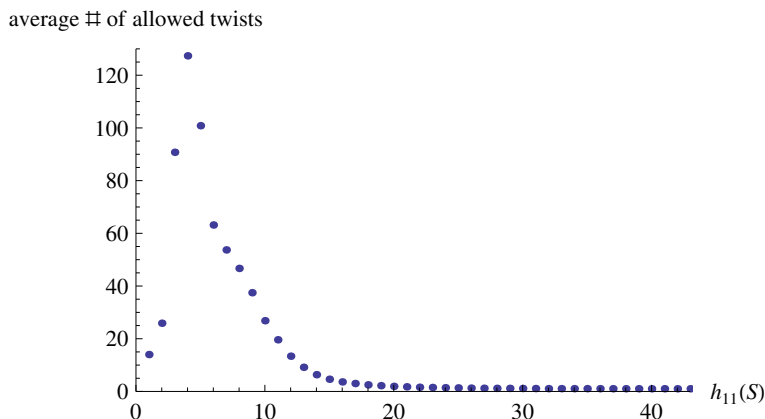
Figure 2. The average number of distinct $\mathbb{P}^1$ bundles B over a toric base surface S in the set found in [18], as a function of $h^{1,1}(S)$.

compute

$$g_5 \in \Gamma(\mathcal{O}_D(-6K_D) \otimes N_{D/B}) = \Gamma(\mathcal{O}_D(12Y)) \tag{3.8}$$

from which we see that $g_5 = 0$ defines a nontrivial curve that is itself a divisor in the surface D. Along this curve, f and g vanish to orders $(4, 6)$. Similarly, any surface S with a curve of -9 or below gives a divisor D carrying an E_8 singularity where the curve $g_5 = 0$ in D is a $(4, 6)$ locus; thus, we need not consider any surfaces S with curves of self-intersection -9 or below.

On the other hand, if S does not admit any curves of self-intersection -9 or below, then there is at least one $\mathbb{P}^1$ bundle B over S that can support a good elliptic Calabi-Yau fourfold, namely the trivial bundle $B = \mathbb{P}^1 \times S$. This gives us a subset of 24,483 surfaces S of the 61,539 found in [18] over which to construct bases B. Of these, the largest has $h^{1,1} = 72$, and is described by a chain of toric curves containing 12 copies of the basic sequence $-8, -1, -2, -3, -2, -1, -8, \ldots$ terminated by a 0 curve, with the next-to-last -8 curve on each end replaced by a -7.

The average number of allowed twists over each base S is graphed as a function of $h^{1,1}(S)$ in figure 2. Note that for $h^{1,1} > 43$, the only $\mathbb{P}^1$ bundle bases B that support elliptic CY fourfolds are trivial bundles $\mathbb{P}^1 \times S$. The largest number of distinct twists possible over a given surface S is 1119, which occurs for the generalized del Pezzo surface with a toric description in terms of curves of self-intersection $((-1, -1, -2, -1, -2, -1, -1))$.[6] This surface can be constructed by blowing up the del Pezzo surface dP_3 at the intersection point between a pair of -1 curves, and the set of base threefolds constructed as $\mathbb{P}^1$ bundles over it was also analyzed in [19].

3.4 Divisors supporting non-Higgsable gauge groups

One particularly interesting aspect of the class of bases we are considering here is that they give a broad set of distinct local structures for divisors and normal bundles that can

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[6]We generally use the notation $((n_1, \ldots, n_N))$ to denote the self-intersections of the divisors d_i in the toric base surface S for the $\mathbb{P}^1$ fibration of a base B.

arise in 3D F-theory bases. Indeed, essentially every base in the class we have constructed represents two distinct combinations of divisor and normal bundle, corresponding to the two sections of the $\mathbb{P}^1$ bundle and $\pm$ the twist, so that we have roughly 200,000 distinct local divisors and normal bundles that can arise in 3D bases. (A very small number of bases formed from nontrivial $\mathbb{P}^1$ bundles have a symmetry under exchanging the two sections so that the local structure on both sides is the same, so the total number of local geometries is slightly smaller than $2 \times 109,158 = 218,316$.) Each base B also contains a set of divisors D_i that each have the topology of a Hirzebruch surface, associated with the restriction of the $\mathbb{P}^1$ bundle to the toric curves d_i in the surface S. Because the range of geometries is much larger for the divisors associated with sections, we focus more on those here.

It is interesting to consider the subset of the local divisor + normal bundle geometries that support non-Higgsable gauge groups. In total we find that only 368 of the 109,158 bases have twists that produce non-Higgsable gauge groups on the divisor associated with either section. The divisor D in each case is topologically described by the base S of the $\mathbb{P}^1$ fibration. The divisors that support non-Higgsable groups all have $h^{1,1}(D) \leq 14$; the three divisors with the largest value of 14 have toric self-intersections

$$((-4,-1,-4,-1,-2,-3,-2,-1,-4,-1,-4,-1,-2,-3,-2,-1))$$
$$((-4,-1,-4,-1,-4,-1,-2,-3,-2,-1,-2,-3,-2,-1,-4,-1))$$
$$((0,-3,-2,-1,-4,-1,-4,-1,-4,-1,-4,-1,-4,-1,-2,-3)).$$

Note that none of these divisors contains any curves of self-intersection below -4. There are some bases/divisors that support NHC's and that contain -5 curves, beginning with $\mathbb{F}_5$, but no divisors supporting a non-Higgsable group arise in our data set that have curves of self-intersection -6 or below. This suggests that no divisor that supports a non-Higgsable gauge group can have any curves of self-intersection below -5, which would dramatically restrict the set of possible divisors supporting non-Higgsable groups.
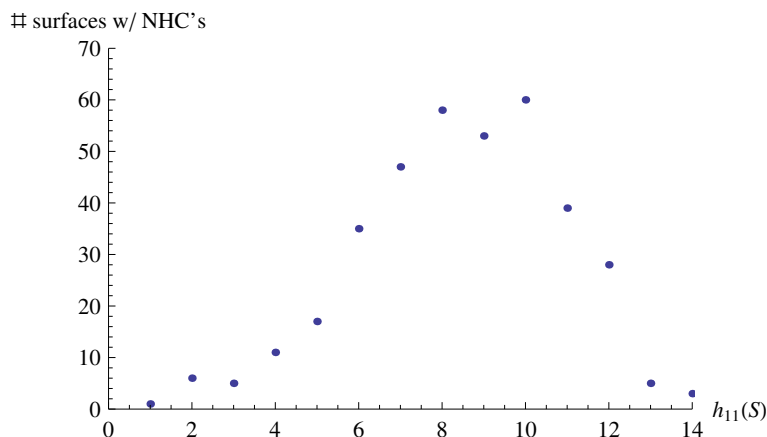
The number of bases B that support non-Higgsable groups on one or both sections is graphed as a function of $h^{1,1}(S)$ in figure 3.

There are a total of 8355 distinct bases B that have non-Higgsable clusters on one or both of the sections, of which 508 have non-Higgsable clusters on both sections. Thus, in the set of bases we have constructed there are close to 9000 distinct divisor + normal bundle combinations that support a non-Higgsable gauge group.

The average number of possible normal bundles per divisor for sections that support a non-Higgsable gauge group factor is graphed as a function of $h^{1,1}(D)$ in figure 4

3.5 The prevalence of non-Higgsable clusters

One basic question regarding threefold bases for 4D F-theory models is the extent to which non-Higgsable clusters are generic features in the sense that they arise for a large fraction of threefold bases. As described in section 2.3, virtually all base surfaces for 6D F-theory compactifications give rise to non-Higgsable gauge groups, with the only exceptions being the *weak-Fano* generalized del Pezzo surfaces. It is natural to ask whether a similar story holds in four dimensions. While the story for base threefolds is more complicated, the

Figure 3. Number of bases S that support non-Higgsable groups on a section of the $\mathbb{P}^1$ bundle over S, as a function of $h^{1,1}(S)$.
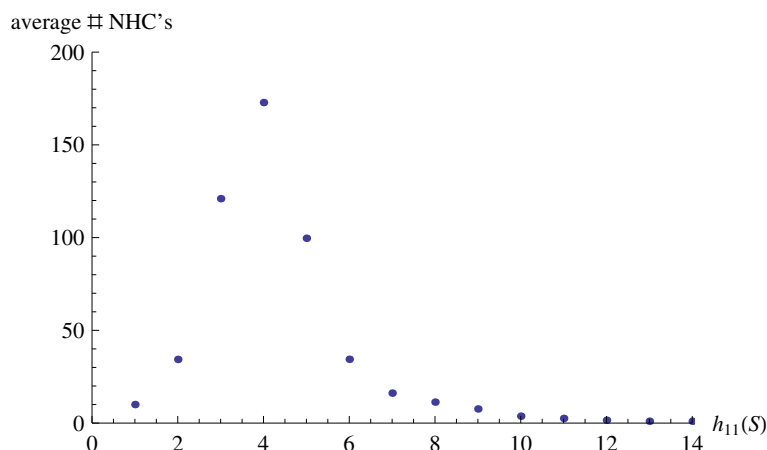
Figure 4. Average number of different normal bundles compatible with a non-Higgsable group on a section divisor, as a function of $h^{1,1}(S)$.

bases we study here provide some evidence that the general picture may be rather similar, at least for toric threefold bases.

We discuss some simple aspects of this question here, leaving further analysis for later work. We begin by discussing weak Fano threefolds; we then describe some explicit results from the classification of $\mathbb{P}^1$-bundle bases that we have carried out, and then examine some general features and analyses that may help both to explain the explicit results found here and lead towards further understanding of the generality of geometrically non-Higgsable structure in 4D models.

3.5.1 Weak Fano threefolds

In any dimension, a weak Fano variety satisfies the condition that $-K \cdot C \geq 0$ for any effective curve C (i.e., $-K$ is *nef*).[7] For surfaces, this condition implies that there are no irreducible effective curves of self-intersection -3 or below. Thus, for 6D F-theory compactifications, weak Fano bases are precisely those that have no non-Higgsable gauge group factors.

For toric threefolds B, the weak Fano condition has a similar but slightly different connection to the singularity structure of an elliptically fibered Calabi-Yau fourfold over B. In particular, the condition that $-K \cdot C \geq 0$ for all effective C is equivalent to the condition that there are no codimension one, two, or three loci in B where f or g are forced to vanish. Mathematically this is encoded in the statement that a divisor on a smooth toric variety is *basepoint free* if and only if it is nef (see [40], Proposition 1.5). At a heuristic level this can be understood from the fact that when $-K \cdot C < 0$, it means that there is no representative of the curve class C that is disjoint from or transversely intersecting $-K$, since such a representative would have a vanishing or positive intersection number. Thus, $-K \cdot C < 0$ implies that C is contained within the locus $-K$, and therefore it is also contained in $-4K$ and $-6K$, so that f and g would both need to vanish on C. We see from this that any base that is not weak Fano must have at least one curve on which f and g are forced to vanish. In the other direction, if $-K \cdot C \geq 0$ for all effective curves, then we expect that all curves can be put in general position so that they are not contained within $-K$. In the situation where the base is toric this expectation is realized and there is no locus on which f, g are forced to vanish when B is weak Fano. When B is not toric, however, this condition is not always correct. In any case, for toric bases such as those we consider here, the weak Fano condition is equivalent to the condition that there are no effective curves C in B where f, g necessarily vanish.

Note, however, that unlike for 6D compactifications, for 4D compactifications the fact that a base is not weak Fano does not necessarily imply the existence of a divisor on which f, g must vanish to at least orders $1, 2$, which is the condition needed for the presence of a geometrically non-Higgsable gauge group factor. In fact, there are threefold bases that are not weak Fano in which there are divisors where (f, g, Δ) vanish only to orders $(1, 1, 2)$. These exhibit a Kodaira type II singularity with no gauge group, which were studied via a local deformation in [14]; the fact that such singularities have no gauge group can be understood from the different vanishing cycles of the colliding branes. There are also threefold bases that are not weak Fano where f, g vanish only on curves. We explore next the cases where this occurs in the bases we have constructed.

3.5.2 Explicit computation of base threefolds without non-Higgsable groups

Of the 109,158 threefold bases that we have constructed, all but 1,824 have a geometrically non-Higgsable gauge factor on at least one divisor. As mentioned above, however, there is a richer structure for those bases without non-Higgsable gauge factors than there is in six-

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[7]A weak Fano variety also has an anti-canonical class $-K$ that is *big*, but $-K$ is big for any toric variety [39], so we focus attention in this paper on the nef condition.

dimensional compactifications. In particular, even in the absence of non-Higgsable gauge groups, a base can have (f, g, Δ) vanishing to order $(1, 1, 2)$ on one or more divisors, or to higher orders on one or more curves.

Of the 1824 threefold bases without non-Higgsable gauge groups, 1788 have no generic vanishing of f, g on any divisor. Of the other 36, there are 35 that have a divisor where (f, g) vanish to orders $(1, 1)$. The simplest of these is a $\mathbb{P}^1$ bundle base B constructed from a toric generalized del Pezzo surface S with curves of self-intersection $((1, -2, -1, -2, -2, 0))$ and twist $T = d_3 + d_4 + 2d_5 + 3d_6$. The resulting threefold has f, g both vanishing to order one on Σ_-, the divisor associated with the toric ray $s_- = (0, 0, 1)$ in (3.2). There is also one case in which f, g vanish to orders $(1, 0)$ on a toric divisor, given by the threefold base associated with a $\mathbb{P}^1$ bundle over the toric surface having self-intersections $((-2, -2, -1, -2, -2, -1, -2, -2, -1))$ and twist $d_3 + d_4 + d_5 + 2d_6 + d_7$.

Of the 1788 bases that have no generic vanishing of f, g on a divisor, all but 80 have the property that f, g vanish on some curves in the base, often to relatively high orders. A typical example is given by the $\mathbb{P}^1$ bundle over $\mathbb{F}_1$, the toric surface with self-intersections $((1, 0, -1, 0))$, with twist $T = 2d_4$. In this case, there is no generic vanishing of f, g on any toric divisor but f, g vanish to orders $(2, 3)$ on the curve defined by $D_3 \cap \Sigma_-$. An interesting open question is the physical nature of F-theory compactifications on bases of this type. While there is no non-Higgsable gauge group at the level of geometry, the higher-order codimension two singularity suggests the presence of some kind of "matter curve". It is possible that these specific singularities may just be a generalization of the kind of cusp singularity that appears at points in the discriminant locus for 6D compactifications, without matter from $7-7$ strings, but it is also possible that these singularities have further physical significance, for example as uncharged matter. This is an interesting question for further investigation.

3.5.3 General features of bases without non-Higgsable structure

One simple feature of the $\mathbb{P}^1$-bundle bases that lack non-Higgsable gauge groups is that they are all constructed from surfaces S that have no curves of self-intersection -4 or below. This is relatively easy to understand. Over a curve d_i of self-intersection $d_i \cdot d_i = n = -4$ or below, from (3.6) the twist must satisfy $|T \cdot d_i| \leq 1$. If $T \cdot d_i = 0$, then the local geometry is simply that of $\mathbb{P}^1 \times S$ in the vicinity of the curve, so there must be a non-Higgsable cluster from the standard 6D analysis. If $T \cdot d_i = 1$ (or equivalently -1), then the local geometry of D_i is that of $\mathbb{F}_1$ with a normal bundle of $\mathcal{O}_{D_i}(nF)$, where X, F are the standard basis of effective divisor curves on $\mathbb{F}_1$ with $X \cdot X = -1, X \cdot F = 1, F \cdot F = 0, -K_{D_i} = 2X + 3F$. In this case from (5.2) we have

$$f_0 \in \Gamma(\mathcal{O}(8X + 12F + 4nF)). \tag{3.9}$$

The divisor $8X + 12F + 4nF$ in D_i is not effective unless $n \geq -3$. Similarly,

$$g_0 \in \Gamma(\mathcal{O}(12X + 18F + 6nF)), \tag{3.10}$$

$$g_1 \in \Gamma(\mathcal{O}(12X + 18F + 5nF)). \tag{3.11}$$

In fact these bundles do not have global sections if $n \leq -4$, in which case f_0, g_0 and g_1 do not exist and there must be a non-Higgsable gauge group on D_i.

We can also consider explicitly the consequences of imposing the weak Fano condition discussed above on the threefold base B. The weak Fano condition states that $-K \cdot C \geq 0$ for any curve C in the base B. For the toric $\mathbb{P}^1$-bundle bases that we are considering, the effective holomorphic curves C are either curves that can be described as intersections of the form $\Sigma_\pm \cdot D_i$, or of the form $D_i \cdot D_{i+1}$ (using the cyclic convention $N + 1 \to 1$). There are linear relations between these curves; in particular, each of the curves $D_i \cdot D_{i+1}$ is simply the fiber F of the $\mathbb{P}^1$ bundle, and the curves $\Sigma_- \cdot D_i$ are essentially the curve classes $d_i \subset \Sigma_-$, which have two relations in homology, while the curves $\Sigma_+ \cdot D_j$ are linear combinations of the $\Sigma_- \cdot D_i$ with the fiber F. It is convenient to keep all of these classes separate, however, as they act as an effective (though redundant) set of generators for the Mori cone (see e.g. [41]). To check the weak Fano condition, therefore, it suffixes to check that $-K \cdot C \geq 0$ for each of the curve classes just mentioned. Since all of these curves may be represented in homology as the intersection of toric divisors, we can assess this condition by computing the triple intersection numbers of toric divisors. The divisors associated with sections satisfy the relation

$$\Sigma_+ = \Sigma_- + \sum_i t_i D_i \, . \tag{3.12}$$

The 3D cones of the fan are all associated with nonzero triple intersection numbers

$$\Sigma_\pm \cdot D_i \cdot D_{i+1} = 1 \tag{3.13}$$

where $i + 1$ is again taken cyclically on the base. From the structure of the base we inherit the intersections

$$\Sigma_\pm \cdot D_i \cdot D_i = n_i \, . \tag{3.14}$$

We have $\Sigma_+ \cdot \Sigma_- = 0$, and

$$\Sigma_+ \cdot \Sigma_+ \cdot D_i = \Sigma_+ \cdot \left(\sum_j t_j D_j \right) \cdot D_i = T \cdot d_i \, . \tag{3.15}$$

with intersections in S implied in the last expression and henceforth. Similarly, we have $\Sigma_- \cdot \Sigma_- \cdot D_i = -T \cdot d_i$.

With this information we can now check the weak Fano condition. Using the fact that the anti-canonical class of B is $-K = \Sigma_+ + \Sigma_- + \sum_j D_j$ we compute

$$- K \cdot \Sigma_\pm \cdot D_i = 2 + n_i \pm T \cdot d_i \, . \tag{3.16}$$

If this expression is negative the weak Fano condition is violated, which (using equation (3.4)–(3.7)) occurs whenever we have a -3 curve in the base, or a -2 curve d with a nonzero twist $T \cdot d \neq 0$, or a -1 curve d with a twist $|T \cdot d| \geq 2$, or more generally a curve in the base of self-intersection m with a twist $|T \cdot d| \geq 3 + m$. We also compute

$$- K \cdot D_i \cdot D_{i+1} = 2 \tag{3.17}$$

and thus any violation of the weak Fano condition must come from curves of the form $\Sigma_\pm \cdot D_i$. We therefore conclude that there is a curve C with $-K \cdot C < 0$ if and only there is a curve d_i with (3.16) negative, and we use this to check whether or not B is almost Fano.

Analyzing the allowed threefold bases B in terms of the base surfaces S and twists T (according to the condition just derived) shows that there are precisely 80 cases that are weak Fano. These are the cases where there is no vanishing of (f, g) on any curve in the threefold base B. In the cases that are not weak Fano, we can carry out an explicit local analysis that demonstrates the appearance of curves where (f, g) vanish. For example, consider the case mentioned above of a $\mathbb{P}^1$ bundle on $\mathbb{F}_1$ with twist $2d_4$. In this case, the section Σ_- is a complex surface $\mathbb{F}_1$ embedded into B with normal bundle $N = -2F$. As in (3.9),

$$f_k \in \Gamma(\mathcal{O}_{\mathbb{F}_1}(8X + 12F - 2(4-k)F)). \tag{3.18}$$

For $k = 0$, the divisor associated with the line bundle is $c = 8X + 4F$, which satisfies $c \cdot X = -4$, so f_0 vanishes on X. Similarly for f_1, g_0, g_1, g_2, so we see that in general this type of weak Fano base with a divisor $\mathbb{F}_1$ having normal bundle $N = \mathcal{O}_{\mathbb{F}_1}(-2F)$ must have a curve with orders of vanishing of (f, g) of at least $(2, 3)$. Similar arguments can be made in other cases where the base is not weak Fano.

Thus, we see that there are essentially 3 distinct classes of bases. For the weak Fano bases, there are no curves in the base on which f, g must vanish. For a somewhat larger class of bases, which are not weak Fano, there are no non-Higgsable gauge groups, but f, g vanish on some curves or possibly divisors in the base. And for the majority of threefold bases, there are divisors that carry non-Higgsable gauge groups.

While there is no complete classification of weak Fano threefolds,[8] we expect that this set of bases forms a very small fraction of the full set of allowed threefold bases for F-theory compactifications to 4D. We also expect that the class of bases without non-Higgsable gauge group factors may be relatively small. In the rest of this paper we give some further evidence for these conclusions.

4 Hodge numbers of elliptic CY fourfolds over $\mathbb{P}^1$-bundle bases

For 6D F-theory models, the Hodge numbers of the generic elliptic Calabi-Yau threefold over each base provide a convenient parameterization of the set of models that gives a simple birds-eye view of the space of allowed theories. Figure 1, for example, shows the Hodge numbers for the generic elliptic CY threefolds over the full set of toric bases S. From the analysis of [18, 25–27, 29], we know that the outline of the set of all possible models is clearly captured in this diagram, with the set of bases that do not give rise to non-Higgsable clusters a small subset confined to the far left of the diagram.

While for 6D models, at least for large $h^{2,1}(X_3)$, rigorous bounds are known for the set of possibilities, the state of knowledge for Calabi-Yau fourfolds is much weaker. Nonetheless, from what is known of the set of toric constructions for CY fourfolds, it seems at least

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[8]Note, however, that there are only about 100 Fano threefolds, which instead satisfy the strict inequality $-K \cdot C > 0$; we consider these further in the following section.

based on our current limited understanding that the story for elliptic CY fourfolds may be not be too wildly different from that for CY threefolds. To get some initial sense of how the models we consider here fit into the broader landscape of possibilities for elliptic Calabi-Yau fourfolds, we describe here some rough aspects of the set of Hodge numbers for elliptically fibered Calabi-Yau fourfolds produced from generic Weierstrass models over the base threefolds we have constructed.

Systematic constructions of a large set of Calabi-Yau fourfolds (which may or may not be elliptically fibered) were carried out using toric and related methods in [33, 58–60]. The distribution of Hodge numbers $h^{1,1}, h^{3,1}$ of these fourfolds has a striking similarity to the "shield" pattern of Hodge numbers for CY threefolds identified in the analysis of Kreuzer and Skarke [30, 31]; as we now understand it, this shield pattern for toric hypersurface CY threefolds defines the outer boundary of the set of Hodge numbers for *all* elliptic CY threefolds, whether toric or not. In the absence of examples to the contrary, it is natural to expect that the Hodge numbers of all elliptic CY fourfolds may similarly be contained within the shield-shaped pattern of known fourfold constructions. In this section we explore how the bases we have constructed fit into the larger class of known Calabi-Yau fourfolds using estimates of the Hodge numbers based on the base of the fibration.

4.1 Computing Hodge numbers of fourfolds from base geometry

For a Calabi-Yau threefold that is a generic elliptic fibration over a complex surface base S, the Hodge numbers of the CY threefold can be determined from the geometry of the base, as described in [25, 26]. In principle, a similar approach can be taken to computing the Hodge numbers of a generic elliptic Calabi-Yau fourfold X over a toric threefold base, though some aspects of such a computation are not yet fully understood. We focus here on the Hodge numbers $h^{1,1}(X), h^{3,1}(X)$. A more complete description of the computation of Hodge numbers from the base geometry will be given elsewhere [61]; here we simply summarize some aspects of that story relevant for the computations here.

We begin with $h^{1,1}(X)$, for which the analysis is the simplest. From the Shioda-Tate-Wazir relation [2, 62],

$$h^{1,1}(X) = h^{1,1}(B) + \operatorname{rank}(G) + 1\,, \tag{4.1}$$

where G is the (non-Higgsable) gauge group associated with the F-theory compactification on the generic elliptic fibration. This relation asserts that the set of divisors on X is spanned by divisors on B lifted to divisors on X, "vertical" (sometimes called Cartan) divisors arising from the resolution of codimension one singularities associated with nonabelian components of G, and "horizontal" linearly independent sections. The contributions to $h^{1,1}(X)$ from $h^{1,1}(B)$ and the nonabelian part of G are easy to compute from the geometry of B, since G for the generic elliptic fibration is simply the non-Higgsable nonabelian gauge group in G. For most bases we expect only a single section for the generic elliptic fibration, so there is no contribution to rank G from abelian factors. In general, the rank of the abelian gauge group is determined by the Mordell-Weil group of the fibration, with the number of U(1) factors given by $r = k-1$, where k is the number of linearly independent global sections of the fibration. For toric base surfaces for 6D compactifications, $h^{1,1}(X_3)$ can be computed

independently using anomaly cancellation conditions, and in all cases matches (4.1) with no contribution from the Mordell-Weil group, so that there are no non-Higgsable U(1) factors over any toric base surface for elliptic Calabi-Yau threefolds. The situation for fourfolds is less clear. We do not have an independent approach analogous to the anomaly cancellation mechanism of 6D theories to verify the absence of non-Higgsable abelian factors, though in all cases where explicit checks have been made there are no such contributions. In 6D, a very small class of non-toric bases has been identified that support non-Higgsable U(1) factors [26, 63]. While similar non-toric bases with non-Higgsable abelian factors are to be expected for 4D models, we have no reason to expect that this can happen over toric bases. In our computation of $h^{1,1}(X)$ for the elliptic fourfolds over base threefolds B we assume the absence of a nontrivial Mordell-Weil group, and simply compute using the non-Higgsable nonabelian gauge group. We leave open the possibility that the resulting Hodge numbers may be slightly off, if non-Higgsable U(1) factors arise in certain cases.

We now turn to $h^{3,1}(X)$. This corresponds to the number of complex structure moduli for the elliptic Calabi-Yau fourfold X. A simple estimate for $h^{3,1}$ can be determined by counting the allowed monomials in the Weierstrass model. The number of such monomials W_f in f is simply the number of vectors $q \in N^*$ such that $\langle q, w_i \rangle \geq -4$ for all rays $w_i \in N$ in the fan defining B as a toric variety, and similarly for W_g with $\langle q, w_i \rangle \geq -6$. As in the 6D case [18], there are several additional terms that must be included. First, there is an overparameterization of the Weierstrass model by the number $w_{\text{aut}} = 3 + w_{\text{polar}}$ of automorphisms, where 3 is the dimension of the base variety and the number of universal toric automorphisms, and w_{polar} is the set of additional automorphisms, which are in one-to-one correspondence with lattice points in the interior of a codimension one face of the polar polytope defined through $\langle q, w_i \rangle \geq -1$ [64]. Second, there are additional degrees of freedom that must be added, associated with each combination of a ray w_i that lies on the interior of a one-dimensional face of the convex polytope constructed from all the w_j's, and a monomial q that lies on the interior of the dual one-dimensional face of the dual polar polytope in N^*. As described in more detail in [61], this term follows from the combinatorial analysis of Batyrev [65] in cases where there is a simple reflexive polytope that can be built from a $\mathbb{P}(2, 3, 1)$ fibration over the toric surface S, and we assume that it holds more generally in other circumstances as well. We denote the number of these additional degrees of freedom by W_{11}. Including these additional terms, and subtracting one for the universal natural scalar hypermultiplet, we have

$$h^{3,1}(X) = W_f + W_g - w_{\text{aut}} + W_{11} - 1 \,. \tag{4.2}$$

As an example of a situation where the extra term W_{11} becomes relevant, consider the $\mathbb{P}^1$ bundle over $\mathbb{F}_0$ with twist $2X$. In this case we have $w_- = (0, 0, 1) = (w_1 + w_3)/2$, where $w_1 = (0, 1, 0)$ and $w_3 = (0, -1, 2)$, so w_- is in the interior of a 1D face of the convex polytope. It is easy to check that there is one monomial in the interior of the dual 1D face of the polar polytope, so in this case we have $W_{11} = 1$.

There are a number of subtleties in the computation of the Hodge numbers of X that force us to treat the formulae (4.1) and (4.2) as only approximate "Hodge numbers". One issue is that unlike for Calabi-Yau threefolds, where it is known that there is a geometric

resolution of all the relevant singularities [66], for fourfolds there are situations where singularities can arise that cannot be resolved. In such cases there is no obvious geometric meaning to the Hodge numbers $h^{1,1}, h^{3,1}$. In [33], the approach taken was to use Vafa's formulae [67] in terms of the chiral ring of Landau-Ginsburg models, with the idea that the Hodge numbers should make sense in this context even in the absence of a geometric resolution. In various simple cases the computation we get here using base geometry matches with that analysis. As mentioned above, we are assuming that there are no generically nontrivial Mordell-Weil groups over any of the bases we consider. There are also issues in applying (4.2) when there is not a direct simple construction of the fourfold using a reflexive polytope, as discussed further in [61], though we expect the formula to still be valid in those cases. In any case, for the purposes of this paper we can simply treat (4.1) and (4.2) as "approximate" computations of the Hodge numbers of the generic elliptic Calabi-Yau fourfold (or its analogue when there is no geometric resolution), as a rough way of characterizing the distribution of bases in the context of the larger set of possible Calabi-Yau fourfold Hodge numbers. In the remainder of this paper we simply use these estimates as "Hodge numbers" with no further apology, keeping in mind the various issues just raised.

4.2 (Approximate) Hodge numbers of elliptic CY's over $\mathbb{P}^1$-bundle bases

Using the approach described in the preceding section, we have computed the approximate "Hodge numbers" of the generic elliptic Calabi-Yau fourfold over each base in our list in terms of the geometric data of the toric base B. The distribution of Hodge numbers for our bases are shown in figure 5, and compared in figure 6 to the Hodge numbers computed in [58] for the full set of Calabi-Yau fourfolds realized as hypersurfaces in weighted projective space using the Batyrev reflexive polytope approach. Note that though the Calabi-Yau fourfolds from [58] are not necessarily elliptically fibered, recent evidence suggests that a large fraction of Calabi-Yau threefolds are in fact elliptically fibered [25, 29], particularly at large Hodge numbers, and along similar lines a recent analysis of complete intersection Calabi-Yau fourfolds [68] shows that over 99.5% are actually elliptically fibered.

In general, the bases we have constructed give fourfolds with relatively small Hodge numbers, in particular with small $h^{1,1}(X)$. As we describe in the remainder of this section, we take this to mean that our data set gives a higher than typical fraction of 4D F-theory models *without* non-Higgsable gauge groups, so that from this set we are actually getting a smaller fraction of models with non-Higgsable gauge factors, and in the full set of threefold bases we might expect non-Higgsable clusters over much more than 98% of the possible bases.

4.3 Hodge numbers and the hierarchy of non-Higgsable groups

There is a natural hierarchy in the complexity of bases, associated with the maximal Kodaira singularity type that is forced to arise for a generic Weierstrass model over any given base B. Lowest in this hierarchy are the Fano and weak Fano bases, which support no generic Kodaira singularities of any codimension, for threefold bases as well as for surfaces. These bases generally give rise to elliptic Calabi-Yau manifolds that have relatively small Hodge numbers; in particular the Picard number $h^{1,1}(X)$ is generally quite low for Fano
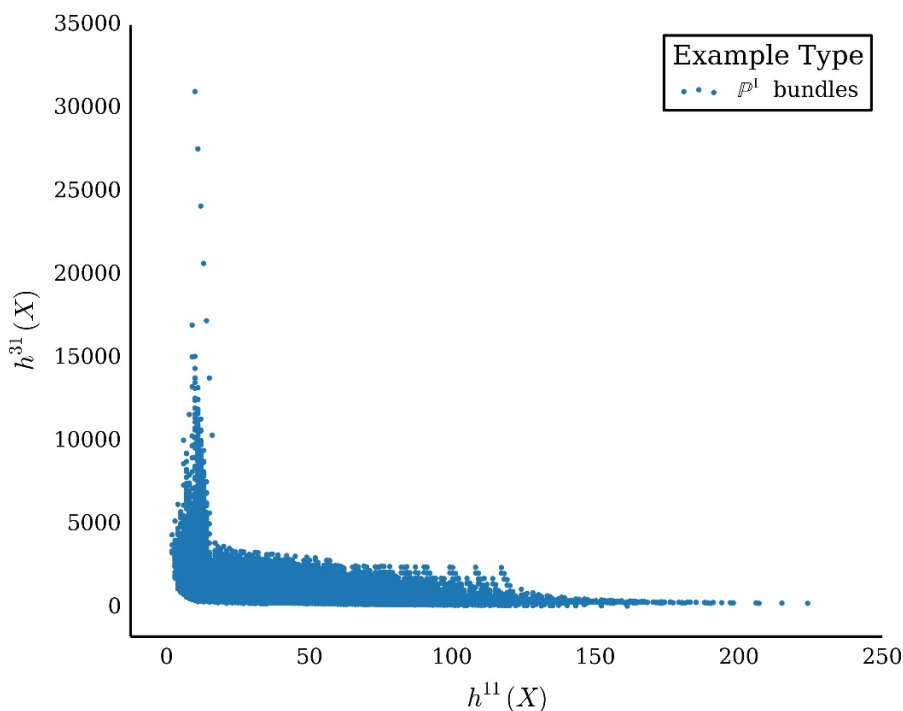
Figure 5. Plotted are the Hodge numbers for each example in our data set, as estimated from the geometry of each base.

and weak Fano bases. As the Kodaira singularity types increase, the Hodge numbers, particularly $h^{1,1}(X)$ increase. We illustrate this hierarchy first for the well-understood case of elliptic Calabi-Yau threefolds, and then compare with the results for our threefold base constructions for elliptic Calabi-Yau fourfolds.

A plot of the set of Hodge numbers that can arise for generic elliptically fibered CY threefolds over a given toric base with each possible maximal Kodaira singularity type is shown in figure 7. We focus primarily on toric bases for simplicity, though consideration of non-toric bases as in [26, 27, 29] shows that essentially the same pattern holds for non-toric bases. Clearly, the Hodge numbers of threefolds over bases that support no singularity (i.e., Fano and weak Fano bases) or small Kodaira singularity types (e.g. type III, IV, I_0^*) are relatively small. The absolute numbers of threefolds that have various automatic Kodaira singularity types — i.e., non-Higgsable clusters — are increasingly large as the singularity type increases. As mentioned earlier, there are only 10 Hodge number pairs associated with generic elliptic fibrations over bases that do not support Kodaira singularities. Of the 7524 distinct Hodge number pairs that are realized by generic elliptic fibrations over toric bases, 7 are associated with such weak Fano bases. Only another 50 Hodge number pairs are associated with bases giving only type III singularities (i.e. with only non-Higgsable SU(3) factors). And indeed, only 426 of the Hodge number pairs (less than 10%) are associated with bases that generate Kodaira singularities that are not worse than I_0^*. 5296
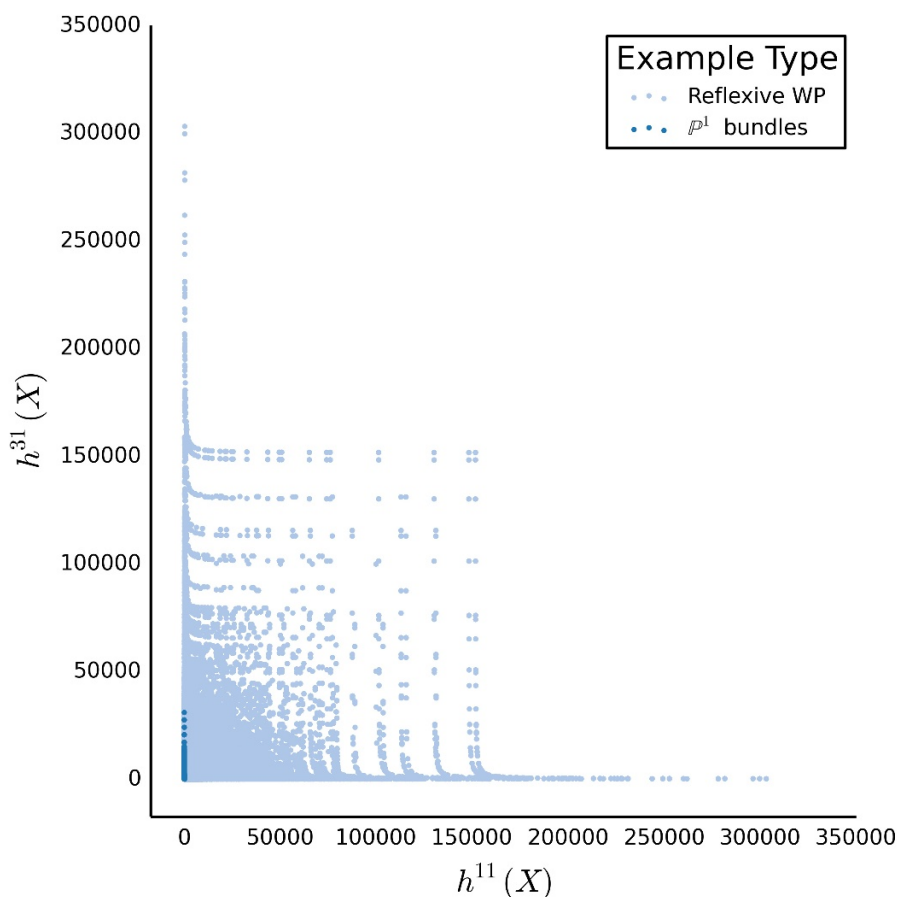
Figure 6. Hodge numbers of the examples in our data set plotted in the context of the set of Hodge numbers from Calabi-Yau fourfolds described through reflexive polytopes as hypersurfaces in 5D weighted projective space [58].

of the Hodge number pairs are associated with E_8 non-Higgsable gauge groups.[9] The upshot of this analysis is that there is a clear picture of the distribution of bases for 6D F-theory compactifications. There is a very small fraction of bases that do not give rise to non-Higgsable gauge groups in the low-energy theory. The majority of bases give rise to non-Higgsable gauge groups, with the rank of the groups generally increasing with the Hodge numbers of the elliptic threefold (and correspondingly with the numbers of fields in the low-energy supergravity theories).

We now consider the distribution of Kodaira singularity types for the $\mathbb{P}^1$-bundle three-fold bases that we have constructed. A graph of the Hodge numbers, again coded by maximum Kodaira singularity type, is shown in figure 8. Again, we see the pattern that the bases that generate smaller Kodaira singularity types are less numerous and produce

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[9]Note that this set of toric data includes bases that have -9, -10, and -11 curves, which lead to bases that are not strictly toric. In our fourfold analysis in this paper we have not included such bases so we miss a large number of theories with geometrically non-Higgsable E_8 gauge groups, as discussed further below.
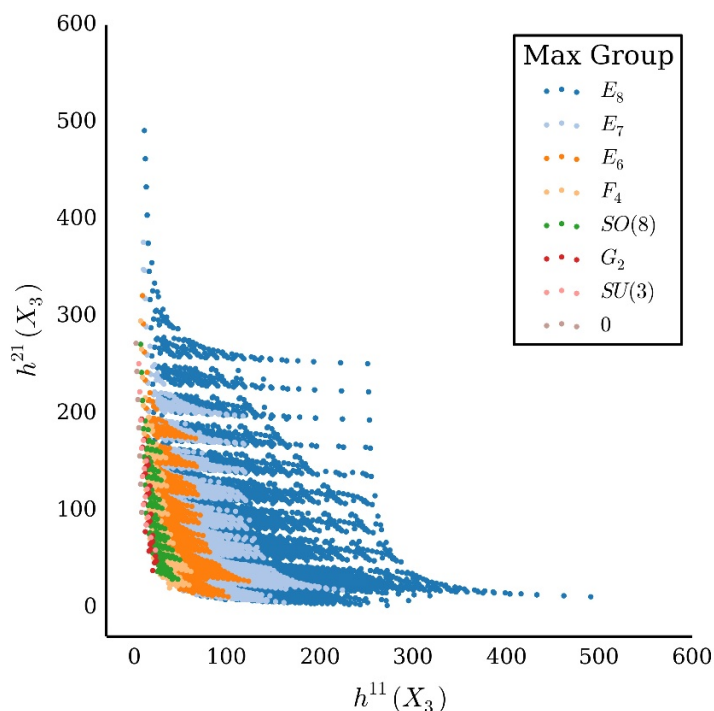
Figure 7. Distribution of Hodge numbers for elliptic Calabi-Yau threefolds over bases that have a given maximum Kodaira singularity type. When different bases give the same Hodge numbers, the color shown is the minimum across bases of the maximum for each base; i.e., all numbers associated with weak Fano bases that produce no singularities are shown, all numbers associated with bases that produce at most type IV singularities but are not weak Fano are shown, etc.

fourfolds with smaller Hodge numbers. As mentioned above, we have only included strictly toric $\mathbb{P}^1$-bundle bases that support E_8 non-Higgsable gauge factors when the E_8 factor arises on a divisor with no codimension two $(4,6)$ singularities, unlike in the 6D plot, figure 7. Thus, we see very few examples with E_8 non-Higgsable groups in the 4D plot. This is presumably an artifact of the class of bases we are considering; we expect that as for 6D, bases that produce at least one non-Higgsable E_8 will dominate the distribution for larger Hodge numbers. In particular, we note that when we include $\mathbb{P}^1$-bundle constructions over toric bases where there is an E_8 on a divisor with a $(4,6)$ codimension two singularity on a curve within the E_8 divisor, analogous to a $-9, -10,$ or -11 curve for a twofold base, we find an additional 68,528 threefolds that support a non-Higgsable E_8. Unlike in the 6D case, however, computing the Hodge numbers of the threefold that results after blowing up the $(4,6)$ curves is not straightforward, so we have not included these bases in our analysis. This does suggest, however, that the domination by E_8 non-Higgsable groups at large Hodge numbers for 4D theories will parallel the structure for 6D theories.

Considering some specific types of bases, we begin with Fano threefold bases. It is known that there are 105 Fano varieties of dimension three [69]. Of these 18 are toric, and were considered as F-theory bases in [33]. Of these 18, 10 are included in our list
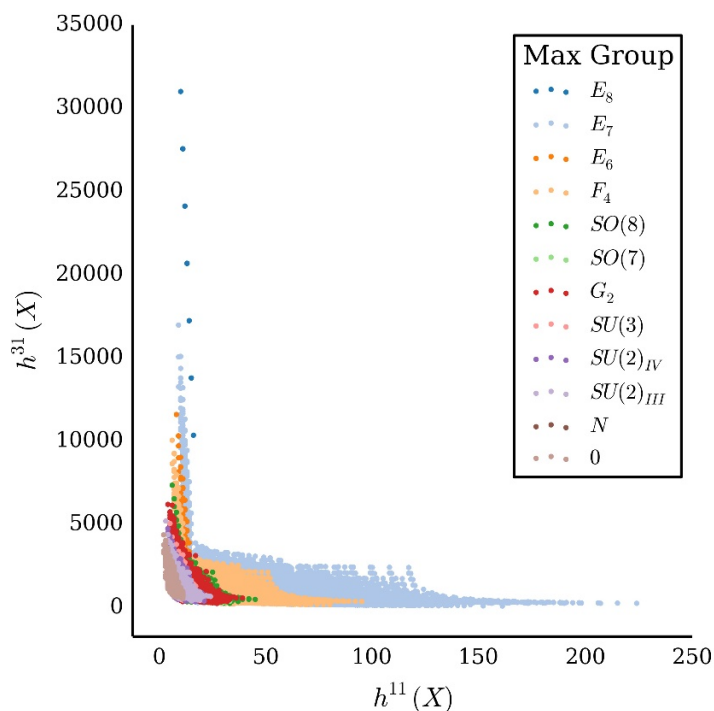
Figure 8. Distribution of Hodge numbers for elliptic Calabi-Yau fourfolds over bases that have a given maximum Kodaira singularity type. When different bases give the same Hodge numbers, the color shown is the minimum across bases of the maximum for each base, as in the corresponding 6D plot.

of $\mathbb{P}^1$-bundle bases, and are included in the 80 weak-Fano bases that produce no generic singularities. As mentioned above, these are a subset of the 1824 bases that produce no non-Higgsable gauge groups (figure 9). The largest value of $h^{1,1}(X)$ for a Fano base in our set is 6 (for $\mathbb{P}^1 \times dP_3$), and the largest value for a weak Fano base is $h^{1,1}(X) = 9$ (for $\mathbb{P}^1 \times gdP_6$, where gdP_6 is the generalized del Pezzo surface of degree 6 given by the toric surface S with self-intersections $((-2,-2,-1,-2,-2,-1,-2,-2,-1)))$. For bases that do not support any non-Higgsable gauge group, the maximum of $h^{1,1}(X)$ is 12, for bases that support at most an SU(2) the maximum is 25, etc. The maximum values of $h^{1,1}(X)$ where X has each possible maximum Kodaira singularity type are listed in table 2. The picture for threefold bases for 4D compactifications thus closely matches that of surface bases for 6D compactifications. In particular, the set of bases that do not give rise to non-Higgsable gauge factors is relatively small and confined to the region of small $h^{1,1}(X)$. We expect that when a much broader class of elliptic Calabi-Yau fourfolds is considered, this pattern will persist, and the large number of fourfolds at larger $h^{1,1}(X)$ that are not given through a $\mathbb{P}^1$-bundle construction will be dominated by those with non-Higgsable gauge groups.
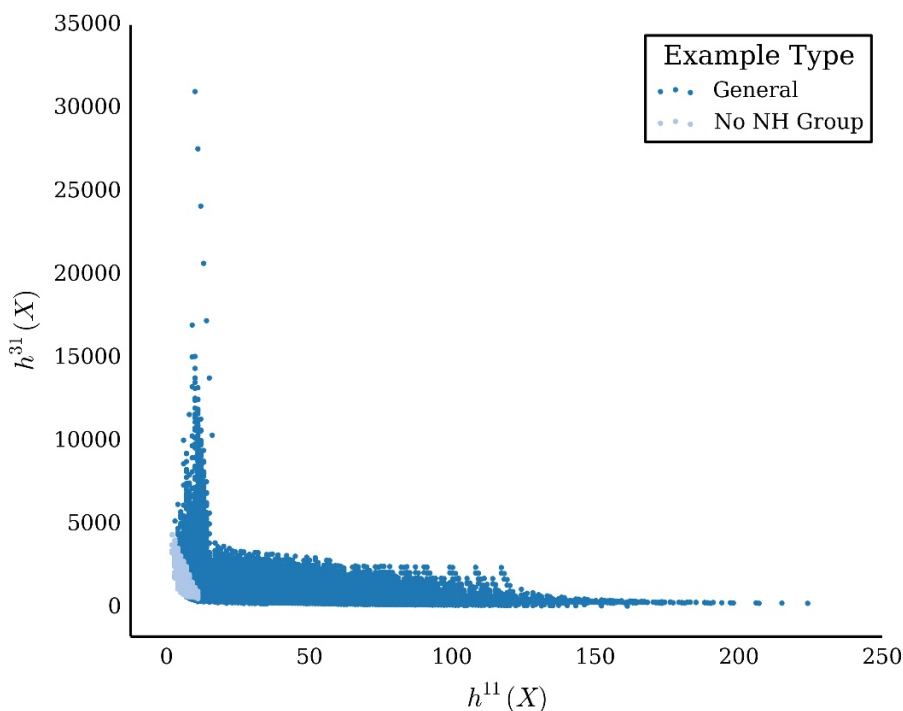
Figure 9. Plotted are the Hodge numbers for each example in our data set, where examples without a non-Higgsable gauge group are denoted in light blue.

G	none	SU(2)	SU(3)	G_2	SO(7)	SO(8)	F_4	E_6	E_7	E_8
sing. type	$\leq II$	III	IV	I_0^*	I_0^*	I_0^*	IV^*	IV^*	III^*	II^*
max $h^{1,1}(X)$	12	25	25	40	40	46	96	96	225	492*

Table 2. Table of maximum value of Hodge number $h^{1,1}(X)$ for any base B in our dataset that supports Kodaira singularities that are no worse than a given type, with associated non-Higgsable gauge group factor G. The starred value 492 for E_8 refers to the maximum of $h^{1,1}(X)$ for a slightly broader class of $\mathbb{P}^1$-bundle base constructions where we allow $(4,6)$ curves on a divisor supporting an E_8; the base with this value is $\mathbb{P}^1 \times S_{491,11}$, where S is the base surface that supports the elliptically fibered Calabi-Yau threefold X_3 with largest known $h^{1,1}(X_3)$ (see e.g. [18, 25]).

4.4 Minimal models

Some further discussion, and comparison to the better-understood scenario in 6D F-theory models, may be helpful in clarifying the role of the bases we construct in this paper in the context of the larger set of all 4D F-theory compactifications.

We begin with a brief summary of the situation in 6D, expanding on the review of section 2.3. For 6D F-theory models, all possible base surfaces (including non-toric bases, excluding only the trivial case of the Enriques surface) are constructed as blow-ups of the minimal bases $\mathbb{P}^2$ and $\mathbb{F}_m, m = 0, \dots, 8, 12$. These blow-ups are performed by blowing up points in the minimal bases, giving additional divisors and increasing $h^{1,1}(B_2)$ and $h^{1,1}(X_3)$.

Any such blow-up leaves fixed or decreases the value of the self-intersection of any given curve, and thus cannot decrease the set of non-Higgsable gauge groups. Thus, the set of all F-theory models associated with generic elliptic fibrations over any base is constructed by blowing up points on the minimal bases, where each blow-up decreases $h^{2,1}(X_3)$, increases $h^{1,1}(X_3)$, and either leaves invariant or increases the non-Higgsable gauge content of the theory. This matches with the structure of figure 7, where the del Pezzo and generalized del Pezzo surfaces are realized by blow-ups of P^2, which has $(h^{1,1}, h^{2,1}) = (2, 272)$ for a generic elliptic fibration, and further blow-ups on -2 curves or below produce bases with larger $h^{1,1}(X_3)$ and non-Higgsable gauge groups. As another illustration, the base $\mathbb{F}_8$ has Hodge numbers $(10, 376)$ and a -8 curve supporting an E_7 non-Higgsable factor. Blow-ups of this base on points that do not lie on the -8 curve produce a new set of bases that support a maximal E_7 gauge group — seen in the family of (light blue) points that sit down and to the right from the point $(10, 376)$ in figure 7. This pattern of E_7 structures was also noted in [70]. In particular, this picture makes it clear that the only bases that support elliptically fibered Calabi-Yau threefolds without non-Higgsable gauge groups are those that come from a limited set of blow-ups on the only minimal bases that do not support non-Higgsable gauge groups (i.e., those without curves of self-intersection -3 or below, which are $\mathbb{P}^2, \mathbb{F}_1$, and $\mathbb{F}_2$.

We expect that a similar story will hold in 4D, though there are many additional complications. The analogue of the minimal model story for surfaces is the Mori program for threefolds [37], which is rather more complicated. We leave a more thorough treatment of this story for future work, and make only a few general heuristic comments on the structure of what is expected as motivation to set the context for the bases we have constructed here. Roughly speaking, the minimal threefolds for the Mori program should be distributed in three general classes: first, Fano bases; second, bases that are $\mathbb{P}^1$ bundles over a base surface S; and third, bases that are surface S bundles over $\mathbb{P}^1$. The Fano bases are a relatively small set including spaces like $\mathbb{P}^3$. The second is essentially the class we are studying here, with various further restrictions such as that S and B are toric. And the third set is another interesting class of constructions that will be described further elsewhere. Note that the third class includes a large number of constructions with much larger $h^{3,1}(X)$ than those considered here. For example, the Calabi-Yau fourfold with the largest known $h^{3,1} = 303,148$, is an S bundle over $\mathbb{P}^1$ with $S = S_{251,251}$ with $S_{251,251}$ the base surface for the elliptic CY threefold with Hodge numbers $(251, 251)$. The third class produces divisors with less interesting non-Higgsable structure, however, since when we restrict to the toric context all divisors are either Hirzebruch surfaces $\mathbb{F}_m$ or the fiber surface S with a trivial normal bundle. For a full analysis of the Mori theory story, one should include the possibility of singularities in the bases. Such singularities can be associated with superconformal field theories coupled to the supergravity theory [20, 71, 72]. In the present work we focus on smooth toric bases B constructed as $\mathbb{P}^1$ bundles over smooth surfaces S in the class constructed in [18].

In this context, we can analyze our $\mathbb{P}^1$-bundle bases and consider which of them are "minimal" in the simple sense that there is no divisor on B that can be blown down (i.e. shrunk to a point) to give another smooth toric base B'. Such bases are not truly minimal in the sense of Mori theory, since we do not consider flips, flops, or blowing down to singular spaces. But they do give a simple picture of how some of the bases we consider may form
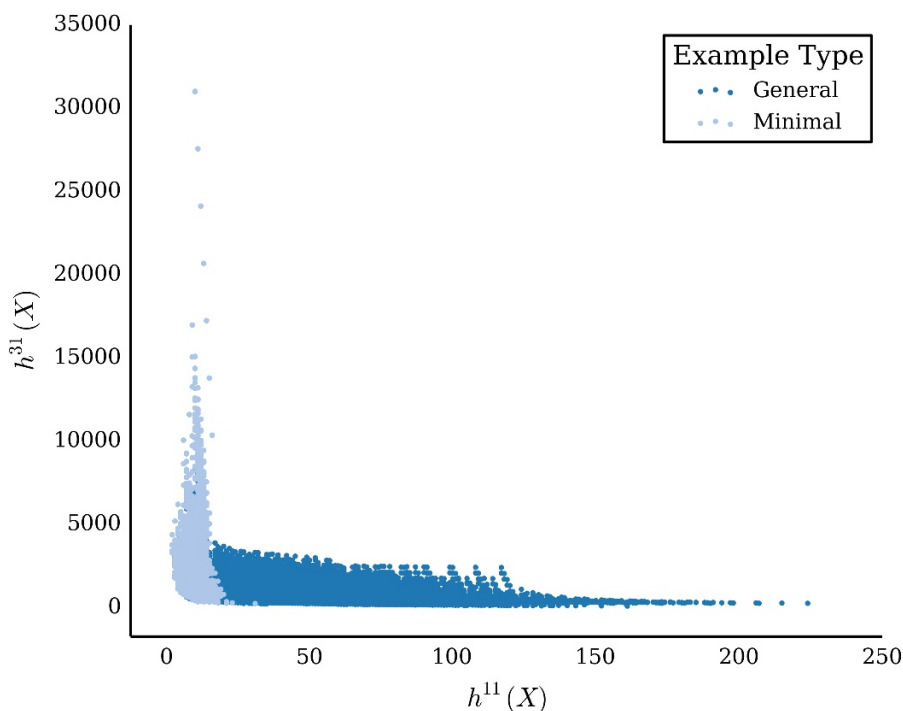
Figure 10. Plotted are the Hodge numbers for each example in our data, where examples that are "minimal", in the sense that there is no divisor that can be blown down to give another smooth toric base, are denoted in light blue.

a set of smooth minimal toric base threefolds from which a wider range of threefold bases, with larger non-Higgsable gauge groups, can be constructed by sequential blow-ups. The criterion that one of the $\mathbb{P}^1$-bundle bases in our set can be blown down on a divisor is very simple. The divisor Σ_- can only be blown down to a point in the case where the base surface is $S = \mathbb{P}^2$ and the twist is $T = H$. Σ_- can be blown down to a curve only if the base is $S = \mathbb{F}_m$ and the twist is $T = 2X + kY$, where $X \cdot X = -m$, Y is the fiber of the Hirzebruch, and $k \leq 2m$ is an integer. The story is identical for Σ_+ with opposite values for the twist. Otherwise, the only way of blowing down a divisor is for a divisor D_i to be associated with a ray $w_i = w_{i-1} + w_{i+1}$ (with i cyclic as usual), which occurs only when $T \cdot d_i = 0$. In such a case, D_i can be blown down to a curve.

The set of Hodge numbers for generic elliptic fibrations over threefold bases that are "minimal" in this minimal smooth toric sense are plotted in figure 10. The point of this plot is that, as for the Calabi-Yau threefold case with base surfaces, the minimal bases lie along the left side of the plot, with very small values of $h^{1,1}(X)$. The other bases, to the right, are formed from blow-ups of this set of minimal bases. We expect that more generally, in the full set of elliptic CY fourfolds, there is a similar structure, with minimal bases lying along the left side of the figure, and bases formed from multiple blow-ups of the minimal bases, with increasing $h^{1,1}$ and increasing non-Higgsable gauge group content,

going to the right. Thus, this analysis reinforces the picture that we expect threefold bases that do not generate non-Higgsable gauge groups to be a relatively small set localized in the region with relatively small $h^{1,1}$.

5 Non-Higgsable clusters

In this section we study the structure of the non-Higgsable clusters that arise in our examples. The most important conclusion that we would like to draw from this set is that geometrically non-Higgsable clusters seem to be generic features of 4D $\mathcal{N} = 1$ F-theory vacua. We have constructed $109, 158$ bases, and $107, 334$ of these have non-Higgsable clusters; that is, 98.3% of the examples exhibit non-Higgsable gauge groups. From figure 11 it can be seen that the likelihood of a non-Higgsable seven-brane configuration appearing in our $\mathbb{P}^1$-bundle bases increases quickly as a function of $h^{1,1}(S)$, plateauing near 100% likelihood for $h^{1,1}(S) \geq 10$; compare to the similar conclusion of figure 9. Furthermore, from figure 12 it can be seen that the *number* of non-Higgsable gauge factors increases with increasing $h^{1,1}(S)$, and the increase is approximately linear for $h^{1,1}(S) \geq 10$.

From the analysis and arguments of the previous section, the evidence we have so far suggests that for more general bases supporting elliptic Calabi-Yau fourfolds, the fraction of bases that give non-Higgsable clusters will be even higher. We do not have a rigorous argument for this conclusion, and it is possible that there is some enormous class of elliptic Calabi-Yau fourfolds that are unrelated to the known CY fourfolds from toric and Landau-Ginsburg constructions. The hypothesis that non-Higgsable clusters are generically more prevalent than in our restricted dataset seems quite plausible, however, based on the parallel with the 6D story and the fact that the weak Fano bases and bases that have no non-Higgsable gauge groups are in a subset with relatively small Hodge numbers among those we have considered, combined with the observation that most of the set of known Calabi-Yau fourfolds have larger values of $h^{1,1}(X)$. This conclusion is also supported by the sense in which many of the bases we have constructed here are essentially "minimal" smooth toric bases, which can be used to construct many other bases with greater non-Higgsable content by consecutive blowing up.

Our goal in this section is to study the detailed structure of the non-Higgsable clusters that arise in the class of bases we have constructed here. We consider the relative frequency of appearance of the various allowed individual non-Higgsable gauge group factors, as well as products, focusing on some structures that may be relevant for producing semi-realistic physics like that of the standard model. One important issue to keep in mind is that the specific class of bases we have chosen to consider here may have a strong influence on the distribution of detailed aspects of the non-Higgsable clusters. Some particular aspects of this are as follows: first, because we have only included strictly toric bases, we have dropped all bases with E_8 non-Higgsable factors on divisors when there is any curve on that divisor where there is a vanishing of f, g to orders $4, 6$. As mentioned in the previous section, there are an additional $68,528$ $\mathbb{P}^1$ bundle bases that would be included in the class under consideration here except for the appearance of such $(4, 6)$ curves. As a consequence
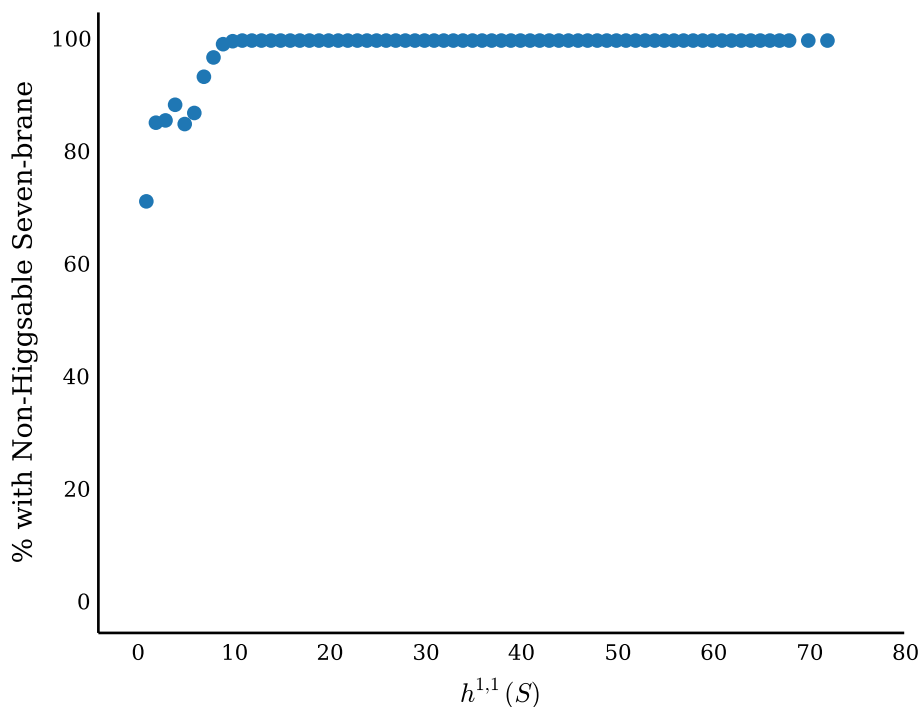
Figure 11. Percentage of bases B with a non-Higgsable cluster as a function of $h^{11}(S)$.

of this restriction, E_8 gauge group factors are artificially suppressed in the set of bases we consider here.

Second, the global structure of the bases we have constructed is quite specialized. In particular, the structure of the set of divisors is such that there is a "ring" of divisors D_i associated with the lifts of the curves d_i in the base; each of the D_i is a Hirzebruch surface. Then there are two sections $\Sigma_\pm$, each with the topology of the base S. This global structure has several consequences. Among other things, the distribution of gauge groups on the sections $\Sigma_\pm$ will be taken from a broader range of local divisor surface + normal bundle geometries, and may be more characteristic of the full range of possibilities for general toric threefold bases. In addition, in terms of global structure there is less opportunity for large structures with loops and branching in the topology of the quiver associated with the non-Higgsable gauge group factors in a given cluster. And of course, we are restricting here to toric bases, so that, for example, all the curves on which divisors intersect are simply $\mathbb{P}^1$'s.

Despite these biases introduced by the choice of bases we use here, we believe that there are some significant general lessons that can be learned from the distribution of gauge groups and matter found in the specific geometries we have considered. In particular, these bases provide a rich range of examples for more detailed exploration of specific features of non-Higgsable clusters that may be relevant both for phenomenological and more theoretical reasons. Also, the general gist of our results, which is that there is a fairly
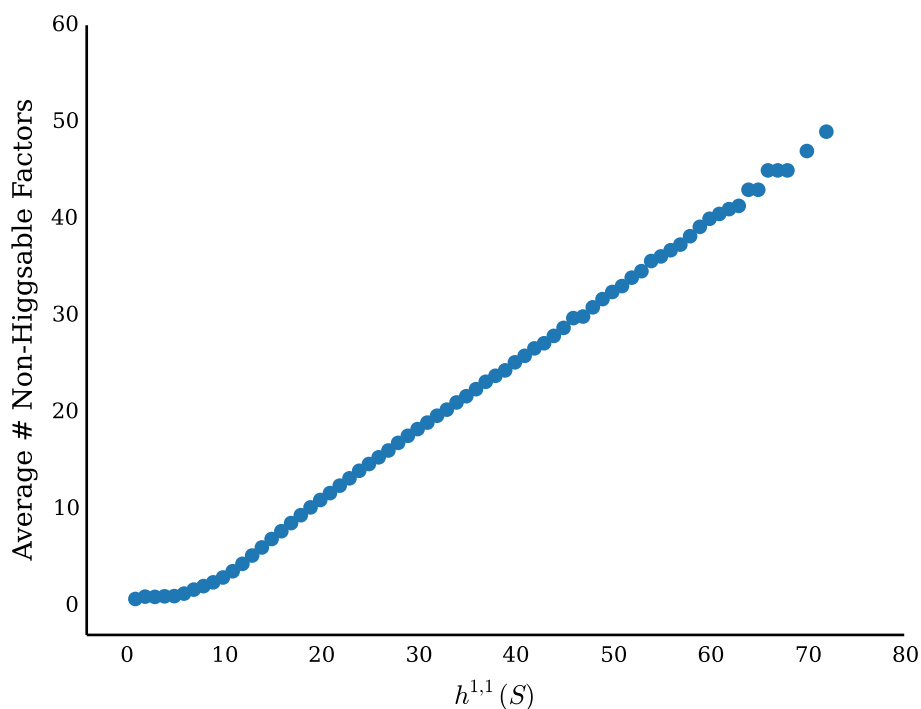
Figure 12. Average number of non-Higgsable gauge factors per example as a function of $h^{11}(S)$.

broad range of possibilities realized across the set of non-Higgsable groups and products that are in principle allowed from F-theory, should be valid in a broader class of bases. In particular, we do not find for example that the product of two gauge group factors $SU(3) \times SU(2)$ is either hugely dominant or completely absent in the distribution of possible non-Higgsable clusters, though it is somewhat suppressed compared to other product groups such as $G_2 \times SU(2)$, and individual factors $SU(3)$ and $SO(7)$ are less frequent than the individual factors $SU(2)$ and G_2. We expect these and other such general patterns to persist over other classes of bases.

To compute the explicit non-Higgsable gauge factors associated with each base we have used the straightforward toric approach in which the Weierstrass functions f, g are defined in terms of the sets of monomials in the dual lattice,

$$\mathcal{F} = \{q \in N^* : \langle q, w_i \rangle \geq -4 \, \forall i\}, \tag{5.1}$$

$$\mathcal{G} = \{q \in N^* : \langle q, w_i \rangle \geq -6 \, \forall i\}. \tag{5.2}$$

The order of vanishing of f on a divisor w is then given by $\mathrm{ord}_w f = min_{q \in \mathcal{F}} \langle q, w \rangle + 4$, and similarly for g. The monodromy determining the precise non-Higgsable gauge algebra can also be read off directly from the set of monomials as described in [16, 19].

Note that in this section, in a more phenomenologically-motivated spirit, we describe the non-Higgsable structure in terms of gauge groups rather than gauge algebras, with the understanding that the group is only explicitly determined from the orders of vanishing of

	E_8	E_7	E_6	F_4	SO(8)	SO(7)	G_2	SU(3)	SU(2)$_{IV}$	SU(2)$_{III}$
Total Percentage	.002	9.93	.028	15.8	.779	6.09	22.5	.470	18.2	26.2
Section Percentage	.204	12.22	2.43	18.9	4.12	.154	20.12	4.27	16.6	21.0

Table 3. Relative frequency of occurrence for each gauge factor, first for the total set of gauge factors, then for those arising on one of the sections. Note that SU(2)$_{III}$ refers to an SU(2) arising on a type III Kodaira fiber, while SU(3)$_{IV}$ refers to an SU(2) arising on a type IV fiber with monodromy.

f, g and monodromy at the level of the algebra, and further analysis can give a quotient of the group by a discrete subgroup in specific cases.

5.1 Single factors

In this subsection we consider the frequency with which any single gauge factor G_i appears in our examples. Figures 13 and 14 display the number of overall occurrences of each factor G_i on a general divisor and on a section, in each case as a function of $h^{11}(S)$. In table 3 we display the relative frequency of each factor, computed by tabulating the number of times each factor occurs in our data set and then computing for each the percentage of the total. First we display the percentage of occurrence amongst all factors, and then the percentage restricted to those factors that appear on one of the sections. As discussed earlier, we expect that the latter may be more accurately demonstrative of the behavior that occurs for broader sets of three-folds, since the sections exhibit a broader set of topologies than the other divisors, which are all Hirzebruch surfaces given by $\mathbb{P}^1$-bundles over base curves. Note, however, that the general pattern in the distribution is not highly sensitive to this distinction. In both cases the most frequently appearing factors are SU(2) and G_2.

Looking at the results of this analysis, one aspect of the frequency of the single factors can be thought of heuristically in terms of an intuitively sensible principle. Namely, if multiple groups may be realized by the same Kodaira singularity type, but with different outer monodromies, then the group associated with the largest monodromy group action is the one that occurs most frequently. In particular, F_4 occurs more frequently than E_6 and both come from Kodaira type IV^*, G_2 occurs more frequently than SO(7) or SO(8) and all come from Kodaira type I_0^*, and SU(2)$_{IV}$ occurs more frequently than SU(3) and both come from Kodaira type IV. The reason that this is intuitively natural is because *not* having the largest monodromy group action requires that the fibration satisfy additional conditions beyond that of the general case. Note that this means that one should expect non-Higgsable SU(3) to be relatively uncommon, because it can only occur from a non-Higgsable cluster realized by a Kodaira type IV fiber *without* outer monodromy.

5.2 Two-factor product subgroups

We now briefly discuss the prevalence of non-Higgsable two-factor products with jointly charged matter. By this we mean that the non-Higgsable group contains a product of factors $G_1 \times G_2$, where the non-Higgsable seven-brane configuration carrying G_1 intersects
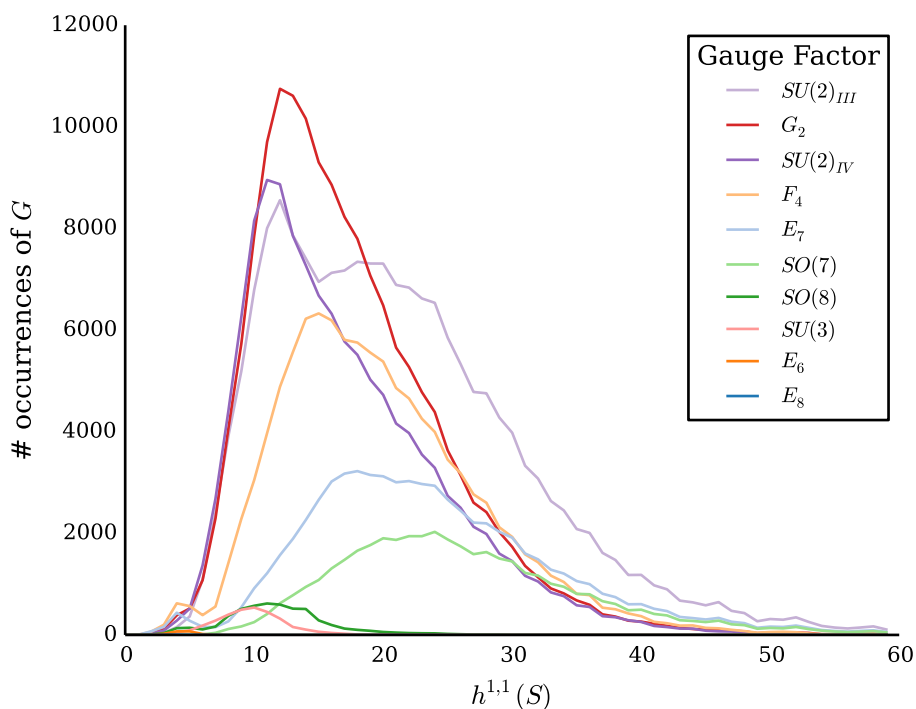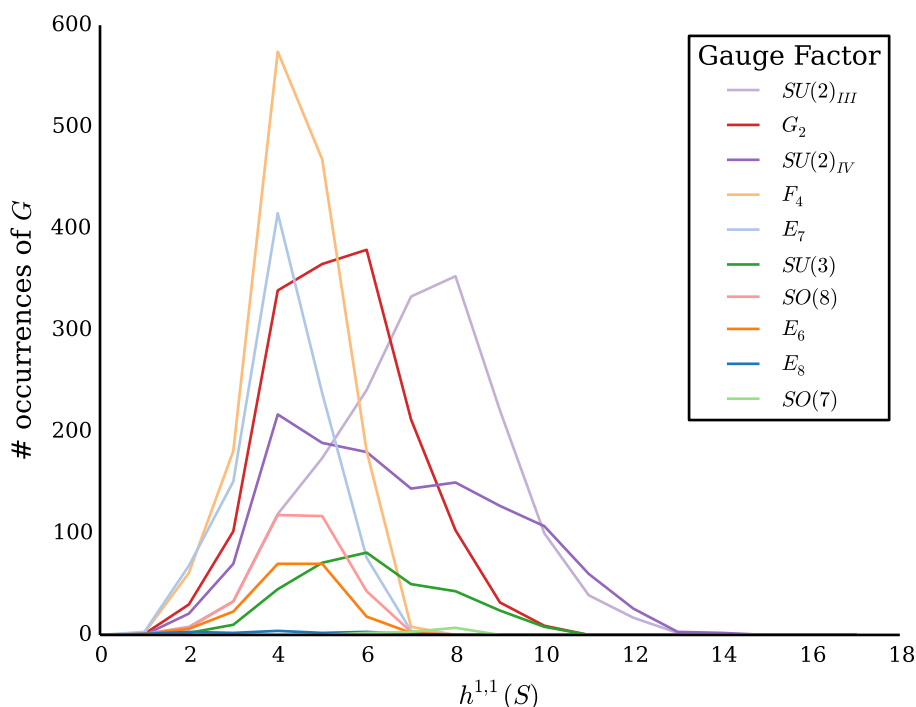
Figure 13. Displayed are the occurrences of a given gauge factor as a function of $h^{11}(S)$.

the non-Higgsable seven-branes carrying G_2. In fact, there are only five possibilities for two-factor products with jointly charged matter, given in (2.6).

The prevalence of two-factor groups can again be understood according to the principle discussed in section 5.1. That principle is simply that for any given Kodaira fiber, the mostly likely group to occur is the one with the largest outer monodromy group action, since this one occurs generally, and the other groups that may be realized by the same Kodaira fiber occur only if additional conditions are satisfied. In particular, this means that SO(7) and SU(3) are relatively unlikely groups compared to the others, since they must satisfy additional conditions. This suggests that $G_2 \times SU(2)$ and $SU(2) \times SU(2)$ should be much more likely than $SU(3) \times SU(3)$, $SU(3) \times SU(2)$, or $SO(7) \times SU(2)$, and in fact this is reflected in our data; see figures 15 and 16. Note by comparing the figures that the prevalence of $G_2 \times SU(2)$ and $SO(7) \times SU(2)$ relative to other factors (including $SU(3) \times SU(2)$) goes down in considering products where one of the factors sits on a section. As discussed throughout the rest of the paper, we believe that the analyses where one group factor sits on a section is more demonstrative of the general 4D story, and thus particular attention should be paid to figure 16.

5.3 Cluster structure

As described in [16], an interesting feature of non-Higgsable clusters in 4D F-theory models is that the associated quiver diagrams can have nontrivial structure, with branchings and

Figure 14. Displayed are the occurrences of a given gauge factor on one of the sections as a function of $h^{11}(S)$.

loops possible. While the connectivity structure of divisors in the bases we consider here is rather simple and does not support arbitrary branching and looping structure, we explore briefly a few aspects of these features of the non-Higgsable clusters that arise in the bases we have constructed here.

First, we consider branching. No branching is possible unless there is a non-Higgsable gauge factor on at least one section. For bases that have a non-Higgsable gauge group on one but not both sections, the number of gauge groups on the divisors D_i associated with curves on S can range from 0 through 7. The largest branching occurs for the base S having self-intersections

$$((0, -3, -2, -1, -4, -1, -4, -1, -4, -1, -4, -1, -4, -1, -2, -3)) \qquad (5.3)$$

and twist

$$T = (t_1, \ldots, t_N) = (0, 0, 1, 1, 0, -1, -1, -3, -2, -5, -3, -7, -4, -9, -5, -2) \,. \qquad (5.4)$$

This has a group SU(2) on the bottom section Σ_+, additional SU(2) factors on vertical divisors D_i with $i = 2, 16$ (associated with the -3 curves d_2, d_{16}), and G_2 factors on the vertical divisors D_i with $i = 5, 7, 9, 11, 13$ (associated with the -4 curves in S). This is the only base with a branching of degree seven. A variety of bases support non-Higgsable clusters where a group factor on a section has branching of degree 3, 4, 5, or 6; a simple example of a case of branching of degree three is described explicitly in [16].
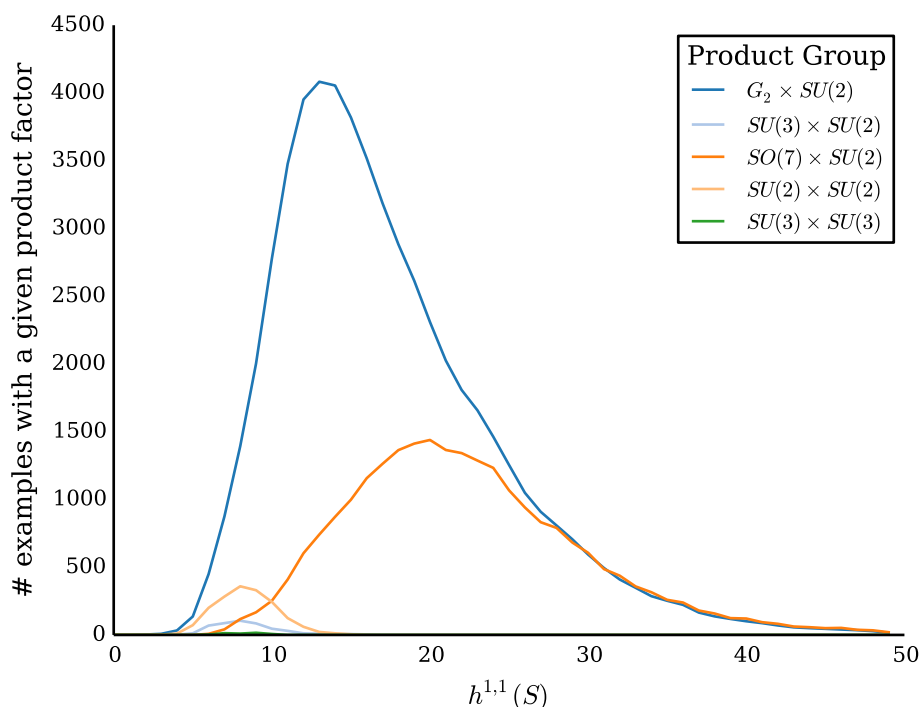
Figure 15. Displayed are the number of bases B with a given product group as a function of $h^{11}(S)$.

Considering bases that have a non-Higgsable gauge group on both sections, there are only two examples that have more than one gauge group on divisors D_i, forming a closed loop in the quiver. One example is on the base with self-intersections

$$((-3,-1,-2,-2,-1,-3,-1,-2,-2,-1)) \tag{5.5}$$

and twist

$$T = (0,0,1,2,3,0,-3,-2,-1,0). \tag{5.6}$$

In this case there are four SU(2) factors on the divisors $\Sigma_\pm$, D_1, and D_6, which intersect pairwise in the topology of a closed loop. In the other example, the base has self-intersections

$$((-1,-2,-2,-1,-4,-1,-4,-1,-2,-2,-1)) \tag{5.7}$$
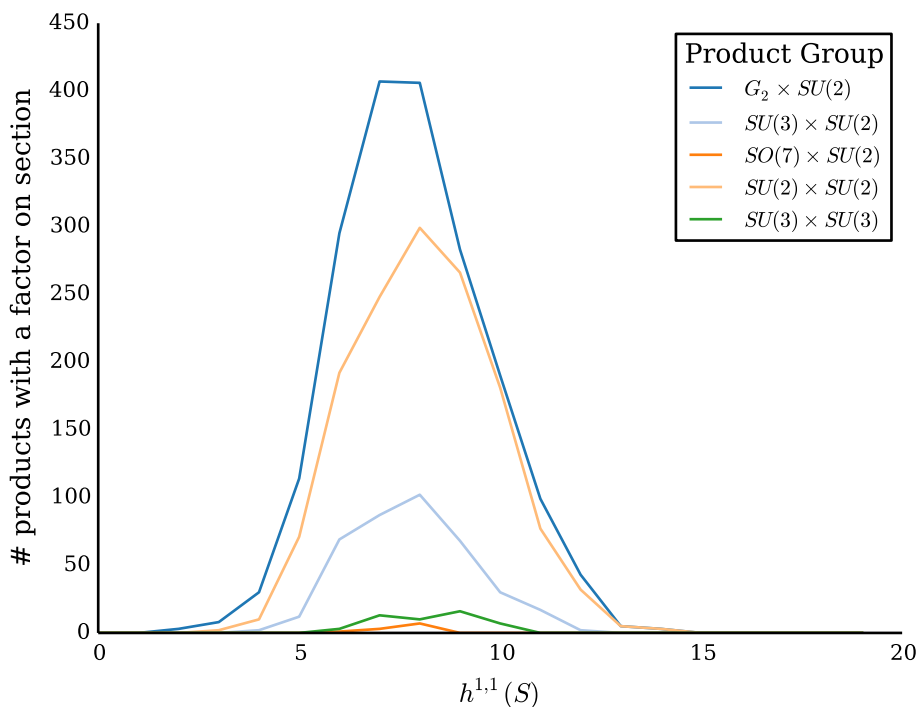
and twist

$$T = (0,0,0,1,1,3,2,5,3,0,-3). \tag{5.8}$$

In this case there are SU(2) factors on $\Sigma_\pm$ and G_2 factors on D_5 and D_7.

5.4 Non-Higgsable QCD

Motivated by the existence of an unbroken QCD sector in nature and the ability of non-Higgsable clusters to avoid a large tuning problem in Weierstrass moduli, with coauthors

Figure 16. Displayed are the number of bases B with a given product group as a function of $h^{11}(S)$, in the case that one of the factors in the product is on a section.

Grassi and Shaneson, we proposed in [12] studying the phenomenological possibility of realizing $SU(3)_c$ from a non-Higgsable seven-brane configuration. Such a configuration necessarily arises from a type IV Kodaira fiber, in which case the electroweak factor $SU(2)_L$ may arise from a type III fiber, a type I_2 fiber, or a type IV fiber with outer monodromy. The type I_2 case necessarily has a geometrically Higgsable $SU(2)_L$, whereas the III and IVm realizations may be either Higgsable or non-Higgsable depending on the base. Given current knowledge, both Higgsable and non-Higgsable $SU(2)_L$ must be considered because the geometric non-Higgsability condition we are using in F-theory is only a statement about one sector of fields in a supersymmetric, ultraviolet theory. The possibility that the SU(2) seen in the electroweak symmetry group in nature is "non-Higgsable" in this sense is not ruled out experimentally. Indeed, supersymmetry breaking and/or other effects could give rise to radiative electroweak symmetry breaking in the infrared, as is common in many phenomenological scenarios, even if there is an obstruction in the supersymmetric ultraviolet theory to a Higgsing deformation, such as by a quartic term arising from an F-term or D-term. Realizing the SU(3) or SU(3)×SU(2) nonabelian factors of the standard model through non-Higgsable gauge groups that appear in generic F-theory models over certain bases presents an alternative to the well-studied GUT F-theory scenario [42–46].

To have matter that is jointly charged under the SU(3) and SU(2) factors, the divisors supporting these factors must intersect. Along the intersection there is an enhanced
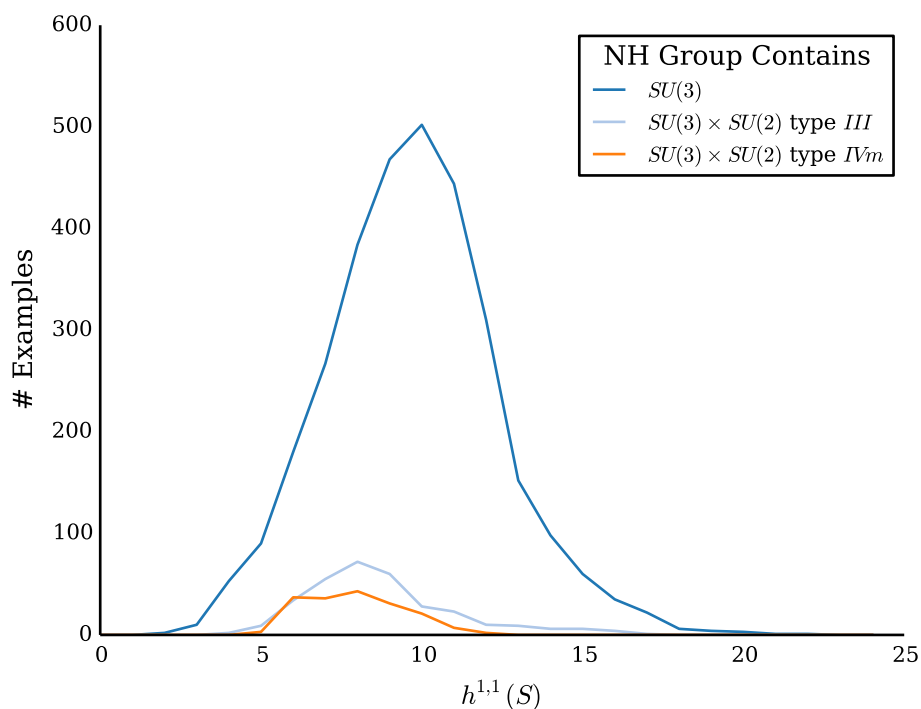
Figure 17. Number of bases B that realize a scenario of Non-Higgsable QCD.

Kodaira singularity type that can encode matter charged under both of the factors. In six-dimensional compactifications these codimension two singularities always correspond to matter in the physical theory. In 4D constructions, however, the details of the matter content, particularly the chirality of the matter, depend upon more detailed considerations involving G-flux and other issues. Only the Lie-algebraic representation content is specified by the geometry of the Weierstrass model. We do not go deeply into this here (see [12] for further comments and references) but simply consider any intersecting branes each carrying a nontrivial gauge group to have "geometric matter". In particular, the intersection of the divisors carrying these gauge factors is a prerequisite for a realization of the standard model, in which quarks are jointly charged under the two factors.

In this section we study the realizations of non-Higgsable QCD that occur in our dataset, always with a non-Higgsable SU(3) factor by definition, but allowing all possibilities for the SU(2)$_L$ sector. The most coarse measure of this phenomenon is presented in figure 17, which plots the number of bases B that contain a non-Higgsable SU(3) factor, as well as the number containing a non-Higgsable SU(3) × SU(2) factor on intersecting divisors, using either a type III or IVm fiber for SU(2)$_L$. Note that this plot allows for double counting: those examples with SU(3) × SU(2) are included in the SU(3) plot, and similarly some examples with SU(3) × SU(2) may realize *both* types of SU(3) × SU(2) in a single example. Note that while for many bases that have an SU(3) factor there will be some divisor that intersects the SU(3) divisor, on which it is possible to tune an SU(2) through an I_2 or

other singularity, there can also be bases where this cannot be done without introducing a $(4,6)$ singularity on a curve. Thus, the set of bases with some divisor having an SU(3) gives an upper bound and a general sense of the range of possibilities, but is not a precise determination of a set of bases that could admit an SU(3) × SU(2) with a Higgsable SU(2).

Let us be more specific about the counts. In our set of 109,158 example bases B there are 3,092 examples that have a non-Higgsable SU(3) factor. Of those, 319 have an intersecting SU(3) × SU(2) factor (with geometric matter) of *IV-III* type and 180 have an SU(3) × SU(2) factor (with geometric matter) of *IV-IVm* type. The *total* number of bases B in our set with an SU(3) × SU(2) factor is 474, which, from the other counts just listed, implies that there are $319 + 180 - 474 = 25$ examples with both a *IV-III* and a *IV-IVm*. There are 313 examples that have a non-Higgsable SU(3) × SU(2) where the seven-branes that support the SU(3) and SU(2) factors do not intersect any other seven-branes. The base structure, twists, and gauge algebras for some of those examples are listed in table 4 and table 5. The examples in table 4 all have neither the SU(3) seven-branes nor the SU(2) seven-branes on a section, and all of those examples have hidden gauge sectors. On the other hand, the examples in table 5 have one of the SU(3) or SU(2) seven-branes on a section, and these examples do not have hidden gauge sectors. In other examples with non-Higgsable SU(3) × SU(2), the SU(3) and SU(2)$_{III}$ seven-branes intersect other seven-branes that carry additional gauge factors not currently observed in nature. Matter that is jointly charged with the extra seven-branes may produce exotic states that in some cases may be identified as exotic WIMPs; see section 5.5.

Some of the examples have a non-Higgsable SU(2) × SU(3) × SU(2), similar to a so-called left-right model, where there is bifundamental matter charged under SU(3) and each of the SU(2) factors separately, but no bifundamental matter charged under both SU(2) factors. Though we will not exhaustively categorize the possibilities, it is illustrative to study this possibility in a simple subset. One such subset is the 25 examples that have both *IV-III* and *IV-IVm* realizations of SU(3) × SU(2); in fact these are typically left-right models. We have displayed their algebraic structure in table 6, where SU(3) denotes non-Higgsable SU(3), SU(2)$_{IV}$ denotes a non-Higgsable SU(2) from a *IVm* fiber, SU(2)$_{III}$ denotes a non-Higgsable SU(2) from a type *III* fiber, and *II* denotes a singular seven-brane configuration with a type *II* fiber, which does not carry a gauge group but might give rise to matter at intersections with other branes. The first two entries represent the sections $\Sigma_\pm$ of the $\mathbb{P}^1$ bundles, whereas the rest of the entries are for the divisors D_i that are $\mathbb{P}^1$ bundles over the curves d_i in the base S. Seven-branes that occur on the sections may not intersect each other, but intersect any seven-brane in the base, whereas seven-branes on the divisors D_i intersect one another only if their respective base curves are adjacent.

There are a number of phenomena that can be seen easily in these 25 examples, given the topological intersections just discussed. First, note that there is always a non-Higgsable SU(3) factor on one of the sections, and the only SU(2) factors arise from divisors D_i. In these examples, any SU(3) factor that is not on one of the sections is not adjacent to any of the SU(2) factors. Therefore, the only SU(3) × SU(2) charged matter occurs at the intersection between an SU(3) factor on a section and an SU(2) factor on one of the divisors D_i. Considering in particular the left-right symmetric SU(2) × SU(3) × SU(2) structure just

$d_i \cdot d_i$ for base curves d_i	Twist $T \cdot d_i$ for each d_i	Gauge Sectors
$(1,-5,-1,-2,-2,-3,-1,-3,-2,0)$	$(0,1,-1,1,0,-1,1,-1,0,0)$	$G_2, \mathrm{SU}(2)_{III} \times \mathrm{SU}(3), \mathrm{SU}(2)_{IV}$
$(1,-5,-1,-2,-3,-1,-3,-2,-2,0)$	$(-1,0,0,0,1,-1,1,0,-1,0)$	$F_4, \mathrm{SU}(2)_{IV}, \mathrm{SU}(3) \times \mathrm{SU}(2)_{III}$
$(1,-5,-1,-2,-3,-1,-3,-2,-2,0)$	$(2,1,0,0,-1,1,-1,0,1,1)$	$G_2, \mathrm{SU}(2)_{IV}, \mathrm{SU}(3) \times \mathrm{SU}(2)_{III}$
$(0,-2,-2,-3,-1,-3,-2,-1,-6,-1,0)$	$(1,1,0,-1,1,-1,0,0,1,0,2)$	$\mathrm{SU}(2)_{III} \times \mathrm{SU}(3), \mathrm{SU}(2)_{IV}, \mathrm{SO}(8)$
$(0,-2,-2,-3,-1,-3,-2,-1,-6,-1,0)$	$(0,1,0,-1,1,-1,0,0,1,-1,2)$	$\mathrm{SU}(2)_{III} \times \mathrm{SU}(3), \mathrm{SU}(2)_{IV}, F_4$
$(0,-2,-3,-1,-3,-2,-2,-1,-6,-1,0)$	$(0,0,1,-1,1,0,-1,1,-1,0,0)$	$\mathrm{SU}(2)_{IV}, \mathrm{SU}(3) \times \mathrm{SU}(2)_{III}, F_4$
$(0,-3,-1,-2,-3,-1,-3,-2,-2,-1,-3)$	$(0,0,0,0,1,-1,1,0,-1,1,-1)$	$\mathrm{SU}(2)_{IV}, \mathrm{SU}(2)_{IV}, \mathrm{SU}(3) \times \mathrm{SU}(2)_{III}$
$(0,-3,-1,-2,-3,-1,-3,-2,-2,-1,-3)$	$(1,1,0,0,-1,1,-1,0,1,0,-1)$	$\mathrm{SU}(2)_{IV}, \mathrm{SU}(3) \times \mathrm{SU}(2)_{III}$
$(0,-4,-1,-2,-2,-3,-1,-3,-2,-1,-2)$	$(0,1,-1,1,0,-1,1,-1,0,0,0)$	$\mathrm{SU}(2)_{IV}, \mathrm{SU}(2)_{III} \times \mathrm{SU}(3), \mathrm{SU}(2)_{IV}$
$(0,-4,-1,-2,-3,-1,-3,-2,-2,-1,-2)$	$(-1,0,0,0,1,-1,1,0,-1,0,1)$	$G_2, \mathrm{SU}(2)_{IV}, \mathrm{SU}(3) \times \mathrm{SU}(2)_{III}$
$(0,-4,-1,-2,-3,-1,-3,-2,-2,-1,-2)$	$(0,0,0,0,1,-1,1,0,-1,1,-1)$	$G_2, \mathrm{SU}(2)_{IV}, \mathrm{SU}(3) \times \mathrm{SU}(2)_{III}$
$(0,-4,-1,-2,-3,-1,-3,-2,-2,-1,-2)$	$(1,1,0,0,-1,1,-1,0,1,0,0)$	$\mathrm{SU}(2)_{III}, \mathrm{SU}(2)_{IV}, \mathrm{SU}(3) \times \mathrm{SU}(2)_{III}$
$(0,-5,-1,-2,-2,-3,-1,-3,-2,-1,-1)$	$(0,1,-1,1,0,-1,1,-1,0,0,0)$	$G_2, \mathrm{SU}(2)_{III} \times \mathrm{SU}(3), \mathrm{SU}(2)_{IV}$
$(0,-5,-1,-2,-3,-1,-3,-2,-2,-1,-1)$	$(-1,0,0,0,1,-1,1,0,-1,0,0)$	$F_4, \mathrm{SU}(2)_{IV}, \mathrm{SU}(3) \times \mathrm{SU}(2)_{III}$
$(0,-5,-1,-2,-3,-1,-3,-2,-2,-1,-1)$	$(0,0,0,0,1,-1,1,0,-1,1,-1)$	$F_4, \mathrm{SU}(2)_{IV}, \mathrm{SU}(3) \times \mathrm{SU}(2)_{III}$
$(0,-5,-1,-2,-3,-1,-3,-2,-2,-1,-1)$	$(1,1,0,0,-1,1,-1,0,1,0,1)$	$G_2, \mathrm{SU}(2)_{IV}, \mathrm{SU}(3) \times \mathrm{SU}(2)_{III}$
$(0,-5,-1,-2,-3,-1,-3,-2,-2,-1,-1)$	$(0,1,0,0,-1,1,-1,0,1,-1,2)$	$G_2, \mathrm{SU}(2)_{IV}, \mathrm{SU}(3) \times \mathrm{SU}(2)_{III}$
$(0,-6,-1,-2,-2,-3,-1,-3,-2,-1,0)$	$(0,1,-1,1,0,-1,1,-1,0,0,0)$	$F_4, \mathrm{SU}(2)_{III} \times \mathrm{SU}(3), \mathrm{SU}(2)_{IV}$
$(0,-6,-1,-2,-3,-1,-3,-2,-2,-1,0)$	$(1,1,0,0,-1,1,-1,0,1,0,2)$	$\mathrm{SO}(8), \mathrm{SU}(2)_{IV}, \mathrm{SU}(3) \times \mathrm{SU}(2)_{III}$
$(0,-6,-1,-2,-3,-1,-3,-2,-2,-1,0)$	$(0,1,0,0,-1,1,-1,0,1,-1,2)$	$\mathrm{SO}(8), \mathrm{SU}(2)_{IV}, \mathrm{SU}(3) \times \mathrm{SU}(2)_{III}$

Table 4. Examples with a non-Higgsable $\mathrm{SU}(3) \times \mathrm{SU}(2)$ sector that do not intersect other seven-branes, where neither $\mathrm{SU}(3)$ nor $\mathrm{SU}(2)$ is on a section. These are all the examples with $h^{11}(S) < 10$. The order of the gauge factors corresponds to the order of the curves d_i, and in general the gauge factors lie on the curves with most negative self intersections; i.e., in the first example the G_2 lies on D_2, $\mathrm{SU}(2)_{III}$ on D_5, $\mathrm{SU}(3)$ on D_6, etc.

| $d_i \cdot d_i$ for base curves d_i	Twist $T \cdot d_i$ for each d_i	Gauge Sectors
$(2,-1,-1,-4,-1,-1)$	$(5,1,1,-1,1,1)$	$\mathrm{SU}(3) \times \mathrm{SU}(2)_{III}$
$(2,-1,-2,-1,-4,0)$	$(5,1,0,1,-1,2)$	$\mathrm{SU}(3) \times \mathrm{SU}(2)_{III}$
$(1,-1,-2,-1,-4,-1,-1)$	$(3,1,0,1,-1,0,2)$	$\mathrm{SU}(3) \times \mathrm{SU}(2)_{III}$
$(1,-1,-2,-1,-4,-1,-1)$	$(4,1,0,1,-1,1,1)$	$\mathrm{SU}(3) \times \mathrm{SU}(2)_{III}$
$(1,-1,-2,-1,-4,-1,-1)$	$(3,2,-1,1,-1,1,1)$	$\mathrm{SU}(3) \times \mathrm{SU}(2)_{III}$
$(1,-1,-2,-2,-1,-4,0)$	$(4,1,0,0,1,-1,2)$	$\mathrm{SU}(3) \times \mathrm{SU}(2)_{III}$
$(1,-1,-2,-2,-1,-4,0)$	$(3,2,-1,0,1,-1,2)$	$\mathrm{SU}(3) \times \mathrm{SU}(2)_{III}$
$(0,-2,-1,-3,-1,-2,0)$	$(3,0,1,0,1,0,3)$	$\mathrm{SU}(2)_{IV} \times \mathrm{SU}(3)$
$(0,-2,-1,-3,-1,-2,0)$	$(-3,1,-2,1,-1,0,-3)$	$\mathrm{SU}(2)_{III} \times \mathrm{SU}(3)$
$(0,-2,-2,-1,-3,-1,0)$	$(3,0,0,1,0,1,3)$	$\mathrm{SU}(2)_{IV} \times \mathrm{SU}(3)$
$(0,-2,-2,-1,-3,-1,0)$	$(3,0,0,1,-1,2,2)$	$\mathrm{SU}(2)_{III} \times \mathrm{SU}(3)$
$(0,-2,-2,-1,-3,-1,0)$	$(-3,0,1,-2,1,-1,-3)$	$\mathrm{SU}(2)_{III} \times \mathrm{SU}(3)$
$(1,-3,-1,-2,-2,-2,0)$	$(4,0,1,0,0,0,3)$	$\mathrm{SU}(2)_{IV} \times \mathrm{SU}(3)$
$(1,-3,-1,-2,-2,-2,0)$	$(-4,1,-2,1,0,0,-3)$	$\mathrm{SU}(2)_{III} \times \mathrm{SU}(3)$

Table 5. Examples with a non-Higgsable $\mathrm{SU}(3) \times \mathrm{SU}(2)$ sector that do not intersect other seven-branes where either the $\mathrm{SU}(3)$ or $\mathrm{SU}(2)$ is on a section. These are all the examples with $h^{11}(S) < 6$. In each case the latter factor in the gauge group lies on a section $\Sigma_\pm$. The first of these examples was also featured in [12].

$$
\begin{array}{c}
(0, \mathrm{SU}(3), 0, 0, \mathrm{SU}(2)_{IV}, 0, II, \mathrm{SU}(2)_{III}, 0, 0) \\
(\mathrm{SU}(3), 0, 0, \mathrm{SU}(2)_{IV}, 0, II, \mathrm{SU}(2)_{III}, 0, 0, 0) \\
(\mathrm{SU}(3), 0, 0, \mathrm{SU}(2)_{IV}, 0, II, \mathrm{SU}(2)_{III}, 0, 0, 0, 0) \\
(\mathrm{SU}(3), 0, 0, \mathrm{SU}(2)_{IV}, 0, II, \mathrm{SU}(2)_{III}, 0, 0, 0, 0) \\
(0, \mathrm{SU}(3), 0, \mathrm{SU}(2)_{III}, II, 0, \mathrm{SU}(2)_{IV}, 0, 0, 0, 0) \\
(0, \mathrm{SU}(3), 0, \mathrm{SU}(2)_{III}, II, 0, \mathrm{SU}(2)_{IV}, 0, 0, 0, 0) \\
(\mathrm{SU}(3), 0, \mathrm{SU}(2)_{IV}, 0, 0, 0, 0, \mathrm{SU}(2)_{III}, 0, 0, 0, 0) \\
(\mathrm{SU}(3), 0, \mathrm{SU}(2)_{IV}, 0, 0, 0, 0, \mathrm{SU}(2)_{III}, 0, 0, 0) \\
(\mathrm{SU}(3), 0, \mathrm{SU}(2)_{IV}, 0, 0, 0, 0, \mathrm{SU}(2)_{III}, 0, 0, 0) \\
(\mathrm{SU}(3), 0, 0, \mathrm{SU}(2)_{IV}, 0, 0, 0, 0, 0, 0, \mathrm{SU}(2)_{III}, 0) \\
(0, \mathrm{SU}(3), 0, \mathrm{SU}(2)_{III}, 0, 0, 0, 0, 0, 0, \mathrm{SU}(2)_{IV}, 0) \\
(0, \mathrm{SU}(3), 0, \mathrm{SU}(2)_{III}, 0, 0, 0, 0, 0, 0, 0, \mathrm{SU}(2)_{IV}) \\
(\mathrm{SU}(3), 0, \mathrm{SU}(2)_{IV}, 0, II, \mathrm{SU}(2)_{III}, 0, 0, 0, 0, 0, 0) \\
(\mathrm{SU}(3), 0, \mathrm{SU}(2)_{IV}, 0, II, \mathrm{SU}(2)_{III}, 0, 0, 0, 0, 0, 0) \\
(\mathrm{SU}(3), 0, \mathrm{SU}(2)_{III}, II, 0, \mathrm{SU}(2)_{IV}, 0, 0, 0, 0, 0, 0) \\
(\mathrm{SU}(3), 0, \mathrm{SU}(2)_{III}, II, 0, \mathrm{SU}(2)_{IV}, 0, 0, 0, 0, 0, 0) \\
(\mathrm{SU}(3), 0, \mathrm{SU}(2)_{III}, II, 0, \mathrm{SU}(2)_{IV}, 0, \mathrm{SU}(2)_{IV}, 0, 0, 0, 0, 0) \\
(\mathrm{SU}(3), 0, \mathrm{SU}(2)_{III}, II, 0, \mathrm{SU}(2)_{IV}, 0, II, 0, 0, 0, 0, 0) \\
(\mathrm{SU}(3), 0, \mathrm{SU}(2)_{III}, II, 0, \mathrm{SU}(2)_{IV}, 0, \mathrm{SU}(2)_{III}, 0, 0, 0, 0, 0) \\
(\mathrm{SU}(3), 0, \mathrm{SU}(2)_{III}, II, 0, \mathrm{SU}(2)_{IV}, 0, 0, \mathrm{SU}(2)_{IV}, 0, 0, 0, 0) \\
(\mathrm{SU}(3), 0, \mathrm{SU}(2)_{III}, II, 0, \mathrm{SU}(2)_{IV}, 0, 0, II, 0, 0, 0, 0) \\
(\mathrm{SU}(3), 0, \mathrm{SU}(2)_{III}, II, 0, \mathrm{SU}(2)_{IV}, 0, 0, \mathrm{SU}(2)_{III}, 0, 0, 0, 0) \\
(0, \mathrm{SU}(3), \mathrm{SU}(2)_{IV}, 0, 0, \mathrm{SU}(2)_{III}, II, 0, \mathrm{SU}(2)_{IV}, 0, 0, \mathrm{SU}(2)_{III}, II, 0) \\
(\mathrm{SU}(3), 0, \mathrm{SU}(3), 0, 0, \mathrm{SU}(2)_{IV}, 0, II, \mathrm{SU}(2)_{III}, 0, 0, 0, 0) \\
(\mathrm{SU}(3), 0, \mathrm{SU}(3), 0, \mathrm{SU}(2)_{IV}, 0, II, \mathrm{SU}(2)_{III}, 0, 0, 0, 0) \\
\end{array}
$$

Table 6. Displayed is the algebra structure of the 25 examples with both IV-III and IV-IVm realizations of $\mathrm{SU}(3) \times \mathrm{SU}(2)$. See the text for a discussion.

discussed, in all of these 25 examples the seven-brane carrying $\mathrm{SU}(3)$ intersects at least two seven-branes carrying $\mathrm{SU}(2)$, none of which intersect each other. When the $\mathrm{SU}(3)$ intersects precisely two seven-branes carrying $\mathrm{SU}(2)$, the $\mathrm{SU}(2) \times \mathrm{SU}(3) \times \mathrm{SU}(2)$ idea is realized. In other examples the seven-brane with $\mathrm{SU}(3)$ on a section intersects three $\mathrm{SU}(2)$'s, none of which intersect each other, in which case it is the central vertex in a quiver that attaches to disjoint $\mathrm{SU}(2)$ nodes. In the last two examples, the $\mathrm{SU}(3)$ on the section intersects two $\mathrm{SU}(2)$ factors and one $\mathrm{SU}(3)$ factor on the D_i, giving another quiver with a central node and three branches.

One obvious question that we do not address in this paper is the potential origin of a $\mathrm{U}(1)$ factor in a model with non-Higgsable $\mathrm{SU}(3)$. The simplest approach to including a $\mathrm{U}(1)$ factor would be to tune it by hand, using for example the general $\mathrm{U}(1)$ Weierstrass form of [47]. It would be desirable, however, in the spirit of this work, to find a more natural way of including the abelian part of the standard model. While for some very special (non-toric) bases [26, 63] there are non-Higgsable $\mathrm{U}(1)$ factors in 6D F-theory compactifications,
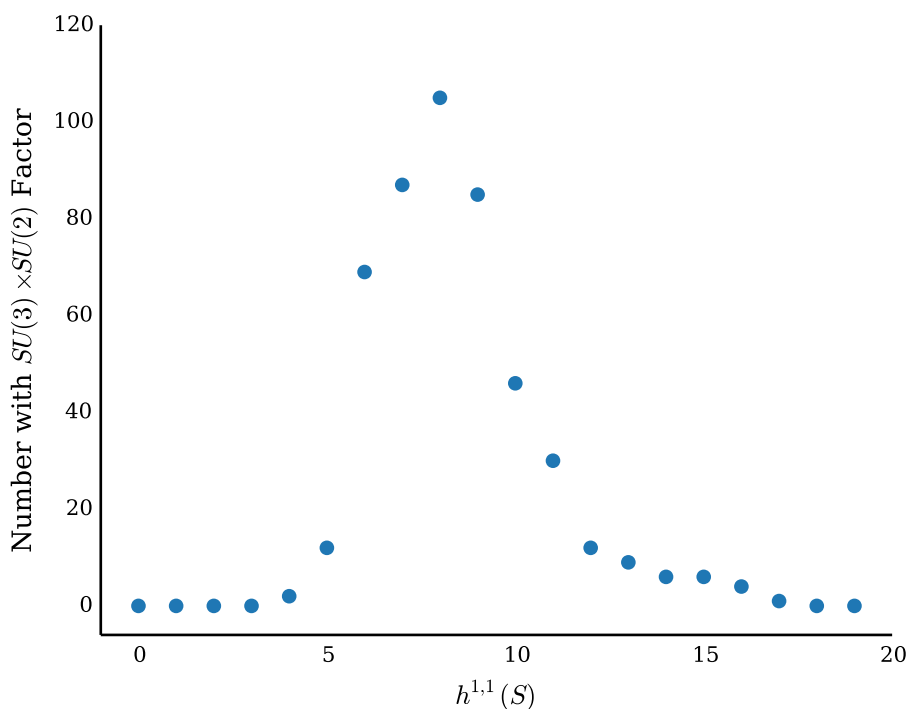
Figure 18. Number of bases B that have a non-Higgsable $\mathrm{SU}(3) \times \mathrm{SU}(2)$ gauge factor as a function of $h^{11}(S)$.

it is not clear how frequently such abelian factors are generic over threefold bases for 4D F-theory constructions; this remains an interesting avenue for further investigation.

5.5 Dark matter

While the appearance of $\mathrm{SU}(3)$ and $\mathrm{SU}(2)$ as the only $\mathrm{SU}(N)$ gauge factors that may be non-Higgsable is suggestive for visible sector phenomenology, there are also at least two distinct ways in which non-Higgsable clusters may be relevant for dark matter phenomenology. In this section and the next we will explore the extent to which these possibilities arise in our classification.

One possibility (also considered in the F-theory context in [43]) is that hidden sector dark matter may arise from gauge sectors that are topologically disconnected from the visible sector. Such dark matter would interact with the visible sector gravitationally but not via gauge forces. Such dark matter can arise naturally in generic F-theory models since non-Higgsable seven-branes may be topologically disconnected from those realizing the standard model. In such a case, the lightest particle charged under the hidden sector gauge group G is stable, and therefore if cosmologically produced it will contribute to the dark matter relic abundance. Even if no matter is charged under the hidden sector group G, uncharged glueballs may play a role as hidden sector dark matter.

The second possibility that may be relevant is that of exotic weakly interacting massive particle (WIMP) dark matter. By an exotic WIMP we mean a a WIMP that arises in an "exotic" multiplet, perhaps most easily defined in an $\mathcal{N} = 1$ theory as a chiral multiplet beyond the MSSM spectrum that contains a WIMP candidate. Such particles are distinct from the neutralinos of the MSSM, but nevertheless give rise to interesting dark matter candidates by virtue of the thermal or non-thermal [48] WIMP miracle. Considered in the context of non-Higgsable clusters such exotic WIMPs arise quite naturally: if a non-Higgsable $SU(2)_L$ seven-brane intersects another non-Higgsable seven-brane carrying a non-abelian gauge theory H then there are exotic multiplets charged under $SU(2)_L \times H$. In fact, since H arises from a non-Higgsable seven-brane configuration it may only be in the set $H \in \{G_2, SU(3), SU(2), SO(7)\}$, though there are more possibilities if H arises on a tuned brane configuration. It is also possible for extra matter to arise as codimension two singularities on a curve within the divisor carrying a gauge group such as the $SU(2)$ factor in the standard model, even without another gauge group on another divisor that passes through that curve. This could also give rise to interesting exotic WIMP candidates.

In this section we present some results from our classification related to both of these scenarios, beginning with the first. We emphasize from the outset here, however, that particularly in this set of considerations regarding the structure of multiple and/or multi-factor non-Higgsable clusters, the kinds of statistical distributions that we find are strongly influenced by the specific structure of the threefold bases we are working with, so the results should be taken only as very qualitative and suggestive. As discussed in section 5.4, there are $3,092$ examples with a non-Higgsable $SU(3)$ factor. These are the ones that may realize the non-Higgsable QCD scenario, and we would like to know whether in these examples there is at least one gauge factor on a divisor that does not intersect the divisors supporting the standard model $SU(3) \times SU(2)$. Note as mentioned above that we have not explicitly identified curves on which Higgsable $SU(2)$ factors can be tuned on all the $SU(3)$ models, so for simplicity we simply look for other clusters disjoint from the $SU(3)$ to get a qualitative sense of the possibilities. In the language of quivers, there must be a quiver component that is disconnected from the component that contains the visible sector gauge node. Given the topology of our B's, if there is a gauge group on a seven-brane on either of the sections of the $\mathbb{P}^1$ bundle then there is a single quiver component. Therefore examples with a non-Higgsable $SU(3)_c$ that have a disconnected hidden sector gauge group must have no gauge group on either of the sections; this restricts us to $2,493$ of the possible $3,092$ examples with a non-Higgsable $SU(3)$ factor. Having disconnected hidden sector gauge groups in these examples is equivalent to having disconnected quiver components arising from base curves; there are $2,239$ such examples.

These $2,239$ examples of non-Higgsable QCD have a relatively short list of disconnected hidden sector gauge groups. The groups of the disconnected components in these examples, as well as their multiplicities, are

$$
\begin{aligned}
SU(2)_{IV} &: & 1753 \\
SU(2)_{III} &: & 905 \\
F_4 &: & 789
\end{aligned}
$$

$$
\begin{aligned}
G_2 : &\quad 762 \\
\mathrm{SU}(2)_{III} \times G_2 : &\quad 292 \\
\mathrm{SO}(8) : &\quad 188 \\
\mathrm{SU}(2)_{IV} \times G_2 : &\quad 137 \\
E_7 : &\quad 91 \\
\mathrm{SU}(2)_{III} \times \mathrm{SU}(2)_{III} : &\quad 7 \\
\mathrm{SU}(2)_{III} \times \mathrm{SO}(7) \times \mathrm{SU}(2)_{III} : &\quad 3 \\
\mathrm{SU}(2)_{IV} \times \mathrm{SU}(2)_{III} : &\quad 3
\end{aligned}
\tag{5.9}
$$

where the subscript on each SU(2) indicates whether it is realized on a Kodaira type III or IV singularity.

For the purposes of studying hidden sector dark matter it is also useful to examine table 4. This table lists some of the 85 examples in our set that have a non-Higgsable $\mathrm{SU}(3) \times \mathrm{SU}(2)$ sector where the SU(3) and SU(2) seven-branes intersect each other, but no other non-Higgsable seven-brane, and furthermore that neither of these seven-branes is on a section. Note that in the other 228 cases where there is a disconnected $\mathrm{SU}(3) \times \mathrm{SU}(2)$ sector, with one factor on a section, there can be no hidden sectors in the theory from non-Higgsable seven-brane configurations. All 85 examples of the type listed in table 4, however, has other non-Higgsable gauge groups that do not intersect the visible sector. In some cases the hidden sectors are single factor super Yang-Mills theories, whereas in other cases they contain product gauge groups with bifundamental matter. We see that rich dark sectors with no gauge interactions with the visible sector are common in this subset of models. It should be emphasized again, however, that the distribution of types of hidden sector dark matter found in the examples we have considered here is likely heavily influenced by our choice of bases. In particular, in general E_8 dark matter sectors are likely much more frequent than in our analysis, where we have only considered strictly toric bases with no $(4,6)$ curves on divisors carrying E_8 groups. Also, the non-Higgsable clusters in the hidden sector dark matter cases described here are heavily influenced by the geometry of the base surface S. In general we might expect more, and more complicated hidden sector dark matter structure for typical threefold bases B, which will have larger Hodge numbers and typically a more complicated topology. Nonetheless, the sample considered here gives some general sense of the kinds of possibilities that may arise, and suggest that SU(2) hidden sector dark matter may be the most common occurrence in generic F-theory models.

We now study the possibility of exotic WIMP dark matter associated with an additional hidden gauge sector; pure matter WIMP contributions are considered in the next subsection. The simplest measure of the possibility of WIMP dark matter associated with a hidden gauge sector is to simply count and categorize the intersection of non-Higgsable SU(2) seven-branes with other seven-branes.

We present three types of counts, with the first being the simplest, all presented in table 7. The first count is a count of the number of times that a non-Higgsable seven-brane with a given gauge group G intersects a non-Higgsable SU(2) seven-brane. In this counting method, we note that G_2 is the most common exotic WIMP hidden sector group, followed

SU(2) Type	$SU(2)_{III}$	$SU(2)_{IV}$	SU(3)	G_2	SO(7)
All	2322	2148	524	94780	83771
Sections	703	1193	276	1522	1
Sections and intersects SU(3)	10	75	9	47	0

Table 7. Frequency of gauge groups under which exotic WIMPs are charged.

by SO(7); this closely mimics what is expected from six-dimensional compactifications, in which case these are the *only* non-Higgsable groups that may intersect a non-Higgsable SU(2). This is again likely because of the structure of our bases and in particular because in that context we have allowed for SU(2) factors that arise on seven-branes on divisors D_i that are always $\mathbb{P}^1$-bundles over curves in S, and which dominate the geometry.

The second type of count is to perform the same type of counting, but to mitigate for the effects of the specific types of base geometry by including only only non-Higgsable SU(2) groups that arise on one of the sections of the $\mathbb{P}^1$-bundle. In this case we see that intersections with SO(7) *almost never* occur, while intersections with G_2 non-Higgsable seven-branes are still the most common. Finally, our third count is to consider only non-Higgsable SU(2) seven-branes on one of the sections that also intersect a non-Higgsable SU(3) seven-brane configuration (other than the instance of the group G being counted, in the case $G = SU(3)$); that is, these SU(2) configurations are more naturally identifiable as $SU(2)_L$ given the intersection with an SU(3) seven-brane. The results of these countings are shown in figure 7.

5.6 Higgs sector fields

To realize electroweak symmetry breaking in a supersymmetric theory, the simplest approach is to have a vector pair of Higgs doublet fields as in the MSSM. We briefly investigate here how this might occur in a scenario where the standard model SU(3) × SU(2) is realized through non-Higgsable factors. Extra matter fields charged under the SU(2) can arise when the divisor supporting the non-Higgsable SU(2) factor has an additional codimension two singularity on a curve where (f, g) vanish to higher order. When the resulting matter is in the fundamental representation of SU(2), the fields (if an appropriate hypercharge is also realized) can give rise to the pair H_u, H_d of the MSSM. Such a Higgs sector could break the SU(2) if radiative corrections due to supersymmetry breaking give the lightest Higgs field a negative quadratic term so that it is forced to acquire a vacuum expectation value; i.e., if radiative electroweak symmetry breaking is realized.

Rather than doing a thorough statistical analysis on the possibilities, we simply describe how extra matter fields can arise from additional codimension two singularities on the SU(2) divisor, and speculate about how such fields may be stabilized in a supersymmetric theory. Such extra matter fields could be part of a Higgs sector, or alternatively could play a role as weakly interacting dark matter as discussed previously. We conclude with an explicit example in which these kinds of additional SU(2)-charged fields arise.

While in any non-Higgsable realization of the product SU(3) × SU(2) there is always a codimension two singularity at the intersection between the SU(3) and SU(2) branes

that can carry matter, additional matter curves also arise on a large number of the SU(2) factors that appear as components of SU(3) $\times$ SU(2) non-Higgsable clusters. In fact, in all the examples in table 4 the residual discriminant intersects the SU(2) seven-brane along a curve (away from the SU(3) locus) that is non-trivial in homology. In future work it may be interesting to study the detailed structure of these curves. We describe one explicit example below.

One natural question is how it could be that a massless Higgs pair (H_u, H_d) that arises in this fashion would not give rise to a flat direction in the supersymmetric $\mathcal{N} = 1$ F-theory supergravity vacuum, since there is no complex structure deformation that breaks SU(2), and yet $H_u H_d$ (with SU(2) indices contracted by ϵ_{ij}) is a gauge invariant holomorphic function and therefore there is a corresponding D-flat symmetry breaking direction. If the obstruction is visible in a weakly coupled Lagrangian description of this $d = 4\ \mathcal{N} = 1$ supergravity theory, the natural possibility is that it is an F-term obstruction. One possible obstruction may arise due to the possible appearance of an SU(3) $\times$ SU(2) singlet field Φ at the same locus as the Higgs fields H_u, H_d. The superpotential may then contain a term $\Phi H_u H_d$; the F-term for Φ then gives rise in turn to quartic terms in the potential $(H_u H_d)(H_u^\dagger H_d^\dagger)$, which in combination with D-terms could stabilize the Higgs field at quartic order, providing a physical mechanism that could explain the non-Higgsable nature of the field in the supersymmetric theory, and yet remaining compatible with the potential for electroweak symmetry breaking after radiative corrections push the quadratic term in the Higgs field negative. Note that such singlets would play the role of the exotic singlet in the NMSSM extension of the minimal supersymmetric standard model. Other such terms could potentially stabilize the squark sector, which would also need to be fixed in a model with non-Higgsable SU(3) $\times$ SU(2). Alternatively, it may also be possible that the obstruction giving rise to non-Higgsability of the SU(2) gauge factor in the supersymmetric ultraviolet theory is due to non-perturbative physics not captured by a Lagrangian description.

As an explicit example, consider the base defined by the toric surface with curves of self-intersection

$$((1, -1, -2, -1, -4, -1, -1)) \tag{5.10}$$

and twist

$$T = (0, 0, 2, 3, 2, 4, 3). \tag{5.11}$$

This base has a non-Higgsable SU(2) (type III) on Σ_- and a non-Higgsable SU(3) on D_5. There is naturally a codimension two singularity associated with potential matter on the curve $C = \Sigma_- \cap D_5$, where (f, g) vanish to orders $(3, 4)$. There is also, however a codimension two singularity on the curve $C' = \Sigma_- \cap D_2$, where (f, g) vanish to orders $(2, 3)$. This curve[10] does not intersect with C, and is naturally associated with additional matter charged under the SU(2) gauge group. There are also codimension two singularities on the curves $\Sigma_- \cap D_4, \Sigma_- \cap D_6$ where (f, g) vanish to orders $(2, 2)$. In other cases, similar matter curves on an SU(2)$_{III}$ locus can have other types of increased orders of vanishing of (f, g). These codimension two curves with increased vanishing of (f, g) potentially

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[10]Another way to think of this singularity is that it is a curve along which the seven-brane carrying SU(2) intersects the residual part of the discriminant, which has an I_1 Kodaira fiber.

give fundamental or other matter representations charged under the SU(2). A complete resolution or deformation analysis of any given model (see, e.g., [49–57] or [13, 14]) would need to be done to determine the specific matter content over any given curve. Matter fields produced in this way, as mentioned above, can potentially play a role as a Higgs field or as weakly interacting dark matter candidates.

6 Conclusions

In this paper we have explored a specific class of complex threefold bases for F-theory compactifications to $\mathcal{N} = 1$ supergravity theories in four space-time dimensions. We have constructed roughly 100,000 bases B that have the form of $\mathbb{P}^1$ bundles over toric base surfaces. We have studied the non-Higgsable cluster content over each of these bases and extracted some general lessons from the analysis.

The general conclusions for which we have found evidence are that:

1) Geometrically non-Higgsable gauge groups and matter are ubiquitous features in the landscape of $\mathcal{N} = 1$ 4D F-theory vacua.

2) The specific group SU(3) and the product SU(3) × SU(2) appear as non-Higgsable gauge group components less frequently than a handful of other possibilities, but they are realized geometrically in a wide range of F-theory vacua.

In addition to analyzing the statistics of non-Higgsable clusters on the specific bases we have constructed we have also made more general arguments, in part by analogy with the better understood 6D story, that support the conclusions that these hypotheses hold in a wide range of F-theory vacua beyond the particular constructions considered here. Note that many of the features found here, including the genericity of non-Higgsable clusters when $h^{1,1}(B)$ is not small, and the general broad distribution of non-Higgsable gauge group factors and clusters, are also confirmed from an analysis of toric bases using a Monte Carlo approach, which will be presented elsewhere [28].

The work presented here suggests that geometrically non-Higgsable gauge groups are a fairly universal feature in nonperturbative F-theory vacua. These structures may play an important role in better understanding the global structure of the space of vacua, and in realizing the observed standard model of particle physics in F-theory. The results of this paper, however, represent only a small first step towards a systematic understanding of these issues. While in 6D, there is a growing body of evidence [26, 27, 29] that known toric constructions of elliptic Calabi-Yau threefolds provide a good qualitative picture of the space of all elliptic CY threefolds even outside the domain of toric geometry, the situation is less clear in 4D. There is no proof that the number of elliptic Calabi-Yau fourfolds is finite, and there are base threefolds that are not rational that support elliptic CY fourfolds, unlike the case of base surfaces. Nonetheless, it seems feasible that all threefold bases that support elliptic Calabi-Yau fourfolds are connected through geometric transitions to the set of toric bases, which may be controllable in a similar fashion to the set of toric bases for threefolds, which have been completely enumerated [18]. Thus, it may be possible to

capture general aspects of the distribution of non-Higgsable clusters in 4D F-theory models by the analysis of simple toric constructions.

One particularly noteworthy aspect of the distribution of elliptic CY fourfolds described here is the increasing probability and number of non-Higgsable clusters as the Hodge numbers increase. From the landscape point of view, the fourfolds with large Hodge numbers are those that give rise to the greatest number of flux vacua [73, 74]. This gives additional weight to the notion that the F-theory landscape is dominated by vacua with geometrically non-Higgsable gauge groups and matter. While recent work has begun to explore some more explicit aspects of the distribution of 4D F-theory vacua [75–77], these analyses have focused primarily on almost-Fano bases, which have no non-Higgsable clusters and as we have found here represent a small and non-representative sample of the set of possible bases. There are many features of the larger set of vacua available, even just in the toric context, which would be interesting to study further in the broader class of vacua that include non-Higgsable structure.

In any event, the constructions considered here provide a rich set of examples with which to further explore the nature and consequences of non-Higgsable gauge groups and matter in 4D supersymmetric F-theory compactifications. The analysis undertaken here relies completely on the geometry and complex structure of the F-theory base threefolds. A major outstanding challenge for F-theory is to systematically understand the role of fluxes and additional degrees of freedom on seven-brane world-volumes, which may among other things enhance or break parts of the geometrically non-Higgsable gauge group. A serious effort to understand the phenomenology of models based on non-Higgsable SU(3) or SU(3) × SU(2) groups in global F-theory compactifications will likely involve significant advancements in our understanding of these issues.

As a resource for the reader, we have provided the details of the set of threefold bases constructed in this paper, and information about the corresponding non-Higgsable structures, in a data file that can be accessed online.[11]

A Improved constraints

In this paper we have used constraints on twists which are stronger than those given in (3.4)–(3.7), which were derived in [19]. In this appendix we briefly describe two different

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[11]The basic data (S, T, and non-Higgsable structure on toric divisors and curves) of the 109,158 threefold bases constructed in this paper is available online in an ancillary mathematica format text file at http://ctp.lns.mit.edu/wati/data/p1-bundles.m.

approaches to deriving stronger constraints. We implemented both of these approaches to give a complete enumeration of allowed $\mathbb{P}^1$ bundles in the desired class, and got identically matching results from both approaches.

A.1 Constraints on twists: I

We begin by reviewing in the toric language the origin of the constraints (3.4)–(3.7), following similar logic to that given in [19]. Consider a local component of the toric geometry of the $\mathbb{P}^1$ bundle. We choose coordinates where

$$w_1 = (1, 0, 0) \tag{A.1}$$
$$w_2 = (0, -1, 0) \tag{A.2}$$
$$s_- = (0, 0, 1). \tag{A.3}$$

These correspond to the divisors D_1, D_2, Σ_-. Now, assume d_2 is a curve of self-intersection $-n$. Then we have

$$w_3 = (-1, -n, t), \tag{A.4}$$

where $t = \tilde{t}_2 = T \cdot d_2 = t_3$. If we choose coordinates (a, b, c) on the dual lattice N^* encoding monomials, the condition that (f, g) do not vanish to orders $(4, 6)$ on $\Sigma_- \cap D_2$ is that there must be some monomial $(a, b, c) \in \mathcal{G}$ such that

$$(a, b, c) \cdot (0, -1, 1) = c - b \leq -7 \tag{A.5}$$

where $(a, b, c) \cdot (-1, -n, t) \geq -6$. We have $a \geq -6, b \leq 6, c \geq -6$ from the definition of $\mathcal{G}$ (5.2) and the rays (A.1)–(A.3). Assuming that $n, t > 0$, we see that the most negative values of (A.5) can come from the monomials $(-6, 6, -1)$ or $(-6, 1, -6)$, depending on whether $t > n$ or $n > t$. If both of these monomials are ruled out then there is a $(4, 6)$ vanishing on $\Sigma \cap D_2$. The condition is then

$$-n - 6t \geq -12 \quad \text{or} \quad -6n - t \geq -12. \tag{A.6}$$

One of these must be satisfied to avoid the $(4, 6)$ curve. It is straightforward to check that this condition precisely gives the rules (3.4)–(3.7). A straightforward generalization of this analysis, however, gives further constraints on the rays $w_i, i > 3$. If any such ray has the form $w_i = (-m, -n, t)$ with $m, n, t > 0$, then there is a similar constraint

$$-n - 6t \geq -6(1 + m) \quad \text{or} \quad -6n - t \geq -6(1 + m). \tag{A.7}$$

These constraints can be interpreted in terms of constraints on sequences of twists that can appear over any given base. For example, if d_2 is a -12 curve, we must have $\tilde{t}_2 = 0$. We must also however then have a -1 curve d_3, so that $w_4 = (-1, -11, \tilde{t}_3)$, from which the above constraints give $\tilde{t}_3 = 0$. Similar arguments place very strong constraints on the kinds of twists that are allowed for a general $\mathbb{P}^1$ bundle base. By implementing these constraints for all curves $\Sigma_\pm \cap D_i$ and all w_i with rays that lie in the appropriate octant in a local coordinate system as above, the problem of classifying the $\mathbb{P}^1$ bundle bases over the toric surfaces from [18] becomes a tractable computational procedure, which we have implemented.

A.2 Constraints on twists: II

The second approach is slightly more abstract and based on the principle of the Zariski decomposition, as used in [18] to classify the non-Higgsable clusters for complex surface bases. This second approach could be used for base surfaces S that are not necessarily toric. The basic idea is that if one has information about connected curves C_i in a surface, certain curve classes X could contain higher multiples of the C_i as components than the number of components that would be derived if one only used knowledge of a single curve. It is straightforward to see why this phenomenon may be possible. Consider an effective irreducible curve L in a surface. We would like to know whether a given effective class X contains L as a component. A fact from the algebraic geometry of surfaces, which underlies the Zariski decomposition on surfaces, is that if $L \cdot L < 0$ and $X \cdot L < 0$, then X contains L, i.e. $X = L + X_1$ for some effective divisor (curve) X_1. Now by the same argument if $X_1 \cdot L < 0$, X_1 contains L and we iterate this process until $X = nL + X_n$ for an effective divisor X_n with $X_n \cdot L \geq 0$. Without further information about the surface and the curves in it, we have pulled out as many copies of L from X as we can.

Suppose, though, that there is another curve R with $R \cdot R < 0$ and $X_n \cdot R < 0$. Then $X = nL + R + X_{n+1}$ and again we recurse until we end up with $X = nL + mR + X_{n+m}$ for some effective divisor X_{n+m} satisfying $X_{n+m} \cdot R \geq 0$. Now note that

$$L \cdot X_{n+m} = L \cdot X - nL \cdot L - nmL \cdot R = L \cdot X_n - nm L \cdot R \qquad (A.8)$$

satisfies $L \cdot X_{n+m} \geq 0$ if $L \cdot R = 0$, but if $L \cdot R \neq 0$ then $L \cdot X_{n+m} < 0$ if $nmL \cdot R > L \cdot X_n$. That is, if L intersects a curve R which is also a component of X, then X may contain additional components of R. To determine the full number of times that a class X contains some negative self intersection curves C_i in a connected set as components, one simply recurses over all curves in the set until

$$X = \sum_i n_i C_i + X \qquad (A.9)$$

with $X \cdot C_i \geq 0$ for all i. This can be quickly implemented on a computer. We will henceforth refer to this method of pulling off as many components as possible based on adjacency data (self-intersections and twists of adjacent curves) as a *recursive pull*.

Now consider the relevance for minimality of the Weierstrass equation. Write

$$f = \sum_{k=0}^{8} z^k w^{8-k} f_k \qquad \text{and} \qquad g = \sum_{k=0}^{12} z^k w^{12-k} g_k, \qquad (A.10)$$

where f and g are sections of $\mathcal{O}(-nK_2 - (n-k)T)$ for $n = 4, 6$, respectively, and define the class $[n_k] \equiv -nK_2 - (n-k)T$. For any genus zero curve C in the twofold base, $\chi(G) = 2 = \int_C c_1(C) = (-K_2 - C) \cdot C$, and therefore

$$[n_k] \cdot C = n (2 + C \cdot C) - (n-k) T \cdot C, \qquad (A.11)$$

where $T \cdot C$ is what we have called $\tilde{t}$ in the toric context. Though the following twist bounds apply beyond the context for toric varieties, we will use the $\tilde{t}$ notation for simplicity. Note

that the only geometric data this depends on is the self-intersection and the twist, which allows for the analysis of twist bounds without needing to specify a base.

Building on the previous ideas, we now present a recursive and relatively efficient algorithm for the classification of allowed $\mathbb{P}^1$-bundles over a toric base. This algorithm could be generalized to a broader class of base surfaces.

Since we are building all allowed $\mathbb{P}^1$ bundles over the toric surfaces of [18], a natural question is how to break this difficult combinatoric problem into manageable parts. In order to tackle the combinatorics, we would like to note the following facts:

- The weakest bounds on twists in (3.4) are for $C^2 = 0$ curves in S. However, since there are at most two 0-curves in any of the bases of [18], the bigger combinatoric penalty comes from the (-1)-curves, which typically occur with higher frequency in the examples we study.

- Since (-1) curves may potentially give rise to additional components when adjacent to other negative self-intersection curves with twists over them, the twists over the (-1) curves may be constrained beyond the results of [19].

- While there are sometimes short chains of (-1)-curves,[12] it is typically the case that (-1)-curves are separated by sequences of curves with negative self-intersection. These are the non-Higgsable clusters of [11]; for simplicity we refer to such a sequence as a *6NHC* since they correspond to a non-Higgsable cluster in the 6d theory obtained from compactifying F-theory on a Calabi-Yau elliptic fibration over that S.

Instead of the costly algorithm for classifying allowed $\mathbb{P}^1$ bundles in which one simply applies the constraints (3.4)–(3.7), the algorithm we employ first classifies the possible allowed twists over each 6NHC where a recursive pull is used to strengthen the bounds. This can be done efficiently by taking the chain of curves and iteratively assigning twists, subject to the condition that the recursive pull on $[n_k]$ does not give rise to a $(4, 6)$ curve. A $(4, 6)$ curve arises whenever $[n_k]$ contains $(n - k)$ components of C for all $k \in 0 \ldots n - 1$ and $n = 4, 6$. This is already a significant improvement, but it does not improve on the combinatorics of (-1)-curves since there is never a (-1) curve in a 6NHC.

It is often the case, however, in the bases of interest that a (-1) curve is adjacent to a 6NHC. So, given the classification of twists over 6NHC's, one can put a (-1)-curve on either end, or on both ends, and study the classification of twists over that sequence of curves — letting "o" denote the presence of a (-1) curve, we call such sequences o6NHC, 6NHCo, and o6NHCo. Similarly, a recursive pull can be used to classify the possible twists over such sequences, and the classification is much smaller than direct application of the constraints (3.4)–(3.7). For example, the longest 6NHC that appears in the bases of interest is a chain of 23 consecutive (-2) curves, from which we can form the o6NHCo

$$[-1, -2, -2, -2, -2, -2, -2, -2, -2, -2, -2, -2, -2, -2, -2, -2, -2, -2, -2, -2, -2, -2, -2, -2, 1].$$
$$(\text{A.12})$$

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[12]The longest such chain occurring in a base of [18] has length 6, and this only occurs in one example, dP_3.

In this case the simple constraints (3.4)–(3.7) would allow for about $13^2 \cdot 3^{23} \sim 16$ trillion possible sets of twists, whereas a recursive pull shows that again there are only 587 possibilities. This is the most dramatic combinatoric gain, but large gains also exist for the other o6NHCo's.

Having classified all allowed twists over 6NHC, o6NHC, 6NHCo, and o6NHCo sequences our algorithm proceeds to construct the complete base surfaces, breaking the bases into building blocks that include these various different types of sequences. Instead of applying the constraints curve-by-curve, we proceed building block by building block, using twists over building blocks which have been classified. For example, instead of viewing one particular S as

$$[-6, -1, -3, -1, -3, -2, -1, -6, -1, -2, -2, -2, -2, -2, -2, -2, -1] \qquad (A.13)$$

the algorithm we wrote (which is efficient but could be made more efficient) views it as

$$[[-6], [-1], [-3, -1], [-3, -2, -1], [-6, -1], [-2, -2, -2, -2, -2, -2, -2], [-1]] \qquad (A.14)$$

where, for example, the allowed twists over $[-6]$ and $[-1]$ were classified by the simple single-curve twist rules but the method described above classifies the twists over the remaining "building blocks" which have more than one curve of negative self interaction. The combinatoric gains for the entire base are compounded from the combinatoric gains of the building blocks. The algorithm is to fix a base surface S and break it into building blocks as above. Then one uses the classified twists over each building block to construct the possible twists over all curves in S, checking for $(4, 6)$ divisors and curves after assigning all twists, and keeping any pair of S and twists that does not have $(4, 6)$ divisors or curves.

The results of implementing this algorithm agree perfectly with those from the more explicit toric method described above. It is interesting to note that while these constraints are clearly necessary for the construction of a consistent base, it turns out that when this set of constraints is implemented completely, these are also sufficient conditions, which we have confirmed by comparison with the bases produced by the direct toric algorithm. This shows that there are no additional nonlocal constraints on the set of allowed bases. This observation may be helpful in construction of more general classes of threefold F-theory bases.

B Additional constraints on E_8 non-Higgsable seven-branes

Throughout this work we have been careful to check whether the elliptic fibration contains curves where (f, g) vanish to orders $(4, 6)$ or higher, which often occurs when two non-Higgsable seven-branes intersect. For non-Higgsable seven-branes that carry an E_8 factor, however, a $(4, 6)$ curve can be obtained from a collision with the I_1 locus in a way that is not manifested on a toric curve. This is the analogue for 4D F-theory compactifications of the $-9, -10,$ and -11 curves that have similar properties in 6D F-theory compactifications. While in the 6D story these are relatively easy to handle by blowing up points on the base, the situation is more complicated for curves on threefolds, so we do not consider threefold bases in this paper that have such $(4, 6)$ curves on E_8 divisors. In this brief appendix we describe the details of how we rule out such bases.

Consider a Weierstrass model on a threefold base B that contains a non-Higgsable E_8 seven-brane configuration along a divisor Z. If the base is toric, we have a homogeneous coordinate z so that $Z = \{z = 0\}$, and the coefficients in the Weierstrass model can always be written in the form

$$f = z^4 \tilde{f}_4 = z^4 f_4 + z^5 f_5 + \cdots \qquad\qquad g = z^5 \tilde{g}_5 = z^5 g_5 + z^6 g_6 + \cdots \qquad \text{(B.1)}$$

The discriminant locus takes the form

$$\Delta = 27 z^{10} g_5^2 + \mathcal{O}(z^{11}) \qquad\qquad \text{(B.2)}$$

Thus, a $(4,6)$ curve arises on the divisor Z whenever g_5 generically vanishes along some curve. In the toric context, g_5 is described by a set of monomials in $\mathcal{G} \subset N^*$. If g_5 contains more than one monomial then there is always a vanishing curve, as can be seen in any local coordinate chart, where g_5 simply becomes a polynomial in two variables. If g_5 contains only one monomial, then the only vanishing curves can be toric curves, which we already check in our analysis. Thus, to rule out the bases that have $(4,6)$ vanishing loci on non-toric curves within E_8 divisors, we can simply restrict attention to bases where g_5 only has one monomial in G.

From a more general point of view, we see that the I_1 locus intersects the E_8 seven-brane if the curve $C \equiv \{z = g_5 = 0\}$ exists in B. Taking $z \in \mathcal{O}_B(Z)$, we have $\tilde{g}_5 \in \mathcal{O}_B(-6K_B - 5Z)$ and therefore the curve C is in the class $[C] = (-6K_B - 5Z) \cdot Z \in H_2(B, \mathbb{Z})$. If $[C]$ is trivial in homology, the intersection doesn't exist and there is no associated $(4,6)$ curve; this happens if $[g_5]$ is itself trivial, i.e. g_5 is a section of $\mathcal{O}_B$, which is sufficient but not necessary for $[C] = 0$. From the point of view of B, the necessary and sufficient condition for avoiding a $(4,6)$ curve from an E_8-I_1 collision on Z is that $[C]$ is trivial. If Z is $\mathbb{P}^1$ bundle over a curve in S, a short calculation shows that $[C] = 0$ only if $\tilde{g}_5 \in \mathcal{O}_B$. On the other hand, if the divisor Z carrying the E_8 is one of the sections of the $\mathbb{P}^1$-bundle, then one might imagine that it could be the case that $[C] = 0$ even if $[g_5] \neq 0$, that is if $\tilde{g}_5$ appears as a section of a non-trivial line bundle on B; one should then explicitly compute the class of $[C]$, which we have done for each of the bundles we have constructed, giving precisely the same results as the monomial analysis above. In general, though, the question from the point of view of B is simply whether or not the curve C exists.

The desired condition can also be stated in terms of the local geometry of Z; this perspective, which is that taken in [16], is particularly useful since it extends to general base threefolds even when they are not toric. By the adjunction formula, when we restrict g_5 to Z we have

$$g_5 = \tilde{g}_5|_{z=0} \in \mathcal{O}_Z(-6K_Z) \otimes N_{Z/B}. \qquad\qquad \text{(B.3)}$$

The condition that this line bundle have no vanishing sections is that $N_{Z/B} = \mathcal{O}_Z(6K_Z)$. This is a more general way of stating the condition that an E_8 divisor have no $(4,6)$ curves, which was also described in [19] in the context of F-theory models with heterotic duals.

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