MINIMAL ORDER WIENER FILTER FOR A SYSTEM WITH EXACT MEASUREMENTS

by

Violet B. Haas*

ABSTRACT

   The minimal order Wiener filter is constructively derived for a linear, time invariant, detectable system some of whose measurements are noiseless, and a separation principle is derived for the general, singular LQG problem.

I. Introduction

Here we consider the system represented by

$$\dot{x}(t) = Ax(t) + Bw(t), \tag{1.1}$$

$$y(t) = Cx(t) + v(t), \quad t \geq 0 \tag{1.2}$$

where $w(\cdot)$ and $v(\cdot)$ are sample vector valued functions of zero mean uncorrelated Gaussian white noise processes and the initial state $x(0)$ is a zero mean Gaussian random variable which is uncorrelated from $w(t)$ and $v(t)$ for all $t > 0$. The vector $x(t)$ belongs to R^m, $m \leq n$, $w(t)$ belongs to R^p, $p \leq n$, the real matrices A, B, C have appropriate dimensions, C has full rank, (C, A) is a detectable pair, $w(\cdot)$ and $v(\cdot)$ have intensities I_p and R respectively, where I_p is the identity matrix in R^p and R is real and positive semidefinite. Let rank $R = r < m$. Then there is a nonsingular transformation $T_0 = (U_0', W_0')'$ in the space of measurement variables such that,

$$T_0 R T_0' = \begin{pmatrix} R_1 & 0 \\ 0 & 0 \end{pmatrix}$$

where $R_1 = U_0 R U_0'$ has full rank, and primes denote matrix transposition. The transformed measurement vector $T_0 y(t)$ consists of two components, one of which has associated noise with intensity R_1 and the other of which contains no noise. We write,

$$y_1 \overset{\Delta}{=} U_0 y = U_0 Cx + U_0 v, \tag{1.3}$$

$$y_2 \overset{\Delta}{=} W_0 y = W_0 Cx \quad . \tag{1.4}$$

In this paper, we show that under certain hypothesis on the matrices A, B, C, there is an $n-m+r^{th}$ order steady-state unbiased optimal state estimator,

which is the Wiener filter for an n-m+r$\underline{\text{th}}$ order dynamical system. The

state of the filter is a linear functional of the measurements y(·) and

a finite number of time derivatives of the exact, or noiseless, measure-

ments. Our hypothesis is precisely the dual of the strengthened,

generalized, Legendre-Clebsch condition for the dual optimal singular

regulator. The strengthened generalized Legendre-Clebsch condition

guarantees that all optimal controls are either regular or have finite

order of singularity. For a definition of order of singularity, see

Krener [1].

Our results will apply, not only to systems with some exact

measurements, but also to systems whose measurements contain colored

noise (signals whose correlation times are not short when compared

with times of interest in the system). Colored noise may be simulated

by a "shaping filter", whose system equations combine with the original

system equations in such a way that the augmented system appears as

though it has some noiseless measurements. Bryson and Johansen [2] in

1965 were the first to derive a Kalman filter for such singular systems,

but they did not carry their development beyond the case of a "singularity

or order one", that is the case when a regular (nonsingular) Kalman filter

results from replacing the exact measurements by their first time derivatives.

They thought there would always be only a finite number of differentiations

and replacements of exact meaurements needed to reduce the problem to a

nonsingular one, an assumption that is erroneous. A number of authors

have attempted to extend the Bryson-Johansen results for both continuous

and discrete systems, but none of them understood how to obtain the

optimal state estimator in case replacement of the noiseless meaurements
by their first derivatives (or differences) did not result in a full rank
measurement noise intensity matrix. See, for example, [3]-[5].

In [6] we considered the singular, finite horizon, optimal state
estimation problem. There we showed that if the measurement noise
intensity matrix is singular then there is no optimal state estimator
(Kalman filter) whose state is a linear functional of only the original
measurements. We also showed that if when a finite number of time
derivatives of exact measurements replace the exact measurements in the
system representation and yield a new measurement noise intensity
matrix with full rank then there is an optimal state estimator which can
be described by the (m-r) exact measurement equations (1.4) together with
a dynamical system of order n-m+r. The optimal estimate is a linear
functional of the original measurements and a finite number of time
derivatives of the exact measurements. In [7] we considered steady-state
optimal state estimators and showed that when some of the dynamical
state equations are not affected by noise, i.e., when some states are
undisturbed by input noise, and when a finite number of time differentiations
and replacements of exact measurements result in a full rank new measure-
ment noise intensity matrix then the order of the optimal state estimator
may be reduced by the codimension of the "disturbable subspace". However,
we did not show how to reduce the order further by using the exact measure-
ments as new states. The difficulty was that detectability may be destroyed
by the needed coordinate transformations and reduction of the state space.
Detectability is of no concern for estimation over a finite time horizon,
but is crucial in the steady state. Lemma 2.2 shows that detectability can

be preserved if we choose our state coordinate transformation wisely.

II. Some Preliminary Lemmas

Lemma 2.1. Let A be a homomorphism of R^n into itself and let $C: R^n \to R^m$ be epic. Then there exists a map V such that the map defined by $(C', V')'$ is an isomorphism and if $(G,H)' = (C',V')^{-1}$ then all eigenvalues of VAH are eigenvalues of A.

Proof. First let us suppose that A has n linearly independent eigenvectors. Then there are (n-m) linearly independent eigenvectors $v_i, i = 1,2,\ldots,$ (n-m) of A* (the complex conjugate transpose of A) for which the matrix V, whose i^{th} row is v_i^* satisfies

$$\text{rank } (C*, V*) = n$$

Define the (mutually orthogonal) unit vectors w_i, $i=1,\ldots,n-m$, by

$$v_i^* = w_i^* V. \tag{2.1}$$

Then since $VH = I_{n-m}$ we have

$$w_i^* = v_i^* H . \tag{2.2}$$

For each i there is an eigenvalue, λ_i, of A satisfying

$$v_i^* A = \lambda_i v_i^* . \tag{2.3}$$

Postmultiplying (2.3) by H yields

$$v_i^* AH = \lambda_i w_i^* \tag{2.4}$$

and for all i these equations may be written as

$$VAH = \text{diag } (\lambda_1,\ldots,\lambda_{n-m}) . \tag{2.5}$$

Thus, the eigenvalues of VAH are precisely $\lambda_1, \ldots, \lambda_{n-m}$. If V and H are complex-valued then there is a nonsingular transformation L so that LV and HL^{-1} are real. We may then substitute LV for V in the construction described above.

Now suppose that A* has fewer than n linearly independent eigenvectors. Then we can augment an appropriate set of linearly independent eigenvectors with enough generalized eigenvectors to yield a total of (n-m) linearily independent vectors which we can stack together in rows to form the matrix V in such a way that (C',V') is an isomorphism. If v_i is an eigenvector then (2.3) holds. If v_j is a generalized eigenvector then either we can choose $v_1, \ldots, v_{n-m}$ so that

$$v_j^* A = \lambda_j v_j^* + v_\ell^* \qquad (2.6)$$

for some $\ell = 1, \ldots, (n-m)$, or

$$v_j^* A = \lambda_j v_j^* + c^* \qquad (2.7)$$

where c* is a linear combination of the rows of C. In the latter case we have c*H = 0, so that if we again define w_i, i=1,2,..., n-m as in (2.1), then from (2.7) we obtain,

$$v_j^* AH = \lambda_j w_j^* \quad . \qquad (2.8)$$

If an equation like (2.7) holds for each j = 1,..., n-m, then again VAH is diagonal with its diagonal entries all eigenvalues of A. Now suppose that exactly one row of V is a generalized eigenvector of A* and that (2.6) holds. Without loss of generality we may suppose that j = n-m and ℓ=n-m-1. In this case we find that

$$\text{VAH} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_2 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdots & & \\ 0 & 0 & \cdots & 1 & \lambda_{n-m} \end{pmatrix},$$

and again the eigenvalues of VAH are eigenvalues of A. It is now clear that if more than one row of V is a generalized eigenvector of A* and if the vectors w_i are defined by (2.1) then VAH is in Jordan normal form and all its diagonal entries are eigenvalues of A.

The matrix pair (C,A) is said to be detectable if there is a matrix F such that A-FC is a stable matrix. A square matrix is stable if all its eigenvalues have negative real parts.

Lemma 2.2. Suppose A and C satisfy the hypothesis of the previous lemma and suppose that (C,A) is a detectable pair. Let $0 < r < m$ and write $C = (C_1', C_2')'$ where C_1 has r rows. Then there exists a map V so that (C_2', V') is an isomorphism, and if $(C_2', V')^{-1} = (G,H)'$ then $(C_1 H, \text{VAH})$ is a detectable pair.

Proof. Choose $F = (F_1, F_2)$ so that $A-FC = A-F_1 C_1 - F_2 C_2$ is stable and let V be a matrix whose existence relative to the pair $(A-FC, C_2)$ is asserted in Lemma 3.1. Let $(C_2', V')^{-1} = (G,H)'$. Then the eigenvalues of V(A-FC)H are all in the open left half plane and

$$V(A-FC)H = V(A-F_1 C_1)H = \text{VAH} - VF_1 C_1 H.$$

Thus $(C_1 H, \text{VAH})$ is a detectable pair.

III. The Optimal Filter in Case of a First Order Singularity

Consider the system described by (1.1) and (1.3)-(1.4) where (C,A) is detectable. Differentiate (1.4) with respect to time noting (1.1) to obtain

$$\dot{y}_2 = W_0 CAx + W_0 CBw \quad . \tag{3.1}$$

If $W_0 CBB'C'W_0'$ has full rank define $D_0 = W_0 C$. Let $z_1 = y_1$ and $z_2 = \dot{y}_2$ be the new set of measurement variables. Our system is now equivalently represented by (1.1) and

$$z_1 = U_0 Cx + U_0 v \tag{3.2}$$

$$z_2 = W_0 Cx + W_0 CBw \quad . \tag{3.3}$$

Define $C_1 = U_0 C$, $C_2 = W_0 C$ and let (V,H) be the matrix pair whose existence is asserted in Lemma 2.2. Define a nonsingular coordinate transformation in the state space by (1.4) and

$$\xi = Vx \, , \tag{3.4}$$

and let $(C'W_0', V')^{-1} = (G, H)'$. Then

$$x = Gy_2 + H\xi \, . \tag{3.5}$$

Pre-multiplying (1.1) by V and substituting (1.4) and (3.5) into the result and also into (3.2)-(3.3), we obtain

$$\dot{\xi} = VAH\xi + VAGy_2 + VBw \tag{3.6}$$

$$\eta_1 \overset{\Delta}{=} z_1 - U_0 CGy_2 = U_0 CH\xi + U_0 v \tag{3.7}$$

$$\eta_2 \overset{\Delta}{=} z_2 - W_0 CGy_2 = D_0 AH\xi + D_0 Bw \quad . \tag{3.8}$$

We must now uncorrelate the state and measurement noise in such a way that the property of detectability is not destroyed. To this end let $k=0$, let $Q = BB'$ and define

$$M_k = QD_k' (D_k QD_k')^{-1} .$$

Add zero to the right hand side of (3.6) in the form

$$VM_k (\eta_2 - D_k AH\xi - D_k Bw) . \tag{3.9}$$

We obtain,

$$\dot{\xi} = V(I - M_k D_k) AH\xi + VAGy_2 + VM_k \eta_2$$

$$+ V(I - M_k D_k) Bw, \tag{3.10}$$

the matrix pair,

$$((H'C'U_0', H'A'D_k')', V(I - M_k D_k) AH) \tag{3.11}$$

is detectable and the noise signal $V(I - M_k D_k) Bw$ is uncorrelated from the measurement noise, $(v'U_0', w'B'D_0')'$. Hence, by earlier results [7] the system represented by (3.10), (3.7) and (3.8) has a Wiener filter whose state $\hat{\xi}$ is described by,

$$\dot{\hat{\xi}} = VAH\hat{\xi} + VAGy_2 + K_1 (\eta_1 - U_0 CH\hat{\xi})$$

$$+ K_2 (\eta_2 - D_k AH\hat{\xi}) \tag{3.12}$$

where,

$$K_1 = PH'C'U_0' (U_0 RU_0')^{-1}, \quad K_2 = (PH'A' + VQV') D_k' (D_k QD_k')^{-1} \tag{3.13}$$

P is the maximal positive semidefinite solution of the algebraic Riccati equation,

$$VAHP + PH'A'V' + VQV' = K_1 U_0 R U_0' K_1' + K_2 D_k Q D_k' K_2' \qquad (3.14)$$

and the restriction of the map

$$(VA - K_1 U_0 C - K_2 D_k A) H \qquad (3.15)$$

to the disturbable (i.e., controllable) subspace of the pair

$$(V(I - M_k D_k) AH, \quad V(I - M_k D_k) B) \qquad (3.16)$$

is stable.

We have found that if $D_0 BB' D_0'$ is nonsingular, then there is a linear state estimator for the system described by (1.1)-(1.2) which minimizes $\lim_{t \to \infty} \text{tr } E[(x(t) - \hat{x}(t))(x(t) - \hat{x}(t))']$. This estimator is described by (1.4), (3.4) and (3.12) - (3.14) and the estimator dynamics are stable on the disturbable subspace of the (n-m+r) - dimensional state space.

The initial condition $\hat{\xi}(0^+)$ may be updated from $V x(0) = 0$ by use of the exact measurement vector $y_2(0^+)$. However, as we shall see in the next section, the best choice of $\hat{\xi}(0^+)$ may indeed by null.

IV. Further Order Reduction of the Filter

Since the map (3.14) may not be stable it is desirable, for computational purposes, to remove from consideration those states which lie in the un-disturbable subspace. To this end, with k=0 define maps A_k and B_k by

$$A_k = V(I-M_kD_k)AH, \quad B_k = V(I-M_kD_k)B. \tag{4.1}$$

With these substitutions (3.10) becomes

$$\dot{\xi} = A_k\xi + VAGy_2 + VM_k\eta_2 + B_kw . \tag{4.2}$$

Let $N = n-m+r$ and define

$$N_k = \bigcap_{i=1}^{N-1} \text{Ker}(B_k'(A')^{k-1}) . \tag{4.3}$$

Let $<A_k|B_k>$ denote the orthogonal complement of N_k in R^N. Then N_k is the undisturbable subspace of the pair (A_k,B_k) and $<A_k|B_k>$ is the disturbable (controllable) subspace of the pair. In a coordinate system compatible with the decomposition

$$R^N = N_k \oplus <A_k|B_k> , \tag{4.4}$$

we have $\xi' = (\xi_1', \xi_2')$,

$$A_k = \begin{pmatrix} A_{1k} & 0 \\ A_{2k} & \bar{A}_k \end{pmatrix} \quad B_k = \begin{pmatrix} 0 \\ \bar{B}_k \end{pmatrix} \quad V = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}, \quad H = (H_1,H_2) .$$

Then,

$$\dot{\xi}_1 = A_{1k}\xi_1 + V_1AGy_2 + V_1M_k\eta_2 \tag{4.5}$$

$$\dot{\xi}_2 = \bar{A}_k \xi_2 + A_{2k} \xi_1 + V_2 AGy_2 + V_2 M_k \eta_2 + \bar{B}_k w \qquad (4.6)$$

$$\zeta_1 \overset{\Delta}{=} \eta_1 - U_0 CH_1 \xi_1 = U_0 CH_2 \xi_2 + U_0 v \qquad (4.7)$$

$$\zeta_2 \overset{\Delta}{=} \eta_2 - D_k AH_1 \xi_1 = D_k AH_2 \xi_2 + D_k Bw \qquad (4.8)$$

Note that (4.5) has no additive noise. The measurements y_2 and η_2 are known functions and so may be considered as inputs. Hence, the best estimate of ξ_1 must be given by

$$\dot{\hat{\xi}}_1 = A_{1k} \hat{\xi}_1 + V_1 AGy_2 + V_1 M_k \eta_2, \quad \hat{\xi}_1 (0) = 0. \qquad (4.9)$$

Then if we define $\tilde{\xi}(t) = \xi(t) - \hat{\xi}(t)$, we have,

$$\dot{\tilde{\xi}}_1 = A_{1k} \tilde{\xi}_1 \ ,$$

and since $E[\tilde{\xi}_1(0)] = 0$ we must have $\tilde{\xi}_1(t) \equiv 0$. Thus, we may suppose that $\xi_1(\cdot)$ is a known function. With this in mind ζ_1 and ζ_2 may be considered as new measurement variables. Since the pair (A_k, B_k) is controllable and the pair $((H_1'C'U_0', H_1'A'D_k'), \bar{A}_k)$ is detectable (see [8]) then by standard theory [9] the Wiener filter for the system described by (4.6)-(4.8) is represented by

$$\dot{\hat{\xi}}_2 = \bar{A}_k \hat{\xi}_2 + A_{2k} \hat{\xi}_1 + V_2 AGy_2 + V_2 M_k \eta_2$$

$$+ K_1 (\zeta_1 - U_0 CH_2 \hat{\xi}_2) + K_2 (\zeta_2 - D_k AH_2 \hat{\xi}_2) , \qquad (4.10)$$

where

$$K_1 = \bar{P} H_2' C' U_0' (U_0 R U_0')^{-1}, \quad K_2 = (\bar{P} H_2' A' + \bar{B}_k \bar{B}_k') D_k' (D_k Q D_k')^{-1}, \qquad (4.11)$$

$\bar{P}$ is the unique, symmetric, positive definite solution of the algebraic Riccati equation,

$$\bar{A}_k \bar{P} + \bar{P}\bar{A}_k' + \bar{B}_k\bar{B}_k' = K_1 (U_0 R U_0') K_1' + K_2 (D_k Q D_k') K_2', \tag{4.12}$$

and the matrix

$$\bar{A}_k - (K_1 U_0 C + K_2 D_k A) H_2 \tag{4.13}$$

is stable. The initial condition on $\hat{\xi}_2$ may be chosen to satisfy,

$$\hat{\xi}_2 (0^+) = 0, \tag{4.14}$$

for since the matrix (4.13) is stable, the steady state value of $\hat{\xi}_2$ is un-affected by the initial condition. The best steady-state estimate $\hat{x}(t)$ of the original state $x(t)$ is given by

$$\hat{x}(t) = G y_2 (t) + H\hat{\xi}(t) . \tag{4.15}$$

Contrary to popular belief, there is no information to help us update the estimate $\hat{\xi}(0)$ once $y_2(0)$ is known. In references [2] and [10] it is suggested that the initial condition $x(0)$ be updated by means of a formula that involves the expression

$$(R+CPC')^{-1} .$$

We note here that the kernel of R must be contained in the kernel of PC' since PC'W_0' must vanish - there can be no estimation error for those (transformed) state variables which are components of the exact measurement $W_0 y$. Thus, if R is singular then so is (R+CPC').

5. Higher Order Conditions for Optimality: $k > 1$.

Suppose $D_0 Q D_0'$ does not have full rank. Then there exists a sequence $\{T_i\}$, $i = 1, \ldots, k$ of transformations,

$$T_i' = (U_i', W_i'),\tag{5.1}$$

such that for $i = 1, 2, \ldots, k-1$,

$$B'(A')^{i-1} C' W_0' W_1' \ldots W_{i-1}' U_i' \text{ has full rank}\tag{5.2}$$

$$B'(A')^{i-1} C' W_0' W_1' \ldots W_i' = 0\tag{5.3}$$

and

$$B'(A')^{k-1} C' W_0' \ldots W_{k-1}' \text{ has full rank}\tag{5.4}$$

For $k > 1$ let U_k be an identity map and define D_k by,

$$D_k' = (C' W_0' U_1', \; A' C' W_0' W_1' U_2', \ldots, (A')^{k-1} C' W_0' W_1' \ldots W_{k-1}' U_k') \; .\tag{5.5}$$

If $D_0 Q D_0'$ does not have full rank, then $k > 1$, and from (3.1) we find

$$U_1 \dot{y}_2 = U_1 W_0 CAx + U_1 W_0 CBw,$$

$$W_1 \dot{y}_2 = W_1 W_0 CAx$$

and

$$W_1 \ddot{y}_2 = W_1 W_0 CA^2 x + W_1 W_0 CABw \; .\tag{5.6}$$

If now $W_1 W_0 CAB$ has maximal rank then the sequence (5.1) stops with $k = 1$ and $D_1' = (C' W_0' U_1', \; A' C' W_0' W_1')$. If $D_1 Q D_1'$ has full rank, we define z_1 as before, and we define $z_2 = U_1 \dot{y}_2$ and $z_3 = W_1 \ddot{y}_2$ as new measurement vectors. The measurement noise now has a full rank intensity matrix. We define V, G, H, η_1 and ξ as before, and define new measurements η_{21}, η_{22} by

$$\eta_{21} \stackrel{\Delta}{=} z_2 - U_1 W_0 CAGy_2 = U_1 W_0 CAH\xi + U_1 W_0 CBw \tag{5.7}$$

$$\eta_{22} \stackrel{\Delta}{=} z_3 - W_1 W_0 CA^2 Gy_2 = W_1 W_0 CA^2 H\xi + W_1 W_0 CABw \quad . \tag{5.8}$$

Let $\eta_2 \stackrel{\Delta}{=} (\eta_{21}', \eta_{22}')'$. Then with k=1 we have,

$$\eta_2 \stackrel{\Delta}{=} D_k AH\xi + D_k Bw \quad . \tag{5.9}$$

Then exactly as before, the system described by (3.10), (3.7) and (5.9) has an optimal steady-state estimator which is described by (1.4), (3.5), (4.9) - (4.12) and (4.14), and the matrix (4.13) is stable. Note that $D_k Q D_k' > 0$ implies (5.4).

It can easily be shown by induction that if (5.2) - (5.3) hold for i=1,...,k-1, and if $D_k Q D_k' > 0$ then the steady state, optimal state estimator is given by (1.4), (3.5), (4.9)-(4.12) and (4.14) and the matrix (4.13) is stable. Note that by the Cayley-Hamilton Theorem, D_{n+j} has the same number of linearly independent rows as D_n for $j \geq 0$. We have thus proved the following theorem.

<u>Theorem 5.1.</u> Let A: $R^n \to R^n$, C:$R^n \to R^m$, B: $R^p \to R^n$, and consider the uncertain system described by (1.1) and (1.3)-(1.4). Suppose the pair (C,A) is detectable and rank R=r. Let $\{T_i\}$, i=1,2,...,k-1 denote a sequence of isomorphisms satisfying (5.1)-(5.3). If $D_k Q D_k'$ has full rank then k$\leq$n, and in a coordinate system compatible with the decomposition (4.4) there exists a Wiener filter with state $\hat{\xi}(t)$ satisfying (4.9)-(4.12) and (4.14) and the matrix (4.13) is stable. The best estimate $\hat{x}(t)$ of the original state x(t) is a linear combination of the exact measurements, $W_0 y(t)$ and of $\hat{\xi}(t)$ and $\hat{\xi}$ is described by an (n-m+r)$\underline{\text{th}}$ order dynamical system. The error vector $\tilde{\xi}(t)$ lives in the disturbable subspace of the pair (A_k, B_k) where A_k and B_k are defined by (4.1).

When $D_k Q D_k'$ has full rank, we shall say that our optimal estimator has order of singularity k+1. The condition $D_k Q D_k' > 0$ is precisely the dual of the condition guaranteeing that each component of the optimal control of a dual singular regulator have order of singularity k or less. This latter condition was derived in earlier work [11].

It may happen that there is no nonnegative integer k for which $D_k Q D_k'$ has full rank. In this event the steady-state, optimal state estimator cannot be derived by our methods and we shall say that this estimator has infinite order of singularity.

VI. The Singular Separation Principle

Here we consider the optimal stochastic control problem defined by

$$\dot{x} = Ax + Bu + w, \tag{6.1}$$

$$y = Cx + v, \quad t \geq 0. \tag{6.2}$$

We seek a control function $u^*(\cdot)$ to minimize the cost functional,

$$J(u) = \lim_{T \to \infty} \frac{1}{T} E\left[\int_0^T (x'Xx + 2x'Su + uUu)\,dt\right] \tag{6.3}$$

where w, v and $x(0)$ are described as before, except that now the semi-definite intensity Q of the process $w(\cdot)$ is not BB'. The weighting matrix U is symmetric and nonnegative definite. When U is not strictly positive definite and the analogous deterministic control problem has finite order of singularity then there is a finite sequence of nonsingular transformations of the control and state variables which will put the problem into a similar form but with a strictly positive definite control weighting matrix. So without loss of generality we may suppose that U is strictly positive definite and that the solution of the deterministic optimal control problem exists. We suppose further that the pair (C,A) is detectable and the pair (A,B) is stabilizable.

Define the transformation $T_0 = (U_0', W_0')'$ as before and by Lemma 3.2 we can find a matrix V such that $(C'W_0', V')$ has full rank and if $(C'W_0';V')^{-1} = (G, H)'$, then the pair $((C'U_0, A'C'W_0')', VAH)$, is detectable. We further suppose that the estimation problem has finite order of singularity and that there exists a nonnegative integer k such that $D_k Q D_k'$ is nonsingular. By standard arguments [12] we obtain a separation principle. The optimal control u^* is given by,

$$u^* = -U^{-1}(B'P + S')\hat{x}, \tag{6.4}$$

$$\hat{x} = GW_0 y + H\hat{\xi} \tag{6.5}$$

$$\dot{\hat{\xi}} = AH\hat{\xi} + V \, AGW_0 y + VBu$$

$$+ K_1(\eta_1 - U_0 CH\hat{\xi}) + K_2(\eta_2 - D_k AH\hat{\xi}), \tag{6.6}$$

where P is the maximal symmetric positive semidefinite solution of the algebraic Riccati equation,

$$(A-BU^{-1}S')'P + P(A-BU^{-1}S') + (X-SU^{-1}S') = PBU^{-1}B'P, \tag{6.7}$$

K_1 and K_2 satisfy

$$K_1 = \pi H'C'U_0' (U_0 RU_0')^{-1}, \tag{6.8}$$

$$K_2 = (\pi H'A' + VQV')D_k' (D_k QD_k')^{-1}, \tag{6.9}$$

π is the maximal symmetric positive semidefinite solution of the algebraic Riccati equation,

$$VAH\pi + \pi H'A'V' + VQV' = K_1(U_0 RU_0')K_1' + K_2(D_k QD_k')K_2', \tag{6.10}$$

and η_1 and λ_2 are observation variables defined by

$$\eta_1 = U_0 CH\xi + U_0 v, \tag{6.11}$$

$$\eta_2 = D_k AH\xi + D_k w. \tag{6.12}$$

Note from equations (6.4)-(6.5) that the necessary condition for optimality derived in [13], namely that if the optimal control is given by $u^* = Z_1\hat{\xi} + Z_2 y$ then trace $(Z_2' UZ_2 R) = 0$, is automatically satisfied since $Z_2 R = -U^{-1}(B'P+S')GW_0 R = 0$.

VII. <u>Conclusion.</u>

We have seen that in case there is a nonnegative integer k for which the matrix $D_k Q D_k'$ has full rank then a steady-state optimal state estimator of reduced order can be designed for the uncertain system described by (1.1)-(1.2). In this case the order of the optimal estimator is reduced by the number of noise-free measurements, and the error vector lives in an even lower dimensional space when not all states are disturbable by input noise. Further-more the solution of this singular estimation problem is the dual of the solution of the dual optimal regulator problem if all optimal controls have finite order of singularity. We have also combined the results of the deterministic optimal regulator problem with the singular state estimation problem to solve the general singular LQG problem and state a singular separation principle.

REFERENCES

[1] A.J. Krener, "The High Order Maximum Principle and its Applications to Sngular Extremals," SIAMJ Control, Vol. 15, pp. 256-293, 1977.

[2] A.E. Bryson and D.E. Johansen, "Linear Filtering for Time-Varying Systems Using Measurements Containing Colored Noise," IEEE Trans. Automat. Contr., Vol. AC-10, pp. 4-10, 1965.

[3] E. Fogel and Y.F. Huang, "Reduced Order Optimal State Estimation for Linear Systems with Partially Noise-Corrupted Measurements," IEEE Trans. Automat. Contr., vol. AC-25, pp. 994-996, 1980.

[4] E. Tse and M. Athans, "Optimal, Minimal-Order Observer-Estimators for Discrete Time-Varying Systems," IEEE Trans. Automat. Contr., Vol. AC-15, pp. 416-426, 1970.

[5] E. Tse, "Observer-Estimators for Discrete-Time Systems," IEEE Trans. Automat. Contr., Vol. AC-18, pp. 10-16, 1973.

[6] Y.-T. Ju and V.B. Haas, "A Duality Principle for State Estimation with Partially Noise-Corrupted Measurements", Int. J. Contr., Vol. 37, pp. 1039-1056, 1983.

[7] V.B. Haas, "Reduced Order State Estimation for Linear Systems with Exact Measurements," Automatica, Vol. 20, to appear, 1984.

[8] W.M. Wonham, Linear Multivariable Control: A Geometric Approach. New York, N.Y.: Springer-Verlag, 1979.

[9] H. Kwakernaak and R. Sivan, Linear Optimal Control Systems, New York, N.Y.: Wiley-Interscience, 1972.

[10] A. Gelb, Applied Optimal Estimation, Cambridge, MA: MIT Press, 1974.

[11] V.B. Haas, "On Optimal Controls Having Finite Order of Singularity," Massachusetts Institute of Technology, Laboratory for Information and Decision Systems, Report LIDS-P-1340, November, 1983.

[12] A.E. Bryson and Y.C. Ho, Applied Optimal Control, Waltham, MA: Blaisdell, 1969.

[13] S. Basuthakur and C.H. Knapp, "Optimal Constant Controllers for Stochastic Linear Systems," IEEE Trans. Automat. Contr., Vol. AC-20, pp. 664-666, 1975.