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# Embedding Stacked Polytopes on a Polynomial-Size Grid

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## Abstract

A stacking operation adds a  $d$ -simplex on top of a facet of a simplicial  $d$ -polytope while maintaining the convexity of the polytope. A stacked  $d$ -polytope is a polytope that is obtained from a  $d$ -simplex and a series of stacking operations. We show that for a fixed  $d$  every stacked  $d$ -polytope with  $n$  vertices can be realized with nonnegative integer coordinates. The coordinates are bounded by  $O(n^{2 \log_2(2d)})$ , except for one axis, where the coordinates are bounded by  $O(n^{3 \log_2(2d)})$ . The described realization can be computed with an easy algorithm.

The realization of the polytopes is obtained with a lifting technique which produces an embedding on a large grid. We establish a rounding scheme that places the vertices on a sparser grid, while maintaining the convexity of the embedding.

MSC: 52B55 and 52B11

Keywords: Stacked Polytopes, Integer Realization, Grid Size, Liftings

## 1 Introduction

Steinitz’s Theorem [23, 29] states that the graphs of 3-polytopes<sup>1</sup> are exactly the planar 3-connected graphs. In particular, every planar 3-connected graph can be realized as a 3-polytope. The original proof is constructive, transforming the graph by a sequence of local operations down to a tetrahedron. Unfortunately, the resulting polytope construction requires exponentially many bits of accuracy for each vertex coordinate. Stated another way, this construction can place the  $n$  vertices on an integer grid, but that grid may have dimensions doubly exponential in  $n$  [14]. The situation in higher dimensions is even worse. Already in dimension 4, there are polytopes that cannot be realized with rational coordinates, and a 4-polytope that can be realized on the grid might require coordinates that are doubly exponential in the number of its vertices [29]. Moreover, it is NP-hard to decide whether a lattice is a face lattice of a 4-polytope [18, 19].

How large an integer grid do we need to embed a given planar 3-connected graph with  $n$  vertices as a polytope? This question goes back at least eighteen years as Problem 4.16 in Günter M. Ziegler’s book [29]; he wrote that “it is quite possible that there is a quadratic upper bound” on the length of the maximum dimension. The best bound so far is exponential in  $n$ , namely  $O(2^{7 \cdot 21n})$  [4, 17]; see below for the long history. The central question is whether a polynomial grid suffices, that is, whether Steinitz’s Theorem can be made efficient. For comparison, a planar graph can be embedded in the plane with strictly convex faces using a polynomial-size grid [3]. In this paper, we give the first nontrivial subexponential upper bound for a large class of 3-polytopes. Moreover, our construction generalizes to higher dimensions and we show that a nontrivial class of  $d$ -polytopes can be realized with integers coordinates, which are bounded by a polynomial in  $n$ .

A  $d$ -dimensional stacked polytope is a polytope that is constructed by a sequence of “stacking operations” applied to a  $d$ -simplex. A stacking operation glues a  $d$ -simplex  $\Delta$  atop a simplicial facet  $f$  of polytope, by identifying  $f$  with a face of  $\Delta$ , while maintaining the convexity of the polytope. Thus a stacking operation removes one facet  $f$  and adds  $d$  new facets having a new common vertex. We call this new vertex *stacked on  $f$* .

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<sup>1</sup>In our terminology a *polytope* is always understood as a *convex* polytope.

**Our results.** We present an algorithm that realizes a stacked polytope on a grid whose dimensions are polynomial in  $n$ . In our presentation  $\log$  denotes the binary logarithm. Our main result is the following:

**Theorem 1** *Every  $d$ -dimensional stacked polytope can be realized on an integer grid, such that all coordinates have size at most  $10d^2R^2$ , except for one axis, where the coordinates have size at most  $6R^3$ , for  $R = dn^{\log(2d)}$ .*

As a corollary of Theorem 1 we obtain that every stacked 3-polytope can be embedded on the grid of dimensions  $270n^{5.17} \times 270n^{5.17} \times 18n^{7.76}$ .

**Related work.** Several algorithms have been developed to realize a given graph as a 3-polytope. Most of these algorithms are based on the following two-stage approach. The first stage computes a plane (*flat*) embedding. To extend the plane drawing to a 3-polytope, the plane drawing must fulfill a criterion which can be phrased as an “equilibrium stress condition”. Roughly speaking, replacing every edge of the graph with a spring, the resulting system of springs must be in a stable state for the plane embedding. Plane drawings that fulfill this criterion for the interior vertices can be computed as barycentric embeddings, i.e., by Tutte’s method [26, 27]. The main difficulty is to guarantee the equilibrium condition for the boundary vertices as well, because in general this goal is achievable only for certain locations of the outer face. The second stage computes a 3-polytope by assigning every vertex a height expressed in terms of the spring constants of the system of springs.

The two-stage approach finds application in a series of algorithms [5, 9, 10, 14, 17, 18, 22]. The first result that improves the induced grid embedding of Steinitz’s construction is due to Onn and Sturmfels [14]; they achieved a grid size of  $O(n^{160n^3})$ . Richter-Gebert’s algorithm [18] uses a grid of size  $O(2^{18n^2})$  for general 3-polytopes, and a grid of size  $O(2^{5.43n})$  if the graph of the polytope contains at least one triangle. These bounds were improved by Ribó Mor [16] and later on by Ribó Mor, Rote, and Schulz [17]. The last paper expresses an upper bound for the grid size in terms of the number of spanning trees of the graph. Using the recent bounds of Buchin and Schulz [4] on the number of spanning trees, this approach gives an upper bound on the grid size of  $O(2^{7.21n})$  for general 3-polytopes and  $O(2^{4.83n})$  for 3-polytopes with at least one triangular face. These bounds are the best known to date for the general case. Very recently, Pak and Wilson proved that every simplicial<sup>2</sup> 3-polytope can be embedded on a grid of size  $4n^3 \times 8n^5 \times (500n^8)^n$  [15].

Zickfeld showed in his PhD thesis [28] that it is possible to embed very special cases of stacked 3-polytopes on a grid polynomial in  $n$ . First, if each stacking operation takes place on one of the three faces that were just created by the previous stacking operation (what might be called a *serpentine stacked polytope*), then there is an embedding on the  $n \times n \times 3n^4$  grid. Second, if we perform the stacking in rounds, and in every round we stack on every face simultaneously (what might be called the *balanced stacked polytope*), then there is an embedding on a  $\frac{4}{3}n \times \frac{4}{3}n \times O(n^{2.47})$  grid. Zickfeld’s embedding algorithm for balanced stacked 3-polytopes constructs a barycentric embedding. Because of the special structure of the underlying graph the plane embedding remarkably fits on a small grid.

Every stacked polytope can be extended to a balanced stacked polytope at the expense of adding an exponential number of vertices. By doing so, Zickfeld’s grid embedding for the balanced case induces a  $O(2^{3.91n})$  grid embedding for general stacked 3-polytopes.

Little is known about the lower bound of the grid size for embeddings of 3-polytopes. An integral convex embedding of an  $n$ -gon in the plane needs an area of  $\Omega(n^3)$  [2, 1, 25]. Therefore, realizing a 3-polytope with an  $(n-1)$ -gonal face requires at least one dimension of size  $\Omega(n^{3/2})$ . For simplicial polytopes (and hence for stacked polytopes), this lower-bound argument does not apply. However, there are planar 3-trees that require  $\Omega(n^2)$  area for plane straight-line drawings [13]. We show in Sect. 4 how to get a lower bound of  $\Omega(n^3)$  volume for stacked 3-polytopes based on this 2d example.

Alternative approaches for realizing general 3-polytopes come from the original proof of Steinitz’s theorem, as well as the Koebe–Andreev–Thurston circle-packing theorem, which induces a particular polytope realization called the *canonical polytope* [29]. Das and Goodrich [6] essentially perform

<sup>2</sup>A polytope is *simplicial* if all its faces are simplices.

many inverted edge contractions on many independent vertices in one step, resulting in a singly exponential bound on the grid size. The proof of the Koebe–Andreev–Thurston circle-packing theorem relies on nonlinear methods and makes the features of the 3-dimensional embedding obtained from a circle packing intractable; see [21] for an overview. Lovász studied a method for realizing 3-polytopes using a vector of the nullspace of a *Colin de Verdière matrix* of rank 3 [12]. It is easy to construct these matrices for stacked polytopes; however, without additional requirements, the computed grid embedding might again need an exponential-size grid.

**Our contribution.** At a high level, we follow the popular two-stage approach: we compute a *flat embedding* in  $\mathbf{R}^{d-1}$  and then lift it to a  $d$ -polytope. Although this two-stage approach is well known for realizing 3-polytopes, it cannot be easily extended to higher dimensions. There exists a generalization of equilibrium stresses for polyhedral complexes in higher dimensions by Rybnikov [20]. However, it is not straightforward to operate with the formalism used by Rybnikov for our purposes. To overcome the difficulties of handling the more complex behavior of the higher-dimensional polytopes and to keep our presentation self-contained, we develop our own specialized methods to study liftings of *stacked* polytopes. Notice that our methods coincides with the approach of Rybnikov, however, we omit the proof, since it is not required for this presentation.

Instead of specifying the stress and then computing the barycentric embedding we construct the “stress” and the barycentric embedding in parallel. To specify the stress we define the heights of the lifting (actually, the vertical movement of the vertices as induced by the stacking operation). In our presentation the concept of stress is therefore reduced to a certificate for the convexity of the lifting, but it is not defining the flat embedding. On the other hand we still use barycentric coordinates to determine the flat embedding. A crucial step in our algorithm is the construction of a balanced set of barycentric coordinates, which corresponds to face volumes in the flat embedding. Initially, all faces have the same volume, but to prevent large heights in the lifting, we increase the volumes of the small faces. To see which faces must be blown up, we make use of a decomposition technique from data structural analysis called *heavy path decomposition* [24]. Based on this decomposition, we subdivide the stacked polytope into a hierarchy of (serpentine) stacked polytopes, which we use to define the barycentric coordinates. At this stage the lifting of the flat embedding would result in a grid embedding with exponential coordinates. But since we have balanced the volume assignments, we can allow a small perturbation of the embedding, while maintaining its convexity. Analyzing the size of the feasible perturbations shows that we can round to points on a polynomially sized grid.

A preliminary version of this work was presented at the 22nd ACM-SIAM Symposium on Discrete Algorithms (SODA) in San Francisco [8]. In the preliminary version we were focused on the more prominent 3-dimensional case and did not present any bounds for higher-dimensional polytopes. For the sake of a unified presentation we changed the construction of the lifting slightly. In the preliminary version we defined the constructed stress as a linear combination of stresses defined on certain  $K_4$ s. In this paper we specify the “movement” for every stacked vertex (its vertical shift) directly. This can be considered as the dual definition of the lifting. Another difference is the more careful analysis of the size of the  $z$ -coordinates in the final embedding. In contrast to the preliminary version, where we presented bound of  $224,000n^{18}$  for the  $z$ -coordinates, we present a different method for bounding the height of the lifting, which yields an upper bound of  $18n^{7.76}$  instead. We remark also that the conference version contained a flaw in the area assignment, which is now fixed by a slightly modified construction (Lemma 7). In a paper that followed the preliminary version Igamberdiev and Schulz introduced a duality transformation for 3-polytopes that allows to control the grid size of the dual polytope [11]. By this our results for stacked polytopes can be transferred to their dual polytopes (truncated 3-polytopes), which shows that also this class can be realized on a polynomial-sized grid.

## 2 Specifying the geometry of stacked polytopes

In this section we develop the necessary tools for defining an embedding of a stacked  $d$ -polytope with the two-stage approach. We present the framework for our algorithm by defining its basic

construction and by identifying certificates to guarantee its correctness. The actual algorithm is then presented in Section 3.

## 2.1 Matrices, determinants, simplices

We start our presentation with introducing some notation. Assume that  $S = (\mathbf{s}_1, \dots, \mathbf{s}_k)$  is a sequence of  $(k-1)$ -dimensional vectors. Then we denote by  $(\mathbf{s}_1, \dots, \mathbf{s}_k)$  the matrix whose row vectors (in order) are  $\mathbf{s}_1, \dots, \mathbf{s}_k$ . We define as

$$[S] = \det \begin{pmatrix} \mathbf{s}_1 & \mathbf{s}_2 & \dots & \mathbf{s}_k \\ 1 & 1 & \dots & 1 \end{pmatrix}.$$

Notice that  $[S]$  equals  $(k-1)!$  times the signed volume of the simplex spanned by  $\mathbf{s}_1, \dots, \mathbf{s}_k$ . When working with sequences we use the binary  $\circ$  operator to concatenate two sequences. If there is no danger of confusion we identify an element with the singleton set (or sequence) that contains this element. If  $f$  is a function on  $\mathcal{U}$  we extend  $f$  in the natural way to act on sequences on  $\mathcal{U}$ , i.e., we set  $f((a_1, \dots, a_z)) := (f(a_1), \dots, f(a_z))$ .

Throughout the paper we denote for a sequence  $X$  of affinely independent points the simplex spanned by  $X$  by  $\Delta_X$ . If this simplex is the realization of some face of an embedding of a polytope we denote this face by  $f_X$ . For a  $d$ -polytope we call a  $(d-1)$ -face a *facet*, and a  $(d-2)$ -face a *ridge*.

By convention, we understand as  $\mathbf{R}^k$  the space spanned by the first  $k$  standard basis vectors. In this sense we consider the subspace  $\mathbf{R}^{k-1}$  as space embedded inside  $\mathbf{R}^k$ . Furthermore, when we speak about hyperplanes in the following, we consider only those hyperplanes in  $\mathbf{R}^k$  that are not orthogonal to  $\mathbf{R}^{k-1}$ .

## 2.2 Flat embedding

An important intermediate step in our embedding algorithm is the construction of a flat embedding, which is a simplicial complex constructed by repeated weighted barycentric subdivisions of a  $(d-1)$ -simplex  $\Delta_B$ . The combinatorial structure of the flat embedding represents the face lattice of the stacked polytope. Let  $\Delta_S$  be a  $(d-1)$ -simplex and let  $\Delta_1, \Delta_2, \dots, \Delta_d$  be the  $(d-2)$ -simplices that define its boundary. A *barycentric subdivision* adds a new point  $\mathbf{p}$  in the interior of  $\Delta_S$  and splits  $\Delta_S$  into  $d$   $(d-1)$ -simplices. For every simplex  $\Delta_i$  we obtain a new simplex  $\Delta'_i$  spanned by  $\Delta_i \cup \{\mathbf{p}\}$ . If the volume of all  $\Delta'_i$ s are given such that they sum up to the volume of  $\Delta_S$ , then there is one unique vertex  $\mathbf{p}$ , such that the subdivision respects this volume assignment. We call the volumes the  $\Delta'_i$ s the *barycentric coordinates* of  $\mathbf{p}$  with respect to  $\Delta_S$ .

We apply repeated barycentric subdivisions to create (a flat) embedding of a stacked polytope. In particular, the subdivisions carry out the stacking operations geometrically (projected in the  $z=0$  hyperplane). We start the subdivision process with two copies of a  $(d-1)$ -simplex in the  $z=0$  hyperplane glued along the boundary. One of these “initial” simplices will be the *base face*  $f_B$ , which remains unaltered during further modifications. The other face will be repeatedly subdivided such that the combinatorial structure of the stacked polytope is obtained. The constructed realization is still flat, which means that it lies in the  $z=0$  hyperplane.

## 2.3 Liftings

Our algorithm for realizing stacked polytopes is based on the *lifting technique*. The assignment of an additional coordinate to every vertex in the flat embedding is called a *lifting*. For the assignment of a new coordinate  $z$  to the point  $\mathbf{p} \in \mathbf{R}^{d-1}$  we use the shorthand notation  $(\mathbf{p}, z)$ . The lifting is complemented by the projection function  $\pi: \mathbf{R}^d \rightarrow \mathbf{R}^{d-1}$  that simply removes the  $d$ -th coordinate. As shortcut notation we write  $\llbracket S \rrbracket$  for the expression  $[\pi(S)]$ . In the following we refer to the  $d$ -th coordinate of a point in  $\mathbf{R}^d$  as its  $z$ -coordinate.

A hyperplane  $h \subset \mathbf{R}^d$  is characterized by a function  $z_h: \mathbf{R}^{d-1} \rightarrow \mathbf{R}$  that assigns to every point  $\mathbf{p} \in \mathbf{R}^{d-1}$  a new coordinate, such that  $(\mathbf{p}, z_h(\mathbf{p}))$  lies on  $h$ . We denote by  $h(S)$  the hyperplane spanned by  $S$  and use as shortcut notation  $z_S$  for  $z_{h(S)}$ . The following lemma gives us a convenient expression for the function  $z_S$ .

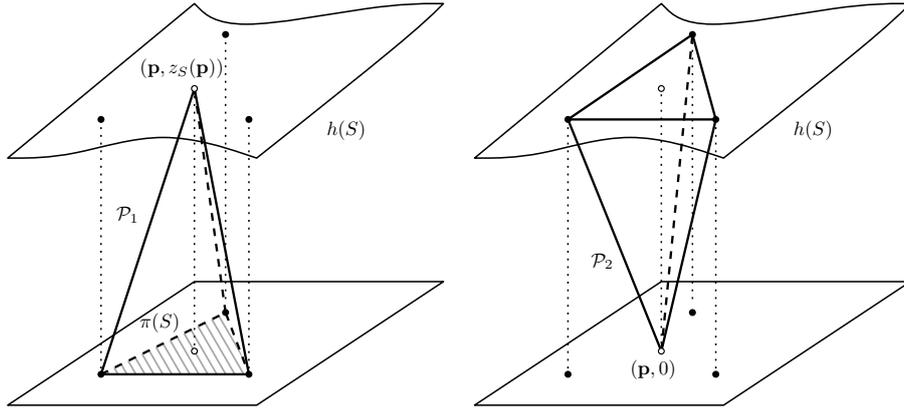


Figure 1: The simplices  $\mathcal{P}_1$  and  $\mathcal{P}_2$  as defined in the proof of Lemma 1. Both pyramids have the same (absolute) volume.

**Lemma 1** *Let  $S$  be a set of  $d$  affinely independent points from  $\mathbf{R}^d$ . For every point  $\mathbf{p} \in \mathbf{R}^{d-1}$  we have*

$$z_S(\mathbf{p}) = \frac{[S \circ (\mathbf{p}, 0)]}{[S]}.$$

**Proof.** We give a geometric proof of the lemma. Consider the  $d$ -simplex  $\mathcal{P}_1$  spanned by  $\pi(S)$  and  $(\mathbf{p}, z_S(\mathbf{p}))$ , and the  $d$ -simplex  $\mathcal{P}_2$  spanned by  $S$  and  $(\mathbf{p}, 0)$  as depicted in Fig. 1. Simplex  $\mathcal{P}_1$  is the affine image of  $\mathcal{P}_2$ , under the mapping  $(\mathbf{q}, z) \mapsto (\mathbf{q}, z_S(\mathbf{q}) - z)$ . Since this mapping is a linear shear, it preserves the volumes of the simplices. The simplex  $\mathcal{P}_1$  can be understood as a pyramid with base  $[S]$  and height  $z_S(\mathbf{p})$  and therefore its absolute volume  $vol$  equals

$$vol(\mathcal{P}_1) = \left| \frac{1}{d(d-1)!} [S] z_S(\mathbf{p}) \right|.$$

On the other hand we can express the absolute volume of  $\mathcal{P}_2$  using the determinant formula and obtain

$$vol(\mathcal{P}_2) = \left| \frac{1}{d!} [S \circ (\mathbf{p}, 0)] \right|.$$

Setting  $vol(\mathcal{P}_1) = vol(\mathcal{P}_2)$  proves  $z_S(\mathbf{p}) = |[S \circ (\mathbf{p}, 0)]/[S]|$ . It remains to check if the sign of the expression in the lemma is correct. By the geometric argument the correctness does not depend on the actual configuration given by  $S$  and  $\mathbf{p}$ . To see this note that for a fixed  $\mathbf{p}$  a sign change in  $[S \circ (\mathbf{p}, 0)]$  occurs if and only if it occurs in  $[S]$ . Moreover, if we keep  $S$  fixed, a sign change of  $[S \circ (\mathbf{p}, 0)]$  can only happen if  $(\mathbf{p}, 0)$  changes its location relative to  $h(S)$ , which on the other hand also changes the sign for  $z_S(\mathbf{p})$ . Hence, it suffices to look at a concrete example. Let  $S$  be given by the standard basis of  $\mathbf{R}^d$ , and let  $\mathbf{p}$  be the origin in  $\mathbf{R}^{d-1}$ . In this case, clearly,  $[S \circ (\mathbf{p}, 0)] = +1$  and  $[S] = +1$ . Since  $z_S(\mathbf{p}) = 1$  the sign is correct.  $\square$

## 2.4 Creases and stresses

Let  $f, g$  be two hyperplanes in  $\mathbf{R}^d$ . Furthermore, let  $S$ , resp.  $T$ , be a sequence of  $d$  affinely independent points of  $f$ , resp.  $g$ , such that deleting the last element in  $S$  and  $T$  gives the same subsequence  $X$ . It follows that  $f$  and  $g$  intersect in a flat of dimension  $d-2$  that contains the simplex  $\Delta_X$ . Furthermore, we denote by  $\mathbf{s}_d$  the last point in the sequence  $S$ , that is the point not in  $T$ , and similarly we denote by  $\mathbf{t}_d$  the point in  $T$  that is not in  $S$ . We also set  $\mathbf{r} = \pi(\mathbf{s}_d)$ . The situation is depicted in Fig. 2. We define as *creasing* on  $X$

$$c(f, g, X) := \frac{z_T(\mathbf{r}) - z_S(\mathbf{r})}{[S]}. \quad (1)$$

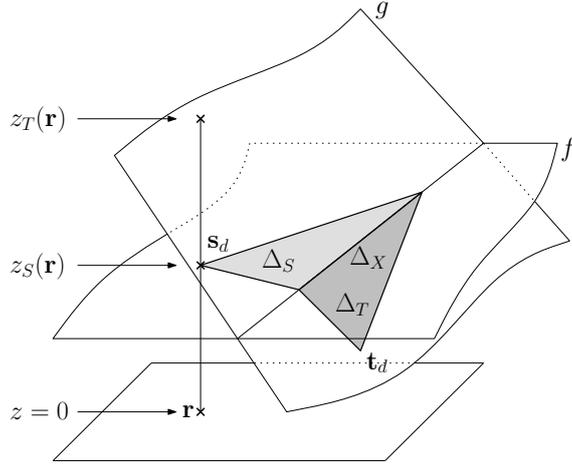


Figure 2: The creasing on  $X$  is defined in terms of the objects depicted in the figure (for  $d = 3$ ).

Roughly speaking, the creasing is a measure for the intersection angle of  $f$  and  $g$  with respect to the volume of  $\Delta_X$ .

The following lemma shows that the function  $c$  is well-defined.

**Lemma 2** *Let  $c(f, g, X)$  be defined as above, i.e.,  $S$  spans  $f$ ,  $T$  spans  $g$ , such that  $S = X \circ \mathbf{s}_d$ , and  $T = X \circ \mathbf{t}_d$ .*

- (a) *It holds that  $c(f, g, X) = -c(g, f, X)$ .*
- (b) *The value of  $c(f, g, X)$  is independent of the choice of  $\mathbf{s}_d$  and  $\mathbf{t}_d$ .*

**Proof.** Let  $T$  be the sequence  $(\mathbf{t}_1, \dots, \mathbf{t}_d)$  and let  $z_i$  be the  $z$ -coordinate of  $\mathbf{t}_i$ . We use Lemma 1 to rephrase  $c(f, g, X)$  by substituting  $z_T(\mathbf{r})$  and obtain

$$c(f, g, X) = \frac{z_T(\mathbf{r}) - z_S(\mathbf{r})}{\llbracket S \rrbracket} = \frac{[T \circ (\mathbf{r}, 0)] - z_S(\mathbf{r})\llbracket T \rrbracket}{\llbracket S \rrbracket\llbracket T \rrbracket}. \quad (2)$$

To further evaluate the last expression we notice that

$$[T \circ (\mathbf{r}, 0)] - [T \circ (\mathbf{r}, z_S(\mathbf{r}))] = z_S(\mathbf{r})\llbracket T \rrbracket.$$

Since  $[T \circ (\mathbf{r}, z_S(\mathbf{r}))] = [T \circ \mathbf{s}_d]$ , we obtain that  $[T \circ (\mathbf{r}, 0)] - z_S(\mathbf{r})\llbracket T \rrbracket = [T \circ \mathbf{s}_d]$ . Plugging the last equation into (2) gives

$$c(f, g, X) = \frac{z_T(\mathbf{r}) - z_S(\mathbf{r})}{\llbracket S \rrbracket} = \frac{[T \circ (\mathbf{r}, 0)] - z_S(\mathbf{r})\llbracket T \rrbracket}{\llbracket S \rrbracket\llbracket T \rrbracket} = \frac{[T \circ \mathbf{s}_d]}{\llbracket T \rrbracket\llbracket S \rrbracket}. \quad (3)$$

Notice that  $[T \circ \mathbf{s}_d] = -[S \circ \mathbf{t}_d]$ , since both underlying matrices differ only by one column swap (the sequence of the first  $d - 1$  members of  $S$  and  $T$  coincide). As a consequence, exchanging  $S$  and  $T$  in the right hand side of (3) changes only the sign, which proves statement (a).

To prove statement (b) we argue as follows. Assume we have replaced  $\mathbf{t}_d$  in  $T$  (that is the only point in  $T$  not in  $X$ ) by some other point on  $h(T)$ . The replacement of  $\mathbf{t}_d$  does not change  $z_T(\mathbf{r})$  in Equation (1), and all other parts of the equation only depend on  $S$ . Therefore, the expression given in (1) does not depend on  $\mathbf{t}_d$ . It remains to show, that also changing the last point in  $S$ , does not change the creasing of  $X$ . However, this follows easily since by statement (a) we can interchange the roles of  $S$  and  $T$  (this results in a sign change), then we apply the above argument, and then we change  $S$  and  $T$  back (which cancels the sign change).  $\square$

Based on the concept of creasing we are now ready to define the crucial concept of stress. Let us first introduce the notion of right and left facet. We assume that we have given a flat

embedding at this point. Let  $X$  be the set of vertices of some ridge  $f_X$  of  $P$ . We fix an arbitrary order for  $X$ , and do so also for the vertex sets of the other ridges. The ridge  $f_X$  separates two facets of  $P$ , say  $f_S$  and  $f_T$ . As usual,  $S = X \cup \mathbf{s}_d$  is the vertex set of  $f_S$ . If  $\llbracket X \circ \mathbf{s}_d \rrbracket > 0$  then  $f_S$  is called *left of  $f_X$* , if  $\llbracket X \circ \mathbf{s}_d \rrbracket < 0$  then  $f_S$  is called *right of  $f_X$* . We have one exception from this rule: for the special (base) face  $f_B$  the notion of left and right is interchanged. Note that the definitions left/right are independent of the choice of the  $z$ -coordinates, and only depend on the flat embedding. Note also, that every ridge has a right and a left incident facet.

**Definition 1 (stress)** *Let  $P$  be stacked  $d$ -polytope obtained by lifting a flat embedding and let  $X$  be a sequence of vertices which defines a ridge. We define as stress on  $X$*

$$\omega_X := c(h_l, h_r, X), \quad (4)$$

where  $h_r$  contains the facet right of  $X$  and  $h_l$  contains the facet left of  $X$ .

We can immediately deduce the following property for the stress along  $X$ .

**Lemma 3** *The stress on  $X$  is not affected by the order of the elements in  $X$ .*

**Proof.** When reordering the elements of  $X$ , only the sign of a creasing along  $X$  might change. When it flips then our notion of left and right face will also be exchanged, which means that the left face becomes the right face with respect to  $X$  and vice versa. Let  $\bar{X}$  be a reordering of  $X$ , such that the creasing along  $X$  changes its sign and the right/left position of the faces is swapped. Suppose that  $f$  contains the face left of  $\bar{X}$  and right of  $X$ , and that  $g$  contains the other face incident to  $X$ . Then we have by Lemma 2

$$\omega_{\bar{X}} = c(f, g, \bar{X}) = -c(f, g, X) = c(g, f, X) = \omega_X.$$

□

We remark that the stress that we define is an equilibrium stress in the classical sense. In fact, in three dimensions it corresponds to the stress that is given by the Maxwell–Cremona correspondence. In particular, the formulation based on (1) can be found in a similar form in Hopcroft and Kahn [10, Equation (11)].

## 2.5 Convexity

The difficult part for the height assignment is to choose the heights such that the resulting lifting gives a *convex* realization of the simplicial complex. To guarantee the convexity of the final embedding we use the stresses induced by the lifting as a certificate. By knowing the signs of all stresses we can determine if the selected  $z$ -coordinates produce a convex embedding with help of the following lemma.

**Lemma 4** *Let  $P$  be the lifting of a flat embedding, such that all vertices have been assigned with a nonnegative  $z$ -coordinate. If*

- for all  $f_X$  incident to  $f_B$  we have  $\omega_X < 0$  and
- for all other  $f_X$  we have  $\omega_X > 0$ ,

then  $P$  is a convex polytope.

**Proof.** As a preliminary step we show that  $P$  is *locally convex*. By this we mean that for every facet  $f_T \neq f_B$  of  $P$  spanned by  $T$ , all adjacent facets lie “below” the hyperplane spanned by  $T$ . Let  $f_S$  be a face that is adjacent to  $f_T$  along the ridge  $f_X$ . The sequences  $S$  and  $T$  are ordered such that they coincide on the first  $d - 2$  elements, that is  $X$ . We assume for now that  $f_T$  is right of  $f_X$ , which implies  $\llbracket T \rrbracket < 0$ .

**Case 1 ( $S \neq B$ ):** We denote by  $\mathbf{r}$  the vertex in  $S$  that is not in  $T$  projected into the  $z = 0$  hyperplane. By definition  $f_S$  is left of  $f_X$  and hence  $\llbracket S \rrbracket > 0$ . Since  $\omega_X > 0$ , we know that  $c(f_S, f_T, X) > 0$ . This leads to

$$\omega_X > 0 \iff c(f_S, f_T, X) > 0 \iff \frac{z_T(\mathbf{r}) - z_S(\mathbf{r})}{\llbracket S \rrbracket} > 0 \iff z_T(\mathbf{r}) > z_S(\mathbf{r}).$$

Hence the interior of  $f_S$  lies below  $h(T)$ .

**Case 2** ( $S = B$ ): The only difference here is that by assumption  $\omega_X < 0$  and that  $\llbracket S \rrbracket$  and  $\llbracket T \rrbracket$  have the same sign, i.e.,  $\llbracket S \rrbracket < 0$ . Therefore,

$$\omega_X < 0 \iff c(f_S, f_T, X) < 0 \iff \frac{z_T(\mathbf{r}) - z_S(\mathbf{r})}{\llbracket S \rrbracket} < 0 \iff z_T(\mathbf{r}) > z_S(\mathbf{r}).$$

Again, the interior of  $f_S$  lies below  $h(T)$ .

For both cases we have assumed that  $f_T$  is right of  $f_X$ . If  $f_T$  would be left of  $f_X$  instead, then the sign of  $\llbracket S \rrbracket$  would change and furthermore,  $\omega_X = c(f_T, f_S, X) = -c(f_S, f_T, X)$  by Lemma 2. Both effects cancel, i.e., for case 1 we have

$$\omega_X > 0 \iff c(f_S, f_T, X) < 0 \iff \frac{z_T(\mathbf{r}) - z_S(\mathbf{r})}{\llbracket S \rrbracket} < 0 \iff z_T(\mathbf{r}) > z_S(\mathbf{r}),$$

and the same ‘‘cancellation’’ happens in case 2.

We have shown that  $P$  is locally convex and will extend this to global convexity. This follows in our setting by Theorem 2.3.20 from the book of De Loera, Rambau and Santos [7]. For completeness we also give the detailed argument in this place. In particular, we show that for every facet  $f_T \neq f_B$  all other vertices of  $P$  lie below  $h(T)$ . We first generalize the observations that proved the local convexity. In the above estimations we can pick as  $\mathbf{r}$  any vertex that lies in the halfspace of  $\mathbf{R}^{k-1}$  that is bounded by the supporting hyperplane of  $\pi(f_X)$  and that does not contain  $\pi(T)$ . It still holds that  $z_T(\mathbf{r}) > z_S(\mathbf{r})$ .

We are now ready to prove that  $P$  is a convex polytope. Let  $\mathbf{p}$  be a vertex of  $P$  not in  $T$  with  $z$ -coordinate  $z_p$  and let  $\pi(\mathbf{p}) = \mathbf{r}$ . Consider a segment  $\ell$  in  $\mathbf{R}^{k-1}$  that connects  $\mathbf{r}$  with a point on  $\pi(T)$ . We can assume that  $\ell$  avoids the vertices of  $\pi(P)$  in its interior, otherwise we perturb  $\ell$  slightly. Let the supported hyperplanes of the facets visited when traversing  $\ell$  be (in order)  $h(T) = h(1), h(2), \dots, h(k)$ . Due to the local convexity we have for  $\mathbf{r} = \pi(\mathbf{p})$  that  $z_{h(k)}(\mathbf{r}) < z_{h(k-1)}(\mathbf{r})$ . Furthermore, due to the generalized observation in the previous paragraph we have for all  $j \leq k$  that  $z_{h(j)}(\mathbf{r}) < z_{h(j-1)}(\mathbf{r})$ . This yields

$$z_T(\mathbf{r}) = z_{h(1)}(\mathbf{r}) > z_{h(2)}(\mathbf{r}) > \dots > z_{h(k)}(\mathbf{r}) = z_p,$$

which proves the assertion for  $f_T \neq f_B$ . The base face  $f_B$  lies in the  $z = 0$  hyperplane, which is clearly a bounding hyperplane since all vertices have positive  $z$ -coordinates. Thus all supporting hyperplanes of the facets of  $P$  are bounding hyperplanes and therefore  $P$  is convex.  $\square$

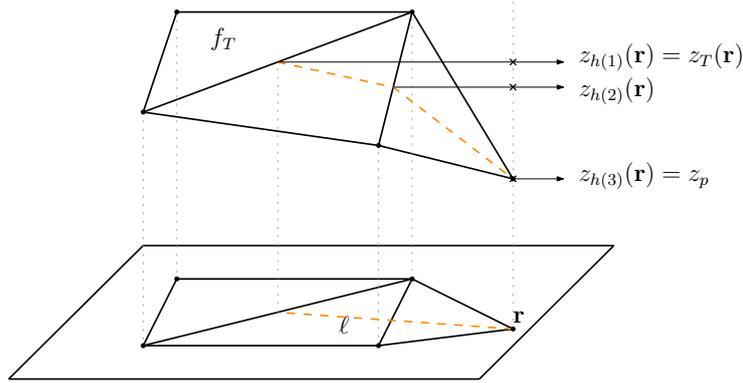


Figure 3: Proof of Lemma 4: A cascading sequence of inequalities (coming from the local convexity of  $P$ ) shows that every vertex of  $P$  lies below the supporting hyperplane of a facet  $f_T \neq f_B$ .

## 2.6 Keeping track of the stresses while stacking

We discuss next how a stacking operation changes the associated stresses in a lifted barycentric subdivision. Let  $\Delta_D$  be a  $(d - 1)$ -simplex of a simplicial complex embedded in  $\mathbf{R}^d$ . We stack

the new vertex  $\mathbf{p}$  on top of  $D$ . To determine the stacking geometrically, we describe the location of  $\mathbf{r} := \pi(\mathbf{p})$  inside  $\pi(\Delta_D)$  with barycentric coordinates, that is we specify the absolute areas of the  $(d-1)$ -simplices containing  $\mathbf{r}$  in the projection. Additionally, we describe how far the  $z$ -coordinate  $z_p$  of  $\mathbf{p}$  lies above  $h(D)$ . This will be denoted by  $\zeta := z_p - z_D(\mathbf{r})$ . We refer to  $\zeta$  as the *vertical shift*. For convenience we assume that the stacking process increases the  $z$ -coordinate of the stacked point, since this will always be the case in the following. However, the following observations can be easily generalized for stacking operations that decrease the  $z$ -coordinates of the stacked point.

The addition of  $\mathbf{p}$  has two effects. First of all, by stacking  $\mathbf{p}$  we create new ridges, furthermore, the existing ridges at the boundary  $f_D$  are now incident to new facets and hence the corresponding stress is altered. We refer to the newly introduced ridges as the *interior ridges*, and to the ridges on the boundary of  $f_D$  as the *exterior ridges* of the stacking operation.

We discuss first how the stresses on the exterior ridge are altered.

**Lemma 5** *Assume that we have stacked a new vertex  $\mathbf{p}$  on a convex polytope realized in  $\mathbf{R}^d$  with stress  $\omega$  as described in Lemma 4. Let  $X$  be the point set of an exterior ridge  $f_X$  (as defined above). After the stacking the new stress  $\hat{\omega}$  equals*

$$\hat{\omega}_X := \omega_X - \frac{\zeta}{\llbracket S \rrbracket},$$

where  $S = X \circ \mathbf{p}$  and  $\zeta$  is the vertical shift of the stacking.

**Proof.** Let  $T$  be the ordered point set, such that  $f_T$  and  $f_S$  are adjacent after stacking  $\mathbf{p}$ , and  $f_X$  is the intersection of  $f_S$  and  $f_T$ . We denote as  $h$  the hyperplane that contains the face  $f_D$  on which  $\mathbf{p}$  was stacked onto; see Fig. 4.

Assume for now that  $f_S$  is left of  $f_X$ , which implies that  $\llbracket S \rrbracket > 0$ . This gives for  $\mathbf{r} = \pi(\mathbf{p})$

$$\hat{\omega}_X = c(f_S, f_T, X) = \frac{z_T(\mathbf{r}) - z_S(\mathbf{r})}{\llbracket S \rrbracket} = \frac{z_T(\mathbf{r}) - (z_h(\mathbf{r}) + \zeta)}{\llbracket S \rrbracket} = \omega_X - \frac{\zeta}{\llbracket S \rrbracket} = \omega_X - \frac{\zeta}{\llbracket S \rrbracket}.$$

If  $f_S$  is however right of  $f_X$  we obtain

$$\hat{\omega}_X = -c(f_S, f_T, X) = -\frac{z_T(\mathbf{r}) - z_S(\mathbf{r})}{\llbracket S \rrbracket} = -\frac{z_T(\mathbf{r}) - (z_h(\mathbf{r}) + \zeta)}{\llbracket S \rrbracket} = \omega_X + \frac{\zeta}{\llbracket S \rrbracket} = \omega_X - \frac{\zeta}{\llbracket S \rrbracket},$$

since  $\llbracket S \rrbracket < 0$ . □

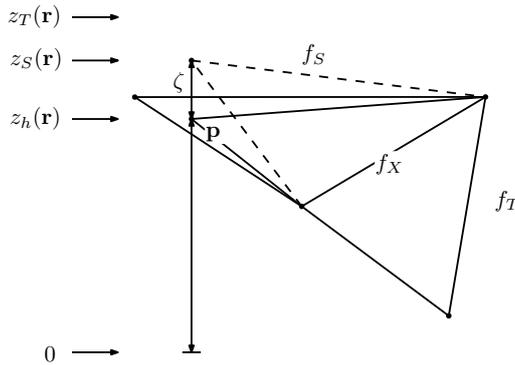


Figure 4: The situation (for  $d = 3$ ) as discussed in Lemma 5: the effect of stacking the vertex  $\mathbf{p}$  for an exterior ridge.

**Lemma 6** *Assume we have stacked a vertex  $\mathbf{p}$  on top of some face  $f_D$ . Let  $X$  be the point set of an interior ridge  $f_X$  as defined above. Furthermore, let  $S := X \circ \mathbf{s}_d$  and  $T := X \circ \mathbf{t}_d$  be the*

sequence of vertices of the faces separated by  $X$ . After the stacking the stress on  $X$  equals the positive number

$$\omega_X := \zeta \left| \frac{[D]}{[S][T]} \right|.$$

**Proof.** Fig. 5 depicts the situation described in the lemma. The facet  $f_S$  might be either left or right of  $f_X$ . Due to Equation (3) we have

$$\omega_X = \pm c(f_S, f_T, X) = \pm \frac{[T \circ s_d]}{[T][S]}.$$

Assume we have doubled  $f_D$  and stack with an  $\varepsilon$ -small  $\zeta$  at the “top side” of the induced complex. This will surely generate a convex  $d$ -simplex, and by the computations in the proof of Lemma 4 the sign of the induced  $\omega_X$  is positive. Clearly, the sign of  $\omega_X$  does not change if we increase the vertical shift.

We know that  $[T \circ s_d]$  is  $d!$  times the volume of the corresponding simplex. This volume on the other hand can be expressed as  $\zeta \cdot |\text{vol}(\pi(\Delta_D))|/d$ , and  $[D]$  equals  $(d-1)!$  times the volume of  $\pi(\Delta_D)$ . Hence we have  $[T \circ s_d] = \zeta \cdot [D]$  and the statement of the lemma follows.  $\square$

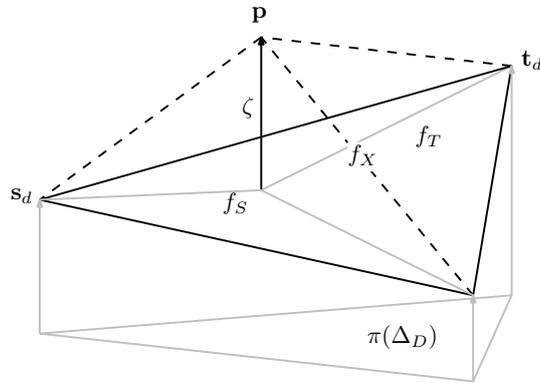


Figure 5: The situation (for  $d = 3$ ) for an interior ridge as discussed in Lemma 6.

## 2.7 The tree-representation of a stacked polytope

The combinatorial structure of a polytope is typically encoded by its face lattice. If the polytope is a stacked polytope, we can also describe its structure by “recording” in which way we have carried out the stacking operations. The most natural way to keep track of the stacking operations is an ordered rooted tree. Let  $P$  be a stacked polytope, then the *tree-representation* of  $P$  is a  $d$ -ary tree  $\mathcal{T}(P)$ , which is defined as follows: The leaves of the trees are in one-to-one correspondence to the facets of  $P$ , with the exception of the face  $f_B$ , which is not present in the tree. Interior nodes correspond to  $d$ -cliques in the graph of  $P$  that used to be facets at some point during the stacking process.<sup>3</sup> The root represents the initial copy of  $f_B$  in the beginning. The children of a node  $v$  represent the faces that were introduced by stacking a vertex onto the face associated with  $v$ . To reproduce the combinatorial structure from  $\mathcal{T}(P)$  we fix an ordering for the edges emanating from an interior node  $v$ , such that it is possible to map the children of  $v$  to the faces generated by stacking onto  $v$ ’s face in a unique way. By this the mapping between the faces of  $P$  and the nodes of  $\mathcal{T}(P)$  can be reproduced. If  $P$  has  $n$  vertices, then  $\mathcal{T}(P)$  has  $(n-d)(d-1)+1$  leaves and  $n-d$  interior nodes.

We will use the tree-representation of  $P$  to specify the geometry of the flat embedding of  $P$ . This can be achieved by assigning for every leaf  $v$  of  $\mathcal{T}(P)$  a rational number (weight), which corresponds to the volume of  $v$ ’s face in the flat embedding of  $P$ . More precisely, the weight

<sup>3</sup>When considering an intermediate configuration in the stacking process, we refer to the  $d$ -cliques of  $P$  that are faces in the intermediate polytope also as *facets*, although they are not necessarily facets of  $P$ .

specifies the volume of  $v$ 's face times  $(d - 1)!$ . To emphasize this relationship, we call the weights *face-weights*. We extend the face-weight assignments for the interior nodes by summing for every node the face-weights of its children recursively up. After we have determined the location of  $f_B$ , such that its volume is  $(d - 1)!$  times the face-weight of the root of  $\mathcal{T}(P)$ , the coordinates of the whole flat embedding are determined. In particular, for every stacking operation, the (normalized) face-weights specify the location of the stacked vertex since they denote its barycentric coordinates. Thus, by traversing  $\mathcal{T}(P)$  we can produce the flat embedding incrementally in a unique way.

### 3 The embedding algorithm

We assume that the combinatorial structure of the stacked polytope is given in form of its tree-representation  $\mathcal{T}(P)$ . The embedding algorithm works in three steps. First we generate the face-weights for  $\mathcal{T}(P)$  and fix the coordinates for the face  $f_B$ . This will give us a flat embedding of the polytope. In the next step we “lift” the polytope, by defining for every vertex  $v_i$  the *vertical shift*  $\zeta_i$ . Assume that we have stacked  $v_i$  onto the face  $f$ . By construction we pick always positive vertical shifts. By carefully choosing the right vertical shifts we obtain an embedding of  $P$  as a *convex* polytope, however this embedding does not fit on a polynomial grid. In the final step we round the points to appropriate grid points, while maintaining convexity.

#### 3.1 Balancing face-weights

We apply a technique from data structure analysis called the *heavy path decomposition* (see Tarjan [24]). Roughly speaking, it decomposes a tree into paths, such that the induced hierarchical structure of the decomposition is balanced. We continue with a brief review of the heavy path decomposition. Let  $u$  be a non-leaf of a rooted tree  $T$  with root  $r$  ( $r = u$  is possible). We denote by  $T_u$  the subtree of  $T$  rooted at  $u$ . Let  $v$  be the child of  $u$  such that  $T_v$  has the largest number of nodes (compared to the subtrees of the other children of  $u$ ), breaking ties arbitrarily. We call the edge  $(u, v)$  a *heavy edge*, and the edges to the other children of  $u$  *light edges*. The heavy edges induce a decomposition of  $T$  into paths, called *heavy paths*, and light edges; see Fig. 6. The node on a heavy path that is closest to the root is called its *top node*. We call a heavy path with its incident light edges (ignoring the possible edge from its top node to its parent) a *heavy caterpillar*. The heavy path decomposition decomposes the edges of  $T$  into heavy caterpillars. We say that two heavy caterpillars are adjacent if their graphs would be adjacent subgraphs in  $T$ . This adjacency relation induces a hierarchy, which we represent as a rooted tree  $\mathcal{H}(T)$ . The nodes in  $\mathcal{H}(T)$  are the heavy caterpillars, and its edges represent the adjacency relation between caterpillars. The root of  $\mathcal{H}(T)$  is the heavy caterpillar that includes the root of  $T$ . When  $(u, v)$  is a light edge and  $u$  is the parent of  $v$  then the size of  $T_v$  is at most half as big as the size of  $T_u$ . Hence, every root-leaf path in  $T$  can visit at most  $\log t$  light edges for  $t$  being the size of  $T$ . Fig. 6 shows an example of a tree-representation and its associated hierarchy as a tree.

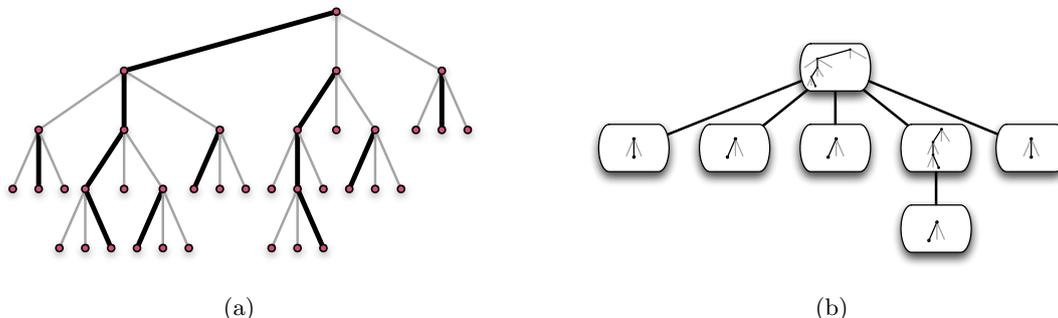


Figure 6: A tree-representation  $\mathcal{T}$  of a stacked 3-polytope (a) and the corresponding hierarchy based on the caterpillars as tree  $\mathcal{H}(\mathcal{T})$  (b).

The assignment of the face-weights is guided by the heavy path decomposition of  $\mathcal{T}(P)$ . We call a node  $v$  *balanced* if (1) the face-weights of  $v$ 's light-edge children are all the same and (2) the face-weight of the heavy edge child is not smaller than the face-weight of every light edge child. If every node is balanced we call the face-weight assignment balanced. An example of a balanced tree-representation is depicted in Fig. 8.

**Lemma 7** *We can find a balanced set of integer face-weights for  $\mathcal{T}(P)$  such that the face-weight associated with the root is at most  $dn^{\log 2d}$  and no face-weight is less than 1. Here,  $n$  denotes the number of vertices in the corresponding graph of the stacked polytope.*

**Proof.** Let  $H = \mathcal{H}(T)$  be the tree representing the hierarchy of the heavy caterpillars for some  $d$ -ary tree  $T$ . The height of  $H$  is the length (number of edges) of its longest root-leaf path. For  $T$  we call the height of  $\mathcal{H}(T)$  its *hpd-height*.

To prove the lemma we use the following claim. Since the proof of the claim will be constructive it will also induce a strategy of how to set the face-weights.

**Claim:** For all  $k \geq 0$  the following holds: If  $T$  is a  $d$ -ary tree with  $m$  leaves and hpd-height  $k$  then we can find a set of balanced face-weights, such that the face-weight of the root is at most  $md^k$ .

We prove the claim by induction on  $k$ . A tree with hpd-height 0 is a heavy caterpillar, whose subtrees on the light edges are leaves. In this case it suffices to set the face-weights of all leaves to 1 and then propagate the face-weights to the interior nodes. Clearly, this face-weight assignment is balanced and the face-weight of the root equals the number of leaves. Thus, the claim holds for this (base) case.

We continue with the induction step. Assume that the claim holds for all trees with hpd-height less than  $k$ . We can now balance the face-weights of any tree of hpd-depth  $k$  as follows (see also Fig. 7 for an illustration): Let the heavy path incident to the root be  $h$ . The subtrees connected to  $h$  via a light edge have all depth less than  $k$ . For these subtrees we use our induction hypothesis and (recursively) determine the corresponding face weights and propagate these weights to  $h$ . If a subtree is a single leaf it will be assigned with a face-weight of 1. We are left with balancing the nodes of  $h$ . So let  $v$  be a node on  $h$  and let  $u_v^+$  be one of its light edge children with the largest face-weight. We will now make each light child  $u \neq u_v^+$  as ‘‘heavy’’ as  $u_v^+$ . To do so, we determine the difference  $\delta$  between the face-weight of  $u_v^+$  and the face-weight of  $u$ . Next, we increase the face-weight of every node on the heavy path with top node  $u$  by  $\delta$ . Notice that this keeps the face-weights in the subtree rooted at  $u$  balanced. To make the updates consistent we also increase the face-weight on  $h$  from  $v$  to its root by  $\delta$ . We continue with the other children of  $u$  in the same fashion and then repeat this for every node on  $h$ . We also have to guarantee that the face-weight of every light edge child is not larger than the face-weight of its heavy edge child. To ensure this, we increase the face-weight of the leaf of  $h$  by  $\delta_r$  and propagate this weight along  $h$ , where  $\delta_r$  is the largest face-weight of one of the light edge subtrees hanging off of  $h$ . Note that this is done only once for the heavy path  $h$ . By this we have obtained a balanced set of face-weights for  $T$ .

We are left with bounding the new face-weight of the root. Let  $S$  be the sum of the face-weights of all the light edge subtrees of  $h$  before balancing the nodes of  $h$ , which is also the face-weight of the root minus 1 at this time. By the induction hypothesis we have  $S \leq md^{k-1}$ . When balancing the light subtrees on a vertex  $v$  we increased the face-weight of the root by some  $\delta$  at most  $d - 2$  times. The ‘‘charge’’  $\delta$  was not larger than the face-weight of the corresponding  $u_v^+$ . The total increment at this stage is therefore less than  $(d - 2)S$ . The final increment by  $\delta_r$  is at most  $S - 1$ . Therefore, we have that the face-weight of the root is at most

$$(S + 1) + (d - 2)S + (S - 1) = dS \leq dm^k$$

and the proof of the claim follows.

The statement of the lemma follows now from the claim. According to Subsec. 2.7 the tree  $\mathcal{T}(P)$  has less than  $(d - 1)n$  leaves. When traversing a light edge  $(u, v)$  the number of interior nodes is at least halved. The tree  $\mathcal{T}(P)$  has less than  $n$  interior nodes. Thus, every path in  $\mathcal{T}(P)$  from the top node of a heavy path to the root of  $\mathcal{T}(P)$  visits at most  $\log n$  light edges. As a



nodes of the faces  $f_{Y_j}$  in  $\mathcal{T}(P)$  that are connected to the node of  $f_D$  via a light edge have all the same face-weight. Let this weight be  $B_i$ . The only (possible) other face-weight for a node associated with one of the faces  $f_{Y_j}$  is denoted by  $A_i$ . We set as the vertical shift for the vertex  $v_i$

$$\zeta_i := A_i \cdot B_i. \quad (5)$$

Note that  $A_i \geq B_i$ , since the face-weights were balanced.

**Lemma 8** *The embedding induced by the face-weights and the vertical shifts  $\zeta_i$  guarantees:*

1. For every ridge  $f_X$  not on  $f_B$  we have  $\omega_X \geq 1$ .
2. For every ridge  $f_X$  on  $f_B$  we have  $0 > \omega_X \geq -R \geq -d \cdot n^{\log(2d)}$ .

**Proof.** We first bound the stresses on faces not on  $f_B$ . Recall that by construction the face-weight of a facet  $f_D$  coincides with  $\llbracket D \rrbracket$ . We study how the stress on  $\omega_X$  evolves during the stacking process. The initial value of  $\omega_X$  is assigned by some stacking operation that introduced  $f_X$ . We assume that this stacking operation stacked a vertex on the face  $f_D$ . The face-weights of the new facets introduced by the stacking are all the same, namely  $B_i$ , except for one possible larger face-weight, namely  $A_i$ . Let  $f_S$  and  $f_T$  be the two facets incident to  $f_X$  such that  $\llbracket S \rrbracket \geq \llbracket T \rrbracket$ . By Lemma 6, we have that at this moment

$$\omega_X = \zeta_i \left| \frac{\llbracket D \rrbracket}{\llbracket S \rrbracket \llbracket T \rrbracket} \right| = A_i B_i \left| \frac{\llbracket D \rrbracket}{\llbracket S \rrbracket \llbracket T \rrbracket} \right| \geq \llbracket D \rrbracket,$$

since  $\llbracket S \rrbracket \leq A_i$  and  $\llbracket T \rrbracket = B_i$ .

This positive initial stress decreases when stacking on a facet that has  $f_X$  on the boundary. So assume we stacked  $\mathbf{p}_k$  on such a facet. Let  $C_X(k)$  denote the amount of the decrement due to this stacking. By Lemma 5 we have  $C_X(k) = \zeta_k / \llbracket S_k \rrbracket$ , where  $S_k = X \circ \mathbf{p}_k$ . Recall that  $\zeta_k = A_k B_k$ , where  $A_k$  and  $B_k$  are the two different values of face-weights of faces introduced by stacking  $\mathbf{p}_k$ . It follows that  $\llbracket S_k \rrbracket \in \{A_k, B_k\}$ , and therefore  $C_X(k) = \{A_k, B_k\} \setminus \{\llbracket S \rrbracket\}$  equals either  $A_k$  or  $B_k$ , and hence, is an integer. We charge the value of  $C_X(k)$  to a face-weight of a face  $f_{Y_k} \neq f_{S_k}$  that was introduced when stacking  $\mathbf{p}_k$ . Stacking operations that decrease  $\omega_X$  further, stack either onto  $f_{S_k}$ , or onto the opposite facet incident to  $f_X$  (see Fig. 9). As a consequence, the projected facets  $\{\pi(f_{Y_j}) \mid \text{stacking of } \mathbf{p}_j \text{ decrements } \omega_X\}$  have disjoint interiors and are properly contained inside  $\pi(\Delta_D)$ . Since it is not possible to cover  $\pi(\Delta_D)$  with these faces completely, the difference  $\llbracket D \rrbracket - \sum_j C_X(j)$  is at least 1 and therefore  $\omega_X \geq \llbracket D \rrbracket - \sum_j C_X(j) \geq 1$ .

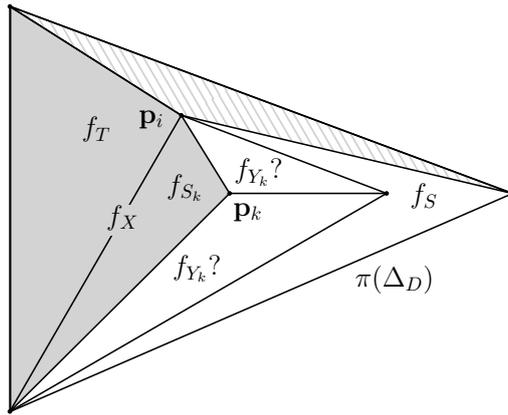


Figure 9: When stacking  $\mathbf{p}_k$  the stress on  $X$  is decreased by at most the face-weight charged to  $f_{Y_k}$ . There are two candidates for  $f_{Y_k}$ . All further negative charges will be attributed to “volumes” contained inside  $S_k$  and  $T$ . The face with the crossed-line pattern will never be charged to  $\omega_X$ .

For the stresses on the boundary we can argue as follows. The stress on  $X$  is a combination of several negative charges, but without having an initial positive charge. By the above argument,

the negative charges are attributed to faces with disjoint interiors. All these faces are contained inside  $\pi(\Delta_B)$  and hence  $|\omega_X|$  is at most  $\llbracket B \rrbracket$ . By Lemma 7 we have that  $\llbracket B \rrbracket = R \leq d \cdot n^{\log(2d)}$ , and the lemma follows.  $\square$

### 3.3 Rounding to grid points

Let us wrap up what we have constructed so far. Based on the tree-representation we have defined weights for each face. This gave rise to a realization of a projection of the stacked polytope. As a next step we have computed a lifting based on the face-weights. This construction produces a convex realization of the desired stacked polytope, we even know that the stresses are polynomially related. However, the realization does not lie on a polynomial grid yet. To obtain an integer realization we round the coordinates of the points down. The rounding will be carried out in two steps. First we perturb all coordinates such that they are multiples of some parameter  $\alpha$ , resp.  $\alpha_z$  for the  $z$ -coordinates. In a second step we scale the perturbed embedding by multiplying all coordinates with  $1/\alpha$ , resp. with  $1/\alpha_z$ , for the  $z$ -coordinates. The details for the rounding procedure are little bit more subtle: We start with rounding the coordinates in the flat embedding, then we update the vertical shifts  $\zeta_i$  slightly. The  $z$ -coordinates are rounded with respect to the lifting defined by the modified vertical shifts.

Since projected volumes play an important role in our construction we discuss as a first step how the rounding will effect these volumes.

**Lemma 9** *If we round the coordinates of the flat embedding down, such that every coordinate is a multiple of  $\alpha$  we have that for every facet  $f_X$*

$$|\llbracket X \rrbracket - \llbracket X' \rrbracket| \leq \alpha d^2 L^{d-2},$$

where  $X'$  denotes the points  $X$  after the rounding.

**Proof.** Remember that the points  $X$  are contained inside the simplex  $\Delta_B$ , which is spanned by the standard basis vectors of  $\mathbf{R}^{d-1}$  scaled by  $L = \sqrt[d-1]{R}$  and the origin. We first show how the rounding of the first coordinate for every  $x \in X$  effects  $\llbracket X \rrbracket$ . The point set  $X$  after rounding the first coordinate is named  $X^1$ . Let  $E(1)$  denote the change of  $\llbracket X \rrbracket$  in the worst case, that is,  $E(1) = \max_X |\llbracket X \rrbracket - \llbracket X^1 \rrbracket|$ . We denote by  $x_i \in \mathbf{R}$  the first coordinate (the  $x$ -coordinates) of the  $i$ th point in  $X$ , and for a point sequence  $X$  we denote by  $X_{-i}$  the set with removed  $x$ -coordinates and without the  $i$ th point of  $X$ . Note that  $X_{-i} = X_{-i}^1$ . Let the change of the coordinates  $x_i$  be  $\varepsilon_i \leq \alpha$ . We estimate  $E(1)$  using the Laplace expansion for the determinants (along the row for the  $x$ -coordinates) by

$$\begin{aligned} E(1) = |\llbracket X \rrbracket - \llbracket X^1 \rrbracket| &\leq \left| \sum_{i=1}^d (-1)^i x_i \llbracket X_{-i} \rrbracket - \sum_{i=1}^d (-1)^i (x_i + \varepsilon_i) \llbracket X_{-i}^1 \rrbracket \right| \\ &\leq \left| \sum_{i=1}^d (-1)^i x_i \llbracket X_{-i} \rrbracket - \sum_{i=1}^d (-1)^i (x_i + \varepsilon_i) \llbracket X_{-i} \rrbracket \right| \\ &\leq \sum_{i=1}^d |\alpha \llbracket X_{-i} \rrbracket| \\ &\leq d\alpha L^{d-2}. \end{aligned}$$

For the last transition we upper bounded the determinants  $\llbracket X_{-i} \rrbracket$  by  $L^{d-2}$ . This bound follows, since the maximal volume of a  $(d-2)$ -simplex in  $\Delta_B$  is spanned by the  $d-1$  points of  $B \setminus \mathbf{0}$ . This point set would also maximize  $\llbracket X_{-i} \rrbracket$  and hence this determinant is at most  $L^{d-2}$ .

Note that there was nothing special in rounding the *first* coordinate (while keeping the others fixed). If we would round only the  $k$ th coordinate, we obtain the same estimation. Thus, we can apply the rounding of a single coordinate one by one for each of the  $(d-1)$ -coordinates of  $\mathbf{R}^{d-1}$ . Every time we round in one coordinate we introduce an additive “error” of  $d\alpha L^{d-2}$  to  $|\llbracket X \rrbracket - \llbracket X' \rrbracket|$ . Thus in total we get the asserted bound of  $|\llbracket X \rrbracket - \llbracket X' \rrbracket| \leq \alpha d^2 L^{d-2}$ .  $\square$

In the following we use  $\llbracket X' \rrbracket$  and similar expressions to denote the corresponding determinants after rounding.

**Corollary 1** *If we round the coordinates of the flat embedding down, such that every coordinate is a multiple of  $\alpha$ , we have for every facet  $f_X$  in the flat embedding*

$$1 - \alpha d^2 L^{d-2} \leq \frac{\llbracket X' \rrbracket}{\llbracket X \rrbracket} \leq 1 + \alpha d^2 L^{d-2}.$$

**Proof.** The statement follows from Lemma 9 and the observation that for any facet  $f_X$  in the flat embedding we have  $1 \leq \llbracket X \rrbracket$ .  $\square$

As a next step we discuss, how to set the parameter  $\alpha$ , such that the perturbed flat embedding will still have a positive stress. To describe the lifting (stress) we defined for every vertex in the flat embedding a vertical shift  $\zeta_i$ , as given in (5). The definition of  $\zeta_i$  was based on the face-weights obtained from the balanced tree-representation. We adjust the vertical shifts after the perturbation slightly. When stacking  $\mathbf{p}_i$  we introduced  $d$  new faces. Let the faces  $f_{A_i}$  and  $f_{B_i}$  be the two faces out of the  $d$  newly introduced faces with the largest volume. We define

$$\zeta'_i := \llbracket A'_i \rrbracket \llbracket B'_i \rrbracket.$$

**Lemma 10** *When we pick as the perturbation parameter  $\alpha = 1/(10d^2 L^{d-2} R)$  then the perturbed flat embedding with the vertical shifts  $\zeta'_i$  induces an embedding, whose interior stresses are at least  $4/5$ , and whose boundary stresses are negative and larger than  $-2R$ .*

**Proof.** We mimic the strategy of the proof of Lemma 8. Again, all faces in this proof are considered as projected into the  $z = 0$  hyperplane. For the proof of the lemma the sign of the determinants  $\llbracket \cdot \rrbracket$  is not important. For the sake of a simple presentation we misuse notation and simply write  $\llbracket \cdot \rrbracket$  instead of  $|\llbracket \cdot \rrbracket|$  in this proof.

Every stress  $\omega_X$  is a combination of a positive stress and several negative stresses attributed to different stacking operations. Let  $\omega_X^+$  be the positive stress, that is the initial nonzero stress with respect to the stacking sequence, and let  $\omega_X^-$  the absolute value of the sum of all negative stresses, such that  $\omega_X = \omega_X^+ - \omega_X^-$ . To bound  $\omega_X$  we derive bounds for  $\omega_X^+$  and  $\omega_X^-$ . We start with the bound on the positive stress. Assume  $\omega_X^+$  was introduced by stacking  $\mathbf{p}_i$  at some face  $f_D$ , such that due to Lemma 6 we have

$$\omega_X^+ = \zeta'_i \left| \frac{\llbracket D' \rrbracket}{\llbracket S' \rrbracket \llbracket T' \rrbracket} \right|.$$

Here,  $f_S$  and  $f_T$  are the two faces introduced by stacking  $\mathbf{p}_i$  that contain  $f_X$ . The height  $\zeta'_i$  is defined as the product of two face-weights corresponding to  $f_{A_i}$  and  $f_{B_i}$ . Assume that  $\llbracket A'_i \rrbracket \geq \llbracket B'_i \rrbracket$  and that  $\llbracket S' \rrbracket \geq \llbracket T' \rrbracket$ . By the definition of  $\zeta'_i$  we have that  $\llbracket A'_i \rrbracket \geq \llbracket S' \rrbracket$  and  $\llbracket B'_i \rrbracket \geq \llbracket T' \rrbracket$ . Therefore,  $\omega_X \geq \llbracket D' \rrbracket$ .

The value of  $\omega_X^-$  is composed of several ‘‘charges’’. Whenever we stack inside a face that contains  $f_X$  we increase  $\omega_X^-$ . Let us study now one of these situations. Assume we stack  $\mathbf{p}_k$  inside a face that contains  $f_X$ . Let  $f_{S_k}$  be the new face that contains  $f_X$ . By Lemma 5 we increase  $\omega_X^-$  by  $|\zeta'_j / \llbracket S'_k \rrbracket| := inc_{X,k}$ . Due to the balanced face-weights we had in the unperturbed setting only two different new ‘‘face volume values’’ when stacking a vertex. Hence for  $\zeta'_i := \llbracket A'_k \rrbracket \llbracket B'_k \rrbracket$ , we had either  $\llbracket S_k \rrbracket = \llbracket A_k \rrbracket$ , or  $\llbracket S_k \rrbracket = \llbracket B_k \rrbracket$ . We define

$$C_k := \begin{cases} B_k & \text{if } \llbracket S_k \rrbracket = \llbracket A_k \rrbracket, \\ A_k & \text{if } \llbracket S_k \rrbracket = \llbracket B_k \rrbracket \end{cases}$$

and set  $\delta_+ := 1 + \alpha d^2 L^{d-2}$  and  $\delta_- = 1 - \alpha d^2 L^{d-2}$ . If  $\llbracket S_k \rrbracket = \llbracket A_k \rrbracket$  we have according to Corollary 1

$$inc_{X,k} = \left| \frac{\llbracket A'_k \rrbracket}{\llbracket S'_k \rrbracket} \llbracket B'_k \rrbracket \right| \leq \left| \frac{\delta_+ \llbracket A_k \rrbracket}{\delta_- \llbracket S_k \rrbracket} \llbracket B'_k \rrbracket \right| = \frac{\delta_+}{\delta_-} \llbracket B'_k \rrbracket = \frac{\delta_+}{\delta_-} \llbracket C'_k \rrbracket.$$

The remaining case  $\llbracket S_k \rrbracket = \llbracket B_k \rrbracket$  is completely symmetric and does also give  $inc_{X,k} \leq \delta_+ \llbracket C'_k \rrbracket / \delta_-$ . Let  $K$  be the set of vertex indices, whose stacking contributed to  $\omega_X^-$ . As noticed in Lemma 8 (see also Fig. 9), for any two distinct  $s, t \in K$  we have that  $f_{C_s}$  and  $f_{C_t}$  have disjoint interiors,

and furthermore the set  $\bigcup_{k \in K} f_{C_k}$  is contained in the perturbed simplex  $\pi(\Delta_D)$ , but “misses” at least one face. By Corollary 1 the face-weight of the missing face is at least  $\delta_-$ . Therefore,

$$\omega_X^- = \sum_{k \in K} inc_{X,k} \leq \sum_{k \in K} \frac{\delta_+}{\delta_-} \llbracket C'_k \rrbracket = \frac{\delta_+}{\delta_-} \sum_{k \in K} \llbracket C'_k \rrbracket \leq \frac{\delta_+}{\delta_-} (\llbracket D' \rrbracket - \delta_-) = \frac{\delta_+}{\delta_-} \llbracket D' \rrbracket - \delta_+.$$

We finish the proof by combining the bounds for  $\omega_X^+$  and  $\omega_X^-$ . When we pick  $\alpha = 1/(10d^2L^{d-1}R)$  as specified in the lemma then we get  $\delta_+ = 1 + 1/(10R)$  and  $\delta_- = 1 - 1/(10R)$ . We can now obtain the following bound for  $\omega_X$  when  $X$  is an interior ridge (note that  $\pi(\Delta_{D'}) \subseteq \pi(\Delta_B)$ ):

$$\begin{aligned} \omega_X &\geq \omega_X^+ - \omega_X^- \\ &\geq \llbracket D' \rrbracket - \left( \frac{\delta_+}{\delta_-} \llbracket D' \rrbracket - \delta_+ \right) \\ &= \delta_+ - \llbracket D' \rrbracket \left( \frac{\delta_+}{\delta_-} - 1 \right) \\ &\geq \delta_+ - \llbracket B \rrbracket \left( \frac{\delta_+}{\delta_-} - 1 \right) = \delta_+ - \delta_+ R \left( \frac{\delta_+}{\delta_-} - 1 \right) \\ &= \frac{80R^2 - 2R - 1}{100R^2 - 10R} \\ &\geq \frac{4}{5} \end{aligned}$$

If  $X$  is a boundary face we have  $\omega_X^+ = 0$ . Notice that for our choice of  $\alpha$  we have  $\delta_+/\delta_- \leq 2$ , for  $R \geq 3$ . We conclude that for a boundary face we have

$$\omega_X \geq - \left( \frac{\delta_+}{\delta_-} \llbracket B' \rrbracket + \delta_+ \right) \geq - \frac{\delta_+}{\delta_-} \llbracket B' \rrbracket \geq -2 \llbracket B' \rrbracket \geq -2R.$$

□

The choice of the parameter  $\alpha = 1/(10d^2L^{d-2}R)$  ensures, that no volume of a face will flip its sign. In particular, by Lemma 9, the change of volume is less than  $1/(10\llbracket B \rrbracket)$ .

To construct an integer realization we need to round the  $z$ -coordinates as well. We round the  $z$ -coordinates such that every  $z$ -coordinate is a multiple of  $\alpha_z$ , where  $\alpha_z$  is some value to be determined later. The final analysis requires an upper bound for the maximal  $z$ -coordinate before rounding, which we give in the following lemma.

**Lemma 11** *For  $z_{\max}$  being the maximal  $z$ -coordinate in the lifting of the perturbed flat embedding induced by the vertical shifts  $\zeta'_i$  we have that*

$$0 < z_{\max} < 2R^2.$$

**Proof.** The  $z$ -coordinates of all vertices not on  $f_B$  are positive since all vertical shifts are positive. We consider the perturbed flat embedding. Take any boundary ridge  $f_X$  of  $f_B$ . By Lemma 10 we have that  $-\omega_X < 2R$ . Let  $\mathbf{p}_i$  be the vertex with the highest  $z$ -coordinate  $z_{\max}$  and set  $\mathbf{r} = \pi(\mathbf{p}_i)$ . Now take the facet  $f_Y$  adjacent to  $f_B$  via  $f_X$  (we let  $Y$  and  $B$  coincide on the first  $d-2$  vertices). Due to the convexity of the lifting  $f_Y$  supports a bounding hyperplane, and therefore  $z_Y(\mathbf{r}) \geq z_{\max}$ . By Equation (1) we have  $\omega_X \llbracket Y \rrbracket = z_B(\mathbf{r}) - z_Y(\mathbf{r}) = -z_Y(\mathbf{r})$ . Since  $f_Y$  is properly contained inside  $f_B$  it follows that

$$z_{\max} \leq z_Y(\mathbf{r}) = -\omega_X \llbracket Y \rrbracket < -\omega_X \llbracket B \rrbracket \leq 2R^2.$$

□

By rounding the  $z$ -coordinates we might violate the convexity of the lifting. The following lemma shows us how to carry out the rounding of the  $z$ -coordinates after we have rounded the coordinates of the flat embedding, such that the resulting embedding remains a convex polytope.

**Lemma 12** *By setting  $\alpha_z = 1/(3R)$  and rounding such that all  $z$ -coordinates are multiples of  $\alpha_z$  the lifting defined by  $\zeta'_i$  and the perturbed flat embedding will remain an embedding of a convex polytope.*

**Proof.** Let  $z'_i$  be the coordinate  $z_i$  after rounding and, similarly, let  $\omega'_X$  be the stress after rounding the  $z$ -coordinates. We have that  $0 \leq z_i - z'_i \leq \alpha_z$ . By Lemma 10 the interior ridges in the perturbed flat embedding have a stress  $\omega_X$  that is at least  $4/5$ . Let  $f_X$  be any interior ridge that is incident to the facets  $f_S$  and  $f_T$ . By Equation (3) we can express  $\omega'_X$  in terms of  $[T']$ ,  $[[S']]$ , and  $[[T']]$ , where  $T'$  is formed by the point set  $S' \cup T'$ . The value of  $[T']$  can be expressed as  $\sum_{i \in I} z'_i [[A'_i]]$ , where  $I$  is the index set of the vertices of  $T'$  and  $A'_i$  is given by  $T' \setminus \{\mathbf{p}'_i\}$  in some appropriate order (Laplace expansion). We split the set  $I$  into  $I_+ := \{i \in I \mid [[A_i]] \geq 0\}$  and  $I_- := I \setminus I_+$ . As usual  $f_B$  denotes the boundary face. Note that the projections of the simplices spanned by the sets  $A'_i$  double-cover the projection of  $T'$  into the  $z = 0$  hyperplane, and therefore  $\sum_{i \in I} [[A'_i]] \leq 2[[B']] \leq 2[[B]] \leq 2R$ . Moreover, by Corollary 1 and our choice of  $\alpha$ , we have  $[[S']][[T']] \geq (1 - 1/(10R))^2$ . The stress on an interior ridge  $f_X$  after rounding ( $\omega'_X$ ) can be bounded as follows

$$\begin{aligned} \omega'_X &= \frac{[T']}{[[S']][[T']]} = \frac{\sum_{i \in I} z'_i [[A'_i]]}{[[S']][[T']]} = \frac{\sum_{i \in I_+} z'_i [[A'_i]]}{[[S']][[T']]} - \frac{\sum_{i \in I_-} z'_i [[A'_i]]}{[[S']][[T']]} \\ &\geq \frac{\sum_{i \in I_+} (z_i - \alpha_z) [[A'_i]]}{[[S']][[T']]} - \frac{\sum_{i \in I_-} z_i [[A'_i]]}{[[S']][[T']]} \\ &= \underbrace{\frac{\sum_{i \in I} z_i [[A'_i]]}{[[S']][[T']]} }_{t_1} - \underbrace{\frac{\sum_{i \in I_+} \alpha_z [[A'_i]]}{[[S']][[T']]} }_{t_2} \end{aligned}$$

We observe that the  $\sum_{i \in I} z_i [[A'_i]] = [T'_z]$ , where  $T'_z$  denotes  $T$  after rounding in the flat embedding, but before rounding the  $z$ -coordinates. This gives us  $t_1 \geq [T'_z]/([S'][[T']]) = \omega_X \geq 4/5$ . Moreover, as already noticed,  $\sum_{i \in I_+} [[A'_i]] \leq \sum_{i \in I} [[A'_i]] \leq 2R$ . After plugging in our choice for  $\alpha_z$  and our bound for  $[[S']][[T']]$  we obtain  $t_2 \leq 2R/(3R(1 - 1/(10R))^2)$ . This yields

$$\omega'_X \geq t_1 - t_2 \geq \frac{4}{5} - \frac{2R}{3(1 - 1/(10R))^2 R} = \frac{4}{5} - \frac{200R^2}{3(10R - 1)^2}.$$

Note that the last expression is a monotone increasing function which is positive for  $R \geq 3$ .

We are left with checking the sign for the stresses on the ridges that define the boundary of  $f_B$ . The stresses for these faces have to remain negative. Note that this is certainly the case, if all  $z$ -coordinates after the rounding remain positive. Before rounding the  $z$ -coordinates, every  $z$ -coordinate of a vertex not on  $f_B$  was at least as large as the smallest vertical shift  $\zeta'_i$ . Since the vertical shifts are defined as the sum of two face-weights, we have that the nonzero  $z$ -coordinates are at least  $(1 - 1/(10R))^2$ . As observed earlier, rounding the  $z$ -coordinates decreases the  $z$ -coordinates by at most  $1/(3R)$ . Therefore we have for every  $\mathbf{p}_i \notin B$

$$z'_i \geq \left(1 - \frac{1}{10R}\right)^2 - \frac{1}{3R} > 1 - \frac{1}{5R} - \frac{1}{3R} > 0,$$

for every  $R \geq 3$ . Hence, after rounding the  $z$ -coordinates the sign pattern of the stresses verifies the convexity of the perturbed realization.  $\square$

We now summarize our analysis and state the main theorem.

**Theorem 1** *Every  $d$ -dimensional stacked polytope can be realized on an integer grid, such that all coordinates have size at most  $10d^2 R^2$ , except for one axis, where the coordinates have size at most  $6R^3$ , for  $R = dn^{\log(2d)}$ .*

**Proof.** To get integer coordinates we multiply all coordinates after the rounding with  $1/\alpha$ , except the  $z$ -coordinates, which we multiply with  $1/\alpha_z$ . Since the maximal  $z$ -coordinate is by Lemma 11 at most  $2R^2$ , and we scale by  $1/\alpha_z = 3R$  the bound for the  $z$ -coordinates in the theorem follows. All other coordinates are positive and smaller than  $L$  before rounding. Hence by scaling with  $1/\alpha = 10d^2 L^{d-2} R$  we get that all coordinates are integers (Lemma 10) and the maximum coordinate has size  $10d^2 L^{d-1} R$ . Plugging in the definition of  $L$  gives as upper bound  $10d^2 R^2$  as asserted.  $\square$

By expressing the quantity  $R$  in terms of  $n$  we can restate Theorem 1 as the following corollary.

**Corollary 2** For a fixed  $d$ , every  $d$ -dimensional stacked polytope can be realized on an integer grid polynomial in  $n$ . The size of the largest  $z$ -coordinate is bounded by  $O(n^{3\log(2^d)})$ , all other coordinates are bounded by  $O(n^{2\log(2^d)})$ .

Table 1 lists the induced grid bounds for  $d = 3, \dots, 10$ .

| $d$ | exponent largest non- $z$ -coordinate | exponent largest $z$ -coordinate |
|-----|---------------------------------------|----------------------------------|
| 3   | 5.17                                  | 7.76                             |
| 4   | 6                                     | 9                                |
| 5   | 6.65                                  | 9.97                             |
| 6   | 7.17                                  | 10.76                            |
| 7   | 7.62                                  | 11.43                            |
| 8   | 8                                     | 12                               |
| 9   | 8.34                                  | 12.51                            |
| 10  | 8.65                                  | 12.97                            |

Table 1: The induced grid bounds in terms of  $n$  up to dimension 10.

## 4 A simple lower bound

In this section we present a simple lower bound. The basis of our construction is the following graph. Take the tetrahedron and stack a vertex in every face, then take the resulting graph and stack again a vertex in every face. We call this graph  $B_3$ ; see Fig. 4 for an illustration.

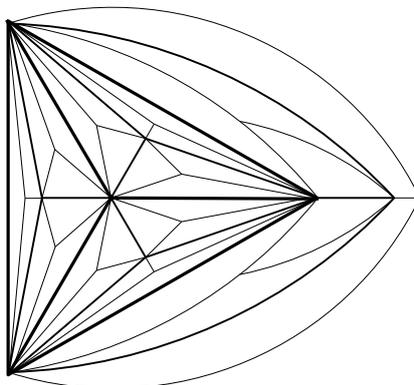


Figure 10: The graph  $B_3$ . It has 36 faces and 20 vertices, 12 of them having degree 3.

**Lemma 13** Let  $P$  be any embedding of  $B_3$  as a stacked 3-polytope. Then there is at least one face with no boundary edge in the orthogonal projection of  $P$  into the  $xy$ -plane.

**Proof.** Let  $\sigma$  be the boundary of  $\pi(P)$ , which is a polygon. We proceed with a case distinction on the size of  $\sigma$ .

**Case 1:**  $\sigma$  contains less than 18 vertices.

Every edge on  $\sigma$  is a boundary edge of two faces. Hence in this case we can have at most 34 faces with a boundary edge, but there are 36 faces in  $B_3$ . So we have at least two faces without a boundary edge in the projection.

**Case 2:**  $\sigma$  contains at least 18 vertices.

The cycle  $\sigma$  splits  $B_3$  into two triangulations (interior+cycle and exterior+cycle). Let us now have a look what happens at a degree 3 vertex of  $B_3$  on  $\sigma$ . Two of its incident edges have to be in  $\sigma$ , which means it is the “tip” of an ear in one of the triangulations. Since we have 12 degree 3 vertices and at most two vertices are not on  $\sigma$ , one of the triangulations, say  $Q$ , has more than

two ears. Now we look at the dual graph of  $Q$  in which we removed the vertex for the outer face. Clearly, this graph is connected (since  $B_3$  is 3-connected) and it has as many degree 1 vertices as  $Q$  has ears. Hence, there has to be a vertex in the dual graph with degree 3. This means that there is a face in  $Q$ , whose adjacent faces are interior faces in  $Q$ . Therefore, we have a face in the projection without a boundary edge as asserted.  $\square$

There are planar 3-trees that require  $\Omega(n^2)$  area for plane straight-line drawings [13]. Let  $G_m$  be such a planar 3-tree with  $m$  vertices that needs  $\Omega(m^2)$  area. For simplicity we assume that  $n$  is a multiple of 36. We glue a copy of  $G_{n/36}$  in each of the 36 faces of  $B_3$ . This yields another planar 3-tree with  $n$  vertices, which we call  $\Gamma_n$ .

**Lemma 14** *The embedding of  $\Gamma_n$  as a convex 3-polytope requires a bounding box of  $\Omega(n^3)$  volume.*

**Proof.** Let  $P$  be an embedding of  $\Gamma_n$  as a 3-polytope. By restricting the 1-skeleton of  $P$  to  $B_3$  and taking the convex hull, we get an embedding of  $B_3$  as a 3-polytope  $P_B$ . Due to Lemma 13, there has to be one face, say  $f$ , which has no boundary edge on  $\pi(P_B)$ . The face  $f$  defines in  $\pi(P)$  a triangle, which contains the graph  $G' = G_{n/36}$ . The supporting planes of the faces adjacent to  $f$  in  $P_B$  define a cone  $C$ . We denote with  $C_0$  the part  $C \setminus P$  that contains the apex of  $C$ . Since none of the edges of  $f$  are on the boundary of  $\pi(P_B)$  we have that  $\pi(C_0) = \pi(f)$ . Hence,  $\pi(P)$  contains a noncrossing drawing of  $G_{n/36}$ , namely  $\pi(G')$ , inside  $\pi(f)$ . Therefore,  $\pi(P)$  needs at least area  $\Omega(n^2)$ .

There was nothing special with choosing the projection into the  $xy$ -plane. By the same arguments we can show, that also the projections into the  $xz$ - and  $yz$ -plane require  $\Omega(n^2)$  area.

Let  $d_x, d_y, d_z$  denote the dimensions of the bounding box of  $P$  along the  $x$ -,  $y$ -, and  $z$ -axis. The volume of the bounding box can be estimated by

$$d_x d_y d_z = \sqrt{d_x^2 d_y^2 d_z^2} = \sqrt{(d_x d_y)(d_x d_z)(d_y d_z)} = \Omega(n^3).$$

$\square$

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