HYPERBOLIC-PSEUDODIFFERENTIAL OPERATORS WITH DOUBLE CHARACTERISTICS

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ABSTRACT

A local parametrix F for hyperbolic pseudodifferential operators P with involutive double characteristics satisfying the Levi condition is constructed. The problem is reduced to construct a parametrix E for the Cauchy problem for a 2×2 symmetric hyperbolic system with characteristic roots of non uniform multiplicity. This is done via the sum of two Fourier Integral Operators and an oscillatory integral with wedges E_3. The wave front set of E is contained in the union of the two canonical relations defined by the Fourier Integral Operators and the "cone generated" by the two canonical relations on the points of double characteristics. Out of the wedge of this cone E_3 is a Fourier Integral Operator and its symbol satisfies a symmetric hyperbolic system. The wave front set of F is contained in the union of the diagonal, the canonical relations defined by H_{p_i} if $P = P_1P_2 + Q$, and the "cone generated" on the points of double characteristics by the canonical relations.

We generalize this construction to get a parametrix for the Cauchy problem of a symmetric hyperbolic system with double characteristics that leads to a parametrix for the Cauchy problem for hyperbolic operators with double characteristics under an assumption that coincides with the Levi condition in the involutive case.

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DEDICATION

A mis padres, A Carolina y al Pueblo Chileno que hicieron de diferente manera esto posible.

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INTRODUCTION

In this paper we study hyperbolic pseudodifferential operators with double characteristics on a C^∞ manifold X. We consider the construction of right and left parametrices for these operators (Chapter I) and we study the Cauchy problem (Chapter II).

Hyperbolic with double characteristics means that the principal symbol p of $P \in L^m(X)$ has the form $p = p_1 p_2$, p_i real valued homogenous functions on T^*X with single characteristics i.e. $d_\xi p_i(x,\xi) \neq 0$ on $p_i(x,\xi) = 0$, $i = 1,2$.

In Chapter I we construct a local parametrix F for P, when the characteristics are involutive i.e.

$$\{p_1, p_2\} = 0 \quad \text{on} \quad \Sigma = \{(x,\xi) \in T^*X - \{0\} \mid p_1(x,\xi) = p_2(x,\xi) = 0\} .$$

We also assume $C_p = 0$ on Σ where C_p is the subprincipal symbol of P. Ivrii-Petkov (see [I-P] and Chapter II.2) have shown that this last assumption is necessary for the well possedeness of the Cauchy problem for P under the involutive assumptions. Also the condition $C_p = 0$ on Σ is equivalent to the local Levi condition (see Chapter I) that it is a necessary and sufficient condition for the well possedeness of the Cauchy problem for hyperbolic operators with characteristic roots of constant multiplicity as it was shown by Flaschka-Strang (see [F-S]). The construction is done first by transforming the operator P to a "simpler" one M in $\mathbb{R}^n$. The principal part of M has the form $D_1 D_2$. Afterwards we consider a system associated to M and we reduce this system to:

$$L = \begin{pmatrix} D_t & 0 \\ 0 & D_t - D_{y_1} \end{pmatrix} + A(t,y,D_y)$$

where the coordinates in $\mathbb{R}^n$ are denoted by $(t,y) \in \mathbb{R} \times \mathbb{R}^{n-1}$, $A \in L^0(\mathbb{R}^{n-1})$ smooth in t. For constructing a parametrix for L we construct a parametrix for the Cauchy problem for L, i.e. an operator $E\colon C_0^\infty(\mathbb{R}^{n-1}) \to C^\infty(\mathbb{R}^n)$ s.t.

$$LE \in C^\infty(\mathbb{R}^{n-1} \times \mathbb{R}^n)$$

$$\gamma_0 E = \mathrm{Id} \bmod L^{-\infty}(\mathbb{R}^{n-1})$$

where $\gamma_0\colon C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^{n-1})$ is the restriction to the hypersurface $t = 0$, i.e. $\gamma_0 f(y) = f(0,y)$, $f \in C^\infty(\mathbb{R}^n)$. E is not a Fourier Integral Operator as it is when P is strictly hyperbolic or P has characteristic roots

of constant multiplicity. (See Chapter II.1 and [CH_1].)
We have

$$E = E_1 + E_2 + E_3 \qquad E_i \in I^{-\frac{1}{4}}(\mathbb{R}^{n-1}, \mathbb{R}^n, C_i(0))$$
$$i = 1,2$$

$$E_3 f(t,y) = \int_{-t}^{t}\int e^{i\varphi_3(\tau,t,y,\theta)} e_3(\tau,t,y,\theta)\hat{f}(\theta)d\theta d\tau$$

where $e_3 \in S^1(\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^{n-1})$. We have

$$WF'E_3 \subseteq C_1(0) \cup C_2(0) \cup C_3(0)$$

where

$$C_3(0) = \{((t,y,r,\xi);(\bar{y},\bar{\xi})) \in T'(\mathbb{R}^n\times\mathbb{R}^{n-1}) \mid \bar{y}_1 = y_1 + \frac{\tau+t}{2}, \ -t \le \tau \le t, \ \bar{y}_j = y_j, \ j = 2,\dots,n-1, \ r = \xi_1 = 0, \ \xi = \bar{\xi}\} .$$

We have $$C_3(0) = \bigcup_{-t \le \tau \le t} \tilde{C}_3(\tau)$$

and $\tilde{C}_3(t) = C_2(0)$, $\tilde{C}_3(-t) = C_1(0)$.

So $C_3(0)$ is the cone generated by $C_1(0)$ and $C_2(0)$ on the points where $r = \xi_1 = 0$ $(p_1 = p_2 = 0)$. The appearence of the extra term E_3 is motivated in Chapter I, Section 4. Its construction was suggested by the Granoff-Ludwig paper (see [G-L]). We show that E_3 is a Fourier Integral Operator out of the "wedge" of the cone mentioned above and its principal symbol satisfies a symmetric hyperbolic system like the wave equation in three dimensions. The "phenomenon" that the transport equation is a symmetric hyperbolic system appears also in conical refraction (see [L]). E_3 is also a Fourier Integral Operator out of Σ . The wave front set of F is contained in the union of the diagonal, $\tilde{C}_i$ where $\tilde{C}_i$ are the canonical relations defined by H_{p_i} , $i = 1,2$ and $\tilde{C}_3$, where $\tilde{C}_3$ is the "Cone generated" by the canonical relations $\tilde{C}_i(0)$, $i = 1,2$, on Σ , (see Chapter I, Section 7 for more precise information).

In Chapter II we construct a parametrix for a symmetric hyperbolic system with double characteristics, reducing this system via a canonical transformation that "preserves" the Cauchy data, to a system of the form:

$$\tilde{L} = \begin{pmatrix} D_t & 0 \\ 0 & D_t - \tilde{\lambda}_2 \end{pmatrix} + \tilde{D}(t,y,D_y) \quad .$$

$\tilde{D}(t,.,.) \in L^0$ smooth in t. A parametrix for $\tilde{L}$ is constructed using a generalization of the idea of the construction of a parametrix for L. This leads to a parametrix for the Cauchy problem for hyperbolic operators with double characteristics under an assumption that coincides with the Levi condition in the involutive case.

Finally in Chapter III we mention some open problems related to this work.

The general emphasis in this thesis is on constructive methods.

CHAPTER I

THE LOCAL PARAMETRIX

If Y is a C^∞ manifold, we are going to denote by $L^m(Y)$ the pseudodifferential operators whose full symbol is an asymtotic sum of homogenous functions. More precisely: $p \approx \sum_{j=-m}^{\infty} p_{-j}$, p_{-j} homogenous function of degree $-j$ on $T'(Y) = T^*(Y) - \{0\}$ and

$$p - \sum_{j<k} p_{-j} \; S^{-k}(T'(Y))$$

and p is the full symbol of the operator.

If $P \in L^m(Y)$ we are going to denote by small p , the principal symbol unless the other thing is stated.

1. Assumptions. Let X be a C^∞ manifold.

$$P \in L^m(X) \qquad P = P_1P_2 + Q \qquad P_i \in L^{m_i}(X) \quad i = 1,2$$

$$Q \in L^{m_1+m_2-1}(X) \quad .$$

p_i homogenous functions of degree m_i on T^*X with real values, $i = 1,2$.

q homogenous of degree m_1+m_2-1 .

Let $\Sigma = \{x, \xi) \in T'(X) | \; p_1(x, \xi) = p_2(x, \xi) = 0\}$.

We will assume

i_1) $\{p_1, p_2\} = 0$ on Σ

i_2) $d_\xi p_i$ linearly independent on Σ, $i = 1,2$.

i_3) $C_p = 0$ on Σ where C_p is the subprincipal symbol of P .

i_4) H_{p_1}, H_{p_2}, V are l.i. on Σ where H_{p_i} is the Hamiltonian vector field of p_i, $i = 1,2$ and V is the cone axis.

i_5) P_i have simple characteristics, $i = 1,2$.

Notation: We will say that P satisfies (I) if

$$P = P_1 P_2 + Q \qquad P_i \in L^{m_i}(X) \quad i = 1,2 \quad Q \in L^{m_1+m_2-1}(X)$$

p_i homogenous of degree m_i, $i = 1,2$, q homogenous of degree m_1+m_2-1 .

2. Reduction of the problem.

Proposition 2.1: Suppose P satisfies (I); P satisfies

i_1). Then P satisfies $i_3) \Leftrightarrow q = 0$ on Σ .

<u>Proof</u>: We have that in local coordinates on X :

$$(1) \quad c_p = p_{m-1} - \frac{1}{2i} \sum_{j=1}^{n} \frac{\partial^2 p}{\partial x_j \partial \xi_j} \qquad \text{(cf. [D]) and}$$

$$(2) \quad p_{m-1} = q + \frac{1}{i} \sum_{j=1}^{n} \frac{\partial p_1}{\partial \xi_j} \frac{\partial p_2}{\partial x_j} \qquad \text{(cf. [D]) . Also}$$

$$(3) \quad p = p_1 p_2 \quad .$$

Then

$$(4) \quad c_p = q + \frac{1}{2i} \sum_{j=1}^{n} \frac{\partial p_1}{\partial \xi_j} \frac{\partial p_2}{\partial x_j} - \frac{1}{2i} \sum_{j=1}^{n} \frac{\partial p_1}{\partial x_j} \frac{\partial p_2}{\partial \xi_j}$$

$$+ \frac{1}{2i} \sum_{j=1}^{n} \frac{\partial p_1}{\partial \xi_j} \frac{\partial p_2}{\partial x_j} - \frac{1}{2i} \sum_{j=1}^{n} \frac{\partial p_1}{\partial \xi_j} \frac{\partial p_2}{\partial x_j}$$

$$- \frac{1}{2i} p_2 \sum_{j=1}^{n} \frac{\partial^2 p_1}{\partial x_j \partial \xi_j} - \frac{1}{2i} p_1 \sum_{j=1}^{n} \frac{\partial^2 p_2}{\partial x_j \xi_j} \quad .$$

So:

$$(5) \quad c_p = q + \frac{1}{2i} \{p_1, p_2\} \quad \text{on} \quad \Sigma \quad .$$

From (5) Proposition 2.1 is trivial.

Q.E.D.

<u>Definition</u>: Let $(x_o, \xi_o) \in N = \{(x,\xi) \in T'(X) | \, p(x,\xi) = 0\}$. Then we say that P as in (I), satisfies the <u>local Levi condition</u> at (x_o, ξ_o) if $\forall \varphi \in C^\infty(X)$ solution of the equation $p_1(x, d_x\varphi) = 0$ (resp. $p_2(x, d_x\varphi) = 0$) in a neighborhood of x_o with $d_x\varphi(x_o) = \xi_o$ and $\forall f \in C_0^\infty(X)$ supported in a neighborhood of x_o where $d\varphi \neq 0$, we have

$$e^{it\varphi}P(fe^{it\varphi}) = O(t^{m_1+m_2-1}) \quad \text{as} \quad t \to \infty \tag{6}$$

in the sense that $\forall N \in \mathbb{N}$, $\exists\, C_{N,f,\varphi} > 0$ and $R \in \mathbb{R}^+$ s.t. $\forall\, t \geq R$

$$|e^{-it\varphi}P(e^{it\varphi})| \leq C_{N,f,\varphi}\, t^{-N} \qquad t \geq R$$

where $|\ |$, denotes the euclidean norm induced by a coordinate system in a neighborhood of x_o (taking that neighborhood sufficiently small), in the case that

$$p_1(x_o, \xi_o) = 0\,,\quad p_2(x_o,\xi_o) \neq 0 \quad (\text{resp.}\ \ p_1(x_o,\xi_o) \neq 0\,,\ \ p_2(x_o,\xi_o) = 0\ ).$$

If $(x_o, \xi_o) \in \Sigma$, let φ be a solution in a neighborhood of x_o, of $p_1(x, d_x\varphi) = p_2(x, d_x\varphi) = 0$ with $d_x\varphi(x_o) = \xi_o$. Let $f \in C_0^\infty(X)$ supported in a neighborhood of x_o where

$d\varphi \neq 0$ then P satisfies the Levi condition at (x_o, ξ_o) if

$$(7) \qquad e^{-it\varphi}P(fe^{it\varphi}) = O(t^{m_1+m_2-2}) \quad \text{as} \quad t \to \infty ,$$

in the same sense than before.

<u>Proposition 2.3</u>: Let P satisfy condition (I) and (i_1). Then (i_3) is equivalent to the local Levi condition.

<u>Proof</u>: (a) We have that in local coordinates

$$(8) \quad e^{-it\varphi}P(e^{it\varphi})(x) = t^{m_1+m_2}p(x, d_x\varphi)f(x) + t^{m_1+m_2-1}\{\frac{1}{i}\sum_{j=1} \frac{\partial p}{\partial \xi_j}(x, d_x\varphi)\frac{\partial f}{\partial x_j}(x) + C_p(x, d_x\varphi)f(x)\} + t^{m_1+m_2-2}h(t,x) ,$$

(cf. [D]). So if $p_1(x, d_x\varphi) = 0 = p_2(x, d_x\varphi) = 0$, $\varphi \in C^\infty(X)$, $d_x\varphi(x_o) = \xi_o$, $(x_o, \xi_o) \in \Sigma$, then clearly $p(x, d_x\varphi) = 0$, and $\frac{\partial p}{\partial \xi_j}(x, d_x\varphi) = 0$ in a neighborhood of x_o. If we suppose that $C_p = 0$ on Σ, then clearly $e^{-it\varphi}P(e^{it\varphi}) = O(t^{m_1+m_2-2})$. In the case that $p_1(x_o, \xi_o) = 0$, $p_2(x_o, \xi_o) \neq 0$, then clearly for the corresponding φ, $e^{-it\varphi}P(e^{it\varphi}) = O(t^{m_1+m_2-1})$ because $p(x,d_x\varphi) = p_1(x,d_x\varphi)\cdot p_2(x,d_x\varphi) = 0$. Same reasoning works for the case $p_1(x_o,\xi_o) \neq 0$, $p_2(x_o,\xi_o) = 0$.

(b) Suppose the Local Levi condition is satisfied at $(x_o,\xi_o) \in \Sigma$. We want to show $C_p(x_o,\xi_o) = 0$. Take $f \in C_o^\infty(X)$ one in a sufficiently small neighborhood of x_o, where $p_1(x_o,d_x\varphi(x_o)) = p_2(x_o,dx\varphi\,(x_o)) = 0$, $d_x\varphi(x_o) = \xi_o$.

Since $\sum_{j=1}^{n} \frac{\partial p}{\partial \xi_j}(x,d_x\varphi)\frac{\partial f}{\partial x_j}(x) = 0$ and (7),

we have that: $C_p(x,d_x\varphi)f(x) = 0$,

then $C_p(x_o,\xi_o) = 0$.

Q.E.D.

<u>Proposition 2.4</u>: Let $(x_o,\xi_o) \in \Sigma$. Then $\exists\, a_i$ $(i=1,2)$ C^∞ functions homogenous of degree $1-m_i$, s.t. $a_i(x_o,\xi_o) \neq 0$ and $\{r_1,r_2\} = 0$ in a conic neighborhood of (x_o,ξ_o) with $r_i = a_i p_i$, $i = 1,2$.

<u>Proof</u>: Let h_i be C^∞ functions homogenous of degree $1-m_i$, in $T'(X)$, $i = 1,2$,

$$\{h_1p_1,h_2p_2\} = h_1h_2\{p_1,p_2\} + p_2\{h_1p_1,h_2\} + p_1\{h_1,h_2p_2\}.$$

Then $\{h_1p_1,h_2p_2\} = 0$ on Σ. Let us consider the operator

$$(9) \quad H = H_1PH_2 = H_1P_1P_2H_2 + H_1QH_2$$

$$H = \widetilde{H}_1\widetilde{H}_2 + \widetilde{R}$$

with $\widetilde{H}_i = H_iP_i$, $i = 1,2$, $\widetilde{R} = H_1QH_2$.

We have that $C_H = 0$ at Σ since $\{\widetilde{h}_1,\widetilde{h}_2\} = 0$ on Σ and $\widetilde{r} = h_1qh_2$ on Σ. Then using Proposition 2.1 and that clearly H satisfies (I) we get $C_H = 0$ on Σ.

Since $\{h_1,h_2\}(x,\xi) = 0 \ \forall \ (x,\xi) \in \Sigma$, we have that

(10) $\{h_1,h_2\} = \lambda_1 h_1 + \lambda_2 h_2$ in a conic neighborhood of (x_o,ξ_o) , $\lambda_i C^\infty$ homogenous of degree 0 , $i = 1,2$.

Let's observe that

$$(11) \quad \{e^{f_1}h_1,e^{f_2}h_2\} = e^{f_1+f_2}\{h_1,h_2\} + e^{f_1+f_2}\{f_1,h_2\}h_1 + e^{f_1+f_2}\{h_1,f_2\}h_2 + e^{f_1+f_2}\{f_1,f_2\}h_1h_2 .$$

We first solve the equation

(12) $$H_{h_2} f_1 = \lambda_1 .$$

We know that there is a unique solution in a conic neighborhood of (x_o, ξ_o) with f_1 homogenous of degree 0, with initial data $f_1 = 0$ on a conic hypersurface transversal to H_{h_2} at (x_o, ξ_o). (cf. [D])

Having determined f_1, we solve for f_2.

(13) $\{e^{f_1} h_1, f_2\} + \lambda_2 e^{f_1} = 0$ in a conic neighborhood of (x_o, ξ_o), f_2 homogenous of degree 0 with initial data 0 in a conic hypersurface transversed to H_{h_1} at (x_o, ξ_o).

Let's take $a_i = e^{f_i} h_i$, $i = 1,2$. Then we have $\{a_1 p_1, a_2 p_2\} = 0$ in a conic neighborhood of (x_o, ξ_o) because by (10) and (11) we have

(14) $$\{e^{f_1} h_1, e^{f_2} h_2\} = e^{f_1+f_2}[\lambda_1 h_1 + \{f_1, h_2\} h_1] + e^{f_2} h_2 [\lambda_1 e^{f_1} + \{e^{f_1} h_1, f_2\}] .$$

Q.E.D.

<u>Remark</u> : Let's consider

(15) $$A_1 P A_2 = A_1 P_1 P_2 A_2 + A_1 Q A_2$$

$$A = A_1 P A_2 = \widetilde{A}_1 \widetilde{A}_2 + B$$

where $\qquad \widetilde{A}_i = A_i P_i \qquad i = 1,2$

$$B = A_i Q A_2 \quad .$$

We have that $\{\widetilde{a}_1, \widetilde{a}_2\} = 0$ in a neighborhood of $(x_o, \xi_o) \in \Sigma$. We have clearly that $H_{\widetilde{a}_1}(x_o, \xi_o)$, $H_{\widetilde{a}_2}(x_o, \xi_o)$, $V(x_o, \xi_o)$ are l.i., because $H_{\widetilde{a}_i}(x_o, \xi_o) = H_{p_i}(x_o, \xi_o)$, $i = 1,2$.

Also to find a local parametrix for $\widetilde{A}$ near (x_o, ξ_o) is clearly equivalent to finding one for P since the A_i are elliptic near (x_o, ξ_o) , $i = 1,2$.

Lemma 2.5: Let $p_1, \dots, p_k$ be real valued C^∞ functions in a conic neighborhood of $(x_o, \xi_o) \in T'(X)$ which are homogenous of degree 1. For the existence of a homogenous canonical transformation χ from a conical neighborhood U of (x_o, ξ_o) to a conical neighborhood V of $(z_o, \theta_o) \in T'(\mathbb{R}^n)$

$$\chi(x, \xi) = (x_1(x, \xi).,.x_n(x, \xi), \theta_1(x, \xi),., \theta_n(x, \xi)) \in T'(\mathbb{R}^n)$$

with $p_j(x, \xi) = \theta_j(x, \xi)$, $j = 1, \dots, k$ it's necessary and sufficient that:

i) $\{p_i, p_j\} = 0$ in a neighborhood of (x_o, ξ_o), $i,j = 1,\dots,k$.

ii) $H_{p_1}(x_o,\xi_o),\dots,H_{p_2}(x_o,\xi_o), V(x_o,\xi_o)$ are l.i.

<u>Proof</u>: See [D-H].

<u>Reduction</u>: Since $\{a_1, a_2\} = 0$ in a conic neighborhood of $(x_o, \xi_o) \in \Sigma$, and the Remark of page 19, we can apply Lemma 5. We choose for later convenience $(z_o, \theta_o) \in T'(\mathbb{R}^n)$ s.t. $z_o^1 \neq 0$ with $z_o = (z_o^1, \dots, z_o^n)$.

Let's choose (see [D-H]) $A \in I^o(X \times \mathbb{R}^n, \Gamma')$ s.t.

i) Γ is a closed conic subset of graph χ.

ii) $(x_o, \xi_o, z_o, \theta_o)$ is a non-characteristic point for A.

Let $B \in I\,(\mathbb{R}^n \times X, (\Gamma^{-1})')$ be s.t.

(15) $(x_o, \xi_o) \notin WF(AB - I_X)$ — I_X is the identity operator in X.

(16) $(z_o, \theta_o) \notin WF(BA - I_{\mathbb{R}^n})$ — $I_{\mathbb{R}^n}$ is the identity operator in $\mathbb{R}^n$.

Now let's consider

$$\widetilde{P} = BPA \ .$$

Proposition 2.6: $\exists\ R, A_1, A_2 \in L^o(\mathbb{R}^n)$ s.t.

(17) $\quad (z_o, \theta_o) \notin WF(\widetilde{P} - (D_1D_2 + A_1D_1 + A_2D_2 + R)) \quad .$

Proof: We know that the principal symbol of $\widetilde{P}$ is $\xi_1\xi_2$ in a conical neighborhood of $(z_o, \theta_o) \in T'(\mathbb{R}^n) - \{0\}$ (cf. [D-H]). We also know that $C_{\widetilde{p}}(z_o, \theta_o) = 0$, because the subprincipal symbol restricted to Σ is invariant under canonical transformations. So $\widetilde{P} = D_1D_2 + S$ with $S \in L^1(\mathbb{R}^n)$ in a conical neighborhood of $(z_o, \theta_o) \in T'(\mathbb{R}^n) - \{0\}$. The fact that $C_{\widetilde{p}}(z_o, \theta_o) = 0 \Rightarrow$ $S = a_1\xi_1 + a_2\xi_2$ in a conic neighborhood of (z_o, θ_o) with a_i C^∞ functions homogenous of degree 0, so taking $A_i \in L^o(\mathbb{R}^n)$ with principal symbol a_i , $i = 1,2$ $\widetilde{P} = D_1D_2 + A_1D_1 + A_2D_2 + R$ in a conical neighborhood of (z_o, θ_o) .

Q.E.D.

Proposition 2.7:

(18) $\quad (x_o, \xi_o, z_o, \theta_o) \notin WF'(PA - AM)$,

(19) $\quad (z_o, \theta_o, x_o, \xi_o) \notin WF'(BP - MB)$,

where $M = D_1D_2 + A_1D_1 + A_2D_2 + R$ with A_1, A_2, R as in Proposition 2.6.

Proof: $PA - A\tilde{P} = (I - AB)PA$.

Since (15), $(x_o, \xi_o, z_o, \theta_o) \notin WF'(PA - A\tilde{P})$ then by (17) we get (18)

$$BP - \tilde{P}B = BP(Id - AB) \quad .$$

So in the same way we get (19).

Q.E.D.

Remarks: (a) Using (18) and (19) and the construction of a parametrix for M we will show in I.7 how to get a parametrix for P .

(b) The observation about the equivalence of the local Levi condition with condition (i_3) if P satisfies (I) and (i_1) will be discussed further in Chapter II .

3. Reduction of simpler case to a system.

We are going to denote the space of vector valued functions or operators with the same notation as in the scalar case.

<u>Proposition 3.1</u>: There exists an elliptic operator $E \in L^o(\mathbb{R}^n)$ with values in 2×2 matrices s.t.

$$(1) \qquad \begin{pmatrix} D_1 & A_1 D_1 \\ 0 & D \end{pmatrix} = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix} E \mod L^o(\mathbb{R}^n) \ .$$

<u>Proof</u>: We have that:

$$\begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}\begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{12} \end{pmatrix} = \begin{pmatrix} D_1 E_{11} & D_1 E_{12} \\ D_2 E_{21} & D_2 E_{22} \end{pmatrix} \ .$$

So taking $E = \begin{pmatrix} Id & A_1 \\ 0 & Id \end{pmatrix} \in L^o(\mathbb{R}^n)$. We get (1) since $A_1 D_1 = D_1 A_1 \mod L^o(\mathbb{R}^n)$. E is clearly elliptic.

Q.E.D.

So we have for the operator:

$$(2) \qquad \tilde{L} = \begin{pmatrix} D_1 & A_1 D_1 \\ 0 & D_2 \end{pmatrix} + \begin{pmatrix} A_2 & R \\ -Id & 0 \end{pmatrix} = \begin{pmatrix} D_1 & A_1 D_1 \\ 0 & D_2 \end{pmatrix} + \tilde{A}$$

that:

$$(3) \qquad \tilde{L} = \left[\begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix} + A\right] E \mod L^{-\infty}(\mathbb{R}^n) \ , \ A \in L^o(\mathbb{R}^n)$$

with $A = \tilde{A}E'$ and $E' \in L^o(\mathbb{R}^n)$ a parametrix for E.

Let's consider

$$(4) \qquad K = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix} + A \qquad A \in L^0(\mathbb{R}^n) \quad .$$

Let's make the change of variables:

$$(5) \qquad \begin{cases} t = x_1 \\ y_1 = x_1 - x_2 \\ y_j = x_{j+1} & j = 2,\dots,n-1 \quad . \end{cases}$$

We are going to denote the new variables by $(t,y) \in \mathbb{R}^n$ and the corresponding dual variables in the cotengent space by (r,ξ) . So in this new variables, K looks like:

$$(6) \qquad \tilde{K} = \begin{pmatrix} D_t & 0 \\ 0 & D_t - D_{y_1} \end{pmatrix} + \tilde{A} \qquad \tilde{A} \in L^0(\mathbb{R}^n) \quad .$$

<u>Proposition 3.2</u>: There exists $C \in L^0(\mathbb{R}^n)$ elliptic, $A(t,y,D_y) \in L^0(\mathbb{R}^{n-1})$ smooth in t s.t.

$$(7) \quad \begin{pmatrix} D_t & 0 \\ 0 & D_t - D_{y_1} \end{pmatrix} + \tilde{A} = \left(\begin{pmatrix} D_t & 0 \\ 0 & D_t - D_{y_1} \end{pmatrix} + A(t,y,D_y) \right) C \qquad \text{mod} \quad L^{-\infty}(\mathbb{R}^n) \ .$$

<u>Proof</u>: We write:

$$C \approx \sum_{k=0}^{\infty} C_{-k} \qquad A \approx \sum_{k=0}^{\infty} A_{-k} \qquad \begin{matrix} A_{-j} \in L^{-j}(\mathbb{R}^n) \\ C_{-j} \in L^{-j}(\mathbb{R}^n) \end{matrix} \qquad j \in \mathbb{N} \cup \{0\}$$

in the sense that $C - \sum_{k=0}^{V} C_{-k} \in L^{-(V+1)}(\mathbb{R}^n)$

$$A - \sum_{k=0}^{V} A_{-k} \in L^{-(V+1)}(\mathbb{R}^n) \quad .$$

First we take $C_o = \mathrm{Id}$. Then C is elliptic. Let us suppose we have chosen A_{-j} , C_{-j-1} for $j \leq k-1$ s.t.

$$(8) \qquad K - \left(\begin{pmatrix} D_t & 0 \\ 0 & D_t - D_{y_1} \end{pmatrix} + A_o + \dots + A_{-k+1}\right)\left(C_o + \dots + C_{-k}\right) \in L^{-k}(\mathbb{R}^n) \ ,$$

then we must find A_{-k} , C_{-k-1} , s.t.

$$(9) \qquad K - \left(\begin{pmatrix} D_t & 0 \\ 0 & D_t - D_{y_1} \end{pmatrix} + A_o + A_{-k+1}\right)\left(C_o + \dots + C_{-k}\right) = A_{-k} + \begin{pmatrix} D_t & 0 \\ 0 & D_t - D_{y_1} \end{pmatrix} C_{-k-1} \mod L^{-k-1} \quad .$$

Calling the principal symbol of left hand side of (o) $h_{-k} \in S^{-k}(\mathbb{R}^n)$ (because of (8)), we know:

$$(10) \quad h_{-k}(t,y,r, \ ) = \begin{pmatrix} h^{11}_{-k}(t,y,0,\xi) & h^{12}_{-k}(t,y,0,\xi) \\ h^{21}_{-k}(t,y,\xi_1,\xi) & h^{22}_{-k}(t,y,\xi_1,\xi) \end{pmatrix} + \begin{pmatrix} r\int_0^1 \frac{\partial}{\partial r} h^{11}_{-k}(t,y,sr,\xi)ds & r\int_0^1 \frac{\partial}{\partial r} h^{12}_{-k}(t,y,sr,\xi)ds \\ (r-\xi_1)\int_0^1 \left(\frac{\partial}{\partial r}\right) h^{21}_{-k}(t,y,s(r-\xi_1),\xi)ds & (r-\xi_1)\int_0^1 \left(\frac{\partial}{\partial r}\right) h^{22}_{-k}(t,y,s(r-\xi_1),\xi)ds \end{pmatrix} .$$

Since $C_{-k} \in S^{-k}(\mathbb{R}^n)$, it is clear that:

$$a^{ij}_{-k} \in S^{-k}(\mathbb{R}^n \times \mathbb{R}^{n-1}) \qquad i,j = 1,2$$
$$C^{ij}_{-k-1} \in S^{-k-1}(\mathbb{R}^n \times \mathbb{R}^n) \qquad i,j = 1,2 \quad .$$

From (10) we get immediately (9) considering

$$\begin{pmatrix} r & 0 \\ 0 & r-\xi_1 \end{pmatrix}\begin{pmatrix} C^{11}_{-k-1} & C^{12}_{-k-1} \\ C^{21}_{-k-1} & C^{22}_{-k-1} \end{pmatrix} = \begin{pmatrix} rC^{11}_{-k-1} & rC^{12}_{-k-1} \\ (r-\xi_1)C^{21}_{-k-1} & (r-\xi_1)C^{22}_{-k-1} \end{pmatrix} .$$

Q.E.D.

4. Construction of fundamental solution for the Cauchy problem for L .

For a fundamental solution of the Cauchy problem for L, we mean an operator $E: C^\infty_o(\mathbb{R}^{n-1}) \to C^\infty(\mathbb{R}^n)$ s.t.

$$(1) \qquad \begin{cases} LE = R \\ \\ \gamma_o E = Id + R' \end{cases}$$

with $R: C^\infty_o(\mathbb{R}^{n-1}) \rightarrowtail C^\infty(\mathbb{R}^n)$ an operator with C^∞ kernel in $\mathbb{R}^{n-1} \times \mathbb{R}^n$, i.e. $Rf(t,y) = \int r(t,y,y')f(y')dy'$, $f \in C^\infty_o(\mathbb{R}^{n-1})$, $r \in C^\infty(\mathbb{R}^n \times \mathbb{R}^{n-1})$

$R' \in L^{-\infty}(\mathbb{R}^{n-1})$ and

$\gamma_o : C_o^\infty(\mathbb{R}^n) \longrightarrow C^\infty(\mathbb{R}^{n-1})$ defined by

$\gamma_o f(y) = f(0,y) \qquad f \in C_o^\infty(\mathbb{R}^n)$.

<u>Remark</u>: A natural idea to consider, for constructing our E , would be

$$E = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} \text{, with}$$

$$E_1 f(t,y) = \int e^{i\langle y,\theta\rangle} e_1(t,y,\theta)\hat{f}(\theta)d\theta$$

for $f \in C_o^\infty(\mathbb{R}^{n-1})$, $e_1 \in S^m(\mathbb{R}^n \times \mathbb{R}^{n-1})$ for some m

$$E_2 f(t,y) = \int e^{i\{(t+y_1,\theta_1)+y_2\theta_2+\dots+y_{n-1}\theta_{n-1}\}} e_2(t,y,\theta)\hat{f}(\theta)d\theta$$

for $f \in C_o^\infty(\mathbb{R}^{n-1})$, $e_2 \in S^{m'}(\mathbb{R}^n \times \mathbb{R}^{n-1})$ for some $m' \in \mathbb{R}$ because this is the form that the fundamental solution for the Cauchy problem for $\begin{pmatrix} D_t & 0 \\ 0 & D_t - D_{y_1} \end{pmatrix}$ has (see Chapter II.1). We have that

$$\left[\begin{pmatrix} D_t & 0 \\ 0 & D_t - D_{y_1} \end{pmatrix} + A\right]\begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} = \begin{pmatrix} D_t E_1 & 0 \\ 0 & (D_t - D_{y_1})E_2 \end{pmatrix} + \begin{pmatrix} A_{11}E_1 & A_{12}E_2 \\ A_{21}E_1 & A_{22}E_2 \end{pmatrix}.$$

So for being able to prove (1) we would need $A_{12}E_2 = 0$, $A_{21}E_1 = 0$, mod $C^\infty(\mathbb{R}^{n-1}\times\mathbb{R}^n)$ and this is in general impossible.

The second idea is to try $E = E_1 + E_2$, E_i matrix valued operators $i = 1,2$

$$E_1 f(t,y) = \int e^{i\langle y,\theta\rangle} e_1(t,y,\theta)\hat{f}(\theta)d\theta \qquad f \in C_0^\infty(\mathbb{R}^{n-1})$$

with $e_1 \in S^m(\mathbb{R}^n \times \mathbb{R}^{n-1})$ for some $m \in \mathbb{R}$.

$$E_2 f(t,y) = \int e^{i\langle t+y,\theta\rangle} e_2(t,y,\theta)\hat{f}(\theta)d\theta$$

with $e_2 \in S^{m'}(\mathbb{R}^n \times \mathbb{R}^{n-1})$ for some $m' \in \mathbb{R}$, and $\langle t+y,\theta\rangle = (t+y_1)\theta_1 + y_2\theta_2 + \ldots + y_{n-1}\theta_{n-1}$. Then we have that

$$\left[\begin{pmatrix} D_t & 0 \\ 0 & D_t - D_{y_1}\end{pmatrix} + A\right](E_1+E_2) = \int e^{i\langle y,\theta\rangle}\begin{pmatrix} D_t & 0 \\ 0 & D_t - D_{y_1}\end{pmatrix} e_1\hat{f}(\theta)d\theta$$

$$+ \int e^{i\langle y,\theta\rangle}\begin{pmatrix} 0 & 0 \\ 0 & -\theta_1\end{pmatrix} e_1\hat{f}(\theta)d\theta$$

$$+ \int e^{i\langle y,\theta\rangle} e^{-i\langle y,\theta\rangle} A(e^{i\langle y,\theta\rangle} e_1)\hat{f}(\theta)d\theta$$

$$+ \int e^{i\langle t+y,\theta\rangle}\begin{pmatrix} D_t & 0 \\ 0 & D_t - D_{y_1}\end{pmatrix} e_2(t,y,\theta)\hat{f}(\theta)d\theta$$

$$+ \int e^{i\langle t+y,\theta\rangle}\begin{pmatrix} \theta_1 & 0 \\ 0 & 0\end{pmatrix} e_2(t,y,\theta)\hat{f}(\theta)d\theta$$

$$+ \int e^{i<t+y,\theta>} e^{-i<t+y,\theta>} A(e^{i<t+y,\theta)} e_2) \hat{f}(\theta) d\theta \ .$$

For getting (1) we should have then either

$$\begin{pmatrix} \theta_1 & 0 \\ 0 & 0 \end{pmatrix} e_2 = 0 \quad \text{or} \quad \begin{pmatrix} 0 & 0 \\ 0 & -\theta_1 \end{pmatrix} e_1 = 0 \quad \text{mod } S^{m'} \text{ or } S^m \ .$$

Then going to the following step we should have

$$a_{12}(t,y,\theta) e_2^{21} = 0 \qquad a_{12}(t,y,\theta) e_2^{22} = 0 \qquad \text{mod } S^{m'-1} \ .$$

$$\text{or} \quad a_{21}(t,y,\theta) e_1^{11} = 0 \qquad a_{21}(t,y,\theta) e_1^{12} = 0 \qquad \text{mod } S^{m-1} \ .$$

This is because

$$e^{-i<y,\theta>} A(e^{i<y,\theta>} e_1) = a(t,y,\theta) e_1 + h$$

$$e^{-i<t+y,\theta>} A(e^{i<t+y,\theta>} e_2) = a(t,y,\theta) e_2 + h'$$

$$h \in S^{m-1}(\mathbb{R}^n \times \mathbb{R}^{n-1}) \qquad h' \in S^{m'-1}(\mathbb{R}^n \times \mathbb{R}^{n-1})$$

(see [D]). If $e_2^{1j} = 0$, $j = 1,2$, there is no contribution in the second row from the term $\begin{pmatrix} D_t & 0 \\ 0 & D_t - D_{y_1} \end{pmatrix} e_1$. So we have again restrictions on A , that are not satisfied in general.

The idea of our construction is to try to annihilate the terms of the form $\begin{pmatrix}\theta_1 & 0\\ 0 & 0\end{pmatrix}e_2$ and $\begin{pmatrix}0 & 0\\ 0 & -\theta_1\end{pmatrix}e_1$ that cause the trouble and don't disturb the initial data, this is accomplished in the following way. We put

(2) $E = E_1 + E_2 + E_3 \qquad E_i\colon C_o^\infty(\mathbb{R}^{n-1}) \longrightarrow C^\infty(\mathbb{R}^n) \quad i = 1,2,3$

(3) $E_1 f(t,y) = \int e^{i<y,\theta>} e_1(t,y,\theta)\hat{f}(\theta)d\theta$

$$f \in C_o^\infty(\mathbb{R}^{n-1})$$

(4) $E_2 f(t,y) = \int e^{i<t+y,\theta>} e_2(t,y,\theta)\hat{f}(\theta)d\theta$

(5) $E_3 f(t,y) = \int_{-t}^{t}\int e^{i<\frac{\tau+t}{2}+y,\theta>} e_3(\tau,t,y,\theta)\hat{f}(\theta)d\theta d\tau$

where $<\frac{\tau+t}{2}+y,\theta> = (\frac{\tau+t}{2}+y_1)\theta_1 + \dots + y_{n-1}\theta_{n-1}$. Let's observe that:

(6) $$\begin{cases} <\frac{\tau+t}{2}+y,\theta> = <y,\theta> & \text{when} \quad \tau = -t \\[2ex] <\frac{\tau+t}{2}+y,\theta> = <t+y,\theta> & \text{when} \quad \tau = t \end{cases}$$

and

$$(7)\quad \begin{cases} \begin{pmatrix} D_t & 0 \\ 0 & D_t \end{pmatrix} e^{i<y,\theta>} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} D_t - D_{y_1} & 0 \\ 0 & D_t - D_{y_1} \end{pmatrix} e^{i<t+y,\theta>} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} D_t - D_\tau & 0 \\ 0 & D_\tau + D_t - D_{y_1} \end{pmatrix} e^{i<\frac{\tau+t}{2}+y,\theta>} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{cases}$$

and

$$(8)\quad \begin{cases} D_t(<\frac{\tau+t}{2}+y,\theta>) = D_\tau(<\frac{\tau+t}{2}+y,\theta>) \\ (D_t - D_{y_1})(<\frac{\tau+t}{2}+y,\theta>) = -D_\tau(<\frac{\tau+t}{2}+y,\theta>) \end{cases} .$$

(6), (7) and (8) play a fundamental role in our construction.

<u>Construction of</u> e_1, e_2 <u>and</u> e_3 . Take $f \in C_o^\infty(\mathbb{R}^{n-1})$. We have

$$(9)\quad LEf(t,y) = \int e^{i<y,\theta>} e^{-i<y,\theta>} \left[\begin{pmatrix} D_t & 0 \\ 0 & D_t \end{pmatrix} + A \right] (e^{i<y,\theta>} e_1) \hat{f}(\theta) d\theta$$

$$+ \int e^{i<y,\theta>} e^{-i<y,\theta>} \left[\begin{pmatrix} 0 & 0 \\ 0 & -D_{y_1} \end{pmatrix} \right] (e^{i<y,\theta>} e_1) \hat{f}(\theta) d\theta$$

$$+\int e^{i<y,\theta>}e_3(-t,t,y,\theta)\hat{f}(\theta)d\theta$$

$$+\int e^{i<t+y,\theta>}e^{-i<t+y,\theta>}\left[\begin{pmatrix} D_t-D_{y_1} & 0 \\ 0 & D_t-D_{y_1}\end{pmatrix}+A\right](e^{i<t+y,\theta>}e_2)\hat{f}(\theta)d\theta$$

$$+\int e^{i<t+y,\theta>}e^{-i<t+y,\theta>}\left[\begin{pmatrix} D_{y_1} & 0 \\ 0 & 0\end{pmatrix}\right]\left(e^{i<t+y,\theta>}e_2\right)\hat{f}(\theta)d\theta$$

$$+\int e^{i<t+y,\theta>}e_3(t,t,y,\theta)\hat{f}(\theta)d\theta$$

$$+\int_{-t}^{t}\int e^{i\varphi_3(\tau,t,y,\theta)}e^{-i\varphi_3(\tau,t,y,\theta)}\left[\begin{pmatrix} D_t & 0 \\ 0 & D_t-D_{y_1}\end{pmatrix}+A\right](e^{i\varphi_3}e_3)\hat{f}(\theta)d\theta d\tau$$

where $\varphi_3(\tau,t,y,\theta) = <\frac{\tau+t}{2}+y,\theta>$. We have used here that A doesn't contain D_t derivatives and (6). Using (7) and (8), integrating by parts in the last integral of (9), we get:

$$(10)\quad LEf(t,y) = \int e^{i<y,\theta>}\begin{pmatrix} D_t & 0 \\ 0 & D_t\end{pmatrix}e_1\hat{f}(\theta)d\theta$$

$$+\int e^{i<y,\theta>}e^{-i<y,\theta>}A(e^{i<y,\theta>}e_1)\hat{f}(\theta)d\theta$$

$$+\int e^{i<y,\theta>}e^{-i<y,\theta>}\begin{pmatrix} 0 & 0 \\ 0 & -D_{y_1}\end{pmatrix}(e^{i<y,\theta>}e_1)\hat{f}(\theta)d\theta$$

$$+\int e^{i<y,\theta>}e_3(-t,t,y,\theta)\hat{f}(\theta)d\theta$$

$$+ \int e^{i\langle y,\theta\rangle} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} e_3(-t,t,y,\theta)\hat{f}(\theta)d\theta$$

$$+ \int e^{i\langle t+y,\theta\rangle} \begin{pmatrix} D_t - D_{y_1} & 0 \\ 0 & D_t - D_{y_1} \end{pmatrix} e_2(t,y,\theta)\hat{f}(\theta)d\theta$$

$$+ \int e^{i\langle t+y,\theta\rangle} e^{-i\langle t+y,\theta\rangle} A(e^{i\langle t+y,\theta\rangle} e_2)\hat{f}(\theta)d\theta$$

$$+ \int e^{i\langle t+y,\theta\rangle} e^{-i\langle t+y,\theta\rangle} \begin{pmatrix} D_{y_1} & 0 \\ 0 & 0 \end{pmatrix} (e^{i\langle t+y,\theta\rangle} e_2)\hat{f}(\theta)d\theta$$

$$+ \int e^{i\langle t+y,\theta\rangle} e_3(t,t,y,\theta)\hat{f}(\theta)d\theta$$

$$+ \int e^{i\langle t+y,\theta\rangle} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} e_3(t,t,y,\theta)\hat{f}(\theta)d\theta$$

$$+ \int_{-t}^{t} \int e^{i\varphi_3} \begin{pmatrix} D_t - D_\tau & 0 \\ 0 & D_t - D_{y_1} + D_\tau \end{pmatrix} e_3(\tau,t,y,\theta)\hat{f}(\theta)d\theta d\tau$$

$$+ \int_{-t}^{t} \int e^{i\varphi_3} e^{-i\varphi_3} A(e^{i\varphi_3} e_3)\hat{f}(\theta)d\theta d\tau \ .$$

From (10) we are going to deduce the transport equations for e_1 , e_2 , and e_3 :

First a remark:

<u>Remark</u>: It is enough for constructing E , to construct E satisfying

$$\text{(11)} \qquad \begin{aligned} &PE \in C^\infty(\mathbb{R}^{n-1} \times \mathbb{R}^n) \\ &\gamma_o E = S \mod L^{-\infty}(\mathbb{R}^{n-1}) \end{aligned}$$

where $S \in L^{o}(\mathbb{R}^{n-1})$, $WFS \subset V$, V a sufficiently small Conic neighborhood of $T'(\mathbb{R}^{n-1})$ (we are going to construct $Ef(t,y)$ for t in a given finite interval of time), because we can take pseudodifferential partitions of the unity.

(a) We will put

$$e_i \approx \sum_{j=0}^{\infty} e_i^{-j} \qquad e_i^{-j} \text{ homogenous of degree } -j \text{ in } T^*X \text{ , } i = 1,2$$

in the sense that

$$e_i - \sum_{j=0}^{V} e_i^{-j} \in S^{-(V+1)}(\mathbb{R}^n \times \mathbb{R}^{n-1}) \quad .$$

Choose e_1^o s.t.

$$(12) \quad \begin{pmatrix} D_t & 0 \\ 0 & D_t \end{pmatrix} e_1^o(t,y,\theta) + a(t,y,\theta) e_1^o(t,y,\theta) = 0$$

$$t \in [0,t_o] \qquad (y,\theta) \in V \quad .$$

Choose e_2^o s.t.

$$(13) \quad \begin{pmatrix} D_t - D_{y_1} & 0 \\ 0 & D_t - D_{y_1} \end{pmatrix} e_2^o(t,y,\theta) + a(t,y,\theta) e_2^o(t,y,\theta) = 0$$

$$t \in [0,t_o] \qquad (y,\theta) \in V \quad .$$

e_1^0 , e_2^0 have to satisfy the initial condition at $t = 0$:

$$(14) \qquad e_1^0(0,y,\theta) + e_2^0(0,y,\theta) = s(y,\theta) \qquad (y,\theta) \in V \ .$$

We extend e_1^0 , e_2^0 to be in $S^{-\infty}([0,t_0] \times \complement\overline{X})$, $\overline{X} \subset V$, X open.

For constructing e_1^{-j} , we solve the equation

$$(15) \quad \begin{pmatrix} D_t & 0 \\ 0 & D_t \end{pmatrix} e_1^{-j}(t,y,\theta) + a(t,y,\theta)e_1^{-j}(t,y,\theta) = \tilde{h}_{-j}(t,y,\theta)$$

with $\tilde{h}_{-j}$ homogenous of degree $-j$ in θ . For e_2^{-j} , we solve:

$$(16) \quad \begin{pmatrix} D_t - D_{y_1} & 0 \\ 0 & D_t - D_{y_1} \end{pmatrix} e_2^{-j}(t,y,\theta) + a(t,y,\theta)e_2^{-j}(t,y,\theta) = \tilde{k}_{-j}(t,y,\theta)$$

with $\tilde{k}_{-j}$ homogenous of degree $-j$ in θ . e_1^{-j} , e_2^{-j} submitted to the initial condition at $t = 0$:

$$(17) \quad e_1^{-j}(0,y,\theta) + e_2^{-j}(0,y,\theta) = s_{-j}(y,\theta), (y,\theta) \in M \subset V$$

$$\text{if} \quad s \approx \sum_{j=0}^{\infty} s_{-j} \ , \quad s_{-j} \ \text{homogenous of degree} \ -j \ .$$

Again e_1^{-j} , e_2^{-j} are extended to be $S^{-\infty}$ out of $H \subset V$, H closed and for $t \in [0,t_o]$. $\tilde{h}_{-j}$, $\tilde{k}_{-j}$ are chosen, so that

$$(18)\quad \left(\begin{pmatrix} D_t & 0 \\ 0 & D_t \end{pmatrix} + A\right)E_1 \in C^\infty(\mathbb{R}^{n-1} \times \mathbb{R}^n)$$

$$\left(\begin{pmatrix} D_t - D_{y_1} & 0 \\ 0 & D_t - D_{y_1} \end{pmatrix} + A\right)E_2 \in C^\infty(\mathbb{R}^n \times \mathbb{R}^{n-1}) \quad .$$

(b) e_3: Let $e_3^1(\tau,t,y,\theta)$ be homogenous of degree 1 s.t.

$$(19)\quad \begin{pmatrix} D_t - D_\tau & 0 \\ 0 & D_t - D_{y_1} + D_\tau \end{pmatrix} e_3^1(\tau,t,y,\theta) + a(t,y,\theta)e_3^1(t,y,\theta) = 0$$

for $t \in [0,t_o]$, $-t < \tau < t$, $(y,\theta) \in V$. Putting $e_3^1 = \begin{pmatrix} {}^{11}e_3^1 & {}^{12}e_3^1 \\ {}^{21}e_3^1 & {}^{22}e_3^1 \end{pmatrix}$, we require (20) and (21) when $\tau = t$ and $\tau = -t$

$$(20)\quad \begin{pmatrix} {}^{11}e_3^1(t,t,y,\theta) & {}^{12}e_3^1(t,t,y,\theta) \\ 0 & 0 \end{pmatrix} = w_1(t,y,\theta)$$

$$t \text{ near } 0 \; ; \; (y,\theta) \text{ near } V$$

where

$$w(t,y,\theta) = e^{-i<t+y,\theta>}\begin{pmatrix} D_{y_1} & 0 \\ 0 & 0 \end{pmatrix}(e^{i<t+y,\theta>}e_1) \approx w_1 + \sum_{j=0}^{\infty} w_{-j}$$

w_1 homogenous of degree 1 in θ , w_{-j} homogenous of degree $-j$ and

$$(21)\quad \begin{pmatrix} 0 & 0 \\ {}^{21}e_3^1(-t,t,y,\theta) & {}^{22}e_3(t,t,y,\theta) \end{pmatrix} = v_1(t,y,\theta)$$

where $$v(t,y,\theta) = e^{-i\langle y,\theta\rangle}\begin{pmatrix} 0 & 0 \\ 0 & -D_{y_1} \end{pmatrix}(e^{i\langle y,\theta\rangle}e_1)$$

$$\approx v_1(t,y,\theta) + \sum_{j=0}^{\infty} v_{-j}(t,y,\theta)$$

v_1 homogenous of degree 1 in θ , v_{-j} homogenous of degree $-j$ in θ .

We extend e_3^1 to be in $S^{-\infty}$ out of V , $t \in [0,t_o]$. (The meaning of this is the same as in the extension of e_1^o , e_2^o .) For the j-th step, we solve

$$(22)\quad \begin{pmatrix} D_t-D_\tau & 0 \\ 0 & D_t-D_{y_1}+D_\tau \end{pmatrix} e_3^{-j}(\tau,t,y,\theta) + a(t,y,\theta)e_3^{-j}(t,y,\theta)$$

$$= \tilde{j}_j(\tau,t,y,\theta) \qquad t \in [0,t_o]\ -t < \tau < t$$
$$(y,\theta) \in W \subset V$$

$$(23)\quad \begin{pmatrix} {}^{11}e_3^{-j}(t,t,y,\theta) & {}^{12}e_3^{-j}(t,t,y,\theta) \\ 0 & 0 \end{pmatrix} = w_{-j}(t,y,\theta)$$
$$t \in [0,t_o]$$
$$(y,\theta) \in W$$

and

$$(24)\quad \begin{pmatrix} 0 & 0 \\ {}^{21}e_3^{-j}(-t,t,y,\theta) & {}^{22}e_3^{-j}(t,t,y,\theta) \end{pmatrix} = v_{-j}(t,y,\theta)$$
$$t \in [0,t_o]$$
$$(y,\theta) \in W$$

$\tilde{j}_{-j}(\tau,t,y,\theta)$ is homogenous of degree $-j$ in θ. $\tilde{j}_{-j}$ is chosen so that

$$(25)\quad \left\{\begin{pmatrix} D_t-D_\tau & 0 \\ 0 & D_t-D_{y_1}+D_\tau \end{pmatrix} + A \quad E_3'\right\} \in C^\infty(\mathbb{R}^{n-1}\times\mathbb{R}^{n+1})$$

where $E_3' : C_0^\infty(\mathbb{R}^{n-1}) \longrightarrow C^\infty(\mathbb{R}^{n+1})$ defined by

$$E_3' f(t,y,\tau) = \int e^{i<\frac{\tau+t}{2}+y,\theta>} e_3(\tau,t,y,\theta)\hat{f}(\theta)d\theta \quad .$$

From (18), (25) considering (19) and (20), (21) we get: $PE \in C^\infty(\mathbb{R}^{n-1}\times\mathbb{R}^n)$.

From (17) we get:

$$\gamma_0 E = S \quad .$$

So we get (1) using the remark (11).

5. Construction of a parametrix for L .

In (4.) we got an operator $E: C_0^\infty(\mathbb{R}^{n-1}) \longrightarrow C^\infty(\mathbb{R}^n)$

$$LE = R \in C^\infty(\mathbb{R}^{n-1}\times\mathbb{R}^n)$$

$$\gamma_0 E = Id + R' \qquad\qquad R' \in L^{-\infty}(\mathbb{R}^{n-1}) \quad .$$

Let $R'': C_0^\infty(\mathbb{R}^{n-1}) \longrightarrow C^\infty(\mathbb{R}^n)$ be defined by

(1) $$R''f(t,y) = R'f(0,y) \quad .$$

Clearly $R'' \in C^\infty(\mathbb{R}^{n-1} \times \mathbb{R}^n)$. We have also from (1) and 4.(1)

(2) $$P(E - R'') \in C^\infty(\mathbb{R}^{n-1} \times \mathbb{R}^n)$$

$$\gamma_0(E - R'') = Id \quad .$$

Notation: We will denote $E - R''$ by E .
Doing the same construction that in 4 we can construct a one parameter family of operators $(E_s)_{s\in\mathbb{R}}$

$$E_s: C_0^\infty(\mathbb{R}^{n-1}) \longrightarrow C^\infty(\mathbb{R}^n)$$

depending smoothly on s , such that:

(3) $$LE \in C^\infty(\mathbb{R}^{n-1} \times \mathbb{R}^n)$$

$$\gamma_s E_s = Id$$

where $\gamma_s: C^\infty(\mathbb{R}^n) \longrightarrow C^\infty(\mathbb{R}^{n-1})$, $\gamma_s f(y) = f(s,y)$.
Let's consider $\widetilde{E}: C_0^\infty(\mathbb{R}^n) \longrightarrow C^\infty(\mathbb{R}^n)$ defined by

(4) $$\tilde{E}f(t,y) = \int_0^t (E_s\gamma_s f)(t,y)ds \quad .$$

We clearly have that

(5) $$L\tilde{E}f(t,y) = f(t,y) + \tilde{R}f(t,y)$$

from (3) and the form of L with

$$\tilde{R}: C_o^\infty(\mathbb{R}^n) \longrightarrow C^\infty(\mathbb{R}^n) \quad \text{of the form}$$

$$\tilde{R}f(t,y) = \int_0^t\int \tilde{r}(t,y,s,\bar{y})f(s,\bar{y})d\bar{y}\,ds$$

with $\tilde{r} \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$.

Definition 5.1: Let $N: C_o^\infty(\mathbb{R}^n) \longrightarrow C^\infty(\mathbb{R}^n)$. We will say $N \in N^{-\infty}$ if

(6) $$Nf(t,y) = \int_0^t\int n(t,y,s,\bar{y})f(s,\bar{y})d\bar{y}ds$$

with $n \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$.

So we get from (5) and definition 5.1 an operator $\tilde{E}$

$$\tilde{E}: C_o^\infty(\mathbb{R}^n) \longrightarrow C^\infty(\mathbb{R}^n) \quad \text{s.t.}$$

(7) $$L\tilde{E} = Id + \tilde{R} \qquad \tilde{R} \in N^{-\infty} \quad .$$

Proposition 5.2:

$$WFN \subseteq \{((t,y,r,\xi);(\bar{t},\bar{y},\bar{r},\bar{\xi})) \in T'(\mathbb{R}^n \times \mathbb{R}^n) \mid \xi = r = \bar{\xi} = 0\}$$

if $N \in N^{-\infty}$.

Proof: Let us denote by K_N the Schwartz kernel of N . Then we have

$$K_N(\varphi \otimes \psi)(e^{-i<\cdot;\alpha(r,\xi)>}e^{-i<\cdot;\alpha(\bar{r},\bar{\xi})>}) =$$

$$\iint_0^t \iint n(t,y,\bar{t},\bar{y})e^{-i<(t,y);\alpha(r,\xi)>}e^{-i<(\bar{t},\bar{y});\alpha(\bar{r},\bar{\xi})>} \varphi(t,y)\psi(\bar{t},\bar{y})\,dy\,dt\,d\bar{t}\,d\bar{y} \ .$$

Take $((t_o,y_o,r_o,\xi_o);(\bar{t}_o,\bar{y}_o,\bar{r}_o,\bar{\xi}_o)) \in T'(\mathbb{R}^n \times \mathbb{R}^n)$ s.t. $r_o \neq 0$. Then for r sufficiently near r_o , so that $|r| \geq c > 0$. We have

$$D_t e^{-i<(t,y);\alpha(r,\xi)>} = -(\alpha r)e^{-i<(t,y);\alpha(r,\xi)>} \ .$$

So applying integration by parts a sufficiently large number of times with respect to the variable t , we get $\forall\, M \in \mathbb{N}$, $\exists\, C_{M,\varphi,\psi}$ s.t.

$$(8) \quad |K_N(\varphi \otimes \psi)(e^{-i<\cdot;\alpha(r,\xi)>}e^{-i<\cdot;(\alpha(r,\xi)>})| \leq C_{M,\varphi,\psi}t^{-M} \quad t \geq 1 \ .$$

The same argument changing D_t for D_{y_i} or $D_{\bar{y}_i}$ proves (8) if $\xi_o^i \neq 0$ or $\bar{\xi}_o^i \neq 0$.

Q.E.D.

Proposition 5.3: Let $N \in N^{-\infty}$, then

$$WFN^t \subseteq \{((t,y,r,\xi);(\bar{t},\bar{y},\bar{r},\bar{\xi})) \in T'(\mathbb{R}^n \times \mathbb{R}^n) \mid r = \bar{\xi} = \xi = 0\} .$$

Proof: Same argument as in Proposition 5.2.

Q.E.D.

So for making the same statement about the right and left parametrices for L , we introduce the class:

Definition 5.4: We will say that $H \in H^{-\infty}$ if $H\colon C_o^\infty(\mathbb{R}^n) \longrightarrow C^\infty(\mathbb{R}^n)$ s.t.

$$WFH \subseteq \{((t,y,r,\xi);(\bar{t},\bar{y},\bar{r},\bar{\xi})) \in T'(\mathbb{R}^n \times \mathbb{R}^n) \mid r = \xi = 0\} \quad \text{or}$$

$$WFH \subseteq \{((t,y,r,\xi);(\bar{t},\bar{y},\bar{r},\bar{\xi})) \in T'(\mathbb{R}^n \times \mathbb{R}^n) \mid \bar{r} = \bar{\xi} = 0\} .$$

Proposition 5.5: Let $A \in L^m(\mathbb{R}^n)$ be a properly supported pseudodifferential operator, $H \in H^{-\infty}$, then

a) $AH \in H^{-\infty}$

b) $HA \in H^{-\infty}$

c) $H^t \in H^{-\infty}$.

Proof: We know (cf. [D]) that

$$(9) \quad WF'(AH) \subseteq WF'(A) \circ WF'(H) \cup (\underset{\pi_1(\mathbb{R}^n)}{WF'A} \times (0)) \cup (0 \times \underset{\pi_2(\mathbb{R}^n)}{WF'H})$$

where $\underset{\pi_1(\mathbb{R}^n)}{WF'A} = \{(t,y,r,\xi) \in T'(\mathbb{R}^n) \mid \exists\ (\bar{t},\bar{y}) \in \mathbb{R}^n$ s.t. $((t,y,r,\xi);(\bar{t},\bar{y},0,0)) \in WFA\}$

$\underset{\pi_2(\mathbb{R}^n)}{WF'H} = \{(\bar{t},\bar{y},\bar{r},\bar{\xi}) \in T'(\mathbb{R}^n) \mid \exists\ (t,y) \in \mathbb{R}^n$ s.t. $((t,y,0,0),(\bar{t},\bar{y},\bar{r},\bar{\xi})) \in WFH\}$.

From (9), we deduce then a) and b). c) is immediately a consequence from

$$K_{N^t}(\varphi \otimes \psi)(e^{-i<\cdot;\alpha(r,\xi)>}e^{-i<\cdot;\alpha(\bar{r},\bar{\xi})}) = <N^t\tilde{\varphi},\tilde{\psi}> = <\tilde{\varphi},N\tilde{\psi}>$$

where $\tilde{\varphi} = \varphi e^{-i<\cdot;\alpha(r,\xi)>}$, $\tilde{\psi} = \psi e^{-i<\cdot;\alpha(r,\xi)>}$.

Q.E.D.

Considering A^t instead of A , we can construct an operator $\widetilde{E}^t: C_o^\infty(\mathbb{R}^n) \longrightarrow C^\infty(\mathbb{R}^n)$ s.t. $\widetilde{E}^t L = Id + H'$, $H' \in H^{-\infty}$. So as a consequence of c), we have operators $\widetilde{E}$, $\widetilde{E}^t$, satisfying

$$(10) \qquad \begin{aligned} L\widetilde{E} &= Id + H \qquad && H \in H^{-\infty} \\ \widetilde{E}^t L &= ID + H' \qquad && H' \in H^{-\infty} . \end{aligned}$$

6. Construction of a parametrix for M .

Notation: In this paragraph, we will denote by $\widetilde{E}$ all the parametrices constructed.

Proposition 6.1: $\exists\ \widetilde{E}: C_o^\infty(\mathbb{R}^n) \longrightarrow C^\infty(\mathbb{R}^n)$ s.t.

$$(1) \qquad \widetilde{K}\widetilde{E} = Id \mod H^{-\infty} .$$

Proof: By 3.(7) we have that

$$\widetilde{K} = LC \text{ with } C \text{ elliptic.}$$

Let C' be a parametrix for C . Using 5.(10) and Proposition 5.5, we get that

$$\widetilde{K}\widetilde{E}C' = \text{ID mod } H^{-\infty} \quad .$$

Q.E.D.

Proposition 6.2: $\exists\ \widetilde{E}: C_o^\infty(\mathbb{R}^n) \longrightarrow C^\infty(\mathbb{R}^n)$ s.t.

(2) $$K\widetilde{E} = \text{Id mod } H^{-\infty} \quad .$$

Proof: We have that K is obtained from $\widetilde{K}$ by the change of variables

3.(5) $$\begin{aligned} t &= x_1 \\ y_1 &= x_1 - x_2 \\ y_j &= x_{j+1} \qquad j = 2,\dots,n-1 \quad . \end{aligned}$$

So we just change the variables in the parametrix $\widetilde{E}$ of $\widetilde{K}$ and we observe that $H^{-\infty}$ is invariant under the change of variables 3.(5), because if $(t,y,r,\xi) \in T'(\mathbb{R}^n)$ are the new variables obtained under the change of variables 3.(5) and $(x,\theta) \in T'(\mathbb{R}^n)$ are the old ones, then $\theta_1 = r$, $\theta_2 = r - \xi_1$, $\theta_{j+1} = \xi_{j-1}$, $j \geq 3$.

Q.E.D.

Proposition 6.3: $\exists\widetilde{E}: C_o^\infty(\mathbb{R}^n) \longrightarrow C^\infty(\mathbb{R}^n)$ s.t.

(3) $$\widetilde{L}\widetilde{E} = \text{Id mod } H^{-\infty}(\mathbb{R}^n) \quad .$$

Proof: We have by 3.(3) that

$$\tilde{L} = KE \quad \text{with} \quad E \quad \text{elliptic.}$$

Let E' be a parametrix for E. Using (2) and Proposition 5.5 we get:

$$\tilde{L}E'\tilde{E} = \mathrm{Id} \bmod H^{-\infty}(\mathbb{R}^n) \ .$$

Q.E.D.

Proposition 6.4: $\exists\ \tilde{E}: C_o^\infty(\mathbb{R}^n) \longrightarrow C^\infty(\mathbb{R}^n)$ s.t.

(4) $$M\tilde{E} = \mathrm{Id} \bmod H^{-\infty}(\mathbb{R}^n) \ .$$

Proof: Using Proposition 5.5, and 5.(10) we can prove Proposition 6.1, Proposition 6.2, Proposition 6.3, with $\tilde{E}$ a left parametrix instead of a right parametrix (taking real transposes).

So $\exists\ \tilde{E}: C_o^\infty(\mathbb{R}^n) \longrightarrow C^\infty(\mathbb{R}^n)$ s.t.

(4) $\tilde{E}\tilde{L} = \mathrm{Id} + H$, $H \in H^{-\infty}(\mathbb{R}^n)$. In this case, we have with

$$\tilde{E} = \begin{pmatrix} \tilde{E}_{11} & \tilde{E}_{12} \\ \tilde{E}_{21} & \tilde{E}_{22} \end{pmatrix} \ .$$

(5) $$\begin{pmatrix} \tilde{E}_{11} & \tilde{E}_{12} \\ \tilde{E}_{21} & \tilde{E}_{22} \end{pmatrix}\begin{pmatrix} D_1 & A_1D_1 \\ 0 & D_2 \end{pmatrix} = \begin{pmatrix} \tilde{E}_{11}D_1 & \tilde{E}_{11}A_1D_1+\tilde{E}_{12}D_2 \\ \tilde{E}_{21}D_1 & \tilde{E}_{21}A_1D_1+\tilde{E}_{22}D_2 \end{pmatrix},$$

and

(6) $$\begin{pmatrix} \tilde{E}_{11} & \tilde{E}_{12} \\ \tilde{E}_{21} & \tilde{E}_{22} \end{pmatrix}\begin{pmatrix} A_2 & R \\ -Id & 0 \end{pmatrix} = \begin{pmatrix} \tilde{E}_{11}A_2-\tilde{E}_{12} & \tilde{E}_{11}R \\ \tilde{E}_{21}A_2-\tilde{E}_{22} & \tilde{E}_{21}R \end{pmatrix}.$$

Let's take $U = \begin{pmatrix} D_2u \\ u \end{pmatrix}$, $u \in C_o^\infty(\mathbb{R}^n)$. Then

(7) $\tilde{E}\tilde{L}U =$

$$\begin{pmatrix} \tilde{E}_{11}D_1D_2u + \tilde{E}_{11}A_1D_1u + \tilde{E}_{12}D_2u + \tilde{E}_{11}A_2D_2u - \tilde{E}_{12}D_2u + \tilde{E}_{11}Ru \\ \tilde{E}_{21}D_1D_2u + \tilde{E}_{21}A_1D_1u + \tilde{E}_{22}D_2u + \tilde{E}_{21}A_2D_2u - \tilde{E}_{22}D_2u + \tilde{E}_{12}Ru \end{pmatrix}$$

$$= \begin{pmatrix} D_2u \\ u \end{pmatrix} + \begin{pmatrix} H_{11}D_2u + H_{12}u \\ H_{21}D_2u + H_{22}u \end{pmatrix} \qquad H_{ij} \in H^{-\infty}(\mathbb{R}^n)$$

$\Rightarrow$

(8) $\tilde{E}_{21}Mu = u + H'u$; $H' \in H^{-\infty}(\mathbb{R}^n)$ by proposition 5.5.

Note that

$$M^t = D_1D_2 + D_1A_1^t + D_2A_2^t + R^t$$

(9) $$M^t = D_1D_2 + A_1^tD_1 + A_2^tD_2 + \tilde{R}$$

with $\quad \widetilde{R} = [D_1, A_1^t] + [D_2, A_2^t] + R^t \in L^o(\mathbb{R}^n)$.

So M^t has the same form as M , $\Rightarrow$

$$\exists\ \widetilde{E}: C_o^\infty(\mathbb{R}^n) \longrightarrow C^\infty(\mathbb{R}^n) \quad \text{s.t.}$$

(10) $$\widetilde{E}M^t = Id + H , \quad H \in H^{-\infty} .$$

Taking tranpose and applying Proposition 5.5 we are done.

Q.E.D.

7. Construction of a local parametrix for P .

Let us consider the operator P satisfying (I) and assumptions (i_1), (i_2), (i_3), (i_4), (i_5).

a) Local right parametrix. See also [D-H].

i) Take $(x_o, \xi_o) \in \Sigma$. We can take $T \in L^o(\mathbb{R}^n)$ s.t. WFT is near (x_o, ξ_o) . Since we have constructed a canonical transformation carrying (x_o, ξ_o) into (z_o, θ_o) , and we have freedom to choose z_o , we will assume $z_o' \neq 0$ (this is for latter convenience) and we can take T with WFT so near (x_o, ξ_o) s.t. $\overline{\chi(WFT)}$

doesn't intersect the surface $z_o' = 0$, where χ is the canonical transformation of Lemma 2.5.

Let $\psi \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ s.t. $\psi = 1$ in a neighborhood of the diagonal Δ in $\mathbb{R}^n \times \mathbb{R}^n$, and $\psi = 0$ outside another sufficiently small neighborhood of Δ .

Let's take

$$(1) \qquad F = A\psi\tilde{E}BT \ , \quad A \text{ and } B \text{ as in } 2.(15) \text{ and } 2.(16).$$

Then we have:

$$(2) \quad PF = PA\psi\tilde{E}BT = (PA - AM)\psi\tilde{E}BT + AM\psi\tilde{E}BT \ .$$

We know that $(x_o, \xi_o, z_o, \theta_o) \notin WF'(PA - AM) \subset \Gamma$ by Proposition 2.7. So we have that $\exists$ a conical neighborhood V of (z_o, θ_o) s.t.

$$(3) \qquad (PA - AM)v \in C^\infty \text{ if } WFv \subset V \ .$$

$WF'(\psi\tilde{E})$ can be chosen arbitrarly close to the Δ in $T'(\mathbb{R}^n \times \mathbb{R}^n)$, by choosing the support of ψ close to the diagonal in $\mathbb{R}^n \times \mathbb{R}^n$.

If $WF(T)$ is so close to (x_o, ξ_o, x_o, ξ_o) s.t. $\chi(WFv) \subset V'$, where V' is a conic neighborhood of (z_o, θ_o) s.t. $WF(\psi\tilde{E}v) \subset V$ if $WFv \subset V'$ $\Rightarrow$

(4) $$(PA - AM)\psi\tilde{E}BT \in C^\infty \quad .$$

We have also that:

(5) $$AM\psi\tilde{E}BT = -AM(1 - \psi)EBT + ABT + AHBT, \quad H \in H^{-\infty}(\mathbb{R}^n)$$

by Proposition 6.4. We have

(6) $$(x_o, \xi_o, x_o, \xi_o) \notin WF(AHBT) \quad .$$

(6) follows from the fact that

$$WF'(C \circ D) \subset WF'(C) \circ WF'(D) \cup WF_X'(A) \times D_{T*}(Z)) \cup O_{T*X} \times WT_Z'(B)$$

where $C: C_o^\infty(Y) \longrightarrow D'(X)$ $\quad$ X, Y, Z C^∞ manifolds

$D: C_o^\infty(Z) \longrightarrow D'(Y)$

(cf. [D])

and the fact that $WF'(A) \subseteq T'(X) \times T'(\mathbb{R}^n)$

$WF'(B) \subseteq T'(\mathbb{R}^n) \times T'(X)$

(7) $$ABT = (AB - I)T + T \quad .$$

Since $(x_o, \xi_o) \notin WF(AB - ID)$, (7) says that:

(8) $$(AB - I)T \in C^\infty$$

if WFT is sufficiently close to (x_o, ξ_o, x_o, ξ_o) .

Then from (4), (5), (6), (7), (8) we get

$$(x_o, \xi_o) \notin WF(PA\tilde{E}B - Id_X)T \quad .$$

If $(x_o, \xi_o) \in T'(X)$ is such that

(a) $p_1(x_o, \xi_o) \neq 0$, $p_2(x_o, \xi_o) \neq 0$, a local right and left parametrix is easily constructed, since in this case p is elliptic at (x_o, ξ_o) .

(b) $p_1(x_o, \xi_o) \neq 0$, $p_2(x_o, \xi_o) = 0$, the construction of a local right and left parametrix is known, since in this case p is with single characteristics at (x_o, ξ_o) , because of $i_5)$.

(c) $p_1(x_o, \xi_o) = 0$, $p_2(x_o, \xi_o) \neq 0$, same argument as in (b).

b) <u>Local left parametrix</u>.

In the proof of Proposition 6.4 it was shown that there exists $\tilde{E}: C_o^\infty(\mathbb{R}^n) \longrightarrow C^\infty(\mathbb{R}^n)$ s.t.

$$(9) \qquad \tilde{E}M = Id + H , \qquad H \in H^{-\infty}(\mathbb{R}^n) .$$

We take now $T' \in L^0(\mathbb{R}^n)$ with $WF(T')$ sufficiently near (z_0, θ_0) such that $WFT \cap \{(z,\theta) \in T'(\mathbb{R}^n) \mid z_1 = 0\} = \emptyset$. We know by Proposition 3.7, that

$$(10) \qquad (z_0, \theta_0, x_0, \xi_0) \notin WF'(BP - MB) \quad .$$

We take $\psi \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, $\psi = 1$ near the diagonal in $\mathbb{R}^n \times \mathbb{R}^n$ and 0 outside another small neighborhood of Δ. We take

$$(11) \qquad F = TB\psi\tilde{E}A \quad .$$

And using that $(z_0, \theta_0) \notin WF(BA - Id_{\mathbb{R}^n})$, we get, using the same proof as in 7. (a) that

$$(12) \qquad (x_0 \xi_0, x_0, \xi_0) \notin WF'(B\psi\tilde{E}AP - Id_X)$$

when $(x_0, \xi_0) \in \Sigma$. The argument for $(x,\xi) \in T'(X)$, $(x,\xi) \notin \Sigma$, is the same as given in 7 a).

-o-

In the following section we will analyze the properties of the local parametrix constructed for P, through the properties of the parametrix constructed for M. We will also analyze the fundamental solution for the Cauchy problem for L.

8. Properties of the parametrix constructed for P,M and the fundamental solution of the Cauchy problem for L .

(i) Singularities of the operators constructed.

We will use the following lemmas:

Lemma 8.1: If $u_1, u_2 \in D'(\mathbb{R}^n)$, $WFu_i \subset \Gamma_i$, $i = 1,2$, Γ_i closed cones in $T'(\mathbb{R}^n)$, $\Gamma_1 \cap (-\Gamma_2) = \emptyset$ where $\Gamma_2 = \{(x,-\xi) \in T'(\mathbb{R}^n) | (x,\xi) \in \Gamma_2\}$ then $u_1u_2 \in D'(\mathbb{R}^n)$ and $WF(u_1u_2) \subset (\Gamma_1 + \Gamma_2) \cup \Gamma_1 \cup \Gamma_2$ where $\Gamma_1 + \Gamma_2 = \{(x, \xi_1+\xi_2) \in T'(\mathbb{R}^n) | (x,\xi_1) \in \Gamma_1 , (x,\xi_2) \in \Gamma_2\}$.

Lemma 8.2: Let $\Pi: \mathbb{R}^m \times \mathbb{R} \longrightarrow \mathbb{R}^m$ the projection, then if $u \in D'(\mathbb{R}^m \times \mathbb{R})$, $\Pi_* u \in D'(\mathbb{R}^m)$ if $\Pi: \text{sup } u \longrightarrow \mathbb{R}^m$ is proper and $WF(\Pi_* u) \subseteq \{(z,\eta) \in T'(\mathbb{R}^m) | \exists \tau \in \mathbb{R} ; (z,\eta,\tau,0) \in WFu\}$.

Proof: See [D].

Note also that $\Pi_* u = \int u(x,\tau) d\tau$ (in formal terms) i.e. integration over the fiber τ .

Remark: Lemmas 8.1 and 8.2 are more general than stated, but we will need them only in this form.

(a) $E_3 : C_o^\infty(\mathbb{R}^{n-1}) \longrightarrow C^\infty(\mathbb{R}^n)$ was defined by

$$E_3 f(t,y) = \int_{-t}^{t} \int e^{i\langle \frac{\tau+t}{2}+y,\,\theta\rangle} e_3(\tau,t,y,\theta)\hat{f}(\theta)\,d\theta\, d\tau \quad .$$

Putting the inner integral as an oscillatory integral, we have:

$$E_3 f(t,y) = \int_{-t}^{t} \int e^{i\langle \frac{\tau+t}{2}+y,\,\theta\rangle - \langle \bar{y},\theta\rangle} e_3(\tau,t,y,\theta) f(\bar{y})\,d\bar{y}\,d\theta\, d\tau \quad .$$

Let

$$E_3' f(t,y) = \int e^{i\langle \frac{\tau+t}{2}+y,\,\theta\rangle - \langle \bar{y},\theta\rangle} e_3(\tau,t,y,\theta) f(\bar{y})\,d\bar{y}\,d\theta \quad .$$

(It makes sense of course as an oscillatory integral.)
We have that since E_3' is a Fourier Integral Operator:

$$(1) \quad WF' E_3' \subseteq \left\{ \begin{array}{l} ((\tau,m);((t,y,r,\xi);(\bar{y},\bar{\xi}))) \in T'(\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^{n-1}) \,| \\ \bar{y}_1 = \frac{\tau+t}{2} + y_1 \ , \quad \bar{y}_j = y_j \ , \quad j = 2,\dots,n-1 \\ \xi = \bar{\xi} \ , \quad m = r = \frac{1}{2}\xi_1 \end{array} \right\} .$$

Let $H(t+\tau) = \begin{cases} 1 & \text{if } -t \le \tau \\ 0 & \text{otherwise} \end{cases}$ in $\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^{n-1}$, and $H(t-\tau) = \begin{cases} 1 & \text{if } \tau \leq t \\ 0 & \text{otherwise} \end{cases}$ in $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n-1}$.

$$WFH(t+\tau) \subseteq \{((\tau,m);(t,y,r,\xi);(\bar{y},\bar{\xi})) \in T'(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n-1} \mid \tau = -t\ ,\quad \xi = \bar{\xi} = 0\ ,\quad m = -r\}\ ,$$

and

$$WFH(t-\tau) \subseteq \{((\tau,m),(t,y,r,\xi);(\bar{y},\bar{\xi})) \in (\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n-1} \mid \tau = t\ ,\quad \xi = \bar{\xi} = 0\ ,\quad m = r\}\ .$$

So, we have that $WFH(t+\tau) \cap (-WFH(t-\tau)) = \emptyset$ and $WF'(H(t+\tau)H(t-\tau)) \cap (-WF'E_3') = \emptyset$. Considering that

$$K_{E_3}(t,y,\bar{y}) = \int H(t+\tau)H(t-\tau)K_{E_3'}(\tau,t,y,\bar{y})d\tau\ .$$

and using Lemmas 8.1 and 8.2 we get:

$$(2) \qquad WF'E_3 \subseteq \left\{ \begin{array}{l} ((t,y,r,\xi);(\bar{y},\bar{\xi})) \in T'(\mathbb{R}^n \times \mathbb{R}^{n-1}) \mid \\ \bar{y}_1 = y_1 + \frac{\tau+t}{2}\ ,\ -t \leq \tau \leq t\ ,\ \bar{y}_j = y_j\ , \\ \qquad\qquad\qquad\qquad j = 2,\ldots,n-1 \\ r = \xi_1 = 0\ ,\quad \xi = \bar{\xi} \end{array} \right\} = C_3(0)$$

$$\cup \left\{ \begin{array}{l} ((t,y,r,\xi);(\bar{y},\bar{\xi})) \in T'(\mathbb{R}^n \times \mathbb{R}^{n-1})| \\ \bar{y}_1 = y_1 + t \ , \ \bar{y}_j = y_j \ , \ j = 2,\ldots,n-1 \\ r = \xi_1 = \bar{\xi}_1 \ , \quad \xi = \bar{\xi} \end{array} \right\} = C_2(0)$$

$$\cup \left\{ \begin{array}{l} ((t,y,r,\xi);(\bar{y},\bar{\xi})) \in T'(\mathbb{R}^n \times \mathbb{R}^{n-1})| \\ \\ \bar{y}_j = y_j \ , \ j = 1,\ldots,n-1 \ , \ r = 0 \ , \ \xi = \bar{\xi} \end{array} \right\} = C_1(0) \ .$$

<u>Remark</u>: Note that we have that $C_1(0)$ and $C_2(0)$ are the canonical relations that appear in the construction of the fundamental solution of the Cauchy problem for D_t and $D_t - D_{y_1}$ respectively. (See Chapter II.1)

$$C_1(0) = \left\{ \begin{array}{l} ((t,y,r,\xi);(\bar{y},\bar{\xi})) \in T'(\mathbb{R}^n) \times T'(\mathbb{R}^{n-1})| \\ (t,y,r,\xi) \text{ is in the same bicharacteristic} \\ \text{strip of } H_{p_1} = D_t \text{ as } (0,\bar{y},\lambda_1(0,\bar{y},\bar{\xi}),\bar{\xi}) \end{array} \right\}$$

$\lambda_1(t,y,\xi) = 0 \ \forall \ (t,y,\xi) \in \mathbb{R} \times T'(\mathbb{R}^{n-1})$ in this case.

$$C_2(0) = \left\{ \begin{array}{l} ((t,y,r,\xi);(\bar{y},\bar{\xi})) \in T'(\mathbb{R}^n) \times T'(\mathbb{R}^{n-1}) \\ (t,y,r,\xi) \text{ is in the same bicharacteristic} \\ \text{strip of } H_{p_2} = D_t - D_{y_1} \text{ as } (0,\bar{y},\lambda_2(0,\bar{y},\bar{\xi}),\bar{\xi}) \end{array} \right\}$$

$\lambda_2(t,y,\xi) = \xi_1 \ \forall \ (t,y,\xi) \in \mathbb{R} \times T'(\mathbb{R}^{n-1})$ in this case.

Note that $\lambda_1(t,y,\xi) = \lambda_2(t,y,\xi) \Leftrightarrow \xi_1 = 0$.
Let $\tilde{C}_i(0) = C_i(0) \cap \{((t,y,r,\xi);(\bar{y},\bar{\xi}) \in T'(\mathbb{R}^n) \times T'(\mathbb{R}^{n-1}) \mid \xi_1 = 0\}$, $i = 1,2$.
We have that $\tilde{C}_i(0)$ are isotropic submanifolds of $T'(\mathbb{R}^n \times \mathbb{R}^{n-1})$ of dimension $2n-2$, $i = 1,2$. Note that

$$(3) \qquad C_3(0) = \bigcup_{-t \leq \tau \leq t} \tilde{C}_3(\tau)$$

where

$$\tilde{C}_3(\tau) = \left\{ \begin{array}{l} ((t,y,r,\xi);(\bar{y},\bar{\xi})) \in T'(\mathbb{R}^n) \times T'(\mathbb{R}^{n-1}) \mid \\ \bar{y}_1 = y_1 + \frac{t+\tau}{2}, \ \bar{y}_j = y_j, \ j = 2,\dots,n-1 \\ \xi = \bar{\xi}, \quad r = \xi_1 = 0 \end{array} \right\}$$

$\tilde{C}_3(\tau)$ is an isotropic submanifold of $T'(\mathbb{R}^n \times \mathbb{R}^{n-1})$ of dimensions $2n-2$ for each fixed τ. Note that

$$(4) \qquad \tilde{C}_3(t) = \tilde{C}_2(0), \quad \tilde{C}_3(-t) = \tilde{C}_1(0)$$

Also $\tilde{C}_1(0) \cap \{((t,y,r,\xi);(\bar{y},\bar{\xi}) \in T'(\mathbb{R}^n) \times T'(\mathbb{R}^{n-1}) \mid t = 0\} =$

$$\tilde{C}_2(0) \cap \{((t,y,r,\xi);(\bar{y},\bar{\xi}) \in T'(\mathbb{R}^n) \times T'(\mathbb{R}^{n-1}) \mid t = 0\}.$$

(3) and (4) explains the sentence: the "cone generated by $\tilde{C}_1(0)$ and $\tilde{C}_2(0)$".

(b) From the construction of the parametrix $\tilde{E}$ for M it is clear that it has the form

$$(5)\quad \tilde{E}f(t,y) = \sum_{\substack{i,j=1 \\ k=1,2,3}}^{2} B_{ij}^{k} \int_0^t (E_{ij}^{k}(s)\gamma_s f)(t,y)ds \quad .$$

Recall that $E = E_1 + E_2 + E_3$ are 2×2 matrices of operators

$$E_k = \begin{pmatrix} E_{11}^k & E_{21}^k \\ E_{21}^k & E_{22}^k \end{pmatrix} \qquad B_{ij}^k \in L^0(\mathbb{R}^n) \quad .$$

So to calculate $WF'\tilde{E}$ it is enough to calculate $WF'\tilde{E}_1$, $WF'\tilde{E}_2$, $WF'\tilde{E}_3$, since $WF'(B_{ij}^k) \subset \Delta$ where Δ is the diagonal of $T'(\mathbb{R}^n)\times T'(\mathbb{R}^n)$.

(b_1) We have that

$$(E_1(s)\gamma_s(f)(t,y) = \int e^{i\langle y-\bar{y},\theta\rangle} e_1(s,t,y,\theta)f(s,\bar{y})d\bar{y}d\theta$$

$$f \in C_0^\infty(\mathbb{R}^n) \quad .$$

We will consider

$$E_1(s)\gamma_s : C_o^\infty(\mathbb{R}^n) \longrightarrow C^\infty(\mathbb{R}^{n+1})$$

$$WF'E_1(s)\gamma_s \subseteq \{((\tau,m);(t,y,r,\xi),(\bar{t},\bar{y},\bar{r},\bar{\xi})) \in T'(\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n) \mid \bar{y}_j = y_j\ ,\ j = 1,\dots,n-1\ ,\ \bar{t} = s\ ,\ \xi = \bar{\xi}\ ,\ m = \bar{r}\ ,\ r = 0\}\ .$$

Taking into account that:

$$\tilde{E}_1 = \int H(t-s)H(s)E_1(s)\gamma(s)ds$$

using Lemmas 8.1, 8.2 and

$$WFH(s) \subseteq \{((s,u);(t,y,r,\xi);(\bar{t},\bar{y},\bar{r},\bar{\xi})) \in T'(\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n) \mid s = 0\ ,\ r = \xi = \bar{r} = \bar{\xi} = 0\}\ ,$$

$$WFH(t-s) \subseteq \{((s,u);(t,y,r,\xi);(\bar{t},\bar{y},\bar{r},\bar{\xi})) \in T'(\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n) \mid t = s\ ,\ u = r\ ,\ \xi = \bar{r} = \bar{\xi} = 0\},$$

we get:

(6)

$$WF'\tilde{E}_1 \subseteq \left\{ \begin{matrix} ((t,y,r,\xi);(\bar{t},\bar{y},\bar{r},\bar{\xi})) \in T'(\mathbb{R}^n\times\mathbb{R}^n) \mid \\ \bar{y}_j = y_j\ ,\ j=1,\dots,n-1\ ,\ \xi = \bar{\xi}\ ,\ r = \bar{r} = 0 \end{matrix} \right\} = C_1$$

$$\cup \left\{ \begin{array}{l} ((t,y,r,\xi);(\bar t,\bar y,\bar r,\bar\xi)) \in T'(\mathbb{R}^n \times \mathbb{R}^n) | \\ \bar y_j = y_j \ , \ j=1,\dots,n-1 \ , \ t=\bar t \ , \ r=\bar r \ , \ \xi=\bar\xi \end{array} \right\} = \Delta$$

$$\cup \left\{ \begin{array}{l} ((t,y,r,\xi);(\bar t,\bar y,\bar r,\bar\xi)) \in T'(\mathbb{R}^n \times \mathbb{R}^n) | \\ \bar t = 0 \ , \ r=0 \ , \ \xi=\bar\xi \ , \ \bar y_j = y_j \ , \ j=1,\dots,n-1 \end{array} \right\} = \tilde C_1(0) \ .$$

Note that C_1 is the canonical relation defined by

$C_1 = \{((t,y,r,\xi);(\bar t,\bar y,\bar r,\bar\xi)) \in T'(\mathbb{R}^n \times \mathbb{R}^n) | \ (t,y,r,\xi)$

and $(\bar t,\bar y,\bar r,\bar\xi)$ are in the same bicharacteristic

strip corresponding to $H_r = D_t\}$.

Δ denotes the diagonal in $T'(\mathbb{R}^n) \times T'(\mathbb{R}^n)$. Notice also, that

$$\tilde C_1(0) \ = \ C_1(0) \bullet R(0)$$

where $R(0)$ is the canonical relation associated to the Fourier Integral Operator:

$\gamma_o: C_o^\infty(\mathbb{R}^n) \longrightarrow C^\infty(\mathbb{R}^{n-1})$, defined by

$\gamma_o f(y) \ = \ f(0,y)$,

$$R(0) = \{((\bar{y},\bar{\xi});(t,y,r,\xi)) \in T'(\mathbb{R}^{n-1} \times \mathbb{R}^n) \mid \xi = \bar{\xi} ,$$
$$y = \bar{y} , \ t = 0\} .$$

<u>Remark</u>: Note that the singularities of $\tilde{E}_1$ that lie in Δ or $\tilde{C}_1(0)$ come from the "wedge" $s = t$ or $s = 0$, because if $\tilde{E}'f(t,y) = \int E^1(s)\gamma(s)f(t,y)ds$, supposing that this would make sense, then we would have

$$WF'\tilde{E}' \subseteq C_1 .$$

More precisely $\tilde{C}_1(0)$ comes from the contribution of $\tau = 0$ $(H(\tau))$ and the Δ comes from the contribution of $\tau = t(H(t-\tau))$. In the same way, we get:

(b_2)

(7) $$WF'(\tilde{E}_2) \subseteq C_2 \cup \Delta \cup \tilde{C}_2(0)$$

where $C_2 = \{((t,y,r,\xi);(\bar{t},\bar{y},\bar{r},\bar{\xi})) \in T'(\mathbb{R}^n \times \mathbb{R}^n) \mid$

$$\bar{y}_1 = y_1 + (t-\bar{t}), \ \bar{y}_j = y_j , \ j = 2,\dots,n-1 ,$$
$$\xi = \bar{\xi} , \ r = \bar{r}\} ,$$

$$\tilde{C}_2(0) = \{((t,y,r,\xi);(\bar{t},\bar{y},\bar{r},\bar{\xi})) \in T'(\mathbb{R}^n \times \mathbb{R}^n) \mid$$
$$\bar{t} = 0 , \bar{y}_j = y_j , \ j = 2,\dots,n-1 , \bar{y}_1 = y_1 + t ,$$
$$\xi = \bar{\xi} , r = \xi_1\} .$$

We notice that

$$C_2 = \{((t,y,r,\xi);(\bar{t},\bar{y},\bar{r},\bar{\xi})) \in T'(\mathbb{R}^n \times \mathbb{R}^n) \mid (t,y,r,\xi)$$

and $(\bar{t},\bar{y},\bar{r},\bar{\xi})$ are in the same bicharacteristic strip of $H_{r-\xi_1} = D_t - D_{y_1}\}$.

We also have $\widetilde{C}_2(0) = C_2(0) \circ R(0)$.

(b_3) We have

$$\widetilde{E}_3 f(t,y) = \int_0^t (E_3(s)\gamma_s f)(t,y)ds \tag{8}$$

where

$$E_3(s)\gamma_s f(t,y) = \int_{-t+s}^{t-s}\int e^{i\langle\frac{\tau-s+t}{2} + y - \bar{y},\, \theta\rangle} e_3(s,\tau,t,y,\bar{y},\theta)\, f(s,\bar{y})d\bar{y}d\theta d\tau$$

where $\langle\frac{\tau-s+t}{2} + y - \bar{y}, \theta\rangle = (\frac{\tau-s+t}{2} + y_1 - \bar{y}_1)\theta_1 + (y_2 - \bar{y}_2)\theta_2 + \dots + (y_{n-1} - \bar{y}_{n-1})\theta_{n-1}$.

We consider the operator

$\widetilde{E}_{3\tau}(s)\colon C_0^\infty(\mathbb{R}^n) \longrightarrow C^\infty(\mathbb{R}^{n+1})$ to be defined by

$$\tilde{E}^{(s)}_{3\tau} f(t,y) = \int e^{<\frac{\tau-s+t}{2} + y - \bar{y}, \theta>} e_3(s,\tau,t,y,\theta) f(s,\bar{y}) d\bar{y} d\theta$$

in the sense of an oscillatory integral. We have that:

$$\begin{aligned} WF'\tilde{E}^{(s)}_{3\tau} \subseteq \{((\tau,m);(s,u);(t,y,r,\xi);(\bar{t},\bar{y},\bar{r},\bar{\xi})) \in \\ T'(\mathbb{R}\times\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n) \mid \ \bar{y}_1 = y_1 + \frac{\tau-s+t}{2} , \\ \bar{y}_j = y_j \ , \ j = 2,\ldots,n-1 \ , \ \bar{t} = s \ , \ m = r = \tfrac{1}{2}\xi_1 , \\ \xi = \bar{\xi} \ , \ \bar{r} = -u = -\tfrac{1}{2}\xi_1\} \end{aligned}$$

and considering that

$$\begin{aligned} WFH(t-s+\tau) \subseteq \{((\tau,m);(s,u);(t,y,r,\xi);(\bar{t},\bar{y},\bar{r},\bar{\xi})) \in \\ T'(\mathbb{R}\times\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n) \mid \ \tau = t-s \ , \ m=-r=u \ , \\ \xi = \bar{\xi} = \bar{r} = 0\} \end{aligned}$$

and

$$\begin{aligned} WFH(-t+s+\tau) \subseteq \{((\tau,m);(s,u);(t,y,r,\xi);(\bar{t},\bar{y},\bar{r},\bar{\xi})) \in \\ T'(\mathbb{R}\times\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n) \mid \ \tau = -t+s \ , \\ \xi = \bar{\xi} = \bar{r} = 0 \ , \ m = r = -u\} \end{aligned}$$

and applying Lemmas 8.1, 8.2 to the fact that:

$$E_3(s)\gamma(s) = \int H(t-s-\tau)H(-t+s+\tau)\tilde{E}^{(s)}_{3\tau}\,d\tau$$

we get

$$WF'E_3^{(s)}\gamma_s \subseteq \left\{\begin{array}{l} ((s,u);(t,y,r,\xi);(\bar{t},\bar{y},\bar{r},\bar{\xi})) \in T'(\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n) \big| \\ \bar{y}_1 = y_1 + \frac{\tau-s+t}{2},\ -t+s \le \tau \le t-s,\ \bar{y}_j = y_j, \\ j=2,\dots,n-1,\ \bar{t}=s,\ \xi=\bar{\xi},\ \xi_1 = r = \bar{r} = u = 0 \end{array}\right\}$$

$$\cup \left\{\begin{array}{l} ((s,u),(t,y,r,\xi);(\bar{t},\bar{y},\bar{r},\bar{\xi})) \in T'(\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n) \big| \\ \bar{y}_1 = y_1 - s + t,\ \bar{y}_j = y_j,\ j=2,\dots,n-1,\ \bar{t}=s \\ u = r = \bar{r} = \xi_1 = 0,\ \xi=\bar{\xi} \end{array}\right\}$$

$$\cup \left\{\begin{array}{l} ((s,u),(t,y,r,\xi);(\bar{t},\bar{y},\bar{r},\bar{\xi})) \in T'(\mathbb{R}\ \ \mathbb{R}^n\ \ \mathbb{R}^n) \big| \\ \bar{y}_j = y_j,\ j=1,\dots,n-1,\ \bar{t}=s, \\ u = r = \bar{r} = \xi_1 = 0,\ \xi=\bar{\xi} \end{array}\right\}$$

<u>Remark</u>: The contribution to $WF'E_3(s)\gamma(s)$ from $\tilde{E}^{(s)}_{3\tau}$ is reflected in the first term.

Considering that

$$\widetilde{E}_3 f(t,y) = \int H(t-s)H(s)(E_3^{(s)}\gamma_s)f)(t,y)ds$$

and applying our lemmas we get:

(9)

$$WF'\widetilde{E}_3 \subseteq \left\{ \begin{array}{l} ((t,y,r,\xi);(\bar{t},\bar{y},\bar{r},\bar{\xi})) \in T'(\mathbb{R}^n \times \mathbb{R}^n) \mid \\ \bar{y}_1 = y_1 + \frac{\tau-\bar{t}+t}{2},\ -t+\bar{t} \leq \tau \leq t-\bar{t},\ \bar{y}_j = y_j \\ j = 2,\ldots,n-1,\ r = \bar{r} = \xi_1 = 0,\ \xi = \bar{\xi} \end{array} \right\} = C_3$$

$$\cup \left\{ \begin{array}{l} ((t,y,r,\xi);(\bar{t},\bar{y},\bar{r},\bar{\xi})) \in T'(\mathbb{R}^n \times \mathbb{R}^n) \mid \\ \bar{y}_1 = y_1 + \frac{\tau+t}{2},\ -t \leq \tau \leq t,\ \bar{y}_j = y_j, \\ j = 2,\ldots,n-1,\ \bar{t} = 0,\ r = \bar{r} = \xi_1 = 0,\ \xi = \bar{\xi} \end{array} \right\} = \widetilde{C}_3(0)$$

$$\cup \left\{ \begin{array}{l} ((t,y,r,\xi);(\bar{t},\bar{y},\bar{r},\bar{\xi})) \in T'(\mathbb{R}^n \times \mathbb{R}^n) \mid \\ \bar{y}_1 = y_1,\ \bar{y}_j = y_j,\ j = 1,\ldots,n-1,\ t = \bar{t}, \\ r = \bar{r} = \xi_1 = 0,\ \xi = \bar{\xi} \end{array} \right\} \subset \Delta$$

$$\cup \left\{ \begin{array}{l} ((t,y,r,\xi);(\bar{t},\bar{y},\bar{r},\bar{\xi})) \in T'(\mathbb{R}^n \times \mathbb{R}^n) \mid \\ \bar{y}_1 = y_1 + t,\ \bar{y}_j = y_j,\ j = 2,\ldots,n-1,\ \bar{t} = 0 \\ \bar{r} = \xi_1 = 0,\ \xi = \bar{\xi} \end{array} \right\} = \widetilde{C}_2^3(0)$$

$$\cup \left\{ \begin{array}{l} ((t,y,r,\xi);(\bar{t},\bar{y},\bar{r},\bar{\xi})) \in T'(\mathbb{R}^n \times \mathbb{R}^n) \\ \bar{y}_j = y_j \ , \ j = 1,\ldots,n-1 \ , \ t = \bar{t} \ , \\ r = \bar{r} = \xi_1 = 0 \ , \ \xi = \bar{\xi} \end{array} \right\} \subset \Delta$$

$$\cup \left\{ \begin{array}{l} ((t,y,r,\xi);(\bar{t},\bar{y},\bar{r},\bar{\xi})) \in T'(\mathbb{R}^n \times \mathbb{R}^n) \\ \bar{y}_j = y_j \ , \ j = 1,\ldots,n-1 \ , \ \bar{t} = 0 \ , \\ r = \bar{r} = \xi_1 = 0 \ , \ \xi = \bar{\xi} \end{array} \right\} = \tilde{C}_1^3(0)$$

Let us denote $H(0) = \tilde{C}_3(0) \cup \tilde{C}_2^3(0) \cup \tilde{C}_1^3(0$. So we get

(10) $$WF'\tilde{E}_3 \subseteq C_3 \cup \Delta \cup H(0) \quad .$$

<u>Remark</u>: Note that C_3 is the "generated cone" by the conical relations C_1 and C_2 in a sense similar to the remark of page 57, since

$$C_3 = \cup\, C_3(\tau) \ , \quad \tilde{\tilde{C}}_3(t) = C_1 \ , \quad \tilde{\tilde{C}}_3(-t) = C_2 \ ,$$
$$-t \leq \tau \leq t \ ,$$

$$\tilde{\tilde{C}}_i = C_2 \cap \{((t,y,r,\xi);(\bar{t},\bar{y},\bar{r},\bar{\xi})) \in T'(\mathbb{R}^n \times \mathbb{R}^n) | \ r = \bar{r} = \xi_1 = 0\} \ , \ i = 1,2 \ .$$

So from (b_1), (b_2), and (b_3) we get:

<u>Proposition 8.3</u>:

$$WF'E \subseteq \Delta \cup C_1 \cup C_2 \cup C_3 \cup \tilde{C}_1(0) \cup \tilde{C}_2(0) \cup H(0) \ .$$

We will get rid of the terms of the form $C_1(0)$, $C_2(0)$ and $H(0)$.

(c) Let us recall that the local parametrix for P near a point $(x_o, \xi_o) \in \Sigma$ was defined by

$$F = A\psi\tilde{E}BT \qquad \text{(See Section 7)}$$

We required for $T \in L^o(\mathbb{R}^n)$ that $\overline{\chi(WFT)}$ does not intersect the surface $z_o' = 0$, where χ is the canonical transformation defined in Lemma 3.5. So by the calculus of wave front sets (see [D]), we get that

$$\text{(11)} \qquad WF'F \subseteq \Delta_{T'(X)} \cup \chi^{-1}(C_1) \cup \chi^{-1}(C_2) \cup \chi^{-1}(C_3)$$

(because of the condition required for T .) We have, because χ (resp. χ^{-1}) preserve Hamiltonian vector fields H_{p_i} , $i = 1,2$ (resp. H_{θ_i} , $i = 1,2$) and the corresponding bicharacteristic strips that

<u>Proposition 8.4</u>: $WF'F \subseteq \Delta_{T'(X)} \cup \tilde{C}_1 \cup \tilde{C}_2 \cup X^{-1}(C_3)$ where $\tilde{C}_i = \{((x,\xi),(y,\eta) \in T'(X \times X) \mid (x,\xi)$ and (y,η) are in the same bicharacteristic strip corresponding to $H_{p_i}\}$, $i = 1,2$.

So the new element in the singularities of the parametrix of F is the term $\chi^{-1}(C_3) \subseteq \Sigma$, and which is sort of a cone with $\tilde{C}_1$ and $\tilde{C}_2$ as wedges. Δ is the diagonal in $T'(X) \times T'(X)$.

(ii). Further description of operators constructed.

(a) It is clear from the construction of Section 4, that E_i are Fourier Integral Operators $i = 1,2$. We have that: $E_3: C_o^\infty(\mathbb{R}^{n-1}) \longrightarrow C^\infty(\mathbb{R}^n)$

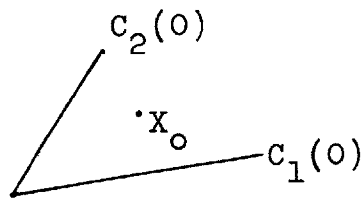
$$E_3 f(t,y) = \int_{-t}^{t}\int e^{i\langle \frac{\tau+t}{2} + y - \bar{y}, \theta\rangle} e_3(\tau,t,y,\theta) \quad f(\bar{y})d\bar{y}d\theta d\tau .$$

Take $X_o = (t_o, y_o, r_o, \xi_o);(\bar{y}_o, \bar{\xi}_o) \in C_3(0)$ (see (2))

and $\qquad \bar{y}_o' \neq y_1^o + t_o , \quad \bar{y}_o' \neq y_1^o , \quad t_o \neq 0 .$

(This means that $X_o \in C_3(0)$ but it is not in the "wedge" of the cone.)

Take $\varphi(t,y,\bar{y}) \in C_o^\infty(\mathbb{R}^n \times \mathbb{R}^{n-1})$, $\varphi = 1$ near $(t_o, y_o, \bar{y}_o)$ but supported in points $(t,y,\bar{y})$ s.t. $\bar{y}_1 \neq y_1 + t$; $y_1 \neq \bar{y}_1$, $t \neq 0$.



Clearly $X_o \notin WF(E_3 - E_3^o)$ with

$$E_3^o f(t,y) = \int_{-t}^{t}\int e^{i\langle \frac{\tau+t}{2} + y - \bar{y}, \theta\rangle} e_3(\tau,t,y,\theta)\varphi(t,y,y)f(\bar{y})d\bar{y}d\theta d\tau$$

So E_3 and E_3^o are equivalent at X_o. Let

$$\theta = (\theta_1,\theta') \in \mathbb{R}\times\mathbb{R}^{n-1}, \quad y = (y_1,y') \in \mathbb{R}\times\mathbb{R}^{n-1},$$
$$\bar{y} = (\bar{y}_1,\bar{y}') \in \mathbb{R}\times\mathbb{R}^{n-1} .$$

Then we have:

$$E_3^o f(t,y) = \int e^{i\langle y' - \bar{y}', \theta'\rangle} h(t,y,\bar{y},\theta')f(\bar{y})d\bar{y}d\theta'$$

where

$$(13)\quad h(t,y,\bar{y},\theta') = \int_{-t}^{t}\int e^{i\langle \frac{\tau+t}{2} + y_1 - \bar{y}_1, \theta_1\rangle} e_3(\tau,t,y,\theta_1,\theta')\,\varphi(t,y,\bar{y})d\theta_1 d\tau .$$

<u>Remark</u>: Note that (13) makes sense as an oscillatory integral (so as an usual integral) using the usual trick

of integration by parts argument, since in this case we have, that the only problem for making sense of (13) using the usual trick of integration by parts (see [H_2]) is the appearence of terms of the form:

$$a_1(t,y,\bar{y},\theta') = \int e^{i(y_1-\bar{y}_1)\theta_1} m_1(t,y,\theta_1,\theta')\varphi(t,y,\bar{y})d\theta_1$$

where $m_1 \in S^m(\mathbb{R}^n \times \mathbb{R}^{n-1})$ for some m or

$$a_2(t,y,\bar{y},\theta') = \int e^{i<t+y_1-\bar{y}_1,\theta_1>} m_2(t,y,\theta_1,\theta')\varphi(t,y,\bar{y})d\theta_1 \ .$$

Note that on $\operatorname{supp}\varphi$, $y_1 \neq \bar{y}_1$ and $\bar{y}_1 \neq t + y_1$, so we have

$$a_1(t,y,\bar{y},\theta') = \int \frac{1}{(y_1-\bar{y}_1)} e^{i(y_1-\bar{y}_1)\theta_1}(D_{\theta_1})m_1(t,y,\theta_1,\theta') \varphi(t,y,\bar{y})d\theta_1 \ .$$

Integrating by parts a sufficiently large number of times, we get $a_1 \in S^{-\infty}(\mathbb{R}^n \times \mathbb{R}^{n-1} \times \mathbb{R}^{n-2})$. In the same way we get $a_2 \in S^{-\infty}(\mathbb{R}^n \times \mathbb{R}^{n-1} \times \mathbb{R}^{n-2})$.

<u>Claim</u>: $h \in S^1(\mathbb{R}^n \times \mathbb{R}^{n-1} \times \mathbb{R}^{n-2})$.

<u>Proof</u>: By the remark, we can consider

$$h(t,y,\bar{y},\theta') = \int_{-t}^{t}\int e^{i\langle \tau+t+y_1+\bar{y}_1,\theta_1\rangle} e_3(\tau,t,y,\theta_1,\theta')$$
$$\varphi(t,y,\bar{y})d\theta_1 d\tau$$

as an usual integral. The result is then trivial considering that $e_3 \in S^1$, $(1+|\theta_1|+|\theta'|) \leq (1+|\theta_1|)\cdot(1+|\theta'|)$ and the term corresponding to $(1+|\theta_1|)$ is taking care by integration by parts in the oscillatory integral.
Also,

$$\frac{1}{(1+|\theta_1|+|\theta'|)^k} \leq \frac{1}{(1+|\theta'|)^k} .$$

So we have that E_3^o is a Fourier Integral Operator and

$$WF'E_3^o \subseteq \{((t,y,r,\xi);(\bar{y},\bar{\xi}) \in T'(\mathbb{R}^n \times \mathbb{R}^{n-1}) \mid y' = \bar{y}' ,$$
$$r = \xi_1 = \bar{\xi}_1 = 0 , \bar{\xi}' = \xi'\} = H_3 .$$

Note that H_3 is a Lagrangian submanifold of $T'(\mathbb{R}^n) \times T'(\mathbb{R}^{n-1})$ and it is obtained from $C_3(0)$ by eliminating the wedge $y_1 \leq \bar{y}_1 \leq y_1 + t$.

<u>Claim</u>:

$$\begin{pmatrix} D_t & 0 \\ 0 & D_t - D_{y_1} \end{pmatrix} h(t,y,\bar{y},\theta') = \int_{-t}^{t}\int e^{i\langle \frac{\tau+t}{2} + y_1 - \bar{y}_1,\theta_1\rangle} \varphi(t,y,\bar{y})$$
$$h_3(\tau,t,y,\theta_1,\theta')d\theta_1 d\tau$$

with

$$h_3(\tau,t,y,\theta_1,\theta') = \begin{pmatrix} D_t-D_\tau & 0 \\ 0 & D_t-D_{y_1}+D_\tau \end{pmatrix} e_3(\tau,t,y,\theta_1,\theta')$$

for $(t,y,\bar{y})$ near $(t_o,y_o,\bar{y}_o)$. The equality is mod $S^{-\infty}(\mathbb{R}^n \times \mathbb{R}^{n-1} \times \mathbb{R}^{n-2})$.

<u>Proof</u>: The proof follows immediately since there are no contributions from derivatives of φ near $(t_o,y_o,\bar{y}_o)$ and

$$D_t\ e^{i\langle \frac{\tau+t}{2} + y_1 - \bar{y}_1,\theta_1\rangle} = D_\tau\ e^{i\langle \frac{\tau+t}{2} + y_1 - \bar{y}_1,\theta_1\rangle}$$

$$(D_t - D_{y_1})e^{i\langle \frac{\tau+t}{2} + y_1 - \bar{y}_1,\theta_1\rangle} = -D_\tau\ e^{i\langle \frac{\tau+t}{2} + y_1 - \bar{y}_1,\theta_1\rangle} .$$

We have by construction that:

$$\begin{pmatrix} D_t-D_\tau & 0 \\ 0 & D_t-D_{y_1}+D_\tau \end{pmatrix} e_3^1 + \sigma_A(t,y,\theta_1,\theta')e_3^1 = 0$$

then using the claim we conclude that:

$$\begin{pmatrix} D_t & 0 \\ 0 & D_t-D_{y_1} \end{pmatrix} h(t,y,\bar{y},\theta') + \int_{-t}^{t}\int e^{i\langle \frac{\tau+t}{2} + y_1 - \bar{y}_1,\theta_1\rangle}\varphi(t,y,\bar{y})$$

$$\sigma_A(t,y,\theta_1,\theta')e_3 d\theta_1 d\tau = 0$$

in a neighborhood of $(t_o, y_o, \bar{y}_o)$. We have

$$h = h_1 + \sum_{j=0}^{-\infty} h_{-j} \mod S^{-\infty}(\mathbb{R}^n \times \mathbb{R}^{n-1} \times \mathbb{R}^{n-2}) .$$

Developing in Taylor series σ_A around $\theta_1 = 0$, we get:

$$(14) \quad \begin{pmatrix} D_t & 0 \\ 0 & D_t - D_{y_1} \end{pmatrix} h_1(t, y, \bar{y}, \theta') + \sigma_A(t, y, 0, \theta')h = 0$$

in a neighborhood of $(t_o, y_o, \bar{y}_o)$, because

$$\sigma_A(t, y, \theta_1, \theta') = (\sigma_A(t, y, 0, \theta') + \theta_1 \widetilde{a}_{-1}(t, y, \theta_1, \theta')$$

with $\widetilde{a}_{-1} \in S^{-1}(\mathbb{R}^n \times \mathbb{R}^n)$ and the term of the form:

$$\int_{-t}^{t} \int e^{i<\frac{\tau+t}{2} + y_1 - \bar{y}_1 > \theta_1} \varphi(t, y, \bar{y}) \theta_1 \widetilde{a}_{-1}(t, y, \theta_1, \theta')$$
$$e_3(\tau, t, y, \theta_1, \theta') d\theta_1 d\tau$$

$$= \int_{-t}^{t} \int 2 \frac{\partial}{\partial \tau} e^{i<\frac{\tau+t}{2} + y_1 - y_1 > \theta_1} \varphi(t, y, \bar{y}) \widetilde{a}_{-1}(t, y, \theta_1, \theta')$$
$$e_3(\tau, t, y, \theta_1, \theta') d\theta \, d\tau .$$

Then integrating by parts, we get:

$$= \int_{-t}^{t}\int 2e^{i<\frac{\tau+t}{2} + y_1 - \bar{y}_1>\theta_1} \varphi(t,y,\bar{y})\tilde{a}_{-1}(t,y,\theta_1,\theta') \frac{\partial}{\partial\tau} e_3\ (\tau,t,y,\theta_1,\theta')d\theta_1 d\tau$$

$$+2\int e^{i(t+y_1-\bar{y}_1)\theta_1}\varphi(t,y,\bar{y})\tilde{a}_{-1}(t,y,\theta_1,\theta')e_3(t,t,y,\theta_1,\theta')d\theta_1$$

$$-2\int e^{i(y_1-\bar{y}_1)\theta_1}\varphi(t,y,\bar{y})\tilde{a}_{-1}(t,y,\theta_1,\theta')e_3(-t,t,y,\theta_1,\theta')d\theta_1$$

$$= I_1 + I_2 + I_3 \quad .$$

Using the same arguments as in the proof of claim we can show

$$I_1 \in S^{o}(\mathbb{R}^n \times \mathbb{R}^{n-1} \times \mathbb{R}^{n-2}) \quad I_j \in S^{-\infty}(\mathbb{R}^n \times \mathbb{R}^{n-1} \times \mathbb{R}^{n-2})$$

$$j = 2,3 \ .$$

<u>Remarks</u>: a) Apparently we would get $h_1 = 0$, since $h_1(0,y,\bar{y},\theta') = 0$, but (14) is only valid in a neighborhood not intersecting $t = 0$.

b) Note that equation (14) says that if $\tilde{h}_1$ is the principal symbol of E_3^o, then

(15) $$\begin{pmatrix} H_{\tilde{p}_1} & 0 \\ 0 & H_{\tilde{p}_2} \end{pmatrix}\tilde{h}_1 + \tilde{C}_{3p}\tilde{h}_1 = 0$$

where $\tilde{p}_1$ is $p_1 = r$ lifted to $T'(\mathbb{R}^n) \times T'(\mathbb{R}^{n-1})$

$\tilde{p}_2$ is $p_2 = r - \xi_1$ lifted to $T'(\mathbb{R}^n) \times T'(\mathbb{R}^{n-1})$

and $\tilde{c}_{3p}$ is the pull back of the subprincipal symbol of

$P = \begin{pmatrix} D_t & 0 \\ 0 & D_t - D_{y_1} \end{pmatrix} + A(t,y,D_y)$ under the projection:

$H_3 \longrightarrow T'(\mathbb{R}^n)$. Then $E_3^o \in I^{1/4}(\mathbb{R}^{n-1}, \mathbb{R}^n, H_3)$.

-o-

We had by construction that

$$\begin{pmatrix} D_t & 0 \\ 0 & D_t \end{pmatrix} e_1^o + \sigma_A(t,y,\theta_1,\theta')e_1^o = 0 .$$

Then calling $\tilde{e}_1^o$ the principal symbol of E_1 and $\tilde{e}_{1p}$ the pull back of the subprincipal of

$P = \begin{pmatrix} D_t & 0 \\ 0 & D_t - D_{y_1} \end{pmatrix} + A(t,y,D_y)$ under the projection

$C_1^o(0) \longrightarrow T'(\mathbb{R}^n)$, $i = 1,2$, we have:

$$\begin{pmatrix} H_{\tilde{p}_1} & 0 \\ 0 & H_{\tilde{p}_1} \end{pmatrix} \tilde{e}_1 + \tilde{c}_{1p}\tilde{e}_1 = 0 .$$

In a similar way we get:

$$\begin{pmatrix} H_{\tilde{p}_2} & 0 \\ 0 & H_{\tilde{p}_2} \end{pmatrix}\tilde{e}_2 + \tilde{C}_{2p}\tilde{e}_2 = 0 .$$

So as a conclusion to 8(ii)(a), we have that E_1 , E_2 , are Fourier Integral Operators, E_3 is a Fourier Integral Operator out of the wedge of $C_3(0)$, the principal symbols of E_i , $i = 1,2$, satisfy the usual transport equation and the principal symbol of E_3^o satisfies a symmetric hyperbolic system. Note also that the order of E_3^o differs by 1 of the order of E_1 , E_2 .

(b) We have

$$\tilde{E}_1 f(t,y) = \int_0^t \int e^{i\langle y - \bar{y}, \theta\rangle} e_1(s,t,y,\bar{y},\theta) f(s,\bar{y}) d\bar{y} d\theta ds .$$

Let $\varphi(t,y,\bar{t},\bar{y}) \in C_o^\infty(\mathbb{R}^n \times \mathbb{R}^n)$

$$\operatorname{Supp} \varphi \subseteq \{(t,y,\bar{t},\bar{y}) \in \mathbb{R}^n \times \mathbb{R}^n \mid \bar{t} \neq 0 , \bar{t} \neq t\}$$

then we consider

$$\tilde{E}_1^o f(t,y) = \iint e^{i\langle y - \bar{y}, \theta\rangle} e_1(s,t,y,\theta)\varphi(t,y,s,\bar{y}) dy d\theta ds \; f(s,\bar{y})$$

It is clear then that if

$$X_o \in C_1 \quad \text{and} \quad X_o = ((t_o, y_o, r_o, \xi_o);(\bar{t}_o, \bar{y}_o, \bar{r}_o, \bar{\xi}_o))$$

with $\bar{t}_o \neq 0$, $t_o \neq \bar{t}_o$, $\varphi = 1$ near $(t_o, y_o, \bar{t}_o, \bar{y}_o)$ then $X^o \notin WF(\widetilde{E}_1 - \widetilde{E}_1^o)$. This is very natural since in the calculation of $WF'\widetilde{E}_1$ the terms Δ and $\widetilde{C}_1(0)$ came from the boundary contributions that are "killed" by φ , $\widetilde{E}_1^o \in I^{-\frac{1}{2}}(\mathbb{R}^n, \mathbb{R}^n, C_1)$.

We have that

$$\widetilde{E}_2^o f(t,y) = \iint e^{i\langle t + y - \bar{y}, \theta\rangle} e_2(s,t,y,\theta)\varphi(t,y,s,\bar{y}) f(s,\bar{y}) d\bar{y} d\theta .$$

Take $X_o = ((t_o, y_o, r_o, \xi_o);(\bar{t}_o, \bar{y}_o, \bar{r}_o, \bar{\xi}_o)) \in C_2$, $\varphi(t,y,\bar{t},\bar{y}) \in C_o^\infty(\mathbb{R}^n \times \mathbb{R}^n) = 1$ near $(t_o, y_o, \bar{t}_o, \bar{y}_o)$ s.t.

$$\operatorname{Supp} \varphi = \{(t,y,\bar{t},\bar{y}) \in \mathbb{R}^n \times \mathbb{R}^n | \ \bar{t} \neq 0 \ , \ \bar{t} \neq t\} .$$

We have again $X_o \notin WF(E_2 - \widetilde{E}_o)$ and $\widetilde{E}_2^o$ is a FIOP with $WF'\widetilde{E}_2^o \subset C_2$,

$$\widetilde{E}_2^o \in I^{-\frac{1}{2}}(\mathbb{R}^n, \mathbb{R}^n, C_2)$$

$$\widetilde{E}_3 f(t,y) = \int_0^t \int_{-t+s}^{t-s} e^{i\langle \frac{\tau - s + t}{2} + y - \bar{y}, \theta\rangle} e_3(s,\tau,t,y,\theta) f(s,y) \, d\bar{y} \, d\theta \, d\tau \, ds .$$

Let $X_o = ((t_o, y_o, r_o, \xi_o; \bar{t}_o, \bar{y}_o, \bar{r}_o, \bar{\xi}_o)) \in C_3$. Let $\varphi(t,y,\bar{t},\bar{y}) \in C_o^\infty(\mathbb{R}^n \times \mathbb{R}^n) = 1$ near X_o, and

$$\operatorname{Supp} \varphi \subseteq \{(t,y,\bar{t},\bar{y}) \mid \bar{t} = 0,\ t = \bar{t},\ t = 0,\ \bar{y}_1 \neq y_1 + t,\ \bar{y}_1 \neq y_1\}.$$

Let us consider

$$\widetilde{E}_3^o f(t,y) = \iint e^{i\langle y' - \bar{y}', \theta'\rangle} h(t,s,y,\bar{y},\theta') f(s,\bar{y}) d\bar{y} d\theta' ds$$

where

$$h(t,s,y,\bar{y},\theta') = \int_{-t+s}^{t-s}\int e^{i\langle \frac{\tau-s+t}{2} + y_1 - \bar{y}_1\rangle \theta_1} e_3(\tau,t,s,y,\theta_1,\theta') \varphi(t,y,s,\bar{y}) d\theta_1 d\tau .$$

With the same arguments as in (b), we have:

$$\widetilde{E}_3^o \in I^o(\mathbb{R}^n, \mathbb{R}^n, C_3) ,$$

and $$X_o \notin WF(E_3 - \widetilde{E}_3^o) .$$

To conclude this section, we have that $\widetilde{E}_1$, $\widetilde{E}_2$, $\widetilde{E}_3$ are FIOPS out of certain regions.

(c) Let $X_o = (t_o, y_o, r_o, \xi_o); (\bar{y}_o, \bar{\xi}_o) \in T'(\mathbb{R}^n) \times T'(\mathbb{R}^{n-1})$. Let us suppose that $\xi_o' \neq 0$. Let g be homogenous of degree 0 in θ , $g = 1$ near ξ_o , $\operatorname{essup} g \subseteq \{\theta \in \mathbb{R}^n | \ \theta_1 \neq 0\}$.

Claim: $X_o \notin WF'(E_3 - \tilde{E}_3')$ where

$$\tilde{E}_3' \colon C_o^\infty(\mathbb{R}^{n-1}) \longrightarrow C^\infty(\mathbb{R}^n) \qquad \text{and}$$

$$\tilde{E}_3' f(t,y) = \int_{-t}^{t} \int e^{i\langle \frac{t+\tau}{2} + y - \bar{y}, \theta\rangle} g(\theta) e_3(\tau, t, y, \theta) f(\bar{y}) d\bar{y} d\theta d\tau \ .$$

Proof: In the calculation of $WF'E_3$ (see 8(i)(a)) we have

$$WF'E_3 \subseteq \{(t,y,r,\xi);(\bar{y},\bar{\xi}) \in T'(\mathbb{R}^n) \times T'(\mathbb{R}^{n-1}) | \ \xi = \bar{\xi} ,$$
$$\xi \in \text{ess sup } e_3\}$$

since $\xi_o \notin \text{ess supp } e_3(1 - g(\theta))$ the claim is trivial. Now

$$\tilde{E}_3' f(t,y) = \int_{-t}^{t} \int \frac{\partial}{\partial \tau} e^{i\langle \frac{t+\tau}{2} + y - \bar{y}, \theta\rangle} 2 \, \frac{g(\theta)}{\theta_1} e_3(\tau, t, y, \theta)$$
$$f(\bar{y}) d\bar{y} d\theta d\tau \ .$$

Integrating by parts, we get:

$$\tilde{E}_3'f(t,y) = \int_{-t}^{t}\int e^{i\langle\frac{t+\tau}{2} + y - \bar{y}, \theta\rangle_2} \frac{g(\theta)}{\theta_1} \frac{\partial}{\partial\tau}e_3 f(\bar{y})d\bar{y}d\theta d\tau$$

$$+ \int e^{i\langle t + y - \bar{y}, \theta\rangle_2} \frac{g(\theta)}{\theta_1} e_3(t,t,y,\theta)f(\bar{y})d\bar{y}d\theta$$

$$- \int e^{i\langle y - \bar{y}, \theta\rangle_2} \frac{g(\theta)}{\theta_1} e_3(-t,t,y,\theta)f(\bar{y})d\bar{y}d\theta \quad .$$

Repeating this procedure a sufficiently large number of times, we get:

$$\tilde{E}_3' = I_1 + I_2 \qquad I_i \in I^{-1/4}(\mathbb{R}^n, \mathbb{R}^{n-1}, C_i(0)) \ , \quad i = 1,2 \quad .$$

CHAPTER II

THE CAUCHY PROBLEM

1. Parametrix for the Cauchy problem for a strictly hyperbolic differential operator.

In this section we only intend to give an outline of the construction in $\mathbb{R}^n$, with the purpose of motivating section 4 of Chapter I and section 3 of this chapter. For further details we refer to [CH_1], [D], and [H_3].

We will denote by $(t,y) \in \mathbb{R} \times \mathbb{R}^{n-1}$ the variables in $\mathbb{R}^n$ and $(r,\xi) \in \mathbb{R} \times \mathbb{R}^{n-1}$ the dual variables in $T^*(\mathbb{R}^n)$.

Let P be a differential operator with C^∞ coefficients of degree $m \geq 1$ and let us assume that its principal symbol p has the form:

$$(1) \quad p(t,y,r,\xi) = (r - \lambda_0(t,y,\xi))\dots(r - \lambda_{m-1}(t,y,\xi))$$

where λ_i are homogenous function of degree 1 in ξ, $i = 0,\dots,m-1$ and

$$(2) \quad \lambda_i(t,y,\xi) \neq \lambda_j(t,y,\xi) \quad \text{for } i \neq j, \ 0 \leq i,j \leq m-1, \ \xi \neq 0 .$$

(2) and (1) imply that $\lambda_i \in C^\infty(\mathbb{R} \times T'(\mathbb{R}^{n-1}))$, $i = 0,\dots,n-1$, because $p \in C^\infty(T'(\mathbb{R}^n))$. Note also that (1) implies that the hypersurface $t = 0$ is non characteristic for P.

Let us denote $p_i(t,y,r,\xi) = r - \lambda_i(t,y,\xi)$, $i = 0,\dots,m-1$ and $P_i \in L^1(\mathbb{R}^n)$ a pseudodifferential operator with principal symbol p_i, $i = 0,\dots,m-1$.

Let us consider:

(3) $$C_i(0) = \{((t,y,r,\xi);(\bar{y},\bar{\xi})) \in T'(\mathbb{R}^n) \times T'(\mathbb{R}^{n-1}) \mid (t,y,r,\xi) \text{ is in the same bicharacteristic strip of } H_{p_i} \text{ as } (0,\bar{y},\lambda_i(\bar{t},\bar{y},\bar{\xi}),\bar{\xi})\},\quad i = 0,\dots,m-1$$

we have $C_i(0) = C_i \circ R(0)$ where $R(0)$ is the canonical relation associated to the Fourier Integral operator γ_0, where γ_0 is the restriction to the hypersurface $t = 0$, and

$$C_i = \{((t,y,r,\xi);(\bar{t},\bar{y},\bar{r},\bar{\xi})) \in T'(\mathbb{R}^n) \times T'(\mathbb{R}^n) \mid (t,y,r,\xi) \text{ is in the same bicharacteristic strip of } H_{p_i} \text{ as } (\bar{t},\bar{y},\bar{r},\bar{\xi})\}, i=0,\dots,m-1 .\}$$

Assumption (1) implies that $\mathbb{R}^n$ is pseudoconvex with respect to P, so we have that C_i are canonical relations $i = 0,\dots,m-1$ (see [D-H]).

$C_i(0)$ are canonical relations $i = 0,\dots,m-1$ and a local coordinate system for $C_i(0)$ is given by:
Let $\varphi_i(t,y,\xi)$ be a C^∞ homogenous of degree 1 function in ξ, solution of:

$$(4)\qquad \begin{cases} \dfrac{\partial\varphi_i}{\partial t} = \lambda_i(t,y,d_y\varphi_i) \\ \\ \gamma_0\varphi_i = \langle y,\xi\rangle \end{cases} \qquad i = 0,\dots,m-1$$

in a conic neighborhood Γ of $(0,y_0,\xi_0) \in \mathbb{R}\times T'(\mathbb{R}^{n-1})$, then

$$(5)\quad F: \Gamma \longrightarrow C_i(0)$$

$$(t,y,\xi) \longrightarrow ((t,y,d_t\varphi_i,d_y\varphi_i);(d_\xi\varphi_i,\xi))$$

is a local diffeomorphism.

<u>Definition 1.1</u>: We call E a parametrix for the Cauchy problem for P, if

$$E = \sum_{j=0}^{m-1} E_j\ , \qquad E_j \text{ are Fourier Integral Operators,}$$

$E_j \in I^{-j-\frac{1}{4}}(\mathbb{R}^n, \mathbb{R}^{n-1}, C_i(0))$, $j = 0,\ldots,m-1$, satisfying

$$(6) \quad \begin{cases} PE_j \in C^\infty(\mathbb{R}^{n-1} \times \mathbb{R}^n) \\ \gamma_0\left(\frac{\partial}{\partial t}\right)^k E_j = \delta_{kj}\,\mathrm{Id} \mod L^{-\infty}(\mathbb{R}^{n-1}) \\ \qquad\qquad j = 0,\ldots,m-1 , \end{cases}$$

$$\delta_{kj} = \begin{cases} 1 & k = j \\ 0 & k \neq j \end{cases} .$$

See $[CH_1]$ for the construction of a solution for the Cauchy problem for P from the E_j satisfying (6) .

We will consider examples to motivate the appearence of the E_j and its relation with the Cauchy problem, as well as to give the main ideas of their construction.

(i) On $\mathbb{R}^2$ let $P = D_t - D_y$. Let us consider the Cauchy problem

$$(7) \quad \begin{cases} Pu = 0 \\ \gamma_0 u = f \end{cases} \qquad f \in C_0^\infty(\mathbb{R}) .$$

The solution u is given by:

(8) $u(t,y) = \int e^{i(t+y)\xi} \hat{f}(\xi)d\xi = f(t+y)$.

Let us consider the operator E that maps the Cauchy data f into the solution u , so we have:

(9) $$Ef(t,y) = \int e^{i(t+y)\xi}\hat{f}(\xi)d\xi \quad .$$

From (9) it is clear that E is a Fourier Integral Operator and

(10) $$\begin{cases} PE = 0 \\ \gamma_o E = \mathrm{Id} \end{cases} .$$

also

$$\begin{cases} \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial y}\right)\varphi = 0 \\ \varphi(0,y,\xi) = \langle y,\xi\rangle \end{cases}$$

with $\varphi(t,y,\xi) = (t+y)\xi$

and $WF'E = \{((t,y,r,\xi);(\bar{y},\bar{\xi})) \in T'(\mathbb{R}^2) \mid \bar{y} = d_\xi\varphi(t,y,\bar{\xi}),$

$$r = d_t\varphi \ , \ \xi = d_y\varphi\} \quad .$$

(Observe relation with (4) and (5) of this section). This example is very particular as it will be shown in example (ii) since (10) is an exact equality and the amplitude of

E is equal to 1.

(ii) Let $P = D_t - \lambda(t,y,D_y)$ on $\mathbb{R}^n$ where $\lambda(t,y,\xi) \in C^\infty(\mathbb{R} \times T'(\mathbb{R}^{n-1}))$ is homogenous of degree 1 in ξ and $\lambda(t,y,D_y)$ is a pseudodifferential operator in $\mathbb{R}^{n-1}$ smooth in t .

Let us try as in (10) to find an operator E of the form

$$Ef(t,y) = \int e^{i\varphi(t,y,\xi)}\, \hat{f}(\xi)d\xi \ , \quad f \in C_o^\infty(\mathbb{R}^{n-1})$$

with φ satisfying:

$$\frac{\partial\varphi}{\partial t} = \lambda(t,y,d_y,\varphi)$$
$$\varphi(0,y,\xi) = \langle y,\xi\rangle$$

We have: $PEf = \int P(e^{i\varphi(\cdot,\cdot,\xi)})\hat{f}(\xi)d\xi$. So we need:

(11) $$P(e^{i\varphi(\cdot,\cdot,\xi)}) = 0 \ .$$

(11) is not satisfied in general, because we may have contributions on S^o or lower order from $\lambda(t,y,D_y)\,(e^{i\varphi})$.

The way to "kill" these lower order terms is introducing an amplituded $a \in S^o(\mathbb{R}^n \times \mathbb{R}^{n-1})$ in E :

$$Ef(t,y) = \int e^{i\varphi(t,y,\xi)} a(t,y,\xi)\hat{f}(\xi)d\xi \quad .$$

We need now to solve:

$$(12) \qquad e^{-i\varphi}P(ae^{i\varphi}) = 0 \ .$$

Because of the asymtotic expansion of $e^{-i\varphi}P(ae^{i\varphi})$ (see [D]) we would have to solve an infinite number of differential equations along characteristics (of $D_t - \lambda(t,y,D_y)$) with initial condition $a(0,y,\xi) = 1$.

For avoiding this we put

$$(13) \qquad a \approx \sum_{j=0}^{\infty} a_{-j} \qquad a_{-j} \in S^{-j}(\mathbb{R}^n \times \mathbb{R}^{n-1}) \quad .$$

Pluging (13) into (12) we have now to solve for each a_{-j}, $j \geq 1$ an inhomogenous differential equation along the characteristics with 0 initial condition. For a_o we have to solve an homogenous differential equation along the characteristics with initial condition: $a_o(o,y,\xi) = 1$.

So what it is possible to find in this example is a Fourier Integral Operator satisfying:

$$\begin{cases} PE \in C^{\infty}(\mathbb{R}^{n-1} \times \mathbb{R}^n) \\ \gamma_o E = Id \bmod L^{-\infty}(\mathbb{R}^{n-1}) \end{cases} \quad .$$

To motivate the appearence of the same number of Fourier Integral Operators as characteristic roots λ_i, $i = 0,\dots,m-1$, in the decomposition (1) of p, we consider:

(iii) Let $P = \frac{\partial^2}{\partial t^2} - \Delta$ where $\Delta = \frac{\partial^2}{\partial y_1^2} + \dots + \frac{\partial^2}{\partial y_{n-1}^2}$ is the Laplacian in $n-1$ dimensional space. $n \geq 2$.

The solution of the Cauchy problem:

$$\begin{cases} Pu = 0 \\ \gamma_o u = 0 \\ \gamma_o \frac{\partial}{\partial t} u = f \end{cases} \qquad f \in C_o^\infty(\mathbb{R}^{n-1})$$

is
$$u(t,y) = \int e^{i\{<y,\xi> + t|\xi|\}} \frac{1}{2i|\xi|} \hat{f}(\xi)d\xi - \int e^{i\{<y,\xi> - t|\xi|\}} \frac{1}{2i|\xi|} \hat{f}(\xi)d\xi .$$

Let
$$\varphi_1(t,y,\xi) = <y,\xi> + t|\xi|$$
$$\varphi_1(t,y,\xi) = <y,\xi> - t|\xi| ,$$

then
$$\frac{\partial\varphi_1}{\partial t} = |\xi| \qquad \frac{\partial\varphi_2}{\partial t} = -|\xi|$$
$$\gamma_o\varphi_1 = <y,\xi> \qquad \gamma_o\varphi_2 = <y,\xi> .$$

$|\xi|$ is the principal symbol of the square root of the Laplacian $\sqrt{\Delta}$ and the principal symbols of P and $(\frac{\partial}{\partial t} - \sqrt{\Delta})(\frac{\partial}{\partial t} + \sqrt{\Delta})$ coincide. We have that in this example $\lambda_1(\xi) = |\xi|$ and $\lambda_2(\xi) = -|\xi|$, and the map that sends the Cauchy data to the solution is a sum of two Fourier Integral Operators.

A strictly hyperbolic operator P (i.e. p satisfies (1)) behaves "essentially" as $\tilde{P}$ with

$$\tilde{P} = (D_t - \lambda_o(t,y,D_y))\dots(D_t - \lambda_{m-1}(t,y,D_y)) \quad .$$

The essential features of the construction of the E_j have been indicated in the examples. We put:

$$E_j f(t,y) = \int e^{i\varphi_j(t,y,\xi)} a_j(t,y,\xi)\hat{f}(\xi)d\xi \ , \ f \in C_o^\infty(\mathbb{R}^{n-1})$$

with φ_j satisfying (4), $j = 0,\dots,m-1$. The principal symbol of the E_j will satisfy the differential equation ("along characteristics")

$$(14)\quad H_{\tilde{p}_j}\tilde{e}_j + \tilde{c}_j\tilde{e}_j = 0 \quad \text{on} \quad C_j(0) \ , \quad j = 0,\dots,m-1 \ ,$$

where $\tilde{p}_j$ is the lifting of p_j to $T'(\mathbb{R}^n)\times T'(\mathbb{R}^{n-1})$ and $\tilde{c}_j$ is the pullback of the subprincipal of P_j to $C_j(0)$ under the projection

$$C_j(0) \longrightarrow T'(\mathbb{R}^n) .$$

Initial conditions for e_j are determined from the condition $\gamma_o(\frac{\partial}{\partial t})^k E_j = \delta_{kj} Id$, and it is possible to satisfy them because the characteristic roots are different.

A very important motivation of the construction of (6) is the paper of Lax (see [La] where an approximative solution is constructed.

Chazarin succeeded in constructing a parametrix for the Cauchy problem for hyperbolic operators P with characteristic roots of constant multiplicity if P satisfies the Levi condition, with a slight modification of (6) (what changes are the number of Fourier Integral Operators and their order) (see [CH_1]).

Flaschka and Strang had shown before [CH_1] that the Levi condition is necessary for the C^∞ well possedeness of the Cauchy problem for hyperbolic differential operators with characteristic roots of constant multiplicity, using a modification of Lax construction in [La] (see [F-S]).

Equations (14) are called transport equations.

2. Ivrii-Petkov result.

We will state in this section a result of Ivrii-

Petkov, related to condition (iii) of Chapter 1, Section 1, and therefore to the Levi condition according to I. Proposition 2.3. For the proofs see [I-V] and also the very nice exposition of Hörmander (see [H_1]) of the Ivrii-Petkov paper.

Let P be a differential operator with C^∞ coefficients in $\Omega \subset \mathbb{R}^n$, Ω open $n \geq 2$. Coordinates are denoted by $x = (t,y) \in \mathbb{R} \times \mathbb{R}^{n-1}$ and dual variables by $\eta = (r,\xi) \in \mathbb{R} \times \mathbb{R}^{n-1}$.

Let $\Omega_{t'} = \{x \in \Omega \mid t < t'\}$.

<u>Definition 2.1</u>: The Cauchy problem is said to be correctly posed in $\Omega_{t'}$ if

(a) $\forall\, f \in C_0^\infty(\Omega)$, $\exists\, u \in \mathcal{E}'(\Omega)$ with $Pu = f$ in $\Omega_{t'}$.

(b) $u \in \mathcal{E}'(\Omega)$ and $Pu = 0$ in $\Omega_{t'} \Rightarrow u = 0$ in $\Omega_{t'}$.

Let $(x_o, \eta_o) \in T'(\Omega)$, and A, B, C the matrices:

$$A = \left(\frac{\partial^2 p}{\partial \eta_i \partial \eta_j}(x_o, \eta_o)\right), \quad B = \left(\frac{\partial^2 p}{\partial x_i \partial \eta_j}(x_o, \eta_o)\right),$$

$$C = \left(\frac{\partial^2 p}{\partial x_i \partial x_j}(x_o, \eta_o)\right), \quad 1 \leq i,j \leq n .$$

Let $u = (x,\eta) \in T^*(\Omega)$, $v = (\bar{x},\bar{\eta}) \in T^*(\Omega)$. Let Q be the symmetric bilinear form on $T^*(\Omega) \times T^*(\Omega)$ defined by:

$$(1) \quad Q(u,v) = \tfrac{1}{2}\langle A\eta,\bar{\eta}\rangle + \tfrac{1}{2}\langle Bx,\bar{\eta}\rangle + \tfrac{1}{2}\langle Cx,\bar{x}\rangle \quad .$$

$\langle\ ,\ \rangle$ is the scalar product in $\mathbb{R}^n$. Let $F\colon T^*\Omega \to T^*(\Omega)$ be the linear map given by

$$(2) \qquad Q(u,v) = \sigma(u,Fv)$$

where σ is the canonical 2-form in $T^*(\Omega)$ i.e. in local coordinates

$$\sigma = \sum_{i=1}^{n} d\eta_i \wedge dx_i \quad .$$

<u>Proposition 2.2</u>: Let $(x_o,\eta_o) \in T^*(\Omega)$ such that:

$$p_1(x_o,\eta_o) = p_2(x_o,\eta_o) = 0 \quad \text{where}$$

$$p(x_o,\eta_o) = p_1(x_o,\eta_o)p_2(x_o,\eta_o)$$

then $\quad \{p_1,p_2\}(x_o,\eta_o) = 0 \Rightarrow F^2 = 0$.

<u>Theorem 2.3</u>: Let Ω be an open set in $\mathbb{R}^n$, let $x_o \in \Omega$, and assume the Cauchy problem is correctly possed in Ω_t

for t near t_o if $x_o = (t_o, y_o)$. Assume that:

$$p(x_o, \eta_o) = 0 \ , \quad \frac{\partial p}{\partial r}(x_o, \eta_o) = 0 \ , \quad \frac{\partial^2 p}{\partial r^2}(x_o, \eta_o) < 0$$

Let F be the linear map corresponding to Q (see (1) and (2)), then if F has no real eigenvalues different from zero, then:

$$|C_p(x_o, \eta_o)| \leq \sum u_j$$

where iu_j are the eigenvalues of F on the positive imaginary axis repeated according to their multiplicity and C_p is the subprincipal symbol of P .

<u>Corollary 2.4</u>: $F^2 = 0 \Rightarrow C_p(x_o, \eta_o) = 0$.

So by Proposition 2.2 and Corollary 2.4 a necessary condition for the well possedeness of the Cauchy problem in the sense of Definition 2.1 is that $C_p(x_o, \eta_o) = 0$ at points where $p_1(x_o, \eta_o) = p_2(x_o, \eta_o) = 0$ if $P = P_1 P_2 + Q$ $\{p_1, p_2\}(x_o, \eta_o) = 0$, P_i with simple characteristics with respect to r i.e. $d_r p_i(x_o, \eta_o) \neq 0$, $i = 1,2.$

3. Cauchy problem for symmetric hyperbolic systems with double characteristics.

Let Y be an open set in $\mathbb{R}^{n-1}$, $n \geq 2$. Let $X = \mathbb{R} \times Y$. Variables in Y will be denoted by (t,y) and dual variables by $(r, \xi) \in \mathbb{R} \times \mathbb{R}^{n-1}$. Let

$$(1) \quad P = \begin{pmatrix} D_t - \lambda_1(t,y,D_y) & 0 \\ 0 & D_t - \lambda_2(t,y,D_y) \end{pmatrix} + D(t,y,D_y)$$

where $\lambda_i(t,y,\xi) \in C^\infty(\mathbb{R} \times T'(Y))$ are real valued homogenous functors of degree 1 in ξ, $i = 1,2$, and $\lambda_i(t,y,D_y)$ are pseudodifferential operators in Y depending smoothly on t. $D(t,.,.)$ is in $L^0(Y)$, smooth in t. All pseudodifferential operators will be assumed to be classical ones and properly supported.

(i) Reduction to simpler case.

Let $\varphi \in C^\infty(\mathbb{R} \times T'(Y))$ satisfy

$$(2) \qquad \begin{cases} \dfrac{\partial \varphi}{\partial t} = \lambda_1(t,y,d_y\varphi) \\ \gamma_0 \varphi = \langle y, \xi \rangle \end{cases}$$

in a conic neighborhood of $(0, y_0, \xi_0) \in \mathbb{R} \times T'(Y)$. Let

$$(3) \qquad \Phi(t,y,r,\xi) = \varphi(t,y,\xi) + tr .$$

Let χ be the canonical transformation defined in a conic neighborhood V of $(0,y_o,r_o,\xi_o) \in T'(X)$ with $\xi_o \neq 0$, to $T'(\mathbb{R}^n)$ defined by

$$(4) \quad \chi(t,y,d_t\Phi(t,y,r,\xi),d_y\Phi(t,y,r,\xi)) = (d_r\Phi,d_\xi\Phi,r,\xi) .$$

Note that

$$(5) \quad \begin{cases} \Phi(0,x,r,\xi) = \langle y,\xi\rangle \\ d_r\Phi(t,y,r,\xi) = t \\ d_y\Phi(0,y,r,\xi) = \xi \\ d_t\Phi(t,y,r,\xi) = r + \lambda_1(t,y,d_y\Phi) . \end{cases}$$

We are going to denote also by (t,y,r,ξ) coordinates in $T'(\mathbb{R}^n)$. Let $\chi(0,y_o,r_o,\xi_o) = (0,\bar{y}_o,\bar{r}_o,\bar{\xi}_o)$. Let $A \in I^o((\mathbb{R}\times Y)\times\mathbb{R}^n,\Gamma')$ where Γ is a closed conic subset of the graph of χ, be defined by:

$$(6) \quad Af(t,y) = \int e^{i\varphi(t,y,\xi)-\langle\bar{y},\xi\rangle} a(t,y,\bar{y},\xi)f(t,\bar{y})d\bar{y}d\xi$$

where $a \in S^o(\mathbb{R}\times Y\times\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$. Let $B \in I^o((\mathbb{R}\times Y)\times\mathbb{R}^n,(\Gamma^{-1})')$ be such that

$$(0,y_0,r_0,\xi_0) \notin WF(AB - Id_X)$$

$$(0,\bar{y}_0,\bar{r}_0,\bar{\xi}_0) \notin WF(BZ - Id_{\mathbb{R}^n}) \ .$$

Because of (6) we have

$$\gamma_0 A = \tilde{A}\gamma_0 \quad \text{where} \quad \tilde{A} \in L^0(X_0) \ ,$$

$X_0 = \{(0,y) \in \mathbb{R} \times Y\}$ and

$$\tilde{A}g(y) = \int e^{i\langle y-\bar{y},\xi\rangle} a(0,y,\bar{y},\xi) g(\bar{y}) d\bar{y} d\xi \ .$$

B can be chosen so that:

(7) $$\gamma_0 B = \tilde{B}\gamma_0 \ , \qquad \tilde{B} \in L^0(\mathbb{R}^{n-1}) \ ,$$

$\mathbb{R}^{n-1} = \{(0,y) |\ y \in \mathbb{R}^{n-1}\}$. Note that we can chose $\tilde{A}$, $\tilde{B}$ elliptic near $(y_0,\xi_0) \in T'(Y)$, $(\bar{y}_0,\bar{\xi}_0)$ respectively.

<u>Proposition 3.1</u>: The principal symbol of

$\tilde{P} = BPA$ is near $(0,\bar{y}_0,\bar{r}_0,\bar{\xi}_0)$.

$$\tilde{P} = \begin{pmatrix} r & 0 \\ 0 & r-\tilde{\lambda}_2(t,y,\xi) \end{pmatrix}$$ for some $\tilde{\lambda}_2 \in C^\infty(\mathbb{R} \times T'(\mathbb{R}^{n-1}))$ homogenous of degree 1 in ξ.

Proof: We have to show:

$$p(t,x,d_t\Phi(t,y,r,\xi),d_y\Phi(t,y,r,\xi)) = \widetilde{p}(t,d_\xi\Phi,r,\xi) .$$

Now

$$p(t,y,d_t\Phi(t,y,r,\xi)d_y\Phi(t,y,r,\xi)) =$$

$$\begin{pmatrix} d_t\Phi-\lambda_1(t,y,d_y\Phi) & 0 \\ 0 & d_t\Phi-\lambda_2(t,y,d_y\Phi) \end{pmatrix} ,$$

$$\widetilde{p}(t,d_\xi\Phi(t,y,r,\xi),r,\xi) = \begin{pmatrix} r & 0 \\ 0 & r-\widetilde{\lambda}_2(t,d_\xi\Phi(t,y,r,\xi),\xi) \end{pmatrix} .$$

Now because of (5) we have our claim.

Q.E.D.

Note that $\widetilde{\lambda}_2(t,d_\xi\Phi(t,y,r,\xi),\xi) = \lambda_2(t,y,d_y\Phi(t,y,r,\xi) - \lambda_1(t,y,d_y\Phi(t,y,r,\xi)$

and we have named $y = d_\xi\Phi$, $\xi = d_y\Phi$.

Remark: $R = \begin{pmatrix} D_t-\lambda_1(t,y,D_y) & 0 \\ 0 & D_t-\lambda_2(t,y,D_y) \end{pmatrix}$ is not a pseudodifferential operator on $\mathbb{R}\times Y$, because it is not pseudolocal (see [N]). However $WF'R$ outside the diagonal in $T'(\mathbb{R}\times Y)$ only contains points of the form $((t,y,r,0);(t,y,r,0))$ but taking BRA those points do not contribute to $WF'(BRA)$ (see [D]) because we have

$\xi_o \neq 0$.

So we are reduced to study the operator

$$(8) \qquad \tilde{P} = \begin{pmatrix} D_t & 0 \\ 0 & D_t - \tilde{\lambda}_2(t,y,D_y) \end{pmatrix} + \tilde{D}(t,y,D_t,D_y)$$

$$\tilde{D} \in L^o(\mathbb{R}^n) \quad ,$$

using the same argument as in Chapter I. Proposition 3.2 developing in Taylor series around $r = 0$ in the first row and $r = \tilde{\lambda}_2(t,y,\xi)$ in the second row, we can find an elliptic operator $C \in L^o(\mathbb{R}^n)$ and $\tilde{D}(t,y,D_y) \in L^o(\mathbb{R}^{n-1})$ smooth in t such that:

$$(9) \qquad \tilde{P} = C\left[\begin{pmatrix} D_t & 0 \\ 0 & D_t - \tilde{\lambda}_2(t,y,D_y) \end{pmatrix} + \tilde{D}(t,y,D_y)\right] \quad .$$

(Indeed in Proposition 3.2 we got a C in the right hand side of the right hand side of (9), but taking real transposes we can get (9).) Let

$$(10) \quad L = \begin{pmatrix} D_t & 0 \\ 0 & D_t - \tilde{\lambda}_2(t,y,D_y) \end{pmatrix} + \tilde{D}(t,y,D_y) \quad .$$

(ii) <u>Construction of parametrix for the Cauchy problem for</u> L .

In this paragraph we are not going to give as many details as in I.4 since the construction is along the same lines.

Let $E\colon C_o^\infty(\mathbb{R}^{n-1}) \longrightarrow C^\infty(\mathbb{R}^n)$ be defined by $E = E_1 + E_2 + E_3$ where:

$$E_1 f(t,y) = \int e^{i\langle y,\theta\rangle} e_1(t,y,\theta)\hat{f}(\theta)d\theta$$

$$E_2 f(t,y) = \int e^{i\varphi_2(t,y,\theta)} e_2(t,y,\theta)\hat{f}(\theta)d\theta$$

$$E_3 f(t,y) = \int_{-t}^{t}\int e^{i\varphi_3(\tau,t,y,\theta)} e_3(\tau,t,y,\theta)\hat{f}(\theta)d\theta d\tau$$

where

(11) $$\begin{cases} \dfrac{\partial\varphi}{\partial t} = \tilde{\lambda}_2(t,y,d_y\varphi_2) \\ \gamma_o\varphi_2 = \langle y,\theta\rangle \end{cases}$$

and

(12) $$\varphi_3(\tau,t,y,\theta) = \varphi_2(\tfrac{t+\tau}{2},y,\theta) \quad .$$

Note that:

(13) $$\begin{cases} \varphi_3(-t,t,y,\theta) = \langle y,\theta\rangle \\ \varphi_3(t,t,y,\theta) = \varphi_2(t,y,\theta) \\ \dfrac{\partial}{\partial t}\varphi_3(\tau,t,y,\theta) = \dfrac{\partial}{\partial\tau}\varphi_3(\tau,t,y,\theta) \\ \dfrac{\partial}{\partial t} - \tilde{\lambda}_2(t,y,d_y\varphi_3) = -\dfrac{\partial}{\partial\tau}\varphi_3 \quad . \end{cases}$$

$e_i \approx \sum_{j=0}^{\infty} e_i^{-j}$, e_i^{-j} homogenous of degree $-j$, $i = 1,2$, are chosen so that:

$$(14) \quad \left(\begin{pmatrix} D_t & 0 \\ 0 & D_t \end{pmatrix} + \tilde{D}(t,y,D_y)\right)E_1 \in C^\infty(\mathbb{R}^{n-1} \times \mathbb{R}^n)$$

$$(15) \quad \left(\begin{pmatrix} D_t-\tilde{\lambda}_2(t,y,D_y) & 0 \\ 0 & D_t-\tilde{\lambda}_2(t,y,D_y) \end{pmatrix} + \tilde{D}(t,y,D_y)\right)E_2 \in C^\infty(\mathbb{R}^{n-1} \times \mathbb{R}^n)$$

$$(16) \qquad \gamma_o(E_1 + E_2) = Id \quad .$$

$e_3 = e_3^1 + \sum_{j=0}^{\infty} e_3^{-j}$, e_3^1 homogenous of degree 1 and e_3^{-j} homogenous of degree $-j$ are chosen so that:

$$(17) \quad \left(\begin{pmatrix} D_t-D_\tau & 0 \\ 0 & D_t+D_\tau-\tilde{\lambda}_2^\infty(t,y,D_y) \end{pmatrix} + \tilde{D}(t,y,D_y)\right)E_3' \in C^\infty(\mathbb{R}^{n-1} \times \mathbb{R}^{n+1})$$

where E_3': $C_o^\infty(\mathbb{R}^{n-1}) \longrightarrow C^\infty(\mathbb{R}^{n+1})$, and

$$E_3'f(\tau,t,y) = \int e^{i\varphi_3(\tau,t,y,\theta)} e_3(\tau,t,y,\theta)\hat{f}(\theta)d\theta \ ,$$

$$f \in C_o^\infty(\mathbb{R}^{n-1}) \ .$$

Let $\tilde{\tilde{P}} = \begin{pmatrix} D_t-D_\tau & 0 \\ 0 & D_t+D_\tau-\tilde{\lambda}_2(t,y,D_y) \end{pmatrix}$ then

$\tilde{\tilde{p}}(\tau, t, y, d_\tau\varphi_3, d_t\varphi_3, d_y\varphi_3) = 0$.

Initial conditions are given for e_3 at $\tau = t$ and $\tau = -t$ by:

$$(18)\quad \begin{pmatrix} e_3^{11}(t,t,y,\theta) & e_3^{12}(t,t,y,\theta) \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \tilde{\lambda}_2(t,y,D_y) & 0 \\ 0 & 0 \end{pmatrix}(e^{i\varphi_2}e_2) \mod S^{-\infty} ,$$

and

$$(19)\quad \begin{pmatrix} 0 & 0 \\ e_3^{21}(-t,t,y,\theta) & e_3^{22}(-t,t,y,\theta) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -\tilde{\lambda}_2 & 0 \end{pmatrix}(e^{i\langle y,\theta\rangle}e_1) \mod S^{-\infty} .$$

The symmetric hyperbolic system that we have to solve for e_3 in order to have (17) is:

$$\left(\begin{pmatrix} D_t - D_\tau & 0 \\ 0 & D_t + D_\tau - \sum_{k=1}^{n-1} \frac{\partial\tilde{\lambda}_2}{\partial\xi_k}\frac{\partial}{\partial y_k} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ q & 0 \end{pmatrix}\right) e_3 + \tilde{d}(t,y,d_y\varphi_3)e_3^1 = 0 ,$$

where:

$$\tilde{\lambda}_2(t,y,D_y)(e^{i\varphi_3}e_3) = \tilde{\lambda}_2(t,y,d_y\varphi_3) + \sum_{k=1}^{n-1} \frac{\partial\tilde{\lambda}_2}{\partial\xi_k}\frac{\partial}{\partial y_k}\, e_3 + qe_3$$

+ lower order terms.

Note also, that $\left(D_t + D_\tau - \sum_{k=1}^{n-1} \frac{\partial\tilde{\lambda}_2}{\partial\xi_k}\frac{\partial}{\partial y_k}\right)(\varphi_3) = 0$.

So the transport equations for the e_i , $i = 1,2$, are the usual ones, i.e. calling $\tilde{e}_i$ the principal symbol of E_i , $i = 1,2$.

$$(20) \qquad H_{\tilde{p}_i}\tilde{e}_i + \tilde{C}_{p_i} e_i = 0 \quad \text{on} \quad C_i(0)$$

where $\tilde{p}_i$ is the lifting of p_i to $T'(\mathbb{R}^{n-1}) \times T'(\mathbb{R}^n)$, $p_1 = r$, $p_2 = r - \tilde{\lambda}_2(t,y,\xi)$. $\tilde{C}_{p_1}$ is the pull back of the subprincipal symbol of

$\begin{pmatrix} D_t & 0 \\ 0 & D_t \end{pmatrix} + \tilde{D}(t,y,D_y)$ under the projection $C_1(0) \to T'(\mathbb{R}^n)$.

$\tilde{C}_{p_2}$ is the pull back of the subprincipal symbol of

$\begin{pmatrix} D_t - \tilde{\lambda}_2 & 0 \\ 0 & D_t - \tilde{\lambda}_2 \end{pmatrix} + \tilde{D}(t,y,D_y)$ under the projection

$C_2(0) \to T'(\mathbb{R}^n)$. $C_i(0)$ as in II.1.(2) with $\lambda_1 = 0$, $\lambda_2(t,y,\xi) = \tilde{\lambda}_2(t,y,\xi)$. Let $\tilde{e}_3$ be the principal symbol of E_3' , then it satisfies:

$$(20) \quad \begin{pmatrix} H_{\tilde{\tilde{p}}_1} & 0 \\ 0 & H_{\tilde{\tilde{p}}_2} \end{pmatrix} + C_{\tilde{\tilde{p}}}\tilde{e}_3 = 0 \quad \text{on} \quad \tilde{C}_3(0)$$

where $\widetilde{C}_3(0)$ is the canonical relation defined by E_3' and $\tilde{\tilde{p}}_1$ is the lifting of $r - m$ to $T'(\mathbb{R}^{n+1}) \times T'(\mathbb{R}^{n-1})$. $\tilde{\tilde{p}}_2$ is the lifting of $r + m - \widetilde{\lambda}_2$ to $T'(\mathbb{R}^{n+1}) \times T'(\mathbb{R}^{n-1})$, where the variables in $T'(\mathbb{R}^{n+1})$ are denoted by (τ, t, y, m, r, ξ). So $\widetilde{e}_3$ satisfies a symmetric hyperbolic system which is an essential difference with the strictly hyperbolic case.

So in conclusion of (ii) we get E s.t.

$$(21) \qquad \begin{cases} \widetilde{P}E \in C^\infty(\mathbb{R}^{n-1} \times \mathbb{R}^n) \\ \gamma_0 E = \mathrm{Id} \bmod L^{-\infty}(\mathbb{R}^{n-1}) \end{cases} .$$

(iii) <u>Construction of a parametrix of the Cauchy problem for</u> P.

Clearly we can choose in 3 (ii) E s.t.

$$(22) \qquad WF(AB - \mathrm{Id}) \circ \chi^{-1} \circ WF'E = \Phi .$$

Now let $\widetilde{E} = BE$. Then by (21) and (7) we get:

$$A P \widetilde{E} \in C^\infty(\mathbb{R}^{n-1} \times \mathbb{R}^n)$$

$$\gamma_0 B \widetilde{E} = \widetilde{B} \gamma_0 E = \widetilde{B} .$$

Let $\widetilde{B}'$ be a parametrix for $\widetilde{B}$, and let $\tilde{\tilde{E}} = \widetilde{E}B'$.

Then we get

$$A\,P\,\tilde{\tilde{E}} \in C^{\infty}(\mathbb{R}^{n-1} \times \mathbb{R}^{n})$$

$$\gamma_o \tilde{\tilde{E}} = \mathrm{Id} \bmod L^{-\infty}(\mathbb{R}^{n-1}) .$$

By (22) finally

$$(23) \quad \begin{cases} P\,\tilde{\tilde{E}} \in C^{\infty}(\mathbb{R}^{n-1} \times \mathbb{R}^{n}) \\ \gamma_o \tilde{\tilde{E}} = \mathrm{Id} \bmod L^{-\infty}(\mathbb{R}^{n-1}) \end{cases} .$$

<u>Remark</u>: Essentially the same construction for the parametrix for the Cauchy problem for P works for an $m \times m$ system of the form:

$$(24)\quad P = \begin{pmatrix} D_t-\lambda_1(t,y,D_y) & & & 0 \\ & D_t-\lambda_2(t,y,D_y) & & \\ & & \ddots & \\ 0 & & & D_t-\lambda_m(t,y,D_y) \end{pmatrix} + D(t,y,D_y)$$

where λ_1 , λ_2 , D satisfy the same hypothesis as in this section, $\lambda_j(t,y,\xi) \in C^{\infty}(\mathbb{R} \times T'(Y))$ homogenous of degree 1 in ξ , $j = 3,\dots,m$, and $\lambda_j \neq \lambda_k$, $j = 3,\dots,m$, $k = 1,2,\dots,m$, $j \neq k$, $\lambda_j(t,y,D_y)$ pseudodifferential operators in Y smooth in t , $j = 3,\ldots,m$.

Let $\Phi(t,y,r,\xi) = \varphi(t,y,\xi) + tr$ as in 3.(4) .

(25) $\Phi_j(t,y,r,\xi) = \varphi_j(t,y,\xi) + tr$ with

$$\begin{cases} \dfrac{\partial\varphi_j}{\partial t} = \lambda_j(t,y,d_y\varphi_j) \\ \varphi_j(0,y,\xi) = \langle y,\xi\rangle \end{cases}$$

and take the associated canonical transformatation χ, χ_j . Let A, A_j be the associated Fourier Integral Operators to Φ, Φ_j as in 3.(6) $j = 3,\ldots,m$, and take

$$\tilde{\tilde{A}} = \begin{pmatrix} A & & & & 0 \\ & A & & & \\ & & A_3 & & \\ & & & \ddots & \\ 0 & & & & A_m \end{pmatrix}$$

$$\tilde{\tilde{B}} = \begin{pmatrix} B & & & & \\ & B & & & \\ & & B_3 & & \\ & & & \ddots & \\ 0 & & & & B_m \end{pmatrix}$$

with B, B_j local inverses of A, A_j . $\tilde{\tilde{A}}$ is a Fourier integral operator, since the canonical relations associated to A, A_j are disjoint $j = 3,\dots,m$. We consider now $\tilde{P} = \tilde{\tilde{A}} P \tilde{\tilde{B}}$ and we are reduced to the study of

$$(26) \qquad \tilde{P} = \begin{pmatrix} D_t & & & & 0 \\ & D_t-\tilde{\lambda}_2 & & & \\ & & D_t & & \\ & & & \ddots & \\ 0 & & & & D_t \end{pmatrix} + \tilde{D}(t,y,D_y) \; ,$$

$$\tilde{D}(t,.,.) \in L^0(Y) \; .$$

The construction of fundamental solution E of the Cauchy problem for (26) has the same form as that for (8), i.e. $E = E_1 + E_2 + E_3$ with E_i as before.

The transport equations for e_1, e_2, e_3 are obtained from:

$$\text{a)} \qquad \left(\begin{pmatrix} D_t & & 0 \\ & \ddots & \\ 0 & & D_t \end{pmatrix} + \tilde{D}(t,y,D_y) \right) E_1 \in C^\infty(\mathbb{R}^{n-1} \times \mathbb{R}^n)$$

$$\text{b)} \qquad \left(\begin{pmatrix} D_t-\tilde{\lambda}_2 & & & 0 \\ & D_t-\tilde{\lambda}_2 & & \\ & & \ddots & \\ 0 & & & D_t-\tilde{\lambda}_2 \end{pmatrix} + \tilde{D}(t,y,D_y) \right) E_2 \in C^\infty(\mathbb{R}^{n-1} \times \mathbb{R}^n).$$

The initial conditions are:

c) $$\gamma_o(E_1 + E_2) = \mathrm{Id} \bmod L^{-\infty}(\mathbb{R}^{n-1})$$

d) $$\left(\begin{pmatrix} D_t - D_\tau & & & & \\ & D_t - \lambda_2 + D_\tau & & & \\ & & D_t - D_\tau & & \\ & & & \ddots & \\ & & & & D_t - D_\tau \end{pmatrix} + D(t,y,D_y)\right) E_3' \in C^\infty(\mathbb{R}^{n-1} \times \mathbb{R}^n)$$

$$e_3^{1j}(-t,t,y,\theta) = -\tilde{\lambda}_2(e^{i\langle y,\theta\rangle} e_1^{1j}) \qquad j = 1,\dots,m \quad \bmod S^{-\infty}$$

$$e_3^{kj}(t,t,y,\theta) = \tilde{\lambda}_2(e^{i\varphi_2} e_2^{kj}) \qquad k > 1,\ j = 1,\dots,m \quad \bmod S^{-\infty}$$

4. Parametrix for the Cauchy problem for hyperbolic operators with double characteristics.

Notation is the same as in section 3 of this Chapter. Let

(1) $$P = (D_r - \lambda_1(t,y,D_y))(D_t - \lambda_2(t,y,D_y)) + S$$

$S \in L^1(X)$ and we assume $s(r,t,y,\xi) = 0$ if $r - \lambda_1(t,y,\xi) = r - \lambda_2(t,y,\xi) = 0$.

<u>Proposition 4.1</u>: If $\widetilde{\lambda}_1(t,y,\xi)$ is the full symbol of $\lambda_1(t,y,D_y)$, $\widetilde{\lambda}_1 = \lambda_1 + \sum_{j=0}^{\infty} \lambda_1^{-j}$, $\lambda_1^{-j} \in S^{-j}(\mathbb{R} \times T'(Y))$, then we can choose λ_0 so that:

$$(2) \qquad P = (D_t - \lambda_1(t,y,D_y))(D_t - \lambda_2(t,y,D_y)) + T$$

with $T \in L^0(X)$.

<u>Proof</u>: Let $\widetilde{p}_1 \in S'(T^*(X))$ be the term of order one in the asymptotic expansion of the full symbol of p , then to have (2) comparing terms of order 1 we must solve for λ_1^0

$$\widetilde{p}_1 = -\lambda_1^0(r - \lambda_2) - D_t\lambda_2 + \sum_{j=1}^{n-1} \frac{\partial\lambda_1}{\partial\xi_j} D_{x_j}\lambda_2 + (r - \lambda_1)h_0$$

where h_0 is the term of order 0 in the asymtotic expansion of the full symbol of $-\lambda_2(t,y,D_y)$. So at $r = \lambda_1(t,y,\xi)$

$$(3) \quad \widetilde{p}_1(t,y,\lambda_1(t,y,\xi),\xi) + D_t\lambda_2 - \sum_{j=1}^{n-1} \frac{\partial\lambda_1}{\partial\xi_j} D_{x_j}\lambda_2 = -\lambda_1^0(\lambda_1 - \lambda_2).$$

But the left hand side of (3) is s and by assumption it vanishes when $\lambda_1 = \lambda_2$, so λ_1^0 is determined.

Q.E.D.

Now let us consider the system:

$$(4)\qquad \tilde{L} = \begin{pmatrix} D_t-\lambda_1(t,y,D_y) & 0 \\ 0 & D_t-\lambda_2(t,y,D_y) \end{pmatrix} + \begin{pmatrix} 0 & T \\ -Id & 0 \end{pmatrix} .$$

Let $\tilde{D} = \begin{pmatrix} 0 & T \\ -Id & 0 \end{pmatrix} \in L^0(X)$. We can choose $C \in L^0(X)$ elliptic and $\tilde{D}(t,y,D_y) \in L^0(Y)$ smooth in t s.t.

$$\tilde{L} = C\left[\begin{pmatrix} D_t-\lambda_1 & 0 \\ 0 & D_t-\lambda_2 \end{pmatrix} + \tilde{D}(t,y,D_y)\right]$$

By Chapter II, Section 3 we can construct E s.t.

$$LE \in C^\infty(\mathbb{R}^{n-1} \times \mathbb{R}^n)$$

$$\gamma_o E = Id \quad \text{mod } L^{-\infty}(\mathbb{R}^{n-1}) .$$

So we have

$$(5)\qquad \begin{cases} \tilde{L}E \in C^\infty(\mathbb{R}^{n-1} \times \mathbb{R}^n) \\ \gamma_o E = Id . \end{cases}$$

From (5) we get:

$i_1)$ $(D_t-\lambda_1(t,y,D_y))E_{11} + TE_{21} \in C^\infty(\mathbb{R}^{n-1} \times \mathbb{R}^n)$

$i_2)$ $(D_t - \lambda_1(t,y,D_y))E_{12} + TE_{22} \in C^\infty(\mathbb{R}^{n-1} \times \mathbb{R}^n)$

$i_3)$ $(D_t - \lambda_2(t,y,D_y))E_{21} - E_{11} \in C^\infty(\mathbb{R}^{n-1} \times \mathbb{R}^n)$

$i_4)$ $(D_t - \lambda_2(t,y,D_y))E_{22} - E_{12} \in C^\infty(\mathbb{R}^{n-1} \times \mathbb{R}^n)$

$i_5)$
$$\begin{cases} \gamma_o E_{11} = Id \\ \gamma_o E_{22} = Id \\ \gamma_o E_{21} = 0 \\ \gamma_o E_{12} = 0 \end{cases} \mod L^{-\infty}(\mathbb{R}^{n-1})$$

From $i_1)$ and $i_3)$ we get:

$$(D_t - \lambda_1(t,y,D_{y1})(D_t - \lambda_2(t,y,D_y))E_{21} + TE_{21} \in C^\infty(\mathbb{R}^{n-1} \times \mathbb{R}^n) .$$

From $i_2)$ and $i_4)$ we get:

$$(D_t - \lambda_1(t,y,D_y)(D_t - \lambda_2(t,y,D\ ))E_{22} + TE_{22} \in C^\infty(\mathbb{R}^{n-1} \times \mathbb{R}^n)$$

Also from $i_5)$ $\begin{cases} \gamma_o E_{22} = Id \\ \gamma_o E_{21} = 0 \mod L^{-\infty} \end{cases}$.

From $i_3)$ and $i_5)$ $\gamma_o(D_t - \lambda_2(t,y,D_y)E_{21} = Id \mod L^{-\infty}$,

But $$\gamma_o \lambda_2(t,y,D_y) = \lambda_2(0,y,D_y)\gamma_o E_{21}$$
$$= 0 \bmod L^{-\infty} .$$

From i_4) and i_5) $\gamma_o D_t E_{22} = 0 \bmod L^{-\infty}$.

Then calling
$$E_1 = E_{22}$$
$$E_2 = E_{21} ,$$

We have
$$\begin{cases} PE_j \in C^\infty(\mathbb{R}^{n-1} \times \mathbb{R}^n) \\ \gamma_o\left(\frac{\partial}{\partial t}\right)^k E_j = \delta_{kj}\, \mathrm{Id} \bmod L^{-\infty}(\mathbb{R}^{n-1}) , \quad k,j = 1,2 . \end{cases}$$

So $E = E_1 + E_2$ is a parametrix for the Cauchy problem for P .

Remark: Using remark of II Section 3, we can construct a fundamental solution of the Cauchy problem for an operator of the form

$$P = (D_t - \lambda_1(t,y,D_y))(D_t - \lambda_2(t,y,D_y))\dots(D_t - \lambda_m(t,y,D_y) + S$$

with λ_1 , λ_2 , S as before and λ_j , $j = 3,\dots,m$, as in the remark in Section 3 of this chapter.

Using a slight modification of Proposition 4.1 we are reduced to consider

$$P = (D_t - \lambda_1(t,y,D_y))(D_t - \lambda_2(t,y,D_y))\dots(D_t - \lambda_m(t,y,D_y)) + T$$

$$T \in L^0(X)$$

because the principal symbol of $(D_t - \lambda_n(t,y,D_y))\dots$ $(D_t - \lambda_m(t,y,D_y))$ is different from zero when $r = \lambda_1 = \lambda_2$ and we can make a reduction to the case

$$L = \begin{pmatrix} D_t-\lambda_1 & & & 0 \\ & D_t-\lambda_2 & & \\ & & \ddots & \\ 0 & & & D_t-\lambda_m \end{pmatrix} + \tilde{D}(t,y,D_y)$$

and continue as before.

CHAPTER III

Open Problems.

The main problem that is implicit in this thesis is to make a general theory of oscillatory integrals with "wedges", i.e. to make sense of expressions of the form

$$(1) \quad \int_{-t}^{t}\int e^{i\varphi_3(\tau,t,y,\bar{y},\theta)} e_3(\tau,t,y,\bar{y},\theta) f(\bar{y}) d\bar{y} d\theta$$

with conditions on e_3 , φ_3 , etc., or more generally to make sense of oscillatory integrals with singular symbols (in (1) we have the term $H(t- \;)H(t+ \;)e_3$ with H the Heaviside function) and I think a generalization of [G] would lead to that. With a functional calculus it could be constructed (maybe) a global parametrix in certain cases and it could lead to results in the asymtotic study on the spectral function of an elliptic system P on which the eigenvalues of p are multiple and in the description of the singularities of the spectral function (see [D-G] and [H_4]).

The problem of conical refraction, that has many relations with this thesis, is very interesting as well (see [L]).

In Chapter II the singularities of the parametrix constructed are not analyzed, since this was done in detail in the involutive case in Chapter I. In the non involutive case (i.e. $\{p_1, p_2\} \neq 0$ on $p_1 = p_2 = 0$) we observe that this condition implies $\frac{\partial^2}{\partial \tau^2} \varphi_3 \neq 0$ where $\frac{\partial}{\partial \tau} \varphi_3 = 0$ with φ_3 as in Chapter II, Section 3 (12); so applying the method of stationary phase to E_3, we get that the "extra" term in the singularities of E_3 are broken bicharacteristics (corresponding to H_{p_i}, $i = 1,2$) starting on points where $p_1 = p_2 = 0$. (See [G_a-L] and [M].)

We do not have definitive results on these problems yet, so we have not included its analysis on this thesis. We will come back to this soon.

NOTATION

(1) 3.(4) for instance means number 4 of section 3 of the same chapter. (4) means 4 of the same section and chapter.

(2) If X is a C^∞ manifold:

i_1) $T'(X) = T^*X - \{0\}$.

i_2) $C^\infty(X)$ is the set of C^∞ function on X .

i_3) $C_o^\infty(X)$ is the set of C^∞ functions on X with compact support.

i_4) $D'(X)$ is the set of distributions on X .

i_5) $\epsilon'(X)$ is the set of distributions on X with compact support.

i_6) $L^m(X)$ denotes the set of properly supported, classical pseudodifferential operators on X .

i_7) $L^{-\infty}(X)$ is the set of pseudodifferential operators with C^∞ kernel.

i_8) $P \in L^m(X)$, p denotes its principal symbol and C_P its subprincipal symbol.

i_9) If Y is a C^∞ manifold and $A: C_o^\infty(X) \rightarrow D'(Y)$ linear map, then $K_A \in D'(Y \times X)$ denotes its Schwartz Kernel.

i_{10}) If $T \in D'(X)$, WFT denotes the wave front set of the distribution T.

If $A: C_o^\infty(X) \to D'(Y)$ continuous linear then $WFA = WFK_A$.

i_{11}) If $A \in C_o^\infty(X) \to D'(Y)$ continuous linear then

$$WF'A = \{((y,\eta);(x,\xi)) \in T'(Y \times X) \mid ((y,\eta);(x,-\xi)) \in WFA\}$$

$$WF_Y'A = \{(y,\xi) \in T'(Y) \mid \exists\, x \in X \text{ such that } ((x,0);(y,\eta)) \in WFA\}$$

$$WF_X'A = \{(x,\xi) \in T'(X) \mid \exists\, y \in Y \text{ such that } ((y,0);(x,\xi)) \in WFA\} .$$

3) i_1) Let X be an open set in $\mathbb{R}^n$, $f \in C_o^\infty(X)$,

$$\hat{f}(\xi) = \int e^{-i\langle x,\xi\rangle} f(x)dx$$

$$\langle x,\xi\rangle = x_1\xi_1 + \ldots + x_n\xi_n .$$

i_2) $f,g \in C^\infty(X)$,

$$\{f,g\} = \sum_{j=1}^{n} \frac{\partial f}{\partial \xi_j}\frac{\partial g}{\partial x_j} - \frac{\partial f}{\partial x_j}\frac{\partial g}{\partial \xi_j} .$$

i_3) $S^m(X \times \mathbb{R}^n) = \{a \in C^\infty(X \times \mathbb{R}^n) \mid$ given $K \subset X$ compact, $\exists\, C_{\alpha,\beta,K} > 0$ such that

$$(D_x^\alpha D_\xi^\beta a(x,\xi)) \leq C_{\alpha,\beta,K}(1 + |\xi|)^{m-|\beta|}$$

$\forall\, \alpha = (\alpha_1,\ldots,\alpha_n)$, $\beta = (\beta_1,\ldots,\beta_n)$ multiindices

$\alpha_i, \beta_i \in \mathbb{N} \cup \{0\}$, $\left(\frac{\partial}{\partial x}\right)^\alpha = \dfrac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1},\ldots,\partial x_n^{\alpha_n}}$

$$|\alpha| = \alpha_1 + \dots + \alpha_n , \quad D_{x_i} = \frac{1}{i} \frac{\partial}{\partial x_i} \} .$$

In general we use the notations of [D].

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BIOGRAPHICAL NOTE

Gunther Uhlmann was born on February 9, 1952 in Quillota Chile. He attended high school in the Instituto Rafael Ariztía of Quillota from which he graduated in 1969.

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