

Essays on Finance

by

Angel Serrat Tubert

Submitted to the Sloan School of Management
in partial fulfillment of the requirements for the degree of
Doctor of Philosophy

at the

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Author
Sloan School of Management
May 2, 1996

Certified by
Jiang Wang
Associate Professor of Finance
Thesis Supervisor

Accepted by
Birger Wernerfelt
Chairman, Departmental Committee on Graduate Students

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Abstract

This thesis is composed of three essays. One of them deals with international portfolio choice issues and the other two deal with the behavior of exchange rates in multilateral target zones. In the first chapter, I develop a dynamic general equilibrium model of a two-country exchange economy with nontraded goods and complete financial markets. In the model, the nontraded goods play the role of state variables that shift the marginal utility of the traded goods. This affects relative prices of goods and assets and generates hedging demands that: (i) provide a rationale for the well known Fama (1984) foreign exchange excess return predictability puzzle, and (ii) help explain the well documented Home Bias puzzle in international equity portfolios. In the second chapter, I construct a model in the spirit of Krugman (1991) that provides a closed form solution for the exchange rate process in a multilateral target zone. New economic insights beyond Krugman's results for the bilateral case are offered and new methodological issues arise from the use of multidimensional reflected diffusion processes. The qualitative predictions of the model are consistent with the empirical evidence, unlike the original bilateral model of Krugman. In the third chapter, the multilateral target zone model developed in chapter 2 is estimated using a simulated method of moments methodology. It is found that the model fits the data from the EMS quite well. The good performance of the model relative to Krugman's is driven by the parameters that reflect the degree of cooperation among monetary authorities in maintaining the system.

Thesis Supervisor: Jiang Wang
Title: Associate Professor of Finance

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Chapter 1

A Dynamic Equilibrium Model of International Risk-Sharing Puzzles

1.1 Introduction

International evidence on exchange rate and interest rate behavior and on international portfolio choice is at odds with the predictions of current dynamic asset allocation models. This is specially true regarding the lack of theoretical explanations of the behavior of exchange rate risk premia and the cross-sectional evidence on international portfolio choice: Exchange rate risk premia exhibits much volatility and investors tend to bias their equity portfolios towards domestic equities.

In this paper I develop a two-agent international real exchange economy with two goods which can be traded and two goods which cannot. In the model, the nontraded goods play the role of the unspecified factors in much of the previous literature on multifactor dynamic asset pricing models. Closed form solutions for exchange rates, interest rates and portfolio policies are obtained. The behavior of equilibrium equity returns, bond returns and option returns are also characterized. The portfolio choice problem is studied and home bias in equity portfolios arises in

equilibrium. This is because the nontraded goods enter nonseparably in the utility function and they therefore shift the marginal utility of the tradable goods randomly over time. Investors will then want to hedge this risk. If the correlation between the output growth rate of tradable and nontradable goods is higher within countries than between countries¹, and under mild conditions on preferences, investors will rationally overload their portfolio with domestic equity, relative to the world portfolio. I show that the presence of nontraded goods also helps explain the so called excess return predictability puzzle of Fama (1984a, 1984b). It will also be shown that, in equilibrium, a multifactor pricing equation determines equity returns in a three-beta fashion. If there are no nontraded goods, this factor pricing equation collapses to the CCAPM factor pricing equation. In addition, many of the empirical regularities observed in exchange rate and forward rate time series, and also in the behavior of bond returns, are accounted for. All results are compared to those of a benchmark economy without nontraded goods, and the economic intuitions as to why differences arise are outlined.

Researchers tend to think of international finance problems using two alternative paradigms, depending on whether frictionless markets are assumed or frictions of some kind are taken into account. In the context of the first paradigm, one tends to extrapolate to an international setting results from three frictionless benchmark models. Two of them are general equilibrium, complete-markets approaches and the third is a partial equilibrium one: (1) A closed single-agent exchange-economy benchmark consisting of an intertemporal model with power utility and lognormal endowment process [Lucas (1978), Svensson (1985, 1987) and Zapatero (1995) for the log-utility case], (2) International extensions of the production-economy of Cox Ingersoll and Ross (1985) representative agent model (Nielsen and Saa-Requejo, 1992) (3) The international CAPM [Solnik (1974), Adler and Dumas (1983), Dumas and Solnik (1995)] and the extensions to multifactor and latent variables models [Connor and Korajczyk (1991), Hansen and Hodrick (1983), Gibbons and Ferson (1985), Campbell and Hamao (1992)]. Although the last approach has seen some success in

¹this will not be strictly necessary, though

explaining the cross-sectional variation of international equity returns, these models fail to account for many of the empirical regularities observed in international data. The multifactor extensions of CIR in (2) account for some of the observed regularities in the movement of exchange rates and the international term structures of interest rates. However, the absence of specific economic identification of the factors in those models makes it difficult to extract new economic insights. In addition, very few of the above lines of research address the problem of international portfolio choice determination.

In the “frictions” paradigm, one can include the equilibrium approaches that incorporate non-nominal frictions in an international setting [such as Dumas (1992,1994) and Sellin and Werner (1993)] and nominal models, normally based on cash-in-advance constraints [Lucas (1982), Svensson (1985)]. However, even though these models can rationalize certain empirical regularities such as deviations from PPP, they cannot explain why home bias in international equity portfolios arises in equilibrium unless one is willing to accept extremely low rates of risk aversion [Uppal (1993)]. On the other hand, the existing nominal models simply don’t address the issue of portfolio choice.

Our first motivation in constructing the model consists of investigating under which conditions national investors would optimally bias their equity portfolios towards domestic equities. All three modelling paradigms above share the same view regarding the distribution of portfolio holdings across countries. Namely, national deviations from investment in the world portfolio will be small or inexistent. Of course, we know from Merton (1973) and Huang (1987), among others, that this is not a necessary prediction of a dynamic asset pricing model. However, no equilibrium model of the international economy has been able to generate the right diversity in portfolio holdings. This is in clear contradiction to the well documented bias towards domestic assets. French and Poterba (1991) report that U.S. investors held 94% of their stock market investments in domestic equities at the end of 1989, Japanese investors held 98% and U.K investors 82% [see also Adler and Jorion (1992), Uppal (1992) and Tesar and Werner, (1992,1995)]. This is a puzzle, not only given the predictions of

standard portfolio choice models, but also given the consensus regarding the potential welfare gains of international diversifications [for instance, Grubel (1968), Van Wincoop (1994) and Obstfeld (1994)]. The home bias puzzle has been addressed from many perspectives, but none has yet provided a convincing explanation. International extensions of the CAPM [Adler and Dumas, (1983)] indicate that home bias can arise, in absence of PPP, because of an inflation hedging mechanism arising from the fact that agents translate returns to different numeraire currencies. However, national deviations from the world portfolio will consist of a portfolio almost entirely composed of bonds, as long as inflation is much less volatile than nominal returns and there are PPP deviations, thus leaving little hope of explaining significant differences in equity portfolios. This is confirmed in Cooper and Kaplanis (1994) empirical work. The puzzle cannot be justified by tax differential treatments either. For one thing this could only affect domestic tax-free institutional investors that cannot benefit from tax credits, and secondly, it is difficult to believe that the maximal differential loss from investing abroad associated to taxes could be large². In addition, Cooper and Kaplanis also calibrate the differential deadweight costs needed to justify the observed holdings using an international extension of the CAPM, and these costs turn out to be, for conventional levels of risk aversion, between four and twenty-five times larger than those that one could justify from the observed tax costs plus transaction costs. It is also difficult to believe explanations fully based on asymmetric information grounds. One reason is simply information arbitrage. International portfolio management seeking diversification benefits could always be delegated to the informed local managers. In addition, if the issue for a domestic investor seeking diversification benefits was an adverse selection problem associated with trading in foreign spot markets against better informed local agents, we should see much more foreign positions in the markets where the index can be traded through a futures contract. Regarding sovereign or confiscation risk, there is no reason to believe that this risk would not be diversifiable.

²For instance, a withholding tax of 15% on a dividend of 4% would yield a loss per year of 60 basis points to a domestic tax-free investor

The association of nontraded goods with the home bias phenomenon has some tradition in international finance. In the first models, nontraded goods affected portfolio choice by introducing price risk: The agents have to purchase their allocation of nontradables period after period. Since nontraded goods enter separably in the utility function, agents can completely insure against this price risk by simply owning the firms that produce nontraded goods located in his country [see Eldor, Pines and Schwartz (1988), Stockman and Dellas (1990) and Tesar (1993)]. In any case, these papers only rationalize why investors will want to bias their portfolio towards domestic nontradable-goods producing firms. However, they will still invest in the same fund of firms that produce tradable goods. As Tesar (1993) and Pesenti and Van Wincoop (1994) pointed out, this result may be due to the assumption made in these models that nontraded goods are separable from traded goods in the utility function, at least in a partial equilibrium framework. However, Baxter, Jermann and King (1995) argue that in a two-period exchange economy nonseparabilities cannot be the reason of home bias in equities of firms that produce tradable goods, in equilibrium. There are then two open questions: First, whether home bias in both tradable and nontradable producing firms arises in a dynamic equilibrium due to the presence of non traded goods. Second, if the answer to the previous question is yes, what is the magnitude of the bias that can be generated by calibrating the model to macroeconomic data. In this paper, I attempt to answer these two questions.³

The mechanism that drives the home bias in the model can be illustrated as follows: Imagine a toy economy with two agents (d and f), two periods (0 and 1), two states (a and b) and three goods (t , n_d and n_f). Consumption takes place in period 1 only. The good t can be consumed and traded by both agents while n_d (resp. n_f) can only be consumed by agent d (resp. f). Preferences over consumption in period 1

³However, Baxter, Jermann and King (1995), by linearizing the Pareto-sharing rules around the competitive allocation and interpreting the coefficients of this expansion as portfolio weights effectively treat the portfolio holdings problem as if preferences were separable, i.e. ignoring cross derivatives. In addition, their results cannot be extrapolated to a dynamic, continuous-time framework since their portfolio policies would not be self-financing in the sense of Harrison and Kreps (1979): portfolio policies not only have to finance the consumption allocation at some period, but also have to provide funds to retrade their portfolio for future consumption.

are identical for both agents and are given by the utility function $U^i(t, n_i) = t^q n_i^p$ for $i = d, f$ and some positive constants q and p . There are two firms (T_d and T_f) that produce the good t . Firm T_d produces in period 1 the quantities \bar{t} in state a and 0 in state b , while firm T_f produces in period 1 the quantities 0 in state a and \bar{t} in state b . In addition, there are two firms (N_d and N_f) that produce the nontradable goods n_d and n_f , respectively. Firm N_d produces in period 1 the quantities \bar{n} in state a and 0 in state b , while firm N_f produces in period 1 the quantities 0 in state a and \bar{n} in state b . Agent d is endowed with all the stock of firms T_d and N_d and agent f is endowed with all the stock of firms T_f and N_f . If agents didn't care about nontradables ($p = 0$), in the competitive equilibrium each agent would consume $\bar{t}/2$ in each state in period 1 and would own 50% of the shares of firms T_d and T_f . Thus each individual portfolio replicates the world market portfolio of firms that produce good t and there is no home bias. Now, if agents care about nontradables ($p \neq 0$), in equilibrium agent d would consume \bar{t} units of the tradable good in state a and 0 units in state b , while agent f would consume 0 units of the tradable good in state a and \bar{t} units in state b . In addition, the equilibrium holdings that finance this allocation imply that agent d would own 100% of firm T_d while agent f would own 100% of firm T_f . However, at period zero, both firms T_d and T_f have equal share in the world market portfolio since they are worth the same⁴. Therefore home bias in equity portfolios arises.

As it is clear in the previous example, for home bias to arise, we need (1) complementary of traded and nontraded goods in the utility function, and (2) a certain correlation structure for the endowment growth rates. In particular, it will be sufficient, although not necessary, that the home endowment of tradable goods be more correlated with the home endowment of nontradable goods than with the endowment of nontradable goods abroad. Empirical justification of (1) will be given in Section 2.3 below. On the other hand, (2) can be justified simply by computing the correlation coefficient of the rate of growth of the production of nontradables and tradables at

⁴Strictly speaking, because the marginal utility in one of the states is zero, the foreign firm is worthless for the domestic agent, while it has some value for the foreign agent, and thus no equilibrium exists. To overcome this, one can let the lower bound on the endowment of nontradables be ϵ instead of 0, and then take the limit.

home. This figure turns out to be 0.91 for the USA, 0.45 for Japan, 0.68 for Canada, 0.75 for Germany, 0.76 for France, 0.55 for the UK and 0.8 for Italy. On the other hand, the correlation between the growth rate of the production of tradables at home and the production of nontradables abroad (an average of the correlations with each of the other G-7 countries) turns out to be 0.52 for the USA, 0.1 for Japan, 0.31 for Canada, 0.52 for Germany, 0.3 for France, 0.25 for the UK and 0.34 for Italy⁵. However, it is worth mentioning at this point that it is not to what extent the endowment processes of tradables are correlated what matters for portfolio choice in a dynamic context, but rather how the correlation structure between the equilibrium prices of these stocks of endowment reflects the underlying correlation of the dividend process.

The idea in the example above needs to be extended to a dynamic setting. This is because a phenomenon that arises in a two period model under certain assumptions need not arise in a dynamic context under equivalent assumptions (the CAPM is an example). One reason is that, in a dynamic context, the equilibrium price processes of the firms will reflect the existence of nontraded goods (that will enter as state variables in the equilibrium asset price processes). This will generate hedging demands in the portfolio choice problems since the supply of nontraded goods will function as additional state variables that induce movements in the investment opportunity set. Note that the nature of this hedging demand is different from the one that arises in the previous example, where agents hedge movements of state variables that enter the utility function directly. This additional hedging demand is genuinely dynamic (i.e. it cannot arise in a two period model) and could have opposite effects to the ones illustrated in the example above. In addition, it will turn out that our model has interesting implications for the behaviour of the forward exchange rate risk premia regarding the well known Fama (1984a) puzzle. Of course, forward rates can only be addressed in a dynamic context. Lastly, we want to study the issue of home bias relative to shares of firms (real assets) and also relative to contingent claims in net supply. This is important because home bias in one kind of traded assets could be

⁵This is computed from the OCDE International Sectoral Databank with data until 1991. Details from the computation of these correlations are available from the author.

offset by positions taken in other kind of assets.

In this paper I set up a dynamic model with four goods (two traded and two non-traded) and complete financial markets. Nontraded goods will affect the equilibrium allocation because they enter nonseparably in the utility function. The assumptions made to introduce nontraded goods as state variables in the economy are supported by the empirical evidence. In addition, the fact that these state variables consist of the output of certain firms that has to be consumed and is subject of market clearing conditions makes the equilibrium problem essentially different from the formulation in Merton (1973). For one thing, the dynamic portfolio policies will have to provide funds to the agents at any time not only to buy their optimal consumption allocation of tradables and recompose their portfolio at that time (as in standard models), but also to purchase their own endowment of nontradables at some equilibrium prices. All the asset pricing results and the competitive consumption allocation of the model will follow by applying standard results of dynamic asset pricing theory in complete markets. However, the technical problem that this paper resolves is a new way is how to identify the portfolio holdings that finance that equilibrium allocation. Certainly this is a problem that does not arise in standard representative agent models since these models are mostly interested in asset pricing results.

To sum up, the contributions of this paper are: (1) To provide an economic rationale for the existence of a systematic domestic bias in equity portfolios, (2) To provide an economic rationale for the observed stochastic behavior of foreign exchange risk premia, (3) To provide a multifactor model of international asset pricing where we can identify explicitly the economic nature of the factors, and therefore it is testable, (4) To analyze the interaction between the structure of the goods markets and the pricing of real and financial assets, and (5) From a methodological point of view, to show how Malliavin calculus techniques can be used to analyze optimal portfolio policies and equilibrium price processes in a dynamic equilibrium framework.

In the next section I define the economy and its equilibrium concept. Existence issues are discussed and a methodology for the solution of the equilibrium is presented. After that, I solve the model in the specific case where there are no nontradables.

Then I solve the model in the case where nontradables are present, and the equilibrium asset returns and portfolio policies are characterized. The equilibrium behavior of interest rates, exchange rates and risk premia is then derived. After that, I discuss the conditions that make home bias in equity portfolios arise and I provide a potential explanation for the well-known Fama (1984) foreign exchange excess return puzzle. Appendix 1 contains the proofs. Appendix 2 contains an introduction to the Malliavin calculus techniques used in proving some of the results.

1.2 The International Economy

1.2.1 The real side

Throughout the following sections, fix a probability space $\mathcal{S} = (\Omega \times [0, T], \mathcal{B} \otimes \mathfrak{B}([0, T]), \mathcal{P} \times \text{Lebesgue})$ with a filtration $\{\mathcal{B}_t^W\}$ generated by an 4-dimensional Wiener processes $\{W_t\}$ defined on such space. Denote by $\{\mathcal{B}_t\}$ the augmentation under \mathcal{P} of the filtration generated by $\{W_t\}$. All processes to follow are assumed to be adapted to $\{\mathcal{B}_t\}$, and this information is common knowledge.

We have an exchange economy with two countries, the *domestic* country and the *foreign* country. In each country lives a different consumer-investor which we will call *agent*. As we will see later, agents are different because their preferences and consumption sets are different. Each agent lives during an interval of time $[0, T]$ and consumes continuously during that period of time. There are four consumption goods. Two of them (from now on *tradables*) are consumed by both agents and they can be traded between countries. They are produced by two firms, each one located in a different country. They will be called the *domestic tradable* and the *foreign tradable*. The domestic tradable will be taken to be the numeraire. The other two consumption goods (from now on *non-tradables*) are country-specific non-traded goods. Each one of them is produced by a different firm located in each country and is consumed by the agent that lives in that country. There are thus three goods markets. The tradables are interchanged in an international goods markets at an equilibrium exchange rate

process $\{\phi_t\}$. Henceforth ϕ_t denotes the price at time t , in units of the domestic tradable, of one unit of the foreign tradable. The nontraded goods can be exchanged for units of the numeraire by both residents and nonresidents in an internal market within each country at the country-specific, equilibrium relative price $\{\eta_t\}$ for the domestic internal market, and $\{\eta_t^*\}$ for the foreign internal market. Thus, η_t (resp. η_t^*) denotes the price at time t , in units of the domestic tradable of one unit of the nontradable good produced in the domestic (resp. foreign) country. Let's define the relative price vector process $p_t = (1, \phi_t, \eta_t, \eta_t^*)'$.

Since this is an exchange economy, we do not model the production decisions of the firm directly. However, we will continue to refer to the endowment flows in each country as *dividends* paid by *firms* (rather than the flow of consumption goods from an endowment stock). Thus, in the domestic (resp. foreign) country, one of the firms produces a stream of units of the traded good given by a progressively measurable dividend process $\{\delta_t\}$ (resp. $\{\delta_t^*\}$), while the firm producing the nontradable lays out an endowment stream given by a progressively measurable process $\{n_t\}$ (resp. $\{n_t^*\}$). Let the endowment processes in logarithms be represented by the vector $e_t = (\log \delta_t, \log \delta_t^*, \log n_t, \log n_t^*)$. We will assume that e_t follows a generalized multivariate Ornstein-Uhlenbeck process of the form⁶:

$$de_t = (a_t - b_t e_t) dt + \sigma_t dW_t \quad (1.1)$$

where e_0 is independent of $\{W_t\}$, and where a_t, b_t and σ_t are (4×1) , (4×4) and (4×4) real matrices respectively $\forall t \in [0, T]$, which are nonrandom and locally bounded. The specification (1.1) is appealing because it allows for mean-reversion in rates of growth (if $\text{diag}[b_t] > 0$) and not in levels (in accordance with business cycle evidence) plus random conditional heteroskedasticity (in levels). It also allows deterministic trends. The usual lognormal case as a special case is contained in (1.1) (if a_t and σ_t are constant matrices and $b_t = 0$). We will assume that a_t, b_t and σ_t are such that the

⁶This class of processes is overly restrictive for many of the purposes of this paper. When appropriate, generalizations will be pointed out.

variance-covariance matrix of Y_t is nonsingular $\forall t \in [0, T]$ (sufficient conditions for these are provided in the second Remark following the proof of Lemma 3 in Appendix 1). Financial markets allow for trade in the equity of the four firms, plus instantaneous borrowing/lending opportunities denominated in units of both tradable goods.

It is worth mentioning that all the results of the model to follow, except those in the portfolio policies section, would go through unchanged with an endowment process much more general than (1.1). In spite of this, we will even assume more structure on the form of a_t, b_t and σ_t in order to simplify the interpretation of the results regarding portfolio holdings in Subsection 5.5., as well as other results in the paper. The matrix b_t is specified to be:

$$b_t = \begin{pmatrix} \bar{0} \\ 0 & 0 & b_t^n & 0 \\ 0 & 0 & 0 & b_t^{n^*} \\ \bar{0} \end{pmatrix} \quad (1.2)$$

where $\bar{0}$ is a 2×4 matrix of zeros, and $b_t^n, b_t^{n^*}$ are deterministic functions of time. We also restrict the first two elements of the column vector a_t to be constants, and the first two row vectors of σ_t to be constant vectors. This is because in this paper we want to focus on the heterogeneity created by the presence of nontraded goods, and therefore we take the most common process for the endowment of tradable goods (i.e. lognormal). Thus, in (1.2), we let the endowment processes of the tradable goods follow lognormal processes, we let the logarithm of the nontraded endowment processes exhibit mean reversion. Finally, we will write the diffusion matrix of (1.1) as $\sigma_t = [\sigma'_\delta : \sigma'_{\delta^*} : \sigma'_{n,t} : \sigma'_{n^*,t}]'$ where, for instance, σ_{δ^*} stands for the 1×4 diffusion vector of the endowment of foreign tradables.

1.2.2 The financial side

The number of assets is such that, in equilibrium, financial markets are complete and no redundant assets exist. The market structure (i.e. the menu of assets that

the agents are allowed to trade) consists of four real assets and trading in a riskless borrowing/lending opportunity in units of the numeraire. The real assets are shares of the tradable-producing firms and shares of the nontradable-producing firms. If a foreign (domestic) investor buys a share of the nontradables producing firms of the domestic (foreign) country, he is entitled to a stream of payments denominated in units of the domestic (foreign) non-tradable. When he receives this payout, he will exchange it for units of one of the numeraire tradable good that he can consume (or exchange for the non-numeraire tradable good at price $\{\phi_t\}$) in the domestic (foreign) goods market at the relevant equilibrium relative price. When no confusion may arise, shares of the firms will be indistinguishably called *assets* and their endowment streams, *dividends*. The equity of the two firms in each country is negotiated in a stock market located in that country.

In addition to borrowing and lending in units of the numeraire at rate $\{r_t\}$, we will occasionally discuss the shadow price of borrowing/lending in units of the foreign tradable, at rate $\{r_t^*\}$. The latter is, of course, locally risky when measured in terms of the numeraire⁷.

We will denote the prices of the stock of the four firms and the traded contingent claims by the 4-dimensional vector $P_t = (P_{\delta,t}, P_{\delta^*,t}, P_{n,t}, P_{n^*,t}) =: P_t^r$ where the price of the stock of each firm and the price vector of the zero net supply contingent claims are given in units of the numeraire. Define also the 1×4 dividend vector process in units of the numeraire as $\delta_t^a := y_t \odot p_t$ (\odot stands for entry-wise product). The gains processes associated with the real assets, measured in units of the numeraire, are defined as the price plus accumulated dividends. Since the information flow is Brownian, it follows from Huang (1985) that the price processes (in units of the numeraire) of the four assets will follow an Itô process. Therefore the gains process

⁷These interest rates, in units of tradables, correspond to the notion of national real interest rates. Note that it does not make sense to define riskless borrowing/lending opportunities in units of the nontradable goods. This is because such investment would always be inherently locally risky to one of the agents. He would be faced to relative price uncertainty since he cannot consume the good in question.

will satisfy a stochastic differential equation of the form:

$$dG_t = dP_t + \delta_t^a dt = I_{P_t} \left(\mu_t^G dt + \sigma_t^G dW_t \right) \quad (1.3)$$

for some processes μ_t^G and σ_t^G , where μ_t^c and σ_t^c are the 2×1 -dimensional vector valued process and 2×4 -dimensional matrix valued process in μ_t^G and σ_t^G standing for the drift vector and diffusion matrix of the gains processes associated with the zero net supply contingent claims. Of course, the processes μ_t^G and σ_t^G , are to be determined in equilibrium. We assume that $\int_0^T \left(\|\mu_t^G\|^2 + \|\sigma_t^G\|^2 \right) dt < \infty$ a.s. where σ_t^G stands for each of the row vectors of σ_t^G and $\|\cdot\|^2$ is the usual \mathcal{R}^d euclidean norm. Finally I_x stands for a 4×4 diagonal matrix with the coordinates of x in the diagonal.

1.2.3 The agents

Each agent is characterized by preferences defined over the set of consumption rates of the tradable goods and the home nontradable good. For notational convenience, we will expand the dimension of these vector processes to account for consumption of the nonnational nontraded good (set equal to zero). Thus we let the vector process $C := (c_\delta, c_{\delta^*}, c_n, 0)$ be a consumption stream for the domestic agent and $C^* := (c_\delta^*, c_{\delta^*}^*, 0, c_n^*)$ a consumption stream for the foreign agent. C and C^* belong to the consumption space $\mathcal{C} := \prod_{k=1}^4 \mathcal{H}_+(\Omega \times [0, T], \mathcal{O}, \mathcal{P} \times Lebesgue)$ where the space \mathcal{H}_+ consists of the positive adapted processes $\{x_t\}$ satisfying $\int_0^T x_t dt < \infty$ $\mathcal{P} \times Lebesgue - a.s.$ Preferences over this consumption set are represented by VN-M time-additive utility functions $U, U^* : \mathcal{C} \rightarrow \mathcal{R}$ given by:

$$\begin{aligned} U &= \frac{1}{q} E \int_0^T e^{-\rho t} \left(c_{n,t}^p c_{\delta,t}^q + c_{n,t}^{\tilde{p}} c_{\delta^*,t}^q \right) dt \\ U^* &= \frac{1}{q} E \int_0^T e^{-\rho^* t} \left(c_{n^*,t}^{p^*} c_{\delta,t}^{*q} + c_{n^*,t}^{\tilde{p}^*} c_{\delta^*,t}^{*q} \right) dt \end{aligned} \quad (1.4)$$

for some constants $p, \tilde{p}, p^*, \tilde{p}^*, \rho, \rho^*, q \in [0, 1]$ such that $p + q \leq 1, \tilde{p} + q \leq 1, p^* + q \leq 1, \tilde{p}^* + q \leq 1$.

The utility functions (1.4) are separable with respect to traded goods and non-separable with respect to non-traded goods. They also exhibit complementarity be-

tween domestic and foreign consumption goods. Evidence that this kind of complementarity in the utility function is in accordance with the empirical evidence is presented in Hansen and Singleton (1983), Kravis and Lipsey (1987), and Stockman and Tesar (1990)⁸. In the utility function above, minus the elasticity of substitution between the consumption of domestic and foreign tradables is constant and equal to $1 - q$. We could easily accommodate shifts in the indifference map that would tilt preferences towards domestic or foreign tradables, even without the presence of nontradables, by multiplying one of the terms in the right hand side of (1.4) by a constant. This would preserve the constant elasticity of substitution while allowing different tastes regarding domestic and foreign tradables. The analysis in this paper would carry through by only substituting in the computations to follow $\beta \exp(\rho t)$ and $\gamma \exp(\rho^* t)$ for $\exp(\rho t)$ and $\exp(\rho^* t)$, respectively. We impose $\beta = 1$ since we want to focus on the heterogeneity in preferences created by the presence of nontraded goods. In the case where $p = \tilde{p}$, $p^* = \tilde{p}^*$ preferences are homogeneous (of degree $q + p$). We will call this case the *homogeneous case* and will often study its properties. In the homogeneous case, the above preferences exhibit constant relative risk aversion with degree of risk aversion is equal to $1 - (q + p)$ for the domestic agent and $1 - (q + p^*)$ for the foreign agent⁹, and the budget shares are $\frac{p}{p+q}$ for the expenditure on the nontrad-

⁸Complementarity arises in a CES utility specification (which has our utility functions as a special case) if the elasticity of intertemporal substitution for a composite good is larger than the elasticity of substitution between traded and nontraded goods. Estimates of the former range in 0.5–2 (Hansen and Singleton (1983) while estimates of the latter are estimated to be 0.44 [Stockman and Tesar, (1990)].

⁹For the measurement of risk aversion with many commodities, see Kihlstrom and Mirman (1974) and Stiglitz (1969). The measure offered by the first authors is valid only if preferences of the two agents represent the same ordinal preferences. The measure offered by the second author allows the comparison of preferences that do not represent the same ordinal preferences, at least for a certain kind of gambles (when income is uncertain and relative prices fixed). In this case, the Arrow-Pratt results for the one-dimensional case apply to the indirect utility function as a function of consumption expenditure at some point in time. The indirect utility function is defined as $V(I; \tilde{p}) = U(x(p, I))$ where $x(p, I) \in \arg \max (U(x) \text{ s.t. } px \leq I)$ where p is the vector of relative prices. In our case the indirect utility function is given by:

$$V(I; \phi, \eta) = \frac{p^p}{q^{p+1}} \left(1 + \frac{p}{q}\right)^{-(q+p)} \eta^{-p} \left(1 + \phi^{\frac{q}{q-1}}\right)^{1-q} I^{(q+p)}$$

for the domestic agent, and similarly for the foreign one. The Arrow-Pratt measure follows as $-\frac{V_{II}}{V_I} I = 1 - (q + p)$, and is independent of relative prices.

able good and $\frac{q}{p+q}$ for the expenditure on tradables (these properties will be useful later to calibrate the model to the data).

The initial wealth of each agent is given by the market value of the stock of the firms located in his country. Denote by X_0 (resp. X_0^*) the initial wealth of the domestic (resp. foreign) agent. Each agent can trade all available securities. We will denote by $\pi_t = (\pi_{t,\delta}, \pi_{t,\delta^*}, \pi_{t,n}, \pi_{t,n^*})$ and $\pi_t^* = (\pi_{t,\delta}^*, \pi_{t,\delta^*}^*, \pi_{t,n}^*, \pi_{t,n^*}^*)$ the dollar holdings (in units of the numeraire) of the domestic and foreign agents of shares of the domestic tradable-producing firms, foreign tradable-producing firms, domestic nontradable-producing firms and foreign-nontradable producing firms. We will assume that $\int_0^T \|\pi_t\|^2 dt < \infty$ *a.s.* where $\|\cdot\|$ stands for the usual euclidean norm in \mathcal{R}^4 .

1.3 Equilibrium

In this section, the consumption and portfolio choice problem that the agent of each country faces is formulated. After that, the equilibrium concept is defined, and it is shown that an equilibrium exists and is unique. We will show how, for the purpose of computation of the equilibrium, we can focus on the consumption choice problem alone while the portfolio choice is uniquely determined once the consumption choice problem is solved. This is possible because markets are dynamically complete¹⁰. However, it will be necessary to obtain a representation of the optimal portfolio policies which is different from the usual one [as in for example, Cox and Huang, (1989)]. This is done in Proposition 1 below. The need for this will become apparent to the reader as our strategy to solve for the equilibrium unfolds.

The strategy to solve the equilibrium consists of using our characterization of portfolio policies to show that equilibrium in the financial markets follows once we choose the state-price density in such a way that the good markets clear. This is done in Theorem 1 below. Moreover, once the equilibrium state-price density has been found, our characterization of the equilibrium portfolio policies will provide a

¹⁰this is a well known result [Cox and Huang (1989)]

particularly suitable computational procedure for the equilibrium portfolio holdings.¹¹

In the next section, we will proceed to compute the equilibrium in the special case where there are no nontradables in the economy, and then we will move to solve the equilibrium in the general case in Section 5. This will include the main methodological contribution of this paper, namely the use of Malliavin calculus techniques to analyze the stochastic properties of equilibrium asset prices and their impact on equilibrium holdings in an heterogeneous agents economy.¹²

1.3.1 The consumption-portfolio choice problems

By standard arguments (Harrison and Kreps (1979)), from the gains processes (1.3) we can construct the price of risk process $\theta_t = (\sigma_t^G)^{-1}(\mu_t^G - r_t - 1)$ and the process:

$$\xi_t = \exp \left(- \int_0^t r_s ds - \frac{1}{2} \int_0^t \|\theta_s\|^2 ds - \int_0^t \theta_s dW_s \right) \quad (1.5)$$

which we *conjecture* to be the (unique) state price density for contingent claims denominated in units of the numeraire, and $\xi \in \mathcal{L}^2(\Omega \times [0, T], \mathcal{P} \times \text{Lebesgue})$. We will deal with the validity of this conjecture later. Let's focus now, for expositional reasons, on the problem of the domestic agent (the problem of the foreign agent being symmetric).

For any $C \in \mathcal{C}_i$ and any portfolio process π_t such that $\int_0^T \|\pi_t\|^2 < \infty$ *a.s.*, define the wealth equation:

$$dX'_t = \left(X'_t - \pi'_t 1 \right) r_t dt - C_t p_t dt + \pi_t \mu_t^G dt + \pi_t \sigma_t^G dW_t \quad (1.6)$$

Denote by $\mathcal{G}(\{\mathcal{B}_t\})$ the set of 4-dimensional, $\{\mathcal{B}_t\}$ -progressively measurable pro-

¹¹This could not be done in the Cox-Huang risk-neutral representation. In this case, we would have an additional dimension in solving the equilibrium: We would have to distinguish between the equilibrium spot interest rates and the equilibrium change of measure (Radon-Nykodim derivative) as two different objects to solve for.

¹²Malliavin calculus has been used to characterize portfolio policies in a partial equilibrium setting in the framework of the Cox-Huang martingale approach by Ocone and Karatzas (1991). Detemple and Zapatero (1991) use Malliavin calculus to analyze the impact of habit formation on equilibrium risk premia in a Lucas-type no-trade economy.

cesses x_t that satisfy $\int_0^T \|x_t\|^2 dt < \infty$ a.s. A portfolio process is called *admissible* if it belongs to \mathcal{G} , and there exists a consumption process C such that the wealth process X' that, for an initial given wealth X_0 , solves the stochastic differential equation (1.6), satisfies $X'_t \geq 0 \forall 0 \leq t \leq T$ and $X'_0 \leq X_0$. Progressive measurability is an elementary information requirement. The second condition guarantees the existence of the stochastic integral in the wealth process (together with the nondegeneracy assumption on σ^G). The third condition is a solvency constraint, and the fourth condition is a budget constraint. Now, if such a process π exists for some consumption process C , one says that π *finances* C .

In these conditions, the problem of the agent consists of choosing a consumption process C and an admissible portfolio process π financing C such that the utility functional (1.4) is maximized, namely, for the domestic agent:

$$\begin{aligned} & \sup_{(C, \pi) \in \mathcal{C} \times \mathcal{G}} \frac{1}{q} E \int_0^T e^{-\rho t} \left(c_{n,t}^p c_{\delta,t}^q + \tilde{c}_{n,t}^p \tilde{c}_{\delta,t}^q \right) dt \quad (P) \\ & \pi \text{ finances } C \\ & \text{given } X_0 \end{aligned}$$

and the problem for the foreign agent is similar (call it problem P^*). Now, we will call a consumption process C (resp. C^*) *admissible* for the domestic (foreign) agent if the consumption processes C and C^* satisfy:

$$\begin{aligned} E \left(\int_0^T \xi_t C_t p_t dt \right) & \leq E \left(\int_0^T (\xi_t (\delta_t + \eta_t n_t)) dt \right) =: X_0 \\ E \left(\int_0^T \xi_t C_t^* p_t dt \right) & \leq E \left(\int_0^T \xi_t (\phi_t \delta_t^* + \eta_t^* n_t^*) dt \right) =: X_0^* \end{aligned} \quad (1.7)$$

respectively, where the state price density is given by (1.5). Denote the sets of admissible consumption processes by $\mathcal{C}(X_0)$ and $\mathcal{C}^*(X_0^*)$.

The following proposition proves that, in our context, a financing portfolio exists for every admissible consumption process, and it is unique. The proof is constructive and provides a computational procedure that we will use later. The spirit of the proposition is familiar from the single-good case (Cox and Huang, 1989). However, it

is expressed in terms different from the usual ones (i.e. not under the equivalent martingale measure). This is so because the object that we solve for in the equilibrium is the state price density. In these conditions, it is easier to construct the optimal holdings under the natural probability measure rather than the equivalent martingale measure, as in Cox and Huang (1989).

Proposition 1: Suppose that the covariance matrix of equilibrium returns σ_t^G has full rank. Then for any $C \in \mathcal{C}(X_0)$ and $C^* \in \mathcal{C}^*(X_0^*)$, there exists a unique admissible portfolio π_t financing C and a unique admissible portfolio π_t^* financing C^* . These portfolios coincide *a.s.* with the processes:

$$\begin{aligned}\pi_t &:= \left(\frac{\psi_t}{\xi_t} + X_t \theta_t \right) \left(\sigma_t^G \right)^{-1} \\ \pi_t^* &:= \left(\frac{\psi_t^*}{\xi_t^*} + X_t^* \theta_t \right) \left(\sigma_t^G \right)^{-1}\end{aligned}\tag{1.8}$$

where X_t and X_t^* stand for the wealth processes of the domestic and foreign agent, respectively, and where ψ_t and ψ_t^* are the adapted processes arising in the martingale representation of the processes

$$\begin{aligned}\zeta_t &:= E\left(\int_0^T \xi_s C_s p_s ds \mid \mathcal{B}_t\right) - E\left(\int_0^T \xi_s C_s p_s ds\right) \\ \zeta_t^* &:= E\left(\int_0^T \xi_s C_s^* p_s ds \mid \mathcal{B}_t\right) - E\left(\int_0^T \xi_s C_s^* p_s ds\right)\end{aligned}\tag{1.9}$$

as stochastic integrals against brownian motion.

Proposition 1 allows us transform the problem (P) (resp. P^*) of the domestic (foreign) agent, which is essentially dynamic, into two problems: (i) a static problem to identify the optimal consumption choice (ii) a martingale representation problem to identify the portfolio process that finances the optimal consumption choice obtained in (i). In these circumstances the agents in our model first solve the problem:

$$\sup_{C \in \mathcal{C}(X_0)} \frac{1}{q} E \int_0^T e^{-\rho t} \left(c_{n,t}^p c_{\delta,t}^q + c_{n,t}^{\bar{p}} c_{\delta,t}^q \right) dt\tag{1.10}$$

for the domestic agent and,

$$\sup_{C^* \in \mathcal{C}^*(X_0^*)} \frac{1}{q} E \int_0^T e^{-\rho^* t} \left(c_{n^*,t}^{*p^*} c_{\delta,t}^{*q} + c_{n^*,t}^{*\bar{p}^*} c_{\delta^*,t}^{*q} \right) dt \quad (1.11)$$

for the foreign agent, and then proceed to identify uniquely the portfolio processes that finance the solutions to (1.10) and (1.11). Let's write the set of solutions to (1.10) as $M(\delta, n, \phi, \eta, \xi, X_0) \subset \mathcal{C}(X_0)$ and the solutions to (1.11) as $M^*(\delta^*, n^*, \phi, \eta^*, \xi, X_0^*) \subset \mathcal{C}^*(X_0^*)$. We will show later that $M(\delta, n, \phi, \eta, \xi) \neq \emptyset$ and $M^*(\delta^*, n^*, \phi, \eta^*, \xi) \neq \emptyset$. The following definition will be useful later when we discuss portfolio policies:

Definition: The process $\pi_t \sigma_t^G$ and $\pi_t^* \sigma_t^{*G}$ obtained from (1.8) in Proposition 1 will be called the *portfolio kernel* processes.

1.3.2 Equilibrium definition, existence and uniqueness

The equilibrium concept used here is an adaptation of the Security Markets Equilibrium concept of Duffie and Huang (1985). In the above situation, our economy is characterized by the primitives $\mathcal{E} = (\mathcal{S}, \{U, U^*\}, Y_t)$. An equilibrium is defined as an array of stochastic processes $\mathcal{H} = (\{C_t\}, \{C_t^*\}, \{\pi_t, \pi_t^*\}, P_t, p_t, \xi_t)$ such that $C \in M(\delta, n, \phi, \eta, \xi, X_0)$ and $C^* \in M^*(\delta^*, n^*, \phi, \eta^*, \xi, X_0^*)$ and markets clear, i.e.:

$$\begin{aligned} c_{\delta,t} + c_{\delta^*,t}^* &= \delta_t \\ c_{\delta^*,t} + c_{\delta,t}^* &= \delta_t^* \\ c_{n,t} &= n_t \\ c_{n^*,t}^* &= n_t^* \\ X_0 &= P_{\delta,0} + P_{n,0} \\ X_0^* &= P_{\delta^*,0} + P_{n^*,0} \\ \pi_t + \pi_t^* &= (P_{\delta,t}, P_{\delta^*,t}, P_{n,t}, P_{n^*,t}) \end{aligned} \quad (1.12)$$

$\forall(\omega, t) \in \Omega \times [0, T]$ $\mathcal{P} \times$ Lebesgue – a.s. Note that, by Walras law, the market for instantaneous riskless borrowing/lending is also in equilibrium if (1.12) holds.

With the intention of simplifying the proof of existence and uniqueness, in the

next theorem, we prove that if the processes $(\xi_t, \phi_t, \eta_t, \eta_t^*)$ are such that the consumption policies arising from the problems (1.10) and (1.11) clear the goods markets, then all financial markets are in equilibrium. This result makes use of the fact that equilibrium portfolio policies are constructed as prescribed in (1.8) of Proposition 1. This theorem is useful since it allows us to concentrate on the clearing of the goods-market exclusively (while clearing of the financial markets is guaranteed), and deal with existence problems there when we actually want to compute the equilibrium \mathcal{H} . We first will need the following lemma:

Lemma 1: Suppose that ξ_t is an equilibrium state price density and p_t is an equilibrium relative price process. Then, in equilibrium, the price processes of the real assets satisfy:

$$P_t^r = \frac{1}{\xi_t} E \left(\int_t^T \xi_s \delta_s^a ds \mid \mathcal{B}_t \right)$$

And now we are in conditions to show how we can reduce the dimensionality of the equilibrium problem:

Theorem 1: Suppose that:

(1) There exist a set of processes $(\hat{\xi}_t, \hat{p}_t) \in \mathcal{L}^2(\Omega \times [0, T], \mathcal{P} \times \text{Lebesgue})$ such that $C \in M(\delta, n, \hat{\phi}, \hat{\eta}, \hat{\xi}, X_0^*)$, $C^* \in M(\delta^*, n^*, \hat{\phi}, \hat{\eta}^*, \hat{\xi}, X_0)$ and $c_{\delta,t} + c_{\delta^*,t} = \delta_t$, $c_{\delta^*,t} + c_{\delta,t} = \delta_t^*$, $c_{n,t} = n_t$ and $c_{n^*,t} = n_t^*$, and

(2) Agents finance their consumption allocations with the portfolio policies prescribed in Proposition 1.

Then the markets for the shares of the productive assets, the markets for the zero net supply contingent claims and the market for riskless borrowing/lending are in equilibrium, i.e. the last equation of (1.12) holds.

In view of Theorem 1, existence and uniqueness of the securities market equilibrium will be shown once we prove that there exist a set of processes $(\hat{\xi}_t, \hat{\phi}_t, \hat{\eta}_t, \hat{\eta}_t^*)$

such that the goods market clear. We will defer this proof until Proposition 2 in Section 5, where we actually solve for the equilibrium in the general case.

We have outlined the structure of our economy and we have reduced the dimension of the equilibrium existence and uniqueness problem. We are now ready to compute the equilibrium and solve for the variables of interest (consumption levels, asset prices, state price density, exchange rate, nontraded good prices, interest rates and equilibrium asset holdings processes). For computational purposes, the way to proceed is to compute the Arrow-Debreu equilibrium that supports the Securities Market equilibrium, and then complete the Securities Market equilibrium using Lemma 1, Proposition 1 and Theorem 1. The program that we need to carry out thus involves: First, compute the equilibrium consumption processes of the traded goods (first and second equations in (1.12)). This produces an equilibrium pricing kernel for contingent claims denominated in units of the numeraire traded good and the equilibrium exchange rate. Second, choose the relative price processes $\{\eta_t^i\}_{i \in I}$ such that each country has its internal nontraded goods markets in equilibrium, given the optimal consumption for the traded good computed above and the non-tradable endowment process. Third, compute the equilibrium price processes of the traded assets using Lemma 1 evaluated at the equilibrium pricing kernel. Fourth, use these price processes (in particular, their diffusion matrix), and Proposition 1, together with the consumption allocations, to identify the portfolio policies at the equilibrium path. This is done by first identifying the portfolio kernel process (Definition 1), the diffusion matrix of equilibrium prices, and then splitting the portfolio kernel into the equilibrium holdings process and the diffusion matrix of equilibrium prices.

In the next section we solve for the equilibrium in the special case where there are no nontradable goods. In that case, the preferences of the agents are identical across countries. We will call this economy the *benchmark economy*. We obtain closed form solutions for all variables of interest and we show how the predictions of the solution of the model which are at odds with the stylized facts. Then we move to an economy with nontradable goods that has the benchmark economy as a special case. In that case we will see how the predictions of the benchmark economy

are modified in the right direction according to the empirical evidence. We will also identify the economic mechanisms that modify the nature of the equilibrium when nontraded goods are present. We will insist particularly on the issues that arise as puzzles in the traditional international finance literature, such as the foreign excess return predictability puzzle and the home bias puzzle in equity portfolios.

1.4 The Benchmark Economy: No Nontradables

In this section we present the equilibrium for our exchange economy when there are no nontradables. As we develop this particular case, we will address the empirical relevance of the predictions obtained from this benchmark economy regarding the behavior of exchange rates, forward rates, interest rates, risk premia, consumption risk sharing and portfolio choice.

Since our general model stresses differences in preferences due solely to the presence of nontradables, here preferences are VN-M with identical felicity function across countries equal to

$$\begin{aligned} U &= \frac{1}{q} E \int_0^T e^{-\rho t} (c_{\delta,t}^q + c_{\delta^*,t}^q) dt \\ U^* &= \frac{1}{q} E \int_0^T e^{-\rho t} (c_{\delta,t}^q + c_{\delta^*,t}^q) dt \end{aligned} \tag{1.13}$$

The utilities (1.13) correspond to the general specification (1.4) once we set $p = p^* = \tilde{p} = \tilde{p}^* = 0$. We also impose equal discount rates, $\rho = \rho^*$. Let the first two row vectors of σ_t be the constant vectors σ_δ and σ_{δ^*} . In addition, let the first two components of the vector a_t in (1.2) be $\mu_\delta + \frac{\|\sigma_\delta\|^2}{2}$ and $\mu_{\delta^*} + \frac{\|\sigma_{\delta^*}\|^2}{2}$, respectively, where μ_δ and μ_{δ^*} are constants¹³. Agents can trade the shares of the two firms that produce the tradable goods, and riskless asset. We will refer to instantaneous borrowing or lending opportunities in units of the foreign (non-numeraire) consumption good at an equilibrium rate $\{r_t^*\}$, which we will interpret as the foreign interest rate¹⁴. The

¹³The dimension of these vectors was set to be 1×7 in the previous section. However, since there are neither nontradables nor relevant state variables in the benchmark economy we could redefine the seven-dimensional Wiener process into a two-dimensional Wiener process, without loss of generality. Of course, two linearly independent risky asset and a riskless asset complete the markets in the benchmark economy.

¹⁴This asset is not explicitly available for trade (since it would be redundant), but agents can

domestic and foreign agents hold the risky portfolios $\pi_t = [\pi_\delta, \pi_{\delta^*}]$ and $\pi_t^* = [\pi_\delta^*, \pi_{\delta^*}^*]$ respectively.

In what follows we describe the competitive equilibrium in the benchmark economy. Solving the optimization problems (1.10) applied to (1.13) and inverting the first order conditions, we obtain:

$$\begin{aligned} c_{\delta,t} &= (e^{\rho t} \xi_t \lambda)^{\frac{1}{q-1}} & c_{\delta^*,t} &= (e^{\rho t} \phi_t \xi_t \lambda)^{\frac{1}{q-1}} \\ c_{\delta^*,t}^* &= (e^{\rho t} \xi_t \lambda^*)^{\frac{1}{q-1}} & c_{\delta,t}^* &= (e^{\rho t} \phi_t \xi_t \lambda^*)^{\frac{1}{q-1}} \end{aligned} \quad (1.14)$$

where we let λ (resp. λ^*) be the lagrange multiplier associated with the budget constraint (1.7) of the domestic (resp. foreign) agent. Now, choose the equilibrium state price density ξ_t and exchange rate $\{\phi_t\}$ so that market clearing in both markets obtains when consumption demands are given in (1.14). Setting $c_{\delta,t} + c_{\delta^*,t}^* = \delta_t$ and $c_{\delta^*,t} + c_{\delta,t}^* = \delta_t^*$ we then have:

$$\xi_t = \delta_t^{q-1} e^{-\rho t} \left[\lambda^{\frac{1}{q-1}} + \lambda^{*\frac{1}{q-1}} \right]^{1-q} \quad \phi_t = \left(\frac{\delta_t}{\delta_t^*} \right)^{1-q} \quad (1.15)$$

while the consumption allocations are given by, substituting (1.15) in (1.14):

$$\begin{aligned} c_{\delta,t} &= c \delta_t & c_{\delta^*,t} &= (1-c) \delta_t \\ c_{\delta^*,t}^* &= c \delta_t^* & c_{\delta,t}^* &= (1-c) \delta_t^* \end{aligned} \quad (1.16)$$

where $c = \left(\frac{\lambda^{\frac{1}{q-1}}}{\lambda^{\frac{1}{q-1}} + \lambda^{*\frac{1}{q-1}}} \right)$. Using the budget constraints, one can solve for the lagrange multipliers, and one can easily show that they satisfy¹⁵ $\lambda^* = \lambda \times \left(\frac{\delta_0^{*q} f^*(0)}{\delta_0^q f(0)} \right)^{q-1}$ for some deterministic functions $f(t)$ and $f^*(t)$.¹⁶ From (1.16), we see that consumption growth

synthetise its payoff with the available assets.

¹⁵If one further normalizes $\xi_0 = 1$, both multipliers can be identified to be $\lambda = \delta_0^{q-1} \left[1 + \frac{\delta_0^{*q} f^*(0)}{\delta_0^q f(0)} \right]^{1-q}$ and $\lambda^* = \delta_0^{q-1} \left[1 + \frac{\delta_0^q f(0)}{\delta_0^{*q} f^*(0)} \right]^{1-q}$.

¹⁶The functions $f(t)$ and $f^*(t)$ are given by:

$$\begin{aligned} f(t) &= \frac{1}{\rho - q\mu_\delta + (q-q^2) \frac{\|\sigma_\delta\|^2}{2}} \left[1 - \exp \left(\left(q\mu_\delta + (q^2 - q) \frac{\|\sigma_\delta\|^2}{2} - \rho \right) (T-t) \right) \right] \\ f(t)^* &= \frac{1}{\rho - q\mu_{\delta^*} + (q-q^2) \frac{\|\sigma_{\delta^*}\|^2}{2}} \left[1 - \exp \left(\left(q\mu_{\delta^*} + (q^2 - q) \frac{\|\sigma_{\delta^*}\|^2}{2} - \rho \right) (T-t) \right) \right] \end{aligned} \quad (1.17)$$

rates are perfectly correlated internationally in our benchmark economy. However, the empirical evidence suggests that their correlation is far from unity. This consumption correlation puzzle is well documented in the international business cycles literature (see, for example Stockman and Tesar (1990, 1995), and Tesar (1993)). As already had been noted by these authors, nontraded goods will introduce a wedge between the correlation of consumption between countries and aggregate endowment.¹⁷

By standard arguments¹⁸, we can identify the equilibrium interest rate and price of risk per unit of volatility¹⁹ vector, for contingent claims denominated in units of the domestic good, from minus the predictable and local martingale parts of the Doob-Meyer decomposition of the equilibrium state price density. By applying Itô's lemma to the first equation in (1.15) we then see that the equilibrium interest rate and price of risk are actually constant and equal to:

$$\begin{aligned} r &= \rho + (1 - q)\mu_\delta - \frac{(1-q)(2-q)}{2} \|\sigma_\delta\|^2 \\ \theta &= (1 - q)\sigma_\delta \end{aligned} \tag{1.18}$$

where $\|\cdot\|$ is the euclidean norm in \mathcal{R}^2 . Applying Itô's lemma to the second equation in (1.15), we see that the equilibrium exchange rate is a lognormal process that solves:

$$\begin{aligned} \frac{d\phi_t}{\phi_t} &= \left[(1 - q)(\mu_\delta - \mu_{\delta^*}) - q(1 - q)\frac{\|\sigma_\delta\|^2}{2} + (1 - q)(2 - q)\frac{\|\sigma_{\delta^*}\|^2}{2} - (1 - q)^2\sigma_\delta\sigma_{\delta^*}' \right] dt + \\ &+ (1 - q)(\sigma_\delta - \sigma_{\delta^*})dW_t \end{aligned} \tag{1.19}$$

The exchange rate process (1.19) exhibits constant conditional variance. This is however at odds with extensive evidence provided by the ARCH literature [see Bollerslev et. al. (1992), Bekaert (1992), Ballie and Bollerslev (1989), Domowitz

¹⁷For a test of international consumption risk-sharing controlling for nontradables, see Lewis (1995).

¹⁸See, for instance, Duffie (1992, pp 98-99).

¹⁹In the context of this paper, by "price of risk" we mean a vector process θ_t such that if a traded asset exhibits expected gross returns μ_t^g and a diffusion vector σ_t^g , then it holds that $\mu_t^g - r_t = \theta_t\sigma_t^{g'}$. By "risk premium" (domestic or foreign) we will refer to the quantity $\mu_t^c - r_t$ where μ_t^c stands for the expected gross return of the firm that produces tradables (domestic or foreign). Occasionally, when no confusion may arise, we will use the two terms interchangeably.

and Hakkio (1985), Hodrick and Srivastava (1986), and Jorion (1995)]. Moreover, the evidence suggests that the conditional volatility of the exchange rate should be dependent on the level of interest rates [Ballie and Bollerslev (1990) and Giovannini and Jorion (1987)]. These are features that will be accounted for once nontradables are introduced in the benchmark economy.

We are now interested in computing the interest rate and stock market risk premium for contingent claims denominated in units of the foreign good. We associate them to the foreign interest rate and foreign stock market risk premium, respectively. To do that, we need to identify the pricing kernel for contingent claims denominated in units of the foreign good. The following lemma allows this identification. It relates the domestic and foreign pricing kernels in a way that will provide useful in relating interest rates, stock market risk premia and the exchange rate. It is completely general and will be indeed used in the general case later:

Lemma 2: The domestic and foreign pricing kernels (state price density in units of the foreign tradable good) are related in the following form:

$$\frac{\phi_t \xi_t}{\phi_s \xi_s} = \frac{\xi_t^*}{\xi_s^*} \quad (1.20)$$

where $T > t > s$.

Now we see that, applying Itô's lemma to the product of the two expressions in (1.15), by virtue of Lemma 2:

$$\begin{aligned} r^* &= \rho + (1 - q)\mu_{\delta^*} - (1 - q)(2 - q)\frac{\|\sigma_{\delta^*}\|^2}{2} \\ \theta^* &= (1 - q)\sigma_{\delta^*} \end{aligned} \quad (1.21)$$

Even though there are no forward contracts in our market structure, we can certainly speak of the shadow price of a forward contract in equilibrium (its fair price if replicated with existing assets). Since interest rates are deterministic, the forward price for delivery at time $\tau > t$ is given by $F_t^\tau = e^{(r-r^*)(\tau-t)}\phi_t$. The instantaneous for-

exchange risk premium is defined as the deviation from instantaneous uncovered interest rate parity and is given by $rp_t := E_t(\frac{d\phi_t}{\phi_t}) + r^* - r$, where $E_t(\frac{d\phi_t}{\phi_t})$ is the term multiplying dt in (1.19). Empirical studies normally focus on two additional quantities: the forward-spot differential at maturity τ defined as $fsd_t^\tau := \ln F_t^\tau - \ln \phi_t$, (i.e. the difference in the continuously compounded domestic and foreign yields for default-free bonds with maturity equal to $\tau - t$), and the forward premium, defined as the deviation from the unbiasedness hypothesis, $fp_t^\tau := \ln F_t^\tau - E_t(\ln \phi_{t+\tau})$. All three quantities are constant in our benchmark economy and equal to:

$$\begin{aligned} rp_t &= (1 - q)^2 \left(\|\sigma_\delta\|^2 - \sigma_\delta \sigma_{\delta^*}' \right) \\ fp_t &= -rp_t(\tau - t) \\ fsd_t^\tau &= (r - r^*)(\tau - t) = (1 - q) \left((\mu_\delta - \mu_{\delta^*}) + (2 - q) \frac{(\|\sigma_{\delta^*}\|^2 - \|\sigma_\delta\|^2)}{2} \right) (\tau - t) \end{aligned} \tag{1.22}$$

If the endowment processes have the same mean and identical instantaneous variance, the instantaneous exchange risk premium, the forward premium and the forward-spot differential are zero at any maturity regardless of the level of risk aversion. If the level of risk aversion is zero, namely $1 - q = 0$, all three quantities are also zero. Of course we could obtain rp_t either by direct summation of its components from (1.19), (1.18) and (1.21) or by finding the equilibrium expected return of an strategy consisting of lending one unit in the foreign-produced good, financing it by borrowing in the domestic good, and liquidating the position one instant afterwards. The uncertainty of this strategy comes exclusively from the exchange rate uncertainty and this is priced at the equilibrium price of risk per unit of volatility (1.18). Thus, the expected return satisfies $E_t(\frac{d\phi_t}{\phi_t}) + r^* - r = \theta(1 - q)(\sigma_\delta - \sigma_{\delta^*})' = (1 - q)^2 \left(\|\sigma_\delta\|^2 - \sigma_\delta \sigma_{\delta^*}' \right)$ using (1.18) and the diffusion component of (1.19). The foreign exchange risk premium in this economy is thus a constant, which contradicts glaring evidence supporting time varying risk premia as reported, for instance, in Cumby and Obstfeld (1984), Fama (1984), Hodrick and Srivastava (1984), and Hodrick (1987). Of course, the failure to account for the Fama (1984) excess return predictability puzzle follows. In addition, the forward rate exhibits the same volatility as the exchange rate. Contrary to this, the stylized fact is rather that the volatility of the forward-spot differential should be

smaller than that of the spot exchange rate changes (see De Vries (1994)).

Asset prices can be found using Lemma 1 and straightforward integration using (1.15):

$$\begin{aligned} P_{\delta,t} &= E \left(\int_t^T \frac{\xi_s}{\xi_t} \delta_s ds | \mathcal{B}_t \right) = \delta_t f(t) \\ P_{\delta^*,t} &= E \left(\int_t^T \frac{\xi_s}{\xi_t} \phi_s \delta_s^* ds | \mathcal{B}_t \right) = \delta_t^{1-q} \delta_t^{*q} f^*(t) \end{aligned} \quad (1.23)$$

Applying Itô's lemma to (1.23), we see that ex-dividend equity returns exhibit the same volatility as the underlying endowment processes. Now, an application of Itô's lemma to (1.23) yields the diffusion matrix of the equilibrium gains process associated with the equilibrium asset prices:

$$\sigma^G = \begin{pmatrix} \sigma_\delta \\ (1-q)\sigma_\delta + q\sigma_{\delta^*} \end{pmatrix} \quad (1.24)$$

In view of (1.24), it is clear that international equity returns are homoskedastic and equity premia are not time varying. However, this contradicts evidence regarding time varying second moments in foreign equity returns as documented in Harvey (1991) and Ferson and Harvey (1991). In addition, evidence in favor of time varying equity risk premia is presented in Harvey (1991) and Dumas and Solnik (1995). We will see that the presence of nontradables will change the pricing of the firms producing tradables, and the presence of hedging demands will make the first and second moments of equity returns time-varying.

Denote X_t and X_t^* the wealth processes of the domestic and foreign agent respectively that solve (1.6). Using (1.16) and (1.15) we see from direct integration that:

$$\begin{aligned} X_t &= E \left(\int_t^T \frac{\xi_s}{\xi_t} (c_{\delta,s} + \phi_s c_{\delta^*,s}) ds | \mathcal{B}_t \right) = c (P_{\delta,t} + P_{\delta^*,t}) \\ X_t^* &= E \left(\int_t^T \frac{\xi_s}{\xi_t} (c_{\delta^*,s}^* + \phi_s c_{\delta,s}^*) ds | \mathcal{B}_t \right) = (1-c) (P_{\delta,t} + P_{\delta^*,t}) \end{aligned} \quad (1.25)$$

Applying Proposition 1 above to identify the portfolio kernel processes for the domestic and foreign agents, we find, after applying the martingale representation

theorem²⁰ to (1.25)²¹:

$$\begin{aligned}\pi_t \sigma_t^G &= \left(c \delta_t f(t) + c \delta_t^{1-q} \delta_t^{*q} f^*(t) (1-q) \right) \sigma_\delta + q c \delta_t^{1-q} \delta_t^{*q} f^*(t) \sigma_\delta \\ \pi_t^* \sigma_t^G &= \left((1-c) \delta_t f(t) + (1-c) \delta_t^{1-q} \delta_t^{*q} f^*(t) (1-q) \right) \sigma_\delta + q (1-c) \delta_t^{1-q} \delta_t^{*q} f^*(t) \sigma_\delta\end{aligned}\tag{1.26}$$

Now, usnig σ_t^G as given in (1.24), we can identify the equilibrium dollar holdings to be:

$$\begin{aligned}\pi_t &= \left(c \delta_t f(t), c \delta_t^{1-q} \delta_t^{*q} f^*(t) \right) = c (P_{\delta,t}, P_{\delta^*,t}) \\ \pi_t^* &= \left((1-c) \delta_t f(t), (1-c) \delta_t^{1-q} \delta_t^{*q} f^*(t) \right) = (1-c) (P_{\delta,t}, P_{\delta^*,t})\end{aligned}\tag{1.27}$$

From (1.25) and (1.27), we have

$$X_t - \pi_t - 1 = X_t^* - \pi_t^* - 1 = 0$$

and no agent will invest in the riskless asset. Note that we could have identified directly the equilibrium holdings (1.27) by noticing that the competitive allocation (1.16) determines linear pareto sharing rules with deterministic coefficients. In these conditions, the portfolio policies that finance the equilibrium allocation imply no trade after date zero, and each agent holds a proportion c -for the domestic agent- and $(1-c)$ -for the foreign agent- of the shares of each firm.

In this benchmark economy, there is no trade in financial markets after period zero, because the Pareto sharing rules are linear in aggregate endowment with constant coefficients. The proportions of the equity portfolio invested in home assets are:

$$\begin{aligned}PW_t &= \frac{c \delta_t f(t)}{c \delta_t f(t) + c \delta_t^{1-q} \delta_t^{*q} f^*(t)} = \frac{P_{\delta,t}}{P_{\delta,t} + P_{\delta^*,t}} \\ PW_t^* &= \frac{(1-c) \delta_t f(t)}{(1-c) \delta_t f(t) + (1-c) \delta_t^{1-q} \delta_t^{*q} f^*(t)} = \frac{P_{\delta^*,t}}{P_{\delta,t} + P_{\delta^*,t}}\end{aligned}\tag{1.28}$$

and every agent replicates the world market portfolio, by following buy-and-hold strategies. Obviously, we were expecting this because the agents are equal. Incidentally, note that there is an embarrassingly simple reason why home bias may arise:

²⁰See for example, Protter (1990), p.54.

²¹or also applying Proposition 5 below.

Imagine two countries that produce and consume two different goods, and each country consumes exclusively his own good. Then there is no trade and there is no room for risk-sharing: every country invests only in his stock. The problem is that one has to accept extreme and ad-hoc differences in the preference of domestic goods relative to foreign goods²². In this paper, we want to focus on the heterogeneity created by the presence of nontraded goods alone. Without nontraded goods, investors do not exhibit home bias in selecting equity portfolios. Rather, they invest in the world market portfolio.

In addition to the fact that the model without nontradables is at odds regarding the basic stylized facts from international financial data, the model without nontradable goods also shares some problems with closed-economy models of similar nature. In particular, the market price of risk is constant, and since the volatility of equilibrium prices is constant, the stock market premium is also constant. This contradicts evidence in favor of a countercyclical stock market premium (Fama and French (1989), Campbell and Cochrane (1994)). Also, stock returns and consumption growth are perfectly correlated. Evidence against this is presented in Cochrane and Hansen (1992). In addition, the model has the unappealing property that real interest rates

²²We could allow differences in tastes towards domestic and foreign goods by tilting the indifference map as $u(c_\delta, c_{\delta^*}) = \frac{1}{q} E \int_0^T e^{-\rho s} \left(\bar{\beta} c_{\delta,s}^q + c_{\delta^*,s}^q \right) ds$ and $u^*(c_\delta^*, c_{\delta^*}^*) = \frac{1}{q} E \int_0^T e^{-\rho s} \left(c_{\delta,s}^{*q} + \bar{\gamma} c_{\delta^*,s}^{*q} \right) ds$ and $\bar{\beta}, \bar{\gamma}$ are constants larger than one. The elasticity of substitution continues to be constant and equal to $1 - q$. In this case the state price density remains the same and the exchange rate process just gets multiplied by the constant $\kappa = \left(\frac{\lambda \frac{1}{q-1} + \gamma \lambda^* \frac{1}{q^*-1}}{\beta \lambda \frac{1}{q-1} + \lambda^* \frac{1}{q^*-1}} \right)^{1-q}$ where $\gamma = \bar{\gamma}^{1-q}$ and $\beta = \bar{\beta}^{1-q}$. In addition, the parameters c (resp. $1 - c$) will not be the same for both goods for the domestic (foreign) agent and they will be $c = \left(\frac{\beta \lambda \frac{1}{q-1}}{\beta \lambda \frac{1}{q-1} + \lambda^* \frac{1}{q^*-1}} \right)$ for the domestic good and $\hat{c} = \left(\frac{\lambda \frac{1}{q-1}}{\lambda \frac{1}{q-1} + \gamma \lambda^* \frac{1}{q^*-1}} \right)$ for the foreign good. Modifications in the wealth process and portfolio holdings follow accordingly. Therefore, when preferences differ, the dynamics of the state price density, interest rate, stock market risk premium and exchange rate are not affected. However we obtain: $PW_t = \frac{P_{\delta,t}}{P_{\delta,t} + \frac{\hat{c}}{c} \phi_t P_{\delta^*,t}}$ and $PW_t^* = \frac{\phi_t P_{\delta^*,t}}{\frac{1-c}{1-\hat{c}} P_{\delta,t} + \phi_t P_{\delta^*,t}}$. If β and γ are larger than 1, investors will bias their portfolios towards domestic equities. However, it is not enough to calibrate observed home bias. For example, suppose two identical countries, except for the fact that the $\beta > 1$ multiplies the power of the consumption of the domestic good in the utility function. In this case, to justify a home bias level of the order of 0.95 (rough estimate for USA) over the world portfolio share of domestic assets of 0.5, we would need $\beta \approx 19$. However, it is easy to see that, for our utility function and in this symmetric case, the budget share of the domestic consumption good is $(1 + \beta \frac{1}{q-1})^{-1}$. For a expenditure share on imports around 10%, this would be consistent with $\beta \approx 4.66$.

are constant. However, not only expected bond returns are time-varying (Ilmanen (1995)), but also term premia may take different signs across maturities and across states of nature (Constantinides 1992) and we would like that the term structure of yield volatilities to be able exhibit different shapes (i.e. stochastic).

We have now a complete description of our benchmark economy. Closed form solutions have been obtained for the equilibrium consumption policies, portfolio holdings, asset prices, interest rates, exchange rate and forward rates. We see how the home bias and foreign exchange predictability puzzles arise, in addition to another set of predictions which are at odds with the data. In the next section, we will introduce nontradable goods, and this will make the preferences and consumption sets of the agents different. All results regarding equilibrium prices, interest rates, exchange rates, forward rates, risk premia, consumption process, wealth processes and portfolio holdings will be substantially affected. Moreover, I will show that the predictions of the model move in the right direction in each and every one of the above puzzles. However, we do not claim here that nontraded goods are the most appropriate mechanism to “match” each and every one of the stylized facts above, and certainly many authors have addressed these problems, specially the closed-economy puzzles (for instance, by changing the preference specification and/or endowment dynamics). Our main purpose is to provide an economic explanation of the Home Bias Puzzle and the Excess Return Predictability Puzzle. It is interesting, though, that the presence of nontraded goods in the international economy not only leads us towards an understanding of these two puzzles, but also of the other stylized facts reported above.

1.5 Solution of the Equilibrium in the General Case

In this section, I will solve for the equilibrium in the economy with nontraded goods. First, the equilibrium Arrow-Debreu state-price density is characterized. This allows the identification of the stochastic properties of the equilibrium exchange rate and

interest rates. After that, Malliavin calculus techniques are used to identify the diffusion matrix of equilibrium asset prices. We will then be in conditions to obtain the optimal portfolio policies at the equilibrium path (optimal holdings), again using Malliavin techniques to solve the martingale representation problem and using the diffusion matrix of equilibrium prices obtained previously.

1.5.1 Equilibrium state-prices and exchange rates

First order conditions of the problem (P) above lead to the necessary condition to be satisfied by the optimal consumption processes of the domestic agent:

$$\begin{aligned} c_{\delta,t} &= \left(\exp(\rho t) \xi_t c_{n,t}^{-p} \lambda \right)^{\frac{1}{q-1}} \\ c_{\delta^*,t} &= \left(\exp(\rho t) \phi_t \xi_t c_{n,t}^{-\bar{p}} \lambda \right)^{\frac{1}{q-1}} \\ \lambda \eta_t \xi_t &= \frac{\bar{p}}{q} \exp(\rho t) c_{n,t}^{p-1} c_{\delta,t}^q + \frac{\bar{p}}{q} \exp(\rho t) c_{n,t}^{\bar{p}-1} c_{\delta^*,t}^q \end{aligned} \quad (1.29)$$

where the lagrange multiplier λ solves $E \int_0^T \xi_s C_s p_s ds = E \int_0^T \xi_s (\delta_s + \eta_s n_s) ds$. Equivalently, for the foreign investor, the first order conditions to problem (P^*) yield the demand functions:

$$\begin{aligned} c_{\delta,t}^* &= \left(\exp(\rho^* t) \xi_t c_{n,t}^{*-p^*} \lambda^* \right)^{\frac{1}{q-1}} \\ c_{\delta^*,t}^* &= \left(\exp(\rho^* t) \phi_t \xi_t c_{n,t}^{*\bar{p}^*} \lambda^* \right)^{\frac{1}{q-1}} \\ \lambda^* \eta_t \xi_t &= \frac{\bar{p}^*}{q} \exp(\rho^* t) c_{n,t}^{*p^*-1} c_{\delta,t}^* + \frac{\bar{p}^*}{q} \exp(\rho^* t) c_{n,t}^{*\bar{p}^*-1} c_{\delta^*,t}^* \end{aligned} \quad (1.30)$$

where the lagrange multiplier λ^* solves $E \int_0^T \xi_s C_s^* p_s ds = E \int_0^T \xi_s (\delta_s + \eta_s^* n_s^*) ds$.

Market clearing in the consumption goods markets requires, (*a.s.*), $c_{\delta,t} + c_{\delta^*,t}^* = \delta_t$ and $c_{\delta^*,t} + c_{\delta^*,t}^* = \delta_t^*$, $c_{n,t} = n_t$ and $c_{n,t}^* = n_t^*$. Imposing this conditions in (1.29) and (1.30) we obtain:

$$\xi_t = \delta_t^{q-1} \left[\lambda^{\frac{1}{q-1}} e^{-\frac{\rho}{1-q} t} n_t^{\frac{p}{1-q}} + \lambda^{*\frac{1}{q-1}} e^{-\frac{\rho^*}{1-q} t} n_t^{*\frac{\bar{p}^*}{1-q}} \right]^{1-q} \quad (1.31)$$

$$\phi_t = \frac{1}{\xi_t} (\delta_t^*)^{q-1} \left[\lambda^{\frac{1}{q-1}} e^{-\frac{\rho}{1-q} t} n_t^{\frac{\bar{p}}{1-q}} + \lambda^{*\frac{1}{q-1}} e^{-\frac{\rho^*}{1-q} t} n_t^{*\frac{\bar{p}^*}{1-q}} \right]^{1-q} \quad (1.32)$$

and

$$\begin{aligned}\eta_t &= \frac{p}{q} e^{-\frac{p}{1-q}t} \lambda^{\frac{1}{q-1}} \xi_t^{-\frac{1}{1-q}} n_t^{\frac{p}{1-q}-1} + \frac{\bar{p}}{q} e^{-\frac{\bar{p}}{1-q}t} \lambda^{\frac{1}{q-1}} \phi_t^{\frac{q}{q-1}} \xi_t^{-\frac{1}{1-q}} n_t^{\frac{\bar{p}}{1-q}-1} \\ \eta_t^* &= \frac{p^*}{q} e^{-\frac{p^*}{1-q}t} \lambda^{*\frac{1}{q-1}} \xi_t^{-\frac{1}{1-q}} n_t^{*\frac{p^*}{1-q}-1} + \frac{\bar{p}^*}{q} e^{-\frac{\bar{p}^*}{1-q}t} \lambda^{*\frac{1}{q-1}} \phi_t^{\frac{q}{q-1}} \xi_t^{-\frac{1}{1-q}} n_t^{*\frac{\bar{p}^*}{1-q}-1}\end{aligned}\quad (1.33)$$

Imposing (1.29) and (1.30) into the budget constraints allows us to express the lagrange multipliers in the following form:

$$\lambda = \left(\frac{E\left(\int_0^T \xi_s (\delta_s + \eta_t n_t) dt\right)}{E\left(\int_0^T e^{-\frac{p}{1-q}t} n_t^{\frac{p}{1-q}} \xi_t^{\frac{q}{q-1}} dt\right) + E\left(\int_0^T e^{-\frac{\bar{p}}{1-q}t} n_t^{\frac{\bar{p}}{1-q}} \phi_t^{\frac{q}{q-1}} \xi_t^{\frac{q}{q-1}} dt\right) + E\left(\int_0^T \xi_t \eta_t c_{n,t} dt\right)} \right)^{q-1}\quad (1.34)$$

$$\lambda^* = \left(\frac{E\left(\int_0^T \xi_s (\delta_s^* \phi_s + \eta_t^* n_t^*) dt\right)}{E\left(\int_0^T e^{-\frac{p^*}{1-q}t} n_t^{*\frac{p^*}{1-q}} \xi_t^{\frac{q}{q-1}} dt\right) + E\left(\int_0^T e^{-\frac{\bar{p}^*}{1-q}t} n_t^{*\frac{\bar{p}^*}{1-q}} \phi_t^{\frac{q}{q-1}} \xi_t^{\frac{q}{q-1}} dt\right) + E\left(\int_0^T \xi_t \eta_t^* c_{n^*,t} dt\right)} \right)^{q-1}\quad (1.35)$$

These expressions highlight the nature of the equilibrium existence problem. In view of Theorem 2, we have reduced the equilibrium existence and uniqueness problem to that of finding two positive constants λ, λ^* and two stochastic processes $\{\xi_t\}, \{\phi_t\}$ that jointly solve (1.34), (1.35), (1.31), and (1.32). Existence and uniqueness of the internal relative price processes $\{\eta_t\}$ and $\{\eta_t^*\}$ follows then directly by identifying the marginal rates of substitution of the felicity function of the agents as in (1.33), at the equilibrium path.²³ The following proposition shows that an equilibrium exists and is unique up to nominal scaling.

Proposition 2: There exist two constants λ, λ^* and an two adapted process ξ_t and ϕ_t that solve (1.34), (1.35), (1.31), and (1.32). In addition, the process ξ_t given in (1.31) is indeed a state price density, in the sense of Harrison and Kreps (1979). Moreover, if $(\lambda, \lambda^*, \{\xi_t\}, \{\phi_t\})$ is a solution to (1.34), (1.35), (1.31), and (1.32), so is $(k\lambda, k\lambda^*, k^{-1}\{\xi_t\}, \{\phi_t\})$ for any positive constant k . Otherwise, the solution is unique.

²³Given that $p, \bar{p} < 1$, monotonicity and strict convexity of $\frac{\partial u(x,y,z)}{\partial z}$ guarantees that $\frac{\partial u(x,y,z)}{\partial z} = \frac{\partial u(x,y,z')}{\partial z} \Rightarrow z = z'$.

Note that (1.31) and (1.32) determine completely the structure of the dynamics of ξ_t since the RHS of (1.31) and (1.32) depend on ξ_t only through constants (i.e. λ, λ^*) and its own starting value, which we normalize to be $\xi_0 = 1$. Thus, once we know that such constants exist, we will proceed without being worried by the fact that, except in the benchmark economy, no closed form solution for such constants λ and λ^* exists as functionals of the primitive processes in our economy \mathcal{E} .

1.5.2 Equilibrium Interest Rates and Equity Premium

From (1.31), and (1.32), we see that ξ_t is an Itô diffusion process. Let's write its dynamics as

$$d\xi_t = \xi_t \mu_{\xi_t} dt + \xi_t \sigma_{\xi_t} dW_t \quad \xi_0 = 1 \quad (1.36)$$

for some processes ξ_t and σ_{ξ_t} . From (1.31) and by standard arguments, we can identify the domestic equilibrium risk premium and the domestic equilibrium interest rate from the negative of the drift and diffusion of the stochastic differential equation (1.36). Define $\hat{p} = \frac{p}{1-q}$, $\hat{p}^* = \frac{p^*}{1-q}$. Applying Itô's lemma to (1.31) and rearranging, we get:

$$\begin{aligned} r_t = -\frac{\mu_{\xi_t}}{\xi_t} = & \rho\alpha_t + \rho^*(1 - \alpha_t) + (1 - q)\mu_\delta - (1 - q)(2 - q)\frac{\|\sigma_\delta\|^2}{2} - \\ & - p\alpha_t \left[\mu_{n,t} - ((1 - \hat{p}) + \hat{p}q\alpha_t)\frac{\|\sigma_{n,t}\|^2}{2} \right] - \\ & - p^*(1 - \alpha_t) \left[\mu_{n^*,t} - ((1 - \hat{p}^*) + \hat{p}^*q(1 - \alpha_t))\frac{\|\sigma_{n^*,t}\|^2}{2} \right] + \\ & + (1 - q) \left[\sigma_\delta \left(p\alpha_t\sigma'_{n,t} + p^*(1 - \alpha_t)\sigma'_{n^*,t} \right) \right] + \frac{q}{1-q} pp^* \alpha_t (1 - \alpha_t) \sigma_{n^*,t} \sigma'_{n,t} \end{aligned} \quad (1.37)$$

where the processes $\mu_{n,t}$ and $\mu_{n^*,t}$ stand for the drift of the endowment of nontradables (in levels) given in (1.1) and where the process $\alpha_t \in [0, 1]$ is the proportion of the domestic-produced tradable consumed by the domestic agent, namely:

$$\alpha_t := \frac{1}{1 + e^{\frac{p-p^*}{1-q}t} \left(\frac{\lambda}{\lambda^*} n_t^{*p^*} n_t^{-p} \right)^{\frac{1}{1-q}}} = \frac{c_{\delta,t}}{c_{\delta,t} + c_{\delta^*,t}} \quad (1.38)$$

The expression (1.38) reflects how the fluctuations of the endowments of nontradables represent uninsurable risks, since the competitive allocation of each agent of the domestic tradable good depends on the realization of his endowment of nontradable. The “friction” that prevents consumption risks pooling is the fact that the consumption sets of the agents are defined over different goods²⁴. For the case where there are no nontradables (or equivalently $p = \tilde{p} = p^* = \tilde{p}^* = 0$) and individuals have the same rate of time preference, we recover our benchmark consumption sharing rules $\alpha_t = \lambda^{\frac{1}{q-1}} \left[\lambda^{\frac{1}{q-1}} + \lambda^{*\frac{1}{q-1}} \right]^{-1}$ and consumption risks are perfectly pooled. Of course, in this case we would also recover the interest rate of our benchmark economy, namely $r_t = \rho + (1 - q)\mu_\delta - (1 - q)(2 - q) \frac{\|\sigma_\delta\|^2}{2}$.

By standard arguments, the price of risk per unit of volatility process is the 7-dimensional process obtained from the local martingale part of the decomposition of ξ_t :

$$\theta_t := -\frac{\sigma_{\xi_t}}{\xi_t} = (1 - q)\sigma_\delta - p\alpha_t\sigma_{n,t} - p^*(1 - \alpha_t)\sigma_{n^*,t} \quad (1.39)$$

The expression (1.39) gives the expected excess return of any asset with associated gains process \hat{G}_t , with diffusion vector $\hat{\sigma}_t$, by taking the inner product $\hat{\sigma}_t\theta'_t$. Thus we see that a three-beta pricing equation arises in our economy. More of this will be discussed in the next section.

Consider now ξ_t^* as the foreign state price density, i.e., the pricing kernel that gives today the Arrow-Debreu prices per unit of probability, in units of the foreign tradable,

²⁴It is interesting to observe the dynamics followed by the consumption share process (1.38). Applying Itô's lemma to (1.38) we get:

$$d\alpha_t = -\alpha_t(1 - \alpha_t) \left[\frac{\rho - \rho^*}{1 - q} + \mu_{x,t} - (1 - \alpha_t) \|(p\sigma_n - p^*\sigma_{n^*})\|^2 \right] dt + \frac{\alpha_t(1 - \alpha_t)}{1 - q} (p\sigma_n - p^*\sigma_{n^*}) dW_t$$

where the process $\mu_{x,t}$ is the rate of growth of the process $\frac{n_t^* p^*}{n_t p}$. If the process n_t and n_t^* have the same law, preferences towards nontradables are identical ($p = p^*$) and the discount rates are the same, α_t is a constant. As expected, the rate of growth in the consumption share of each country is inversely related to its discount rate. However, if a country has a larger intensity of preference towards nontradables (a larger p or p^*), we cannot say that the rate of growth of its consumption share will be uniformly larger (path by path).

of contingent claims with payoff denominated in units of the foreign tradable. From Lemma 2 and (1.32) we can apply Itô's lemma to the equality:

$$\xi_t^* = c\delta_t^{*q-1} \left[\lambda^{\frac{1}{q-1}} e^{-\frac{\rho}{1-q}t} n_t^{-\frac{\bar{p}}{1-q}} + \lambda^{*\frac{1}{q-1}} e^{-\frac{\rho^*}{1-q}t} n_t^* \frac{\bar{p}^*}{1-q} \right]^{1-q} \quad (1.40)$$

where the constant $c = \frac{\xi_0^*}{\xi_0\phi_0}$ depends on the starting value of the primitive processes. By identifying the drift of (1.40), we see that the structure of the foreign interest rate and risk premia is the same as the domestic one, where the state variable process $\alpha_t^* \in [0, 1]$ will now be the proportion of the foreign tradable that the domestic agent consumes, namely:

$$\alpha_t^* := \frac{1}{1 + e^{\frac{\rho-\rho^*}{1-q}t} \left(\frac{\lambda}{\lambda^*} n_t^{*\bar{p}^*} n_t^{-\bar{p}} \right)^{\frac{1}{1-q}}} = \frac{c_{\delta^*,t}}{c_{\delta^*,t} + c_{\delta,t}} \quad (1.41)$$

and the same comments in the paragraph below (1.38) apply here. Note that only in the case where preferences are homogeneous and the endowment processes have the same drift and diffusion across countries, interests rates will be identical.

1.5.3 Exchange rate volatility and risk premium

The relationship between the domestic and foreign stock market risk premia has been obtained in Lemma 1. Let's write the process for the exchange rate in differential form $d\phi_t = \phi_t\mu_{\phi_t}dt + \phi_t\vartheta_t'dW$, where the process μ_{ϕ_t} and ϑ_t follow from applying Itô's lemma to (1.32). By Itô's lemma applied to (1.20) we have, *a.s.*:

$$\theta_t^* = \theta_t - \vartheta_t$$

The price of risk process abroad (i.e. for contingent claims denominated in units of the foreign tradable) is then:

$$\theta_t^* = -\frac{\sigma_{\xi_t^*}}{\xi_t^*} = (1-q)\sigma_{\delta^*} - \bar{p}\alpha_t^*\sigma_{n,t} - \bar{p}^*(1-\alpha_t^*)\sigma_{n^*,t} \quad (1.42)$$

and consequently the diffusion vector of the exchange rate process is:

$$\vartheta_t = \theta_t - \theta_t^* = (1 - q)(\sigma_\delta - \sigma_{\delta^*}) - (p\alpha_t - \bar{p}\alpha_t^*)\sigma_{n,t} - (p^*(1 - \alpha_t) - \bar{p}^*(1 - \alpha_t^*))\sigma_{n^*,t} \quad (1.43)$$

In expression (1.43), we see that the difference in the degree of complementarity of the domestic and foreign tradables with the nontradables (p and \bar{p} for the domestic country and p^* and \bar{p}^* for the foreign country) makes the exchange rate process differ from that of the benchmark economy. However, the volatility level of the exchange rate (as given by the norm of the vector (1.43)) will not be affected in any systematic way, except in some special cases²⁵. Also, in this case, the price of risk is identical in both stock markets.

We can now use the pricing kernel to find the deviation from uncovered interest rate parity in equilibrium. We do that by finding the expected excess return of a strategy that consists of borrowing the exact number of units of the numeraire necessary to purchase one unit of the foreign consumption good, lending the proceeds in the foreign instantaneous riskless market and liquidating the proceeds an instant afterwards. The uncertainty rewarded in this strategy is given by the covariation of the exchange rate and the price of risk processes. In the case with homogeneous preferences ($p = \bar{p}$ and $p^* = \bar{p}^*$), the instantaneous exchange rate risk premium is given by:

$$\begin{aligned} \mu_{\phi_t} + r_t^* - r_t &= (1 - q)^2(\sigma_\delta - \sigma_{\delta^*})\sigma'_\delta - p(1 - q)\alpha_t(\sigma_\delta\sigma'_{n,t} - \sigma_{\delta^*}\sigma'_{n,t}) \\ &\quad + p^*(1 - \alpha_t)(1 - q)(\sigma_{\delta^*}\sigma'_{n^*,t} - \sigma_\delta\sigma'_{n^*,t}) \end{aligned} \quad (1.44)$$

which should be compared with the benchmark case (1.22). To see this, consider a strategy consisting on borrowing one unit of the domestic tradable in the instanta-

²⁵for example, suppose that $\sigma_n \cong \sigma_{n^*}$, agents exhibit identical preferences, namely $p = p^*$ and $\bar{p} = \bar{p}^*$ with $p \neq \bar{p}$, and the covariance between the nontradables endowment growth and the tradables endowment growth is close to being equal in every country. Then the volatility (i.e. rate of change of the conditional variance) of the rate of change in the exchange rate is always larger. This is because positive (negative) shocks in the nontradables endowments will shift the demand towards (away from) the same tradable good in each country. This will amplify the fluctuations in the exchange rate because the nontradables endowments are positively correlated.

neous market, converting it into units of foreign tradable at the prevailing interest rate, lending this amount and liquidating the position one instant afterwards, by converting the proceeds into units of the domestic tradable and repaying the loan. Now it is easy to see that the net payout of this strategy is positively correlated with the exchange rate. The exchange rate depends positively on the endowment of domestic tradable and negatively on the endowment of foreign tradable, and does not depend on the endowment of nontradables in the case with homogeneous preferences.

Now suppose that the home endowment of nontradable is more correlated with the home endowment of tradable than with the foreign one. Then the domestic agent will be interested in taking a long position in the above strategy to hedge the impact of shocks of his endowment of nontradables on the marginal utility of tradables. The size of his hedging demand will depend on the intensity of his need for hedging (as measured by p) and his size (measured by his consumption share, α_t). This demand will drive the foreign premium down, and this explains the negative sign in front of the second term in (1.44). The foreign agent will be interested in the opposite position for symmetric reasons, under the equivalent assumption that his endowment of nontradable is more correlated with his endowment of tradable than with that of the domestic agent. He will want to enter that strategy on the short side, which will drive the premium up, depending on the intensity of his hedging needs (p^*) and his relative size ($1 - \alpha_t$). This explains the positive sign in front of the third term in (1.44). It is noteworthy that the relevant size of the country for the determination of the exchange rate premium is given by its consumption share, and not its wealth. This is explained by the nature of the hedging needs of the agents: The determination of the instantaneous exchange rate risk premium does not reflect the need to hedge movements in the investment opportunity set but rather the instantaneous consumption risk that the fluctuations in the endowment of nontradables induce in the marginal utility of the tradable goods. And since the consumption share is stochastic, the foreign exchange risk premium must be stochastic. Not however, that the exchange rate process is still lognormal in this homogeneous preference case. This means that the expected rate of change of the

exchange rate is a constant, and goes in the direction of explaining the results in Fama (1984a). We will deal with the Fama puzzle later.

1.6 On Equilibrium Price Processes

In this section, I characterize the stochastic processes followed by the equilibrium prices of the shares of the four firms and the two bonds. To do that, I use a purely mathematical argument to identify the diffusion process of equilibrium prices plus a purely economical result to identify the drift process of equilibrium price processes. The Clark representation formula of Malliavin calculus is used to identify the diffusion processes associated with the equilibrium asset price processes when written as functional integrals of the primitives in the model, as in Lemma 1. The identification of the drift processes follows then directly by computing its equilibrium expected excess return (by taking the inner product between the equilibrium price of risk per unit of volatility as given in (1.39) and the already found diffusion process of equilibrium prices), and then adding to it the equilibrium spot interest rate (1.37). However, we will not be directly interested in the drift processes of equilibrium asset prices because, for the portfolio choice problem, only the diffusion components matter (locally). We will first solve for the prices of the four real assets in the economy, and then the two long-term bonds. The 4 risky assets traded in our economy will have a diffusion matrix σ_t^G in their gains process. This random matrix is formed by stacking the row vectors given in the matrices (1.45), and (1.48) below.

The equilibrium asset prices of the domestic and foreign technologies are given in Lemma 2 as functionals of the equilibrium state-price density. For our portfolio holdings problem, we are mostly interested in identifying the diffusion vector processes of these equilibrium price processes (or equivalently the corresponding gains processes, since dividends are of locally bounded variation). The following Proposition allows such identification:

Proposition 3: The diffusion matrix of the equilibrium price processes for the

domestic and foreign stock of the tradable-producing firms is given by:

$$\sigma_{i,T}^G = \begin{bmatrix} \sigma_\delta + p(u_t - \alpha_t)\sigma_{n,t} + p^*(v_t - (1 - \alpha_t))\sigma_{n^*,t} \\ (1 - q)\sigma_\delta + q\sigma_{\delta^*} + (\tilde{p}u_t^* - p\alpha_t)\sigma_{n,t} + (\tilde{p}^*v_t^* - p^*(1 - \alpha_t))\sigma_{n^*,t} \end{bmatrix} \quad (1.45)$$

where:

$$u_t := \frac{E\left(\int_t^T \alpha_s \delta_s \xi_s \Lambda^n(t, s) ds | \mathcal{B}_t\right)}{E\left(\int_t^T \delta_s \xi_s ds | \mathcal{B}_t\right)} \quad v_t := \frac{E\left(\int_t^T (1 - \alpha_s) \delta_s \xi_s \Lambda^{n^*}(t, s) ds | \mathcal{B}_t\right)}{E\left(\int_t^T \delta_s \xi_s ds | \mathcal{B}_t\right)} \quad (1.46)$$

and

$$u_t^* := \frac{E\left(\int_t^T \alpha_s^* \phi_s \xi_s \delta_s^* \Lambda^n(t, s) ds | \mathcal{B}_t\right)}{E\left(\int_t^T \phi_s \xi_s \delta_s^* ds | \mathcal{B}_t\right)} \quad v_t^* := \frac{E\left(\int_t^T (1 - \alpha_s^*) \delta_s^* \phi_s \xi_s \Lambda^{n^*}(t, s) ds | \mathcal{B}_t\right)}{E\left(\int_t^T \phi_s \xi_s \delta_s^* ds | \mathcal{B}_t\right)} \quad (1.47)$$

where the processes u, v and u^*, v^* take values in $[0, 1]$ and the deterministic functions $\Lambda^n(t, s)$ and $\Lambda^{n^*}(t, s)$ are given in Lemma 3 in Appendix 1.

From Proposition 3 we see that, in equilibrium, an endogenous correlation²⁶ arises between assets prices and the endowment processes of the nontraded assets. Note that u, v (resp. u^*, v^*) can be interpreted as an average of the future realizations of α_t and α_t^* (resp. $(1 - \alpha_t)$). When all endowment processes are lognormal, u_t, u_t^*, v_t and v_t^* coincide with the proportion of respectively domestic and foreign firms producing tradables of respectively the domestic and foreign agent. Consider for a moment the homogeneous case with $p = \tilde{p}$ and $p^* = \tilde{p}^*$. In this case one can prove that $u_t > u_t^*$ and $v_t < v_t^*$ if the covariation between the domestic tradables and domestic nontradables endowments is sufficiently strong²⁷. This implies that when the world consumption share of the domestic (foreign) country α_t ($1 - \alpha_t$) is “unusually low” as compared with u_t (“unusually high” as compared with v_t) this extra-correlation piece will make the price of the domestic (foreign) tradable producing firm more (less) correlated with

²⁶i.e. not derived from the exogenous correlation between σ_δ and σ_n

²⁷if $\frac{\sigma_\delta \sigma_n'}{\sigma_\delta \sigma_n} > \frac{p^*}{p}$ and $\frac{\sigma_\delta \sigma_n'}{\sigma_\delta \sigma_n} > \frac{p}{p^*}$ (see Lemma 5 below).

the domestic (foreign) endowment of nontradable goods.

By using the same techniques of Proposition 3, we can identify the volatility of the price of the shares of the nontradable producing firms, as follows:

Proposition 4: The diffusion matrix of the stock prices of the firms producing nontradables in the domestic and foreign country is given by:

$$\sigma_{t,N}^G = \begin{bmatrix} (1 - q(1 - a_t)) \sigma_\delta + qb_t \sigma_{\delta^*} + \left(\frac{1}{1-q} c_t - p\alpha_t\right) \sigma_{n,t} - \left(\frac{q}{1-q} d_t + p^*(1 - \alpha_t)\right) \sigma_{n^*,t} \\ (1 - q(1 - a_t^*)) \sigma_\delta + qb_t^* \sigma_{\delta^*} + \left(\frac{1}{1-q} c_t^* - p^*(1 - \alpha_t)\right) \sigma_{n^*,t} - \left(\frac{q}{1-q} d_t^* + p\alpha_t\right) \sigma_{n,t} \end{bmatrix} \quad (1.48)$$

where the processes a_t and b_t , take values on $[0, 1]$ and the processes $c_t, d_t, a_t^*, b_t^*, c_t^*, d_t^*$ take values on $[0, 1]$ and are given in closed form in the proof in Appendix 1.

One can show easily that the terms multiplying σ_n and σ_{n^*} in (1.48) are always positive. Now suppose that the domestic (foreign) endowment of tradables are uncorrelated with the foreign (domestic) endowment of nontradables. Suppose also that the endowments of nontradables are positively correlated and have the same variance. Then (1.48) would imply that the equilibrium prices of the domestic (foreign) nontradable-producing firms are positively correlated with the domestic (foreign) endowment of nontradables and *negatively* correlated with the foreign (domestic) endowment of nontradables. To understand this last result, recall from the first order conditions (1.29) that $\lambda\eta\xi$ is equal to the marginal rate of substitution between the nontradable and the numeraire tradable good. Now, a positive shock in the path of the endowment of foreign tradables will shift the path of the equilibrium Arrow-Debreu state prices ξ upwards (because the foreign agent enjoys more the consumption of an additional unit of tradables because of complementarity). However, the equilibrium consumption allocation of tradable goods to the domestic agent will be smaller (from a central planner point of view, it is now optimal to reallocate the consumption of tradables since the marginal utility for tradables of the foreign agent is higher, path by path). This implies that the marginal rate of substitution between nontradables

and tradables of the domestic agent will be smaller. Thus, η must decrease more than ξ increases so that $\lambda\eta\xi$ is smaller, path by path (see (1.90) in Appendix 1). This implies that, holding constant the path for the endowment of domestic nontradables, the value (in terms of the numeraire) of the revenues generated by the domestic firm producing nontradables, $\xi_t\eta_t n_t$ is smaller, path by path. This explains the negative correlation obtained above.

It is worth noting that in the symmetric case where $p = p^*$, and where the drift and diffusion vectors of the domestic and foreign nontradables are the same, the terms multiplying $\sigma_n (= \sigma_{n^*})$ in (1.45) and (1.48) cancel out. This means, first, that the price processes of the tradable producing firms will be the same as in the benchmark economy. Second, the price processes of the firms producing nontradables will be correlated with their own output only through the difference of the correlation of their output with the endowment of tradables of the two countries. Thus, it may well be a situation where the endowment of nontradables is uncorrelated with the endowment of both domestic and foreign tradables and thus the prices of the firms producing the nontradable goods are uncorrelated with its output. This is because no differential demand for these assets is generated since they are equally attractive to both investors for hedging purposes, and therefore the internal relative prices of nontraded goods will adjust in such a way that the diffusion matrix of the price of the nontradable-producing firms makes the share price of the firm reflect this symmetry. This implies that their local stochastic behavior will not depend on the nontradables endowment through their local stochastic behavior directly, but only through the impact of their levels on the budget shares between tradables (i.e. the components of a_t and b_t).

1.7 Computation of optimal portfolio policies

The optimal portfolio process is the adapted process Φ_t obtained from the representation of the martingale ζ_t arising in Proposition 1 as a stochastic integral against brownian motion. We use Malliavin calculus as the technique to identify the pro-

cess arising in the martingale representation of the optimal consumption processes in Proposition 1²⁸. By the Clark-Ocone formula (Section 2 in Appendix 2),

$$\begin{aligned} \zeta_t &= E\left(\int_0^T \xi_s C_s p_s ds \mid \mathcal{B}_t\right) - E\left(\int_0^T \xi_s C_s p_s ds\right) = \\ &= \int_0^t E\left(\mathcal{D}_s\left(\int_0^T \xi_v (c_{\delta,v} + \phi_v c_{\delta^*,v} + \eta_v n_v) dv\right) \mid \mathcal{B}_s\right) dW_s \end{aligned} \quad (1.49)$$

where $C_s = \{c_{\delta,s}, c_{\delta^*,s}, n_s\}$ solves the static consumption choice problem (1.10), at equilibrium prices. By substituting in (1.98) for the optimal consumption choice, the equilibrium state-price density, the equilibrium exchange rate and the equilibrium internal relative price processes, as functionals of the primitives of our economy and computing the above Malliavin derivative we can obtain the optimal portfolio policies (at the equilibrium path). Once we obtain the process in the representation (1.98) above, we will use the diffusion matrix of equilibrium prices to identify the equilibrium portfolio holdings.

The following proposition provides the portfolio policies explicitly when the endowment processes follow the dynamics (1.1).

Proposition 5: If the endowments follow the process (1.1) the portfolio kernel

²⁸However, in the proof of Proposition 5, we will have to compute the Malliavin derivative of a class of random variables that does not belong to the domain of the Malliavin derivative operator as stated in the literature of the field. Lemma 4 in Appendix 1 provides this extension.

process is given by, for the domestic agent:

$$\begin{aligned}
\Phi_t := \pi_t \sigma_t^G = & \left[q E \left(\int_t^T \frac{\xi_s}{\xi_t} (c_{\delta,s} + \varphi_s \eta_s n_s) ds \mid \mathcal{B}_t \right) + \right. \\
& + (1-q) E \left(\int_t^T \frac{\xi_s}{\xi_t} (c_{\delta,s} + \phi_s c_{\delta^*,s} + \eta_s n_s) ds \mid \mathcal{B}_t \right) \sigma_{\delta} + \\
& + q E \left(\int_t^T \frac{\xi_s}{\xi_t} (\phi_s c_{\delta^*,s} + (1-\varphi_s) \eta_s n_s) ds \mid \mathcal{B}_t \right) \sigma_{\delta^*} + \\
& + \left[\frac{1}{1-q} E \left(\int_t^T \frac{\xi_s}{\xi_t} (p(1-q\alpha_s) (c_{\delta,s} + \varphi_s \eta_s n_s) + \right. \right. \\
& + \tilde{p}(1-q\alpha_s^*) (\phi_s c_{\delta^*,s} + (1-\varphi_s) \eta_s n_s) \Lambda^n(t,s) ds \mid \mathcal{B}_t + \\
& \left. \left. - p\alpha_t E \left(\int_t^T \frac{\xi_s}{\xi_t} (c_{\delta,s} + \phi_s c_{\delta^*,s} + \eta_s n_s) ds \mid \mathcal{B}_t \right) \right] \sigma_{n,t} - \right. \\
& - \left[\frac{q}{1-q} E \left(\int_t^T \frac{\xi_s}{\xi_t} (p^*(1-\alpha_s) (c_{\delta,s} + \varphi_s \eta_s n_s) + \right. \right. \\
& + \tilde{p}^*(1-\alpha_s^*) ((1-\varphi_s) \eta_s n_s + \phi_s c_{\delta^*,s}) \Lambda^{n^*}(t,s) ds \mid \mathcal{B}_t + \\
& \left. \left. + p^*(1-\alpha_t) E \left(\int_t^T \frac{\xi_s}{\xi_t} (c_{\delta,s} + \phi_s c_{\delta^*,s} + \eta_s n_s) ds \mid \mathcal{B}_t \right) \right] \sigma_{n^*,t} \right.
\end{aligned} \tag{1.50}$$

where $\varphi_t := \frac{pc_{\delta,t}}{pc_{\delta,t} + \tilde{p}\phi_t c_{\delta^*,t}} \in [0, 1]$.

Substituting α_t for $(1-\alpha_t)$, α_t^* for $(1-\alpha_t^*)$, p^* and \tilde{p}^* for p and \tilde{p} , φ_t^* for φ_t , n_t^* for n_t (consequently $\sigma_{n^*,t}$ for $\sigma_{n,t}$), and η_t^* for η_t in (1.50) we obtain the portfolio policies for the foreign investor. In the lognormal case with homogeneous preferences and no mean reversion ($b=0$) it is easy to see that the term multiplying $\sigma_{n,t}$ in (1.50) is bounded below by $p(1-\alpha_t)X_t > 0$ and the term multiplying $\sigma_{n^*,t}$ is bounded below by $p^*(1-\alpha_t)X_t > 0$ where X_t is the wealth of the domestic agent. For the foreign investor, the bounds are $p\alpha_t X_t^*$ and $p^*\alpha_t X_t^*$ for the terms multiplying σ_n and σ_{n^*} , respectively, for the wealth process X_t^* . Thus, relative to a benchmark economy without nontradables, we see that the optimal policy (1.50) will additionally charge the portfolio with assets that in equilibrium are positively correlated with the endowment process of domestic nontradables and will partially unload the portfolio from assets positively correlated with the foreign endowment of nontradables. In other words, another way to understand the finding of the optimal portfolio process is to realize that the portfolio $\pi_t = \Phi_t (\sigma_t^G)^{-1}$ is collinear with the solution of the problem:

$$\begin{aligned}
\max_{v \in \mathcal{R}^d} v \sigma_t^G \Phi_t' \quad s.t. \\
v \sigma_t^G \sigma_t^{G'} v' = k
\end{aligned} \tag{1.51}$$

The problem (1.51) consists of finding a portfolio v such that its instantaneous covariance with the pricing kernel process (1.50) is maximal for any given variance k of such portfolio. Since k is arbitrary, we can foresee from (1.50) that the proportion of domestic tradable producing firms in the portfolio of the domestic agent will be larger than in the portfolio of the foreign agent because the portfolio kernel is positively correlated with the domestic endowment of nontradables and negatively correlated with the foreign endowment of nontradables. If the home endowment of tradables is more correlated with the home endowment of nontradables for each country, home bias will follow.

Remark: The domestic and foreign agents will share the same fund (they invest in the world market portfolio) if and only if the vectors $\pi_t \sigma_t^G$ and $\pi_t^* \sigma_t^G$ are collinear $\forall (\omega, t) \in \Omega \times [0, T]$. In our benchmark economy, this was indeed the case since, in that case we had $\pi_t \sigma_t^G = \left(\frac{\lambda^*}{\lambda}\right)^{\frac{1}{1-q}} \pi_t^* \sigma_t^G$ and home bias didn't arise (see Section 4). In the case with nontraded goods, the portfolio kernels can never be collinear (the coefficients multiplying σ_n and σ_{n^*} in $\pi_t \sigma_t^G$ and $\pi_t^* \sigma_t^G$ have opposite sign, while the coefficients multiplying σ_δ and σ_{δ^*} in $\pi_t \sigma_t^G$ and $\pi_t^* \sigma_t^G$ have the same sign).

To obtain the optimal holdings of risky assets π_t in equilibrium, we have to split the diffusion matrix σ_t^G of equilibrium prices from the pricing kernel Φ_t given in (1.50). The following theorem uses Propositions 3,4 and 5 to solve this linear system.

Theorem 2: In equilibrium, the agents will not invest in any of the zero net supply assets. The portfolio holdings of real assets for the domestic agent π_t^r and the

foreign agent $\pi_t^{r,*}$ are given respectively by the vector processes:

$$\begin{aligned} \pi_t^r &= \begin{pmatrix} \pi_{\delta,t} \\ \pi_{\delta^*,t} \\ \pi_{n,t} \\ \pi_{n^*,t} \end{pmatrix} = \begin{pmatrix} E_t \left(\int_t^T \alpha_s \delta_s \frac{\xi_s}{\xi_t} ds \right) \\ E_t \left(\int_t^T \alpha_s^* \phi_s \delta_s^* \frac{\xi_s}{\xi_t} ds \right) \\ E_t \left(\int_t^T \eta_s n_s \frac{\xi_s}{\xi_t} ds \right) \\ 0 \end{pmatrix} \\ \pi_t^{r,*} &= \begin{pmatrix} \pi_{\delta,t}^* \\ \pi_{\delta^*,t}^* \\ \pi_{n,t}^* \\ \pi_{n^*,t}^* \end{pmatrix} = \begin{pmatrix} E_t \left(\int_t^T (1 - \alpha_s) \delta_s \frac{\xi_s}{\xi_t} ds \right) \\ E_t \left(\int_t^T (1 - \alpha_s^*) \phi_s \delta_s^* \frac{\xi_s}{\xi_t} ds \right) \\ 0 \\ E_t \left(\int_t^T \eta_s^* n_s^* \frac{\xi_s}{\xi_t} ds \right) \end{pmatrix} \end{aligned} \quad (1.52)$$

Adding the components of the vectors in (1.52), we have $\pi_t - 1 = X_t$ and $\pi_t^* - 1 = X_t^*$ and therefore there is no investment in the riskless asset in equilibrium. Note that no agent will invest in the nontradables-producing firm of the other country. As one can see in (1.52), home bias will arise in tradable producing firms since the share of consumption in world endowment of tradable goods (α_t, α_t^*) depends positively on the domestic endowment of nontradables and negatively on the foreign endowment of nontradables, if by assumption δ_t is more correlated with n_t than with n_t^* .

The conditions on the correlation structure of the endowment processes for home bias to follow depend on the relative intensity of the national preference for nontradables. If both countries have the same relative preference for nontradables ($p = p^*$), the condition amounts to the fact that the covariance between each country's nontradables endowment growth rate with his tradables endowment growth rate is larger than the one with the foreign tradables endowment growth rate. If one country likes nontradables more than the other, home bias may arise even if the domestic endowment growth rate of tradables is less correlated with the domestic endowment growth rate of nontradables than with the foreign one.

1.8 Discussion

We will divide this section into three subsections. First, a general discussion of the results of the model is presented. We aim at comparing the economies with and without nontraded goods relative to the pricing variables, namely interest rates, exchange rate, market risk premia and the factor pricing equation that arises in the model. Second, we will invest two sections in the specific discussion of how the home bias puzzle and the Fama excess return predictability puzzle are resolved in our economy.

1.8.1 General Discussion

The Exchange Rate: To compare our results so far with those in the benchmark economy, note that when preferences are homogeneous (of degree $p+q$), the exchange rate (1.32) collapses to that of the no-nontradables benchmark economy (1.15). This is so even though preferences towards nontradables may be different across countries (i.e. $p \neq p^*$). The reason is that, in this case, tradable goods can be aggregated into a single composite tradable good in the utility function of both agents. In other words, shocks in the endowment of nontradables do not shift the marginal rate of substitution between tradable goods, and therefore their relative price need not change.

These two arguments imply that the relative price of the tradables (the exchange rate) cannot depend on the endowment of nontradables. This helps us understand how nontradables matter in setting the exchange rate: they reflect the *differential* impact of a shock in nontradables in the marginal utility of the domestic tradable versus the foreign tradable for both agents.

In the absence of non-traded goods, the volatility of the exchange rate is proportional to the volatility of the domestic and foreign endowment processes of tradable goods, and it will be smaller the more correlated these endowment processes are. With nontraded goods, if all endowment processes are mutually uncorrelated, the volatility of the exchange rate is always larger in the presence of nontraded goods. Other things equal, a higher variance of the endowment processes of nontradables means

higher exchange rate volatility, and a higher correlation of domestic endowment of tradables with that of nontradables together with a higher domestic preference for nontradables and relatively higher preference for domestic tradables means a higher exchange rate volatility, while the same for the foreign country means less exchange rate volatility. Lastly, if each country prefers to consume his nontraded good together with consumption of his home-produced tradable, a higher covariance between the nontradable endowment of both countries will induce a lower level of exchange rate volatility. This is clear because demand shocks produced by nontradables shocks will tend to compensate for both the domestic and foreign tradables.

The Interest Rates: In the presence of nontradables, and unlike the benchmark economy, interest rates are stochastic. The presence of nontradables induces not only an intratemporal redistribution effect on tradables (reflected in the exchange rate), but also an intertemporal redistribution of resources. This is so because agents will anticipate higher or lower level of nontradables in the future, and consequently higher or lower future marginal utility of one unit of tradables. They will want to invest today to guarantee that additional unit of tradable tomorrow, when its marginal utility is higher than today (of course, intertemporal discounting plays against this). This will drive interest rates down. As one can see in (1.37), when the drift of the endowments of nontradables is higher, they are more predictable and consequently interest rates are lower because of this intertemporal marginal utility arbitrage mechanism just described. However, a larger instantaneous conditional variance of the endowment of nontradables will make investing in the riskless asset a poor hedge against the agent's consumption risk. Rather, the agents will prefer to borrow and invest the proceeds in a portfolio most correlated with variations in their marginal utility of tradables. This will drive interest rates up. A higher covariance between the domestic endowment of tradables and both endowments of nontradables will also push interest rates up. This is because if the agents anticipate that the aggregate endowment of tradables will be high when its marginal utility is high, they will feel less the need of investing today in the riskless asset to benefit from that state of higher marginal utility of tradables.

Rather, they will want to borrow to invest directly in shares of the tradable-producing firms.

If preferences are homogeneous (of degree $p+q$), although not necessarily identical, then the share of consumption of any country over total endowment is the same for each traded good (i.e. $\alpha_t^* = \alpha_t$). If in addition the drift and diffusion processes of all underlying endowments are constant, then domestic and foreign interest rates will be (locally) perfectly correlated-since they are nonlinear functions of the same state variable-. In the general case, interest rates will not be locally perfectly correlated. Consumption growth rates will not be perfectly correlated either.

The exchange rate volatility depends in a nonlinear fashion on the domestic and foreign interest rates. This is because both interest rates and the exchange rate depend nonlinearly on the state variables α_t^* and α_t . This is consistent with the results reported by Giovannini and Jorion (1987) and Ballie and Bollerslev (1990). In addition, these authors detect a component in the volatility of exchange rates which is orthogonal to the level of interest rates (see also Saa-Requejo (1993)). Our model yields this result, even in the case where all endowment processes are lognormal, because the parabola defined by r_t as a function α_t need not be one-to-one in the range $\alpha_t \in [0, 1]$. We can also see that we can also write the spot risk premium in the foreign exchange market in the general, nonhomogeneous case, as a nonlinear function of the domestic and foreign interest rates, in accordance with the evidence documented in, for instance, Hansen and Hodrick (1993), Hsieh (1984), and Hodrick and Srivastava (1984).

A Three-factor Pricing Equation: Equation (1.39) yields the price of risk per unit of volatility in equilibrium. This determines the expected excess return in equilibrium of every tradable asset (which we obtain by computing the inner product between θ_t and the diffusion vector of such asset). In this way, the model prescribes a *three beta* pricing equation. When there are no nontradables, this equation collapses to the CCAPM. appear in the price of risk. The two new terms or risk factors imply a lesser risk premium for assets positively correlated with the domestic or foreign en-

dowment. These additional factors are proportional to the intensity of preferences for nontradable goods and also to the size of the country (as measured by its consumption share of tradables). The size of the country depends, in a nonlinear way, on the state variables (endowments of nontradables) of the model. This provides a rationale for a *size effect* in equity returns: If the home country is big relative to the foreign one, and if the price of a domestic asset is correlated with the domestic endowment of tradables, its excess return will be lower than in an identical asset but with the opposite correlation structure. Note also that if the endowments of non-tradables are locally riskless, the equilibrium risk premium will be the same that would obtain in a benchmark, and the CCAPM will hold. However, in this case, the equilibrium interest rate would still be lower (as long as the endowment processes of non-tradables have positive expected growth rates). Note also that this result is independent of the dynamics that we have assumed for the tradables and non-tradables endowment processes.

The Market Risk Premium: Since shares of the equity of nontradable-producing firms are not traded in equilibrium, we can identify each national stock market risk premium with the return, in excess of the riskless rate, of the firms that produce tradables, in each respective country. We can compute the risk premium using the equilibrium diffusion matrix (1.45) and the equilibrium price of risk per unit of volatility (1.39). For convenience, we will focus on the domestic stock market. Denote by $\sigma_{t,T}^{G(1)}$ the first row vector of σ_t^G . To shorten the expressions, suppose that the endowment of nontradables is only correlated with the endowment of tradables of the same country. We have then that the domestic market risk premium rp_t is given by:

$$rp_t = \theta_t \sigma_{t,T}^{G(1)} = (1 - q)^2 \|\sigma_\delta\|^2 - p^2 (u_t - \alpha_t) \alpha_t \|\sigma_{n,t}\|^2 - p^{*2} (v_t - (1 - \alpha_t)) \times \\ \times (1 - \alpha_t) \|\sigma_{n^*,t}\|^2 + ((1 - q)pu_t - \alpha_t p(2 - q)) \sigma'_\delta \sigma_{n,t} \quad (1.53)$$

From (1.53) we see that a higher covariance between the home domestic tradable and nontradable endowment processes will always lower the domestic stock market risk

premium. The intuition is obvious: a higher hedging ability of the shares of the domestic tradable-producing firm yield more demand in order to hedge the fluctuations in the marginal utility of tradables due to shocks in nontradables, and therefore command a lesser premium.

An interesting question is whether the market risk premium (1.53) can explain the countercyclical behavior of risk premium as reported, for instance, in Campbell and Cochrane (1994). Note that $\frac{\partial^2 r_{p_t}}{\partial \alpha_t^2} > 0$. The risk premium is decreasing with α_t in accordance with the countercyclical behavior of the risk premium reported by Fama and French (1989)²⁹ if the solution to $\frac{\partial r_{p_t}}{\partial \alpha_t} = 0$, which is always positive, lies to the right of the interval $[0, 1]$. One can check that this is the case if the covariance of the domestic endowment processes of tradables and nontradables is sufficiently high.³⁰ The reason is related to the reason why home bias arises: when the consumption share in world endowment of tradables is low, the importance of nontraded goods shocks in the utility function of the agents is bigger. More home bias will arise and the nontraded goods sector will become more attractive for domestic residents. Therefore, the expected excess return for investments in the traded goods sectors rises.

1.8.2 On Home Bias in portfolio selection

In equilibrium, no agent will invest in the foreign firm that produces nontradables. Regarding the portfolio mix of firms that produce tradables, the agent in each country will charge his portfolio with assets most highly correlated with his endowment of nontradables and will unload assets which better serve as a hedge for the other agent, i.e., assets with a higher covariance with the endowment of the other agent. Agents

²⁹However, here we understand the business cycle a relative one (relative to the other country). Thus our RBC is not the usual RBC.

³⁰The condition is $cov(d\delta_t, dn_t) > \frac{v_t(p^*)^2 var(dn_t^*) + (2 - u_t)p^2 var(dn_t)}{p(2 - q)}$. As an example, consider the lognormal case with identical preferences across two countries of similar size. Assume that the standard deviations of the tradables and nontradables endowment process are of similar magnitude. Assume also $q = p = 0.5$ (in line with observed expenditure share on nontradables). In this case, the above condition indicates that a correlation coefficient between the domestic endowment of tradables and nontradables of 0.65 is sufficient.

will not purchase the same fund of risky assets because the portfolio kernels $\pi_t \sigma_t^G$ and $\pi_t^* \sigma_t^{G^*}$ are not collinear. Note also that home bias does not arise because the domestic agent consumes more of the domestic tradable than the foreign tradable. To see this, note that in the case with homogeneous preferences ($p = \tilde{p}$ and $\tilde{p} = \tilde{p}^*$) the consumption shares of the domestic agent for both tradables is the same ($\alpha_t = \alpha_t^*$, see (1.38) and (1.41)). However, given the conditions on preferences and the correlation structure, home bias still arises. Thus we see that home bias is a phenomenon driven by rational dynamic hedging mechanisms, and not arbitrary differences in preferences.

As before, in the symmetric lognormal case with $p = p^*$ and $\sigma_{nt} = \sigma_{n^*t}$, the terms multiplying σ_{nt} and σ_{n^*t} cancel out and no additional differential demand appears, in equilibrium, for hedging purposes (the two last terms on the RHS of (1.50) both for the domestic and the foreign agent cancel out). In this situation, the diffusion matrix of the equilibrium asset prices is identical to the one in the benchmark economy and although the optimal holdings and equilibrium prices will be different, home bias will not obtain and investors will trade shares of the world market portfolio. The reason is clear: For home bias to follow, it is necessary that investors view the same asset returns differently. In our model, they do so because the consumption sets are different since they are subject to shocks of idiosyncratic state variables. An asset may be worth more to an agent relative to the other because (1) the ability of the asset to hedge his idiosyncratic state variable is bigger than the ability to hedge the state variable of the foreign investor and/or (2) that investor cares more about hedging his idiosyncratic state variable than the foreign one, and (3) An asset serves to hedge movements in the investment opportunity set caused by the impact of (1) and (2) on equilibrium prices. In the special lognormal case with $p = p^*$ and $\sigma_{nt} = \sigma_{n^*t}$, agents cannot view any traded asset differently because the reasons (1) and (2) do not apply. Thus, even when nontradable goods exist and matter, and they indeed affect equilibrium asset prices, consumption allocations, risk premia and interest rates, they may not distort equilibrium asset holdings away from the no-home bias result.

Up to this point, it is difficult to provide necessary and sufficient conditions for

home bias to arise, given the closed form results for the optimal policies. A sufficient condition that one could think for home bias to arise is that the correlation between the endowment process within countries is larger than across countries. This condition is too strong, however. For example, imagine the case where one of the agents does not care about nontradables. In that case, it is irrelevant what the correlation structure is regarding to his endowment of nontradables. The condition ensures that the quadratic covariation of the equilibrium consumption share with the domestic endowment of tradables is higher than with the foreign endowment of tradables.

Result 1: If preferences are homogeneous ($p = \tilde{p}$ and $p^* = \tilde{p}^*$), then if $p\sigma_\delta\sigma'_{n,t} - p^*\sigma_\delta\sigma'_{n^*,t} > 0$ and $p^*\sigma_\delta\sigma'_{n^*,t} - p\sigma_\delta\sigma'_{n,t} > 0$, the quadratic covariation of the equilibrium consumption share with the domestic endowment of tradables is higher than with the foreign endowment of tradables.

Extensive simulations have shown that, under Condition 1, each agent will own a higher proportion of the tradable-producing firm located in his country than the tradable-producing firm located abroad.

In table 1.1, we computed the proportion of investment in the domestic equities of tradable-producing firms for two identical economies with the same preferences and the extreme assumptions that the endowment of nontradables and the endowment of tradables are perfectly correlated within each country, and uncorrelated internationally³¹.

As it is apparent from Table 1.1, the degree of home bias generated is very large and quite sensitive to the value parameters p and q . In the empirical section of this paper, the elasticity of substitution between tradables has been calibrated to 0.7

³¹The functional integrals involved in this computation were estimated by Monte Carlo using a second order weak scheme to approximate the integrals (see Kloeden et al (1994) pp.248-250). The size of the drift and volatility of the endowment processes was set to the estimates obtained by fitting our endowment dynamics to US data from the OCDE International Sectoral Databank, using an exact Maximum Likelihood procedure. The codes for the simulations are written in MATLAB and are available from the author upon request.

	q=0.01	q=0.11	q=0.21	q=0.31	q=0.41	q=0.51
p=0	0.5001	0.4999	0.5003	0.5003	0.5000	0.5001
p=0.1	5.5491	1.0117	0.8016	0.7347	0.7077	0.7007
p=0.2	10.5855	1.5213	1.1034	0.9680	0.9128	0.9007
p=0.3	15.6721	2.0337	1.4014	1.2056	1.1219	1.0988
p=0.4	20.7312	2.5258	1.7020	1.4364	1.3301	1.2978
p=0.5	25.7899	3.0471	2.0001	1.6723	1.5373	1.5035

Table 1.1: Percentage invested in domestic equities of tradable-producing firms

	cd=0	cd=0.25	cd=0.375	cd=0.5	cd=0.625	cd=0.75
	cf=1	cf=0.75	cf=0.625	cf=0.5	cf=0.375	cf=0.25
bsn=0	0.5008	0.5005	0.4982	0.4999	0.5012	0.4993
bsn=0.25	0.2614	0.3776	0.4341	0.4916	0.5493	0.6046
bsn=0.4	0.0253	0.2533	0.3663	0.4808	0.5948	0.7106
bsn=0.5	-0.2136	0.1295	0.3025	0.4721	0.6427	0.8155
bsn=0.57	-0.4540	0.0021	0.2344	0.4587	0.6900	0.9242
bsn=0.62	-0.6808	-0.1219	0.1666	0.4560	0.7379	1.0250
bsn=0.67	-0.9409	-0.2466	0.1054	0.4448	0.7820	1.1331

Table 1.2: Percentage invested in domestic equities of tradable-producing firms, varying cross-country correlation

which implies $q = 0.3$. By letting $p = 0.2$ (which is a very conservative estimate of the budget share of nontradables), we see that our simulation yields an investment in home equities of tradable goods of 97%. As expected, when the intensity of the preference for nontradable goods (p) increases, the degree of home bias increases dramatically. Also, when the elasticity of substitution between tradables increases, the level of home bias increases, keeping p fixed. The reason is that, when we increase the curvature of the function with which the tradables enter the utility function (and the Arrow-Pratt measure of risk aversion), we magnify the importance of shocks in the nontradables endowment on the marginal utility of tradables, relative to shocks in the endowment of tradables themselves. This makes the pure diversification of tradables motive less important relative to the marginal utility of tradables hedging mechanism described above.

In Table 1.2 we compare different intensities of preference towards nontradables (as measured by the budget share of nontradables³²) with different degree of corre-

³²These correspond to the following values for $p = 0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6$ for a fixed value of

lation between the home tradables and nontradables. The elements in any row of Table 1.2 are as follows: the first element in the left ($cf=1, cd=0$) represents the case where the *home* nontradables are perfectly correlated with the *foreign* tradables and uncorrelated with the *home* tradables. As we move to the right, the correlation of the home nontradables with the home tradables increases and its correlation with the foreign tradables decreases. When $cf=cd=0.5$, both correlations are the same, and in the elements most to the right, the *home* nontradables are more correlated with the *home* tradables and uncorrelated with the *foreign* tradables (this is the assumption under which Table 1.1 was constructed). In Table 1.2 we see how we move from inverse home bias to positive home bias (from left to right) as we change the correlation structure between endowment processes. In particular, when the correlation is equal within countries than between countries, the agents invest in the world portfolio ($cd=cf=0.5$ is an approximation, the actual number represents slightly less correlation with the domestic endowment than the foreign one). For a realistic budget share of nontradables (around 57%) we see that a correlation structure such that the correlation within countries is double that of between countries ($cd=0,615$ and $cf=0,375$) we generate a home bias in the equity portfolio of tradable-producing firms of 0.69. Since each agent owns all equity of nontradable producing firms, this means, as a rough average, that the agent would invest around 85% of his portfolio of real assets in domestic equity, while the share in the world portfolio of the domestic firms is only 50%. This is a sizable home bias, and is comparable to the magnitudes that we observe (see the Introduction section).

Additional simulations proved that there is no systematic effect on the level of the home bias of changing the volatility of the endowment processes (keeping the structure of correlations fixed) or by changing the degree of mean reversion in the nontradables endowment processes.

$q = 0.3$.

1.8.3 On the Fama foreign exchange excess return puzzle

In an influential paper, Fama (1984a) pointed out one puzzle that has attracted much attention over the last decade and has not yet been fully rationalized in a general equilibrium model. He defines the forward premium as $P_t^\tau = f_t^\tau - E_t(s_{t+1})$ where f_t^τ is the logarithm of the forward price F_t^τ of the foreign currency in units of the domestic currency for delivery at $\tau > t$, and s_t stands for the logarithm of the spot exchange rate at time t . He then reports that if we split the current forward-spot differential as $f_t^\tau - s_t = P_t^\tau + E_t(s_{t+1} - s_t)$, the unconditional variance of the forward premium component³³ P_t^τ turns out to be higher than the unconditional variance of the predictable part of the change in the logarithm of the exchange rate component $E_t(s_{t+1} - s_t)$. This is regarded as a puzzle since the ex-post variability of the exchange rate changes $s_{t+1} - s_t$ is much larger than that of the forward-spot differential³⁴ $f_t^\tau - s_t$. In his paper, Fama questions whether this result can be generated in a rational expectations equilibrium model.³⁵ As Lewis (1994) points out, the puzzle in the Fama (1984a) regression can be written as the condition that (assuming that the ergodic densities exist):

$$\text{cov}(E(\Delta s_{t+\tau} | \mathcal{B}_t), f_t^\tau - s_t) < \frac{1}{2} \text{var}(f_t^\tau - s_t) \quad (1.54)$$

In the Fama study, τ equals thirty days. Covered interest rate parity requires $F_t^\tau = \phi_t \frac{B^*(t, \tau)}{B(t, \tau)}$ where $B(t, \tau)$ is the price at time t of a zero coupon bond promising

³³Recall that by the conditional exchange rate risk premium we understand the expected return of a strategy that invests one unit of the foreign good in a foreign bond of a certain maturity and finances the position by issuing a bond, denominated in units of the domestic good, of the same maturity, and liquidates both positions at the time of maturity of the bonds. This strategy leaves the exchange rate risk unhedged, although it carries no interest rate risk.

³⁴This is unlike the US Treasury bill market. Fama (1984b) finds that the variation of the difference between the current one month yield and the current one month forward rate for the period beginning one month ahead splits roughly equally between the variation of the expected one month yield to hold one month from now and the premium component. Moreover, the variance of the difference between the current one month yield and the current one month forward rate for the period beginning one month ahead is comparable to the variance of the actual changes in the one month yield.

³⁵The original formulation of the forward-discount puzzle is in nominal terms. The fact that the puzzle persists in real terms is argued in McCulloch (1975) Cumby (1988) and Backus, Gregory and Telmer (1993).

one unit of domestic good at time τ and $B^*(t, \tau)$ is the price at time t , in units of the domestic consumption good, of a zero coupon foreign bond promising one unit of the foreign good at time τ . Of course, these assets exist in the sense that they can be replicated. Taking logarithms,

$$f_t^\tau - s_t = (y_{t,\tau}^d - y_{t,\tau}^f)(\tau - t) - \log \phi_t \quad (1.55)$$

Where $y_{t,\tau}^d$ ($y_{t,\tau}^f$) is the continuously compounded yield at time t of a zero coupon domestic (foreign) bond maturing at time τ . Consider now the case with homogeneous preferences. Then we have, from (1.31) and (1.32), that $\phi_t = \left(\frac{\delta_t}{\delta_t^*}\right)^{1-q}$ as in the benchmark economy (i.e. the exchange rate follows a lognormal process), which means that $E(s_{t+\tau} - s_t | \mathcal{B}_t)$ is a nonrandom affine function of τ , $\forall t < \tau < T$ which means that the right hand side of (1.54) is zero. Now we show that (1.55) is random. This is not surprising since it is easy to see from (1.44) that, in the homogeneous case, the interest rate differential is random (it is a function of α_t if $p\sigma_n \neq p^*\sigma_n^*$). To see formally that (1.55) is random, using the definition of a forward contract and the Clark-Ocone representation formula, one can write the diffusion coefficient of the right hand side of (1.55), in our homogeneous case, as:

$$diff(f_t^\tau - s_t) = (z_t^{*\tau} - z_t^\tau)(p\sigma_n - p^*\sigma_n^*) \quad (1.56)$$

where z_t^τ and $z_t^{*\tau}$ have been defined in the Section on bond prices above. Note that from (1.31) and (1.32), $z_t^{*\tau} \neq z_t^\tau$ unless δ_t is a version of δ_t^* ³⁶. In the benchmark economy, with $p = p^* = 0$, we recover, of course, the result that the forward premium $f_t^\tau - s_t$ is nonrandom. Now, if $p\sigma_n \neq p^*\sigma_n^*$ we see that (1.55) is stochastic, regardless of the maturity of the forward contract τ . Note that this is the same condition for the instantaneous interest rate differential to be stochastic. Consequently the right hand side of (1.54) is positive and the Fama result follows. Note that this explanation of the puzzle has been provided for the quite natural case where preferences are

³⁶This means that $P \times Lebesgue(\delta_t = \tilde{\delta}_t) = 1 \forall t \in [0, T]$.

homogeneous (of degree $p + q$) (this is the case assumed in the empirical section of this paper).

To see why the above result obtains, recall the above reasoning justifying that when preferences are homogeneous (of degree $p + q$) although not necessarily identical, the exchange rate does not depend on the nontradables endowment. However, interest rates certainly still depend on the realization of the nontradables. This is because the intertemporal substitution effects described above hold independently of the intratemporal effects of nontradables shocks in the redistribution of consumption between domestic and foreign tradables. In addition, the uncertainty in domestic and foreign interest rates does not cancel out in their differential. Thus, all the volatility of the forward-spot differential will come from the interest rate uncertainty and therefore, from the forward premium component of the forward-spot differential.

1.9 Conclusions

We have constructed a dynamic model of an international economy with two traded goods and two nontraded goods where home bias in equity portfolio arises in equilibrium and where the behavior of exchange rates, bond prices and both exchange rate and stock market risk premia reproduce a set of stylized facts.

In the model, the consumption sets of the agents are different because they consume nontraded goods. Agents have different preferences, too. Since nontraded goods and traded goods are nonseparable in the utility function, the marginal utility of an additional unit of tradable goods varies randomly with the realizations of his endowment of nontraded goods. An asset may be worth more to an agent relative to the other because (1) the ability of the asset to hedge the variations of his marginal utility is bigger than the ability to perform the same job for the other agent and/or (2) one agent cares more about hedging his idiosyncratic state variable than the foreign one and/or (3) the ability of an asset to hedge movements in the investment opportunity set induced by the impact of (1) and (2) on equilibrium prices. In either case, agents view the same asset returns differently, and thus have different portfolios. Agents will

hedge not only the consumption risk induced by nontradables which enter the utility function as state variables, but also the movements of the investment opportunity set induced by these state variables on equilibrium prices. To the extent that (1) traded and nontraded goods are complements in the utility function and (2) the correlation of the home endowment of tradables with the home endowment of nontradables, weighted by the domestic intensity of preference for nontradables, is larger than its correlation with the endowment of nontradables abroad, weighted by the foreign intensity of preference for nontradables, agents will optimally bias their portfolio towards domestic equities. In particular, each agent will own all equity of the firm that produces his nontradable good. In addition, he will own a larger proportion of the equity of the firm that produces the tradable good located in his country, compared to the weight of this firm in the world market portfolio. We have also shown that the market price of risk will be countercyclical, thus in accordance to the empirical evidence, if the endowments of tradables and nontradables are more correlated within countries than between countries. Regarding the determination of equilibrium returns, the model has produced a three-beta international pricing equation that rationalizes a size effect in equity returns. This size effect captures a differential discount on the premium of assets most correlated with the nontradables endowment of the bigger country. When there are no nontradables, this pricing equation collapses to the CCAPM.

The presence of nontraded goods has helped to rationalize the Fama (1984) exchange rate excess return predictability puzzle. The idea is that while the impact of nontradables on exchange rates cancels out if agents don't distinguish much between home and foreign tradable goods, the impact on the interest rate differential will persist. Assuming that preferences are homogeneous (of degree $p + q$), although not necessarily identical, the unconditional variance of the exchange rate risk premium will be much larger than that of the conditional rate of change of the exchange rate. This is so because nontradables cannot induce any substitution or income effect on tradable goods and therefore will not affect the exchange rate. However, the intertemporal substitution effect of nontradables will affect interest rates even when preferences are homogeneous and the interest rate differential will exhibit additional

volatility relative to a no-nontradables economy. This generates enough volatility in the foreign exchange premium to explain the excess return predictability puzzle. In addition and in accordance to the evidence, to the extent that there is mean reversion in the underlying endowment processes of nontradables, the forward prices will be less volatile than the spot exchange rates, as one can see easily by applying the Malliavin calculus techniques to the definition of forward prices. The reason is the impact of a shock in nontradables today on the path of future exchange rates can be made arbitrarily small by making the degree of mean reversion bigger, but the impact on today's exchange rate will persist. This will magnify the conditional volatility of the spot exchange rate relative to that of the forward exchange rate.

1.10 References

Adler, M. and B. Dumas (1983) "International Portfolio Choice and Corporation Finance: a Synthesis," *Journal of Finance* 38, 925-84

Backus, D. Gregory, A. and Telmer, C. (1993): "Accounting for Forward Rates in Markets for Foreign Currency" *Journal of Finance* 48, p.1887-908.

Ballie, R.T. and T. Bollerslev (1989) "Common Stochastic Trends in a System of Exchange Rates," *Journal of Finance* 44, 167-81

Ballie, R.T. and T. Bollerslev (1990) "A Multivariate Generalized ARCH Approach to Modelling Risk Premia in Forward Foreign Exchange Rate Markets," *Journal of International Money and Finance* 9, 309-24.

Baxter, M. Jermann, U. and King, R. (1995): "Nontraded Goods, Nontraded Factors and International Non-diversification" NBER WP.

Bekaert, G. (1992) "The Time Variation of Risk and Return in Foreign Exchange Markets: A General Equilibrium Perspective," Mimeo Stanford University

Billingsley, P. "*Probability and Measure*" Wiley, NY 1986.

Bollerslev, T., R. Chou, and K. Kroner (1992) "ARCH Modelling in Finance," *Journal of Econometrics* 52, 5-59.

Brainard, W.C. and J. Tobin (1992) "On the Internationalization of Portfolios,"

Oxford Economic Papers 44, 533-565, April.

Campbell, J.Y. and J.H. Cochrane (1994) "By Force of Habit: A Consumption Based Explanation of Aggregate Stock Market Behavior," Working Paper

Campbell, J.Y. and Y. Hamao (1992) "Predictable Stock Returns in the United States and Japan: A Study of Long-Term Capital Market Integration," *Journal of Finance* 47, 43-70

Cochrane, J.H. and L.P. Hansen (1992) "Asset Pricing Lessons for Macroeconomics," NBER Macroeconomics Annual

Connor, G. and R. Korajczyk (1991) "Risk and Return in an Equilibrium APT: Application of a New Test Methodology," *Journal of Financial Economics* 21, 255-89

Constantinides, G. (1992) "A Theory of the Nominal Structure of Interest Rates," *Review of Financial Studies* Vol 5 n.4, 531-52.

Cooper, I. and E. Kaplanis (1994) "Home Bias in Equity Portfolios, Inflation Hedging, and International Capital Market Equilibrium," *Review of Financial Studies* Vol 7 n.1, 45-60.

Cox, J. , J. Ingersoll and S. Ross (1985) "An Intertemporal General Equilibrium Model of Asset Prices," *Econometrica* 53, 363-84.

Cumby, R.E. (1988): "Is it risk? Explaining Deviations from Uncovered Interest Rate Parity" *Journal of Monetary Economics* 22, p.279-99.

Cumby, R. and M. Obstfeld (1984) "International Interest Rate and Price Level Linkages Under Flexible Exchange Rates: A Review of Recent Evidence," in J.F.O. Bilson and R.C. Marston (eds) *Exchange Rate Theory and Practice*, Chicago, IL, University of Chicago Press.

DeVries, C. (1994) "Stylized Facts of Nominal Exchange Rate Returns," in F. Van Der Ploeg (ed.) *The Handbook of International Economics*, Blackwell

Domowitz, I. and C. Hakkio (1985) "Conditional Variance and the Risk Premium in the Foreign Exchange Market," *Journal of International Economics* 19, 47-66.

Dugundji, J. and Granas, A. (1982): "*Fixed Point Theory*" PWN Polish Scientific Publishers, Warszawa

Duffie, D. (1992) *Dynamic Asset Pricing Theory*, Princeton University Press

Duffie, D. and Huang, C.F. (1985): "Implementing Arrow-Debreu Equilibria by Continuous Trading of Few Long-Lived Securities" *Econometrica* 53, 1337-56

Dumas, B. (1992): "Dynamic Equilibrium and the Real Exchange Rate in a Spatially Separated World" *Review of Financial Studies* 5,2 , 153-180.

Dumas, B. "Partial Equilibrium versus General Equilibrium Models of the International Capital Market" in F., Van der Ploeg, ed. *The Handbook of International Macroeconomics* Basil Blackwell Ltd.

Dumas, B. and B. Solnik (1995) "The World Price of Foreign Exchange Risk," *Journal of Finance* Vol 50 n.2, 445-79.

Fama, E.F. (1984a) "Forward and Spot Exchange Rates," *Journal of Monetary Economics* 14, 319-38

Fama, E.F. (1984b) "The Information in the Term Structure," *Journal of Financial Economics* 13

Fama, E.F. and K. French (1989) "Business Conditions and Expected Returns on Stocks and Bonds," *Journal of Financial Economics* 25, 23-49.

Ferson, W.E. and C.R. Harvey (1991) "The Variation of Economic Risk Premiums," *Journal of Political Economy* 99, 385-415.

French, K. and J. Poterba (1991) "Investor Diversification and International Equity Markets," *American Economic Review* 81, 222-6

Gibbons, M. and W.E. Ferson (1985) "Tests of Asset Pricing Models with Changing Expectations and an Unobservable Market Portfolio," *Journal of Financial Economics* 14, 217-36.

Giovannini, A. and P. Jorion (1987) "Interest Rates and Risk Premia in the Stock Market and in the Foreign Exchange Market," *Journal of International Money and Finance* 6, 107-23.

Hansen, L.P. and R.J. Hodrick (1983) "Risk Averse Speculation in the Forward Foreign Exchange Market: An Econometric Analysis of Linear Models," in Jacob A. Frenkel (ed.) *Exchange Rates and International Macroeconomics*, Chicago IL, University of Chicago Press.

Hansen, L.P. and R. Jagannathan (1991) "Implications of Security Market Data

for Models of Dynamic Economies,” *Journal of Political Economy* 99, 225-62.

Hansen, L.P. and Singleton, K.J. (1983): “Stochastic Consumption, Risk Aversion and the Temporal Behavior of Asset Returns” *Journal of Political Economy* 91, 249-265

Harrison, M. and Kreps, D. (1979): “Martingales and Arbitrage in Multiperiod Securities Markets,” *Journal of Economic Theory* 20, 381-408.

Harvey, C. (1991) “The World Price of Covariance Risk,” *Journal of Finance* 46, 111-58.

Hodrick, R.J. (1987) *The Empirical Evidence on the Efficiency of Forward and Futures Foreign Exchange Markets*, Chur. Harwood Academic.

Hodrick, R.J. and S. Srivastava (1984) “An Investigation of Risk and Return in Forward Foreign Exchange,” *Journal of International Money and Finance* 3, 5-29.

Hodrick, R.J. and S. Srivastava (1986) “The Covariation of Risk Premiums and Expected Future Spot Rates,” *Journal of International Money and Finance* 5, 5-21.

Hsieh, D. (1984) “Tests of Rational Expectations and No Risk Premium in Forward Exchange Markets,” *Journal of International Economics* 24, 129-45.

Huang, C.F. (1985): “Information Structures and Viable Price Systems” *Journal of Mathematical Economics* 14, 215-40.

Ilmanen, A. (1995) “Time-Varying Expected Returns in International Bond Markets,” *Journal of Finance* Vol.50 n.2, 481-506.

Jorion, P. (1995) “Predicting Volatility in the Foreign Exchange Market,” *Journal of Finance* Vol.50 n.2, 507-28.

Karatzas, I. and S. Shreve (1988) *Brownian Motion and Stochastic Calculus*, Springer-Verlag, New York.

Kihlstrom, R. and L. Mirman (1974) “Risk Aversion with Many Commodities,” *Journal of Economic Theory* 8, 361-388.

Kloeden, P.E. Platen, E. and Schurz, H. (1994): “*Numerical Solution of SDE through Computer Experiments*” Springer-Verlag.

Lewis, K. (1994) “Puzzles in International Financial Markets,” NBER Working Paper 4951

- Lewis, K. (1995) "What Can Explain the Apparent Lack of International Consumption Risk Sharing?" NBER Working Paper 5203
- Lucas, R.E. (1978) "Asset Prices in an Exchange Economy," *Econometrica* 46, 1429-46
- Merton, R.C. (1973): "An Intertemporal Capital Asset Pricing Model," *Econometrica* 41, 867-88.
- McCulloch, J.H. (1975): "Operational Aspects of the Siegel's Paradox" *Quarterly Journal of Economics* 89, p.170-72.
- Nielsen, L. and J. Saa-Requejo (1992) "Exchange Rates and Term Structure dynamics and the Pricing of Derivative Securities," INSEAD Working Paper
- Nualart, D. (1995) *The Malliavin Calculus and Related Topics*, Springer-Verlag, New York
- Obstfeld, M.(1994): "International Capital Mobility in the 90's" in P. Kennen ed. *Understanding*
- Ocone, D. and Karatzas, I. (1991): "A Generalized Clark Representation Formula, with Application to Optimal Portfolios," *Stochastics and Stochastic Reports*, 34, 187-220.
- Pesenti, P. and Van Wincoop, E. (1994): "Non-traded Goods and International Diversification" mimeo Inozencio Gaspari Institute, Florence.
- Protter, P. (1990) *Stochastic Integration and Differential Equations: A New Approach*, Springer-Verlag, Berlin.
- Saá-Requejo, Jesús (1994): "The Dynamics and the Term Structure of Risk Premia in Foreign Exchange Markets," mimeo, University of Chicago GSB.
- Solnik, B. (1974) "An Equilibrium Model of the International Capital Market," *Journal of Economic Theory* 8, 500-24.
- Stiglitz, J. (1969) "Behavior Towards Risk with Many Commodities," *Econometrica* 37, 660-67.
- Stockman A.C. and Dellas H. (1989) "International Portfolio Nondiversification and Exchange Rate Volatility," *Journal of International Economics* 26, 271-89
- Stockman, A.C. and Tesar, L. (1995) "Tastes and Technology in a Two-Country

Model of the Business Cycle: Explaining International Comovements," *American Economic Review* Vol.85 n.1, 168-85.

Stultz, R.M. (1981): "A Model of International Asset Pricing," *Journal of Financial Economics* 9, 383-406.

Stulz, R.M. (1987): "An Equilibrium Model of Exchange Rate Determination and Asset Pricing with Nontraded Goods and Imperfect Information," *Journal of Political Economy* 95, 1024-1041

Tesar, L.(1993): "International Risk-Sharing and Nontraded Goods," *Journal of International Economics* 35, 69-89.

Tesar, L. and I.M. Werner (1992) "Home Bias and the Globalization of Securities Markets," Working Paper, Graduate School of Business, Stanford University, July

Tesar, L. and I.M. Werner (1995) "Home Bias and High Turnover," *Journal of International Money and Finance* Vol 14 n.4, 467-92.

Uppal, R.(1993): "A General Equilibrium Model of International Portfolio Choice," *Journal of Finance* 48,2 529-553.

Van Wincoop, E.(1994): "Welfare Gains From International Diversification," *Journal of Monetary Economics*

Zapatero, F. (1995): "Equilibrium Asset Prices and Exchange Rates" *Journal of Economic Dynamics and Control* 19, 787-811.

1.11 Appendix 1: Proofs

The following condition on the equilibrium diffusion matrix σ_t^G will processes (1.1) will be assumed throughout the following proof. Later when σ_t^G is solved for, one can check *a fortiori* that this matrix verifies this condition:

Assumption: The vector valued functions $\sigma_{i,t} : \mathcal{R}^+ \mapsto \mathcal{R}^{1 \times d}$ where $\sigma_{i,t}$ stands for the row vectors of σ_t^G is bounded in $[0, T]$ and strongly nondegenerate, i.e. $\zeta' \sigma_{i,t} \sigma_{i,t}' \zeta \geq \epsilon \|\zeta\|^2 \forall \zeta \in \mathcal{R}^N, \forall t \in [0, T]$ for some $\epsilon > 0$.

Proof of Proposition 1:

Applying Itô's lemma to the process $\xi_t X_t$, we obtain:

$$\begin{aligned} d(\xi_t X_t) &= \xi_t \left[(X_t - \pi_t) r_t + \pi_t \mu_t^G - r_t X_t + \pi_t \sigma_t^G \theta_t' - c_{\delta,t} - \phi_t c_{\delta^*,t} - \eta_t c_{n,t} \right] dt + \\ &+ \xi_t \left[\pi_t \sigma_t^G - X_t \theta_t \right] dW_t = -\xi_t (c_{\delta,t} + \phi_t c_{\delta^*,t} + \eta_t c_{n,t}) dt + \xi_t \left[\pi_t \sigma_t^G - X_t \theta_t \right] dW_t \end{aligned} \quad (1.57)$$

since, by standard properties of the state price density, $\mu_t^G - r_t - 1 = \sigma_t^G \theta_t'$. We see then that the wealth process associated with the pair $(\{c_{\delta,t}, c_{\delta^*,t}, c_{n,t}\}, \pi_t)$ satisfies³⁷:

$$\xi_t X_t = X_0 - \int_0^t \xi_s (c_{\delta,s} + \phi_s c_{\delta^*,s} + \eta_s c_{n,s}) ds + \int_0^t \xi_s \left[\pi_s \sigma_s^G - X_s \theta_s \right] dW_s \quad (1.58)$$

Now, given a consumption process $\{c_{\delta,t}, c_{\delta^*,t}, c_{n,t}\} \in \mathcal{C}(X_0)$, let's introduce the $(\{\mathcal{B}_t\}, P)$ -

³⁷This obviously is not the solution to 1.57. The solution actually is as follows, although it does not matter for our purposes except for its mere existence:

$$\begin{aligned} \xi_t X_t &= X_0 + \int_0^t \xi_s \exp \left(-\frac{1}{2} \int_s^t \theta_v' \theta_v dv - \int_s^t \theta_v dW_v \right) \left(-C_s - \phi_s \tilde{C}_s - \eta_s C_s^n + \pi_s \sigma_s^G \theta_s' \right) ds + \\ &+ \int_0^t \exp \left(-\frac{1}{2} \int_s^t \theta_v' \theta_v dv - \int_s^t \theta_v dW_v \right) \xi_s \pi_s \sigma_s^G dW_s \end{aligned}$$

martingale:

$$\zeta_t := E\left(\int_0^T \xi_s(c_{\delta,s} + \phi_s c_{\delta^*,s} + \eta_s c_{n,s}) ds \mid \mathcal{B}_t\right) - E\left(\int_0^T \xi_s(c_{\delta,s} + \phi_s c_{\delta^*,s} + \eta_s c_{n,s}) ds\right) \quad (1.59)$$

for $0 \leq t \leq T$. Clearly, one can recognize ζ_t as the gains process, deflated by the state price density, associated with a trading strategy whose payoff is a cash stream equal to the (deflated) optimal consumption stream. Now, since ζ_t is adapted to the brownian filtration, it is continuous, hence it is locally square integrable. In these conditions, from the Martingale Representation Theorem (e.g. Protter, p.154) we can write ζ_t as a stochastic integral as $\zeta_t = \int_0^t \psi_s dW_s$ where ψ_t is a progressively measurable process on $\{\mathcal{B}_t\}$, such that $\int_0^T \|\psi_s\|^2 ds < \infty$. Defining the progressively measurable process (here we use the fact that σ_t^G has full rank):

$$\pi_t := \left(\frac{\psi_t}{\xi_t} + X_t \theta_t\right) (\sigma_t^G)^{-1} \quad (1.60)$$

we see that, for some constant κ ,

$$\int_0^T \|\pi_t\|^2 dt = \int_0^T \left\| \left(\frac{\psi_t}{\xi_t} + X_t \theta_t\right) (\sigma_t^G)^{-1} \right\|^2 dt \leq \frac{1}{\kappa} \int_0^T \left(\left\| \frac{1}{\xi_t} \right\|^2 \|\psi_t\|^2 + \|X_t\|^2 \|\theta_t\|^2 \right) dt < \infty$$

where for the first inequality we applied Minkowski's inequality and Hölder's inequality (twice) and the fact that the covariance matrix of the gains processes $\sigma_t^G \sigma_t^{G'}$ is strongly nondegenerate $\forall (\omega, t) \in \Omega \times [0, T]$ ³⁸. For the second inequality, we needed the assumption of θ_t (and therefore ξ_t) bounded, and that $\int_0^t \|X_t\|^2 < \infty$ since $E(\|X_t\|^2) \leq Qe^{Q \times t}$ for some constant Q due to the fact that the drift and diffusion coefficients of the stochastic differential equation (1.57) satisfy a Lipschitz and growth condition in X_t .

³⁸If A is strongly nondegenerate, then AA' has an inverse and $\left\| (AA')^{-1} \zeta \right\| \leq \frac{1}{\kappa} \|\zeta\|$ for some number $\kappa > 0$ and $\forall \zeta \in \mathcal{R}^N$ and $\forall (t, \omega) \in [0, T] \times \Omega$ (Karatzas and Shreve 1988, Problem 8.1)

We see then, from (1.60) and (1.59) that the wealth process (1.61) is given by:

$$\begin{aligned} \xi_t X_t &= X_0 - \int_0^t \xi_s (c_{\delta,s} + \phi_s c_{\delta^*,s} + \eta_s c_{n,s}) ds + \int_0^t \xi_s [\pi_s \sigma_s^G - X_s \theta_s] dW_s, = \\ &= X_0 - E\left(\int_0^T \xi_s (c_{\delta,s} + \phi_s c_{\delta^*,s} + \eta_s c_{n,s}) ds \mid \mathcal{B}_t\right) \end{aligned} \quad (1.61)$$

for $0 \leq t \leq T$: Note that (1.61) shows that the process defined as $\Lambda_t := \xi_t X_t + \int_0^t \xi_s (c_{\delta,s} + \phi_s c_{\delta^*,s} + \eta_s c_{n,s}) ds$ is a $(\{\mathcal{B}_t\}, P)$ -local martingale. Also from (1.61), it is clear that X_t has continuous sample paths. Now, since $c_{\delta,t}, c_{\delta^*,t}, c_{n,t}, \phi_t \geq 0$, Λ_t is a nonnegative local martingale from (1.61), by Fatôu's lemma it is also a supermartingale, and thus $E(\Lambda_T) \leq X_0$. This implies, from the budget constraint, that $E(\xi_T X_T) \leq X_0 - E\left(\int_0^T \xi_s (c_{\delta,s} + \phi_s c_{\delta^*,s} + \eta_s c_{n,s}) ds\right) = 0$. Since both ξ_T and X_T are positive, this implies $X_T = 0$. Therefore $E(\Lambda_T) = E\left(\int_0^T \xi_s (c_{\delta,s} + \phi_s c_{\delta^*,s} + \eta_s c_{n,s}) ds\right) = X_0 = \Lambda_0$ and thus the supermartingale Λ is indeed a martingale (it has constant expectation at time zero). Since Λ_t is a martingale and $X_T = 0$, we have $E(\Lambda_T \mid \mathcal{B}_t) = E\left(\int_0^T \xi_s (c_{\delta,s} + \phi_s c_{\delta^*,s} + \eta_s c_{n,s}) ds \mid \mathcal{B}_t\right) = \Lambda_t$ which implies, since $\xi > 0$ a.s.,

$$X_t = \frac{1}{\xi_t} E\left(\int_t^T \xi_s (c_{\delta,s} + \phi_s c_{\delta^*,s} + \eta_s c_{n,s}) ds \mid \mathcal{B}_t\right) \geq 0 \quad (1.62)$$

Thus π is admissible and finances $\{c_{\delta,t}, c_{\delta^*,t}, c_{n,t}\}$ and we have established existence.

To see that π it is the unique admissible process financing $\{c_{\delta,t}, c_{\delta^*,t}, c_{n,t}\}$ for a given initial wealth X_0 , suppose that there exists another admissible portfolio $\hat{\pi}_t$ that finances $\{c_{\delta,t}, c_{\delta^*,t}, c_{n,t}\}$ with wealth process $\xi_t \hat{X}_t = X_0 - \int_0^t \xi_s (c_{\delta,s} + \phi_s c_{\delta^*,s} + \eta_s c_{n,s}) ds + \int_0^t \xi_s [\hat{\pi}_s \sigma_s^G - \hat{X}_s \theta_s] dW_s$, subtracting 1.61 from both sides, of this equation, we can write:

$$M_t := \xi_t (\hat{X}_t - X_t) = \int_0^t \xi_s [(\hat{\pi}_s - \pi_s) \sigma_s^G + (X_s - \hat{X}_s) \theta_s] dW_s \quad (1.63)$$

which shows that M_t is a local martingale and $M_T = \xi_T (\hat{X}_T - X_T) = 0$. Since π and $\hat{\pi}$ are admissible process, $\xi_t \hat{X}_t + \int_0^t \xi_s (c_{\delta,s} + \phi_s c_{\delta^*,s} + \eta_s c_{n,s}) ds$ and $\xi_t X_t + \int_0^t \xi_s (c_{\delta,s} + \phi_s c_{\delta^*,s} + \eta_s c_{n,s}) ds$ are both P -martingales (the proof above that X_t is a martingale and that $X_T = 0$ does not depend on the particular construction of π , and therefore these properties hold for \hat{X}_t , as long as $\hat{\pi}$ is finances $\{c_{\delta,t}, c_{\delta^*,t}, c_{n,t}\}$). Then M_t is also

a martingale, and since $M_T = 0$, we have $M_t = 0$ a.s. Because any martingale which is a constant has zero quadratic variation, we have

$$[M]_t = \int_0^t \xi_s^2 \left\| (\hat{\pi}_s - \pi_s) \sigma_s^G + (X_s - \hat{X}_s) \theta_s \right\|^2 ds = 0$$

which implies that $\hat{\pi}_t = \pi_t$ and $\hat{X}_s = X_s$ or $(\hat{\pi}_s - \pi_s) = (\hat{X}_s - X_s) \theta_s (\sigma_s^G)^{-1}$ ($\mathcal{P} \times Lebesgue$) – a.s. on $\Omega \times [0, T]$. However, if this second option is the one that holds, then from (1.63) we see that $\hat{X}_t = X_t$ must hold $\forall t \in [0, T]$ and therefore $\hat{\pi}_t = \pi_t$ ($\mathcal{P} \times Lebesgue$) – a.s. on $\Omega \times [0, T]$. \square

Proof of Lemma 1:

Absence of arbitrage oportunites implies that the deflated gains process of the real assets $\hat{G}_t := \xi_t P_t^r + \int_0^t \xi_s \delta_s^a ds$ must be a martingale. Indeed, by Itô's lemma applied to the gains processes:

$$d\hat{G}_t = \xi_t \left(I_{P^r} \sigma_t^G - P_t^r \theta_t \right) dW_t = \left(I_{\hat{P}^r} \sigma_t^G - \hat{P}_t^r \theta_t \right) dW_t \quad \hat{G}_0 = G_0 \quad (1.64)$$

where we have defined $\hat{P}_t^r := \xi_t P_t^r$. Since at time T the assets produce no dividend any more and all processes involved in the analysis are continuous, we must have $P_T^r = 0$ a.s. and therefore $\hat{G}_T = \int_0^T \xi_s \delta_s^a ds$. We then have, using the fact that \hat{G}_t is a martingale,

$$\xi_t P_t^r = \hat{G}_t - \int_0^t \xi_s \delta_s^a ds = E(\hat{G}_T | \mathcal{B}_t) - \int_0^t \xi_s \delta_s^a ds = E\left(\int_t^T \xi_s \delta_s^a ds \mid \mathcal{B}_t\right)$$

\square

Proof of Theorem 1:

By assumption, the goods markets are in equilibrium. Write the portfolio (dollar) holdings of the domestic and foreign technologies as the $1 \times d$ -dimensional vectors π_s and π_s^* of dollar holdings at time s where $\pi_s^{d,T} := [\pi_{s,\delta}, \pi_{s,\delta^*}, \pi_{s,n}, \pi_{s,n^*}]$ and $\pi_s^{f,T} := [\pi_{s,\delta}^*, \pi_{s,\delta^*}^*, \pi_{s,n}^*, \pi_{s,n^*}^*]$, for the domestic and foreign agents, respectively. Denote also by σ_s the 4×4 diffusion matrix of the equilibrium price processes. To see that the

risky asset markets are in equilibrium, we have to check that:

$$\begin{aligned}\pi_{s,\delta} + \pi_{s,\delta}^* &= P_{\delta,t} & \pi_{s,n} + \pi_{s,n}^* &= P_{n,t} \\ \pi_{s,\bar{\delta}} + \pi_{s,\bar{\delta}}^* &= P_{\delta^*,t} & \pi_{s,n^*} + \pi_{s,n^*}^* &= P_{n^*,t}\end{aligned}\tag{1.65}$$

From (1.62), in the proof of Proposition 1 and the assumptions on goods market clearing, we have that:

$$\begin{aligned}X_t + X_t^* &= \frac{1}{\xi_t} E \left(\int_t^T \xi_s (c_{\delta,s} + \phi_s c_{\delta^*,s} + \eta_s c_{n,s}) ds \mid \mathcal{B}_t \right) + \\ &+ \frac{1}{\xi_t} E \left(\int_t^T \xi_s (c_{\delta^*,s}^* + \phi_s c_{\delta^*,s}^* + \eta_s^* c_{n^*,s}^*) ds \mid \mathcal{B}_t \right) = \\ &= \frac{1}{\xi_t} E \left(\int_t^T \xi_s \delta_s + \xi_s \phi_s \delta_s^* + \eta_s n_s + \eta_s^* n_s^* \mid \mathcal{B}_t \right) = P_{\delta,t} + P_{\delta^*,t} + P_{n,t} + P_{n^*,t}\end{aligned}\tag{1.66}$$

where we applied the fact that the consumption processes (1.29) clear the tradable goods markets at the pricing kernel and exchange rate (1.31) and (1.32). Now from (1.61), and applying (1.66), we have

$$\begin{aligned}\xi_t (P_{\delta,t} + P_{\delta^*,t} + P_{n,t} + P_{n^*,t}) &= P_{\delta,0} + P_{\delta^*,0} + P_{n,0} + P_{n^*,0} - \\ &- \int_0^t \xi_s (\delta_s + \phi_s \delta_s^* + \eta_s n_s + \eta_s^* n_s^*) ds + \\ &+ \int_0^t \xi_s [(\pi_s + \pi_s^*) \sigma_s - (P_{\delta,s} + P_{\delta^*,s} + P_{n,s} + P_{n^*,s}) \theta_s] dW_s\end{aligned}\tag{1.67}$$

Write the equilibrium price process as $P_t^G = [P_{\delta,t}, P_{\delta^*,t}, P_{n,t}, P_{n^*,t}]$. Denote by G, G^*, G^n, G^{n^*} the deflated gains process of the shares of the firms. Now, from Lemma 1 we can write:

$$\begin{aligned}\xi_t (P_{\delta,t} + P_{\delta^*,t} + P_{n,t} + P_{n^*,t}) &= P_{\delta,0} + P_{\delta^*,0} + P_{n,0} + P_{n^*,0} - \\ &- \int_0^t \xi_s (\delta_s + \phi_s \delta_s^* + \eta_s n_s + \eta_s^* n_s^*) ds + \\ &+ \int_0^t \xi_s [P_s^T \sigma_s^T - (P_{\delta,s} + P_{\delta^*,s} + P_{n,s} + P_{n^*,s}) \theta_s] dW_s\end{aligned}\tag{1.68}$$

Subtracting (1.67) from (1.68) we see that the martingale:

$$N_t := \int_0^t \xi_s [(\pi_s + \pi_s^*) \sigma_s - P_s^T \sigma_s] dW_s\tag{1.69}$$

is identical to a constant (zero). This implies that its quadratic variation must be

zero, i.e.

$$[N]_t = \int_0^t \left\| \xi_s \left((\pi_s + \pi_s^*) \sigma_s - P_s^T \sigma_s^T \right) \right\|^2 ds = 0 \quad (1.70)$$

$\forall t \in [0, T]$. This implies, a.s., $(\pi_s + \pi_s^*) \sigma_s = P_s^T \sigma_s^T$. But since σ_t is nonsingular by assumption and $\xi_t > 0$ a.s. it must be that $\pi_s + \pi_s^* = P_s^T$; or equivalently, (1.65) holds a.s. and thus the markets of risky assets clear. By Walras law, the market for instantaneous riskless borrowing/lending is also in equilibrium. \square

Proof of Lemma 2:

Absence of arbitrage and market completeness implies that for any square integrable, \mathcal{B}_t -measurable random variable φ - a contingent claim that pays in units of the domestic tradable- we have, for $s < t$:

$$\frac{1}{\xi_s} E(\xi_t \varphi \mid \mathcal{B}_s) = \phi_s \frac{1}{\xi_s^*} E\left(\xi_t^* \frac{1}{\phi_t} \varphi \mid \mathcal{B}_s\right)$$

take $\varphi = 1_{\{\mathcal{A}\}}$ for every $\mathcal{A} \in \mathcal{B}_t$, using the definition of conditional expectation the above equality becomes:

$$\frac{1}{\xi_s} \int_{\mathcal{A}} \xi_t P(d\omega) = \phi_s \frac{1}{\xi_s^*} \int_{\mathcal{A}} \xi_t^* \frac{1}{\phi_t} P(d\omega) \quad \forall \mathcal{A} \in \mathcal{B}_t$$

which implies (Billingsley 1986, p.216):

$$\frac{\xi_t}{\xi_s} = \frac{\xi_t^* \phi_s}{\xi_s^* \phi_t}$$

$\mathcal{P} \times \text{Lebesgue}$ - a.s., as desired. \square

Proof of Proposition 2:

For any $(x, y) \in \mathcal{R}_{++}^2$, define the processes:

$$\begin{aligned} \Delta_t(x, y) &= n_t^{-\frac{p}{q-1}} e^{\frac{p}{q-1}t} x + n_t^*^{-\frac{p^*}{q^*-1}} e^{\frac{p^*}{q^*-1}t} y \\ \Delta_t^*(x, y) &= n_t^{-\frac{p}{q-1}} e^{\frac{p}{q-1}t} x + n_t^*^{-\frac{p^*}{q^*-1}} e^{\frac{p^*}{q^*-1}t} y \end{aligned} \quad (1.71)$$

Define also the map $\Lambda : \mathcal{R}_+^2 \rightarrow \mathcal{R}_+^2$ with $\Lambda[(x, y)] = (\Lambda^1(x, y), \Lambda^2(x, y))$ where the functions Λ^1 and Λ^2 are given by:

$$\Lambda^1(x, y) = \frac{E \left(\int_0^T \delta_t^q \Delta_t(x, y)^{1-q} dt \right)}{E \left(\int_0^T e^{\frac{p}{q-1}t} n_t^{-\frac{p}{q-1}} \delta_t^q \Delta_t(x, y)^{-q} dt \right) + E \left(\int_0^T e^{\frac{p}{q-1}t} n_t^{-\frac{p}{q-1}} \tilde{\delta}_t^q \Delta_t^*(x, y)^{-q} dt \right)} \quad (1.72)$$

$$\Lambda^2(x, y) = \frac{E \left(\int_0^T \tilde{\delta}_t^q \Delta_t^*(x, y)^{1-q} dt \right)}{E \left(\int_0^T e^{\frac{p^*}{q-1}t} n_t^{*- \frac{p^*}{q-1}} \delta_t^q \Delta_t(x, y)^{-q} dt \right) + E \left(\int_0^T e^{\frac{p^*}{q-1}t} n_t^{*- \frac{p^*}{q-1}} \tilde{\delta}_t^q \Delta_t^*(x, y)^{-q} dt \right)} \quad (1.73)$$

Defining $\gamma := \lambda^{\frac{1}{q-1}}$ and $\gamma^* := \lambda^{*\frac{1}{q-1}}$ and substituting (1.31) and (1.32) into (1.34) and (1.35), it is clear that we look for a fixed point (γ, γ^*) of Λ . Consider the partially ordered set $\{\mathcal{R}_+^2; \geq\}$. Since the processes $\{\delta_t\}$, $\{\delta_t^*\}$, $\{n_t\}$ and $\{n_t^*\}$ are positive, the processes in (1.71) are positive. Thus if $(x', y') \leq (x, y)$ (memberwise), and since $0 < q < 1$, we have $\Lambda[(x', y')] \leq \Lambda[(x, y)]$ which implies that the map Λ is isotone. Now consider the strip $s_\epsilon = \{(\epsilon, 0) \in \mathcal{R}^2 \mid \epsilon \geq 0\}$. Then $\Lambda[s_\epsilon] = \epsilon K$ where $K \in \mathcal{R}_{++}^2$ is a constant vector. Since s_ϵ cannot be larger than $\Lambda[s_\epsilon]$ for any $\epsilon \geq 0$, this establishes that there exists a vector $a \in \mathcal{R}_+^2 \setminus \{+\infty\}$ such that $a \leq \Lambda[a]$. In these conditions, and since every subset of \mathcal{R}^2 , linearly ordered by \geq , has a supremum, we can apply the Knaster-Tarski fixed point theorem (e.g. Dugundji and Granas (1982)) to conclude that Λ has a fixed point. If (γ, γ^*) is a fixed point of Λ , then $\lambda = \gamma^{q-1}$, $\lambda^* = \gamma^{*q-1}$ and the process defined in (1.31), (1.32) with these values are a solution to (1.31), (1.32), (1.34) and (1.35). This proves the first claim.

Now if $(\lambda, \lambda^*, \{\xi_t\}, \{\phi_t\})$ is a solution to (1.31), (1.32), (1.34) and (1.35), then (γ, γ^*) with $\gamma = \lambda^{\frac{1}{q-1}}$ and $\gamma^* = \lambda^{*\frac{1}{q-1}}$ is a fixed point of Λ . By direct substitution we can verify that, for any $k \geq 0$, $(\hat{\gamma}, \hat{\gamma}^*) := (k^{\frac{1}{q-1}}\gamma, k^{\frac{1}{q-1}}\gamma^*)$ is also fixed point of Λ . and therefore $(k\lambda, k\lambda^*, \{k^{-1}\xi_t\}, \{\phi_t\})$ is also a solution to (1.31), (1.32), (1.34) and (1.35). This proves the second claim.

To show uniqueness, up to a line, suppose that both $\Gamma = (\gamma, \gamma^*)$ and $\underline{\Gamma} =$

$\underline{\gamma}, \underline{\gamma}^* \in \mathcal{R}_+^2$ are fixed points of Λ , and therefore both $(\Gamma^{q-1}, \{\xi_t^{(\Gamma)}\}, \{\phi_t^{(\Gamma)}\})$ and $(\underline{\Gamma}^{q-1}, \{\xi_t^{(\underline{\Gamma})}\}, \{\phi_t^{(\underline{\Gamma})}\})$ are solutions to (1.31), (1.32), (1.34) and (1.35), where $\Gamma^{q-1} := (\gamma^{q-1}, \gamma^{*q-1})$, $\underline{\Gamma}^{q-1} = (\underline{\gamma}^{q-1}, \underline{\gamma}^{*q-1})$. Set, (compare with (1.31) and (1.32)):

$$\begin{aligned}\xi_t^{(\Gamma)} &= \delta_t^{q-1} \left[n_t^{-\frac{p}{q-1}} e^{\frac{p}{q-1}t} \gamma + n_t^{*- \frac{p^*}{q-1}} e^{\frac{p^*}{q-1}t} \gamma^* \right]^{1-q} \\ \phi_t^{(\Gamma)} &= \frac{1}{\xi_t^{(\Gamma)}} \tilde{\delta}_t^{q-1} \left[n_t^{-\frac{\bar{p}}{q-1}} e^{\frac{\bar{p}}{q-1}t} \gamma + n_t^{*- \frac{\bar{p}^*}{q-1}} e^{\frac{\bar{p}^*}{q-1}t} \gamma^* \right]^{1-q}\end{aligned}$$

and similarly for $\xi_t^{(\underline{\Gamma})}$ and $\phi_t^{(\underline{\Gamma})}$. If $\Gamma = \kappa \underline{\Gamma}$ for some positive constant κ , then Γ and $\underline{\Gamma}$ are on a line and we are done by the previous result. Suppose otherwise that Γ and $\underline{\Gamma}$ are not on a line. Define $\nu := \max \left[\frac{\underline{\gamma}}{\underline{\gamma}}, \frac{\underline{\gamma}^*}{\underline{\gamma}^*} \right]$ and $\Upsilon := \nu \underline{\Gamma}$. Then we have $\Upsilon \geq \Gamma$ with strict inequality for one component and equality for the other. W.l.o.g., assume that $\nu = \gamma$, $\nu^* > \gamma^*$. Define the state-price density process $\xi_t^{(\Upsilon)}$ and relative price process $\phi_t^{(\Upsilon)}$ associated with $\Upsilon = (\nu, \nu^*)$ as:

$$\begin{aligned}\xi_t^{(\Upsilon)} &= \delta_t^{q-1} \left[n_t^{-\frac{p}{q-1}} e^{\frac{p}{q-1}t} \nu + n_t^{*- \frac{p^*}{q-1}} e^{\frac{p^*}{q-1}t} \nu^* \right]^{1-q} \\ \phi_t^{(\Upsilon)} &= \frac{1}{\xi_t^{(\Upsilon)}} \delta_t^{*q-1} \left[n_t^{-\frac{\bar{p}}{q-1}} e^{\frac{\bar{p}}{q-1}t} \nu + n_t^{*- \frac{\bar{p}^*}{q-1}} e^{\frac{\bar{p}^*}{q-1}t} \nu^* \right]^{1-q}\end{aligned}\tag{1.74}$$

Since Υ lies on a line through $\underline{\Gamma}$, $(\Upsilon^{q-1}, \{\xi_t^{(\Upsilon)}\}, \{\phi_t^{(\Upsilon)}\})$ is a solution to (1.31), (1.32), (1.34) and (1.35), by the previous result. In addition, from (1.34) and (1.35), we see that if $(\Upsilon^{q-1}, \{\xi_t^{(\Upsilon)}\}, \{\phi_t^{(\Upsilon)}\})$ is a solution to (1.31), (1.32), (1.34) and (1.35), we then must have,

$$\nu E \int_0^T e^{\frac{p}{q-1}t} n_t^{-\frac{p}{q-1}} (\xi_t^{(\Upsilon)})^{\frac{q}{q-1}} dt + \nu E \int_0^T e^{\frac{p^*}{q-1}t} n_t^{*- \frac{p^*}{q-1}} (\phi_t^{(\Upsilon)} \xi_t^{(\Upsilon)})^{\frac{q}{q-1}} dt = E \left(\int_0^T \delta_t \xi_t^{(\Upsilon)} dt \right)$$

which can be written as:

$$\int_0^T \xi_t^{(\Upsilon)} (c_{\delta,t}^{(\Upsilon)} + \phi_t^{(\Upsilon)} c_{\delta^*,t}^{(\Upsilon)}) dt - E \left(\int_0^T \delta_t \xi_t^{(\Upsilon)} dt \right) = 0\tag{1.75}$$

where $c_{\delta,t}^{(\Upsilon)}$ and $c_{\delta^*,t}^{(\Upsilon)}$ are the optimal consumption choices associated with the state-price density $\xi_t^{(\Upsilon)}$ and relative price process $\phi_t^{(\Upsilon)}$ according to (1.29).

Now from (1.74), note that $\xi_t^{(\Upsilon)} > \xi_t^{(\Gamma)}$ *a.s.* in $\Omega \times [0, T]$. Therefore:

$$\int_0^T \delta_t \xi_t^{(\Gamma)} dt < \int_0^T \delta_t \xi_t^{(\Upsilon)} dt \quad \text{a.s.} \quad (1.76)$$

and

$$\begin{aligned} E \int_0^T \xi_t^{(\Upsilon)} (c_{\delta,t}^{(\Upsilon)} + \phi_t^{(\Upsilon)} c_{\delta^*,t}^{(\Upsilon)}) dt &= v E \int_0^T e^{\frac{\rho}{q-1}t} n_t^{-\frac{p}{q-1}} \left(\xi_t^{(\Upsilon)} \right)^{\frac{q}{q-1}} dt + \\ &\quad + v E \int_0^T e^{\frac{\rho}{q-1}t} n_t^{-\frac{p}{q-1}} \left(\phi_t^{(\Upsilon)} \xi_t^{(\Upsilon)} \right)^{\frac{q}{q-1}} dt = \\ &= E \int_0^T v e^{\frac{\rho}{q-1}t} \delta_t^q \left[v n_t^{-\frac{p}{q}} + v^* n_t^{-\frac{pq}{1-q}} n_t^{*-\frac{p}{q-1}} \right]^{-q} dt + \\ &\quad + E \int_0^T v e^{\frac{\rho}{q-1}t} \delta_t^{*q} \left[v n_t^{-\frac{p}{q}} + v^* n_t^{-\frac{pq}{1-q}} n_t^{*-\frac{p}{q-1}} \right]^{-q} dt < \\ &< E \int_0^T \gamma e^{\frac{\rho}{q-1}t} \delta_t^q \left[\gamma n_t^{-\frac{p}{q}} + \gamma^* n_t^{-\frac{pq}{1-q}} n_t^{*-\frac{p}{q-1}} \right]^{-q} dt + \\ &\quad + E \int_0^T \gamma e^{\frac{\rho}{q-1}t} \delta_t^{*q} \left[\gamma n_t^{-\frac{p}{q}} + \gamma^* n_t^{-\frac{pq}{1-q}} n_t^{*-\frac{p}{q-1}} \right]^{-q} dt = \\ &= \gamma E \int_0^T e^{\frac{\rho}{q-1}t} n_t^{-\frac{p}{q-1}} \left(\xi_t^{(\Gamma)} \right)^{\frac{q}{q-1}} dt + \gamma E \int_0^T e^{\frac{\rho}{q-1}t} n_t^{-\frac{p}{q-1}} \left(\phi_t^{(\Gamma)} \xi_t^{(\Gamma)} \right)^{\frac{q}{q-1}} dt = \\ &= E \int_0^T \xi_t^{(\Gamma)} (c_{\delta,t}^{(\Upsilon)} + \phi_t^{(\Gamma)} c_{\delta^*,t}^{(\Upsilon)}) dt \end{aligned} \quad (1.77)$$

where the inequality comes from the fact that $v = \gamma$, $v^* > \gamma^*$. Combining (1.76) and (1.77) we obtain:

$$\begin{aligned} E \int_0^T \xi_t^{(\Upsilon)} (c_{\delta,t}^{(\Upsilon)} + \phi_t^{(\Upsilon)} c_{\delta^*,t}^{(\Upsilon)}) dt - E \int_0^T \gamma_t \xi_t^{(\Upsilon)} dt &< \\ < E \int_0^T \xi_t^{(\Gamma)} (c_{\delta,t}^{(\Upsilon)} + \phi_t^{(\Gamma)} c_{\delta^*,t}^{(\Upsilon)}) dt - E \int_0^T \delta_t \xi_t^{(\Gamma)} dt &= 0 \end{aligned} \quad (1.78)$$

which contradicts (1.75). If, alternatively, $v > \gamma$, $v^* = \gamma^*$, we will obtain the same contradiction as in (1.78), except by starring the multipliers. This establishes that Γ and $-\Gamma$ must be on a line and therefore we have proved uniqueness (up to a line).

Now, to proof viability of the state price density, consider the $(P, \{\mathcal{B}_t\})$ -local martingale η_t defined as: $\eta_t := \exp\left(\int_0^t r_s ds\right) \xi_t = \exp\left(-\int_0^t \theta_s dW_s - \frac{1}{2} \int_0^t \|\theta_s\|^2 ds\right)$ where θ_t is the process (1.39). By Minkowski's inequality, the fact that $\alpha_t \in [0, 1]$ and given Assumption 1, we have, $P - a.s.$

$$\begin{aligned} \|\theta_t\|^2 &\leq (1-q)^2 \|\sigma_\delta\|^2 + p^2 \|\alpha_t \sigma_n\|^2 + p^{*2} \|(1-\alpha_t) \sigma_{n^*}\|^2 \leq \\ &\leq (1-q)^2 \|\sigma_\delta\|^2 + p^2 \|\sigma_n\|^2 + p^{*2} \|\sigma_{n^*}\|^2 < \infty \end{aligned}$$

and therefore $E\left(\exp\left(\frac{1}{2}\int_0^T\|\theta_t\|^2dt\right)\right) < \infty$. Since Novikov's condition is satisfied, the local martingale η_t is indeed a martingale (Karatzas and Shreve, Corollary 5.13) and $E(\eta_t) = 1 \forall t \in [0, T]$. Since η is positive and its expectation is one, it is a Radon-Nikodym derivative and the probability measure $Q(\mathcal{A}) := \int_{\mathcal{A}} \eta_T P(d\omega)$ is a well defined equivalent martingale measure (Harrison and Kreps, 1979) which is a sufficient condition for the absence of arbitrage opportunities and thus ξ is a viable state price density. \square

For the ensuing proofs we will need the following technical results, involving some results from Malliavin calculus (see Appendix 2 for a crash introduction to this technique):

Lemma 3: Let $Z_t = \exp Y_t$ (where the exponential is taken component-wise). The Malliavin derivative process $\mathcal{D}_s Z_t : \Omega \times [0, T] \rightarrow \mathcal{R}^{4 \times 4} \forall 0 \leq s \leq T$ is given by:

$$\begin{aligned} \mathcal{D}_s Z_t &= I_{Z_t} \Lambda(s, t) \sigma_s & \text{for } 0 < s \leq t \leq T \\ \mathcal{D}_s Z_t &= 0 & \text{for } 0 < t < s \leq T \end{aligned} \tag{1.79}$$

where the symbol \mathcal{D} stands for the Malliavin-Fréchet derivative operator (see \spadesuit 1, Appendix 2) and where $\Lambda(s, t) = \Phi(t)\Phi(s)^{-1}$ where the deterministic, matrix-valued function $\Phi(t)$ is the solution of the differential equations system $\frac{d\Phi}{dt} = -b(t)\Phi$ with initial condition $\Phi(0) = I_{4 \times 4}$ (the 4×4 identity matrix). I_{Z_t} is a 4×4 diagonal matrix with the elements of the vector Z_t in the diagonal.

Proof: The solution to the linear stochastic differential equation (1.1) is given by:

$$Y_t = \Phi(t) \left[Y_0 + \int_0^t \Phi^{-1}(s) a(s) ds + \int_0^t \Phi^{-1}(s) \sigma(s) dW_s \right]$$

where $\Phi(t)$ is the 4×4 matrix that solves the differential equation $\frac{d\Phi}{dt} = -b(t)\Phi dt$ with initial condition $\Phi(0) = I_{4 \times 4}$ (Karatzas and Shreve, 1988). The process in levels

is:

$$Z_t := (\delta_t, \delta_t^*, n_t, n_t^*) = \exp \left(\Phi(t) \left[Y_0 + \int_0^t \Phi^{-1}(s) a(s) ds + \int_0^t \Phi^{-1}(s) \sigma(s) dW_s \right] \right) \quad (1.80)$$

Where the exponential is taken term by term. Now since Y_t is a smooth random variable $\forall t \in [0, T]$ (i.e. it belongs to the space \mathcal{S} , see Appendix 2), it belongs to the domain of the Malliavin derivative operator, $D^{1,2}$. The Malliavin derivative process $\mathcal{D}_s Y_t : \Omega \times [0, T] \rightarrow \mathcal{R}^{7 \times 7} \forall 0 \leq s \leq T$ is then given by:

$$\begin{aligned} \mathcal{D}_s Y_t &= \Lambda(s, t) \sigma_s & \text{for } 0 < s \leq t \leq T \\ \mathcal{D}_s Y_t &= 0 & \text{for } 0 < t < s \leq T \end{aligned} \quad (1.81)$$

with $\Lambda(s, t) = \Phi(t) \Phi(s)^{-1}$ where the symbol \mathcal{D} stands for the Malliavin-Fréchet derivative operator (see Appendix 2). To compute the first line of (1.81) we applied the Chain rule of Malliavin calculus to (1.80) and used the fact that, for $0 < s \leq t \leq T$:

$$\mathcal{D}_s \int_0^t \Phi(t) \Phi^{-1}(v) \sigma_v dW_v = \int_0^t \left(\mathcal{D}_s \Phi(t) \Phi^{-1}(v) \sigma_v \right) dW_v + \Phi(t) \Phi^{-1}(s) \sigma_s = \Lambda(s, t) \sigma_s$$

where the first equality is a special case of Proposition A.6 of Ocone and Karatzas (1991) and the second follows since the matrix $\Phi(t) \Phi^{-1}(v) \sigma_v$ is deterministic $\forall t, v \in [0, T]$. The second line of (1.81) from the fact that Y_t is progressively measurable and definition of the Malliavin-Fréchet operator.

Now since $Z_t = \exp(Y_t)$ and $Y_t \in D^{1,2}$, $\exp(\cdot) \in C^\infty(\mathcal{R})$, $Z_t = \exp(Y_t)$ has finite moments of all orders (it is lognormal), and in view of (1.81), we have

$$E \left\{ |Z_t^i| + \left\| \frac{\partial Z}{\partial Y} \mathcal{D} Y_t^i \right\| \right\} = E \left\{ |\exp(Y_t^i)| + \left\| \exp(Y_t^i) 1_{[0,t]}(\cdot) \Lambda(\cdot, t) \sigma^i(\cdot) \right\| \right\} < \infty \quad (1.82)$$

for all components $i = 1, \dots, 4$ and where the norm used is $\|f\| = \int_{[0,t]} f(s) ds$. Then, in view of (1.82), by lemma A.1 of Ocone and Karatzas (1991) $Z_t^i \in D^{1,2}$ and $t \in [0, T]$, and the Chain rule of Malliavin calculus applies to $Z_t^i \exp(Y_t^i)$. Thus we have:

$$\begin{aligned} \mathcal{D}_s Z_t &= I_{Z_t} \Lambda(s, t) \sigma(s) & \text{for } 0 < s \leq t \leq T \\ \mathcal{D}_s Z_t &= 0 & \text{for } 0 < t < s \leq T \end{aligned}$$

as desired. \square

Remark : Note that the matrix function $\Phi(t)$ is nonsingular $\forall t \in [0, T]$ (otherwise the initial condition $\Phi(0) = I_{4 \times 4}$ would be violated), and therefore the above inversion is licit. Sufficient conditions for Assumption 2 above regarding the nondegeneracy of the covariance matrix of Y_t to hold is that the matrix $\int_0^T \Phi^{-1}(t) \sigma(t) (\Phi^{-1}(t) \sigma(t))' dt$ be nonsingular. If b and σ are constant, this is equivalent to the requirement that the matrix $[\sigma, b\sigma, b^2\sigma, b^3\sigma, b^4\sigma]$ be of rank 4 (Karatzas and Shreve, Propositions 6.4 and 6.5), and in this case we have $\Phi(t) = e^{-tb} = \sum_{n=1}^{\infty} \frac{-t^n}{n!} b^n$.

Lemma 4: Denote the domain of the Malliavin differential operator by $D^{1,2}$ ($\spadesuit 1$, Appendix 2). Suppose that $F \in D^{1,2}$ is a \mathcal{B} -measurable random variable, and let $\varphi(F)$ be a \mathcal{R}^m -valued function. If $\varphi(F)$ is locally Lipschitz, then $\varphi(F) \in D^{1,2}$ and the chain rule of Malliavin calculus ($\spadesuit 3$, Appendix 2) applies.

Proof of Lemma 4:

We produce a sequence that localizes $\varphi(F)$ in $D^{1,2}$ as follows. Define the set function $\varphi_F^{-1} : \mathcal{R}^m \rightarrow \Omega$ as $\varphi_F^{-1}(\Gamma) = \{\omega \in \Omega \mid \varphi(F(\omega)) \in \Gamma\}$ and the sequence $\{\varphi_F^{-1}(A_k)\}_{k \in \mathcal{N}} \subset \Omega$. Clearly, $\varphi_F^{-1}(A_k) \uparrow \Omega$ since $A_k \uparrow \mathcal{R}^m$ and $\varphi_F^{-1}(A_k) \in \mathcal{B}$ since F is measurable. Since φ is Lipschitz on each A_k , we have $\varphi_k := 1_{\{\varphi_F^{-1}(A_k)\}} \times \varphi(F) \in D^{1,2}$ by proposition 1.2.3 of Nualart (1995), and of course $\varphi_k(F) = \varphi(F)$ on $\varphi_F^{-1}(A_k)$. Thus $\{\varphi_F^{-1}(A_k), \varphi_k\}_{k \in \mathcal{N}} \subset \mathcal{B} \times D^{1,2}$ is a localizing sequence for $\varphi(F)$ in $D^{1,2}$ and therefore $\varphi(F) \in D_{loc}^{1,2}$. Since the operator \mathcal{D} has the local property (Proposition 1.3.7 of Nualart), $\mathcal{D}\varphi(F)$ is defined without ambiguity on each $\varphi_F^{-1}(A_k)$ as $\mathcal{D}\varphi(F) = \mathcal{D}\varphi_k$ for all $k \in \mathcal{N}$. Since $\mathcal{D}\varphi_k \in L^2(\Omega \times [0, T])$ for all $k \in \mathcal{N}$ and L^2 is complete, $\mathcal{D}\varphi(F)$ is well defined everywhere and we are done in proving that the object $\varphi(F)$ belongs to the domain of the derivative operator \mathcal{D} .

The proof of the Chain rule for Lipschitz functions of Nualart (Proposition 1.2.3) now follows identically except the fact that the random coefficients of the summation of Malliavin differentials of the components of the random vector F will not be necessarily bounded. Also, the proof of the Clark-Ocone representation for our case follows identically as in Proposition 1.3.5 in Nualart. \square

Proof of Proposition 3:

Let's write:

$$\xi_t P_{\delta,t} = E \left(\int_t^T \delta_s \xi_s ds \mid \mathcal{B}_t \right) = E \left(\int_0^T \delta_s \xi_s ds \mid \mathcal{B}_t \right) - \int_0^t \delta_s \xi_s ds \quad (1.83)$$

Using the definition of the deflated gains process and applying Clark's formula to the first term on the RHS of (1.83) we get:

$$\begin{aligned} \hat{G}_t &= \xi_t P_{\delta,t} + \int_0^t \xi_s \delta_s ds = E \left(\int_0^T \delta_s \xi_s ds \mid \mathcal{B}_t \right) = \\ &= E \left(\int_0^T \delta_s \xi_s ds \right) + \int_0^t E \left(\mathcal{D}_s \int_s^T \delta_v \xi_v dv \mid \mathcal{B}_s \right) dW_s \end{aligned} \quad (1.84)$$

Write the diffusion matrix of equilibrium prices as $\sigma_t^G = [\sigma_{\delta,t}^G : \sigma_{\delta^*,t}^G : \sigma_{n,t}^G : \sigma_{n^*,t}^G]'$ where, for instance, $\sigma_{\delta,t}^G$ stands for the 1×4 diffusion vector of the equilibrium price of the firm producing the domestic tradable. Recall from the proof of Lemma 1 that the following stochastic differential equations hold:

$$\begin{aligned} d\hat{G}_t &= \xi_t P_{\delta,t} \left(\sigma_{\delta,t}^G - \theta_t \right) dW_t \\ d\hat{G}_t^* &= \xi_t P_{\delta^*,t} \left(\sigma_{\delta^*,t}^G - \theta_t \right) dW_t \end{aligned} \quad (1.85)$$

We will use (1.85) and an explicit expression for the Malliavin derivative in (1.84) to identify the equilibrium volatility matrix of the traded assets, σ_t^G . Now, for $0 \leq s \leq v \leq T$,

$$\begin{aligned} \mathcal{D}_s \delta_v \xi_v &= \xi_v \mathcal{D}_s \delta_v + \delta_v \mathcal{D}_s \xi_v = \delta_v \xi_v \sigma_\delta - (1 - q) \delta_v \xi_v \sigma_\delta + \\ &+ (1 - q) \delta_v \xi_v \left[\hat{p} \alpha_v \Lambda^n(s, v) \sigma_t^{n'} + \hat{p}^* (1 - \alpha_v) \Lambda^{n^*}(s, v) \sigma_t^{n'^*} \right] \end{aligned} \quad (1.86)$$

Integrating (1.86), multiplying and dividing by $P_{\delta,t}$ and $P_{\delta^*,t}$ respectively, and using Lemma 1 we can thus express (1.84) for the domestic and foreign risky assets

in the following differential form:

$$\begin{aligned} d\hat{G}_t &= \xi_t P_{\delta,t} [q\sigma_\delta + pu_t\sigma_{n,t} + p^*v_t\sigma_{n^*,t}] dW_t \\ d\hat{G}_t^* &= \xi_t P_{\delta^*,t} [q\sigma_{\delta^*} + \tilde{p}u_t^*\sigma_{n,t} + \tilde{p}^*v_t^*\sigma_{n^*,t}] dW_t \end{aligned} \quad (1.87)$$

Now equating the diffusion vectors of the endowment process and (2.62) and using (1.39) we obtain:

$$\begin{aligned} \sigma_{\delta,t}^G &= q\sigma_\delta + pu_t\sigma_{n,t} + p^*v_t\sigma_{n^*,t} + \theta_t = \\ &= \sigma_\delta + p(u_t - \alpha_t)\sigma_{n,t} + p^*(v_t - (1 - \alpha_t))\sigma_{n^*,t} \end{aligned} \quad (1.88)$$

and similarly for the risky asset of the foreign country:

$$\begin{aligned} \sigma_{\delta^*,t}^G &= q\sigma_{\delta^*} + \tilde{p}u_t^*\sigma_{n,t} + \tilde{p}^*v_t^*\sigma_{n^*,t} + \theta_t = \\ &= (1 - q)\sigma_\delta + q\sigma_{\delta^*} + (\tilde{p}u_t^* - p\alpha_t)\sigma_{n,t} + (\tilde{p}^*v_t^* - p^*(1 - \alpha_t))\sigma_{n^*,t} \end{aligned} \quad (1.89)$$

where x, u, v, x^*, u^* and v^* are given in (1.46) and (1.47). We then obtain the diffusion matrix of the equilibrium prices for the productive processes by stacking (1.88) and (1.89) as row vectors in the matrix $\sigma_t^G I_P$ as given by (1.45). \square

Proof of Proposition 4:

The internal relative price of nontraded goods in terms of the numeraire is given by:

$$\begin{aligned} \eta_t &= \frac{p}{q} e^{-\frac{p}{1-q}t} \lambda^{\frac{1}{q-1}} \xi_t^{-\frac{1}{1-q}} n_t^{\frac{p}{1-q}-1} + \frac{\tilde{p}}{q} e^{-\frac{\tilde{p}}{1-q}t} \lambda^{\frac{1}{q-1}} \phi_t^{\frac{q}{q-1}} \xi_t^{-\frac{1}{1-q}} n_t^{\frac{\tilde{p}}{1-q}-1} \\ \eta_t^* &= \frac{p^*}{q} e^{-\frac{p^*}{1-q}t} \lambda^{*\frac{1}{q-1}} \xi_t^{-\frac{1}{1-q}} n_t^{*\frac{p^*}{1-q}-1} + \frac{\tilde{p}^*}{q} e^{-\frac{\tilde{p}^*}{1-q}t} \lambda^{*\frac{1}{q-1}} \phi_t^{*\frac{q}{q-1}} \xi_t^{-\frac{1}{1-q}} n_t^{*\frac{\tilde{p}^*}{1-q}-1} \end{aligned} \quad (1.90)$$

This implies that the dividend process, in terms of the numeraire, for the share prices of the nontraded firms follow the processes-from Lemma 1-:

$$\begin{aligned} P_{n,t} &= \frac{1}{\xi_t} E \left(\int_t^T n_s \eta_s \xi_s ds \mid \mathcal{B}_t \right) \\ P_{n^*,t} &= \frac{1}{\xi_t} E \left(\int_t^T n_s^* \eta_s^* \xi_s ds \mid \mathcal{B}_t \right) \end{aligned} \quad (1.91)$$

in line with the proof of Proposition 3, the deflated gains processes

$$\begin{aligned}\hat{G}_t^n &= \xi_t P_{n,t} + \int_0^t \eta_s \xi_s n_s ds = E \left(\int_0^T \eta_s \xi_s n_s ds \right) + \int_0^t E \left(\mathcal{D}_s \int_s^T \eta_v \xi_v n_v dv \mid \mathcal{B}_s \right) dW_s \\ \hat{G}_t^{n^*} &= \xi_t P_{n^*,t} + \int_0^t \eta_s^* \xi_s n_s^* ds = E \left(\int_0^T \eta_s^* \xi_s n_s^* ds \right) + \int_0^t E \left(\mathcal{D}_s \int_s^T \eta_v^* \xi_v n_v^* dv \mid \mathcal{B}_s \right) dW_s\end{aligned}$$

follow the dynamics:

$$\begin{aligned}d\hat{G}_t^n &= \xi_t P_{n,t} \left(\sigma_{n,t}^G - \theta_t \right) dW_t \\ d\hat{G}_t^{n^*} &= \xi_t P_{n^*,t} \left(\sigma_{n^*,t}^G - \theta_t \right) dW_t\end{aligned}$$

To compute the Malliavin derivative $\mathcal{D}_s \int_s^T \eta_v \xi_v n_v dv$, use (1.90) to write:

$$\eta_v \xi_v n_v = \frac{\lambda^{\frac{1}{q-1}}}{q} \left[p \xi_t^{\frac{q}{q-1}} e^{-\frac{p}{1-q}t} n_t^{\frac{p}{1-q}} + \bar{p} e^{-\frac{p}{1-q}t} \xi_t^{*\frac{q}{q-1}} n_t^{*\frac{p}{1-q}} \right] \quad (1.92)$$

then, applying the Chain rule for Malliavin calculus for (1.92), for $s < v$, we get:

$$\mathcal{D}_s \eta_v \xi_v n_v = \frac{\eta_v \xi_v n_v}{1-q} \left[(p\varphi_v + \bar{p}(1-\varphi_v)) \frac{\mathcal{D}_s n_v}{n_v} - q \left(\varphi_v \frac{\mathcal{D}_s \xi_v}{\xi_v} + (1-\varphi_v) \frac{\mathcal{D}_s \xi_t^*}{\xi_t^*} \right) \right] \quad (1.93)$$

where $\varphi_v := \frac{p e^{-\frac{p}{1-q}t} \xi_t^{\frac{q}{q-1}} n_t^{\frac{p}{1-q}}}{p e^{-\frac{p}{1-q}t} \xi_t^{\frac{q}{q-1}} n_t^{\frac{p}{1-q}} + \bar{p} e^{-\frac{p}{1-q}t} \xi_t^{*\frac{q}{q-1}} n_t^{*\frac{p}{1-q}}} = \frac{p c_{\delta,t}}{p c_{\delta,t} + \bar{p} \phi_t c_{\delta^*,t}} \in [0, 1]$. Note that in the case of homogeneous preferences we have $p = \bar{p}$ and φ coincides with the budget shares of the individual consumption of tradables. Using previous results for the derivatives $\mathcal{D}_s \xi_v$, $\mathcal{D}_s \xi_v^*$ and $\mathcal{D}_s n_v$, we can write (1.93) as:

$$\begin{aligned}\mathcal{D}_s \eta_v \xi_v n_v &= \eta_v \xi_v n_v \left[\varphi \sigma_\delta + (1-\varphi) q \sigma_{\delta^*} - \frac{q}{1-q} (p^* \varphi_v (1-\alpha_t) \right. \\ &\quad \left. + \bar{p}^* (1-\varphi_v) (1-\tilde{\alpha}_t)) \Lambda^{n^*}(s, v) \sigma_{n^*,t} + \right. \\ &\quad \left. + \frac{1}{1-q} (p \varphi_v (1-q\alpha_v) + \bar{p} (1-\varphi_v) (1-q\tilde{\alpha}_v)) \Lambda^n(s, v) \sigma_{n,t} \right]\end{aligned} \quad (1.94)$$

and therefore, integrating (1.94) and taking conditional expectations, we conclude that:

$$\begin{aligned}\sigma_{n,t}^G &= q(a_t \sigma_\delta + b_t \sigma_{\delta^*}) + \frac{1}{1-q} c_t \sigma_{n,t} - \frac{q}{1-q} d_t \sigma_{n^*,t} + \theta_t = \\ &= (1-q(1-a_t)) \sigma_\delta + q b_t \sigma_{\delta^*} + \left(\frac{1}{1-q} c_t - p \alpha_t \right) \sigma_{n,t} - \left(\frac{q}{1-q} d_t - p^* (1-\alpha_t) \right) \sigma_{n^*,t}\end{aligned} \quad (1.95)$$

where the processes a, b, c and d take values on $[0, 1]$ and are defined as:

$$a_t = \frac{E \left(\int_t^T \varphi_v \eta_v \xi_v n_v \Lambda^\delta(t, v) dv \mid \mathcal{B}_t \right)}{E \left(\int_t^T \eta_v \xi_v n_v dv \mid \mathcal{B}_t \right)} \quad b_t = \frac{E \left(\int_t^T (1 - \varphi_v) \eta_v \xi_v n_v \Lambda^{\delta^*}(t, v) dv \mid \mathcal{B}_t \right)}{E \left(\int_t^T \eta_v \xi_v n_v dv \mid \mathcal{B}_t \right)}$$

$$c_t = \frac{E \left(\int_t^T \eta_v \xi_v n_v (p\varphi_v(1 - q\alpha_v) + \tilde{p}(1 - \varphi_v)(1 - q\tilde{\alpha}_v)) \Lambda^n(t, v) dv \mid \mathcal{B}_t \right)}{E \left(\int_t^T \eta_v \xi_v n_v dv \mid \mathcal{B}_t \right)}$$

$$d_t = \frac{E \left(\int_t^T \eta_v \xi_v n_v (p^*\varphi_v(1 - \alpha_v) + \tilde{p}^*(1 - \varphi_v)(1 - \tilde{\alpha}_v)) \Lambda^{n^*}(t, v) dv \mid \mathcal{B}_t \right)}{E \left(\int_t^T \eta_v \xi_v n_v dv \mid \mathcal{B}_t \right)}$$

Similarly, for the foreign firm producing the nontraded good, we get:

$$\begin{aligned} \sigma_{n^*,t}^G &= q(a_t^* \sigma_\delta + b_t^* \sigma_{\delta^*}) - \frac{1}{1-q} c_t^* \sigma_{n^*,t} - \frac{q}{1-q} d_t^* \sigma_{n,t} + \theta_t = \\ &= (1 - q(1 - a_t^*)) \sigma_\delta + q b_t^* \sigma_{\delta^*} + \left(\frac{1}{1-q} c_t^* - p^*(1 - \alpha_t) \right) \sigma_{n^*,t} - \left(\frac{q}{1-q} d_t^* - p\alpha_t \right) \sigma_{n,t} \end{aligned}$$

where the processes $a_t^*, b_t^*, c_t^*, d_t^*$ are identical to a_t, b_t, c_t, d_t except by substituting the process φ_t for $\varphi_t^* := \frac{p^* c_{\delta,t}^*}{p^* c_{\delta,t}^* + \tilde{p}^* \phi_t c_{\delta^*,t}^*}$, $(1 - \alpha_t)$ for α_t , $(1 - \alpha_t^*)$ for α_t^* , n_t^* for n_t , η_t^* for η_t , p and \tilde{p} for p^* and \tilde{p}^* , respectively. \square

Proof of Proposition 5:

We want to apply the Clark-Ocone formula to the process $\{\xi_t(c_{\delta,t} + \phi_t c_{\delta^*,t} + \eta_t n_t)\}_{t \in [0, T]}$. To check that these process belongs to the domain of the Malliavin derivative operator, note that, $\forall t \in [0, T]$ a.s.,

$$\xi_t(c_{\delta,t} + \phi_t c_{\delta^*,t} + \eta_t n_t) = \xi_t c_{\delta,t} + \xi_t^* c_{\delta^*,t} + \eta_t n_t = \eta_t n_t + \varphi(n_t, n_t^*, \delta_t, t) + \varphi^*(n_t, n_t^*, \delta_t^*, t) \quad (1.96)$$

where the functions φ and φ^* are defined as:

$$\begin{aligned}\varphi(x, y, z, t) &= \lambda^{\frac{1}{q-1}} e^{-\frac{p}{1-q}t} x^{\frac{p}{1-q}} z^q \left[\lambda^{\frac{1}{q-1}} e^{-\frac{p}{1-q}t} x^{\frac{p}{1-q}} + \lambda^{*\frac{1}{q-1}} e^{-\frac{p^*}{1-q}t} y^{\frac{p^*}{1-q}} \right]^{-q} \\ \varphi^*(x, y, z, t) &= \lambda^{\frac{1}{q-1}} e^{-\frac{p}{1-q}t} x^{\frac{p}{1-q}} z^q \left[\lambda^{\frac{1}{q-1}} e^{-\frac{p}{1-q}t} x^{\frac{p}{1-q}} + \lambda^{*\frac{1}{q-1}} e^{-\frac{p^*}{1-q}t} y^{\frac{p^*}{1-q}} \right]^{-q}\end{aligned}\quad (1.97)$$

Since the functions (1.97) do not have bounded partial derivatives and do not satisfy a Lipschitz condition, we cannot apply the usual versions of the Chain rule of Malliavin calculus. We cannot directly apply the version of the Chain rule used in Lemma 3 due to Ocone and Karatzas because the condition (1.82) is difficult to verify here. However we can use Lemma 4 since the functions defined in (1.97) are locally Lipschitz. Thus we can apply Lemma 4 and use the Clark-Ocone formula (\spadesuit 3, Appendix 3) applied to (1.96) to write the process ψ from Proposition 3 explicitly:

$$\begin{aligned}\zeta_t &:= \int_0^T \psi_t dW_t = E\left(\int_0^T \xi_s (c_{\delta,s} + \phi_s c_{\delta^*,s} + \eta_s n_s) ds \mid \mathcal{B}_t\right) - \\ &\quad - E\left(\int_0^T \xi_s (c_{\delta,s} + \phi_s c_{\delta^*,s} + \eta_s n_s) ds\right) = \\ &= \int_0^t E\left(\mathcal{D}_s\left(\int_0^T \xi_v (c_{\delta,v} + \phi_v c_{\delta^*,v} + \eta_v n_v) dv\right) \mid \mathcal{B}_s\right) dW_s\end{aligned}\quad (1.98)$$

Substituting for the optimal consumption choice, the equilibrium state-price density and the equilibrium exchange rate, as functionals of the primitives of our economy, we obtain:

$$\begin{aligned}\xi_t(c_{\delta,s} + \phi_s c_{\delta^*,s}) &= \lambda^{\frac{1}{q-1}} \xi_t^{\frac{q}{q-1}} \left(e^{-\frac{p}{1-q}t} n_t^{\frac{p}{1-q}} + e^{-\frac{p}{1-q}t} n_t^{\frac{\hat{p}}{1-q}} \phi_t^{\frac{q}{q-1}} \right) = \\ &= \lambda^{\frac{1}{q-1}} \delta_t^q \left[\lambda^{\frac{1}{q-1}} e^{\frac{p}{q}t} n_t^{-\frac{p}{q}} + \lambda^{*\frac{1}{q-1}} e^{\frac{p-p^*}{(1-q)q}t} n_t^{-\frac{p}{q(1-q)}} n_t^{*\frac{p^*}{1-q}} \right]^{-q} + \\ &\quad + \lambda^{\frac{1}{q-1}} \tilde{\delta}_t^q \left[\lambda^{\frac{1}{q-1}} e^{\frac{p}{q}t} n_t^{-\frac{\hat{p}}{q}} + \lambda^{*\frac{1}{q-1}} e^{\frac{p-p^*}{(1-q)q}t} n_t^{-\frac{\hat{p}}{q(1-q)}} n_t^{*\frac{\hat{p}^*}{1-q}} \right]^{-q}\end{aligned}\quad (1.99)$$

Computing the Malliavin derivative on the right-hand-side of (1.99), we first note that the integral and the Malliavin differential operator commute in a random Stieljes integral, i.e. $\mathcal{D}_s\left(\int_0^T (c_{\delta,v} + \phi_v c_{\delta^*,v} + \eta_v n_v) dv\right) = \int_s^T \mathcal{D}_s \xi_v (c_{\delta,v} + \phi_v c_{\delta^*,v} + \eta_v n_v) dv$. Define $\hat{p} := \frac{p}{1-q}$; $\hat{q} := \frac{q}{q-1}$; $\hat{p}^* := \frac{p^*}{1-q}$; $\hat{p}^- := \frac{\hat{p}}{1-q}$; $\hat{p}^{*-} := \frac{\hat{p}^*}{1-q}$. Computing the integrand explicitly by applying the Chain rule of Malliavin calculus, by virtue of Lemma 4,

we obtain:

$$\begin{aligned}
\mathcal{D}_s \xi_t(c_{\delta,v} + \varphi_v c_{\delta^*,v}) &= q \lambda^{\frac{1}{q-1}} \delta_t^{q-1} \left[\lambda^{\frac{1}{q-1}} e^{\frac{\rho}{q} t} n_t^{-\frac{\rho}{q}} + \lambda^{*\frac{1}{q-1}} e^{\frac{\rho^* - \rho^* q}{(1-q)q} t} n_t^{-\frac{\rho^*}{q}} n_t^{*\tilde{p}^*} \right]^{-q} \mathcal{D}_s \delta_t - \\
&\quad - q \tilde{p} \lambda^{\frac{1}{q-1}} \delta_t^q \left[\lambda^{\frac{1}{q-1}} e^{\frac{\rho}{q} t} n_t^{-\frac{\rho}{q}} + \lambda^{*\frac{1}{q-1}} e^{\frac{\rho^* - \rho^* q}{(1-q)q} t} n_t^{-\frac{\rho^*}{q}} n_t^{*\tilde{p}^*} \right]^{-q-1} \times \\
&\quad \times \left[\lambda^{\frac{1}{q-1}} M \frac{1}{q} e^{\frac{\rho}{q} t} n_t^{-\frac{\rho+q}{q}} - \lambda^{*\frac{1}{q-1}} \frac{1}{q} e^{\frac{\rho^* - \rho^* q}{(1-q)q} t} n_t^{-\frac{\rho^*}{q}-1} n_t^{*\tilde{p}^*} \right] \mathcal{D}_s n_t - \\
&\quad - q \tilde{p}^* \lambda^{\frac{1}{q-1}} \delta_t^q \left[\lambda^{\frac{1}{q-1}} e^{\frac{\rho}{q} t} n_t^{\frac{\rho}{q}} + \lambda^{*\frac{1}{q-1}} e^{\frac{\rho^* - \rho^* q}{(1-q)q} t} n_t^{-\frac{\rho^*}{q}} n_t^{*\tilde{p}^*} \right]^{-q-1} \times \\
&\quad \times \left[\lambda^{*\frac{1}{q-1}} e^{M \frac{\rho^* - \rho^* q}{(1-q)q} t} n_t^{-\frac{\rho^*}{q}} n_t^{*(\tilde{p}^* - 1)} \right] \mathcal{D}_s n_t^* \\
&+ q \lambda^{\frac{1}{q-1}} \delta_t^{*q-1} \left[\lambda^{\frac{1}{q-1}} e^{\frac{\rho}{q} t} n_t^{-\frac{\rho}{q}} + \lambda^{*\frac{1}{q-1}} e^{\frac{\rho^* - \rho^* q}{(1-q)q} t} n_t^{-\frac{\rho^*}{q}} n_t^{*\tilde{p}^*} \right]^{-q} \mathcal{D}_s \delta_t^* - q \tilde{p}^* \lambda^{\frac{1}{q-1}} \delta_t^{*q} \times \\
&\quad \times \left[\lambda^{\frac{1}{q-1}} e^{\frac{\rho}{q} t} n_t^{-\frac{\rho}{q}} + \lambda^{*\frac{1}{q-1}} e^{\frac{\rho^* - \rho^* q}{(1-q)q} t} n_t^{-\frac{\rho^*}{q}} n_t^{*\tilde{p}^*} \right]^{-q-1} \times \\
&\quad \times \left[\lambda^{\frac{1}{q-1}} \frac{1}{q} e^{\frac{\rho}{q} t} n_t^{-\frac{\rho+q}{q}} - \lambda^{*\frac{1}{q-1}} \frac{1}{q} e^{\frac{\rho^* - \rho^* q}{(1-q)q} t} n_t^{-\frac{\rho^*}{q}-1} n_t^{*\tilde{p}^*} \right] \mathcal{D}_s n_t \\
&\quad - q \tilde{p}^{*'} \lambda^{\frac{1}{q-1}} \delta_t^{*q} \left[\lambda^{\frac{1}{q-1}} e^{\frac{\rho}{q} t} n_t^{\frac{\rho}{q}} + \lambda^{*\frac{1}{q-1}} e^{\frac{\rho^* - \rho^* q}{(1-q)q} t} n_t^{-\frac{\rho^*}{q}} n_t^{*\tilde{p}^*} \right]^{-q-1} \times \\
&\quad \times \left[\lambda^{*\frac{1}{q-1}} e^{\frac{\rho^* - \rho^* q}{(1-q)q} t} n_t^{-\frac{\rho^*}{q}} n_t^{*(\tilde{p}^{*' } - 1)} \right] \mathcal{D}_s n_t^*
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{D}_s \eta_v \xi_v n_v &= \eta_v \xi_v n_v \left[\varphi_v q \sigma_\delta + (1 - \varphi_v) q \sigma_{\delta^*} - \frac{q}{1-q} (p^* \varphi_v (1 - \alpha_t) \right. \\
&\quad \left. + \tilde{p}^* (1 - \varphi_v) (1 - \tilde{\alpha}_t)) \Lambda^{n^*}(s, v) \sigma_{n,t} + \frac{1}{1-q} (p \varphi_v (1 - q \alpha_t) + \right. \\
&\quad \left. + \tilde{p} (1 - \varphi_v) (1 - q \tilde{\alpha}_v)) \Lambda^n(s, v) \sigma_{n,t} \right]
\end{aligned}$$

We can now use Lemma 4 to substitute in the Malliavin derivatives of (1.1). Then, integrating the above result, taking expectations and substituting for the optimal

consumption policies we obtain, for the domestic investor:

$$\begin{aligned}
\psi_t = & qE \left(\int_t^T \frac{\xi_s}{\xi_t} (c_{\delta,s} + \varphi_s \eta_s n_s) ds \mid \mathcal{B}_t \right) \sigma_\delta + \\
& + qE \left(\int_t^T \frac{\xi_s}{\xi_t} (\phi_s c_{\delta^*,s} + (1 - \varphi_s) \eta_s n_s) ds \mid \mathcal{B}_t \right) \sigma_{\delta^*} \\
& + \left[\frac{1}{1-q} E \left(\int_t^T \frac{\xi_s}{\xi_t} (p(1 - q\alpha_s) (c_{\delta,s} + \varphi_s \eta_s n_s) + \right. \right. \\
& \left. \left. + \tilde{p}(1 - q\alpha_s^*) (\phi_s c_{\delta^*,s} + (1 - \varphi_s) \eta_s n_s) \Lambda^n(t, s) ds \mid \mathcal{B}_t \right) \right. \\
& \left. + \sigma_n - \right. \\
& \left. - \left[\frac{q}{1-q} E \left(\int_t^T \frac{\xi_s}{\xi_t} (p^*(1 - \alpha_s) (c_{\delta,s} + \varphi_s \eta_s n_s) + \right. \right. \right. \\
& \left. \left. \left. + \tilde{p}^*(1 - \alpha_s^*) ((1 - \varphi_s) \eta_s n_s + \phi_s c_{\delta^*,s}) \Lambda^{n^*}(t, s) ds \mid \mathcal{B}_t \right) + \sigma_n \right. \right.
\end{aligned}$$

Now we can construct the optimal portfolio as prescribed by Proposition 3, namely $\pi_t \sigma_t^G = \frac{\psi_t}{\xi_t} + X_t \theta_t$ to obtain the portfolio kernel $\pi_t \sigma_t^G$ reported in the text.

Performing the same computations for the foreign investor, or directly substituting the parameters $\gamma, \gamma^*, \rho, \rho^*, p, p^*, n, n^*, \rho, \rho^*, \tilde{p}, \tilde{p}^*, \alpha_t$ for $\gamma^*, \gamma, \rho^*, \rho, p^*, p, n^*, n, \rho^*, \rho, \tilde{p}^*, \tilde{p}, (1 - \alpha_t)$ and the processes $c_{\delta,t}, c_{\delta^*,t}$ for $c_{\delta^*,t}^*, c_{\delta,t}^*$ in the expression for ψ^* from that of ψ , we finally find the foreign investor portfolio policies. \square

Proof of Theorem 2:

Recall that the problem consisted of finding the portfolio holdings π_t and π_t^* once we know the form of the diffusion matrix of equilibrium prices σ_t^G from Propositions 3 and 4, and the portfolio kernels $\pi_t \sigma_t^G$ and $\pi_t^* \sigma_t^G$ arising in Proposition 5. Solving this 4×4 systems of equations is not a straightforward matter, since analytical inversion of σ_t^G is not feasible. However, we can proceed as follows.

A change of base from the canonical basis in \mathcal{R}^4 to the basis formed by the vectors $\{\sigma_\delta, \sigma_{\delta^*}, \sigma_{n,t}, \sigma_{n^*,t}\}$ (which are linearly independent by assumption) allows to express the diffusion matrix of equilibrium prices and the portfolio kernels as $\Lambda' \sigma_t = \sigma_t^G$ and

$\Phi'_t \sigma_t = \pi_t \sigma_t^G$ and $\Phi_t^* \sigma_t = \pi_t^* \sigma_t^G$ respectively³⁹, where

$$\Lambda_t = \begin{pmatrix} 1 - q(1 - x_t) & 1 - q & 1 - q(1 - a_t) & 1 - q(1 - a_t^*) \\ 0 & qx_t^* & qb_t & qb_t^* \\ p(u_t - \alpha_t) & \tilde{p}u_t^* - p\alpha_t & \frac{1}{1-q}c_t - p\alpha_t & -\frac{q}{1-q}d_t^* - p\alpha_t \\ p^*(v_t - (1 - \alpha_t)) & \tilde{p}^*v_t^* - p^*(1 - \alpha_t) & -\frac{q}{1-q}d_t - p^*(1 - \alpha_t) & \frac{1}{1-q}c_t^* - p^*(1 - \alpha_t) \end{pmatrix}$$

$$\pi_t' = \begin{pmatrix} \pi_{\delta,t} \\ \pi_{\delta^*,t} \\ \pi_{n,t} \\ \pi_{n^*,t} \end{pmatrix} \quad \pi_t^{*\prime} = \begin{pmatrix} \pi_{\delta,t}^* \\ \pi_{\delta^*,t}^* \\ \pi_{n,t}^* \\ \pi_{n^*,t}^* \end{pmatrix}$$

(1.100)

$$\Phi_t =$$

$$\begin{pmatrix} (1 - q)X_t + qE_t \left(\int_t^T \frac{\xi_s}{\xi_t} (c_{\delta,s} + \varphi_s \eta_s n_s) ds \right) \\ qE_t \left(\int_t^T \frac{\xi_s}{\xi_t} (\phi_s c_{\delta^*,s} + (1 - \varphi_s) \eta_s n_s) ds \right) \\ \left[\frac{1}{1-q} E_t \left(\int_t^T \frac{\xi_s}{\xi_t} (p(1 - q\alpha_s)(c_{\delta,s} + \varphi_s \eta_s n_s) + \right. \right. \\ \left. \left. + \tilde{p}(1 - q\alpha_s^*)(\phi_s c_{\delta^*,s} + (1 - \varphi_s) \eta_s n_s) \Lambda^n(t, s) ds \right) \right. \\ \left. - p\alpha_t X_t \right] \\ \left[-\frac{q}{1-q} E_t \left(\int_t^T \frac{\xi_s}{\xi_t} (p^*(1 - \alpha_s)(c_{\delta,s} + \varphi_s \eta_s n_s) + \right. \right. \\ \left. \left. + \tilde{p}^*(1 - \alpha_s^*)(\phi_s c_{\delta^*,s} + (1 - \varphi_s) \eta_s n_s) \Lambda^{n^*}(t, s) ds \right) \right. \\ \left. - p^*(1 - \alpha_t) X_t \right] \end{pmatrix}$$

$$\Phi_t^* =$$

$$\begin{pmatrix} (1 - q)X_t^* + qE_t \left(\int_t^T \frac{\xi_s}{\xi_t} (c_{\delta^*,s}^* + \varphi_s^* \eta_s^* n_s^*) ds \right) \\ qE_t \left(\int_t^T \frac{\xi_s}{\xi_t} (\phi_s c_{\delta,s}^* + (1 - \varphi_s^*) \eta_s^* n_s^*) ds \right) \\ \left[-\frac{q}{1-q} E_t \left(\int_t^T \frac{\xi_s}{\xi_t} (p\alpha_s(c_{\delta,s}^* + \varphi_s^* \eta_s^* n_s^*) + \tilde{p}\alpha_s^*(\phi_s c_{\delta^*,s}^* + (1 - \varphi_s^*) \eta_s^* n_s^*)) \Lambda^n(t, s) ds \right) \right. \\ \left. - p\alpha_t X_t^* \right] \\ \left[\frac{1}{1-q} E_t \left(\int_t^T \frac{\xi_s}{\xi_t} (p^*(1 - q(1 - \alpha_s))(c_{\delta,s}^* + \varphi_s^* \eta_s^* n_s^*) + \right. \right. \\ \left. \left. + \tilde{p}^*(1 - q(1 - \alpha_s^*))(\phi_s c_{\delta^*,s}^* + (1 - \varphi_s^*) \eta_s^* n_s^*)) \Lambda^{n^*}(t, s) ds \right) \right. \\ \left. - p^*(1 - \alpha_t) X_t^* \right] \end{pmatrix}$$

³⁹Recall that $\sigma_t = (\sigma'_\delta : \sigma'_{\delta^*} : \sigma'_n : \sigma'_{n^*})'$ is the diffusion matrix of the endowment process and is assumed to be nonsingular.

and where

$$X_s = E_t \int_t^T \frac{\xi_s}{\xi_t} (c_{\delta,s} + \phi_s c_{\delta^*,s} + \eta_s n_s) ds \quad X_s^* = E_t \int_t^T \frac{\xi_s}{\xi_t} (c_{\delta,s}^* + \phi_s c_{\delta^*,s}^* + \eta_s^* n_s^*) ds \quad (1.101)$$

The auxiliary processes $a_t, b_t, c_t, d_t, a_t^*, b_t^*, c_t^*, d_t^*, \dot{x}_t, x_t^*, u_t^*, u_t, v_t$ and v_t^* are given in the statement of Proposition 3 and in the proof of Proposition 4. This implies that we can write the linear systems to solve for the portfolio holdings π_t and π_t^* in the form:

$$\Lambda_t \pi_t' = \Phi_t \quad (1.102)$$

$$\Lambda_t \pi_t^{*'} = \Phi_t^* \quad (1.103)$$

Now, by virtue of Theorem 1, we have $\pi_t + \pi_t^* = (P_{\delta,t}, P_{\delta^*,t}, P_{n,t}, P_{n^*,t}) = P_t^r$. We are going to use this to express the system of equations (1.102) and (1.103) above in a more convenient way. First, write $\Lambda_t = \begin{bmatrix} \Lambda_{1,t} \\ \Lambda_{2,t} \end{bmatrix}$ where $\Lambda_{1,t}$ is a 2×4 matrix containing the two upper rows of Λ_t and $\Lambda_{2,t}$ is a 2×4 matrix containing the two lower rows of Λ_t . Denote also $\Phi_t' = (\Phi_{1,t}, \Phi_{2,t}, \Phi_{3,t}, \Phi_{4,t})$ and $\Phi_t^{*'} = (\Phi_{1,t}^*, \Phi_{2,t}^*, \Phi_{3,t}^*, \Phi_{4,t}^*)$. Finally denote by $\tilde{0}$ a 2×4 matrix of zeros and consider the identity matrix I_4 in $\mathcal{R}^{4 \times 4}$. Then the first two equations of respectively the systems (1.102) and (1.103) together with the market clearing condition in the financial markets can be expressed as the following system of linearly independent equations:

$$\begin{pmatrix} \Lambda_{1,t} & \tilde{0} \\ \tilde{0} & \Lambda_{2,t} \\ I_4 & I_4 \end{pmatrix} \begin{pmatrix} \pi_t' \\ \pi_t^{*'} \end{pmatrix} = \begin{pmatrix} \Phi_{1,t} \\ \Phi_{2,t} \\ \Phi_{1,t}^* \\ \Phi_{2,t}^* \\ P_t^r \end{pmatrix} \quad (1.104)$$

where we recall that $P_t^r = E_t \left(\int_t^T \delta_s^a \frac{\xi_s}{\xi_t} ds \right)$. The system (1.104) is exactly identified for the vector (π_t, π_t^*) . The solution to (1.104) is then (1.52). The first four equations in (1.104) hold clearly by direct inspection, and one can check using the definitions of

$a_t, b_t, a_t^*, b_t^*, x_t$ and x_t^* and rearranging, to see that the equations in (1.104) also hold when the portfolio holdings are those given in (1.52). Incidentally, one can check, using the definitions of these processes, that Λ (and therefore σ_t^G)⁴⁰ is singular if $p = \tilde{p}$ and $p^* = \tilde{p}^*$ and when the endowment processes are lognormal. This is because, in this case, one can aggregate the two tradables into a single composite good and this reduces by one the dimension of the consumption process that the agents want to finance. \square

Proof of Result1: We claim $u_t > u_t^*$ and $v_t < v_t^*$, where $u_t = E \left(\int_t^T \alpha_s \omega_s ds \mid \mathcal{B}_t \right)$ and $u_t^* = E \left(\int_t^T \alpha_s \omega_s^* ds \mid \mathcal{B}_t \right)$, where $\omega_t = \frac{\xi_t \delta_t}{E \left(\int_t^T \xi_s \delta_s ds \mid \mathcal{B}_t \right)}$ and $\omega_t^* = \frac{\phi_t \xi_t \delta_t^*}{E \left(\int_t^T \phi_s \xi_s \delta_s^* ds \mid \mathcal{B}_t \right)}$. To see $u_t > u_t^*$, define the functions $g^t(s) := E(\alpha_s \mid \mathcal{B}_t)$ and $f^t(s) := E(\omega_s - \omega_s^* \mid \mathcal{B}_t)$. Then, by Fubini's theorem,

$$u_t - u_t^* = \int_t^T E(\alpha_s(\omega_s - \omega_s^*) \mid \mathcal{B}_t) ds = \int_t^T g^t(s) f^t(s) ds + \int_t^T \text{cov}(\alpha_s, \omega_s - \omega_s^* \mid \mathcal{B}_t) ds$$

We want to prove that the first term in the RHS is negligible while the second term in the RHS above is positive. Note that this is only natural because it is equivalent to the fact that the consumption share of the domestic country is more correlated with positive shocks in the (normalized) value of the domestic investor's marketable endowment than with the value of the foreign investor's marketable endowment. Use Hölder's inequality to bound the absolute value of the first member of the RHS above as $\int_t^T g^t(s) f^t(s) ds \leq \int_t^T |g^t(s) f^t(s)| ds \leq \left(\int_t^T |g^t(s)|^q ds \right)^{\frac{1}{q}} \left(\int_t^T |f^t(s)|^p ds \right)^{\frac{1}{p}}$ for $\frac{1}{q} + \frac{1}{p} = 1$. If we let $p \rightarrow 1$, we have $\|f^t\|_p^{t,T} \downarrow 0$ since, by Fubini's theorem again, using the definition of f^t , $p \downarrow 1 \lim \left(\int_t^T |f^t(s)|^p ds \right)^{\frac{1}{p}} = E \left(\int_t^T (\omega_s - \tilde{\omega}_s) ds \mid \mathcal{B}_t \right) = 0$ and $\|g^t\|_q^{t,T}$ is bounded above by an arbitrarily small number $\epsilon > 0$ as $q \uparrow \infty$ since $g^t(s) < 1 \forall s > t$. Now the differential of the martingale part in the decomposition of α, ω and ω^* is given by, applying Itô's lemma to (1.38) and the definition of ω_t and ω_t^* , and taking into account that we are in the homogeneous preferences case:

⁴⁰Recall that, for any nonsingular matrices A, B we have, for any square matrix C , we have $\text{rank}(ACB) = \text{rank}(C)$. In our case, take $A = I$, $B = \sigma_t$ and $C = \Lambda_t'$.

$$\begin{aligned}
d\{\alpha\}_t^M &= \frac{1}{1-q}\alpha_t(1-\alpha_t)(p\sigma_{n,t} - p^*\sigma_{n^*,t})dW_t \\
d\{\omega\}_t^M &= \omega_t(q\sigma_\delta dW_t + (1-q)d\{S\}_t^M) \\
d\{\tilde{\omega}\}_t^M &= \tilde{\omega}_t(q\sigma_\delta dW_t + (1-q)d\{S\}_t^M)
\end{aligned} \tag{1.105}$$

where $S_t := \left[n_t^{-\frac{p}{q-1}} e^{\frac{p}{q-1}t} \lambda^{\frac{1}{q-1}} + n_t^*{}^{-\frac{p^*}{q-1}} e^{\frac{p^*}{q-1}t} \lambda^{*\frac{1}{q-1}} \right]^{1-q}$. From (1.105) we can obtain the instantaneous correlation coefficient between the growth rates of α_t and $\omega_t, \tilde{\omega}_t$ writing (heuristically):

$$\begin{aligned}
&\rho\left(\frac{d\alpha_t}{\alpha_t}, \frac{d\omega_t}{\omega_t}\right) - \rho\left(\frac{d\alpha_t}{\alpha_t}, \frac{d\tilde{\omega}_t}{\tilde{\omega}_t}\right) = \\
&= \frac{q}{1-q}(1-\alpha_t)(p\sigma_{n,t}\sigma'_\delta + p^*\sigma_{n^*,t}\sigma'_\delta - p^*\sigma_{n^*,t}\sigma'_\delta - p\sigma_{n,t}\sigma'_\delta) > 0
\end{aligned} \tag{1.106}$$

by hypothesis. An identical procedure leads to the conclusion that $v_t < v_t^*$. \square

1.12 Appendix 2: Facts from Malliavin Calculus

This appendix presents the basics of Malliavin calculus in a nutshell. It does not contain an explanation of all the Malliavin calculus results used in the paper, but only the essential ones. It is intended for the reader who is unfamiliar with this area of stochastic analysis. Essentially, Malliavin calculus is to Itô calculus the same that the calculus of variations is to differential calculus in ordinary analysis. In other words, loosely speaking, if the Itô derivative corresponds to the ordinary derivative in infinitesimal calculus, the Malliavin derivative on Wiener space corresponds to the Fréchet derivative on a function space.

♠1. The Malliavin Derivative Operator: Fix a complete probability space (Ω, \mathcal{B}, P) with a N -dimensional brownian motion $\{W_t\}_{t \in [0, T]}$ defined on it. Assume that \mathcal{B} is generated by W . We want to introduce the derivative of a square integrable

random variable $F : \Omega \rightarrow \mathcal{R}$ with respect to the chance parameter $\omega \in \Omega$, once Ω is endowed with some topological structure. Denote by $C_p^\infty(\mathcal{R}^N)$ the set of C^∞ functions $f : \mathcal{R}^M \rightarrow \mathcal{R}$ with f and its partial derivatives satisfy polynomial growth conditions. Denote by \mathcal{S} the class of smooth random variables of the form $F = f(W(t_1), \dots, W(t_n))$ where $t_i \in [0, T]$, ($i = 1, \dots, n$) and $f(x^{11}, \dots, x^{N1}, \dots, x^{1M}, \dots, x^{NM}) \in C_p^\infty(\mathcal{R}^{NM})$. The Malliavin-Fréchet derivative of a random variable $F \in \mathcal{S}$ is the stochastic process $\{\mathcal{D}_t F\}_{t \in [0, T]}$ or equivalently, the random gradient $\mathcal{D}F = (\mathcal{D}^1 F, \dots, \mathcal{D}^N F)$ with components

$$\mathcal{D}^j F(\omega)(t) = \sum_{i=1}^M \frac{\partial}{\partial x^{ji}} f(W(\omega, t_1), \dots, W(\omega, t_M)) 1_{[0, t_i]}(t)$$

for $j = 1, \dots, N$. For example, $\mathcal{D}_t W^i(\tau) = (0, \dots, 1_{[0, \tau]}(t), \dots, 0)$ for any $1 < i < N$. We always have $\mathcal{D}F \in L^2(\Omega \times [0, T])$. As another example, consider the canonical case where Ω is the Fréchet space $C_{co}(\mathcal{R}_+; \mathcal{R}^N)$. Introduce the space $L^2(\Omega; \mathcal{R}^M)$ and the scalar product on L^2 defined by $\langle f, g \rangle = \int_0^T f g ds$. Assume $N = 1$. Then the scalar product $\langle \mathcal{D}F, h \rangle$ coincides with the directional derivative of F in the direction of $\int h(s) ds$, i.e., $\langle \mathcal{D}F, h \rangle = \sum_{i=1}^M \frac{\partial}{\partial x^i} f(W(t_1), \dots, W(t_M)) \langle 1_{[0, t_i]}(t), h \rangle = \sum_{i=1}^M \frac{\partial}{\partial x^i} f(W(t_1), \dots, W(t_M)) \int_0^{t_i} h(s) ds = \frac{d}{dc} F \left(\omega + \epsilon \int_0^{(\cdot)} h(s) ds \right) |_{\epsilon=0}$. Now if F is in fact a Fréchet differentiable functional of the Wiener path on $[0, T]$ with λ^F being the signed measure associated with the Fréchet derivative of F , then $\mathcal{D}_t F = \lambda^F((t, T])$.

We can view \mathcal{D} as a closed and unbounded operator $\mathcal{D} : \text{dom}(\mathcal{D}) \subset L^2(\Omega) \rightarrow L^2(\Omega \times [0, T])$, that maps a certain class of square integrable random variables into square integrable processes (not necessarily adapted). This operator has an adjoint given by the Skorohod integral (a generalization of the stochastic integral to non-adapted processes, see Nualart (1995) section 1.3).

Finally, the domain of \mathcal{D} in $L^p(\Omega)$ is denoted by $D^{1,p}$, the Banach space which is the closure of the random variables in \mathcal{S} with respect to the norm

$$\|F\|_{1,p} := \left(E(|F|^p) + \left(\sum_{i=1}^N \|\mathcal{D}^i F\|^2 \right)^{\frac{p}{2}} \right)^{\frac{1}{p}} \text{ where } \|\cdot\| \text{ stands for the usual } L^2(\Omega \times [0, T]) \text{ norm.}$$

♠2. The Clark-Ocone Representation Formula: From the Martingale Rep-

resentation Theorem (i.e. Protter (1990), p.54), we know that any square integrable random variable $F \in L^2(\Omega)$, measurable with respect to \mathcal{B} (the terminal sigma algebra generated by the Wiener process), can be written as $F = E(F) + \int_0^T \phi_t dW_t$ where $\{\phi_t\}$ is a square-integrable, $\{\mathcal{B}_t\}$ -adapted process. Now, if the random variable F belongs to $D^{1,2}$, the process $\{\phi_t\}$ can be identified as the projection of the Malliavin derivative of F on the sigma algebra generated by the Wiener process, namely, if $F \in D^{1,2}$, we then have the representation:

$$F = E(F) + \int_0^T E(\mathcal{D}_t F | \mathcal{B}_t) dW_t \quad (1.107)$$

For a proof, see Nualart (1995, Proposition 1.3.5). The representation (1.107) also holds for the case with $F \in D^{1,1}$ as shown by Ocone, Karatzas and Li (1991), although we will not need to deal with this case in this paper.

♠3. The Chain Rule of Malliavin Calculus: Let $F = (F^1, \dots, F^m)$ be a random vector whose components belong to $D^{1,p}$, for $p \geq 1$. Let $\varphi : \mathcal{R}^m \rightarrow \mathcal{R}$ be a continuously differentiable function with bounded partial derivatives. Then $\varphi \in D^{1,p}$ and

$$\mathcal{D}\varphi(F) = \sum_{i=1}^m \frac{\partial \varphi}{\partial x_i}(F) \mathcal{D}F^i \quad (1.108)$$

The proof follows by approximating the random variables in F by random variables in \mathcal{S} , and approximating the function φ by convolutions of φ with truncated, C^∞ functions (see, for example, Ocone and Karatzas (1991), Lemma A.1). The assumption of bounded derivatives can be substituted by the assumption that

$$E \left(|\varphi(F)| + \left\| \sum_{i=1}^m \frac{\partial \varphi}{\partial x_i}(F) \mathcal{D}F^i \right\| \right) < \infty \quad (1.109)$$

In addition, in the case with $p = 2$, it is enough that φ satisfies a Lipschitz condition in \mathcal{R}^m for (1.108) to follow. This extension of the chain rule to locally Lipschitz functions is provided in Lemma 4 and used in the Proof of Proposition 5.

Chapter 2

Exchange Rate Dynamics in a Multilateral Target Zone

2.1 Introduction

The exchange rate target-zone literature originated with Krugman's (1991) seminal paper and soon experienced a rapid growth. This literature characterized exchange rate behavior in a two-country world when the range of variation of the exchange rate is limited to a certain nominal band by the monetary authorities of both countries. To date, we know many features about the two-country case thanks to the many extensions of the basic bilateral model pursued by several authors (see below). However, virtually nothing is known about the nature of the equilibrium exchange rates when the currency agreement involves more than two currencies (as real world target zones usually do). This paper provides this extension. In this paper, I provide a closed form solution for the exchange rate in a multilateral target zone. Although this is an important methodological issue on its own, new economic insights appear in our case relative to the traditional bilateral model. In particular, the predictions of the basic bilateral model are substantially affected. Naturally, I also show under which circumstances the general model collapses to the traditional bilateral model of Krugman (1991) in a multicurrency context.

The traditional two country target-zone model unfolds as follows: beginning with

a minimalist monetary model, the exchange rate in a fully credible bilateral target-zone is expressed as a function of a single aggregate state variable, denominated “fundamentals”. Monetary authorities control these “fundamentals” by manipulating the money supply so as to keep them within a certain range, thus bounding the movements of the exchange rate. This function exhibits a stabilizing effect near the target-zone boundaries and two conclusions arise: the conditional volatility of exchange rates is lower the closer the exchange rate to the limits of the band and the unconditional density of the exchange rate is \mathcal{U} -shaped. Extensions of the basic model appeared rapidly, for example, Delgado and Dumas (1991) and (1993)), Krugman and Rotemberg (1991), Froot and Obstfeld (1991), Miller and Weller (1991), and Svensson (1991). However, evidence presented in Koedjick et. al. (1990), Flood, Rose and Mathieson (1991), Lindberg and Söderlin (1991), Beetsma and Van der Ploeg (1994) and Pill (1994) shows that the above basic predictions of the theory as it stands turn out to be at odds with the empirical evidence. Further partially rejecting evidenced was presented by Smith and Spencer (1991) and De Jong (1994). Extensions of the basic model by making several fundamental changes led researchers to reconcile the predictions of the model with the data¹. On one hand, the assumption of perfect credibility of the target zone was relaxed by Bertola and Caballero (1992), and Bertola and Svensson (1993). On the other hand, different (although still *ad hoc*) intervention rules were allowed in Delgado and Dumas (1991, 1993), Flood and Garber (1991) and Lindberg and Söderlin (1991). In all, the scientific community seems to have accepted that the two crucial assumptions of Krugman’s model (perfect credibility and marginal interventions as leading intervention tool) are overwhelmingly rejected by the data. This paper focuses on the effect on these issues that the explicit incorporation of the multilateral nature of real-world target zones might have.

Our intuition would suggest that modelling explicitly a multilateral target zone would generate an exchange rate solution with a hump-shaped steady state distribution within the band. This is because the exchange rate would hit its “effective”

¹Of course, these extensions are justified on their own sake (i.e., there are realignments in real world).

band within the “notional” band before this “notional” band is actually reached. For example, imagine a world with three currencies, A, B and C in a trilateral target zone. The above phenomenon would trigger interventions before the exchange rate between currencies A and B hits its “nominal” band because the exchange rate between currency A and currency C hits its nominal band first, and thus the monetary authorities of country A intervene “intramarginally” if we are only observing the bilateral band between currency A and currency B. This phenomenon would also tend to “brake” the exchange rate strictly within the band and thus its probability mass inside the band would be larger. This intuition has not been proved formally, and indeed it might be wrong: not only is the exchange rate “braked” only on a measure zero set,² but also the “effective” band moves randomly within the “nominal” band. This suggests that the interventions might still be too weak to reverse the \mathcal{U} -shape in the steady-state distribution within the “notional” band. In this paper I resolve this issue.

In this paper I obtain a closed form solution of the exchange rate in a multilateral target zone in the spirit of Krugman (1991). The long-run dynamics of the exchange rate are also characterized. In addition, I will show that the two above problematic predictions of the bilateral original model (vanishing conditional volatility at the edges and \mathcal{U} -shaped unconditional density) do not carry over when the target zone involves several currencies. More importantly, it will be shown that spillover effects derived from interventions by particular monetary authorities can account for these new distributional implications, even without the necessary existence of cross-currency constraints that make the “effective” band of variation of an exchange rate differ from the “notional” band. This implies that the predictive power Krugman’s bilateral model (1991) is, in a way, more limited than expected. For example, in the context of the European Monetary System (EMS), the bilateral model of Krugman is unable to describe the exchange rate dynamics in both a genuine multilateral band where “nominal” and “notional” bands differ, and a *de facto* Deutsche-Mark (DM)

²This is unlike models with explicit intramarginal interventions such as those of Delgado and Dumas (1991), and Lindberg and Sorderlin (1991).

zone (i.e. a set of bilateral bands with the DM as anchor currency). It is also shown that Krugman's solution coincides with our solution for a *currency influence area* (such as a DM zone, or an implicit dollar zone for NAFTA countries) only when the anchor currency monetary authorities never intervene in maintaining the system (or they sterilize their interventions), and all the burden of intervention is placed on the "weaker" currencies.

The results of this paper indicate that Krugman's (1991) model has not yet been tested (or that it has been rejected too quickly), due to the lack of genuine bilateral bands in the real world. In other words, some of the rejecting evidence presented in the literature against the traditional target zone model using data between realignments from existing target zones need not be due to wrong assumptions (full credibility and marginal interventions) but rather to the fact that the model has been tested on the wrong data. The intuition for this result is as follows: consider a currency influence area involving $n + 1$ currencies and recall the basic Krugman's model as a candidate to explain such regime as a combination of n bilateral target zones. Suppose that the monetary authorities of every country i in the band set a limit to the variation of its fundamentals (call it fundamentals i) with the anchor country. The fundamentals i are linear in the difference of the log supplies of money of country i and the anchor country. These fundamentals are regulated by manipulating the money supply of country i and the anchor country. Each time the boundaries for the i^{th} fundamentals are hit, to the extent that the anchor country manipulates its money supply to some degree to manage the i^{th} bilateral band with country i , the fundamentals of all other j members of the area versus the anchor country will be affected. To the extent that the anchor country shares the intervention burden to some extent, spillover effects on the money supplies from interventions will have an impact on the steady-state distribution of *each and every* exchange rate in the band. This provides a rationale for the existence of intramarginal interventions, in a sense to be made precise. If the anchor country never intervenes, we should recover the basic predictions of Krugman's model, which ignores spillover effects of any sort. Obviously, when we move from a currency area to a genuine multilateral target zone (where "effective" and "notional"

bands may differ) this effect can only be strengthened. In addition, the size of the “honeymoon effect”³ that the theoretical model will predict depends on the path that the fundamentals may follow when the exchange rate approaches the edges of the band. This is unlike the bilateral model, in which the amount of “concavity” in the exchange rate as a function of the fundamentals is deterministic (because the path with which the fundamentals approach the edges of the band is unique). This might explain the mixed support to the “honeymoon effect” found in the empirical literature (for example, Flood, Rose and Mathieson (1991)).

In summary, this paper provides an extension of the benchmark target zone model (Krugman (1991)) to the case where the target zones involves an arbitrary number of currencies. A closed form solution for the exchange rate in a genuine multilateral target zone, where “effective” bands can differ from “notional” bands for any currency, is obtained. Furthermore, its steady-state distributional implications will be analyzed, and the reader will find that the model produces predictions much more in accordance with empirical evidence than the traditional bilateral model. I proceed as follows:

Section 2 lays out the standard starting point of the literature, i.e. the basic exchange rate pricing equation and the identification of the fundamentals as the state variables in our model. Section 3 obtains a closed form solution for the exchange rate in a trilateral target zone, when “effective” bands coincide everywhere with “nominal” bands. I first show that the relevant concept of marginal interventions must be specified in the “fundamentals” space rather than the exchange rate space. Then I show that the solution to a certain boundary value problem must coincide with the solution for the exchange rate as a function of the fundamentals. I pay particular attention to the definition of the domain for the fundamentals process and to the economic interpretation of the relevant boundary conditions. The problem is then solved by rotating the axes where the state variables are defined so that an economically meaningful interpretation of the boundary conditions can be given in

³The so-called “honeymoon effect” denotes the property of Krugman’s model that the instantaneous rate of change in the conditional volatility of the exchange rate tends to zero when it gets closer to the limits of the band. Thus the likelihood of future interventions tends to dampen today’s exchange rate volatility.

the new space. This allows one to solve the problem in closed form, and then the solution can be mapped back to the original space. I then discuss the nature of the solution and relate it to the benchmark bilateral model of Krugman (1991). In Section 4 I extend the result to an arbitrary number of currencies and then incorporate cross-currency restrictions that make the “effective” band differ from the “notional” one. This is done by showing that the problem with “effective” bands differing from “notional” bands is equivalent to the previous problem (when “notional” bands are equal to “effective” bands) with an increased dimension. In Section 5, the steady-state density of the exchange rate will be obtained. I first obtain a closed form solution for the steady-state density of the fundamentals by imposing mild additional but economically meaningful restrictions on the structure of the problem, and then the dynamics for the exchange rate process can be easily simulated. Distributional issues regarding the trilateral target zone case of Section 3 are then discussed. Section 6 concludes. Proofs not found in the text are in Appendix 1.

2.2 The Basic Model

Our starting point will be the classic exchange rate asset pricing equation (see Krugman (1991)). In the literature, this equation is commonly written as:

$$s_t = k_t + \gamma \frac{E_t(ds_t)}{dt} \quad (2.1)$$

where s_t stands for the nominal exchange rate and k_t stands for an exogenous stochastic process driving the uncertainty in the model⁴. In this paper, 2.1 is meant to hold

⁴The precise meaning of 2.1 is as follows: Suppose that the processes $\{s_t\}_{t \in [0, T]}$ and $\{k_t\}_{t \in [0, T]}$ are continuous special semimartingales on a given probability space. Then the following equality holds almost everywhere:

$$BV_t^s = \frac{1}{\gamma} \int_0^t (s_v - k_v) dv \quad (2.2)$$

where the process BV_t^s is the bounded variation process arising in the canonical Doob-Meyer decomposition of $\{s_t\}_{t \in [0, T]}$ (see Protter (1990) p.107 for a definition of this concepts). In what follows, we will continue to write expressions like 2.1, while it is clear that what we mean is 2.2. Thus we write, heuristically, the differential representation of the process BV_t^s as the instantaneous conditional expectation operator per unit time, $E_t(ds)dt$. This is done because the finite variation process BV_t^s is

for every pair of countries in the target zone, namely:

Assumption 1: The following asset pricing differential relationship holds for any two different i, j belonging to a currency target zone agreement:

$$s_t^{ij} dt = k_t^{ij} dt + \gamma E_t(ds_t^{ij}) \quad (2.3)$$

where s^{ij} stands for the “nominal” exchange rate between currency i and currency j (quantity of currency i per unit of currency j) and k^{ij} is an aggregate variable called “fundamentals”. In particular,

$$k^{ij} = m^i - m^j + \alpha^i y^i - \alpha^j y^j \quad (2.4)$$

where m^l and y^l stands for the logarithm of the money supply and aggregate endowment of country l . α^l, α^l and γ are constants, for $l = i, j$. I will normalize our notation by setting $s^{i0} = s^i$ for a certain currency indexed by 0.

Note also that if we normalize currency 0 as being our reference and thus $k^{i0} \equiv k^i$ for $i \neq 0$, we then have

$$k^{ij} = k^i - k^j$$

Equation 2.3 is usually obtained from a minimalist monetary model in the following way. Let the money demand equations satisfy $\frac{M_t^i}{P_t^i} = y_t^{\alpha^i} \exp(-\gamma r_t^i)$ for every country i in the target zone, where M_t^i , P_t^i , y_t^i , and r_t^i stand for the money supply, price level, aggregate endowment and nominal instantaneous interest rate processes for country i . α^i and γ (the semi-elasticity of money demand with respect to the interest rate), are constants. We assume that y is positive. Let purchasing power parity (PPP) and a logarithmic version of uncovered interest rate parity hold, namely $P_t^i = S_t^{ij} P_t^j$ and $(r_t^i - r_t^j) dt = E_t(ds_t^j)$ where $S_t^{ij} = \exp s_t^{ij}$. Taking logarithms of both sides of the money demand equation for countries i and j , subtracting them

continuous, and therefore predictable (see Dellacherie and Meyer, (1982), VII.25).

and using the last two equations, we obtain 2.3. Appendix 2 contains an alternative derivation of 2.3 from first principles (i.e. assumptions on endowment and monetary shocks dynamics only plus a Clower-Tsiang constraint).

2.3 The Solution for a Trilateral Target Zone

2.3.1 Definition of the Problem

A Free-Boundary Problem

In this section, a closed form solution for the exchange-rate dynamics will be obtained in the above set-up. This result will be generalized to an n -dimensional target zone with bands of arbitrary size in Section 4. The program that we want to carry out in this section begins with a definition for the target-zone modelling problem in precise mathematical terms. This involves redefining the concept of intramarginal interventions. The dynamics of the exogenous variables will be further specialized to obtain a generalized version of the familiar regulated brownian motion case of the bilateral model. Then we will be in condition to show that our redefinition of marginal interventions was indeed necessary. I will show that it is necessary and sufficient to solve a certain boundary value problem to obtain a solution to the target-zone modelling problem. The boundary value problem will be given an economic interpretation. Once this is done, the boundary value problem will be solved explicitly by first performing a change in the variable space and then using a separation of variables technique. Once the solution is obtained, I will analyze its properties and compare them with the solution for the bilateral case of Krugman (1991).

Note first that it is not possible to construct a multidimensional target zone model in the spirit of Krugman (1991) with only marginal interventions in the conventional sense. In a multidimensional context, it is impossible to construct a model where marginal interventions take place if and only if all currencies involved are at the boundaries of the target zone. This is because such model would impose too stringent conditions on the form of the solution so that the exchange rate functions would

not be able to satisfy the basic pricing equation 2.3. The reader will find a formal proof in Proposition 1 below. Thus, I will redefine the relevant notion of marginal interventions as interventions in the space of the fundamentals rather than the space for the exchange rates. The conventional definition of marginal interventions asserts that interventions are “marginal” when monetary authorities manipulate the fundamentals (i.e. the money supply) precisely when the exchange rate hits the boundaries of the target zone. My definition posits that marginal interventions occur when monetary authorities intervene only when the fundamentals hit certain boundaries, and no intervention takes place in the interior of the domain of the fundamentals⁵.

Recall the basic problem of the bilateral model of Krugman: given a process for the fundamentals and given an exchange rate band of a certain width, an interval is found for the fundamentals process to take values on, and a function is obtained mapping the fundamentals on this domain to the exchange rates. Once this is clear, we can set-up the problem of modelling a three-lateral target zone with marginal interventions as follows. Given a set of exchange rates with their respective nominal bands, given a fundamentals vector process and given a specification of the intervention sharing rules by a reflection vector field (this will become clear soon) we want to find a domain for the fundamentals process such that this processes is reflected in a certain direction towards the interior of this domain every time the boundaries are hit, and we also want to find a set of bounded functions taking values on this domain and expressing the exchange rate as a function of the fundamentals process. This is stated more precisely as follows.

Definition 1: Given a set of 3 exchange rates $\{S^i\}_{i=1}^3$ (S^3 correspond to the cross exchange rate) a set of nominal exchange rate bands $\{[\underline{S}^i, \bar{S}^i]\}_{i=1}^3$, a bivariate diffusion process⁶ and a reflection vector field $\theta : \mathcal{R}^2 \rightarrow \mathcal{R}^2$, a *trilateral target-zone*

⁵More precisely, interventions take place when the fundamentals, as a \mathcal{R}^n -valued stochastic process, hit a certain $(n - 1)$ -dimensional manifold. This definition captures the spirit that interventions only occur on a set of (Lebesgue) measure zero. Both definitions coincide in the unidimensional case (bilateral target zone).

⁶A diffusion process is a progressively measurable stochastic process with continuous sample paths and with the strong markov property.

modelling problem consists of finding a compact planar region $G \subset \mathcal{R}^2$, $\overset{\circ}{G} \neq \emptyset$, a bivariate process k taking values in G , a probability measure P_{k_0} on (Ω, \mathcal{B}) such that k behaves as a diffusion process in $\overset{\circ}{G}$ and is reflected at ∂G in the direction indicated by θ operating of ∂G , 2.3 holds when the expectation is taken with respect to P_{k_0} , and $\sup_{k \in \bar{G}} s^i(k) \leq \log(\bar{S}^i)$ and $\inf_{k \in \bar{G}} s^i(k) \geq \log(\underline{S}^i)$ P_{k_0} a.s. for $i = 1, 2, 3$. A *solution* to the target zone modelling problem is a triplet (s, G, P_{k_0}) where $s = [s^1, s^2, s^3]$ is a vector function expressing the logarithm of the exchange rate as a function of the fundamentals, $s^3 = s^1 - s^2$ and $G \subset \mathcal{R}^2$ and P_{k_0} stand for the domain and the probability law of the fundamentals, respectively.

For now I concentrate on the simplest case possible, namely a trilateral target zone constructed from the model in the previous section, with a restriction on the size of the bands to make “nominal” and “effective” bands coincide everywhere. Thus I pose an auxiliary assumption:

Assumption 2: The exchange rate processes S^1 and S^2 are kept within a target zone, i.e., $(S^1, S^2) \in [\underline{S}^1, \bar{S}^1] \times [\underline{S}^2, \bar{S}^2]$ by manipulating the money supply so as to keep the fundamentals processes within a given subset of \mathcal{R}^2 . Now, if there exists a target zone within currencies #1 and #2, it is assumed to be at least as wide as $[\frac{\underline{S}^1}{\underline{S}^2}, \frac{\bar{S}^1}{\bar{S}^2}]$. I will call this latter restriction an *inactive cross-currency constraint*. Thus, inactive cross-currency constraints mean that “notional” bands and “effective” bands always coincide. I will maintain this assumption until subsection 4.2.

I will further simplify the above problem by restricting my attention to quadrilaterals in \mathcal{R}^2 . Thus the equivalent of the vector field θ will be a 2×4 reflection matrix R , where each column vector expresses the direction of reflection at each one of the four sides of the quadrilateral⁷ In particular:

Assumption 3: The fundamentals process $k_t = (k_t^1, k_t^2)$ is a progressively measur-

⁷See Dai and Harrison (1991) for an application of these modelling techniques to queueing theory.

able bidimensional process on $(\Omega, \mathcal{B}, \{\mathcal{B}\}_{t \geq 0}, P)$ taking values on a quadrilateral $G \subset \mathcal{R}^2$ with vertices $[k^1, k^2]$, $[k^3, k^4]$, $[k^5, k^6]$ and $[k^7, k^8]$, such that, P -almost surely:

$$k_t = k_0 + \mu t + AW_t + \sum_{i=1}^4 R^i \Lambda_t^i \quad (2.5)$$

with k_0 given and where $k_t = (k_t^1, k_t^2)'$, μ , A are a $\mathcal{R}^{2 \times 1}$ vector and a $\mathcal{R}^{2 \times 2}$ non-singular matrix, respectively. The process $\{W_t\}$ is a two-dimensional Brownian motion on (Ω, \mathcal{B}, P) that generates the filtration $\{\mathcal{B}\}_{t \geq 0}$. The initial conditions $k^1(0)$ and $k^2(0)$ are also given and $\Lambda_t = [\Lambda_t^1, \Lambda_t^2, \Lambda_t^3, \Lambda_t^4]$ is a 4-dimensional *regulator* process and whose components Λ_t^i are nondecreasing (2×1) -vector processes with $\Lambda_0^i = 0$ $i = 1, 2, 3, 4$, that increase only if (k_t^1, k_t^2) hits the side $i = 1, 2, 3, 4$ of the quadrilateral G^8 .

The direction of reflection in each side is constant and is given by R^i , $i = 1, 2, 3, 4$, which are 2×1 reflection vectors (see Figure 1). The angle of reflection does not need to be the same in each side, though). The instantaneous covariance matrix of k^1 and k^2 is given by AA' . Denote a_1 and a_2 the row vectors of A . Then $\sigma_1^2 = \|a_1\|^2$, $\sigma_2^2 = \|a_2\|^2$ and $\rho_{12} = \frac{\|a_1 a_2\|^2}{\|a_1\| \|a_2\|}$ stand for the instantaneous conditional variances and the coefficient of instantaneous correlation of k_t^1 and k_t^2 in the interior of G , respectively.

Remark 1: The following characterization of the domain of the fundamentals will be useful. Note that we can write the quadrilateral $G \subset \mathcal{R}^2$ as the set $G = \{k \in \mathcal{R}^2 \mid Nk \geq f\}$, for some vector $f \in \mathcal{R}^4$, and N is a 2×4 matrix whose columns consist of unit vectors perpendicular to each side of the quadrilateral G that point towards the interior of G . The sides of the quadrilateral can thus be written as $F_i = \{k \in \bar{G} \mid k' n_i = f_i\}$, for $i = 1, 2, 3, 4$, where n_i is the i^{th} column of N .

Regarding the definition of the fundamentals as an aggregate macroeconomic variable in Section 2, the regulator process will be interpreted as manipulations of the money supplies of countries i and 0 that take place through interventions in the foreign exchange market.

⁸One can show that the process k has the Markov property and is kept almost surely inside the tetrahedron: see Harrison and Reiman (1981) and Dai and Harrison (1992).

Assumption 4: Let F_i denote the i^{th} boundary face of G . There exist positive constants α_i and β_i , $i = 1, 2, 3, 4$ such that $\alpha_i R_i + \beta_i R_{i+1}$ points into the interior of G from the vertex where F_i and F_{i+1} meet.

Assumption 5: The intervention rules are symmetric: $R^1 = -R^3$ and $R^2 = -R^4$. Moreover, $[R]_{i,j} \neq 0 \forall i = 1, 2 \forall j = 1, 2, 3, 4$. Also let $M = [R^1 : R^2]$ (M is assumed to be nonsingular).

Remark 2: The 2×4 reflection matrix $R = [R^1 : R^2 : R^3 : R^4]$ specifies the direction of reflection of the fundamentals process at the boundaries of its domain. In other words, $R = [R^1 : R^2 : R^3 : R^4]$ where R^i is a 2×1 vector expressing the direction of reflection of the fundamentals at each side $F^i \subset \mathcal{R}^2$ $i = 1, 2, 3, 4$ of the quadrilateral G (see Figure 1). Thus R reflects the agreement among central banks regarding intervention when the fundamentals hit a certain boundary. For example, $R^1 = (1, 0)'$ would imply that when the fundamentals hit the side L^1 of ∂G , only the first component of the fundamentals k_t^1 is reflected; therefore, all the burden of the intervention falls on the central bank of country #1 while country #0 does not intervene (if it did, k_t^2 would also vary as m_t^0 varies in 2.4). Note also that, in practice, a target zone regime is rarely defended by one central bank alone. This implies that the direction of reflection at the boundaries R will consist of column vectors nonorthogonal to the axes.

Remark 3: Assumption 4 guarantees the existence and uniqueness of the process 3.26, and it also assures that it has the strong Markov property. It also guarantees that the vertices are not absorbing points. This result for the current polygonal case follows by piecing together the Williams (1987) and Varadhan and Williams (1985) (for the general result in convex polyhedrons see also Dai and Williams (1995)). Assumptions 4 and 5 follow from natural economic reasons. When the fundamentals hit faces F_1 and F_2 in Figure 1, they always reflect to the northeast. This is because, when F_1 is hit for example, monetary authorities of country 1 increase m^1 while monetary

authorities of country 0 decrease m^0 thus increasing their relative fundamentals $k^1 = m^1 - m^0 + \{\text{other terms}\}$ and also increasing $k^2 = m^2 - m^0 + \{\text{other terms}\}$, because m^0 decreases. By Assumption 5, they will reflect to the southwest at F_3 and F_4 for the same reasons. This will guarantee that any positive linear combination of the reflection vectors at the vertices where F_1 and F_2 meet, and F_3 and F_4 meet, points into G . Also by the above argument, note that k^1 will reflect more intensively than k^2 at F_1 (and F_3) than at F_2 (and F_4). This guarantees that the condition in Assumption 4 is met at the vertices where both F_1 and F_4 meet, and F_2 and F_3 meet.

Remark 4: Assumption 5 says that the agreement among central banks specifies that if the burden of intervention is shared in a certain proportion between Central Bank A and Central Bank B when currency A is too strong relative to Currency B (it hits the limits of the band), the roles are switched when currency B is strong relative to currency A.

Let us conjecture that one can write the functions $s^1(k^1, k^2)$ and $s^2(k^1, k^2)$, namely, the exchange rate processes can be written as a twice differentiable function of the state variables. That this is indeed possible this will be shown below within the proof of Proposition 2. By Itô's lemma we can rewrite 2.3 as:

$$s^1(k^1, k^2)dt = k_1dt + \mathcal{L} [s^1(k^1, k^2)] dt + \sum_{i=1}^4 \gamma \nabla s^1(k^1, k^2)' R^i d\Lambda_t^i \quad (2.6)$$

$$s^2(k^1, k^2)dt = k_2dt + \mathcal{L} [s^2(k^1, k^2)] dt + \sum_{i=1}^4 \gamma \nabla s^2(k^1, k^2)' R^i d\Lambda_t^i \quad (2.7)$$

where \mathcal{L} is the characteristic operator of the two dimensional diffusion process 3.26:

$$\mathcal{L} = \gamma\mu_1 \frac{\partial}{\partial k_1} + \gamma\mu_2 \frac{\partial}{\partial k_2} + \gamma \frac{\sigma_1^2}{2} \frac{\partial^2}{\partial k_1^2} + \gamma \frac{\sigma_2^2}{2} \frac{\partial^2}{\partial k_2^2} + \gamma\sigma_1\sigma_2\rho_{12} \frac{\partial^2}{\partial k_1\partial k_2} \quad (2.8)$$

We now prove that our redefinition of “marginal interventions” as interventions that take place exclusively at the boundaries of the fundamentals domain rather than the exchange rates themselves was indeed necessary. Basically, the result establishes that there cannot exist two functions $s^i: C^2(G) \subset \mathcal{R}^2 \rightarrow \mathcal{R}$, taking values on a domain with nonempty interior G , that reach their maximum and minimum values along the

frontier of such domain (i.e. interventions -in the fundamentals space- are marginal in the exchange-rate space) and satisfy 2.3 on the interior of such domain. The reason is that conditions on the candidate solutions to the fundamental equation 2.3 are too stringent so that this function can be massaged in a way that all points of the boundary of ∂G are extremum points.

Proposition 1: A multilateral target zone modelling problem (see definition 1) such that $s^i(k) = \log(\bar{S}^i)$ or $s^i(k) = \log(\underline{S}^i) \Leftrightarrow k \in \partial G$; for $i = 1, 2$ is not well posed (i.e. admits no solution).

Remark 5: Note that the above proposition provides a rationale for the existence of intramarginal interventions in the exchange-rate space, which are marginal, of course, when observed in the fundamentals space.

The next proposition allows us to actually solve the trilateral target zone modelling problem in closed form, by proving the equivalence between the probabilistic problem and a related analytical problem. More precisely, it presents a necessary and sufficient condition for a function $g : C^2(G') \rightarrow \mathcal{R}$, where G' is an open set containing G , to coincide almost everywhere with the exchange rate solution that we are interested in. Thus, the solution of the target zone modelling problem (s, G, P_{k_0}) is a function $s \in C^2(\mathcal{R}^2)$ for any given compact domain $G \subset \mathcal{R}^2$. Later we will choose such domain G so that this function can be expressed in a simple closed form.

Proposition 2: Given any closed domain with non-empty interior for the fundamentals $G \subset \mathcal{R}^2$, any process s_t^i that solves 2.3 is a function $s^i : G \rightarrow \mathcal{R}$, for $i = 1, 2$. In addition, let $g_j : \mathcal{R}^2 \rightarrow \mathcal{R}$, $j = 1, 2$ be two real valued functions, continuously differentiable in an open set containing G . Then a necessary and sufficient condition for $g_1(k^1, k^2) = s^1(k^1, k^2)$ and $g_2(k^1, k^2) = s^2(k^1, k^2) \forall (k^1, k^2) \in G$ is that g_j ($j = 1, 2$) solve the boundary value problem:

$$\begin{aligned} g_1(x, y) &= x + \mathcal{L}g_1(x, y) \quad (x, y) \in \overset{\circ}{G} \\ g_2(x, y) &= y + \mathcal{L}g_2(x, y) \quad (x, y) \in \overset{\circ}{G} \end{aligned} \tag{2.9}$$

$$\nabla g_j(x, y)' R_i = 0 \quad \text{if } (x, y) \in F_i; \quad j = 1, 2; \quad i = 1, 2, 3, 4 \tag{2.10}$$

where \mathcal{L} is the operator defined in 2.8.

Under these conditions, and following Proposition 2, the boundary value problem that we are going to solve consists of equations 2.9 with the boundary conditions given by 2.10 and

$$\sup_{(k^1, k^2) \in G} \{s^i(k^1, k^2)\} = \log(\bar{S}^i) \quad (2.11)$$

$$\inf_{(k^1, k^2) \in G} \{s^i(k^1, k^2)\} = \log(\underline{S}^i)$$

$$i = 1, 2.$$

in addition to condition 2.17 below.

Remark 6: Note that the vertices of the fundamentals domain G are not yet given, rather they will also be chosen as part of the solution of our analytical problem. In fact, their determination is not trivial since the full solution to the target zone modelling problem (the triplet (s, G, P_{k_0})) must satisfy another boundary condition in addition to 2.10 and 2.11 (namely 2.17 below). Of course, in general we could choose any domain G after meeting the necessary consistency conditions. However, we want to choose G such that a simple closed form solution comes up, and this solution can be easily related to the solution for the bilateral model of Krugman (1991). This is done by imposing that G be a quadrilateral.

A Note on No-arbitrage Consistency

Let now $S^{12} = \frac{s^1}{s^2}$ be the exchange rate between currencies 1 and 2 (quantity of currency 1 per unit of currency 2). Maintaining assumption 2, namely that $S^{12} \in [\underline{S}^{12}, \bar{S}^{12}] \subset [\frac{s^1}{s^2}, \frac{\bar{s}^1}{\bar{s}^2}]$. the range for the exchange rates, in logarithms, becomes:

$$s^1 \in [\underline{s}^1, \bar{s}^1] \quad s^2 \in [\underline{s}^2, \bar{s}^2] \quad s^{12} = s^1 - s^2 \in [\underline{s}^{12}, \bar{s}^{12}] \quad (2.12)$$

where $\bar{s}^{12} = \log(\bar{S}^{12})$, $\underline{s}^{12} = \log(\underline{S}^{12})$, $\bar{s}^1 = \log(\bar{S}^1)$, $\bar{s}^2 = \log(\bar{S}^2)$, $\underline{s}^1 = \log(\underline{S}^1)$, and $\underline{s}^2 = \log(\underline{S}^2)$, and equation 2.3, if derived for s^{12} , becomes:

$$s^{12}dt = (k^1 - k^2)dt + \gamma E_t[ds_t^{12}] \quad (2.13)$$

We need to check now that if we use Proposition 2 to obtain the functions $s^1(x, y)$ and $s^2(x, y)$, and we construct $s^{12}(x, y)$ as $s^{12}(x, y) = s^1(x, y) - s^2(x, y)$, then $s^{12}(k^1, k^2)$ actually solves 2.13. In other words, since we have three exchange rates (s^1, s^2, s^{12}) related by a no-arbitrage condition, we could in fact pick any two of them, solve the boundary value problem 2.9 with 2.10 and construct the third one by using $s^{12} = s^1 - s^2$. Thus, the final solutions that we get must be independent of the way in which we choose such exchange rates. This no-arbitrage consistency of the solution method outlined in proposition 2 is proven next. Note also that this consistency requirement is independent of whether the cross-currency constraints for s^{12} are binding (genuine target zone) or not (currency influence area).

Proposition 3: If $s^1(x, y)$ and $s^2(x, y)$ solve 2.9 with 2.10 then $s^{12}(x, y) = s^1(x, y) - s^2(x, y)$ solves 2.13. Conversely, if $s^{12}(x, y)$ solves 2.13, then $s^{12}(x, y) = s^1(x, y) - s^2(x, y)$.

Remark 7: Note that the boundary conditions 2.11 have not been explicitly taken into account in Proposition 3. This is because by Assumption 2, $\underline{s}^{12} \geq \underline{s}^1 - \bar{s}^2$ and $\bar{s}^{12} \leq \bar{s}^1 - \underline{s}^2$. Note however that this condition can be further weakened to $\underline{s}^{12} \leq \inf_G \{s^1 - s^2\} \leq \sup_G \{s^1 - s^2\} \leq \bar{s}^{12}$.

Economic Interpretation of the Boundary Conditions

After the mathematical statement of the problem, we seek an economic interpretation of the relevant boundary conditions 2.10 (in addition to 2.11) for our boundary value problem 2.6 and 2.7. First, we will check that these conditions meet an elementary no-arbitrage requirement. Then in the next subsection the analytic solution of the problem is derived and the rationale for such conditions will become clear.

Since $g^i : \mathcal{R}^2 \rightarrow \mathcal{R}$ is continuous and G compact, we can define the (non-empty) \mathcal{R}^2 subset:

$$E = \arg \max_G \{s^i(k^1, k^2)\} \cup \arg \min_G \{s^i(k^1, k^2)\} \quad i = 1, 2$$

Because of 2.11, the exchange rate $s^i(k^1, k^2)$ will hit the edges of the band if and only if $(k^1, k^2) \in E$, for $i = 1, 2$. Also, because of 2.10 and by Itô's lemma, the exchange rate will not be a regulated process. This implies that when the fundamentals hit E , the exchange rate becomes locally predictable⁹, and therefore foreign bonds become locally riskless. Naturally, this implies that the boundary conditions imposed in Proposition 2 must be compatible with the requirement that the exchange rate risk premium must vanish at the boundaries of the target zone.

Denote B_t^i the price process for bank accounts denominated in units of consumption good for $i = 0, 1, 2$, i.e., $B_t^i = \frac{\hat{B}_t^i}{P_t^i}$ for $i = 1, 2, 0$ where \hat{B}_t^i is the nominal price process for bank accounts denominated in units of currency $i = 1, 2, 0$, that grow at instantaneously compounded nominal rates r_t^i . Without loss of generality, normalize $P_t^0 = 1$. Observe that, by construction, \hat{B}_t^0 is already denominated in consumption units and therefore r_t^0 coincides with the instantaneous real interest rate in our economy. Under these conditions, markets are complete and a unique equivalent martingale measure (in the sense of Harrison and Kreps (1979)) exists. In other words, because the traded assets (instantaneous bonds) become locally riskless when the state variable process hits E , the drift of the foreign bonds B_t^i measured in domestic currency units, and therefore that of s_t^i , must be the same at the edges of the band both under the natural and the equivalent martingale measures¹⁰.

The above discussion implies that the “smooth-pasting” conditions must then be as follows:

⁹Here I use the term “locally predictable” not in the mathematical definition of predictable process (i.e. measurable w.r.t. the σ -algebra generated by left continuous processes) but rather in the sense that the expected rate of change of the process is equal (a.s. and in \mathcal{L}^2 -norm) to the actual rate of change of the process. That is to say, the exchange rate process is not subject to any Wiener shock when the fundamentals hit E .

¹⁰If the world is not risk neutral $\mathcal{L} \neq \mathcal{L}^*$ (where \mathcal{L}^* is the characteristic operator of the diffusion process for k under an equivalent martingale measure). Precisely \mathcal{L} and \mathcal{L}^* differ only on the term multiplying the gradient of their argument.

$$\nabla' s^i(k^1, k^2) \Phi(k^1, k^2) = 0 ; i = 1, 2, 12; \quad \forall (k^1, k^2) \in E \quad (2.14)$$

where $\Phi : \Omega \times [0, \infty) \rightarrow \mathcal{R}^2$ is the equilibrium risk premium process for the state variables (k^1, k^2) in \bar{G} .

Note that, in general, the risk premium process Φ will not vanish, not even at E ¹¹. Since the risk premium process can be constructed arbitrarily and is always nonzero, for 2.22 to hold we must have:

$$\nabla' s^i(k^1, k^2) = 0 ; i = 1, 2; \quad \forall (k^1, k^2) \in E \quad (2.16)$$

That 2.16 must be satisfied is an economic restriction that has to be met by the solution to 2.9.

To see that conditions 2.10 imply 2.16, I conjecture the following:

Conjecture 1: The solution to 2.9 subject to 2.10 and 2.11 reaches its extrema at the vertices of G .

Later we will solve the problem explicitly and we will confirm that this is indeed the case. Now, if the solution to the above partial differential equation subject to 2.10 is monotone in both arguments and the extrema of g (or s) on G lie on the vertices, and then it is immediate to see that 2.10 implies 2.16 given that M is nonsingular by assumption.

The above discussion leads to an additional restriction related to the cross-currency exchange rate between currencies 1 and 2. Since s^{12} is bounded, by the same reasons

¹¹This is because the quadratic covariation terms in 2.15 below are always nonzero. The regulator process Λ vanishes when we apply the covariation process operator because it is a continuous process of bounded variation. The risk premium process cannot be obtained, though, without further structure in the model. If we assume that the economy considered admits a representative agent, with the maintained assumptions so far we can apply theorem 2 of Cox, Ingersoll and Ross (1985) to deduce that Φ has the form

$$\Phi'(k^1, k^2) = \frac{-J_{WW}}{J_W} d[W, k']_t - \frac{J'_{Wk}}{J_W} d[k, k]_t = \frac{-J_{WW}}{J_W} d[W, k]_t - \frac{J'_{Wk}}{J_W} AA' dt \quad (2.15)$$

for $i = 1, 2$ where J is the indirect utility function for the representative agent and W is aggregate wealth.

as above to avoid arbitrage opportunities we must have

$$\begin{aligned} \nabla s^{12}(k^1, k^2) &= 0 \quad \forall (k^1, k^2) \in E' \\ E' &= \arg \max_{(k^1, k^2) \in G} \{s^{12}(k^1, k^2)\} \cup \arg \min_{(k^1, k^2) \in G} \{s^{12}(k^1, k^2)\} \\ s^{12}(k_1, k_2) &= s^1(k_1, k_2) - s^2(k_1, k_2) \end{aligned} \quad (2.17)$$

2.17 constitutes an additional boundary condition to be satisfied by any candidate solution to the trilateral target zone modelling problem. Note that we arrived at this condition by purely economic arguments and, unlike 2.16 under Conjecture 1, this condition is not embedded in the boundary conditions 2.10 and 2.11 of the analytical problem 2.9 of Proposition 2. This is because the arguments presented in proposition 2 are essentially bivariate (in the space of the exchange rate) and the above no arbitrage condition $s^{12} = s^1 - s^2$ was not taken into account.

I have shown that the boundary conditions 2.10 and 2.11 imposed in the partial differential equations 2.9 actually meet a necessary economic consistency condition. An additional necessary condition 2.17 to be satisfied by any candidate solution to the trilateral target zone modelling problem has also been identified. However, I still have not proposed an economic interpretation for the conditions 2.10 on their whole domain (i.e. the whole set ∂G , not only the vertices of G). This is done next, where I solve the analytical problem 2.9 and 2.10 using a factorization technique that allows an economic identification of the boundary conditions 2.10 by indirect methods.

2.3.2 Transforming the problem

The strategy that I follow consists of changing the space of state variables to a new space such that, with the new state variables, the exchange rate function solves the analytical problem 2.9 with 2.10 and 2.11 derived from the probabilistic problem 2.3. Basically, we want to construct a bivariate stochastic process such that there exists a one-to-one mapping between the fundamentals process and such an auxiliary process. For reasons that will become clear soon, we want this process to have a domain that is a rectangle in \mathcal{R}^2 . We also want this process to behave like an arithmetic Brownian

motion in the interior of its domain, and to be reflected at the boundaries with an orthogonal reflection matrix. We also want such process to have the strong Markov property. Under these conditions, we will be able to write the exchange rate as a function of this process, at any time. Since our auxiliary process is Markov, such function will solve 2.9 subject to 2.10 and 2.11. The domain of the auxiliary process will be chosen in such a way that a simple closed form solution meeting the relevant restrictions arises. After that, the solution will be mapped from the auxiliary space to the original space of the fundamentals. Also, the domain for the fundamentals will be constructed by mapping back the domain for the auxiliary process.

Recall $M = [R^1 : R^2]$ ($M \in \mathcal{R}^{2 \times 2}$ assumed to be nonsingular). Define two auxiliary stochastic processes on the space $(\Omega, \mathcal{B}, \{\mathcal{B}\}_{t \geq 0}, P)$, X and Y , as follows:

$$\begin{pmatrix} X_t \\ Y_t \end{pmatrix} = \begin{pmatrix} \mu_X \\ \mu_Y \end{pmatrix} t + B \begin{pmatrix} W_1(t) \\ W_2(t) \end{pmatrix} + \begin{pmatrix} \Lambda_t^1 - \Lambda_t^3 \\ \Lambda_t^2 - \Lambda_t^4 \end{pmatrix} \quad (2.18)$$

where (W_1, W_2) is the same Wiener process as before, and $(\mu_X, \mu_Y)' = M^{-1}\mu$, $B = M^{-1}A$. Obviously:

$$\begin{pmatrix} k_t^1 \\ k_t^2 \end{pmatrix} = M \begin{pmatrix} X_t \\ Y_t \end{pmatrix} \quad (2.19)$$

a.s. (P^{k_0}) for $(k^1, k^2) \in G$ and obviously since (k^1, k^2) is a Markov process so is (X, Y) . Denote $M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$. For a certain function of the pair (X, Y) , we can certainly make the following identification, for $i = 1, 2$,

$$s^i(k^1, k^2) =: e^i(X, Y) \quad (2.20)$$

Applying Itô's lemma to 2.20, and in view of 3.27 and 4.19, our boundary value problem 2.6 with 2.7 becomes:

$$\begin{aligned} & \gamma \left(\mu_X \frac{\partial}{\partial X} e^i + \mu_Y \frac{\partial}{\partial Y} e^i + \frac{\sigma_X^2}{2} \frac{\partial^2}{\partial X^2} e^i + \frac{\sigma_Y^2}{2} \frac{\partial^2}{\partial Y^2} e^i + \sigma_X \sigma_Y \rho_{XY} \frac{\partial^2}{\partial X \partial Y} e^i \right) + \\ & + m_{i1} X + m_{i2} Y = e^i(X, Y) \end{aligned} \quad (2.21)$$

where

$$\begin{aligned} \begin{pmatrix} \sigma_X^2 \\ \sigma_Y^2 \end{pmatrix} &= \text{diag}(M^{-1}AA'M^{-1}) \\ \begin{pmatrix} \mu_X \\ \mu_Y \end{pmatrix} &= M^{-1} \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \\ \rho_{XY} &= B_{11}B_{21} - B_{12}B_{22} \end{aligned}$$

At this point, we will let (X, Y) vary within a rectangle in \mathcal{R}^2 ; that is to say, $(X, Y) \in S = [\underline{X}, \bar{X}] \times [\underline{Y}, \bar{Y}]$. Later we will determine the appropriate vertices for the quadrilateral G such that the above rectangle S is obtained by mapping G with the transformation 2.19, i.e., $S = \{(x, y) \in \mathcal{R}^2 \mid \exists(a, b) \in G \ni: (x, y)' = M(a, b)'\}$.

We will solve 2.21 by separation of variables. We will impose that e^i be additively separable. I turn now to the economic characterization of its boundary conditions.

Boundary Conditions for the Transformed Problem

To impose the boundary conditions to 2.21 I use the same arbitrage argument as that in subsection 3.1.2. I continue to conjecture that s will reach its extrema at the vertices. To rule out arbitrage opportunities, and in the same line of reasoning as that of subsection 3.1.2, we must have:

$$\begin{aligned} \nabla e^i(X^*, Y^*) &= 0 \quad \forall (X^*, Y^*) \in E^* \\ E^* &= \{x \in \mathcal{R}^2 \mid \exists a \in E \subset \mathcal{R}^2 \ni: x = Ma\} \end{aligned} \tag{2.22}$$

for example, if we assume $[m]_{ij} > 0$ then by conjecture 1 $M(\bar{X}, \bar{Y})' = (\bar{k}_1, \bar{k}_2)$ and $M(\underline{X}, \underline{Y})' = (\underline{k}_1, \underline{k}_2)$ for $\underline{X} < \bar{X}$, $\underline{Y} < \bar{Y}$, $\underline{k}_1 < \bar{k}_1$ and $\underline{k}_2 < \bar{k}_2$ and therefore condition 2.22 reads $\nabla e^i(\bar{X}, \bar{Y}) = \nabla e^i(\underline{X}, \underline{Y}) = 0$.

Also in the same line of reasoning as that of subsection 3.1.2, 2.22 holds necessarily since the risk premium process for (X, Y) does not vanish at the vertices where $e(X, Y)$ reaches an extremum¹².

¹²Let Ψ denote the risk premium process for the transformed state variables (X, Y) on S . Then

However, as (X, Y) moves precisely in a rectangle S , and given that we have imposed that e be additively separable in (X, Y) , 2.22 implies that at least one partial derivative vanishes at the sides of S . Thus our boundary conditions on ∂S read:

$$\frac{\partial e^i(\bar{X}, Y)}{\partial X} = \frac{\partial e^i(\underline{X}, Y)}{\partial X} = \frac{\partial e^i(X, \bar{Y})}{\partial Y} = \frac{\partial e^i(X, \underline{Y})}{\partial Y} = 0 \quad (2.23)$$

for $i = 1, 2$. The relevant boundary conditions for the boundary value problem of subsection 3.1.2., namely 2.10, follow from the conditions 2.23 and the transformation 2.19. To see this, note from 2.19 that

$$\nabla' e^i(X, Y) = \nabla' s^i(k^1, k^2)M \quad (2.24)$$

Now because of 2.23 and the fact that, by construction, the domain S of (X, Y) is a rectangle, 2.24 implies that $\nabla' s^i(k^1, k^2)R_j = 0$ where R_j is the $j = 1, 2, 3, 4$ column vector of the reflection matrix $R = [M : -M]$. Therefore, by absence of arbitrage, the relevant boundary conditions for the transformed problem are precisely that the gradient of the exchange rate as a function of (k^1, k^2) be orthogonal to the reflection vector at each side of the quadrilateral G , precisely as we stated in Proposition 2 by essentially analytical reasons. We have thus obtained the same relevant boundary conditions through an economic argument¹³. The conditions above are the equivalent of the "smooth-pasting"¹⁴ conditions of Krugman (1991) in a multidimensional context.

Ψ can only vanish iff Φ does. In particular $\Psi > 0$ at the vertices of the rectangle S if $\Phi \neq m_1, m_2$ where m_1, m_2 are the column vectors of M . To see this, note that $\Psi(X, Y) = M^{-1}\Phi(k_1, k_2)$ where Φ is the risk premium process for (k_1, k_2) and then

$$\begin{aligned} \Psi'(X, Y)dt &= \frac{-J_{Ww}}{J_w} d[W, (X, Y)]_t - \frac{J'_{W(\varepsilon, \nu)}}{J_w} d[(X, Y), (X, Y)]_t = \\ &= \frac{-J_{Ww}}{J_w} d[W, k' M^{-1}]_t - \frac{J'_{Wk} M}{J_w} M^{-1} d[k, k]_t M^{-1} = \Phi'(k_1, k_2)M^{-1} dt \end{aligned}$$

¹³Dumas (1992) obtained similar boundary conditions (i.e. orthogonality of the gradient to the reflection vector field) for a control problem involving a bidimensional regulated brownian motion on a planar region through a completely different procedure.

¹⁴Or, more properly, "value matching" conditions. See Dumas (1991).

2.3.3 Solving the problem

Now I solve 2.21 subject to 2.23 by separation of variables. Recall that we imposed e^i to be additively separable, i.e. $e^i(X, Y) = C_i(X) + D_i(Y)$. This reduces 2.21 to the following differential equations:

$$\begin{aligned}\gamma\mu_X C_i'(X) + \frac{\gamma\sigma_X^2}{2} C_i''(X) + m_{i1}X &= C_i(X) \\ \gamma\mu_Y D_i'(Y) + \frac{\gamma\sigma_Y^2}{2} D_i''(Y) + m_{i2}Y &= D_i(Y)\end{aligned}\quad (2.25)$$

with boundary conditions $C_i'(\bar{X}) = C_i'(\underline{X}) = D_i'(\bar{Y}) = D_i'(\underline{Y}) = 0$ for $i = 1, 2$. The final solution for $e^i(X, Y)$ is:

$$\begin{aligned}e^i(X, Y) &= \gamma(m_{i1}\mu_X + m_{i2}\mu_Y) + m_{i1}X + m_{i2}Y + K_1^i \exp(\lambda_+^1 X) + K_2^i \exp(\lambda_-^1 X) + \\ &+ K_3^i \exp(\lambda_+^2 Y) + K_4^i \exp(\lambda_-^2 Y)\end{aligned}\quad (2.26)$$

$i = 1, 2$, where

$$\begin{aligned}\lambda_+^1 &= \frac{-\mu_X + \sqrt{\mu_X^2 + 2\frac{\sigma_X^2}{\gamma}}}{\sigma_X^2} & \lambda_+^2 &= \frac{-\mu_Y + \sqrt{\mu_Y^2 + 2\frac{\sigma_Y^2}{\gamma}}}{\sigma_Y^2} \\ \lambda_-^1 &= \frac{-\mu_X - \sqrt{\mu_X^2 + 2\frac{\sigma_X^2}{\gamma}}}{\sigma_X^2} & \lambda_-^2 &= \frac{-\mu_Y - \sqrt{\mu_Y^2 + 2\frac{\sigma_Y^2}{\gamma}}}{\sigma_Y^2}\end{aligned}$$

$$\begin{pmatrix} \sigma_X^2 \\ \sigma_Y^2 \end{pmatrix} = \text{diag}(M^{-1}AA'M^{-1})$$

$$\begin{pmatrix} \mu_X \\ \mu_Y \end{pmatrix} = M^{-1} \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}$$

The constants $K_1^i, K_2^i, K_3^i, K_4^i$ and $\underline{X}, \bar{X}, \underline{Y}, \bar{Y}$ are determined as the solution of the following system of equations:

$$\begin{aligned}m_{i1} + \lambda_+^1 K_1^i \exp\{\lambda_+^1 \bar{X}\} + \lambda_-^1 K_2^i \exp\{\lambda_-^1 \bar{X}\} &= 0 \\ m_{i1} + \lambda_+^1 K_1^i \exp\{\lambda_+^1 \underline{X}\} + \lambda_-^1 K_2^i \exp\{\lambda_-^1 \underline{X}\} &= 0 \\ m_{i2} + \lambda_+^2 K_3^i \exp\{\lambda_+^2 \bar{Y}\} + \lambda_-^2 K_4^i \exp\{\lambda_-^2 \bar{Y}\} &= 0 \\ m_{i2} + \lambda_+^2 K_3^i \exp\{\lambda_+^2 \underline{Y}\} + \lambda_-^2 K_4^i \exp\{\lambda_-^2 \underline{Y}\} &= 0\end{aligned}\quad (2.27)$$

$$\begin{aligned}
e^1(\bar{X} - 1_{(m_{i1} < 0)}(\bar{X} - \underline{X}), \bar{Y} - 1_{(m_{i2} < 0)}(\bar{Y} - \underline{Y})) &= \log(\bar{S}^1) \\
e^1(\underline{X} + 1_{(m_{i1} < 0)}(\bar{X} - \underline{X}), \underline{Y} + 1_{(m_{i2} < 0)}(\bar{Y} - \underline{Y})) &= \log(\underline{S}^1) \\
e^2(\bar{X} - 1_{(m_{i1} < 0)}(\bar{X} - \underline{X}), \bar{Y} - 1_{(m_{i2} < 0)}(\bar{Y} - \underline{Y})) &= \log(\bar{S}^2) \\
e^2(\underline{X} + 1_{(m_{i1} < 0)}(\bar{X} - \underline{X}), \underline{Y} + 1_{(m_{i2} < 0)}(\bar{Y} - \underline{Y})) &= \log(\underline{S}^2)
\end{aligned} \tag{2.28}$$

The boundary conditions in 2.28 are justified by the fact that our solution $e^i(X, Y)$ is monotone in X (in $[\underline{X}, \bar{X}]$) and Y (in $[\underline{Y}, \bar{Y}]$). For example, if $m_{i1}, m_{i2} > 0$, for $i = 1, 2$, then $E^* = \{(\underline{X}, \underline{Y}), (\bar{X}, \bar{Y})\}$. The following lemma verifies this monotonicity result (and therefore our conjecture 1 is verified *a fortiori*):

Lemma 1: $e^i(X, Y)$ is monotonically increasing (decreasing) in $X \in [\underline{X}, \bar{X}]$ if $m_{i1} > 0$ ($m_{i1} < 0$) and monotonically increasing (decreasing) in $Y \in [\underline{Y}, \bar{Y}]$ if $m_{i2} > 0$ ($m_{i2} < 0$).

By lemma 1, the upper and lower limits for each exchange rate e^i will be reached at the vertices of $S = [\underline{X}, \bar{X}] \times [\underline{Y}, \bar{Y}]$ for $i = 1, 2$. Thus, by construction, the boundary conditions automatically impose that $(X, Y) \in [\underline{X}, \bar{X}] \times [\underline{Y}, \bar{Y}] \sup\{e^i(X, Y)\} = \log(\bar{S}^i)$ and $(X, Y) \in [\underline{X}, \bar{X}] \times [-Y, \bar{Y}] \inf\{e^i(X, Y)\} = \log(\underline{S}^i)$, $i = 1, 2$ as desired. Note also that, because of 2.19, $e(X, Y)$ reaches its extrema at the vertices of the (X, Y) -domain S if and only if $s(k^1, k^2)$ reaches its extrema at the vertices of the (k^1, k^2) -domain G . In addition, if $e^{12}(X, Y) = e^1(X, Y) - e^2(X, Y)$ reaches its extrema at $\overset{\circ}{S}$, then clearly $\nabla e^{12}(X, Y) = 0$. Now if $e^{12}(X, Y)$ reaches its extrema at ∂S , a straightforward modification of the proof of lemma 1 indicates that the extrema must lie on the vertices of ∂S , and so by 2.23, $\nabla e^{12}(X, Y) = 0$ at $(X, Y) \in E' = \{\arg \max_S\{e^{12}(X, Y)\} \cup \arg \min_S\{e^{12}(X, Y)\}\}$ and therefore the boundary condition 2.17 is satisfied by the solution 2.26 on the whole domain S .

Once the problem has been solved in the domain of (X, Y) , we can solve for the original range of the fundamentals by finding the vertices of the quadrilateral G using the transformation 2.19 and the vertices of the rectangle $S = [\underline{X}, \bar{X}] \times [\underline{Y}, \bar{Y}]$. Thus we can express the exchange rate as a function of the underlying fundamentals:

Theorem 1: The log-exchange rate process as a function of the fundamentals in

a trilateral target zone without active cross-currencies constraints is given by:

$$s^i(k^1, k^2) = \gamma\mu_i + k^i + K_1^i \exp\left(\lambda_+^1(v_{11}k^1 + v_{12}k^2)\right) + K_2^i \exp\left(\lambda_-^1(v_{11}k^1 + v_{12}k^2)\right) + K_3^i \exp\left(\lambda_+^2(v_{21}k^1 + v_{22}k^2)\right) + K_4^i \exp\left(\lambda_-^2(v_{21}k^1 + v_{22}k^2)\right) \quad (2.29)$$

for $i = 1, 2$, and $M^{-1} = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix}$ where the constants K_j^i and λ_+^i, λ_-^i $j = 1, 2, 3, 4$, $i = 1, 2$ are determined above in 2.27.

Proof: Use 2.19 to rewrite 2.26 in terms of the fundamentals. \square

2.3.4 The Nature of the Solution

Note that the structure of the solution looks remarkably similar to the solution for the traditional bilateral model. In particular, it has the same form for the linear part as Krugman's solution. The difference lies in the nonlinear part. The exponential terms contain a linear combination of the fundamentals of the countries in the target zone weighted by the inverse of the reflection matrix, and there are two of them more, respective to the bilateral case. The solution 2.29 also preserves the characteristic *S*-shaped graph of the solution of the traditional model. Its distributional implications, however, will be shown to be very different.

In particular, note that for M symmetric (take it to be the 2×2 identity matrix), we will have $v_{12} = v_{21} = K_3^i = K_4^i = 0$ and 2.29 reduces to Krugman's solution. This is indeed not surprising: given that if active cross-currency constraints are ruled out by assumption, the only link among two currencies and third currencies in the target zone will be the common rules of intervention as expressed by the reflection matrix M . Thus, intervention spillovers occur only when the reflection matrix M is not diagonal. Furthermore, we will see in subsection 4.2 that when cross-currency constraints matter, there are additional nonlinear sources of discrepancy of the exchange rate solution relative to Krugman's model.

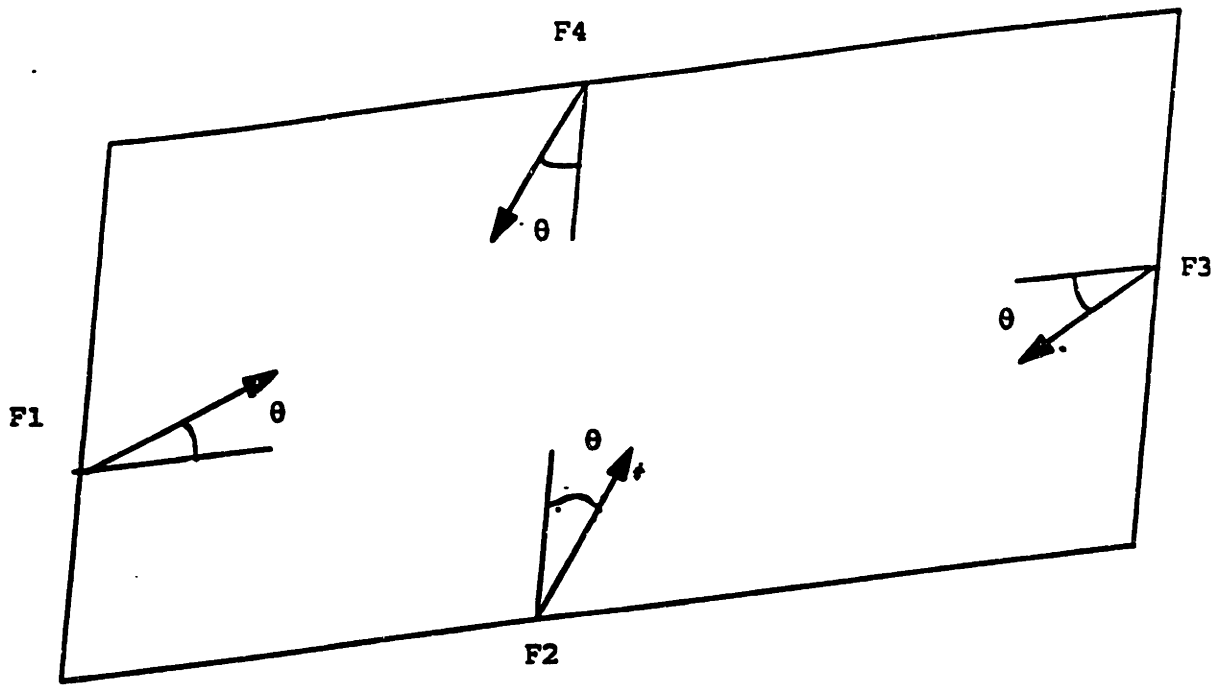


Figure 1: Fundamentals Domain

Fig. 2: $m_1=0.75$ $m_2=0.25$

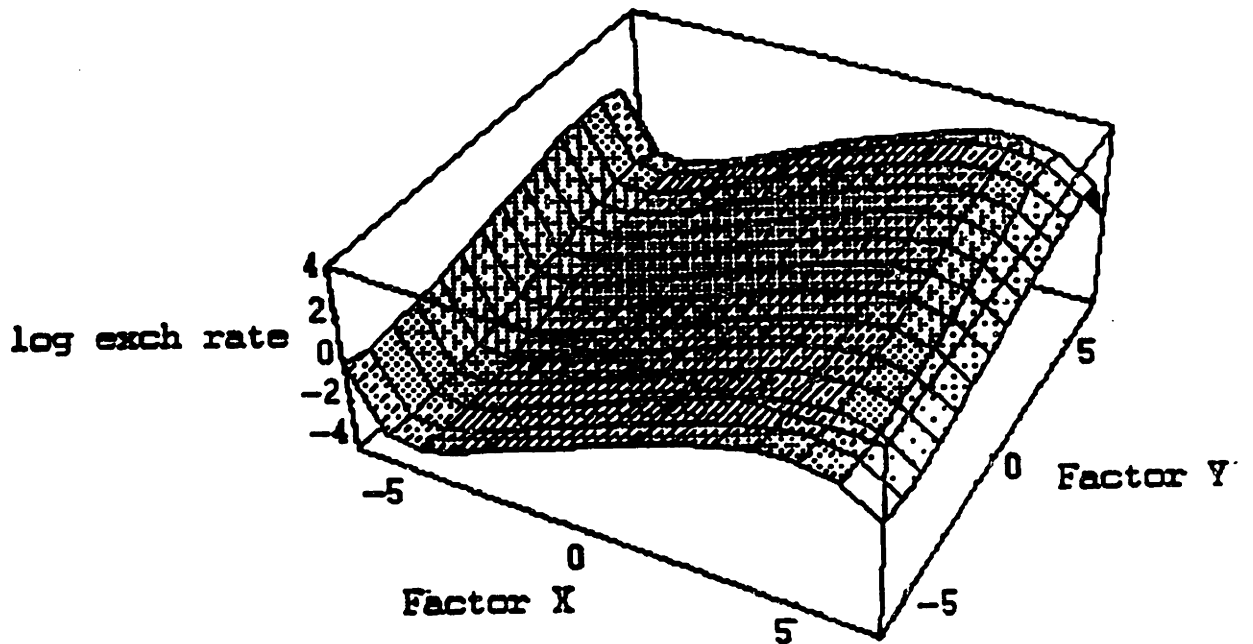


Fig.3:m1=0.5 m2=0.5

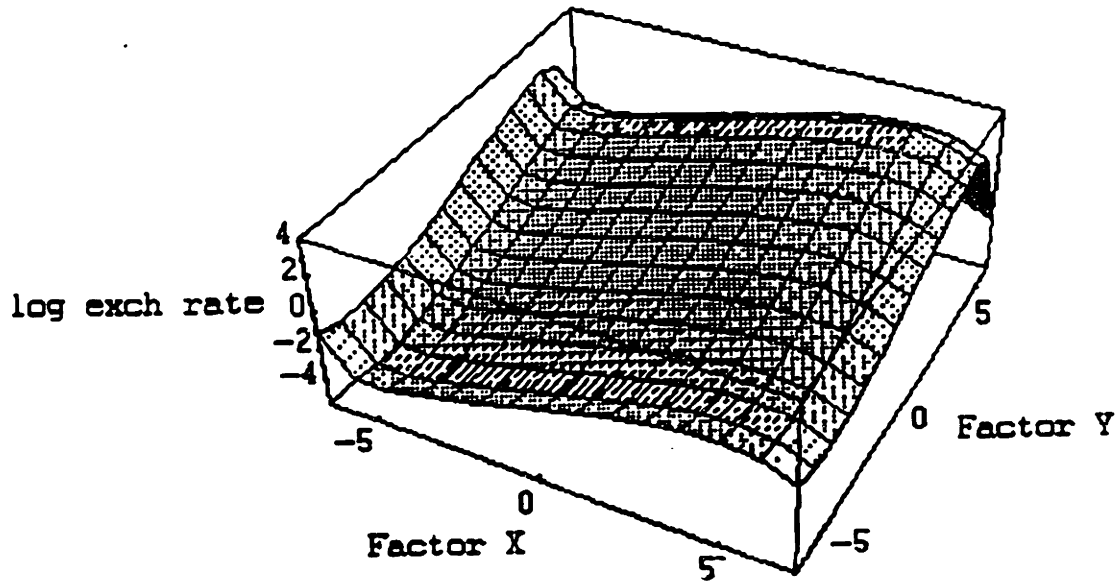
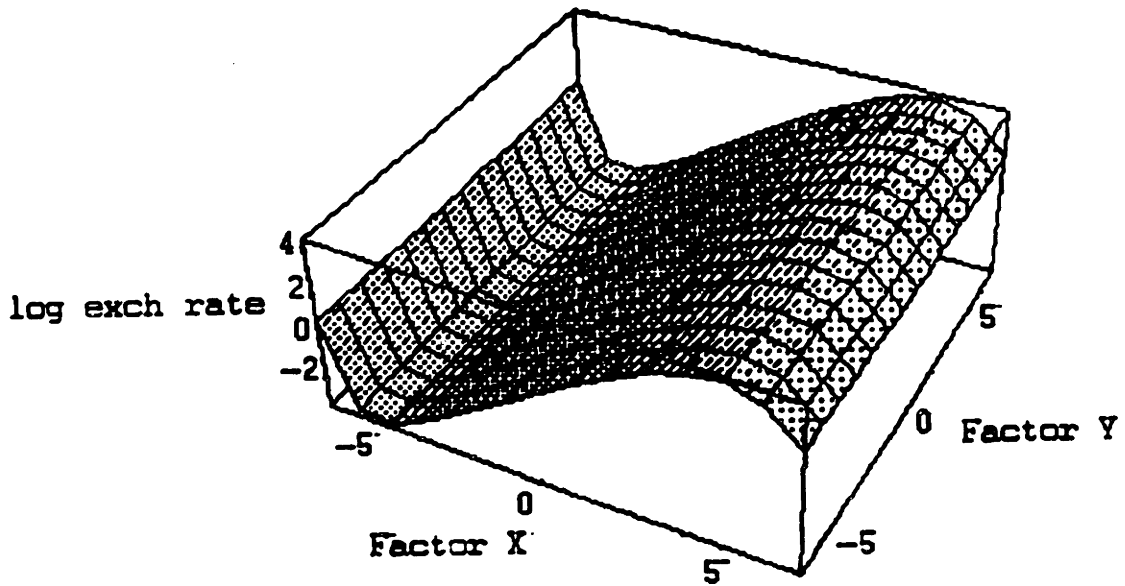


Fig.4:m1=1 m2=0



In addition, contrary to Krugman's model, it is no longer true that the conditional volatility of the exchange rate is always less than the conditional volatility of the underlying fundamentals. It is immediate to see this in Krugman's case (we set M to be diagonal in 2.26). Consider the norm $\|A\|_M = [tr(AA')]^{\frac{1}{2}}$ in the space of real matrices. $\|A\|_M$ expresses the sum of the instantaneous variances of a diffusion process with diffusion matrix A . In our case the instantaneous rate of change in the conditional variance is, by Itô's lemma, the conditional variance of the exchange rate process is:

$$\|\nabla e(X, Y)M^{-1}A\|_M^2 = e^{1'}(X)^2 \|a_1\|^2 + e^{2'}(Y)^2 \|a_2\|^2 \leq \|a_1\|^2 + \|a_2\|^2 = \|A\|_M^2 \quad (2.30)$$

where a_i denotes the i^{th} column vector of A , so that $\|a_i\|^2$ consequently denotes the instantaneous variance of the first component of the fundamentals vector. The equality follows from the assumption that M is the 2×2 identity matrix, (we specialize our model to Krugman's model), and the weak inequality follows from the fact that $e^{i'}(j) \in [0, 1]$ for $i = 1, 2$ and $j = X, Y$, as can be easily deduced from the proof of lemma 1. If M is not diagonal, and thus the model does not collapse to Krugman's, this is no longer necessarily true. In this case, the volatility of the exchange rate can be larger than the one of the underlying fundamentals process somewhere since $\|\nabla e(X, Y)M^{-1}A\|_M^2$ is not necessarily bounded above by the instantaneous variance of the fundamentals process, $\|A\|_M^2$, (although it is so, of course, at the edges of the band for the exchange rate).

What is then the nature of the so called "stabilization effect" on the exchange rate of a target zone regime? In the bilateral model, this effect is evidenced by a slope which is less than one, if we plot the exchange rate against the fundamentals process. This is equivalent to the fact that the exchange rate process exhibits less conditional volatility than the fundamentals process. We see now that the "stabilization effect" must correspond purely to the "honeymoon effect", or the fact that the conditional volatility of the exchange rate vanishes when the exchange rate approaches the edges of its band. However, the exchange rate can be more volatile, when in the interior

of the band, than the fundamentals process, or equivalently, than the exchange rate under a free-float regime with the same underlying fundamentals.

Figures 2 to 4 plot the exchange rate as a function of the transformed state variables X and Y . For ease of presentation¹⁵, the plots of the exchange rate are presented in the (X, Y) space rather than in the (k^1, k^2) space. In all of them the following parameters have been used: $\gamma = 1, \mu = (1, -1)'$, and a diffusion matrix $A = \begin{pmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{pmatrix}$. This matrix corresponds roughly to an instantaneous standard deviation of 0.8 for both factors X, Y and a correlation coefficient of 0.6. Three cases are plotted corresponding to different assumptions about the share of the intervention burden among central banks (the matrix M). In Figures 2 and 3 we recognize in the plotted surface the characteristic Krugman's \mathcal{S} -shape when we let X (Y) vary for any fixed Y (X). The two-dimensional S-shaped surface is more pronounced when the burden of intervention falls mainly on country 0, both in front of currencies 1 and 2.

In contrast, when the boundary of intervention falls entirely on the monetary authorities of countries 1 and 2 in the management of their respective bands against currency 0, as in Figure 4, the exchange rate solution collapses to that of Krugman (1991). This is because the exchange rate becomes insensitive to one of the state variables, and the system becomes *de facto* a combination of two independent bilateral target zones, with our maintained assumption of inactive cross-currency constraints. The reason for this is simply that the manipulation of the money supply in the management of the band between currency 1 and currency 0 never affects the fundamentals of country 2 versus country 0 because of cross-monetary effects. This same result is compatible with the situation in which country 1 (country 2) sterilizes completely the effects of the interventions of country 2 (country 1) on the fundamental k^1 (k^2). Note that this result is independent of whether the fundamentals in countries 1 and 2 are correlated or not. This should come as no surprise in view of the second part of assumption 2.

To summarize, there are two conditions under which the features of a multilateral target zone differ essentially from those on a bilateral target zone: First, the exis-

¹⁵It is easier to plot a surface on a rectangle rather than on a quadrilateral.

tence of active cross-currency constraints that place constraints on the range of the bivariate process for the fundamentals. Second, the fact that intervention of country 0 for maintenance purposes of its band with country 1 will have an impact on the fundamentals with country 2. The first condition was ruled out by assumption 2, and in the case depicted in Figure 3 precisely the second condition is neutralized, and thus Krugman's original solution is obtained. We have seen that the previous second condition basically "added" additional nonlinear terms to Krugman's solution. Now I turn to the relaxation of assumption 5 to take the first condition into account.

2.4 General Solution for Multilateral Target Zones

2.4.1 $n+1$ Currencies, Inactive Cross-currency Constraints

The extension to a model of an arbitrary number of currencies and bands of arbitrary size will be obtained by first extending theorem 1 to a multicurrency context with the assumption that the cross-currencies constraint are nowhere active (as in assumption 5). In other words, we seek for a formula that expresses the exchange rate between currency i ($i = 1, 2, \dots, n$) and currency 0 as a function of an n -dimensional vector of fundamentals, and the band between currency i and currency j is at least as wide as $[\frac{S^i}{S^j}, \frac{\bar{S}^i}{\bar{S}^j}]$ for $i \neq j$, $i, j = 1, 2, \dots, n - 1$ where S^i expresses the # of units of currency i per unit of currency 0. In the next subsection, it will be shown that the case where the cross-currencies constraint is active can be reduced to this case.

In the general n -lateral case the uncertainty is driven by an n -dimensional Wiener process. The fundamentals is an \mathcal{R}^n -valued process K that behaves like an arithmetic brownian motion with drift vector $\mu \in \mathcal{R}^n$ and instantaneous covariance matrix $AA' \in \mathcal{R}^{n \times n}$ in the interior of a convex polyhedral domain in \mathcal{R}^n , and is reflected instantaneously on each of the $2n$ closed subsets of an $(n - 1)$ -dimensional Euclidean space that constitute the boundary of such domain. The direction of reflection is given by a matrix $R \in \mathcal{R}^{n \times 2n}$ which will be assumed to satisfy the symmetry condition $R = [M : -M]$ for some nonsingular matrix $M \in \mathcal{R}^{n \times n}$. As before, this

restriction reflects the assumption that the intervention rules are symmetric. Under these circumstances the partial differential equation 2.21 becomes:

$$e^i(x_1, x_2, \dots, x_n) = m_i X + \gamma \left(\nabla e^i(X) \mu_X + \frac{1}{2} \text{tr} [H_e B B'] \right) \quad (2.31)$$

where $K = MX$, $X' = (x_1, \dots, x_n)$, $e(X) := s(K)$, m^i is the i^{th} row vector of M , H_e is the Hessian matrix of $e(X)$, $B = M^{-1}A$ and $\mu_X = M^{-1}\mu$.

The equivalent boundary conditions for 2.31 here are given by $\frac{\partial e^i(X)}{\partial X_i} \Big|_{X_i \in \{\underline{X}_i, \bar{X}_i\}} = 0$ $\forall i = 1, 2, \dots, n$, as we had seen in proposition 2. As before, we will let X vary within a rectangle in \mathcal{R}^n , and we will map out the vertices of the polyhedron for K using the linear transformation $K = MX$. It is then immediate to obtain:

Theorem 2: The log-exchange rate process as a function of the fundamentals in a $(n + 1)$ -country target zone without active cross-currencies constraints is given by:

$$s^i(k^1, \dots, k^n) = \gamma \mu_i + k^i + \sum_{j=1}^n C_{1j}^i \exp \left(\lambda_+^j \left(l \stackrel{n}{=} 1 v_l^{(j)} k^l \right) \right) + \sum_{j=1}^n C_{2j}^i \exp \left(\lambda_-^j \left(l \stackrel{n}{=} 1 v_l^{(j)} k^l \right) \right) \quad (2.32)$$

where $M^{-1} = [v_{ij}]$ and we take $v^{(j)}$ to be the j^{th} row vector from M^{-1} . The constants λ_+^j, λ_-^j $j = 1, \dots, n$ are given by:

$$\lambda_+^j = \frac{-\theta_j + \sqrt{\theta_j^2 + 2 \frac{\phi_j^2}{\gamma}}}{\phi_j^2} \quad \lambda_-^j = \frac{-\theta_j - \sqrt{\theta_j^2 + 2 \frac{\phi_j^2}{\gamma}}}{\phi_j^2} \quad (2.33)$$

where $\theta = M^{-1}\mu$ and $\phi \odot \phi = \text{diag} (M^{-1}AA'M^{-1})$ are $n \times 1$ vectors, $[v_{ij}] = [M^{-1}]_{ij}$, and where \odot indicates member-wise multiplication. The constants C_{1j}^i, C_{2j}^i $j = 1, \dots, n$ are obtained as the solution of a $2n$ -dimensional system similar to 2.27:

$$\begin{aligned} m_{ij} + \lambda_+^j C_{1j}^i \exp \left(\lambda_{1j} \left(\sum_{l=1}^n v_l^{(j)} \bar{k}^l \right) \right) + \lambda_-^j C_{2j}^i \exp \left(\lambda_{2j} \left(\sum_{l=1}^n v_l^{(j)} \bar{k}^l \right) \right) &= 0 \\ m_{ij} + \lambda_+^j C_{1j}^i \exp \left(\lambda_{1j} \left(\sum_{l=1}^n v_l^{(j)} \underline{k}^l \right) \right) + \lambda_-^j C_{2j}^i \exp \left(\lambda_{2j} \left(\sum_{l=1}^n v_l^{(j)} \underline{k}^l \right) \right) &= 0 \end{aligned} \quad (2.34)$$

and lastly $\bar{K} = (\bar{k}_1, \dots, \bar{k}_n)$, $\underline{K} = (\underline{k}_1, \dots, \underline{k}_n)$, together with the auxiliary variables

vectors $(\bar{X}_1, \bar{X}_2, \dots, \bar{X}_n)$ and $(\underline{X}_1, \underline{X}_2, \dots, \underline{X}_n)$, solve the $4 \times n$ system of equations:

$$\begin{aligned}
s^i \left(M \left[\bar{X}_1 - 1_{(m_{i,1} < 0)}(\bar{X}_1 - \underline{X}_1), \dots, \bar{X}_n - 1_{(m_{i,n} < 0)}(\bar{X}_n - \underline{X}_n) \right] \right) &= \log(\bar{S}^i) \\
s^i \left(M \left[\underline{X}_1 + 1_{(m_{i,1} < 0)}(\bar{X}_1 - \underline{X}_1), \dots, \underline{X}_n + 1_{(m_{i,n} < 0)}(\bar{X}_n - \underline{X}_n) \right] \right) &= \log(\underline{S}^i) \\
(\bar{k}_1, \bar{k}_2, \dots, \bar{k}_n)' &= M(\bar{X}_1, \bar{X}_2, \dots, \bar{X}_n)' \\
(\underline{k}_1, \underline{k}_2, \dots, \underline{k}_n)' &= M(\underline{X}_1, \underline{X}_2, \dots, \underline{X}_n)' \\
i &= 1, 2, \dots, n
\end{aligned} \tag{2.35}$$

Proof: Proceed as above: after factorizing from K on a polyhedron to X in a n -dimensional rectangle with orthogonal reflection at the boundaries, solve 2.31 by separation of variables imposing the no-arbitrage boundary constraints and use lemma 1 to impose the boundary conditions 3.12. Then map the solution back to the domain for K . \square

2.4.2 Active Cross-currency Constraints

Preliminary Discussion

In this subsection I will consider a trilateral target zone as in the previous section, but unlike section 3, we will drop assumption 2. Let $S^{12} = \frac{S^1}{S^2}$ be the exchange rate between currencies 1 and 2 (quantity of currency 1 per unit of currency 2). I obtained the solution reported in theorem 1 by assuming that $[\underline{S}^{12}, \bar{S}^{12}]$ was wider than $[\frac{S^1}{S^2}, \frac{\bar{S}^1}{\bar{S}^2}]$. However, note that this assumption turns out to be unnecessarily strong because, in the solution, S^1 and S^2 are correlated and therefore cannot vary independently. Thus it is enough that the inequalities:

$$\underline{S}^{12} \leq \inf_{(k^1, k^2) \in G} \frac{s^1(k^1, k^2)}{s^2(k^1, k^2)} \leq \sup_{(k^1, k^2) \in G} \frac{s^1(k^1, k^2)}{s^2(k^1, k^2)} \leq \bar{S}^{12} \tag{2.36}$$

be satisfied, where the functions $s^i(k^1, k^2)$ are provided in theorem 1. Thus, just because the parameters of the problem happen to satisfy 3.13, in many applications with active cross-currency constraints, theorem 1 may provide a valid solution to the trilateral target zone modelling problem according to Definition 1.

However, the solution in theorem 1 is not valid for arbitrary parameters that may fail to satisfy 3.13. In this section I will provide a general solution that covers this case. Thus we will assume here that the band for $S^{12} \in [\underline{S}^{12}, \bar{S}^{12}] \subset [\frac{S^1}{S^2}, \frac{\bar{S}^1}{\bar{S}^2}]$ is arbitrary. In other words, there are limits to the range of variation of the exchange rate between currency 1 and currency 0 and between currency 2 and currency 0 that are due not to the limits of the two respective bands, but rather to the fact that the implied exchange rate between currency 1 and currency 2 (i.e. $\frac{S^1}{S^2}$) hits its own band before either S^1 or S^2 reach theirs (see diagram in Figure 4). It will be shown how the general solution collapses back to theorem 1 when the cross-currency band is made wide enough.

A Solution Method

When 3.13 is not satisfied automatically by the solution presented in theorem 1, we must modify our arguments in an essential way, and the problem becomes inevitably more involved. Our solution method consists of expanding the state space by considering the cross currency exchange rate *as if* it were an additional exchange rate in the problem without active cross-currency constraints. A change of variables will be identified that allow us to rotate the state space as before. However, technical issues arise because some matrices involved in the problem are singular. In particular, the variables resulting from rotating the space are a three-dimensional stochastic process that takes values in a closed subset of a plane, and such plane exhibits parallel shifts every time that the boundaries are hit. Details are supplied in the proof of Theorem 3 in Appendix 1. In the space of the fundamentals, this implies that the solution is similar to the one in Theorem 1 except for the facts that two more exponential terms arise and the constants in the solution are “constant” only in the interior of the domain of the fundamentals and depend explicitly on the cumulative foreign exchange intervention processes.

In logarithms, the range for the exchange rates becomes:

$$\begin{aligned}
s^1 \in [\underline{s}^1, \bar{s}^1] \quad s^2 \in [\underline{s}^2, \bar{s}^2] \quad s^{12} = s^1 - s^2 \in [\underline{s}^{12}, \bar{s}^{12}] \\
\underline{s}^{12} \geq \underline{s}^1 - \bar{s}^2 \quad \bar{s}^{12} \leq \bar{s}^1 - \underline{s}^2
\end{aligned} \tag{2.37}$$

and equation 2.3 for s^{12} reads:

$$s^{12} dt = (k^1 - k^2) dt + \gamma E_t[ds_t^{12}]$$

We now make the identifications $k^3 \equiv k^1 - k^2$, $s^{12} \equiv s^3$ and our problem of a trilateral target zone with a cross-currency constraint becomes a problem of a quadrilateral target zone without active cross currency constraints. Note that a new constraint in the range of variation of (k_1, k_2) appears, namely, $\underline{k}_3 = \underline{(k^1 - k^2)} \leq k_3 = k_1 - k_2 \leq \bar{k}_3 = \overline{(k^1 - k^2)}$. Thus, (k_1, k_2) will no longer vary within a quadrilateral G , but on an hexagon¹⁶ $H \subset \mathcal{R}^2$. We artificially expand the dimension of our problem to reduce its nature to the previous case, whose solution is known. Of course, we will pay a price for this artificial manipulation of the problem. We will have to deal with a singular diffusion process and, although we will obtain a closed form solution for the exchange rate, such solution will depend explicitly on the cumulative foreign exchange interventions in addition to the fundamentals process. However, the Markov property of such solution will not be lost.

Consider the expanded state-variable vector $\hat{k}' = (\hat{k}_1, \hat{k}_2, \hat{k}_3) \equiv (k_1, k_2, k_1 - k_2)$ and let $k' = (k_1, k_2)$ be the original state variable vector. The above expansion of the state

implies $\hat{k} = Pk$ where $P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -1 \end{pmatrix}$. The reflection matrix in the three-lateral case

with active constraints is given by the 3×6 matrix $\hat{R} = [\hat{M} : -\hat{M}]$ where the relevant submatrix of the state-expanded system $\hat{M} \in \mathcal{R}^{3 \times 3}$ is given by $\hat{M} = PM$ with M as the 2×3 matrix composed by the 2×1 reflection vectors of the original state variables (k^1, k^2) on three nonparallel sides of a three dimensional polygon. In these conditions

¹⁶Recall that an hexagon is the two-dimensional projection of a three-dimensional cube. This helps to visualize the problem. We artificially create a new dimension but “everything goes on” in two dimensions.

and by construction, the dynamics of the expanded state vector $(\hat{k}_1, \hat{k}_2, \hat{k}_3)$ will be:

$$d \begin{pmatrix} \hat{k}^1 \\ \hat{k}^2 \\ \hat{k}^3 \end{pmatrix} = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_1 - \mu_2 \end{pmatrix} dt + \begin{pmatrix} \hat{A} & 0 \\ (1, -1)\hat{A} & 0 \end{pmatrix} \begin{pmatrix} dW_1 \\ dW_2 \\ dW_3 \end{pmatrix} + \begin{pmatrix} \hat{M} & -\hat{M} \\ (1, -1)\hat{M} & -(1, -1)\hat{M} \end{pmatrix} d\Lambda_t \quad (2.38)$$

where $\hat{A} \in \mathcal{R}^{2 \times 2}$ is our previous diffusion matrix, W_3 is an independent Wiener process (which plays no formal role in the analysis), $\hat{M} \in \mathcal{R}^{2 \times 3}$ is a full row rank reflection matrix of the independent fundamentals, and Λ_t is a 6×1 matrix whose components are the local time process of the fundamentals at each side of the hexagon. The reflection matrix in 2.38 will be denoted by $R = [M : -M] \in \mathcal{R}^{3 \times 6}$.

The construction of the solution will be as before. We must construct a one-to-one mapping between the fundamentals process and an auxiliary process whose domain is a rectangle in \mathcal{R}^3 , that behaves like an arithmetic brownian motion in the interior of its domain, which is reflected at the boundaries of the domain with an orthogonal reflection matrix and which possesses the strong markov property.

Remark 1: Note, however, that there are two additional considerations here: first, the instantaneous conditional covariance matrix AA' of $(\hat{k}^1, \hat{k}^2, \hat{k}^3)$ is singular by construction. This does not cause problem because the diffusion matrix need not be inverted anywhere. Second, any square 3×3 submatrix of the reflection matrix R will be singular. This can be amended by using the generalized inverse matrix M^+ of M instead of M^{-1} in the definition 3.27 and the transformation 2.19. Thus, these features do not prevent getting a solution to the boundary value problem 2.31.

More precisely, the generalized inverse M^+ of M can be used to characterize the solutions to the system $Mx_t = \hat{k}_t$ (where, for our purposes $x_t = (x_t^1, x_t^2, x_t^3)'$, $k_t = (\hat{k}^1, \hat{k}^2, \hat{k}^3)$, and we let M be a singular square matrix)¹⁷. If M is not invertible,

¹⁷The generalized inverse or pseudoinverse of a square matrix M is then constructed as $M^+ = \sum_{j=1}^k \lambda_j^{-1} C_j C_j'$ where λ_j is the j -th nonzero eigenvalue of the matrix $M'M$, and C_j its corresponding eigenvector, for $j = 1, 2, \dots, k = \text{rank}(M)$.

there are either zero or an infinite number of solutions to the above system. All solutions can be parametrized as $x_t = M^+ \hat{k}_t + (I - M^+ M) d_t$, for an arbitrary vector process d_t (if at least one solution to the above system $Mx_t = \hat{k}_t$ exists¹⁸). What is the correct solution to single out? We need the process for x_t to have a diagonal reflection matrix to apply our solution method to the relevant boundary value problem. Thus the dynamics for x_t must have the form $x_t = \Pi_0 t + \Pi_1 W_t + P \Lambda_t$ where Π_0 is a 2×1 (constant) drift vector, Π_1 is a 3×3 diffusion matrix and $P = [I : -I]$ where I is the 3×3 identity matrix. Since $R = PM$, we can single out a unique solution using the above formula involving the generalized inverse matrix by letting $d_t = P \Lambda_t$. Since Λ_t is the unique nondecreasing, continuous, bounded variation process which is singular with respect to Lebesgue measure such that $d\Lambda_t = 0$ in $\overset{\circ}{G}$ and increases in ∂G , note that there is a unique process that solves $Mx_t = \hat{k}_t$ that behaves like an arithmetic brownian motion on $\overset{\circ}{G}$ with drift vector $\Pi_0 = M^+ \mu$, diffusion matrix $\Pi_1 = M^+ A$ and reflection matrix P .¹⁹

Remark 2: The connections between the boundary conditions in our artificially expanded case and the previous case must still be clarified. Make the identification $s^i(k_1, k_2) = s_e^i(\hat{k}_1, \hat{k}_2, \hat{k}_3)$, where s_e^i is the expanded form of the exchange rate function for currency $i = 1, 2$ with respect to currency 0, or for currency 1 with respect to currency 2, $i = 1, 2$. The extended form depends on \hat{k}_1, \hat{k}_2 and \hat{k}_3 , while s is a reduced form of the exchange rate function that depends only on k_1 and k_2 . We want to show that the no-arbitrage interpretation of the boundary conditions also hold for the reduced form solution, i.e., that the gradient of the reduced form solution is orthogonal to the reflection vector for k_1 and k_2 at the boundaries. This is important

¹⁸In our case, the system $Mb = y$ always has a solution for $b = (x_1, x_2, x_3)'$, $y = (k_1, k_2, k_1 - k_2)'$ and $M = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{11} - m_{21} & m_{12} - m_{22} & m_{13} - m_{23} \end{pmatrix} \equiv \begin{pmatrix} M^* \\ m^* \end{pmatrix}$ where we decompose the matrix M in $M^* \in \mathcal{R}^{2 \times 3}$ and $m^* \in \mathcal{R}^{1 \times 3}$. By assumption, $(k_1, k_2) \in \text{Span}(M^*)$, and for every vector $x \in \mathcal{R}^{3 \times 1}$ such that $(k_1, k_2)' = M^* x$ we have $k_1 - k_2 = (1, -1) M^* x = m^* x$ and therefore the third equation is satisfied. Thus the solution set for our equation system is non-empty.

¹⁹Note also that the final solution for the exchange rate as a function of the increased state vector of fundamentals $\hat{k} = (\hat{k}_1, \hat{k}_2, \hat{k}_3)$ would not change had we chosen different values for Π_0 and Π_1 , preserving an orthogonal reflection matrix for x , by changing d_t accordingly, because of Proposition 2 the one-to-one mapping between the process \hat{k} and x .

because k_1 and k_2 are the state variables and the solution for the exchange rate must satisfy the no-arbitrage boundary conditions at the edges of the band for the exchange rate. In other words, the gradient of the reduced form must be orthogonal to the reflection vector (and thus to the risk premium vector) for k_1 and k_2 , at each of the six sides of the extended domain. Observe that, by the chain rule, we have $\nabla s^i(k_1, k_2) = P' \nabla s_e^i(\hat{k}_1, \hat{k}_2, \hat{k}_3)$. Recall that the relevant boundary conditions for the multidimensional case without cross-currency constraints had been shown to be $\nabla' s_e^i(\hat{k}_1, \hat{k}_2, \hat{k}_3) \hat{M} = 0$. Since $\hat{M} = PM$ and $\nabla s^i(k_1, k_2) = P' \nabla s_e^i(\hat{k}_1, \hat{k}_2, \hat{k}_3)$ we have $\nabla' s^i(k_1, k_2) M = 0$. Thus, the relevant boundary conditions for the reduced problem are automatically satisfied once we impose the relevant boundary conditions for the expanded system.

Remark 3: In solving the trilateral target zone problem with active cross-currency constraints we obtain the solutions $s_e^1(\hat{k}_1, \hat{k}_2, \hat{k}_3)$, $s_e^2(\hat{k}_1, \hat{k}_2, \hat{k}_3)$ and $s_e^3(\hat{k}_1, \hat{k}_2, \hat{k}_3) = s_e^{12}(\hat{k}_1, \hat{k}_2, \hat{k}_3)$ by solving the boundary value problem 2.9 (with three dimensions here) subject to the boundary conditions equivalent to 2.10 and 2.11. Then we evaluate the solution functions at $(\hat{k}_1, \hat{k}_2, \hat{k}_3) = (k_1, k_2, k_1 - k_2)$ and we make the identification $s^{12} = s^3$. Note that this clearly preserves the no-arbitrage relationship $s^{12} = s^1 - s^2$ almost everywhere. This is because by proposition 2, the solution to the above boundary value problem is unique and satisfies $s^i = \frac{1}{\gamma} \int_t^\infty e^{-\frac{(t-s)}{\gamma}} E_t(\hat{k}_i(s)) ds$; $i = 1, 2, 3$. Then, in particular, $s^{12} = s^3 = \frac{1}{\gamma} \int_t^\infty e^{-\frac{(t-s)}{\gamma}} E_t(\hat{k}_3(s)) ds = \frac{1}{\gamma} \int_t^\infty e^{-\frac{(t-s)}{\gamma}} E_t(\hat{k}_1(s) - \hat{k}_2(s)) ds = s^1 - s^2$, where we applied the definition of \hat{k}_3 and proposition 2 again.

Thus, it has been proven that, in general, our problem with $n + 1$ currencies with fully active constraints²⁰ becomes a problem of a target zone without cross-currency constraints involving $\frac{n(n+1)}{2} + 1$ currencies, which has been solved above. As an example, the solution for a trilateral target zone with an active cross currency constraint is given in the theorem below:

Theorem 3: The solution to the trilateral target zone modelling problem with

²⁰This would be the case of symmetric bands in percentage terms, like the EMS.

an active cross-currency constraint is given by the function $s \in C^2 (H \times \mathcal{R}_+^6 \rightarrow [\underline{s}, \bar{s}])$ for $H \subset \mathcal{R}^2$:

$$\begin{aligned} s^i(k^1, k^2, \Lambda) = & \gamma I^i M M^+ \mu + k^i + \\ & + \sum_{j=1}^2 C_{1j}^i A_t^{1j} \exp(\lambda_{1j} ((v_{11} + v_{13})k^1 + (v_{12} - v_{13})k^2)) + \\ & + \sum_{j=1}^2 C_{2j}^i A_t^{2j} \exp(\lambda_{2j} ((v_{21} + v_{23})k^1 + (v_{22} - v_{23})k^2)) + \\ & + \sum_{j=1}^2 C_{3j}^i A_t^{3j} \exp(\lambda_{3j} ((v_{31} + v_{33})k^1 + (v_{32} - v_{33})k^2)) \end{aligned}$$

for $i = 1, 2, 12$; where I^i is a 1×3 row vector of zeros except a 1 in the i^{th} position, where $[v_{ij}] = [M^+]_{ij}$, the constants $\lambda_{11}, \lambda_{21}, \lambda_{31}, \lambda_{12}, \lambda_{22}, \lambda_{32}$ satisfy 3.6 with $\lambda_{j1} \equiv \lambda_+^j$, $\lambda_{j2} \equiv \lambda_-^j$ for $j = 1, 2, 3$ and where we replace the 3×1 vectors $\theta = M^+ \mu$ and $\phi \odot \phi = \text{diag}(M^+ A A' M^+)$ for $\theta = M^{-1} \mu$ and $\phi \odot \phi = \text{diag}(M^{-1} A A' M^{-1})$ respectively, and where the constants C_{jl}^i ($j = 1, 2, 3; l = 1, 2; i = 1, 2, 12$) are given in closed form in the proof below. The domain for the fundamentals will be an hexagon in \mathcal{R}^2 given by $H = \{(x, y) \in \mathcal{R}^2 \mid \exists v \in [\underline{X}_1, \bar{X}_1] \times [\underline{X}_2, \bar{X}_2] \times [\underline{X}_3, \bar{X}_3] \ni: Mv = (x, y, x - y)\}$ where the constants $(\underline{X}_1, \bar{X}_1, \underline{X}_2, \bar{X}_2, \underline{X}_3, \bar{X}_3)$ are also given in the proof below. The processes A_t^{ij} for $j = 1, 2, i = 1, 2, 3$ are given by:

$$A_t^{ij} = \exp \left\{ \lambda_{ij} I^i (I - M^+ M) P \Lambda_t \right\} \quad (2.39)$$

where $P = [I, -I]$ where I is the 3×3 identity matrix, I^i is a 1×3 vector with 1 in the i^{th} position and zeros elsewhere, and Λ_t is the 6×1 dimensional regulator process of cumulative interventions. Thus A^j are continuous processes of bounded variation, which are zero at $t = 0$, that vary only if the boundaries for the fundamentals domain (∂H) are hit and are constant in the interior of the domain for the fundamentals. In addition, $s^i(k_t^1, k_t^2, \Lambda_t)$ is a Markov process.

Remark 4: The solution for the active cross-currency constraints case is similar to the case with inactive constraints, except for the fact that it contains two additional exponential terms and that there is a shift in the coefficients of all exponential terms every time that there is an intervention. The presence of these shifts is due to the singular nature of the diffusion process used to solve the problem with active cross-

currency constraints, and they are not present in the previous cases of theorem 1 and 2. With active cross-currency constraints, these terms effectively keep the exchange rates against the anchor currency within a hexagon rather than a rectangle in \mathcal{R}^2 . Note however that the constants in the first four exponential terms in the solution in theorem 3 are not directly comparable to those in the solution presented in theorem 1.

Remark 5: Note that the solution given in theorem 3 reduces to theorem 1 when the cross-currency constraints are not active. In this case the effective domain for the fundamentals is again a quadrilateral, and not an hexagon, because no intervention takes places to prevent s^{12} to exceed its bounds. This can be encompassed in theorem 3 by letting the reflection matrix of the expanded state vector be $M = \begin{pmatrix} \hat{M} & 0_* \\ 0'_* & 0 \end{pmatrix}$

where \hat{M} is a nonsingular 2×2 matrix, and 0_* is a 2×1 vector of zeros. Effectively this matrix indicates that no reflection takes place when $\overline{(k^1 - k^2)} = k_3 = k_1 - k_2$ or $k_3 = \overline{(k^1 - k^2)}$. It is easy to check that, in this case, the pseudoinverse of M is given by²¹ $M^+ = \begin{pmatrix} \hat{M}^{-1} & 0_* \\ 0'_* & 0 \end{pmatrix}$. Thus the local time terms 2.39 vanish, namely $A_t^{1j} = A_t^{2j} = 1$, for $j = 1, 2$ and $v_{3l} = 0$ for $l = 1, 2, 3$ and, from the expressions given in Appendix 1, $C_{3j}^i = 0$ for $i = 1, 2$. Finally, $\gamma I^i M M^+ \mu = \gamma \mu_i$ for $i = 1, 2$ and the expression for $s^i(k^1, k^2, \Lambda)$, $i = 1, 2$, given in theorem 3 collapses to the solution for $s^i(k^1, k^2)$, $i = 1, 2$, given in Theorem 1.

2.5 Implications for Exchange Rate Dynamics

In this section the steady state distribution for the exchange rate in a trilateral target zone as that of section 3 is obtained. The extension to a target zone of arbitrary dimension is straightforward, and the basic point can be made in the simplest three-lateral case without active cross-currency constraints. I will argue that the feature of the simple bilateral model which has been overwhelmingly rejected by the data,

²¹One can see this by checking the conditions that define uniquely the generalized inverse of a matrix M , namely $M M^+ M = M$, $M^+ M M^+ = M^+$, $M^+ M$ symmetric and $M M^+$ also symmetric.

i.e. an unconditional \mathcal{U} -shaped density of the exchange rate within the band, does not necessarily carry over to the three-dimensional case. In addition, I will show that the conditional volatility is no longer a deterministic function of the exchange rate level as in the bilateral case. This will imply that the “honeymoon effect” (Krugman) is not a deterministic but rather a stochastic phenomenon, in the sense that its intensity will vary according to the path which the fundamentals follow that make the exchange rate approach the edges of the band. This may therefore explain the inconclusive empirical evidence about the existence of a “honeymoon effect”.

2.5.1 The Steady-state Density for the Fundamentals

In this section we characterize the steady state density of the fundamentals in a special case. Consider a linear transformation of the fundamentals process as $k' = A^{-1}k$. First we decompose the 2×4 matrix $X = A^{-1}R$ as follows:

$$X' = N + Q \quad (2.40)$$

where N and Q are 4×2 matrices with $diag(NN') = 1$ and $diag(QN') = 0$. We will assume that there exist two 2×2 invertible submatrices of N and Q , namely \bar{N} and \bar{Q} . Thus we have performed a decomposition of the reflection vector on each side of the quadrilateral $A^{-1}G$ for (k^1, k^2) , as $R^i = n_i + q_i$ where n_i and q_i are (1×2) orthogonal vectors and n_i has unit norm, for $i = 1, 2, 3, 4$ ²². In addition we assume that NQ is skew symmetric, i.e.,

$$n_i q'_j + q_i n'_j = 0 \quad \forall i, j = 1, 2, 3, 4 \quad (2.41)$$

We call a decomposition as in 2.40 with the property 2.41, an *oblique skew-symmetric decomposition* of the matrix X . Condition 2.41 can be visualized as follows: from elementary geometry, for each side of the quadrilateral $A^{-1}G$, the cosine of the angle between the reflection vector of the transformed fundamentals k' and the

²²This decomposition is always possible because, as it can be shown, the norm of the vector R^i , ($i = 1, 2, 3, 4$) does not matter for our purposes.

projection of the reflection vector in X on the space orthogonal to the side of the quadrilateral must be the same for all sides. In addition, the directions of reflection must point the same way relative to their normals (i.e. all to the right or all to the left of the vector orthogonal to each side of the quadrilateral).

Remark 1: The economic meaning of this latter assumption is that in the case where the variance covariance matrix of the fundamentals is diagonal, the intervention rules are common for all central banks participating in the currency agreement, Thus, if central bank A shares the burden of intervention with central bank B in a certain way when currency A is weak with respect to currency B, not only the roles between central bank A and central bank B will be switched when currency B is too weak with respect to currency A (assumption 5), but also all others central banks follow the same rule to share the burden of intervention with their respective currencies.

Now given that the vector field $e^i(k^1, k^2)$ $i = 1, 2$ is globally invertible²³ within the range of (k^1, k^2) (because the saddle points of e^i are at the vertices of our quadrilateral²⁴) the following proposition gives the steady-state density of the fundamentals. The density for the exchange rate then follows immediately. Here we will assume that the covariance matrix of the fundamentals is positive definite.

Proposition 4: The steady state density of the fundamentals $k' = (k^1, k^2)$ on its domain G is given by:

$$Dens(k) = \alpha \exp \left(2\mu' A^{-1} (I_2 - \bar{N}'_{\Omega} \bar{Q}'_{\Omega}) A^{-1} k \right) \quad (2.42)$$

where α is a constant, I_2 is the identity matrix in $\mathcal{R}^{2 \times 2}$ and $\bar{N}'_{\Omega}, \bar{Q}'_{\Omega}$ are invertible submatrices of a skew symmetric decomposition of the matrix $\Omega = A^{-1} R$.

Remark 2: The analytical solution for the density of the fundamentals exists thanks to the assumption of skew-symmetry of the orthogonal components of the reflection matrix. Without this assumption, one would have to rely on numerical

²³Taking into account that Krugman solution for the bilateral case is invertible within the band (Delgado and Dornbusch (1991)) it is immediate from 2.26 and 2.19 that our solution is also invertible. The inverse cannot be expressed in a closed analytic form, however.

²⁴Actually, with our latter assumption of skew symmetry the quadrilateral specializes to a parallelepiped.

methods to approximate this density (for this purpose, an algorithm is proposed in Dai and Harrison (1991)).

Once the analytic solution for the density of the fundamentals has been obtained, it is conceptually straightforward to obtain the density for the exchange rate process $e^i(k^1, k^2)$. However an explicit expression for such density is impossible because the inverse of the vector field $\{e^1(k^1, k^2), e^2(k^1, k^2)\}$ cannot be expressed in closed form. Therefore one has to rely on numerical methods or Montecarlo simulation to study the properties of that density. The second method is more efficient in this case as we have a closed form for the density of the fundamentals. In what follows, the steady-state density of the exchange rate under a three-lateral target zone and under a bilateral target zone is obtained by simulating its density. The size of the simulations is 10.000 replications (using the rejection method).

2.5.2 The Steady-state Density for the Exchange Rate

In what follows the results for the steady-state density of the exchange rate are discussed. The benchmark case considered is that of a reflection matrix of $R = [M : -M]$ with $M = \begin{pmatrix} 0.875 & 0.625 \\ 0.625 & 0.875 \end{pmatrix}$, a diffusion matrix $A = \begin{pmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{pmatrix}$ a drift vector $\mu = [1, -1]$, and $\gamma = 1$. These assumptions correspond to a correlation between fundamentals of approximately 0.6. The exchange rate function under these circumstances is plotted in Figure 5.

Fig. 5: $m_1=0.875$ $m_2=0.675$

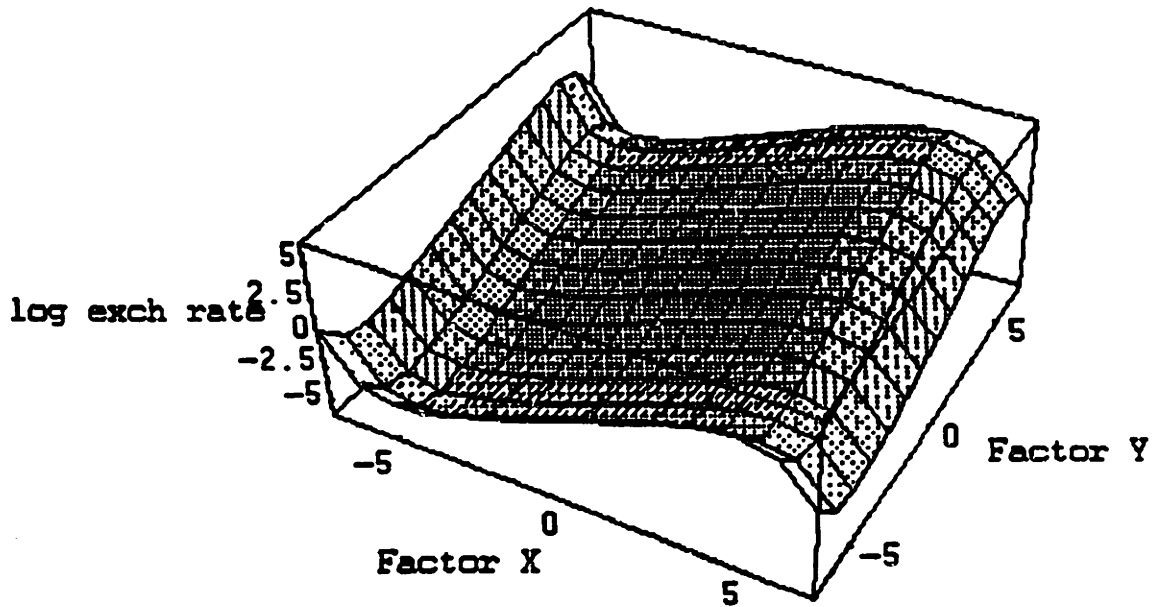
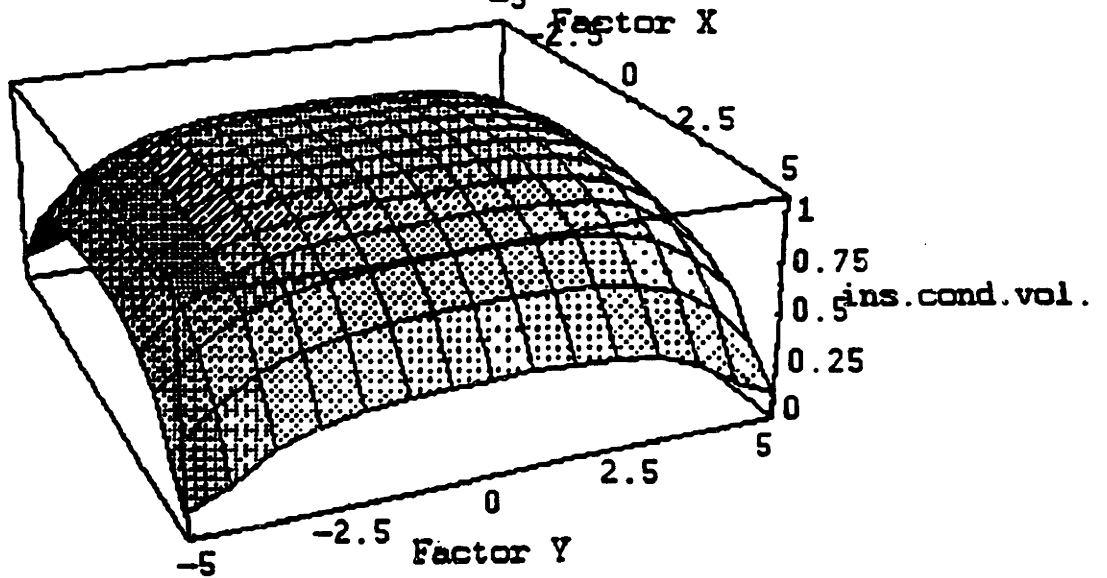
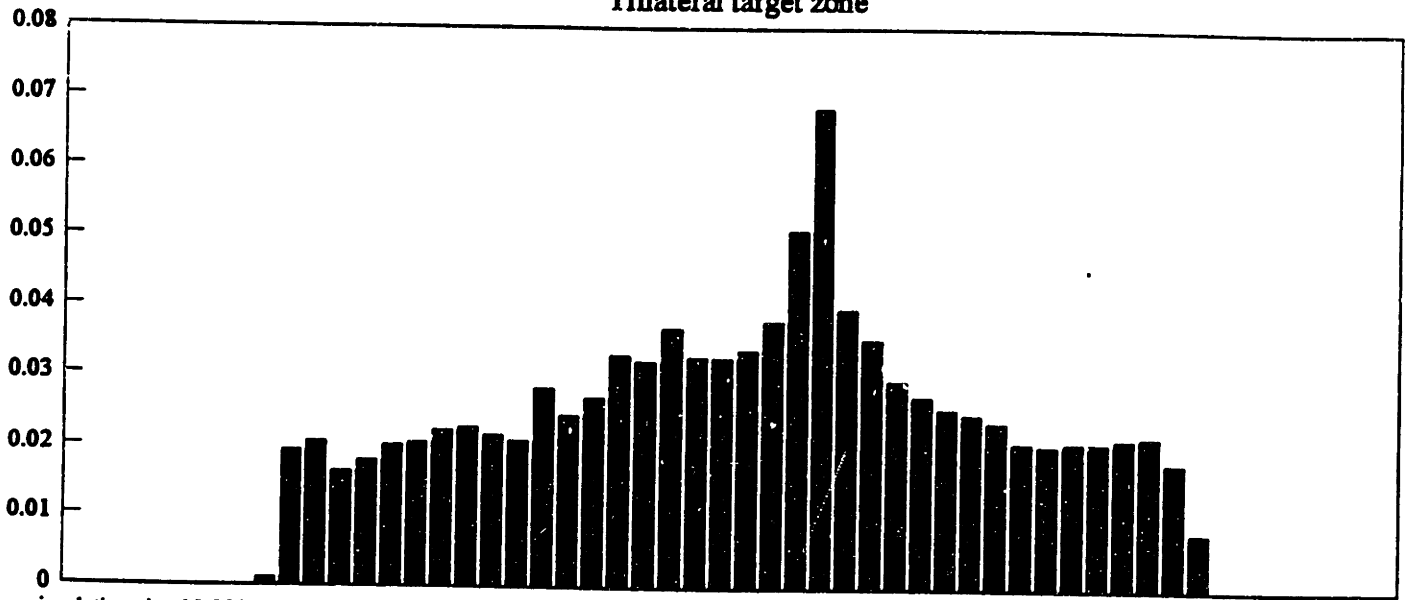


Fig. 6: $m_1=0.875$, $m_2=0.625$



Exchange rate simulated density

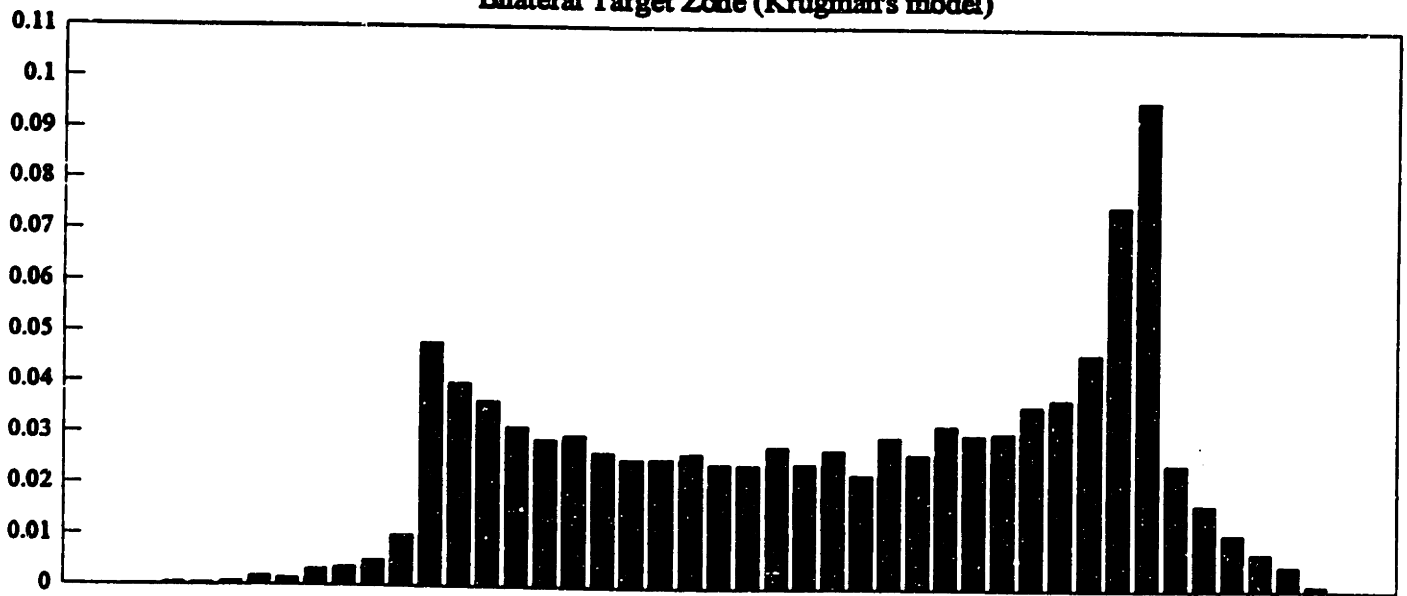
Trilateral target zone



simulation size:10.000
using rejection method

Exchange Rate Simulated Density

Bilateral Target Zone (Krugman's model)



simulation size:10.000
using rejection method

In Figure 6 I plot the conditional instantaneous volatility of the (log) exchange rate. Although the “honeymoon effect” is present, its observed magnitude varies significantly according to the path that the fundamentals follow to approach the exchange rate to the edges of its band. Thus, in the graph, if we approach the point $(-5, -5)$ (that is, where the exchange rate hits its lower limit) from the strip $(X = x, -5)$ the magnitude of the honeymoon effect is very large and one should expect to find it present in the observed data. However, if we approach the same limit from the strip $(-5, Y = y)$, the honeymoon effect, although still present, is much less strong. Therefore, if the real world data correspond to a realization for the sample path of (X, Y) of this type, the honeymoon effect will be difficult to detect by conventional statistical methods. Thus, the solution presented shows that perfect credibility is compatible with conflicting empirical evidence about the “honeymoon effect”. This is why we can say that the “honeymoon effect” is rather a stochastic phenomenon than a deterministic one.

In the plot of the steady state density for the exchange rate under the trilateral target zone one can see that a hump-shape can indeed be obtained, unlike the bilateral model. The density also exhibits fat tails, which is compatible with the empirical evidence relative to the high degree of kurtosis in exchange rate data. For comparison purposes, the simulated density for the exchange rate under a bilateral target zone (Krugman’s model) is also plotted and the typical \mathcal{U} -shape is obtained. Thus, the most striking distributional implication of the bilateral model does not necessarily carry over to the multilateral case. Contrary to the opinion that the failure of the bilateral Krugman (1991) model in explaining real world exchange rate target zone data may lie exclusively in the presence of “effective” bands different from the nominal ones (Pill, (1994)), we see that intervention spillovers alone are enough to obtain a hump-shaped distribution for the steady-state density of the logarithm of the exchange rate within the band.

2.6 Empirical Evidence

The contribution of this paper is mainly methodological. Namely, we develop a theory of multilateral target zones using new techniques that provides new economic insights relative to existing work. However, the issue addressed in this paper can be regarded as an empirical one, and therefore econometric work is called for. Empirical work has been undertaken in a separate paper: Schulstad and Serrat (1995) apply a simulated method of moments technique to estimate trilateral and five-lateral versions of the multilateral exchange rate model developed in this paper. They use EMS data from the so-called hard-EMS period (1987-92).

In their paper, the trilateral model derived from Theorem 1 is estimated ten times using different three-currency combinations. However, due to its computational complexity (27 parameters to estimate), the five-lateral model is estimated only once. The evidence strongly supports the model (overidentifying restrictions tests do not reject the model and the parameter estimates are significant and reasonable magnitudes), while Krugman's model is rejected in almost all cases (for the trilateral model, it is rejected in 8 out of 10 specifications and is rejected strongly in the five-currency specification). In addition, the overidentifying restrictions tests exhibit high power against reasonable alternative hypotheses. In all cases studied, it is the parameters that capture the degree of symmetry in the maintenance of the regime (the reflection matrix), and which have been neglected by the previous literature, that drive the good empirical performance of the model.

Some authors (Lindberg and Sörderlin 1991, Delgado and Dumas (1991,1993) have suggested that introducing *ad-hoc* intramarginal interventions quite improves the empirical fit of the bilateral model. The empirical results of Schulstad and Serrat explain these results since, in the multilateral model, marginal interventions rules in the space of the fundamentals look like intramarginal in the space of the exchange rates, since they induce spillover effects among money supplies.

In addition, they claim that taking into account the spillover effects among money supplies derived from intervention agreements, and made explicit in the multilateral

model, is what actually improves the performance of the model dramatically. Capturing this effect is thus more important to explain the evidence than taking into account explicitly active cross-currency constraints. This enhances the relevance of the theoretical analysis of this paper and stands in contrast to the common belief as to why the multilateral target zone model predictions could be different from the bilateral one: that there are cross-currency constraints that the bilateral model ignores.

2.7 Conclusions

In this paper, the exchange rate in a multilateral target zone is obtained in closed form from a model following the spirit of Krugman (1991). The steady-state dynamics of the exchange rate have also been characterized. The techniques used consist mainly of the use of multidimensional reflected diffusion process. To my knowledge this is the first application of such processes in economics that is able to obtain a closed form solution for problems of interest with an arbitrary dimension.

Although the problem appears quite involved in the beginning, the solution turns out to be very simple. It also exhibits similar features to Krugman's original solution for the bilateral case, in particular, an extension to an n -dimensional surface of a \mathcal{S} -shaped graph. However, two main features distinguish the multilateral case from the bilateral one: first, the spillover effects of individual central bank interventions on the fundamentals of third currencies; second, the restriction on the feasible range of the exchange rate due to active cross-currency constraints (i.e. the range of the exchange rate of currency 0 versus several currencies $i = 1, \dots, n$ is no longer a rectangle in \mathcal{R}^n , or that "notional" bands" no longer coincide with "effective" bands"). In particular, it has been shown that only the first distinguishing feature is necessary to "fix" the two most problematic predictions of the basic model, namely, a \mathcal{U} -shaped distribution of the exchange rate within the band and a non-stochastic "honeymoon effect". Hump-shaped densities can arise in the multilateral model depending on the value of the parameters of the underlying process for the fundamentals. Thus, it has been shown that the original Krugman (1991) model is not a good description of a

fully credible currency influence area (i.e. a set of bilateral bands of several currencies against a common “anchor” currency without “effective” cross-currency constraints), if that the intervention burden is shared to some extent by the monetary authorities of the “anchor” country. Another new result relative to the bilateral case is that the stabilizing effect of a target zone regime appears with certainty only when the exchange rate is close to the edges of the band. In the interior of the band, the volatility of the exchange rate may be larger than the volatility of the fundamentals process.

In my opinion, the empirical work to date does not present evidence against the two crucial assumptions of Krugman’s model, namely perfect credibility and marginal interventions as the leading intervention tool, when the Krugman model is tested on target zone data without realignments. Rather, such empirical results are inconclusive unless they incorporate explicitly the multilateral nature of real-world target zones.

Many extensions and uses of the model are possible. A complete research agenda can be set up based on the results of this paper in the same way that many extensions of the original bilateral model were made. On one hand, further analytical refinements are possible in the direction of allowing an exogenous the domain for the fundamentals, and thus distinguishing explicitly two elements of policy: namely, *when* to intervene (fundamentals domain) and *how* to intervene (reflection matrix M)²⁵. On the other hand, more comprehensive empirical work can be undertaken, possibly using a different methodology than Simulated Method of Moments as in Schulstad and Serrat (1995),

The model can also be extended to include asset pricing issues, and further analysis of its implications on exchange-rate dynamics and interest rate differentials will be possible. This is probably one of the most fruitful extensions of the model. Also, the model can be used to study issues of practical interest such as the effects on exchange rate and interest rate dynamics of increasing the number of currencies in a target zone, and also the cross-effects on third currencies of changing the band size

²⁵It is my feeling that the solution to the target zone modelling problem *given* an arbitrary fundamentals domain is possible, although the form of the solution will be much less “user-friendly” than the solutions provided in theorems 1 to 3. Therefore I think it is better to sacrifice generality for tractability on this issue.

for a particular subset of currencies in the band. In addition, the model can be used to examine the impact of the target zone on the dynamics of the exchange rate of a currency inside the target zone regime with a currency outside the regime.

2.8 References

Beetsma, R. and Van Der Ploeg, F. (1991): "Exchange Rate Bands and Optimal Monetary Accumulation under a Dirty Float" mimeo, University of Amsterdam.

Bertola, G. and Caballero, R. (1992): "Target Zones and Realignment" *American Economic Review* 82, p.520-36.

Bertola, G. and Svensson, L.E.O. (1993): "Stochastic Devaluation Risk and the Empirical Fit of Target Zone Models" *Review of Economic Studies*.

Cox, J., Ingersoll, J. and Ross, S. (1985): "An Intertemporal General Equilibrium Model of Asset Prices" *Econometrica* 53, p.363-384.

Dai, J. and Harrison, M. (1991): "Steady-State Analysis of Reflected Brownian Motion in a Rectangle: Numerical Methods and a Queueing Application" *The Annals of Applied Probability* 1, p.16-35.

Dai, J. and Williams, R. (1995): "Existence and Uniqueness of Semimartingale Reflecting Brownian Motions in Convex Polyhedrons" forthcoming in *Theory of Probability and Its Applications*.

De Jong, F. (1994): "A Univariate Analysis of EMS Exchange Rates Using a Target Zone Model" *Journal of Applied Econometrics* 9, p.31-45.

Delgado, F. and Dumas, B. (1991): "Target Zones: Broad and Narrow" in Krugman and Miller eds. *Exchange Rate Targets and Currency Bands* Cambridge University Press, p.35-54.

Delgado, F. and Dumas, B. (1993): "Monetary Contracting Between Central Banks and the Design of Sustainable Target Zones" *Journal of International Economics*, p.201-224.

Dellacherie, C. and Meyer, P.A. (1982): "*Probabilities and Potential*" North Hol-

land.

Dumas, B. (1991): "Super Contact and Related Optimality Conditions" *Journal of Economic Dynamics and Control* 15, p.675-85.

Dumas, B. (1992): "Dynamic Equilibrium and the Real Exchange Rate in a Spatially Separated World" *Review of Financial Studies* 5, p.153-80.

Flood, R. and Garber, P. (1991): "The Linkage Between Speculative Attack and Target Zone Models of Exchange Rates" *Quarterly Journal of Economics* 106, p.1367-72.

Flood, R., Rose, A., and Mathieson, D. (1991): "An Empirical Exploration of Exchange Rate Target Zones" *Carnegie-Rochester Series on Public Policy* 35, p.7-65.

Froot, K. and Obstfeld, M. (1991): "Exchange Rate Dynamics under Stochastic Regime Shifts: A Unified Approach" *Journal of International Economics* 31, p.203-29.

Gihman, I. and Skorohod, A. (1972): "Stochastic Differential Equations" Springer-Verlag, New York.

Harrison, M. (1985): "*Brownian Motion and Stochastic Flow Systems*" Wiley, New York.

Harrison, M. and Kreps, D. (1979): "Martingales and Arbitrage in Multiperiod Securities Markets" *Journal of Economic Theory*, 20, p.381-408.

Harrison, M. and Williams, R. (1987): "Multidimensional Reflected Brownian Motions Having Exponential Stationary Distributions" *The Annals of Probability* 15, p.115-37.

Harrison, M. and Reiman, M. (1981): "Reflected Brownian Motion on an Orthant" *The Annals of Probability*, 9, p.302-308.

Huang, C. and Pagès, H. (1992): "Optimal Consumption and Portfolio Policies with an Infinite Horizon: Existence and Convergence" *The Annals of Applied Probability* 2, p.36-64.

Koedijk, K.G., Schafgans, M.A., De Vries, G. (1990): "The Tail Index of Exchange Rate Returns" *Journal of International Economy* 29, p.93-108.

Krugman, P. (1991): "Target Zones and Exchange Rate Dynamics" *Quarterly*

Journal of Economics 106, p.669-82.

Krugman, P. and Rotemberg, J. (1991): "Speculative Attacks on Target Zones" in Krugman and Miller eds. *Exchange Rate Targets and Currency Bands* Cambridge University Press, p.117-133.

Krylov, N. (1980): "*Controlled Diffusion Processes*", Springer-Verlag, New York.

Lindberg, H. and Söderlin, P. (1991): "Testing the Basic Target Zone Model on Swedish Data" Seminar Paper #488 Institute for International Studies, Stockholm

Lindberg, H. and Söderlin, P. (1992): "Target Zone Models and the Intervention Policy: The Swedish Case" Seminar Paper #496 Institute for International Studies, Stockholm.

Miller, M. and Weller, P. (1991): "Exchange Rate Bands with Price Inertia" *Economic Journal* 101, p.1380-99.

Pill, H. (1994): "Target Zones & The European Monetary System: A Reconciliation" mimeo, Dep. of Economics, Stanford University. November 1994.

Schulstad, P.M. and Serrat, A. (1995): "An Empirical Examination of a Multilateral Target Zone Model" M.I.T. mimeo.

Smith, G. and Spencer, M. (1991): "Estimation and Testing in Models of Exchange Rate Target Zones and Process Switching" in Krugman and Miller eds. *Exchange Rate Targets and Currency Bands*. Cambridge University Press, p.211-39

Svensson, L.E.O. (1991) : "Target Zones and Interest Rate Variability" *Journal of International Economics* 31, p.655-65.

Svensson, L.E.O. (1992): "An Interpretation of Recent Research on Exchange Rate Target Zones" *Journal of Economic Perspectives* 6, p.119-144.

Varadhan, S.R.S. and Williams, R.J. (1985): "Brownian Motion in a Wedge with Oblique Reflection" *Comm. Pure Appl. Math.*, 38, p.405-443.

Vasicek, O. (1977): "An Equilibrium Characterization of the Term Structure" *Journal of Financial Economics* 5, p.177-88.

Williams, R.J. (1987): "Reflected Brownian Motion with Skew Symmetric Data in a Polyhedral Domain" *Probab. Theory Related Fields* 75, p.459-485.

2.9 Appendix 1: Proofs

Proof of Proposition 1: Assume, w.l.o.g. that $\log(\bar{S}^1) = \log(\bar{S}^2)$. Now suppose that there exists a curve $\Gamma(k_1, k_2) \subset \partial G$ such that $s^1(k_1, k_2) = s^2(k_1, k_2) = \log(\bar{S}^1) \forall (k_1, k_2) \in \Gamma$. Now from 2.6 and 2.7 $\log(\bar{S}^1) = k_1 + \mathcal{L}s^1 = k_2 + \mathcal{L}s^2 \forall (k_1, k_2) \in \Gamma$ where \mathcal{L} is the linear operator $\mathcal{L} : C^2(G) \rightarrow C^2(G)$ defined in the text. First I claim that \mathcal{L} is locally invertible on the set $\Xi = \{f \in C^2(G) \mid f \text{ constant on } \Gamma\}$. To see this, let $f \in \Xi$ and suppose $\mathcal{L}(f) = 0$. For all pairs $(x, y) \in \Gamma$, and for some constants C , $\log(\bar{S}^1)$, we have $s^1(x, y) = f(x, y) + C \Leftrightarrow \mathcal{L}s^1(x, y) = \mathcal{L}f(x, y) \Rightarrow \log(\bar{S}^1) - x = \mathcal{L}f(x, y) = 0$. By doing the same exercise with s^2 we also get $\log(\bar{S}^1) - y = 0 \forall (x, y) \in \Gamma$. This is a contradiction unless f is a constant. Therefore \mathcal{L} is locally invertible on Γ . Now since \mathcal{L} is linear and locally invertible on the set $\Xi = \{f \in C^2(G) \mid f \text{ constant on } \Gamma\}$, we have $k_2 - k_1 = \mathcal{L}(s^1 - s^2) \forall (k_1, k_2) \in \Gamma$ which implies that $\mathcal{L}^{-1} |_{\Xi} (k_2 - k_1) = s^1 - s^2 = 0 \forall (k_1, k_2) \in \Gamma$ as $s^1 - s^2 \in \Xi$. But since $\mathcal{L}^{-1} |_{\Xi}$ is injective, $(k_2 - k_1) \in \text{Ker}(\mathcal{L}^{-1} |_{\Xi}) \Rightarrow k_2 = a + k_1 \forall (k_1, k_2) \in \Gamma$ where a is a constant. This contradicts the assumption that G is a compact set with nonempty interior, and thus the problem is not well posed. \square

Proof of Proposition 2: Sufficiency: The proof consists of two steps: first I represent the solution equation 2.62 as a functional of the state variables. This representation will turn out to be unique in its class and will prove the first statement in the proposition. Then I prove the direct part of the proposition by showing that g coincides with the exchange rate function as represented by our functional equation. I will write the proof for $j = 1$ (since for $j = 2$ the proof is identical).

Step 1: From 2.3 is clear that s_t^i is a semimartingale adapted to the original brownian filtration. Write its Doob-Meyer decomposition as $s_t^i = VF_t^i + M_t^i$ where VF_t^i is a bounded variation process and M_t^i is a local martingale. Since s_t^i is a continuous semimartingale, its bounded variation component is predictable by VII.25 of Dellacherie and Meyer (1982), and thus one can write, heuristically, $dVF_t^i = E_t(ds_t^i)$. Multiply both sides of 2.3 by the process $\exp\left(-\frac{\sigma-t}{\gamma}\right)$ and integrate by parts to obtain,

in shorthand differential notation,

$$d(e^{-\frac{s-t}{\gamma}} s_s^1) = e^{-\frac{s-t}{\gamma}} ds_s^1 - \frac{1}{\gamma} e^{-\frac{s-t}{\gamma}} s_s^1 ds \quad (2.43)$$

We can rewrite 2.43 in integral form as:

$$e^{-\frac{(T-t)}{\gamma}} s_T^1 = s_t^1 + \int_t^T e^{-\frac{(s-t)}{\gamma}} dM_s^i - \frac{1}{\gamma} \int_t^T e^{-\frac{(s-t)}{\gamma}} k_s^1 ds \quad T > s > 0 \quad (2.44)$$

where we used the fact that $dVF_t^i = E_t(ds_t^1)$ in 2.3. Now, since s_t^1 is bounded above and below (by Assumption 2), M_t^i is indeed an L^2 -martingale since any bounded local martingale is a martingale. Thus the stochastic integral and the right hand side of 2.44 is an L^2 -martingale. We can thus take expectations on both sides of 2.44, apply Fubini's theorem and rearrange to obtain:

$$s_t^1 = \frac{1}{\gamma} \int_t^T e^{-\frac{(s-t)}{\gamma}} E_t(k_s^1) ds + e^{-\frac{(T-t)}{\gamma}} E_t(s_T^1) \quad T > s > 0 \quad (2.45)$$

where $E_t(\cdot)$ is a shorthand notation for the expectation operator $E^{P^{k_0}}(\cdot | \mathcal{B}_t)$ and where $\mathcal{B}_t \in \sigma((k_v^1, k_v^2) | 0 \leq v \leq t)$. Again since s_t^1 is bounded, the second term in the right hand side of 2.45 vanishes as we let $T \uparrow \infty$, and since the process $P\{(k_v^1, k_v^2) | 0 \leq v \leq t\}$ is bounded, we can express the limit of the RHS of 2.45 as:

$$s^1(k_t^1, k_t^2) = \frac{1}{\gamma} \int_t^\infty e^{-\frac{(t-s)}{\gamma}} E(k_s^i | k_t^1, k_t^2) ds \quad (2.46)$$

and this establishes the existence of a function $s : G \rightarrow \mathcal{R}$ that represents the exchange rate, thanks to the Markov property of the state variables (k^1, k^2) , and actually one can also prove that such function is continuous since the regulated process $\{k_v^1, k_v^2\}$ has the Feller property (Dai and Harrison (1991), Proposition 1). This proves the first part of the proposition.

Step 2: Let $g^1 : \mathcal{R}^2 \rightarrow \mathcal{R}$ be a twice continuously differentiable function satisfying 2.9 and 2.10. Apply Itô's lemma to the process $e^{-\frac{t}{\gamma}} g^1(k^1, k^2)$ to obtain, in shorthand differential notation,

$$d\left(e^{-\frac{t}{\gamma}}g^1(k_t^1, k_t^2)\right) = \left(-\frac{1}{\gamma}e^{-\frac{t}{\gamma}}g^1(k_t^1, k_t^2) + \frac{1}{\gamma}e^{-\frac{t}{\gamma}}\mathcal{L}g^1(k_t^1, k_t^2)\right) dt + e^{-\frac{t}{\gamma}}\nabla'g^1(k_t^1, k_t^2)AdW_t + \sum_{j=1}^4 e^{-\frac{t}{\gamma}}\nabla'g^1(k_t^1, k_t^2)R_jd\Lambda_j \quad (2.47)$$

where \mathcal{L} is the operator defined in 2.8 and Λ_j is the local time process of k at each face of G , F_j as defined in assumption 4 above. Now substitute 2.9 and 2.10 in 2.47 to obtain:

$$d\left(e^{-\frac{t}{\gamma}}g^1(k_t^1, k_t^2)\right) = -\frac{1}{\gamma}e^{-\frac{t}{\gamma}}k_t^1 dt + e^{-\frac{t}{\gamma}}\nabla'g^1(k_t^1, k_t^2)AdW_t \quad (2.48)$$

rearranging and expressing 2.48 in integral notation, for $T > t > 0$,

$$g^1(k_t^1, k_t^2) = e^{-\frac{T-t}{\gamma}}g^1(k_T^1, k_T^2) + \frac{1}{\gamma}\int_t^T e^{-\frac{s-t}{\gamma}}k_s^1 ds - \int_t^T e^{-\frac{t-s}{\gamma}}\nabla'g^1(k_s^1, k_s^2)AdW_s \quad (2.49)$$

Now, as before, take expectations of both sides of 2.49. Since g is continuously differentiable by construction, $\nabla'g^1(k^1, k^2)$ is bounded since G is compact and the stochastic integral vanishes when we take expectations. As we let $T \uparrow \infty$, the first term on the right-hand side vanishes as g is bounded on G , therefore we end up with:

$$g^1(k_t^1, k_t^2) = \frac{1}{\gamma}\int_t^\infty e^{-\frac{(t-s)}{\gamma}}E(k_s^1 | k_t^1, k_t^2)ds \quad (2.50)$$

and in view of 2.46, we can establish $s^1(k_t^1, k_t^2) = g^1(k_t^1, k_t^2)$ *a.s.* (P^{k_0})

Necessity: Assume that $s^1(k_t^1, k_t^2)$ is a twice continuously differentiable function. Since, by 2.3 and the argument beginning Step 1 above, its bounded variation part is absolutely continuous, it must be that $\nabla's^1R_j = 0 \forall j$ *a.s.*. This is because otherwise, by Itô's lemma, a component $\int \nabla's^1(k_t^1, k_t^2)R_jd\Lambda_j$ would enter the bounded variation part of s^1 . However, the processes $\Lambda_j(t)$ are not absolutely continuous since they are of bounded variation and singular with respect to Lebesgue measure. Thus the boundary conditions 2.10 must be satisfied. Now one can apply Itô's lemma to s^1 and use 2.3 and 2.10 to conclude that 2.9 is also satisfied. \square

Proof of Proposition 3: The first part of the proposition follows immediately from Itô's lemma. For the second assertion note that if s^{12} solves 2.13, then, by proposition 2 it solves the boundary value problem 2.9 with 2.10. By Proposition 2, this solution is unique and satisfies $s^{12} = \frac{1}{\gamma} \int_t^\infty e^{\frac{(t-s)}{\gamma}} E_t(k^{12}(s)) ds = \frac{1}{\gamma} \int_t^\infty e^{\frac{(t-s)}{\gamma}} E_t(k^1(s) - k^2(s)) ds = s^1 - s^2. \square$

Proof of Lemma 1: Since $e^i(X, Y)$ is additively separable, $e^i(X, Y) = C(X) + D(Y)$, -we drop the index i for convenience- it will suffice to prove the result for $C(X)$, the case for $D(Y)$ being identical. Differentiate 2.26 with respect to X , $e_x(X, Y) = C'(X) = m_{i1} + \lambda_+^1 K_1^i e^{\lambda_+^1 X} + \lambda_-^1 K_2^i e^{\lambda_-^1 X}$. First we show that $K_1^i < 0$ and $K_2^i > 0$ if $m_{i1} > 0$ and $K_1^i > 0$ and $K_2^i < 0$ if $m_{i1} < 0$. From the boundary conditions 2.27, we have $0 = m_{i1} + \lambda_+^1 K_1^i e^{\lambda_+^1 \bar{X}} + \lambda_-^1 K_2^i e^{\lambda_-^1 \bar{X}}$ and $0 = m_{i1} + \lambda_+^1 K_1^i e^{\lambda_+^1 \underline{X}} + \lambda_-^1 K_2^i e^{\lambda_-^1 \underline{X}}$. This implies $\lambda_+^1 K_1^i [e^{\lambda_+^1 \bar{X}} - e^{\lambda_+^1 \underline{X}}] = \lambda_-^1 K_2^i [e^{\lambda_-^1 \underline{X}} - e^{\lambda_-^1 \bar{X}}]$, which in turn implies, since $\lambda_+^1, -\lambda_-^1 > 0$ and $\underline{X} < \bar{X}$, that $\text{sgn} K_1^i = -\text{sgn} K_2^i$. Now, if $m_{i1} > 0$, it cannot be that $K_1^i > 0$ and $K_2^i < 0$ because this would violate both boundary conditions, therefore $K_1^i < 0$ and $K_2^i > 0$. If $m_{i1} < 0$. Parallel reasoning leads to the conclusion that $K_1^i < 0$ and $K_2^i > 0$ must hold. Note that, given this result, $C''(X)$ is monotonically decreasing on \mathcal{R} if $m_{i1} > 0$, and monotonically increasing if $m_{i1} < 0$, since $C'''(X) = (\lambda_+^1)^3 K_1^i e^{\lambda_+^1 X} + (\lambda_-^1)^3 K_2^i e^{\lambda_-^1 X} \leq 0$ if $m_{i1} > 0$ and $C'''(X) < 0$ if $m_{i1} < 0$. Now, since $\lambda_+^1, -\lambda_-^1 > 0$, we see that $\lim_{X \downarrow -\infty} C(X) = \infty$ and $\lim_{X \uparrow \infty} C(X) = -\infty$ if $m_{i1} > 0$ and $\lim_{X \downarrow -\infty} C(X) = -\infty$ and $\lim_{X \uparrow \infty} C(X) = \infty$ if $m_{i1} < 0$. Since $C'(X)$ has at most two real roots $\{\underline{X}, \bar{X}\}$ (because $C''(X)$ is either monotonically decreasing or monotonically increasing), it is clear that $0 \leq C'(X) \forall X \in [\underline{X}, \bar{X}]$ if $m_{i1} > 0$ and $C'(X) \leq 0 \forall X \in [\underline{X}, \bar{X}]$ if $m_{i1} < 0. \square$

Proof of Theorem 3:

Step 1: Construction of a one-to-one mapping between the fundamentals and an auxiliary space

Given the considerations of Remark 4.2.1, let's construct the transformed state

vector as

$$x_t = M^+ \hat{k}_t + (I - M^+ M) P \Lambda_t \quad (2.51)$$

with $P = [I : -I]$ where $x_t = (X_1, X_2, X_3)$ and where I is the 3×3 identity matrix and Λ_t is the local time process of \hat{k} at ∂G . Consequently $Mx' = M(X_1, X_2, X_3)' = (\hat{k}_1, \hat{k}_2, \hat{k}_3) = (k_1, k_2, k_1 - k_2)$. Now, given the dynamics of \hat{k} from 2.38, we can write x_t as $x_t = M^+ \mu t + M^+ A W_t + P \Lambda_t$.

Having chosen this one-to-one mapping between x and k . I will let the state vector take values on a \mathcal{R}^3 domain $(X_1, X_2, X_3) \in S = [\underline{X}_1, \bar{X}_1] \times [\underline{X}_2, \bar{X}_2] \times [\underline{X}_3, \bar{X}_3]$, which, for now, we leave unspecified. Given the linear mapping that defines the domain for the fundamentals G , namely, $G = \{a \in \mathcal{R}^3 \mid \exists b \in S \subset \mathcal{R}^3 \ni a = Mb\}$, it is clear that x hits ∂S if and only if \hat{k} hits ∂G . This implies that Λ_t increases only if x hits ∂S and Λ_t is also the local time process of x at ∂S . With these specifications, x_t behaves as an arithmetic brownian motion on $\overset{\circ}{S}$ with drift $M^+ \mu$ and diffusion matrix $M^+ A$, and is reflected at ∂S with an orthogonal reflection matrix P . Also, by uniqueness of the regulator process Λ_t (see Harrison (1985), ch.2), there is no other stochastic process that solves the system $Mx_t = \hat{k}_t$ (everywhere on $\Omega \times [0, \infty)$) that behaves as an arithmetic brownian motion on $\overset{\circ}{S}$ with drift $M^+ \mu$ and diffusion matrix $M^+ A$, and is reflected at ∂S with an orthogonal reflection matrix P . In addition, since Λ_t is the local time process of x at ∂S , it is known that x has the strong Markov property (see Harrison and Reiman (1981) and Dai and Harrison (1992)).

Note that the components of the stochastic processes in $x_t = (X_1(t), X_2(t), X_3(t))$ are linearly independent. This property is exclusively due to the presence of the local time process component $(I - M^+ M) P \Lambda_t$ in 2.51 that, by construction, is a linearly independent process. This can also be seen if we write $x_t = M^+ \mu t + M^+ A W_t + [I : -I] \Lambda_t$ and note that the components of Λ_t generically increase at different regions of the state space $\Omega \times [0, \infty)$ ²⁶. A way to visualize this mapping is that x moves in a compact region of a “random” hyperplane in \mathcal{R}^3 that exhibits parallel shifts every

²⁶That is to say, Λ_t^1 increases only if X_1 hits \bar{X}_1 , Λ_t^2 increases only if X_2 hits \bar{X}_2 , Λ_t^3 increases only if X_3 hits \bar{X}_3 , Λ_t^4 increases only if X_1 hits \underline{X}_1 , Λ_t^5 increases only if X_2 hits \underline{X}_2 , and Λ_t^6 increases only if X_3 hits \underline{X}_3 .

time that the boundaries of such region are hit. Note that had we constructed another process for the auxiliary state vector like, for example, $\hat{x} = M^+ \hat{k}$, then the components of \hat{x} would be linearly dependent processes because $(\hat{k}_1, \hat{k}_2, \hat{k}_3) \equiv (k_1, k_2, k_1 - k_2)$. Thus the transformation 2.51 has three crucial properties: first, it defines a unique one-to-one mapping between the processes x and k . Second, x is a Markov process that behaves like an arithmetic brownian motion when $x \in \overset{\circ}{S}$ and is reflected at ∂S with an orthogonal reflection matrix. Third, the components of x_t are linearly independent stochastic processes. The first property is important in order to write the exchange rate solution indistinguishably in terms of the fundamentals process and the auxiliary process x (and thus to assure that the arbitrage condition $S^{12} = \frac{S^1}{S^2}$ is met everywhere (see remark 4.2.3)). The second property is necessary to guarantee that the boundary value problem 2.9 with 2.10 and 2.11 is well posed when the exchange rate is written as a function of x . The third property is necessary to guarantee that the boundary conditions for the relevant partial differential equation are well posed.

Step 2: Solution to the boundary value problem in the new space

As in 2.26, denote by $e^i(X_1, X_2, X_3)$ the exchange rate solution as a function of the transformed state vector for $i = 1, 2, 12$. Then, a specialization of theorem 2 for $n = 3$ yields:

$$e^i(X_1, X_2, X_3) = \gamma \sum_{j=1}^3 m_{ij} \theta_j + \sum_{j=1}^3 m_{ij} X_j + \sum_{j=1}^3 (C_{j1}^i \exp(\lambda_{j1} X_j) + C_{j2}^i \exp(\lambda_{j2} X_j)) \quad (2.52)$$

for $i = 1, 2, 12$ and where the constants $\lambda_{11}, \lambda_{21}, \lambda_{12}, \lambda_{22}, \lambda_{13}, \lambda_{23}$ satisfy 3.6 with $\lambda_{1j} \equiv \lambda_+^j$, $\lambda_{2j} \equiv \lambda_-^j$ for $j = 1, 2, 3$, and replacing the 3×1 vectors $\theta = M^+ \mu$ and $\phi \odot \phi = \text{diag}(M^+ A A' M^+)$ for $\theta = M^{-1} \mu$ and $\phi \odot \phi = \text{diag}(M^{-1} A A' M^{-1})$, and $C_{11}^i, C_{12}^i, C_{21}^i, C_{22}^i, C_{31}^i, C_{33}^i$ and $\bar{X} \equiv (\bar{X}_1, \bar{X}_2, \bar{X}_3)$ and $\underline{X} \equiv (\underline{X}_1, \underline{X}_2, \underline{X}_3)$ solve the 24×24 equation system:

$$m_{i1} + \lambda_{11} C_{11}^i \exp(\lambda_{11} \bar{X}_1) + \lambda_{12} C_{12}^i \exp(\lambda_{12} \bar{X}_1) = 0 \quad (2.53)$$

$$m_{i2} + \lambda_{21} C_{21}^i \exp(\lambda_{21} \bar{X}_2) + \lambda_{22} C_{22}^i \exp(\lambda_{22} \bar{X}_2) = 0$$

$$m_{i3} + \lambda_{31} C_{31}^i \exp(\lambda_{32} \bar{X}_3) + \lambda_{32} C_{32}^i \exp(\lambda_{32} \bar{X}_3) = 0$$

$$m_{i1} + \lambda_{11} C_{11}^i \exp(\lambda_{11} \underline{X}_1) + \lambda_{12} C_{12}^i \exp(\lambda_{12} \underline{X}_1) = 0$$

$$m_{i2} + \lambda_{21} C_{21}^i \exp(\lambda_{21} \underline{X}_2) + \lambda_{22} C_{22}^i \exp(\lambda_{22} \underline{X}_2) = 0$$

$$m_{i3} + \lambda_{31} C_{31}^i \exp(\lambda_{31} \underline{X}_3) + \lambda_{32} C_{32}^i \exp(\lambda_{32} \underline{X}_3) = 0$$

$$\begin{aligned} \sup_{x \in S} e^1(X_1, X_2, X_3) &= \log(\bar{S}^1) & \inf_{x \in S} e^1(X_1, X_2, X_3) &= \log(\underline{S}^1) \\ \sup_{x \in S} e^2(X_1, X_2, X_3) &= \log(\bar{S}^2) & \inf_{x \in S} e^2(X_1, X_2, X_3) &= \log(\underline{S}^2) \\ \sup_{x \in S} e^{12}(X_1, X_2, X_3) &= \log(\bar{S}^{12}) & \inf_{x \in S} e^{12}(X_1, X_2, X_3) &= \log(\underline{S}^{12}) \end{aligned}$$

The only difference in the boundary conditions in the case with active cross-currency constraints is reflected in the definition of Γ_1 and Γ_2 above. It arises from the requirement that $\sup_{(k_1, k_2) \in H} \{s^{12}(k_1, k_2)\} = \log(\bar{S}^{12})$ and $\inf_{(k_1, k_2) \in H} \{s^{12}(k_1, k_2)\} = \log(\underline{S}^{12})$ and the application of lemma 1: since the elements of the matrix $\hat{M} \in R^{2 \times 3}$ are positive, the function $e^i(X_1, X_2, X_3)$ is, for $i = 1, 2$, monotonically increasing in $(X_1, X_2, X_3) \in S = [\underline{X}_1, \bar{X}_1] \times [\underline{X}_2, \bar{X}_2] \times [\underline{X}_3, \bar{X}_3]$. However, given that the third 1×3 row vector of M is the difference of the first row vector minus the second, it is no longer guaranteed that it will be a positive vector and thus the function $e^{12}(X_1, X_2, X_3)$ is monotonically increasing (decreasing) in $X_j \in [\underline{X}_j, \bar{X}_j]$ if $m_{1j} - m_{2j} > 0$ (< 0) for $j = 1, 2, 3$. This, together with the fact that $e^i(X_1, X_2, X_3)$ is additively separable implies that the boundary conditions that achieve

$$\sup_{(X_1, X_2, X_3) \in [\underline{X}_1, \bar{X}_1] \times [\underline{X}_2, \bar{X}_2] \times [\underline{X}_3, \bar{X}_3]} \{e^i(X_1, X_2, X_3)\} = \log(\bar{S}^i) \quad (2.54)$$

and

$$\inf_{(X_1, X_2, X_3) \in [\underline{X}_1, \bar{X}_1] \times [\underline{X}_2, \bar{X}_2] \times [\underline{X}_3, \bar{X}_3]} \{e^i(X_1, X_2, X_3)\} = \log(\underline{S}^i), \quad (2.55)$$

for $i = 1, 2$ and 12 are precisely 2.53 evaluated at some vertices.

Step 3: Mapping back the solution into the domain of the fundamentals

Now we can write the solution 2.52 in terms of the fundamentals instead of the transformed variables x_t using the transformation $x_t = M^+ \hat{k}_t + (I - M^+ M) P \Lambda_t$ where $M^+ = [v_{ij}] \in \mathcal{R}^{3 \times 3}$, and we take $v^{(j)}$ to be the j -th row vector from M^+ , for $j = 1, 2, 3$. The resulting expression is given in the statement of theorem 3. The form of the domain of the fundamentals G follows clearly from the linear relationship between the fundamentals and the auxiliary variables in the interior of the domain for the auxiliary variables and the fact that Λ_t increases only when the fundamentals hit the boundary of their domain. Now, taking into account that $(\hat{k}_1, \hat{k}_2, \hat{k}_3) \equiv (k_1, k_2, k_1 - k_2)$ we can embed G on a domain $H \subset \mathcal{R}^2$ by letting $H = \{(x, y) \in \mathcal{R}^2 \mid (x, y, x - y) \in G \subset \mathcal{R}^3\}$ where $G = \{a \in \mathcal{R}^3 \mid \exists b \in S = [X_1, \bar{X}_1] \times [X_2, \bar{X}_2] \times [X_3, \bar{X}_3] \subset \mathcal{R}^3 \ni a = Mb\}$. Finally, $s^i(k_1, k_2, \Lambda)$ is Markov because $s^i(k_1, k_2, \Lambda) = e^i(X_1, X_2, X_3)$, $\Omega \times [0, \infty)$ -everywhere, and x has the strong Markov property. \square

Proof of Proposition 4: Define $k' = A^{-1}k$. Then z behaves like a two-dimensional reflected brownian motion in the interior of the quadrilateral with covariance matrix $I_{2 \times 2}$ and reflection matrix $\Omega = A^{-1}R \in \mathcal{R}^{2 \times 4}$. The regulator processes of k and k' are collinear by uniqueness of Λ in 3.26 (Harrison and Reiman (1981) theorem 1, and Harrison (1985) proposition 2.6). With such transformation of the problem, the result 2.42 follows from theorem 6.1 in Harrison and Williams (1987)²⁷, and the application of the Jacobian method. \square

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²⁷Harrison and Williams (1987) prove that the steady-state distribution of a reflected brownian motion with diagonal covariance matrix on a polyhedral domain with an oblique, skew-symmetric-decomposable reflection matrix belongs to the exponential family.

2.10 Appendix 2. Derivation of the Fundamental Exchange Rate Asset Pricing Equation from First Principles.

The objective of this appendix is to arrive to the basic exchange rate pricing equation relating exchange rate and some aggregate composite of macroeconomic variables denominated fundamentals. This equation is $s_t^i = k_t^i + \gamma E_t(\frac{ds_t^i}{dt})$ and is typical of the target zone literature. We will base our whole analysis on this equation holding for every pair of countries in a multilateral target zone. This equation has seen several justifications in the literature (see Krugman (1991) and Svensson (1991) and (1992), for example). However, unlike previous derivations, we want to obtain it without imposing restrictions on naturally endogenous variables (i.e. the nominal interest rate differential) but rather on processes which are usually part of the primitives of any equilibrium model (i.e. endowment process). The payoff is that we will be able to generalize our results to arbitrary preference structures that lead to different real implications (in asset pricing, for example). In essence, we will model the target zone problem as a *nominal* phenomenon and we will make it compatible with any representative agent exchange economy provided that the endowment dynamics and money supply shocks satisfy certain restrictions.

I insist that this derivation is not the unique way to get to the basic asset pricing equation. Manipulation of a monetary model plus an assumption of (log) uncovered interest rate parity would also work. To my knowledge the derivation here is the only one that only makes assumptions on primitives of a general equilibrium model. However, it makes very strong assumptions on money demand (unit velocity) and therefore it is a matter of taste how seriously it can be taken as a foundation of what is presented in the text. In no way, however, do the results of the paper depend on the following.

I consider a trilateral exchange rate target zone. The following assumptions are the standard assumptions in the target zone literature following Krugman (1991):

First fix a probability space $(\Omega, \mathcal{B}, \{\mathcal{B}\}_{t \geq 0}, P)$. The uncertainty in the economy is driven by a bivariate Wiener process and $\{\mathcal{B}\}_{t \geq 0}$ is the filtration generated by this process.

Assumption 1: There exists one perishable consumption good and two currencies in the economy. On the previous space let S_t^1, S_t^2 , r_t^1, r_t^2 and r_t^0 , Y_1 , Y_2 , and Y_0 , M_t^1 , M_t^2 and M_t^0 , P_t^1 , P_t^2 , and P_t^0 , be progressively measurable stochastic processes that denote the exchange rate levels (currency of country #1 and #2 per unit of currency of country #0), nominal interest rates, domestic endowment flows, money supplies, and price levels in countries 1,2, and 0, respectively. Goods markets and financial markets are frictionless, therefore PPP holds:

$$P_t^i = S_t^i P_t^0 \quad i = 1, 2 \quad (2.56)$$

Assumption 2: Residents of each country must use their own currency to finance purchases of the consumption good. All currency is made available immediately to agents in each country. In particular, in our frictionless set-up the following-cash-in advance (Clower) constraints must hold:

$$\frac{M_t^i}{P_t^i} = Y_t^i \quad i = 1, 2, 0. \quad (2.57)$$

Thus money could not be introduced in simpler way in the economy²⁸. Note, however, that the money supply process in our economy will be endogenous. More precisely, it will be decomposed in an endogenous and an exogenous part.

Assumption 4: The endowment and money supply processes follow the dynamics (expressed in integral form), for $t > s > 0$:

²⁸We can dress-up the above assumption with a little story as in Lucas (1982): Imagine an agent as a household composed by two members. The first member sells the whole household endowment at period $t - \delta$ to strangers (or the government) in exchange for money. At period t , second member uses the money balances to acquire the consumption goods for the household. In a single household economy with random endowment flow Y_t , the household sells his endowment for $p_{t-\delta} Y_{t-\delta} = M_{t-\delta}$ where M_t stand for the total nominal money balances standing at time t . At t , the household buys consumption goods using the money, and can thus finance C_t consumption units where $p_t C_t = M_{t-\delta}$. Now let $\delta \downarrow 0$ and if all the processes involved are left-continuous, in equilibrium we get $C_t = Y_t = \frac{M_t}{P_t}$.

$$\begin{aligned}
M_t^i &= M_s^i + \int_s^t (a^{m^i} - b_v^{m^i} \log(M_v)) M_v dv + \int_s^t M_v \sigma^{m^i} dW_v + \int_s^t M_v dR_v^i \\
Y_t^i &= Y_s^i + \int_s^t (a^{y^i} - b_v^{y^i} \log(Y_v)) Y_v dv + \int_s^t Y_v \theta^{y^i} dW_v
\end{aligned} \tag{2.58}$$

where a^{m^i}, a^{y^i} are scalars, $\sigma^{m^i}, \theta^{y^i}$ are 2×1 vectors, and $b^{y^i}, b^{m^i} : \Omega \times [0, \infty) \rightarrow \mathcal{R}$ are $\{\mathcal{B}\}_{t \geq 0}$ -progressively measurable processes with the Markov property, and R_t^i is a continuous, finite variation process which is $\{\mathcal{B}\}_{t \geq 0}$ -progressively measurable and that, for now, we leave unspecified. We will later obtain it endogenously. Thus, the money supply consists of an exogenous part ($dM_t - M_t dR_t$) and an endogenous one, $M_t dR_t$, which will be interpreted as unsterilized interventions in the FX market by the monetary authorities.

Both the exogenous component of the money supply and the endowment processes follow a generalized mean reverting process. Note that these processes for the endowment dynamics are sufficiently general to allow decreasing, constant or increasing stochastic returns to scale²⁹. To motivate further the use of this kind of processes, note that the endowment process in 2.58 combined with a log-utility representative agent economy would produce, as the real interest rate equilibrium process, the popular Vasicek instantaneous interest rate process³⁰. It will be convenient to write the dynamics of the processes in 2.58 in logarithms, using Itô's lemma:

$$\begin{aligned}
dm_t^i &= \left(c^{m^i} + \left(\frac{1}{\gamma}\right)(1 - \alpha_t^i) m_t^i \right) dt + dR_t^i + \sigma^{m^i} dW_t \\
dy_t^i &= \left(c^{y^i} + \left(\frac{1}{\gamma}\right)(1 - \beta_t^i) y_t^i \right) dt + \theta^{y^i} dW_t \\
i &= 1, 2, 0.
\end{aligned} \tag{2.59}$$

where lower cases denote logarithms, and for $i = 1, 2, 0$; $c^{m^i} = a^{m^i} - \frac{\|\sigma^{m^i}\|^2}{2}$, $c^{y^i} =$

²⁹A physical production processes is said to exhibit constant returns to scale if the distribution of its return process is independent of the size at which the production process is operated. It will have increasing (decreasing) stochastic returns to scale if the realization of its return process is uniformly larger (smaller) across all states of nature when the process is operated at a larger scale.

³⁰That is to say, a gaussian, mean reverting process for the instantaneous real interest rate (see Vasicek (1977)). To see this, compute the expected rate of change in the marginal utility of a log-utility investor using Itô's lemma and substitute the endowment process 2.58 as the equilibrium consumption process.

$a^{y^i} - \frac{\|\sigma^{y^i}\|^2}{2}$, $\alpha_t^i = 1 + \gamma b_t^{m^i}$, $\beta_t^i = 1 + \gamma b_t^{y^i}$. We impose $\alpha^i, \beta^i \in \Pi$ where Π denotes the set of diffusions on our probability space such that the (4×1) vector diffusion obtained by stacking the processes 2.59 and α^i, β^i satisfies the appropriate Lipschitz and growth conditions such that a unique strong solution to 2.59 (and therefore 2.58) exists.³¹

I insist that what we are in fact specifying is a money supply process that has two components. The first are monetary shocks that follow an exogenous stochastic process, and the second is an endogenous smooth process that stands for the interventions of the monetary authorities in the FX market.

Taking logarithms in 2.56 and 2.57 and subtracting the result evaluated at $i = 1, 2$ and 0 we obtain:

$$s_t^i = m^i - m^0 + y^0 - y^i \quad (2.60)$$

where $s^i = \log(S^i)$. From 2.60 it follows that the exchange rate is an Itô diffusion with dynamics:

$$ds_t^i = dA_t + \sigma_{s^i}' dW_t \quad (2.61)$$

where $A : \Omega \times [0, \infty) \rightarrow \mathcal{R}$ is a Markov, finite variation process and $\sigma_{s^i}(\cdot, \cdot) : \Omega \times [0, \infty) \rightarrow \mathcal{R}^2$ is a progressively measurable processes. Applying Itô's lemma to 2.60 using 2.59 and rearranging, we obtain that the following relationships holds almost surely³²:

³¹Note that Π is not empty as it contains the processes α^i, β^i constant, as the reader can immediately verify, the solution in this case is

$$\begin{aligned} m_t &= m_s \exp\left[-\left(\frac{\sigma^{m^2}}{2} - \left(\frac{1}{\gamma}\right)(1 - \alpha^i)\right)t + \sigma^{m^1} W_t\right] + \\ &+ e^{m^1} \int_s^t e^{-\left(\frac{\sigma^{m^2}}{2} - \left(\frac{1}{\gamma}\right)(1 - \alpha^i)\right)(t-v) + \sigma^{m^1} (W_t - W_v)} dv + \\ &+ \int_s^t e^{-\left(\frac{\sigma^{m^2}}{2} - \left(\frac{1}{\gamma}\right)(1 - \alpha^i)\right)(t-v) + \sigma^{m^1} (W_t - W_v)} dR_v \end{aligned}$$

the solution for y in this case is identical except for the last integral on the RHS.

³²Here we replace, heuristically, the finite variation part of the Doob-Meyer decomposition of the semimartingale s^i by the instantaneous conditional expectation operator $\frac{E_t(\cdot)}{dt}$. This is possible because the finite variation process I is continuous, and therefore predictable (see Dellacherie and Meyer, (1982), VII.25).

$$s_t^i dt = k_t^i dt + \gamma E_t(ds_t^i) \quad i = 1, 2 \quad (2.62)$$

where k_t^i is an aggregate, progressively measurable, process such that:

$$k_t^i dt = (c^{y^i} - c^{y^0} - c^{m^i} + c^{m^0})\gamma dt + (dR_t^0 - dR_t^i)\gamma + (\alpha_t^i m_t^i - \alpha_t^0 m_t^0 + \beta_t^0 y_t^0 - \beta_t^i y_t^i) dt \quad (2.63)$$

Equation 2.62 is the usual starting point of the literature. Note that we obtained it without any particular assumption relating exchange rates and interest rates. $k_t = (k_t^1, k_t^2)$ will be the state variable vector that drives the uncertainty in the model. Note also that the dynamics of k will depend now on the specification of the dynamics for the exogenous processes α^i , and β^i $i = 0, 1, 2$. In the next section we will specialize these processes to obtain particularly convenient dynamics for k .

We have arrived at equation 2.62 without imposing *ad hoc* restrictions in naturally endogenous variables in a general equilibrium model (i.e. the interest rate differential). Rather, we obtained 2.62 with assumptions on the primitives of a general equilibrium exchange economy, namely the endowment process, the exogenous part of the money supply process plus a cash-in-advance (or Clower) constraint. Therefore, the predictions obtained regarding the exchange rate behavior will carry through no matter how we close the model by imposing some kind of preference structure of a representative agent, for example, because of asset pricing purposes. Thus the model offers great analytical tractability and considerable generality. The price we paid consists of imposing a rather *ad hoc* structure on the exogenous monetary shocks.

Note now from 2.63 that the dynamics of the fundamentals will depend on the processes $\alpha^i, \beta^i \in \Pi$ that we choose. Suppose that we restrict ourselves to bounded variation process in Π , i.e. process that can be written $\alpha_t^i = \int_0^t \mu_{\alpha_i}(s) ds$, $\beta_t^i = \int_0^t \mu_{\beta_i}(s) ds$ for some progressively measurable processes μ_{α_i} and μ_{β_i} . Applying Itô's lemma to 2.63, we observe that the fundamentals process minus the finite variation process

$(dR_t^0 - dR_t^i)\gamma$ will be a diffusion given by:

$$\begin{aligned}
dk_t = & [\alpha_t^i (c^{m^i} + (\frac{1}{\gamma})(1 - \alpha_t^i)m_t^i) + m_t^i\mu_{\alpha^i} - \alpha_t^0 (c^{m^0} + (\frac{1}{\gamma})(1 - \alpha_t^0)m_t^0) - m_t^0\mu_{\alpha^0} \\
& - \beta_t^i (c^{y^i} + (\frac{1}{\gamma})(1 - \beta_t^i)y_t^i) - y_t^i\mu_{\beta^i} + \beta_t^0 (c^{y^0} + (\frac{1}{\gamma})(1 - \beta_t^0)y_t^0) + y_t^0\mu_{\beta^0}]dt + \\
& + (\alpha_t^i\sigma^{m^i} - \alpha_t^0\sigma^{m^0} - \beta_t^i\sigma^{y^i} + \beta_t^0\sigma^{y^0}) dW_t
\end{aligned}
\tag{2.64}$$

Now note that in 2.64, for example if we set the processes α and β to be constant and identical to one, the process k_t behaves like an arithmetic brownian motion on the subset of $\Omega \times [0, \infty)$ where $R_t^0 - R_t^i$ is constant (note also that this also implies that the endowment processes and the exogenous part of the money supply process follow lognormal processes)³³.

³³With drift $\mu_k = c^{m^i} + c^{y^0} - c^{y^i} - c^{m^0}$ and diffusion vector $(\sigma^{m^i} + \sigma^{y^0} - \sigma^{m^0} - \sigma^{y^i})'$.

Chapter 3

An Empirical Examination of a Multilateral Target Zone Model

(with Paul M. Schulstad)

3.1 Introduction

The goal of the target zone literature has been to characterize the behavior of exchange rates in the context of a two-country monetary model where the range of variation of the exchange rate is bounded by a currency band agreement among monetary authorities. The original model of Krugman (1991) was constructed under the assumptions that the target zone is perfectly credible (the commitment of monetary authorities to keep the exchange rate within the band is complete, and they have the ability to do so) and that marginal interventions are the only intervention tool (monetary authorities intervene by manipulating the relative money supplies if and only if the exchange rate hits the limits of the band). The empirical performance of the model has been, however, very poor (see for example Flood, Rose and Mathieson (1991), Lindberg and Söderlin (1991), DeJong (1994) and Smith and Spencer (1992)). The reasons for the model's rejection lie in the clearcut but counterfactual predictions of the bilateral model. First, the exchange rate should exhibit an unconditional U-shaped distribution within the band; second, the conditional volatility of inter-

est rates should approach zero when the exchange rate approaches the limits of the band. Extensions of the basic bilateral model, which relax the assumptions of perfect credibility and marginal interventions as the leading intervention tool, reconcile the predictions of the model with the evidence, although we are not aware of formal tests of such models.

There is a feature, however, whose theoretical and econometric analysis has been completely neglected by the literature and is actually relevant in real-world target zones: currency bands agreements usually involve more than two currencies. In Serrat (1994), a multilateral target zone with an arbitrary number of currencies is modeled in the spirit of Krugman (1991). This model is reviewed below. He obtains closed form solutions expressing the exchange rate as a function of vector of underlying aggregate macroeconomic state variables, denominated fundamentals. The predictions of the multilateral model seem to be much more in accordance with the empirical evidence than those of the bilateral model. In particular, a hump-shaped steady-state distribution of the exchange rate can be recovered and the amount of curvature in the conditional volatility of the exchange rate as it approaches the limits of the band (the size of “honeymoon effect”) is allowed to vary randomly. Following Krugman’s original assumptions, the model is constructed maintaining the assumptions of perfect credibility and marginal interventions, in the space of the fundamentals, as the unique intervention tool.

There are two main features that make the multilateral target zone model different from the bilateral model. First, the existence of cross-currency constraints make the range of variation of one currency versus another tighter than the official bilateral one because the bilateral band with a third currency becomes binding while the exchange rate between the first two currencies is in the interior of its official band. Second, recall from the bilateral model of Krugman (1991), that the fundamentals process is constructed as a linear combination of macroeconomic processes from the *two* countries involved in the target zone agreement. An intervention designed to manipulate the fundamentals between two currencies involved in the system will inevitably affect the fundamentals processes of other currency pairs. In the model it

can be shown that even if we choose the parameters such that the first feature above is nowhere binding (such a case corresponds to a *currency influence area*), the second feature alone makes the predictions of the model differ substantially from those of the bilateral model. They actually provide a rationale for a certain type of intramarginal interventions in the space of the exchange rates, not in the space of the fundamentals. In particular, the predictions that change are the problematic ones (unconditional U-shaped distribution and an always strong “honeymoon effect”) while the appealing ones are kept (namely the stabilizing effect on the exchange rate of the currency agreement). The purpose of this paper is to conduct an econometric evaluation of the multilateral model using a simulated GMM technique, which is the predominant approach for parameter estimation and hypothesis testing in latent variable models (see, for instance, Ferson and Foerster (1994)). In Section 3.1.1, we review the empirical evidence regarding Krugman’s target zone model. In Section 3.2, we outline the multilateral target zone model of Serrat (1994). In Section 3.3 we present the data to be used in the analysis. Section 3.4 discusses the econometric methods used. Section 3.5 presents the results for the versions of the model estimated, and the results of the Monte-Carlo analysis to approximate the power of our test. In Section 3.5, we also present and discuss the results of performing an out-of-sample comparison of the predictive power of the model versus a random walk. Section 3.6 concludes.

3.1.1 Empirical Evidence on the Bilateral Model

Empirical work on Krugman’s model can be classified into papers that restrict themselves to testing the specific nonlinear specification of the model and papers that use more general methods, including nonparametric methods, to test the qualitative implications of the model.

Flood, Rose and Mathieson (1991) conducted the first extensive empirical analysis of Krugman’s model. Assuming uncovered interest rate parity, they construct an instrument for the fundamentals which is equivalent to a weighted sum of the forward and spot rates. They conclude that evidence for the presence of non-linearities in exchange rates within target zones is weak and that their signs are not those pre-

dicted by the basic model. In addition, evidence regarding the stabilization effect predicted by Krugman's model (the "honeymoon effect") is ambiguous. Gourinchas (1994) finds evidence of nonlinearities using a semiparametric approach with a sensible instrumentation of the fundamentals process. Moreover, he uncovers the nature of the nonlinearities as possibly due to a phenomenon of asymmetric credibility, not captured by the bilateral model.

A significant degree of mean-reversion in exchange rates within the band was found in Rose and Svensson (1991) (estimating the Bertola and Svensson (1993) model). Pesaran and Samiei (1992), in a discrete time rational expectations model, found that explicitly incorporating beliefs about stabilizing marginal interventions helps in fitting data for the German Mark/French Franc exchange rate. However, contrary to the predictions of the bilateral model (and consistent with those of the multilateral model) they find that the S -shaped relationship between exchange rates and fundamentals is stochastic rather than deterministic. Several studies have applied simulated method of moments estimation, with different variations, to test Krugman's model: Lindberg and Söderlin (1992) propose a mean-reverting specification for the fundamentals process that matches Swedish data better than Krugman's model. Smith and Spencer (1992) estimate Krugman's model for the Italian Lira/German Mark exchange rate series, although they report convergence problems and many relevant statistics are not reported in their paper. De Jong (1994) applies both maximum likelihood and simulated moments methods to study the fit of the bilateral model for several EMS currencies for the same time period as ours. He finds that the parameters estimates differ substantially across countries. In addition, precise parameter estimates are not, in general, obtained. He rejects Krugman's model for the Dutch Guilder, the French Franc and the Italian Lira against the German Mark using an overidentifying restrictions test. Overall he concludes that Krugman's model is misspecified and suggests extensions of the model in terms of the underlying dynamics of the state variable process.

As mentioned in the previous section, the multilateral target zone model can potentially explain several of the empirical regularities at odds with Krugman's model,

in the context of a fully credible model with marginal interventions. The fact that notional bands can differ from nominal bands could account for reversing one of the most problematic predictions of Krugman's model, namely, an unconditional steady-state density for the exchange rate within the band. This point has been noted elsewhere in the literature (e.g. Pill (1994)). In Serrat (1994) it is shown that not only cross-currency restrictions but also spillover effects from foreign exchange interventions on third currency fundamentals can reverse Krugman's theoretical predictions. If we accept the multilateral model as a good description of reality, then we will have to conclude that the previous empirical literature has been flawed in attempting to test Krugman's model. The poor empirical results found in the literature may not be due to the inappropriateness of Krugman's assumptions of perfect credibility and marginal interventions as the exclusive intervention tool, but rather to the fact that the tests have been performed on an overly restrictive model on data from multilateral target zones. The multilateral nature of real-world target zones makes Krugman's bilateral model inappropriate even under the most generous assumptions (i.e. that cross currency constraints do not matter and thus a multilateral target zone is, in fact, a currency influence area).

In this paper we apply a simulated method of moments technique (MSM) to estimate the model. One of the advantages of this technique is that instrumentation of the fundamentals process is not needed in deriving testable hypotheses from the model. This is important since no particular construction of the fundamentals process is neither assumed nor implied by the model. Naturally, we choose a time span for which Krugman's assumption of perfect credibility may not be far from reasonable. The results strongly support the multilateral model, not only against Krugman's bilateral model, but also against a reasonable alternative hypothesis, as explicit power computations show. We find that the multilateral feature of real-world target zones is crucial in understanding the results of target zone models and that the pessimistic opinion about the poor performance of target zone models must be reconsidered.

This paper presents an ambitious implementation of the MSM technique, in terms of computational demands. In the next section we outline the model and the equation

that we estimate. We first estimate a trilateral version of the model for ten groups of currencies with each group consisting of the German Mark and other two EMS currencies. We then generalize further and estimate a five-currency version of the multilateral model.

We find that both the three-currency (for all groups of currencies) and the five-currency versions of the model are not rejected at the usual levels of significance. In addition, when the parameters of the multilateral target zone model are restricted so as to obtain a multilateral target zone as a combination of simple bilateral (Krugman) versions of the model, we find that these restrictions are strongly rejected by the data. We also conduct an experiment to approximate the power of the overidentifying restrictions test associated with the MSM technique under reasonable specifications for the data generating process under the alternative hypothesis. The model is correctly rejected in most of the cases. It turns out that the estimates of the reflection matrix drive the good performance of the model which is precisely the aspect of reality neglected by the previous literature.

3.2 The Multilateral Target Zone Model

The multilateral model has different theoretical implications than the bilateral model due to the existence of an additional set of parameters that capture the degree of cooperation among central banks in sharing the intervention burden. In the multilateral model, the state variable consists of an n -dimensional vector of fundamentals that is reflected at each side of the fundamentals domain, (an n -dimensional polyhedron). From the $n + 1$ countries involved in the target zone, set a reference (or anchor) country 0.

The underlying theory of target zone modelling is usually obtained from a minimalist monetary model in the following way. Let the money demand equations satisfy $\frac{M_t^i}{P_t^i} = y_{it}^{\alpha^i} \exp(-\gamma r_t^i)$ for each country i ($i = 1, \dots, n$) in the target zone, where M_t^i , P_t^i , y_{it} , and r_t^i denote the money supply, price level, aggregate endowment and nominal instantaneous interest rate processes for country i at time t . α^i and γ (the

semi-elasticity of money demand with respect to the interest rate) are constants. Let purchasing power parity and a logarithmic version of uncovered interest rate parity hold; thus, $P_t^i = S_t^{ij} P_t^j$ and $r_t^i - r_t^j = \frac{E_t(ds_t^{ij})}{dt}$ where $S_t^{ij} = \exp(s_t^{ij})$ is the nominal exchange rate of country i with respect to country j . Taking logarithms of both sides of the money demand equation for countries i and j , subtracting them and using the last two equations, we obtain:

$$s^i = k^i + \gamma \frac{E_t(ds_t^i)}{dt} \quad (3.1)$$

where k^i , the i^{th} component of the fundamentals process, is constructed as a function of underlying macroeconomic variables:

$$k^i = m^i - m^0 + \alpha^i y^i - \alpha^0 y^0 \quad (3.2)$$

where m^l and y^l are respectively the logarithms of the money supply and aggregate endowment of country l . α^l, α^l and γ are constants, for $l = i, 0$.

The fundamentals vector is a regulated n -dimensional reflected arithmetic brownian motion that takes values in an n -dimensional polyhedron, $G \subset \mathcal{R}^n$, with a $n \times 2n$ reflection matrix called R whose column vectors are the reflection vectors at each side of the polyhedron. The concept of reflection vector is explained below. These dynamics are described by the stochastic differential equation:

$$dk_t = \mu dt + A dW_t + \sum_{i=1}^{2n} R^i d\Lambda_t^i \quad (3.3)$$

$$k_0 \in \overset{\circ}{G} \subset \mathcal{R}^n \quad (3.4)$$

where μ is a $n \times 1$ drift vector, A is a $n \times n$ diffusion matrix and W_t is a $n \times 1$ Wiener process. The regulator process $\Lambda_t = \{\Lambda_t^i\}_{i=1}^{2n}$ is a nondecreasing $2n$ -dimensional process that increases if and only if the fundamentals hits the m^{th} side of the polyhedron that constitutes their domain, for $m = 1, \dots, 2n$. It can be proved that the process in

(3.3) is Markov.

The matrix composed of the vectors that indicate the direction of reflection at each side of the fundamentals domain, namely $R = [R^1 : \dots : R^{2n}]$ where R^i is a n -dimensional vector, can be interpreted as a measure of the degree of cooperation among central banks in maintaining the currency band agreement (see Serrat (1994) for details). Thus, when the fundamentals process hits any of the $2n$ sides of its domain, the relative money supplies $m^i - m^0$ are adjusted by the monetary authorities to keep the fundamentals process within its domain. The direction in which the fundamentals process is reflected back is related to the relative intensity with which the relative money supplies (or, in this context, foreign exchange interventions) vary. This is exactly what is captured by the reflection matrix R . By assumption, the intervention rules are symmetric, in the sense that $R^i = -R^{n+i}$ for $i = 1, \dots, n$. Thus, if two central banks share the intervention burden in a certain way when the fundamentals hits a certain face of its domain, the roles are switched when the fundamentals hit the opposite face of their domain.¹ It will be convenient to write $R = [M : -M]$ where $M = [R^1 : \dots : R^n]$ is a $n \times n$ matrix. Hereafter, we denote M the reflection matrix. It is also important to note that the size of the reflection vector does not matter, only its direction. Thus, we can normalize the column vectors of M to have unit norm.² This is important from an econometric point of view; otherwise, the model would not be identified.

The model collapses naturally to Krugman's model if we impose that cross-currency constraints are not binding and that the reflection matrix is the identity matrix.³ This corresponds to a currency influence area in which the anchor currency never collaborates in the foreign exchange interventions by manipulating its own money supply.

Note from equation (3.1) that the fundamentals of country i are modified by the

¹Note that this is a reasonable assumption, the formal design of EMS intervention rules is entirely symmetrical by the 'Belgian Compromise', see Vehrkamp (1994, page 28).

²With this normalization, if the reflection matrix is diagonal it is normalized to be the identity matrix.

³I.e. $R = [I_{n \times n} : -I_{n \times n}]$.

monetary authorities of country i and/or those of country 0. Thus, if the monetary authority of the anchor currency never intervenes and the burden of intervention falls on non-anchor currencies, then there are no spillover effects on third country fundamentals. This corresponds to a diagonal M matrix. In general, if country i intervenes by (say) decreasing its money supply at some point on the boundary of the fundamentals and country 0 increases its money supply to help decrease country i fundamentals, then the fundamentals of country j will also decrease, although by a smaller amount than country i 's fundamentals. Thus, the further any particular column vector of M is from being parallel to any of the axes (i.e. M diagonal), the higher is the involvement of the anchor currency in the intervention operations and thus the higher is the degree of real symmetry in the system. Note also that each column vector of M can be associated with a particular currency.

We always expect that the elements of each column vector of the reflection matrix have the same sign: central bank interventions should work towards the same goal. We also expect the diagonal elements of the M matrix to be larger than the off-diagonal elements, because they indicate a more pronounced direction of reflection for the fundamentals process of the country whose currency is the weakest (or strongest) in that particular region of the domain for the fundamentals. This arises because in our application, we order the data such that the elements of the diagonal of the matrix M correspond to the fundamentals of the currency with the weakest (or strongest for the complementary reflection matrix, $-M$) position. For example, suppose that M is a 2×2 matrix (corresponding to a three country target zone). If the interventions of the anchor currency are of smaller magnitude than the interventions of the weakest (or strongest) currency at some point on one of the sides of the domain for the fundamentals, then the ratio that we should observe between the largest and smallest component of each column vector of M should be larger than 2. This is precisely what we obtain in section 3.5 for most of the cases (i.e. different combinations of trilateral target zones).

The expression for the logarithm of the exchange rate of country i , as a function of the fundamentals in a n -lateral target zone, when cross currency constraints are

nowhere binding, is obtained in Serrat (1994) using equations (3.1) and (3.2) and is given by:

$$s^i(k^1, \dots, k^n) = \gamma\mu_i + k^i + \sum_{j=1}^n C_{1j}^i \exp\left(\lambda_+^j \left(\sum_{l=1}^n v_l^{(j)} k^l\right)\right) + \sum_{j=1}^n C_{2j}^i \exp\left(\lambda_-^j \left(\sum_{l=1}^n v_l^{(j)} k^l\right)\right) \quad (3.5)$$

where $v^{(j)}$ is the j^{th} row vector of M^{-1} . The constants λ_+^j, λ_-^j $j = 1, \dots, n$, are given by

$$\lambda_+^j = \frac{-\theta_j + \sqrt{\theta_j^2 + 2\frac{\phi_j^2}{\gamma}}}{\phi_j^2} \quad \lambda_-^j = \frac{-\theta_j - \sqrt{\theta_j^2 + 2\frac{\phi_j^2}{\gamma}}}{\phi_j^2} \quad (3.6)$$

where $\theta = M^{-1}\mu$ and $\phi \odot \phi = \text{diag}(M^{-1}AA'M^{-1})$ are $n \times 1$ vectors, $[v_{ij}] = [M^{-1}]_{ij}$ and where \odot indicates member-wise multiplication. The constants, C_{1j}^i, C_{2j}^i are obtained as the solution of a $2n$ -dimensional system:

$$m_{ij} + \lambda_+^j C_{1j}^i \exp\left(\lambda_{1j} \left(\sum_{l=1}^n v_l^{(j)} \bar{k}^l\right)\right) + \lambda_-^j C_{2j}^i \exp\left(\lambda_{2j} \left(\sum_{l=1}^n v_l^{(j)} \bar{k}^l\right)\right) = 0 \quad (3.7)$$

$$m_{ij} + \lambda_+^j C_{1j}^i \exp\left(\lambda_{1j} \left(\sum_{l=1}^n v_l^{(j)} \underline{k}^l\right)\right) + \lambda_-^j C_{2j}^i \exp\left(\lambda_{2j} \left(\sum_{l=1}^n v_l^{(j)} \underline{k}^l\right)\right) = 0. \quad (3.8)$$

Finally, $\bar{K} = (\bar{k}_1, \dots, \bar{k}_n)$, $\underline{K} = (\underline{k}_1, \dots, \underline{k}_n)$, together with the auxiliary variables vectors $(\bar{X}_1, \dots, \bar{X}_n)$ and $(\underline{X}_1, \dots, \underline{X}_n)$, solve the $4 \times n$ system of equations:⁴

$$s^i(M[\bar{X}_1, \dots, \bar{X}_n]) = \log(\bar{S}^i), i = 1, \dots, n \quad (3.9)$$

$$s^i(M[\underline{X}_1, \dots, \underline{X}_n]) = \log(\underline{S}^i), i = 1, \dots, n \quad (3.10)$$

$$(\bar{k}_1, \dots, \bar{k}_n)' = M(\bar{X}_1, \dots, \bar{X}_n)' \quad (3.11)$$

$$(\underline{k}_1, \dots, \underline{k}_n)' = M(\underline{X}_1, \dots, \underline{X}_n)'. \quad (3.12)$$

⁴Note that we are implicitly assuming that each element of M is greater than or equal to zero.

It is important to note that if we impose that M be a diagonal matrix (and thus normalize it to be the identity matrix) then the solution in (3.5) collapses to Krugman's solution. The matrix M controls for the amount of spillover effects of foreign exchange interventions on third-currencies fundamentals. Therefore, we insist on this point: it is not the fact that the components of the fundamentals vector are correlated, but rather that M is non-diagonal (i.e. there is collaboration among monetary authorities for intervention purposes) that makes the multilateral solution differ in an essential way from the bilateral solution of Krugman.

Let $S^{12} = \frac{S^1}{S^2}$ be the cross-exchange rate between currencies 1 and 2 (quantity of currency 1 per unit of currency 2). The equation (3.5) is obtained in Serrat (1994), theorem 1, by assuming that the nominal target zone for the cross exchange rate, namely, $[\underline{S}^{12}, \bar{S}^{12}]$ is wide enough so that the cross-currency constraint is never binding. In other words, (3.5) is valid as long as:

$$\underline{S}^{12} \leq \inf_{(k^1, k^2) \in G} \frac{s^1(k^1, k^2)}{s^2(k^1, k^2)} \leq \sup_{(k^1, k^2) \in G} \frac{s^1(k^1, k^2)}{s^2(k^1, k^2)} \leq \bar{S}^{12} \quad (3.13)$$

is satisfied. In Serrat (1994) theorem 3, a closed form solution is provided when (3.13) is not satisfied. However, such solution is much more difficult to implement econometrically. Thus, if the parameters of the problem happen to satisfy (3.13), (3.5) provides a valid solution to the trilateral target zone modelling problem even when $[\underline{S}^{12}, \bar{S}^{12}] \subset [\frac{S^1}{\underline{S}^2}, \frac{\bar{S}^1}{\bar{S}^2}]$. In other words, there are limits to the range of variation of the exchange rate between currency 1 and currency 0 and between currency 2 and currency 0 that are due not to the limits of the two respective bands, but rather to the fact that the implied exchange rate between currency 1 and currency 2 (i.e. $\frac{S^1}{S^2}$) hits its own band before either S^1 or S^2 reach theirs.

Our objective in this paper is to estimate (3.5) using the simulated method of moments on data from the European Monetary System (EMS). Given the dimensionality of the problem, we will limit ourselves to the estimation of (3.5) for the three-currency case (the trilateral model) and the five-currency case (the five-lateral model). Additional tests will be conducted to test Krugman's model as a special case.

In addition, the power of our model specification test (the test overidentifying restrictions associated with minimum distance estimators) will be tested against reasonable specifications for the alternative hypothesis. Regarding the cross-currency constraints restrictions, in all estimations we will assume that (3.13) is satisfied. Once we obtain our estimates, we will check whether this is indeed the case.

3.3 Data

We use weekly observations of exchange rates data from the EMS, in particular, the Belgian Franc (BFr), Dutch Guilder (DFl), Danish Krone (DKr), French Franc (FFr) and Irish Pound (IP) versus the German Mark (DM). The sample consists of 189 observations from January 14, 1987 to August 22, 1990. We choose this sample period for two reasons. First, the multilateral model assumes perfect credibility which means that realignments are not allowed. Thus, in an effort to isolate the ideal credibility conditions of the model, we draw our data from the longest period in which the EMS did not experience any realignments, namely the so-called “hard-EMS” period (January 1987-September 1992)⁵. Within that time period, we choose the period from the beginning until the collapse of the eastern bloc (September 1990), because we cannot control for the effect of such events on the overall credibility of the system. Note that this exercise is not evidence of a sample selection bias, but rather the selection of a valid sample -one that does not boldly contradict the basic assumptions, i.e. no jumps, of the model-. Second, to facilitate comparison with previous results for the bilateral model, our time span coincides with that used in the latest empirical study of Krugman’s model (De Jong, 1994) and partially coincides with the sample of Flood, Rose and Mathieson (1991)⁶. In addition, given the high computational costs of estimating the model, we are forced to economize in choosing the currencies in our study. For ease of comparison, we choose the same currencies as in the study of De Jong (1994) (except for the Italian Lira).

⁵With the exception of a realignment of the central parity against DM for the Italian Lira of 3.7% on the January 5, 1990. However, we do not use the Lira in this paper.

⁶Namely their regimes 12 and 13.

During our sample period, each of the currencies in our sample were restricted to lie within 2.2753% of their central parities with respect to the DM and each other currency within our sample of currencies.⁷ Our data have been transformed such that our exchange rate variable for country i is the logarithm of the ratio of the exchange rate divided by the central parity of currency i with respect to the DM. Table 3.2 presents the descriptive statistics for the logarithm (net of central parity) of the exchange rates in our sample. In Figures 3-1, 3-2, 3-3, 3-4 and 3-5, we plot the exchange rates in our sample.

3.4 Econometric Methodology

In this section, we outline the simulated method of moments (MSM) technique for a time series estimation problem. The spirit of this technique is to ask the model to come as close as possible to predicting the observed moments of the exchange rate series. Briefly, given a candidate parameter vector, we draw a random path, much longer than the sample size, from the distribution of fundamentals paths using the dynamics (3.3) evaluated at our candidate parameter vector. We then compute the exchange rate series that corresponds to the simulated fundamentals series using our candidate parameter vector and (3.5) and then we compute the value of certain loss function. The procedure is repeated for many different candidate parameter vectors.⁸ The simulated method of moments estimator is the parameter vector that minimizes the loss function. Under certain conditions, reviewed below, the MSM estimator is consistent and asymptotically normally distributed and a diagnostic test of the overall fit of the model can be constructed. The essential difference between this method and Hansen's GMM (1982) lies in the fact that an analytical solution for the relevant Euler equations as a function of the parameter vector does not exist. On the other hand, the implementation of MSM requires a specific assumption of the economic environment,

⁷See Grabbe (1991) for a full explanation of the formula for the upper and lower limits of the target zone.

⁸Although the noise used in the Monte-Carlo generation of the fundamentals path is the same across candidate parameter vectors.

i.e. the data generating process, which is not required in a GMM investigation of Euler equations. This allows a direct examination of the implications of the model for several moments of exchange rates. In addition, the ability of the model to fit these empirical moments is easy to interpret. In our case, MSM is particularly well suited because an analytical form for the transition density of the exchange rates is unknown (and thus maximum likelihood is not implementable) while we have a full description of the data generating mechanism imposed by the model. The properties of MSM estimators in a time series context have been studied by Duffie and Singleton (1993) and Lee and Ingram (1991). The difference between our problem and McFadden's (1989) lies in the parameter dependency of the simulated series for the fundamentals.

More precisely, suppose we are given a method of moments problem

$$E[g(s, \theta_0)] = 0 \tag{3.14}$$

where s is a $T \times 1$ vector of data, g expresses the moment conditions generated by a distributional theory on the data parametrized by the $p \times 1$ vector θ_0 belonging to some compact parameter set $\Theta \subset \mathcal{R}^p$.⁹ Depending upon the model, it may be impossible to obtain a closed form expression for the $m \times 1$ vector g . This is our case, since we do not know the ergodic distribution or transition densities of the diffusion (3.3) except for very special cases.

Suppose we can identify a measurable transition function $T : \mathcal{R}^n \times \mathcal{R}^n \times \Theta \rightarrow \mathcal{R}^n$ that describes the dynamics of the state variable $k_{i+1} = T(k_i, \varepsilon_{i+1}, \theta)$, where $\{\varepsilon_i\}_{i=1}^{T \times l}$ is an *i.i.d.* sequence of random variables on some probability space and l is the length of the simulation divided by the length of the time series of data, T . Suppose also that we also have a measurable observation function $f : \mathcal{R}^n \times \Theta \rightarrow \mathcal{R}^n$ mapping the range of the state vector to the moments constructed from the dependent variable. In our case, the transition function T corresponds to a discretization of the integral representation of (3.3), which is shown to converge weakly to the true dynamics in

⁹In our application s is an $T \times n$ matrix. The method of moments problem we outline here generalizes easily to this case.

Appendix 3.8. Also, the observation function corresponds to moments constructed from observations of the exchange rate. The MSM estimator circumvents the difficulty of obtaining analytical expressions for the moments by assuming that we have an \mathcal{R}^n -valued sequence of random variables $\{\hat{\varepsilon}_i\}_{i=1}^{T \times l}$, identically distributed but independent from $\{\varepsilon_i\}_{i=1}^{T \times l}$. Then, for any initial point k_0 and parameter vector $\theta \in \Theta$, we can construct inductively a simulated state variable (fundamentals) process by letting $k_0^\theta = k_0$ and

$$k_{j+1}^\theta = T(k_j^\theta, \hat{\varepsilon}_{j+1}, \theta) \quad (3.15)$$

while the simulated \mathcal{R}^n -valued observation process (the moments of the exchange rates) is constructed as $h_j^\theta = h(s(k_j^\theta, \theta))$. Denote by $\{h\}_{t=1}^T$ the $m \times 1$ real valued sequence of moments constructed from the data.

It is convenient to produce a replication of size $T \times l$, where l is an integer, and thus we obtain the series $\{h^\theta\}_{j=1}^{T \times l}$. In this case we can match l replications to each data observation and reorder the above series as $\left\{ \left\{ h_{ti}^\theta \right\}_{i=1}^l \right\}_{t=1}^T$. Now, for any parameter vector $\theta \in \Theta$, we can construct the sequences:

$$\hat{g}_i(s_i, \theta) = \frac{1}{l} \sum_{i=1}^l (h_t - h_{ti}^\theta) \quad (3.16)$$

$$\hat{g}_T(\theta) = \frac{1}{T} \sum_{i=1}^T \hat{g}_i(h_i, \theta). \quad (3.17)$$

The MSM estimator is then a Generalized Method of Moments (GMM) estimator using $\hat{g}_T(\theta)$ as moment conditions. An estimate, $\hat{\theta}$, for the parameter vector θ is obtained by solving:

$$\min_{\theta \in \Theta} \hat{g}_T(\theta)' \hat{W} \hat{g}_T(\theta) \quad (3.18)$$

where \hat{W} is a weighting matrix with rank of at least p . It is useful to note that if the model is correct, and under the assumptions stated and checked below, the statistic

$$\mathcal{X}_{m-p}^\epsilon = T \hat{g}_T(\hat{\theta})' \hat{W} \hat{g}_T(\hat{\theta}) \quad (3.19)$$

is distributed asymptotically as a Chi-squared variate with $m - p$ degrees of freedom.

We now check a list of regularity conditions needed to ensure consistency and asymptotic normality of the MSM estimator. In addition, we will obtain its asymptotic distribution.

3.4.1 Regularity Conditions

In addition to the common regularity conditions assumed to obtain consistency and asymptotic normality of the GMM estimator, there are two additional problems in the MSM estimation. We need to ensure that the dependence of the estimator of an initial arbitrary state used to initialize the simulated series (3.15) fades away as the simulation size increases and that the simulated state process converges to its stationary distribution¹⁰. Also, a perturbation in the parameter vector affects the whole history of transitions, not only the current observation. This is unlike the usual GMM problem or even McFadden's (1989) MSM problem. Thus we need conditions to insure that this effect is damping rather than exploding and we have some kind of uniform continuity necessary to obtain asymptotic results. In what follows we present a list to insure that in our case, the proper convergence is attained. We follow Duffie and Singleton (DS) (1993).

It is easy to check that Assumptions 1 and 7 of DS (1993) involving Lipschitz conditions uniformly in probability are satisfied in our case because the state space is compact and the observation function is continuously differentiable. In addition, the state vector (the fundamentals) is geometrically ergodic (DS, 4.1) because it has full support and is aperiodic while the transition function is bounded above and below. Now, if the minimizer of (3.18) is unique, the MSM estimator $\hat{\theta}$ converges to θ_0 in probability as $T \rightarrow \infty$ given that the above assumptions are satisfied in our case (DS theorem 1).

¹⁰Of course, a stationary distribution for the fundamentals vector process exists because it is bounded.

3.4.2 The Asymptotic Distribution

4 To obtain asymptotic normality, it is necessary that the estimator $\hat{\theta}$ belongs to the interior of Θ . In addition, $h_j^\theta = h(s(k_j^\theta, \theta))$ must be continuously differentiable (which is our case from (3.5)) and $E\left(\lim_{j \rightarrow \infty} \partial h_j^{\theta_0} / \partial \theta\right)$, where ∂ is shorthand notation for the Jacobian matrix, must exist and be finite with full rank. But, since h is continuously differentiable and k ergodic, this follows from Fatôu's lemma in our case.

It follows from the geometric ergodicity assumptions that the simulated series is independent of the data and that $\sqrt{T}(\hat{\theta} - \theta_0)$ converges in distribution to a normal random vector with mean zero and covariance matrix:

$$avar1(\hat{\theta}) = (\tilde{G}'W\tilde{G})^{-1}\tilde{G}'W\tilde{\Omega}W\tilde{G}(\tilde{G}'W\tilde{G}) \quad (3.20)$$

$$\tilde{G} = E[\partial \hat{g}_i(s_i, \theta_0) / \partial \theta] \quad (3.21)$$

$$W = \text{plim } \hat{W} \text{ as } T \rightarrow \infty \quad (3.22)$$

$$\tilde{\Omega} = var(\hat{g}_i(s_i, \theta_0)) \quad (3.23)$$

where the expectations are taken with respect to the density of the stationary distribution of the variate $\partial h_\infty^{\theta_0} / \partial \theta$, which, again exists because s is continuously differentiable with a bounded derivative and k is bounded by construction. Thus we may also have written $\tilde{G} = E\left(\lim_{j \rightarrow \infty} \partial h_j^{\theta_0} / \partial \theta\right)$.

We have two approaches to estimate (3.20). First, we can estimate $\tilde{\Omega}$ using simulated data. This alternative may prove useful since one has control over the size of the simulations. In this case we can estimate the functions in (3.20) by replacing \tilde{G} by $\hat{G} = \frac{\partial \hat{g}(s_i, \hat{\theta})}{\partial \theta}$, W by \hat{W} and $\tilde{\Omega}$ by $\hat{\Omega} = \frac{1}{T} \sum_{i=1}^T \hat{g}_i(s_i, \hat{\theta}) \hat{g}_i(s_i, \hat{\theta})'$.

Second, Hansen (1982) showed that the choice of $W = \Omega_0^{-1}$, where Ω is given below in (3.24), leads to the most efficient GMM estimator among those with a positive definite distance matrix. By using this weighting matrix, the model is asked to come as close as possible to predicting the observed moments, but to weight more heavily those moments that are estimated more accurately. In particular, we may assume that $\hat{W} \rightarrow \Omega_0^{-1}$ *a.s.* where:

$$\Omega_0 = \sum_{j=-\infty}^{\infty} E \left([h_t - E(h_t)] [h_{t-j} - E(h_{t-j})]' \right) \quad (3.24)$$

and where Ω_0 is a function of the moments calculated from the data alone (see Lee and Ingram, 1991). Thus Ω_0 can be estimated using the Newey-West (1987) approach. In this case we would use as weighting matrix an estimate of Ω_0 and the covariance formula (3.20) simplifies and thus $\sqrt{T}(\hat{\theta} - \theta_0)$ converges in distribution as $T \rightarrow \infty$ to a normal random vector with mean zero and covariance matrix¹¹

$$avar2(\hat{\theta}) = (1 + \tau) \left(\tilde{G}' \Omega_0^{-1} \tilde{G} \right)^{-1} \quad (3.25)$$

where $\tau = 1/l$ (this is corollary 3.1 of DS (1993)). We can implement this formula replacing \tilde{G} by $\hat{G} = \frac{\partial \hat{g}(s, \hat{\theta})}{\partial \theta}$ and Ω_0 by a consistent estimator $\hat{\Omega}$ of (3.24). Only in this case, where we have an asymptotically efficient estimator, the diagnostic statistic in (3.19) has the stated asymptotic distribution.

In this paper we will report asymptotic standard errors computed from the second covariance matrix, (3.25). Standard error calculations using the first covariance matrix were essentially the same. We use as our weighting matrix the Newey-West estimate of (3.24) with truncation lag length equal to three. This reduces the computational demands of our approach as we need not recalculate the matrix for each candidate parameter vector. We choose moments which are most informative about the parameters that we estimate. We use moments of both the level and first differences of the exchange rates in general using moments (generalized appropriately for the greater dimensionality of our estimation problem) used in the other papers which apply MSM to target zone models.

¹¹Since our data are not i.i.d over time, for this result to hold it is necessary that the simulated series have the same frequency as the true series. See Appendix A.

3.4.3 Implementation

Since it is not possible to simulate a continuous record of observations for the fundamentals path, we must discretize the model. An issue arises as to whether the discretization chosen converges weakly to the true continuous time process in the limit as the time interval approaches zero. In addition, there is another issue arising from the approximation of multidimensional reflected processes. It is difficult to approximate a diffusion process with an arbitrary reflection matrix on the boundaries of its domain directly (i.e. on the state variable space) through a discrete time processes, i.e. accommodate oblique reflections. This is because one does not know the properties of the graph of the stochastic process with oblique reflections. However, this is quite tractable if the reflection matrix is diagonal. In this case, the regulator is a buffer stock and can be written explicitly as a functional of the Wiener sample paths¹² and thus the discrete time simulator which converges weakly to the true process can be constructed easily. Thus, to solve this problem we perform a linear transformation of the state space, i.e. from the space of the fundamentals to an auxiliary space of identical dimension. Under this transformation, the auxiliary variables exhibit, by construction, a diagonal reflection matrix.¹³ We will actually perform the simulations and estimation on the transformed space and then we will back up the results to read them in terms of the fundamentals processes. This transformation is the same one performed in Serrat (1994) to obtain (3.5). Details about our simulation of the exchange rate process are explained in Appendix 3.8. The specific moments that we used, for both the three country and five country models, are listed in Appendix 3.9. In Appendix 3.10, we explain the numerical procedures that we used in the paper.

¹²See for instance chapter 2 of Harrison's 1985 book.

¹³Note that this reflection matrix is not the matrix M , which is the reflection matrix in the space of the untransformed fundamentals.

3.5 Empirical Results

3.5.1 Results for Three Currency Case

The trilateral model according to (3.5) can be applied to an arbitrary number of currencies. Given the computational demands of the MSM method we first estimate the model as a trilateral target zone applied to subsets of two elements of the set of five currencies that we work with (the third currency is always the DM).¹⁴ Thus we first estimate the model for a total of ten trilateral target zones formed from our data. Each trilateral target zone model requires the estimation of eight parameters. This number incorporates the restriction that the elements of the column vectors of the reflection matrix sum to one.¹⁵ In addition, the vertices for the domain of the fundamentals are calculated directly from the parameters and thus do not constitute parameters to be estimated in its pure sense. We then take the estimated parameters and use them to construct an initial guess for the estimation of a five-lateral target zone involving the currencies: Dutch Guilder (DFl), Danish Krone (DKr), French Franc (FFr), Irish Pound (IP) along with the German Mark (DM). This involves the estimation of a vector of twenty-seven parameters. Of these parameters, twelve are associated with estimation of the 4×4 M matrix as for the five currency case.

Tables 3.3, 3.4, 3.5, 3.6 and 3.7 report the estimates for the trilateral target zones. Each table corresponds to a country matched with each of the other four currencies as well as the DM. To help with the reading of the tables, we explain the results contained in the first column of Table 3.3 which contains the results of the estimation of the trilateral target zone involving Germany, Belgium and the Netherlands. The parameters μ_i and σ_i are the estimates of the drift and diffusion terms for the fundamentals of countries $i = 1$ (Belgium) and $i = 2$ (Netherlands). ρ is the estimated correlation coefficient of the fundamentals processes for the two countries, while γ is the semi-elasticity of money demand with respect to the interest rate. m_{11} and m_{22}

¹⁴This is the simplest generalization of the bilateral model.

¹⁵The choice of normalization is unimportant, alternatively we could have imposed the norm of each column vector to be one.

are the two estimated diagonal elements of the M . Recall that our normalization is such that the sum of each column of M is one; thus, the two column vectors of M are $[m_{11}, 1 - m_{11}]'$ and $[1 - m_{22}, m_{22}]'$, respectively.

The overidentifying restrictions test (3.19) indicates that the trilateral model is not rejected for any of the ten trilateral target zone models. Although some parameters are estimated imprecisely,¹⁶ the diagonal elements of the reflection matrix M are always significant. We performed Wald tests of the restriction that M is the identity matrix, which is a test whether Krugman's model is not rejected as a restriction on the trilateral model, for each of the ten models.¹⁷ In eight of the ten cases these restrictions were rejected. Only for the target zone formed by the FFr, DKr and DM and for the target zone formed by the IP, DFl and DM, do we not reject that each of these trilateral combinations can be explained by two bilateral Krugman models. Unlike previous applications of MSM to bilateral target zone data (see subsection 3.1.1), our parameter point estimates are of the same order of magnitude across currencies and trilateral target zone combinations. The range for the estimated components of the drift vector for the fundamentals process oscillate between $-.021$ and $.0053$, while the estimated diffusion coefficients vary between $.0032$ and $.024$ across all trilateral combinations.¹⁸ The point estimates for the interest rate semielasticity of money γ varies between $.02$ and $.5$ (with one outlier, $.8$), although this parameter is estimated imprecisely. In Figures 6 and 7 we present the simulated steady-state densities of the exchange rates using the parameter estimates. We can see that the multilateral model is able to generate a variety of shapes for the unconditional density, and is not restricted to hump-shaped distribution like the bilateral model of Krugman (1991).

¹⁶However, recall that our model is a nonlinear model and GMM standard errors perform poorly in small samples. It is the global overidentifying restrictions test that is important, even if the point estimates are imprecise. It is important to note that our results do not conform to the overrejection phenomenon derived from the performance of the overidentifying restrictions test in small samples (see Hansen, Heaton and Yaron, 1994).

¹⁷Recall that if the M matrix is the identity matrix the multilateral model collapses to two bilateral models.

¹⁸Our estimates of the standard deviation of the diffusion process look reasonable when compared with the actual data, with the Netherlands (with its highly stable exchange rate) with the lowest estimated standard deviation and with France and Denmark with the highest estimated standard deviations. See Figures 3-1, 3-2, 3-3, 3-4 and 3-5.

This is one of the reasons why the empirical fit of the multilateral features improve dramatically the empirical fit of the target zone model.

In Table 3.8 we show the results of the examination of whether the cross-currency restrictions (3.13) are violated in the estimated models. The cross currency constraints are not violated in six of the ten models, while in two of the models (DM-BFr-IP) and (DM-DFI-DKr) the cross currency constraints are violated trivially. We will see, however, in the results for the five-currency case that this problem is strongly mitigated. Only for the zones comprised of DM-DFI-IP and DM-DKr-FFr are the constraints rejected significantly. This set of trilateral target zones are characterized by reflection matrices that are very close to being diagonal. Thus the result is not surprising: when the reflection matrix is diagonal, the trilateral target zone collapses to two bilateral target zones. Because, in this case the range of variation of the exchange rate is independent they can indeed reach their maximum distance equal to twice the size of the bilateral zones. However, in the EMS, the official bilateral bands between currencies other than the DM are of the same size as the individual bands with respect to the DM. Thus we see another effect of a non-diagonal reflection matrix, namely, it places restrictions on the range of variation of the cross-exchange rate, *independent* of whether the fundamentals processes are correlated or not.

3.5.2 Results for the Five Currency Case

Table 3.9 reports results for the multilateral target zone involving five currencies (DFI, DKr, FFr, IP and DM).¹⁹ The point estimates of the parameters characterizing the dynamics of the fundamental process are similar to the corresponding estimates for the set of trilateral target zones except for the correlation coefficient among the components of the fundamentals vector. The overidentifying restrictions test does not reject the model at a 96% confidence level.²⁰ The correlation coefficients exhibit less

¹⁹The information in this table is presented as in the tables for the trilateral models (generalized appropriately for increased dimensionality). ρ_{ij} is the estimated correlation coefficient of the fundamentals of countries i and j .

²⁰The calculated test statistic is 17.03 which is slightly above the critical value of the overidentifying restrictions test at the 95% confidence level: $\chi^2_{0.05} = 16.8$.

Table 3.1: Estimated M matrix in the Five Country Case.

$$M = \begin{bmatrix} .970 & .370 & .121 & .353 \\ .001 & .838 & .121 & .148 \\ .001 & .295 & .881 & .225 \\ .244 & .271 & .442 & .896 \end{bmatrix}$$

As noted in the text, the F test of the restriction that $M = I(4)$ is soundly rejected by the model. Due to the ordering of the countries in our data set, the columns of M correspond to the Netherlands, Denmark, France and Ireland respectively.

variation than in the previous cases and they range between -0.39 and 0.63 . However, many parameters continue to be estimated quite imprecisely. The point estimate for the interest rate semielasticity of money demand is 0.167 . This is consistent with previous estimates in the literature (Diebold, 1986) and is also very close to the estimate used by Flood, Rose and Mathieson (1991).

We present in Table 3.1 the point estimate for the reflection matrix M in the five currency case.²¹ The restriction that M is diagonal is strongly rejected by a conventional Wald Test.²² In addition, the cross-currency variation restrictions, if violated, are violated by trivial amounts. In Table 3.10 we report the maximum and minimum values of each cross-currency exchange rate implied by the estimated five country model. Note that the maximum violation of the cross-currency constraint, $.0252$ for the French Franc-Danish Krone exchange rate, exceeds the actual constraint by only 10%. In addition, we conducted a simulation of length 30,000 for each cross-currency pair to assess empirically how frequently the model predicts cross-currency constraint violations. In the simulation run, the cross currency restrictions were violated for only $.292\%$ of the simulated sample.²³ In Figures 3-6 and 3-7, we show the histogram of the four simulated exchange rates. Note that the histograms, appear

²¹In the five currency case, we estimated M imposing that the norm of each column vector be exactly one. In practice, we thus did not estimate the 16 elements of the M matrix directly. Rather, we estimated 12 fundamental parameters from which the M matrix is obtained. The estimates of these parameters are not reported as it is the implied estimate of the M matrix which is relevant.

²²The F test value for this test was 744.3 which is much larger than the critical value obtained from the F distribution, 2.36, with degrees of freedom, 162 and 12.

²³In the simulation, the cross currency restrictions for the DKr-FFr, FFr-DKr and DKr-IP cross currency rates were exceeded 327, 193 and 6 times respectively. The other cross-currency rates were not violated.

to be characterized by the hump shaped distribution not found in simulations of the bilateral model.

3.5.3 Power Analysis

The satisfactory performance of the model in both the three-currency and five-currency case leads us to question whether our application of the simulated method of moments has statistical power to reject data constructed under a reasonable alternative data generating process. In this subsection, we explore the power of our method for the trilateral case of Section 3.5.1. As noted above, econometric estimation of the model is very computer intensive; therefore, we examine the power of the trilateral model for one set of countries: Germany, Belgium and France.²⁴

The results of any power calculation are highly sensitive to the specification of the alternative data generating process. We considered generating exchange rate data with Ornstein-Uhlenbeck process: however, a such mean reverting process is not adequate for our data since, as is reported in Table 3.2, for four of the five countries the first autocorrelation coefficient of the first difference of the exchange rate in logarithms is positive.²⁵ Instead, we chose the most natural alternative data generating process: unregulated arithmetic Brownian motion which corresponds to the 'free float' solution of the model. We choose the drift and variance parameters as (roughly) an average of the fundamental parameter estimates for Belgium and France for the trilateral model.²⁶ Note that the data generating process under the alternative hypothesis is not regulated at the boundaries. However, it is constructed with parameters such that the exchange rate sample path remains within the band in practically all our simulations. This alternative data generating process can also be

²⁴We chose the Belgian Franc and the French Franc because those currencies have a larger share in world markets than are the Danish Krone and Irish Pound. The Dutch Guilder was not chosen because it is atypical of other currencies in our sample and exhibits a relatively low variance with respect to the German Mark.

²⁵one can check that for an Ornstein-Uhlenbeck process, the first order autocorrelation coefficient must lie between $-1/2$ and 0 .

²⁶The exact parameters (for Belgium and France respectively) were $-.007$ and $-.007$ for the drift terms and $.008$ and $.014$ for the variance terms. We assumed zero correlation between the Wiener processes driving each exchange rate.

interpreted as a test of the hypothesis that gamma is zero, and therefore the model collapses to a free-float solution of equation (3.1). This is particularly useful since the confidence interval for gamma is big due to a small sample phenomenon. Thus, since the model exhibits high power against this alternative, we conclude that if in the true data generating process gamma is zero, we would most probably reject the model using the overidentifying restrictions test.

Our results strongly suggest that the model has much power to reject data generated under a reasonable alternative hypothesis. We simulated 100 random exchange rate paths. In only 5 cases did the target zone model incorrectly not reject the data generated under the alternative hypothesis. In most cases, the generated data strongly rejected the target zone model: in 61 simulations out of 100, the model converged to an overidentifying restrictions test statistic of over 75 which is well above the critical value $\chi^2_{.05} = 14.1$.

3.5.4 Meese-Rogoff Horseraces

In an influential paper, Meese and Rogoff (1983) compared the out-of-sample forecasts produced by various exchange rate models with forecasts produced by a random walk model of the exchange rate. Even though actual future values of the righthandside variables were allowed in the dynamic forecasts (thus bestowing an informational advantage upon the exchange rate models), the random walk performed as least as well as the other nonlinear models tested, particularly in short-run predictions. Since Meese and Rogoff (1983), forecasting better than the random walk has become a standard metric by which one can judge models of the exchange rates. However we will argue that, for the reasons outlined below (mainly related to the power of random walk tests) these horserace exercises have little relevance in our context (and perhaps in other contexts as well). As it has become standard in the literature to report the results of such an exercise, below we report results of our own horserace calculations even though the multilateral target zone model has not been constructed for purposes of prediction.

In the “races” performed, we used a trilateral version of the model applied to

the FF/DM, BF/DM and FF/BF exchange rates.²⁷ We reestimated the trilateral model on a subset of our data leaving out the last τ data points where $\tau = \{5, 10, 15, 20, 25, 30, 35\}$. For each value of τ , we compute the implied fundamentals at the last point of the subsample by numerically inverting the closed form solution for the exchange rate. Then, using the estimates for the dynamics of the fundamentals process, we forecast the fundamentals τ periods ahead. We then evaluate the nonlinear functional form expressing the exchange rate vector as a function of the fundamentals vector at the predicted fundamentals vector, using the estimates obtained in the subsample. To this we add an extra term correcting for Jensen's inequality to form the forecasted exchange rate. The error made by this forecast is then compared to the error made by using a random walk with drift model directly estimated on the exchange rate series. In this way we eliminate the informational advantage given to the model by the previous studies, i.e., by not using future information in the forecasts.

Table 3.11, columns 1 and 2, exhibit the ratio of the mean squared error (MSE) of the forecast obtained with the multilateral model over the MSE of the forecast obtained with the random walk, for each size of the out-of-sample data set. Two conclusions arise from our analysis: first, both forecasts are very poor. Second, the random walk model does roughly as well as the multilateral model in predicting future exchange rates. Thus, it seems that the results widely reported in the literature about the robustness of the random walk model are reproduced here. However, in Table 3.11, columns 3 and 4, we report the results of the following experiment: we generate data using the multilateral model evaluated at the estimated parameters for the trilateral case of FF-BF-DM and then perform the same prediction exercise with the simulated data as we did with the true data. Surprisingly enough, the random walk model again does as well as the multilateral model, even when the multilateral model is true by construction.²⁸ The above result led us to examine the power of the random walk tests

²⁷This is the same set of currencies that we used for the power calculations reported above.

²⁸We do not obtain a perfect forecast with the multilateral model because the fundamentals have to be predicted.

against the multilateral model. Toward this end we chose the variance ratio test of Lo and MacKinlay (1988) whose finite sample power advantages (against a wide range of alternatives) over other random walk tests such as Dickey-Fuller or Box-Pierce are well known (Hausman (1988) and Lo and MacKinlay (1989)).²⁹ We performed the variance ratio test on 10,000 simulations of the trilateral model previously estimated for the FF-BF-DM case. In Figures 3-8 and 3-9 we plot the histograms of the test statistic which under the null that the simulated series is a random walk is distributed as a Chi-Squared variate with two degrees of freedom.³⁰ There are four histograms plotted (corresponding to four values of q ($q = \{2, 4, 8, 16\}$) where q is the window used in the variance ratio test (see Lo and MacKinlay (1988))). We find that the maximum power of the test to correctly reject the random walk hypothesis is less than 8%.

We also performed the random walk test on the true data. Similar to the simulated data, the null hypothesis that the true data is a random walk again is not rejected. This is not surprising as our simulation results show that random walk tests have low power. This is consistent with the fact that the random walk predicts quite well even when it is false by construction. With this evidence we conclude that conducting Meese-Rogoff horseraces does not make sense in our context, and that the lack of clear forecasting superiority of the trilateral model versus a random walk model is not evidence against the multilateral model. Our Monte-Carlo evidence against “horse-race” tests to evaluate nonlinear models of exchange rates in our context likely has implications for the empirical literature outside the scope of this paper.

3.6 Conclusions

In this paper we apply a simulated method of moments technique to estimate a multilateral target zone model for which we have closed form solutions. This model is

²⁹This test is essentially a Hausman-type specification test built around the idea that, under the random walk hypothesis, the variance of the increments of the process are linear in the length of the observation interval (we direct the reader’s attention to the papers cited in the text for details).

³⁰At the 5% level of significance the critical value is 5.99.

presented in Serrat (1994) and is based on full credibility assumptions and marginal interventions on the space of the fundamentals (and thus endogenous intramarginal interventions in the space of the exchange rates). The theoretical model has Krugman's (1991) bilateral model as a special case, although, in general, its implications are very different. In this paper, we investigated whether taking into account the multilateral feature of multilateral target zones is important for empirical purposes.

Although the econometric problem is computationally demanding, we estimate the model for a three-currency version (for ten different sets of data) and a five-currency version of the model. We use data from the so-called "hard-EMS" period and we compare the model to Krugman's (1991) bilateral model and to other non-target zone alternatives. The model performs very well when measured with conventional goodness-of-fit criteria and also when explicit power computations are carried out with a reasonable data generating process as an alternative hypothesis. Moreover, the parameter restrictions which make the multilateral model collapse to a superposition of bilateral models of the Krugman-type are strongly rejected. Our positive results are a sharp contrast not only to the previous empirical target zone literature but also to most literature on empirical nonlinear models of exchange rates.

We claim that the good empirical performance of the multilateral model is driven by the parameters that capture the degree of cooperation in maintaining the exchange rate regime, the reflection matrix, that have been neglected by previous literature. We can explain the success of the model and the well-documented empirical failure of the bilateral model as follows. First, allowing the reflection matrix to be non-diagonal permits spillover effects of the foreign exchange interventions of the monetary authorities of one country on all the other's fundamentals. This is why the interventions "look like" intramarginal interventions in the space of the exchange rates (but not in the space of the fundamentals). If we impose that the reflection matrix be diagonal (Krugman's restriction) then these spillover effects are lost and interventions are marginal both in the space of the fundamentals and the space of the exchange rates. In this way, we can explain not only the negative results of the empirical literature on the bilateral model reported in Section 3.1.1, but also the relative success

in reconciling the model with the data that some authors have achieved by introducing ad-hoc intramarginal interventions into Krugman's model (e.g. Lindberg and Soderlin, 1991). Second, the nature of the relationships among exchange rates and fundamentals in the versions of the multilateral model estimated here imply that the cross-currency restrictions derived from differing nominal and effective exchange rate bands are rarely violated and that such violations are of a small size. We insist that this is an empirical result that we have achieved without imposing the cross-currency restrictions explicitly, unlike in Serrat (1994). However, direct estimation of Krugman's model applied to several exchange rates would imply, by cross-arbitrage restrictions, more frequent violations of the cross-currency restrictions and of larger size. Since the estimated reflection matrix is far from diagonal, we detect a high degree of cooperation in the maintenance of the exchange rate system during the period studied.

Thus we conclude that the profession perhaps has discarded full credibility target zone models of the Krugman-type too quickly. Even though realignments exist, our results indicate that a full credibility model of exchange rate dynamics that explicitly takes into account the multilateral nature of real world target zones can perform well during "calm" periods on EMS data.

3.7 References

Bertola, G. and Svensson, L. (1993) "Stochastic Devaluation Risk and the Empirical Fit of Target Zone Models" *Review of Economic Studies* 60, pages 689-712.

De Jong, F. (1994) "A Univariate Analysis of EMS Exchange Rates Using a Target Zone Model" *Journal of Applied Econometrics* 9, pages 93-108.

Diebold, F. (1986) "Rational Expectations, Random Walks and Monetary Models of the Exchange Rate" *Proceedings of the Business and Economics Statistics Section of the American Statistical Association*, pages 101-106.

Duffie, D. and Singleton, K. (1993) "Simulated Moments Estimation of Markov Models of Asset Prices" *Econometrica* 61, pages 929-952.

Ferson, W.E. and Foerster, S.R. (1994) "Finite Sample Properties of Method of Moments in Latent Variable Tests of Asset Pricing Models" *Journal of Financial Economics* 36, pages 29-55.

Flood, R., A. Rose and D. Mathieson (1991) "An Empirical Exploration of Exchange Rate Target Zones" *Carnegie-Rochester Series on Public Policy* 35, pages 7-65.

Gourrinchas, P.O. (1994) "Non-Linearities in Target Zones Using a Non-Linear Instrumental Variables Estimator" M.I.T. mimeo.

Hansen, L.P. (1982) "Large Sample Properties of Generalized Method of Moments Estimators" *Econometrica* 50, pages 1049-54.

Hansen, L.P. Heaton, J. and Yaron, A. (1994) "Finite Sample Properties of Some Alternative GMM Estimators" WP# 3728-94 Sloan School of Management, MIT.

Hausman, J. (1988) "The Optimality of Autocorrelation-Based Tests of the Random Walk Hypothesis" WP, Department of Economics, M.I.T.

Krugman, P. (1991) "Target Zones and Exchange Rate Dynamics" *Quarterly Journal of Economics* 106, pages 669-82.

Lee, B.S. and Ingram, B.F. (1991): "Simulated Estimation of Time Series Models" *Journal of Econometrics*, 47, pages 197-205.

Lindberg, H. and Söderlin, P. (1991) "Testing the Basic Target Zone Model on

Swedish Data” Seminar Paper #488 Institute for International Studies, Stockholm.

Lo, A.W. and MacKinlay, A.C. (1988) “Stock Market Prices Do Not Follow a Random Walk: Evidence from a Simple Specification Test” *Review of Financial Studies*, 1, pages 41-66.

Lo, A.W. and MacKinlay, A.C. (1989) “The Size and Power of the Variance Ratio Test in Finite Samples” *Journal of Econometrics* 40, pages 203-238.

McFadden, D. (1989) “A Method of Simulated Moments for Estimation of Discrete Response Models without Numerical Integration” *Econometrica* 57, pages 995-1026.

Meese, R.A. and Rose, A.K. (1990) “Nonlinear, Nonparametric, Nonessential Exchange Rate Estimation” *American Economic Review*, AEA papers and proceedings, May 1990.

Pesaran, M. and Samiei, H. (1992) “An Analysis of the Determination of the FF/DM Exchange Rate in a Discrete Time Target Zone Model” *Economic Journal*, 102, pages 388-401

Pill, H. (1994) “Target Zones and The European Monetary System: A Reconciliation” mimeo, Department of Economics, Stanford University.

Rose, A. and Svensson, L. (1991) “Expected and Predicted Realignments: the FF/DM exchange rate during the EMS” NBER Working paper 3685.

Serrat, A. (1994) “Exchange Rate Dynamics in a Multilateral Target Zone” M.I.T. mimeo, chapter 2 of this PhD. thesis.

Smith, G. and Spencer, M. (1992) “Estimation and Testing in Models of Exchange Rates Target Zones and Process Switching” in Krugman and

Miller, eds. *Exchange Rate Targets and Currency Bands* Cambridge University Press, pages 211-39.

Vehrkamp. R. (1994) “The European Monetary System During the Phase of Transition to European Monetary Union” *Economic Papers* #101, European Commission, Brussels.

3.8 Appendix 1: Simulation of the Model in Discrete Time

In this appendix we explain our discrete-time approximation of the regulated diffusion process. We prove that the discrete-time approximation converges weakly to the continuous time regulated diffusion process. We present our results for the trilateral model; the generalization to a five-currency case is straightforward.

The state variables consist of the fundamentals process $k_t = (k^1, k^2)$ taking values on a quadrilateral $G \subset \mathcal{R}^2$. On this domain, k follows the dynamics:

$$dk_t = \mu dt + AdW_t + \sum_{i=1}^4 R^i d\Lambda_t^i \quad (3.26)$$

where $k_t = (k_t^1, k_t^2)'$, μ is a 2×1 vector and A is a 2×2 matrix. W_t is a bivariate Wiener process on some probability space, and the initial conditions k_0^1 and k_0^2 are given. The process $\Lambda_t = [\Lambda_t^1, \Lambda_t^2, \Lambda_t^3, \Lambda_t^4]$ is a 2×4 dimensional *regulator* process which is a continuous process, whose increment set is singular with respect to the Lebesgue measure, and whose components Λ_t^i are bidimensional nondecreasing processes with $\Lambda_0^i = 0 (i = 1, 2, 3, 4)$ that increase if and only if (k_t^1, k_t^2) hits side i of the fundamentals domain.³¹ Let $R^i (i = 1, 2, 3, 4)$ be 2×1 reflection vectors. We will decompose the instantaneous covariance matrix of k^1 and k^2 , AA' as the product of two triangular matrices with $A = \begin{pmatrix} \sigma^1 & 0 \\ \sigma^2 \rho & \sigma^2 \sqrt{1 - \rho^2} \end{pmatrix}$. With this decomposition, we are able to identify the instantaneous correlation coefficient between k^1 and k^2 as ρ and the conditional variances of k^1 and k^2 , as σ^1 and σ^2 , respectively.

In the solution method outlined in the theoretical paper, two auxiliary processes are defined by changing the axes of the state variable space. This simplifies the partial differential equation to be solved. It is also convenient to perform the same rotation for empirical purposes. In our simulations we work with the exchange rate as a function of the transformed state variables rather than the fundamentals, for

³¹The processes Λ^i coincide with the local time process of the fundamentals at each side the quadrilateral.

reasons that will become clear.

Define two auxiliary stochastic processes, X and Y , as follows:

$$\begin{pmatrix} X_t \\ Y_t \end{pmatrix} = \begin{pmatrix} \mu_X \\ \mu_Y \end{pmatrix} t + B \begin{pmatrix} W_1(t) \\ W_2(t) \end{pmatrix} + \begin{pmatrix} \Lambda_t^1 - \Lambda_t^3 \\ \Lambda_t^2 - \Lambda_t^4 \end{pmatrix} \quad (3.27)$$

where (W_1, W_2) is the same Wiener process as before, and $(\mu_X, \mu_Y)' = M^{-1}\mu$, $B = M^{-1}A$. Obviously $[k_t^1, k_t^2]' = M[X_t, Y_t]'$ a.s. for $(k^1, k^2) \in G$.

Given the dynamics (3.27), we can restate the properties of this process as:

1. $\Lambda_t^1, \Lambda_t^2, \Lambda_t^3$ and Λ_t^4 are increasing and continuous with $\Lambda_0^1 = \Lambda_0^3 = \Lambda_0^2 = \Lambda_0^4 = 0$
2. $X_t = X'_t + (\Lambda_t^1 - \Lambda_t^3) \in [\underline{X}, \bar{X}]$ and $Y_t = Y'_t + (\Lambda_t^2 - \Lambda_t^4) \in [\underline{Y}, \bar{Y}]$ for all $t \geq 0$, where $X'_t = X_0 + \mu_X t + [1, 0]BW_t$ and $Y'_t = Y_0 + \mu_Y t + [0, 1]BW_t$.
3. Λ_t^1 (Λ_t^3) increases if and only if $X_t = \underline{X}$ ($X_t = \bar{X}$). Λ_t^2 (Λ_t^4) increases if and only if $Y_t = \underline{Y}$ ($Y_t = \bar{Y}$).

Lemma: There exist a unique set of processes $\{\Lambda_t^1, \Lambda_t^2, \Lambda_t^3, \Lambda_t^4\}$ which satisfy 1-3.

These processes are given implicitly by:

$$\Lambda_t^3 = \sup_{0 \leq s \leq t} (X'_s - \underline{X} - \Lambda_s^1)^- \quad (3.28)$$

$$\Lambda_t^1 = \sup_{0 \leq s \leq t} (\bar{X} - X'_s - \Lambda_s^3)^- \quad (3.29)$$

$$\Lambda_t^4 = \sup_{0 \leq s \leq t} (Y'_s - \underline{Y} - \Lambda_s^2)^- \quad (3.30)$$

$$\Lambda_t^2 = \sup_{0 \leq s \leq t} (\bar{Y} - Y'_s - \Lambda_s^4)^- \quad (3.31)$$

where $f^- = -\min(f, 0)$.

Proof: available upon request.

For simulation purposes, we fix $t \geq 0$ and construct the discrete time process:

$$\begin{aligned} X'_n &= X'_0 + \mu_X n t + (1, 0)B \left[\sum_{i=1}^{nt} a_i, \sum_{i=1}^{nt} b_i \right] ; & X'_0 \text{ given} \\ Y'_n &= Y'_0 + \mu_Y n t + (0, 1)B \left[\sum_{i=1}^{nt} a_i, \sum_{i=1}^{nt} b_i \right] ; & Y'_0 \text{ given} \end{aligned} \quad (3.32)$$

where $\{a_i\}$ and $\{b_i\}$ are two independent sequences of independent standard normal

random variables and $[x]$ denotes the integer part of a real number x . For now we consider $t \geq 0$ to be fixed. From the processes (3.32) we construct the following bounded processes recursively:

$$\begin{aligned} X_n &= \min \left\{ \bar{X}, \max \{ \underline{X}, X_{n-1} + (X'_n - X'_{n-1}) \} \right\}, \\ Y_n &= \min \left\{ \bar{Y}, \max \{ \underline{Y}, Y_{n-1} + (Y'_n - Y'_{n-1}) \} \right\}. \end{aligned} \quad (3.33)$$

Now from (3.33) by induction on n we obtain an expression for X_n, Y_n in terms of the path of X' and Y' :

$$\begin{aligned} X_n &= X'_n + Z_n^1 - Z_n^2 \\ Y_n &= Y'_n + V_n^1 - V_n^2 \end{aligned}$$

where Z_n^i, V_n^i solve

$$Z_n^2 = \max_{0 \leq i \leq n} -(X'_i - \underline{X} - Z_i^1) \quad (3.34)$$

$$Z_n^1 = \max_{0 \leq i \leq n} -(\bar{X} - X'_i - Z_i^2) \quad (3.35)$$

$$V_n^2 = \max_{0 \leq i \leq n} -(Y'_i - \underline{Y} - V_i^1) \quad (3.36)$$

$$V_n^1 = \max_{0 \leq i \leq n} -(\bar{Y} - Y'_i - V_i^2) \quad (3.37)$$

Now for each n , let $X_{t,n}, X'_{t,n}$ and $Y_{t,n}, Y'_{t,n}$ be the processes defined by

$$\begin{aligned} X'_{t,n} &= (\sqrt{n})^{-1} X'_{[nt]} & X_{t,n} &= (\sqrt{n})^{-1} X_{[nt]} \\ Y'_{t,n} &= (\sqrt{n})^{-1} Y'_{[nt]} & Y_{t,n} &= (\sqrt{n})^{-1} Y_{[nt]} \end{aligned} \quad (3.38)$$

for each $t \geq 0$.

Theorem: For each t , $(X_{t,n}, Y_{t,n})$ converges in distribution to (X_t, Y_t) .³²

Proof: By the functional central limit theorem, as $n \rightarrow \infty$:

³²A sequence (x_n, y_n) of pairs of random variables converges in distribution (or in law) to a pair of random variables (x, y) if $\lim_{n \rightarrow \infty} P(x_n \leq a, y_n \leq b) = P(x \leq a, y \leq b)$ for each $(a, b) \in \mathcal{R}^2$ which is a point of continuity of the function $f(a, b) = P(x \leq a, y \leq b)$.

$$\sqrt{\frac{[nt]}{n}} \left(\sqrt{[nt]} \right)^{-1} \sum_{i=1}^{nt} a_i \xrightarrow{\mathcal{L}} W_t^1 \quad (3.39)$$

$$\sqrt{\frac{[nt]}{n}} \left(\sqrt{[nt]} \right)^{-1} \sum_{i=1}^{nt} b_i \xrightarrow{\mathcal{L}} W_t^2 \quad (3.40)$$

where W_t^1 and W_t^2 are independent Wiener processes. It follows from (3.32) that, for each $t \geq 0$, $(X'_{t,n}, Y'_{t,n}) \xrightarrow{\mathcal{L}} (X'_t, Y'_t)$. In fact, $\{(X'_{t,n}, Y'_{t,n}) ; 0 \leq t \leq k\}$ converges weakly on $D[0, k] \times D[0, k]$ to $\{(X'_t, Y'_t) ; 0 \leq t \leq k\}$ for all $k \in \mathcal{R}^+$. By the continuous mapping theorem applied to (3.34)-(37) and (3.28)-(31) $(Z^1_{t,n}, Z^2_{t,n}, V^1_{t,n}, V^2_{t,n})$ converges weakly to $\{\Lambda_t^1, \Lambda_t^3, \Lambda_t^2, \Lambda_t^4\}$ and the same theorem applied to (3.34), (3.28) and (3.27) indicates that $\{(X_{t,n}, Y_{t,n}) ; 0 \leq t \leq k\}$ converges weakly on $D[0, k] \times D[0, k]$ to $\{(X_t, Y_t) ; 0 \leq t \leq k\}$ for all $k \in \mathcal{R}^+$. Letting $k \rightarrow \infty$ yields the desired result.

In view of the above theorem, our approximation scheme (3.34) is justified as it converges weakly to the continuous stochastic processes upon which the model is built.

Now it is also clear the reason why we simulate the model in the transformed space. The regulator process in (34)-(37) only adjusts along one dimension at a time, without using the reflection matrix.

In our simulations, we set $\Delta t = 1/(52 \times 5)$. We thus interpret $[e^1(X, Y), e^2(X, Y)]$ as a simulated series of daily exchange rates from which we sample one out of five observations to obtain a series of simulated exchange rates comparable to our weekly exchange rate data. An issue also arises as to how to generate the first observation of the simulated series. For the trilateral models estimated in Section 3.5.1 and for the power calculations reported in Section 3.5.3, we began each simulated series by matching the first observation of the simulated series with the first observation in the actual data. For the five country model estimated in Section 3.5.2, we set the first observation of the vector X (resp. Y) to be $(\underline{X} + \bar{X})/2$ (resp. $(\underline{Y} + \bar{Y})/2$). This is roughly similar to fitting the first simulated observation to that of the actual data (the first observations of each of the series are roughly in the middle of the bands).

3.9 Appendix 2: Moments

In this appendix we list the moments that we used during estimation. Let the logarithm of the level of currency i in German Marks divided by central parity at time t be $e_{i,t}$. In addition, denote $\Delta e_{i,t} \equiv e_{i,t} - e_{i,t-1}$ and $\rho_k(x)$ the k^{th} autocorrelation coefficient of x .

In the trilateral case we match the following fifteen moments:

$$\begin{aligned} & \text{mean}(e_{i,t}), i = 1, 2 \\ & \text{var}(e_{i,t}), i = 1, 2 \\ & \text{mean}(\Delta e_{i,t}), i = 1, 2 \\ & \text{var}(\Delta e_{i,t}), i = 1, 2 \\ & \rho_k(\Delta e_{i,t}), i = 1, 2; k = 1, 2 \\ & \text{cov}(\Delta e_{1,t}, \Delta e_{2,t}) \\ & \text{cov}(\Delta e_{1,t-1}, \Delta e_{2,t}) \\ & \text{cov}(\Delta e_{1,t}, \Delta e_{2,t-1}). \end{aligned}$$

In the case with five country case we match the following thirty six moments:

$$\begin{aligned} & \text{mean}(e_{i,t}), i = 1, 2, 3, 4 \\ & \text{var}(e_{i,t}), i = 1, 2, 3, 4 \\ & \text{mean}(\Delta e_{i,t}), i = 1, 2, 3, 4 \\ & \text{var}(\Delta e_{i,t}), i = 1, 2, 3, 4 \\ & \rho_k(\Delta e_{i,t}), i = 1, 2, 3, 4; k = 1, 2 \\ & \text{cov}(\Delta e_{i,t}, \Delta e_{j,t}), i = 2, 3, 4; j = 1, 2, 3; i > j \\ & \text{cov}(\Delta e_{i,t-1}, \Delta e_{j,t}), i = 2, 3, 4; j = 1, 2, 3; i > j \\ & \text{cov}(\Delta e_{i,t}, \Delta e_{j,t-1}), i = 2, 3, 4; j = 1, 2, 3; i > j. \end{aligned}$$

3.10 Appendix 3: Numerical Methods

In this appendix, we summarize the numerical procedures used in Chapter 3. Given a candidate parameter, we first must solve for the vectors $\bar{X} = [\bar{X}_1, \dots, \bar{X}_n]$ and $\underline{X} = [\underline{X}_1, \dots, \underline{X}_n]$. This necessitates solving a system of $2n$ non-linear equations. We then simulate the transformed fundamentals as discussed in Appendix 3.8 and calculate the criterion function given by (3.19). The simulated method of moments estimator is the parameter vector, θ^* , which minimizes (3.19). We first discuss the methods we used to estimate the trilateral models. Then, we discuss minimization of the criterion function in the five currency case. Finally, we conclude with a description of the power calculations reported in Section 3.5.3.

For the trilateral model, for those parameters which are in theory unbounded we conducted an initial exploratory search of a relatively wide parameter space to discover the empirically relevant parameter space. We then discretized the parameter space (as is done in the simulated annealing algorithm for example) so that along each dimension the parameter space contained 50 equally spaced points. For each trilateral model, we then searched randomly 500 times over this restricted parameter space in search of parameter vectors yielding the three lowest levels of the criterion function. We used the parameter vectors associated with the three lowest levels of the criterion function as starting values for our minimization algorithm.

Our algorithm is a variant of the gradient descent method of numerical optimization. In gradient descent, the gradient at a candidate vector is first calculated. The next candidate vector is chosen by moving in the direction of the gradient with the minimization routine ending when a local minimum is reached. Our algorithm is similar except that we do not use information about the gradient to calculate the next candidate vector. (In our problem, calculation of the gradient is costly while use of the information contained by the gradient is in any case problematic when the state space is discrete.) Instead, we calculate the next candidate parameter by randomly choosing a direction in which to move. If search in the direction is successful then the program descends along that dimension until the objective function increases again.

If search is unsuccessful another direction (randomly chosen) is searched. In eight dimensions, there are of course a multiplicity of directions to search amongst neighbouring points. We defined convergence to a local minimum to be reached when each of the sixteen directions defined by perturbing each of the eight parameters up and down within the grid had been searched and found to yield higher objective functions around the objective function minimizing parameter vector. For each of the trilateral models (and for each of the three starting values at which we began the algorithm) at convergence the test statistic (proportional to the value of the criterion function) was always found to be under 35 and in most cases was below 20. In general, the parameter estimates across local minima were comparable to each other and of the same magnitude. From the parameter vector which yielded the lowest criterion function, we undertook a further local search (with a more fine grid) to obtain a better fit of the model and more precise parameter estimates. The final estimates for each of the trilateral models are the resulting parameter estimates obtained from this search.³³

Our method to estimate the five currency case is similar to that of the three currency case. One important difference is the much larger number of parameters to estimate and greater number of exchange rate series to simulate. This increased dimensionality restricted us to estimating the five currency model from only one set of starting parameters. We chose our starting values with guidance from the results from the three currency estimation. We also in our minimization algorithm we allowed the candidate parameter vector to be different along more than one dimension from the previous best candidate vector. (We adopted this strategy in order to obtain faster convergence.) Our strategy of choosing starting values for our algorithm (for both the three and five country cases) in the region of the parameter space most likely to yield the lowest value of the criterion function maximizes the probability that our algorithm reaches a global minimum. It is important to point out that in any case if we did not reach a global minimum this only hurts us as the probability of rejecting the overidentifying restrictions is correspondingly higher.

For the power calculations reported in Section 3.5.3 we adopted the same min-

³³This second stage typically reduced the criterion function by a further 10-15%.

imization routine as in estimation of the other trilateral models. There were two important differences to note. First, as we were generating the data under a known alternative hypothesis we used this information to provide starting estimates for the minimization algorithm. This led to relatively fast convergence. Second, we did not undertake a local search of a finer grid around the best parameter estimates found for each sample path of generated data. In practice, however, we found in the estimation of the original trilateral models that the local search of a finer grid reduced the test statistics by roughly 10-15%, a number which is relatively small. Given the level at which convergence was usually achieved, this local search would prove immaterial in most cases.

Table 3.2: Descriptive Statistics of the Exchange Rate Data

	Belgian Franc	Dutch Guilder	Danish Krone	French Franc	Irish Pound
maximum	.00513	.00594	.0147	.00816	.00562
minimum	-.0198	-.00313	-.0224	-0.0194	-.0180
mean	-.0107	-.000250	-.00647	-.00682	-.00302
variance	.0000367	.00000362	.000114	.0000644	.0000193
mean of first difference	.0000444	.00000503	-.0000805	-.0000366	.0000694
variance of first difference	.00000119	.000000326	.00000470	.00000414	.00000287
autocorrelation coefficient of first difference	.0918	.0366	.0614	.0990	-.0759

Exchange rate variable is the logarithm of the ratio of the exchange rate divided by the central parity.

Table 3.3: Estimates of the Trilateral Model: Belgium

	Netherlands	Denmark	France	Ireland
μ_1	-.00408 (.00234)	-.0211 (.00680)	-.00736 (.00170)	-.0126 (.00661)
μ_2	.00148 (.00167)	-.0186 (.00969)	-.00728 (.00458)	-.00496 (.00246)
m_{11}	.651 (.210)	0.704 (.0419)	.640 (.0661)	.714 (.134)
m_{22}	.881 (.324)	.645 (.0274)	.782 (.196)	.8364 (.213)
σ_1	.00863 (.00227)	.00321 (.00755)	.00706 (.00165)	.0112 (.00377)
σ_2	.00401 (.00136)	.0189 (.0113)	.0151 (.00736)	.0145 (.0142)
ρ	.353 (.129)	-.947 (.147)	.515 (.0819)	.184 (.218)
γ	.0456 (.282)	.159 (.126)	.0808 (.313)	.801 (1.61)
Test Statistic	6.59	9.88	8.19	10.14
Krugman F Test	82.0	196.7	34.55	4.99
l	5	5	5	5
m	5	5	5	5

l is the number of simulations of the exchange rate. m is the number of simulated observations of the exchange rate per week. Test Statistic is the value of the Overidentifying Restrictions Test. Krugman F test is the Wald Test of the restriction that the M matrix is the 2×2 identity matrix. * indicates that Wald Test does not reject the restriction that $M = I(2)$. Standard errors are in parentheses.

Table 3.4: Estimates of the Trilateral Model: Netherlands

	Belgium	Denmark	France	Ireland
μ_1	.00148 (.00167)	.00204 (.00491)	.00188 (.00246)	.000400 (.000554)
μ_2	-.00408 (.00234)	-.00732 (.00440)	-.00736 (.00278)	.0000400 (.00304)
m_{11}	.881 (.324)	.989 (.245)	.792 (.169)	.772 (1.89)
m_{22}	.651 (.210)	.679 (.159)	.628 (.131)	1.00 (.699)
σ_1	.00401 (.00136)	.00395 (.00152)	.00349 (.00154)	.00359 (.000374)
σ_2	.00863 (.00223)	.0187 (.0114)	.0143 (.00476)	.01128 (.00825)
ρ	.355 (.129)	.358 (.752)	.0796 (.141)	.0996 (.252)
γ	.0456 (.282)	.141 (.883)	.0328 (.167)	.390 (2.38)
Test Statistic	6.59	3.70	5.08	11.83
Krugman F Test	82.01	8.98	361.5	.082*
l	5	5	5	5
m	5	5	5	5

l is the number of simulations of the exchange rate. m is the number of simulated observations of the exchange rate per week. Test Statistic is the value of the Overidentifying Restrictions Test. Krugman F test is the Wald Test of the restriction that the M matrix is the 2×2 identity matrix. * indicates that Wald Test does not reject the restriction that $M = I(2)$. Standard errors are in parentheses.

Table 3.5: Estimates of the Trilateral Model: Denmark

	Belgium	Netherlands	France	Ireland
μ_1	-.0186 (.00969)	-.00732 (.00440)	-.0108 (.00680)	-.0115 (.00542)
μ_2	-.0211 (.00680)	.00204 (.00491)	-.00884 (.00756)	.00121 (.00833)
m_{11}	.645 (.0274)	.679 (.159)	.9556 (.515)	.606 (.098)
m_{22}	.704 (.0419)	0.989 (.245)	1.00 (.579)	.866 (.220)
σ_1	.00189 (.0113)	.0187 (.0114)	.0240 (.0120)	.0183 (.00608)
σ_2	.00321 (.00755)	.00395 (.00152)	.0214 (.00613)	.01369 (.00593)
ρ	-.947 (.147)	.358 (.752)	.569 (.142)	.0658 (.212)
γ	.159 (.126)	.141 (.883)	.517 (.577)	.0217 (.119)
Test Statistic	9.88	3.70	11.66	6.17
Krugman F Test	196.7	8.98	.364*	16.08
l	5	5	5	5
m	5	5	5	5

l is the number of simulations of the exchange rate. m is the number of simulated observations of the exchange rate per week. Test Statistic is the value of the Overidentifying Restrictions Test. Krugman F test is the Wald Test of the restriction that the M matrix is the 2×2 identity matrix. * indicates that Wald Test does not reject the restriction that $M = I(2)$. Standard errors are in parentheses.

Table 3.6: Estimates of the Trilateral Model: France

	Belgium	Netherlands	Denmark	Ireland
μ_1	-.00728 (.00458)	-.00736 (.00278)	-.00884 (.00756)	-.0122 (.00355)
μ_2	-.00736 (.00170)	.00188 (.00246)	-.01076 (.00680)	.00532 (.00471)
m_{11}	.782 (.196)	.628 (.131)	1.00 (.579)	.594 (.0691)
m_{22}	.640 (.0661)	.792 (.169)	.956 (.515)	.694 (.138)
σ_1	.0151 (.00736)	.0143 (.00476)	.02141 (.00613)	.0183 (.00683)
σ_2	.00706 (.00165)	.00349 (.00154)	.0240 (.0120)	.0147 (.0102)
ρ	.5152 (.082)	.0796 (.141)	.569 (.142)	.143 (.180)
γ	.0808 (.313)	.0328 (.167)	.517 (.577)	.101 (.309)
Test Statistic	8.18	5.08	11.66	6.57
Krugman F Test	34.55	361.5	.364*	8.74
l	5	5	5	5
m	5	5	5	5

l is the number of simulations of the exchange rate. m is the number of simulated observations of the exchange rate per week. Test Statistic is the value of the Overidentifying Restrictions Test. Krugman F test is the Wald Test of the restriction that the M matrix is the 2×2 identity matrix. * indicates that Wald Test does not reject the restriction that $M = I(2)$. Standard errors are in parentheses.

Table 3.7: Estimates of the Trilateral Model: Ireland

	Belgium	Netherlands	Denmark	France
μ_1	-.00496 (.00246)	.0000400 (.00304)	.00121 (.00833)	.00532 (.00471)
μ_2	-.0126 (.00661)	.0004 (.000554)	-.0115 (.00542)	-.0122 (.00355)
m_{11}	.836 (.213)	1.00 (.699)	.866 (.220)	.694 (.138)
m_{22}	.714 (.134)	0.772 (1.89)	.606 (.0984)	.594 (.0691)
σ_1	.0145 (.0142)	.0113 (.00825)	.0137 (.00593)	.0147 (.0102)
σ_2	.0112 (.00377)	.00359 (.000373)	.0183 (.00608)	.0183 (.00683)
ρ	.184 (.218)	.0996 (.252)	.0658 (.212)	.143 (.180)
γ	.801 (1.61)	.390 (2.38)	.0217 (.119)	.101 (.309)
Test Statistic	10.14	11.83	6.17	6.57
Krugman F Test	4.99	.0823*	16.1	48.74
n	5	5	5	5
m	5	5	5	5

l is the number of simulations of the exchange rate. m is the number of simulated observations of the exchange rate per week. Test Statistic is the value of the Overidentifying Restrictions Test. Krugman F test is the Wald Test of the restriction that the M matrix is the 2×2 identity matrix. * indicates that Wald Test does not reject the restriction that $M = I(2)$. Standard errors are in parentheses.

Table 3.8: Range of Cross-currency Variation in the Trilateral Model

Cross-currency exchange rate	Maximum Value	Minimum Value
Dutch Guilder-Belgian Franc	.0195	-.0195
Danish Krone-Belgian Franc	.0152	-.0152
French Franc-Belgian Franc	.0169	-.0169
Irish Pound-Belgian Franc	.0235	-.0235
Danish Krone-Dutch Guilder	.0236	-.0236
French Franc-Dutch Guilder	.0160	-.0160
Irish Punt-Dutch Guiler	.0317	-.0317
French Franc-Danish Krone	.0429	-.0429
Irish Punt-Danish Krone	.0148	-.0148
Irish Punt-French Franc	.0114	-.0114

The maximum value of the simulated exchange rate refers to the maximum value taken over the domain of the regulated fundamentals process. Note that the maximum value is the negative of the minimum value only because we have included only three significant digits in the table.

Table 3.9: Estimates of the Five Currency Model

parameter	estimate	parameter	estimate
μ_1	.00315 (.0109)	ρ_{12}	.642 (.382)
μ_2	-.0176 (.00795)	ρ_{13}	-.362 (.295)
μ_3	-.0134 (.0191)	ρ_{14}	.165 (.336)
μ_4	-.00526 (.0346)	ρ_{23}	.425 (.310)
σ_1	.000681 (.00443)	ρ_{24}	.198 (.253)
σ_2	.0217 (.00868)	ρ_{34}	.307 (.163)
σ_3	.0198 (.0119)	γ	.193 (.446)
σ_4	.0197 (.0130)		
Test Statistic	17.03	l	5
Krugman F Test	744.3	m	5

l is the number of simulations of the exchange rate. m is the number of simulated observations of the exchange rate per week. The estimated M matrix is presented in the text. Krugman F test is the Wald Test of the restriction that the M matrix is the 4×4 identity matrix. * indicates that Wald Test does not reject the restriction that $M = I(4)$. Standard errors are in parentheses.

Table 3.10: Range of Cross-currency Variation in Five Country Model.

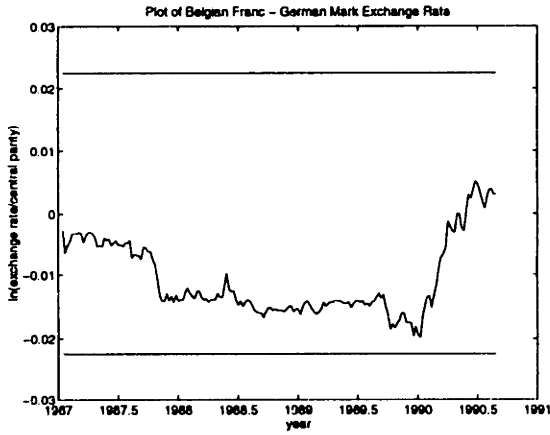
Cross-currency exchange rate	Maximum Value	Minimum Value
Danish Krone-Dutch Guilder	.0217	-.0217
French Franc-Dutch Guilder	.0243	-.0207
Irish Pound-Dutch Guilder	.0181	-.0166
French Franc-Danish Krone	.0252	-.0228
Irish Pound-Danish Krone	.0255	-.0239
Irish Pound-French Franc	.0133	-.0155

The maximum value is defined as the maximum value of the second currency in units of the first currency within the regulated domain of the fundamentals; the minimum is defined in the opposite manner.

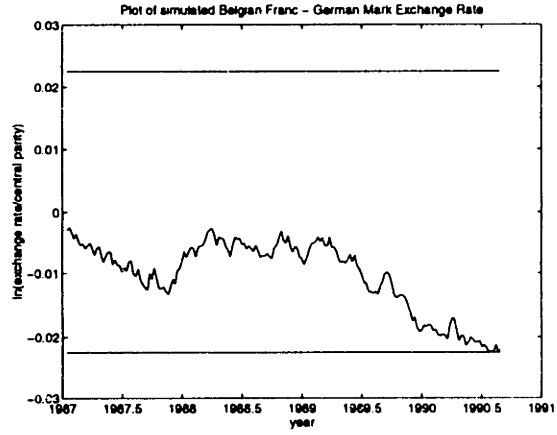
Table 3.11: Horserace Results for DM-BFr-FFr Target Zone

Sample Size	True Data		Simulated Data	
	One Period Error	Cumulative Error	One Period Error	Cumulative Error
$T - 5$	1.17	1.19	1.02	1.05
$T - 10$	1.30	.93	.90	.92
$T - 15$.98	1.36	1.34	1.20
$T - 20$.88	1.50	.95	.95
$T - 25$	1.06	1.19	1.03	.82
$T - 30$.83	1.01	1.20	.78
$T - 35$	1.17	.79	.44	.73

With both the true data and simulated data and for sample sizes $T - \tau$ ($\tau = 5, 10, 15, 20, 25, 30, 35$) we estimate the parameters of a random walk model and the target zone model. For each sample size, we then forecast one period and τ periods ahead using both the random walk and target zone models. The numbers reported above are the ratios of the mean squared error of the forecast of the target zone model over the mean squared error of the random walk model. A number greater than one thus implies the target zone model (given the data and the sample size) does not fit as well as a random walk.

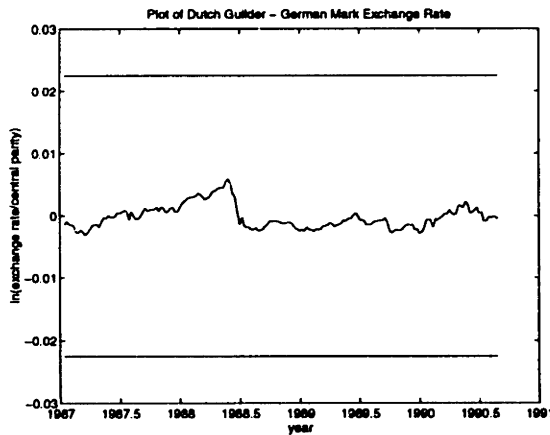


(a)

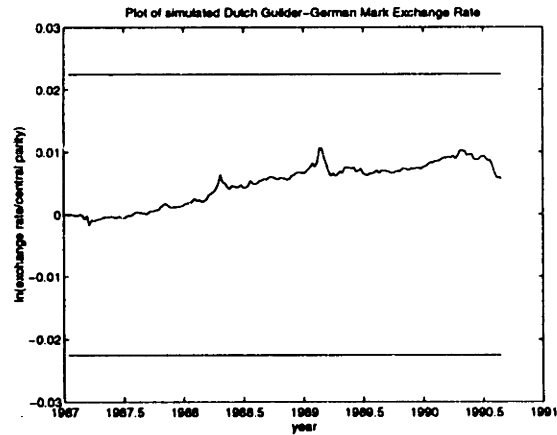


(b)

Figure 3-1: (a) BFr-DM Exchange Rate (b) Simulated BFr-DM Exchange Rate

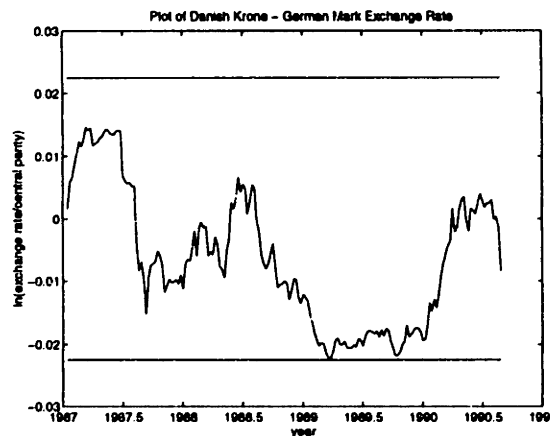


(a)

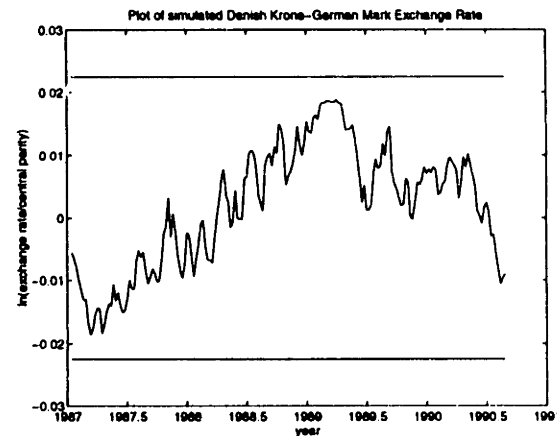


(b)

Figure 3-2: (a) DFl-DM Exchange Rate (b) Simulated DFl-DM Exchange Rate

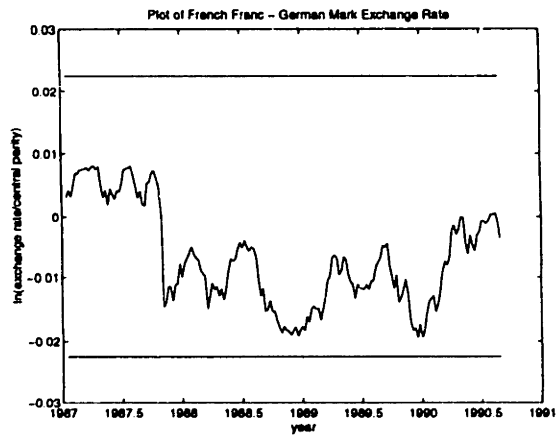


(a)

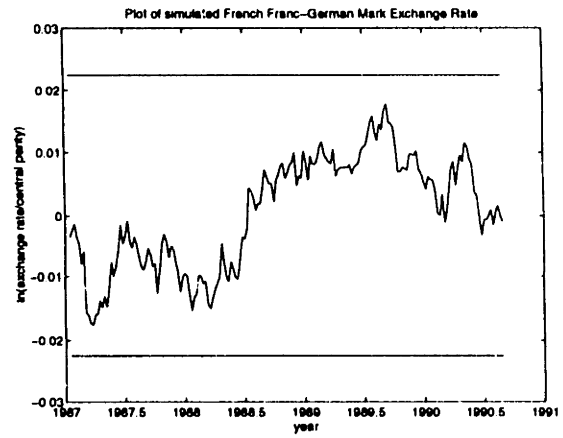


(b)

Figure 3-3: (a) DKr-DM Exchange Rate (b) Simulated DKr-DM Exchange Rate

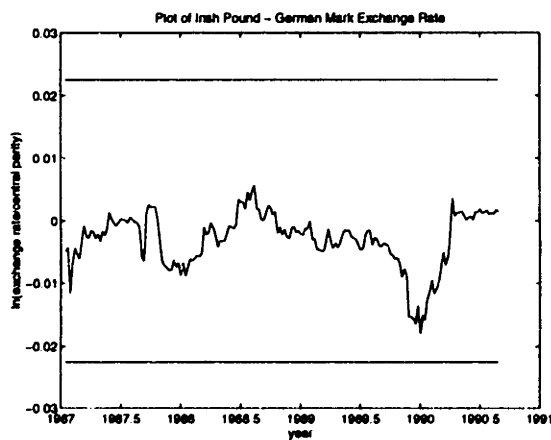


(a)

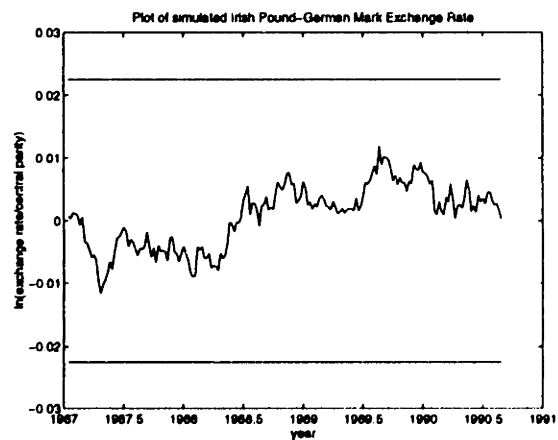


(b)

Figure 3-4: (a) FFr-DM Exchange Rate (b) Simulated FFr-DM Exchange Rate

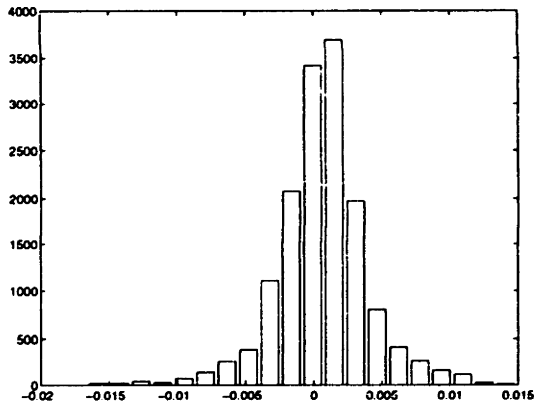


(a)

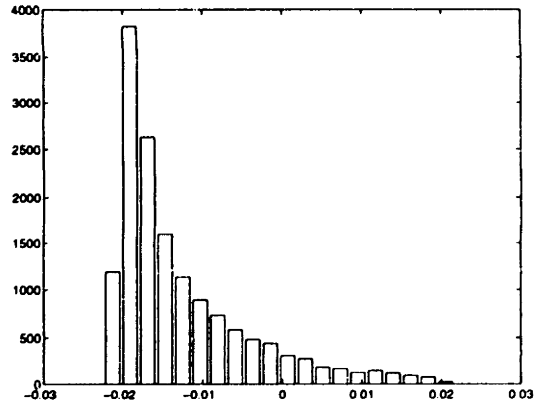


(b)

Figure 3-5: (a) IP-DM Exchange Rate (b) Simulated IP-DM Exchange Rate

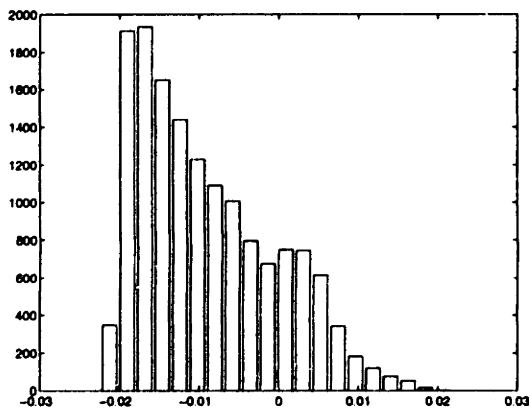


(a)

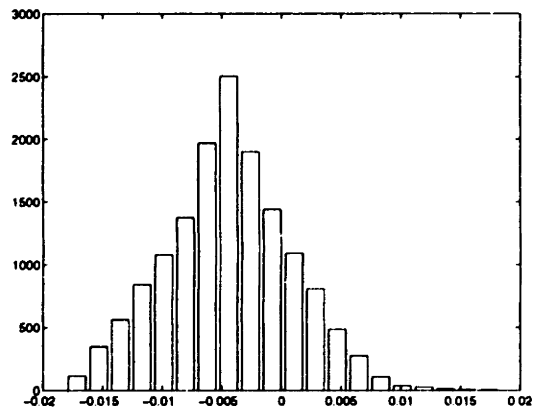


(b)

Figure 3-6: Histogram of (a) Simulated DFl-DM and (b) Simulated DKr-DM Exchange Rates

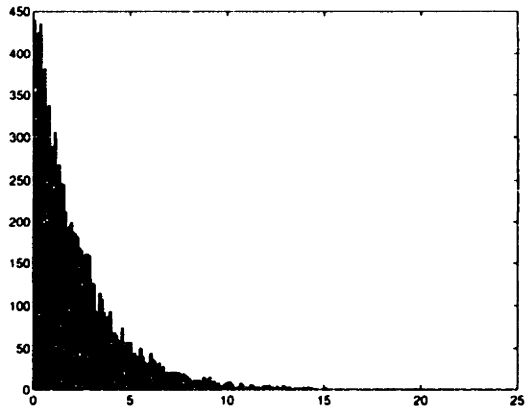


(a)

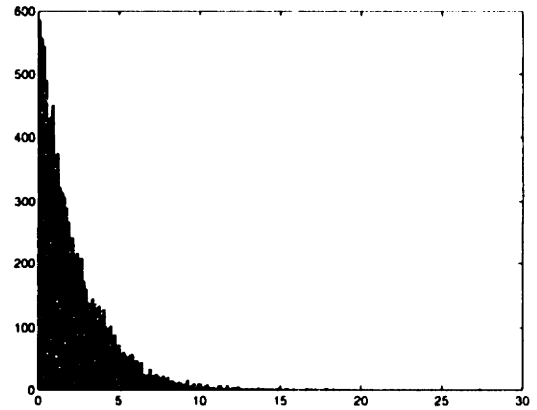


(b)

Figure 3-7: Histogram of (a) Simulated FFr-DM and (b) Simulated IP-DM Exchange Rates

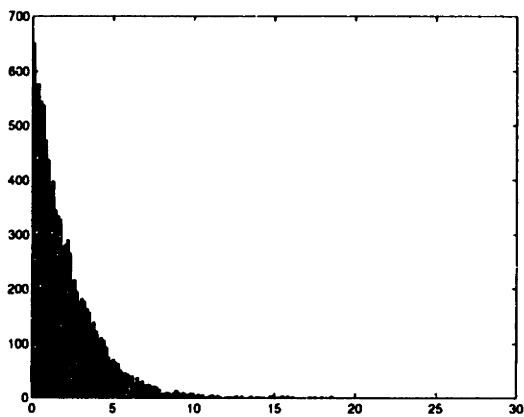


(a)

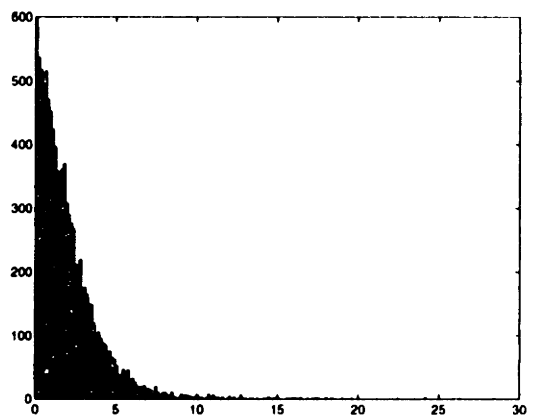


(b)

Figure 3-8: Histogram of Test Statistics of Random Walk Tests (a) $q=2$ (b) $q=4$



(a)



(b)

Figure 3-9: Histogram of Test Statistics of Random Walk Tests (a) $q=8$ (b) $q=16$