ON THE INTEGRAL EXTENSIONS OF ISOMETRIES OF QUADRATIC FORMS OVER LOCAL FIELDS

by

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ABSTRACT

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By Allan Trojan

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Let F be a local field in which 2 is a prime element. Let L be a regular lattice over F , and v,w any two elements in L such that $v^2 = w^2$. In this thesis we develop necessary and sufficient conditions for the existence of an isometry on L which maps v onto w.

Use is made of a mapping, T , from modular lattices onto the residue class field of F. It is found that two maximal vectors, v and w , in a modular lattice L are isometrically equivalent if and only if they have the same length and T(v) = T(w).

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INTRODUCTION

In 1923, Hasse, in a famous paper [3] proved the following theorem: two quadratic forms over the rational numbers are equivalent if and only if they are equivalent over the p-adic numbers (for all primes p) and the real numbers. This result stimulated the study of quadratic forms over the p-adic numbers and local fields in general as a means of examining many of the questions concerning the theory of quadratic forms over algebraic number fields.

Much of the theory of quadratic forms over local fields is now well known. Necessary and sufficient conditions for the representation of one form by another were discovered by Hasse [4]. Such conditions for integral equivalence were obtained by Durfee [1], and O'Meara [6], [7]. O'Meara [8] also found necessary and sufficient conditions for the integral representation of one form by another, provided that 2 is either a unit or a prime element of the local field. The cases where 2 is ramified is still under investigation.

Another unsolved problem is that of finding necessary and sufficient conditions for the integral extension of an isometry acting between two subspaces of a regular quadratic space. That is, given a quadratic space V and two isometric subspaces V_1 and V_2 where V has as a basis $\{x_1, \ldots, x_n\}$, find conditions for the existence of an isometry φ which maps V_1 onto V_2 such that the matrix representation of φ with respect to the given basis consists of integers in the local field. (The existence of some extension of the isometry follows from Witt's theorem [10].) Rosenzweig [9] and James [2] have solved this problem for local fields in which 2 is a unit (so called non-dyadic local fields). The dyadic case, however, is much more difficult, and conditions are known only for a few specialized cases. In this thesis a solution is found for the existence of an integral isometry mapping a one-dimensional subspace onto another one-dimensional subspace of a regular quadratic space over a local field in which 2 is a prime element.

We examine the problem from a geometrical point of view, that is, we consider integral isometries to be isometries on lattices with a quadratic structure. A lattice L is a finite dimensional module over the ring of integers Z of the local field. An isometry on L is, of course, a linear mapping of L onto L which preserves the inner product. The problem which we shall solve may then be stated as follows: given two vectors, v and w, in a lattice L on which we have defined a regualr symmetric product, find necessary and sufficient conditions for the existence of an isometry on L which maps v onto w.

Much use will be made of Rosenzweig's ideas of dividing an arbitrary vector into critical components. As in Rosenzweig's Thesis [9], the problem will be solved by first examining vectors with one critical component and then doing an induction on the number of components. Aside from this general schemata, however, the proofs of most of the theorems are quite different. The most important theorem used is the one proved by O'Meara on the necessary and sufficient conditions for isometry of lattices (Theorem 1.10, Chapter 1). It is interesting to note that many of the methods used, most notably those in Chapter 4, may be used to simplify existing proofs for the non-dyadic case.

Chapter One: Preliminaries

Definition: 1. A local field is a complete field under a non-archimedian valuation with a finite residue class field. We shall denote by Z and P the ring of integers of the field F and the maximal ideal in this ring. π will be used to denote a generating element of P.

2. A dyadic local field is a local field in which $|2| < 1$. Let α be any element of the local field F. Then $\partial(\alpha)$ (the quadratic defect of α) is the ideal generated by the β such that $\alpha - \beta$ is a square and $|\beta|$ is minimal.

Theorem 1.1: a) The quadratic defect of a unit always is one of the following ideals:

Z or 0 when F is non-dyadic l.f.

$4Z, \ldots, P^5, P^3, P, 0$ when F is dyadic l.f.

Furthermore, each of the above ideals actually appears as the quadratic defect of some element of F.

b) In particular, $1 + 4\pi$ is always a square.

c) If $\alpha = 1 + \beta$ where $|4| < |\beta| < 1$, then $\partial(\alpha) = \beta Z$.

d) If α, β are units with $\partial(\alpha) = \partial(\beta) = 4Z$ then $\alpha\beta$ is a square

Definition: Let α, β be elements in an arbitrary field F, then: $(\alpha,\beta) = 1$ when there exist x and y such that $\alpha x^2 + \beta y^2 = 1$.

$= -1$ otherwise.

(α,β) is called the <u>Hasse Symbol</u>.

<u>Theorem</u> 1.2: a) F is a local field. Let Δ be a unit such that $\partial(\Delta) = 4Z$, E an arbitrary unit. Then $(\Delta,E) = 1$. $(\Delta,\pi) = -1$.

b) $(\alpha,\beta\gamma) = (\alpha,\beta)(\alpha,\gamma)$ for any α,β,γ in F.

<u>Definition</u>: 1. A <u>quadratic space</u> over a field F is a vector space V over F with a symmetric inner product $(x,y) \in F$. The quadratic space is called <u>regular</u> if $(x,y) = 0$ for all y in F implies that $x = 0$.

2. Let V, $\overline{V}$ be two quadratic spaces over the field F. Then we say that V and $\overline{V}$ are <u>isometric</u> (written $V \simeq \overline{V}$) if there exists a linear map φ mapping V onto $\overline{V}$ such that $(x,y) = (\varphi(x),\varphi(y))$. φ is called an <u>isometry</u>.

<u>Proposition</u> 1.1: Let V be a quadratic space over F. Assume the characteristic of F is not 2. Then V has an orthogonal basis x_i, $i = 1, \ldots, n$. That is $(x_i,x_j) = 0$ if $i \neq j$. We write $V = \Sigma \oplus \langle x_i \rangle$.

Furthermore, if V is regular, and F is a local field

$$S(V) = \prod_{1 \le i \le j \le n} (x_i^2, x_j^2)$$

Then $S(V)$ is independent of the orthogonal basis chosen.

<u>Definition</u>: Let V be a quadratic space, with basis x_i and let A be the matrix defined by $A_{ij} = (x_i,x_j)$. Then we define $d(V) = \det A$. Note that $d(V)$ is only

defined up to squares of elements in F.

<u>Theorem</u> 1.3: Let U, V be regular quadratic spaces over a local field F. Then $U \simeq V$ if and only if:

1. Dim U = Dim V.
2. d(U) = d(V).
3. S(U) = S(V).

<u>Definition</u>: Let U, V be quadratic spaces. We say V represents U (written $U \to V$) if there is an isometry from U onto a subspace of V.

<u>Theorem</u> 1.4: Let U, V be regular quadratic spaces over a local field F. Then V represents U if and only if:

1. $U \simeq V$ when dim U = dim V
2. $U \oplus \langle dU \cdot dV \rangle \simeq V$ when dim U + 1 = dim V
3. $U \oplus H \simeq V$ when dim U + 2 = dim V and dU=-dV.

where H is the quadratic space denoted by the following matrix:

$$\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \qquad \text{(hyperbolic plane)}$$

The fundamental theorem for all work on quadratic forms is the following well known theorem of Witt:

<u>Witt's Theorem</u>: If V is a regular quadratic space over a field of characteristic not equal to 2, and W, $\overline{W}$ are isometric subspaces, then there is an isometry on V which maps W onto $\overline{W}$.

Lattices

Definition: a) Let F be a field, Z its ring of integers defined by some dedekind domain of prime spots. Let V be a vector space over F, M a submodule of V with respect to the ring of integers Z. Then M is called a lattice on V if there is a basis $\{x_1, \ldots, x_n\}$ for V such that $M = x_1 Z + \ldots + x_n Z$

b) $FM = \{\alpha x : \alpha \in F \text{ and } x \in M\}$

Theorem 1.5 a) Let L be a lattice over V. Then there is a basis $\{y_i\}$ for V and fractional ideals $\mathfrak{a}_i$ such that

$$L = \mathfrak{a}_1 x_1 + \ldots + \mathfrak{a}_n x_n$$

b) In particular, if Z is a principal ideal ring,

$$L = Z x_1 + \ldots + Z x_n$$

for some basis $\{x_i\}$

Definition: Let L, M be two lattices on the same quadratic space. We say that $L \simeq M$ if there is an isometry φ on V such that $\varphi(L) \subseteq M$.

From this point on, we will assume that F is a local field, and all quadratic spaces are regular, unless it is explicitly stated that they are not.

Definition: Let $L = Z x_1 + \ldots + Z x_n$. Then we make the definition $d(L) = \det (x_i, x_j)$, mod $(\text{units})^2$.

Definition: Let $\{x_i\}$ be a basis for L. Then we say that

L is <u>unimodular</u> if $x_i \cdot x_j \in Z$ and $d(L)$ is a unit of Z. We also make the definition V^{π^r} = the quadratic space given by the matrix $\pi^r \cdot (x_i, x_j)$. Then we say L is π^r-modular if L is unimodular over the quadratic space $V^{\pi^{-r}}$.

<u>Definition</u>: 1) Let q be a fractional ideal of F, L a lattice on V. Then $L^q = \{x : x \in L \text{ and } x \cdot y \in q \text{ for all } y \in L\}$.

2) If J is a sublattice of L, we write

$$(J, L) = \{(x,y) : x \in J, y \in L\}$$

3) H(0) is the lattice $xZ + yZ$ with the multiplication table $x^2 = 0$, $x \cdot y = 1$, $y^2 = 0$.

<u>Definition</u>: Let K and $\overline{K}$ be sublattices of L. Then we say $L = K \oplus \overline{K}$ if L is a direct sum of K and $\overline{K}$ considered as modules, and $(x,y) = 0$ if $x \in K$ and $y \in \overline{K}$. We say that K <u>splits</u> L.

<u>Theorem</u> 1.6: Let L be a lattice over the quadratic space V. J a unimodular sublattice. Then J splits L if and only if $(J,L) \subseteq Z$.

<u>Definition</u>: $s(L)$ = the ideal generated by (L,L).

$n(L)$ = the ideal generated by (x,x) where x L.

The following theorem defines an important splitting of L known as the <u>Jordan Decomposition</u> of L.

<u>Theorem</u> 1.7: Let L be a lattice over a quadratic space V. Then L may be written $L = L_1 \oplus \ldots \oplus L_t$ where

the L_i are modular and $s(L_1) \supset s(L_2) \supset \dots \supset s(L_t)$. Furthermore, if we have a second such splitting $L = \overline{L}_1 \oplus \dots \oplus \overline{L}_{\overline{t}}$ then we have the following facts:

1. $t = \overline{t}$
2. $s(L_i) = s(\overline{L}_i)$
3. $\dim L_i = \dim \overline{L}_i$.
4. $n(\overline{L}_i) = s(\overline{L}_i)$ if and only if $n(L_i) = s(L_i)$

From here we shall assume F is a dyadic local field in which 2 is a prime element. The great difficulty in dealing with Lattices over these fields lies in the fact that $n(L) \subseteq s(L)$ but equality does not necessarily hold. Thus we cannot necessarily find a diagonal basis for L. The best possible basis for L is given by the following theorem.

Theorem 1.8: Let L be modular. If L is proper ($n(L) = s(L)$) then L has an orthogonal basis. If L is improper, then L can be written as an orthogonal sum of two-dimensional sub-lattices.

Another important fact about lattices over dyadic local fields is that the cancellation theorem does not always hold. That is, if $L = M \oplus N = \overline{M} \oplus \overline{N}$ and $M \cong \overline{M}$, it is not necessarily true that $N \cong \overline{N}$. However, we do have the following special cases.

Theorem 1.9: a) If $L = K \oplus M = \overline{K} \oplus N$ where $K \cong \overline{K} \cong H(0)$, then $M \cong N$.

b) If $L = K \oplus M = \overline{K} \oplus N$ with $K \simeq \overline{K}$.

Furthermore, if K is π^r-modular and $n(K) \subseteq n(M^{\pi^r Z})$, $n(\overline{K}) \subseteq n(N^{\pi^r Z})$, then $M \simeq N$.

Isometry of Lattices

Definition: a) Let $L = L_1 \oplus \ldots \oplus L_t$ be a Jordan splitting for L. Then t, $\dim L_i$, $s(L_i) = s_i$, $n(L_i) = n_i$ are called the Jordan Invarients of L.

b) $L_{(i)} = L_1 \oplus \ldots \oplus L_i$.

Theorem 1.10: (O'Meara) Let K, L be lattices over the same regular quadratic space over a 2-adic local field F. Then $L \simeq K$ if and only if:

1. K and L have the same Jordan invarients.
2. $dL_{(i)}/dK_{(i)} \equiv 1 \mod n_i n_{i+1}/s_i^2$.
3. $FL_{(i)} \rightarrow FK_{(i)} \oplus \langle 2^{u_i} \rangle$ when $n_{i+1} \subseteq 4n_i$ where $n_i = 2^{u_i} Z$.

We shall have occasion to deal with non-regular lattices. We make the following definition.

Definition: $\text{Rad } L = \{x : x \in L \text{ and } x \cdot v = 0 \text{ for all } v \in L\}$

Theorem 1.11: Rad L splits L and if $L = L_1 \oplus \text{Rad } L$, $\overline{L} = L_2 \oplus \text{Rad } \overline{L}$ where $L \simeq \overline{L}$ then $L_1 \simeq L_2$.

The following theorem is useful in the construction of all possible lattices.

Theorem 1.12: Let L be unimodular. Then L is split by a hyperbolic plane, $H(0)$ if: a) $\dim L \geq 5$

b) dim $L \geq 4$ and L is improper.

Definition: 1. We say the two vectors v, w are equivalent (written $v \sim w$) if there is an isometry on L such that $\varphi(v) = w$.

2. Let $v \in L$. Then $\langle v \rangle^{\perp} = \{x \in L : x \cdot v = 0\}$

The preceding definitions and theorems may be found in O'Meara [5].

Here is a brief outline of the following chapters. We first examine the problem for modular lattices. Given two vectors v and w, with the same length, the problem is to find conditions for equivalence of these vectors. We first find conditions for the two-dimensional case by finding when the natural isometry from $\langle v \rangle \oplus \langle v \rangle^{\perp}$ onto $\langle w \rangle \oplus \langle w \rangle^{\perp}$ is actually an isometry on L. We then reduce the case where $\dim L > 2$ to the two dimensional case by finding when we can imbed v and w in isometric two dimensional lattices which split L and have isometric perpendicular components. It turns out that no more conditions are necessary in this case.

The next case examined, after modular lattices, is that in which two vectors have only one critical index, that is each vector can be imbedded in a modular lattice which splits L. Once again we examine $\langle v \rangle^{\perp}$ and $\langle \bar{v} \rangle^{\perp}$ to see whether it is possible to extend the isometry.

Finally we examine the general case. Here we write

$v = v_1 \oplus \dots \oplus v_n$, $\bar{v} = \bar{v}_1 \oplus \dots \oplus \bar{v}_n$ where v_i and $\bar{v}_i$ have only one critical index. Let $v_{(i)} = v_1 \oplus \dots \oplus v_i$. In this case we find a necessary and sufficient congurence relation between $v^2_{(i)}$ and $\bar{v}^2_{(i)}$ to permit the existence of an isometry which maps v onto $w = w_1 \oplus \dots \oplus w_n$ where the w_i each have one critical index and where $w^2_{(i)} = \bar{v}^2_{(i)}$. We then apply the results obtained for vectors with one critical index.

Chapter Two: Equivalence of Vectors Over Modular Lattices

Definition: 1. $v \in L$ is called maximal or primitive if $2^{-1}v \notin L$.

2. L is 2^k-modular. Then $v \in L$ is called saturated iff $L = \langle x_1 \rangle \oplus \dots \oplus \langle x_n \rangle$ (i.e. proper) $v = 2^n v$ with v maximal and $v = \sum_i \alpha_i x_i$ and $\alpha_i^2 x_i^2 \equiv \alpha_j^2 x_j^2 \bmod 2^{k+1}$ for all i, j. Otherwise v is called unsaturated.

3. L is 2^i-modular. $v, w \in L$ are said to be of the same type if:

1. v, w are both saturated or unsaturated.

2. If v, w are saturated then $v \cdot x_I \equiv w \cdot x_I \bmod 2^{i+1}$, $\forall I$.

We obviously want the definition of saturation and type to be independent of the orthogonal basis chosen. The following lemma establishes that fact.

Lemma 2.1: Let L be unimodular. $v, \bar{v} \in L$.

1. v is saturated iff L is proper and $y \in \langle v \rangle^{\perp}$ implies $\operatorname{ord} y^2 \geq 1$.

2. $v, \bar{v}$ are saturated then $v, \bar{v}$ are of the same type iff for every $y \in L$ we have $v \cdot y \equiv \bar{v} \cdot y \bmod 2$.

3. $v, \bar{v}$ saturated. $\dim L$ is odd. $v^2 = \bar{v}^2$, then v and $\bar{v}$ are of the same type.

4. $v \sim \bar{v}$ implies v and $\bar{v}$ are of the same type.

Proof: We use the following notation throughout the proof of the theorem. $L = \langle x_1 \rangle \oplus \dots \oplus \langle x_n \rangle \quad x_i^2 = \delta_i$

$v = \sum_i \alpha_i x_i \ , \ \overline{v} = \sum_i \overline{\alpha}_i x_i \ , \ y = \sum_i \beta_i x_i .$

1. Let $v \cdot y = 0$ then $\Sigma\, \alpha_i \beta_i \delta_i = 0$. Squaring this we have $\alpha_i^2 \beta_i^2 \delta_i^2 \equiv 0 \bmod 2$. But $\alpha_i^2 \delta_i \equiv \alpha_1^2 \delta_1 \bmod 2$. Thus, $\alpha_1^2 \delta_1 (\Sigma\, \beta_i^2 \delta_i) \equiv 0 \bmod 2$. Therefore $\Sigma\, \beta_i^2 \delta_i = y^2 \equiv 0 \bmod 2$ since $\alpha_1^2 \delta_1 \not\equiv 0 \bmod 2$.

Conversely: Let v be unsaturated. We may assume without loss of generality $\alpha_1^2 \delta_1 \not\equiv \alpha_2^2 \delta_2 \bmod 2$.
Let $y = \delta_2 \alpha_2 x_1 - \delta_1 \alpha_1 x_2$ then $y \cdot v = 0$.
But $y^2 = \delta_1 \delta_2 (\alpha_2^2 \delta_2 - \alpha_1^2 \delta_1) \not\equiv 0 \bmod 2$.

2. Let $\alpha_i \delta_i \equiv \overline{\alpha}_i \delta_i \bmod 2. \quad \forall i.$

$v \cdot y = \Sigma\, \alpha_i \beta_i \delta_i \equiv \Sigma\, \overline{\alpha}_i \beta_i \delta_i \bmod 2. = \overline{v} \cdot y.$
Conversely: $v = \Sigma\, \alpha_i x_i \ , \ \overline{v} = \Sigma\, \overline{\alpha}_i x_i$. If we let $y = x_i$ then by hypothesis $\alpha_i \delta_i \equiv \overline{\alpha}_i \delta_i \bmod 2$. thus $\alpha_i \equiv \overline{\alpha}_i \bmod 2$.

3. $n\alpha_n^2 \delta_n \equiv \Sigma\, \alpha_i^2 \delta_i \bmod 2 \equiv \Sigma\, \overline{\alpha}_i^2 \delta_i \bmod 2 \equiv n\overline{\alpha}_n^2 \delta_n \bmod 2$.

But n is odd. Thus $\alpha_k^2 \equiv \overline{\alpha}_k^2 \bmod 2$. Therefore $\left(\frac{\alpha_k}{\overline{\alpha}_k}\right)^2 \equiv 1 \bmod 2$. Therefore $\alpha_k \equiv \overline{\alpha}_k \bmod 2$.

4. a) v is saturated iff $\overline{v}$ is saturated follows from (1).

b) We must show: $v \cdot x_i \equiv \overline{v} \cdot x_i \bmod 2$ if v is saturated.

Assume $v = \Sigma\, x_i$ where $L = \Sigma \oplus \langle x_i \rangle$ with $x_i^2 = \delta_i \equiv \delta_j \bmod 2$, φ is an isometry with $\varphi(v) = \overline{v}$ and $\varphi(x_i) = \sum_j b_{ij} x_j$.

Thus: $\delta_i \equiv \sum_j b_{ij}^2 \delta_j \bmod 2$. Since isometries preserve inner products.

$$1 \equiv \sum_j b_{ij}^2 \equiv \{\sum_j b_{ij}\}^2 \bmod 2, \text{ and therefore}$$

$$1 \equiv \sum_j b_{ij} \bmod 2.$$

Now: $v \cdot x_i = \delta_i$

$$\overline{v} \cdot x_i = \sum_{j,k} (x_j \cdot x_i) b_{kj} = \sum_i \delta_i b_{ki} \equiv \delta_1 \{\sum_i b_{ki}\} \bmod 2.$$

Thus: $\overline{v} \cdot x_i \equiv \delta_1 \equiv \delta_i \bmod 2$.

Theorem 2.1: (O'Meara) The only two-dimensional modular lattices over an unramified dyadic local field are:

$$H(0) = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \qquad B(0) = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} \qquad H(\rho) = \begin{vmatrix} 2\rho & 1 \\ 1 & 2 \end{vmatrix}$$

$$B(\rho) = \begin{vmatrix} 4\rho & 1 \\ 1 & 1 \end{vmatrix} \qquad E(\epsilon, \partial) = \begin{vmatrix} \epsilon & 1 \\ 1 & 2\partial \end{vmatrix}$$

where ϵ, ∂, ρ are units and $1 + 4\rho$ is not a square. (Note: ρ will always stand for a unit such that $1 + 4\rho$ is not a square.)

Proof: See [8].

Proposition 2.1: $x^2 \equiv 1 \bmod 2^n$. Then $x \equiv 1 \bmod 2^{n-1}$ or $x \equiv -1 \bmod 2^{n-1}$.

Proof: $(x - 1)(x + 1) \equiv 0 \bmod 2_n$. Suppose $x-1 \equiv 0 \bmod 4$. Then $x + 1 \equiv 2 \bmod 4$. Thus $x - 1 \equiv 0 \bmod 2^{n-1}$. Now if $x - 1 \equiv 2 \bmod 4$, then $x + 1 \equiv 0 \bmod 2^{n-1}$.

Theorem 2.2: L is two-dimensional unimodular. v, $\overline{v}$ are maximal. $v^2 = \overline{v}^2$. Then $v \sim \overline{v}$ iff v and $\overline{v}$ have the same type.

<u>Proof</u>: Necessity has been proven (Lemma 2.1 |4|).

Sufficiency: There are three cases to consider.

<u>Case 1</u>. ord $v^2 = 0$. Then $L = \langle v\rangle \oplus \langle v\rangle^{\perp} = \langle\bar{v}\rangle \oplus \langle\bar{v}\rangle^{\perp}$. Then $\langle v\rangle \simeq \langle\bar{v}\rangle$, $\langle v\rangle^{\perp} \simeq \langle\bar{v}\rangle^{\perp}$. Combine these two isometries to form an isometry on L mapping v onto $\bar{v}$.

<u>Case 2</u>. ord $v^2 \geq 1$. $v^2 \neq 0$. We let $L = vZ + wZ$ with matrix representation $\begin{vmatrix} \delta & 1 \\ 1 & E \end{vmatrix}$. Similarly, $L = \bar{v}Z + \bar{w}Z$ with matrix representation $\begin{vmatrix} \delta & 1 \\ 1 & \bar{E} \end{vmatrix}$.

We first show that we may choose $E \equiv \bar{E} \bmod 2$.

If L is improper the result is trivial.

If L is proper, then v and $\bar{v}$ must be saturated. Thus by Lemma 2.1 part 2, $\bar{v}\cdot w = 1 \bmod 2$. Therefore $\bar{v}\cdot w = 1 + 2\eta$. Let $\bar{w} = \dfrac{w}{1 + 2\eta}$. Then $\bar{w}^2 \equiv w^2 \bmod 2$.

Now $(1 - \delta E)/(1 - \delta\bar{E}) = \mu^2$ with μ a unit

$$\equiv 1 + \delta(\bar{E} - E) \bmod 2\delta$$

$$\equiv 1 \bmod 2\delta.$$

By Proposition 2.1 we may choose $\mu = 1 + \delta\lambda$ $\lambda \in Z$.

Let $\varphi(v) = \bar{v}$, $\varphi(v - \delta w) = \mu(\bar{v} - \delta\bar{w})$. This is an isometry on FL since $\langle v - \delta w\rangle \simeq \langle v\rangle^{\perp}$ and $\langle\bar{v} - \delta\bar{w}\rangle \simeq \langle\bar{v}\rangle^{\perp}$ and $\mu^2(\bar{v} - \delta\bar{w}) = \mu^2(-\delta + \delta^2\bar{E}) = \delta(-1 + \delta E) = (v - \delta w)^2$. We must show that φ is an isometry on L, that is, $\varphi(L) \subseteq L$. We have $\varphi(v) \in L$. We need to show $\varphi(w) \in L$. But $\varphi(w) = \varphi\{\frac{1}{\delta}(v - (v - \delta w))\}$

$$= \frac{1}{\delta}(\bar{v} - (\bar{v} - \delta\bar{w}) - \delta(\bar{v} - \delta\bar{w}))$$

$$= \frac{1}{\delta}(\delta\bar{w} - \delta\bar{v} + \delta^2\bar{w})$$

$$= \bar{w} - \bar{v} + \delta\bar{W} \in L.$$

Case 3. $v^2 = 0$. Then $L \simeq B(0)$ or $H(0)$. First let $L \simeq B(0)$, $L = xZ + yZ$, $x^2 = 1$ $x \cdot y = 1$ $y^2 = 0$.

Then $v = \epsilon(y)$ or $\epsilon(2x - y)$
$\bar{v} = \bar{\epsilon}(y)$ or $\bar{\epsilon}(2x - y)$ $\epsilon, \bar{\epsilon}$ are units.

Now by Lemma 2.1 part 2, $\epsilon \equiv \bar{\epsilon} \bmod 2$.
Let $u = \frac{\bar{\epsilon}}{\epsilon} \equiv 1 \bmod 2$. We define the linear map φ_u as follows. $\varphi_u(y) = uy$

$$\varphi_u(2x - y) = \frac{(2x - y)}{u}$$

It is easily checked that φ_u is an isometry on FL. But $\varphi_u(2x) = u2x + (\frac{1}{u} - u)y$ and $\frac{1}{u} - u \equiv 1 \bmod 2$.

Therefore $\varphi_u(x) \in L$, $\varphi_u(y) \in L$, and $\varphi_u(L) \subseteq L$.

Thus φ_u is an isometry on L.

Also let ψ be the isometry such that $\psi(y) = 2x - y$, $\psi(2x - y) = y$ some product of ψ and φ_u will map v onto $\bar{v}$. A similar method also works when $L \simeq H(0)$.

Proposition 2.2: Let $v \in L$ where L is unimodular and v maximal. Then if ord $v^2 \geq 1$, we may write $L = R \oplus S$ with R two-dimensional and $v \in R$, such that:

1. S improper if v saturated and L proper.
2. R improper if v is unsaturated.

Proof: Case 1, L improper. Then we may write $L = \Sigma \oplus L_i$ where the L_i are two-dimensional. Write $L_i = x_iZ + y_iZ$ with $x_i \cdot y_i = 1$. Write $v = \sum_i \alpha_i x_i + \sum_j \beta_j y_j$. Assume $|a_k| = 1$. Then letting $R = vZ + y_nZ$ we have R unimodular and $L = R \oplus S$.

Case 2, L proper, v unsaturated. It is easily seen that we may write $L = \sum \oplus \langle x_i \rangle$ with $x_i^2 = \delta_i$ and $v = \alpha_i x_i$ with α_1 and α_2 both units and $\alpha_1^2 \delta_1 \not\equiv \alpha_2^2 \delta_2 \mod 2$. Now every unit of F is a square mod 2. Thus we may choose β such that $\beta^2 \equiv \delta_2/\delta_1 \mod 2$. Let $y = \beta x_1 + x_2$. It is easily checked that $y^2 \equiv 0 \mod 2$, $v \cdot y \not\equiv 0 \mod 2$. Letting $R = vZ + yZ$ we have R improper and unimodular. Thus R splits L.

Case 3, L proper, v saturated. We again let $L = \sum \oplus \langle x_i \rangle$ Let $R = x_1 Z + vZ$. Write $L = R \oplus S$. Then S is improper by Lemma 2.1.

Proposition 2.3: 1. $B(\rho) \oplus H(\rho) \not\cong B(0) \oplus H(0)$

2. $B(\rho) \oplus H(0) \not\cong B(0) \oplus H(\rho)$

3. $H(\rho) \oplus \langle \varepsilon \rangle \not\cong H(0) \oplus \varepsilon(1+4\rho)$.

4. $H(\rho) \oplus \langle \varepsilon \rangle$ is not isotropic.

5. $H(\rho) \oplus B(\rho)$ is not isotropic.

Proof. See Proposition 9 of [8].

Theorem 2.3: Let L be unimodular $v^2 = \bar{v}^2$ $v, \bar{v}$ maximal. Then $v \sim \bar{v}$ iff v has the same type as $\bar{v}$.

Proof: Necessity: Already done.

Sufficiency:

Case 1. $|v^2| = 1$. Let $L = \langle v \rangle \oplus \langle v \rangle^{\perp} = \langle \bar{v} \rangle \oplus \langle \bar{v} \rangle^{\perp}$ Now $f\, \langle v \rangle^{\perp} \cong F \langle \bar{v} \rangle^{\perp}$, by Witt's Theorem. Furthermore, by Lemma 2.1 $\langle v \rangle^{\perp}$ and $\langle \bar{v} \rangle^{\perp}$ are both proper or both improper and hence have the same Jordan Invariants. Thus they are isometric by Theorem 1.10. We extend the isometry by mapping

v onto $\bar{v}$.

<u>Case 2</u>. ord $v^2 \geq 1$. v unsaturated.

By Proposition 2.2 we may write $L = R \oplus S = \bar{R} \oplus \bar{S}$. R, $\bar{R}$ improper two-dimensional $v \in R$. $\bar{v} \in \bar{R}$. Furthermore $n(S) = n(\bar{S})$ by Lemma 2.1 (1). If $R \simeq \bar{R}$ then $FS \simeq F\bar{S}$ by Witt's Theorem and $S \simeq \bar{S}$ by Theorem 1.10. Thus we may apply Theorem 2.2 to R and $\bar{R}$ and we are through.

So now we must show that we may choose $R \simeq \bar{R}$.

If L is 3-dimensional, the result follows from Proposition 2.3 (3). If ord $v^2 \geq 2$ then $R \simeq \bar{R} \simeq H(0)$ since $H(\rho)$ does not represent any integers of order ≥ 2 by a maximal vector. Thus we may assume $\dim L \geq 4$, ord $v^2 = 1$. Suppose $R \not\simeq \bar{R}$. We may assume without loss of generality that $R \simeq H(0)$ and $\bar{R} \simeq H(\rho)$.

Write $R = vZ + wZ$ with $v \cdot w = 1$ $w^2 = 0$

$\bar{R} = \bar{v}Z + \bar{w}Z$ with $\bar{v} \cdot \bar{w} = 1$ $\bar{w}^2 = 2\eta$

S is proper and $\dim S \geq 2$. Thus there is a vector $y \in S$ with $y^2 \equiv 2\eta \mod 4$.

Let $R' = vZ + (w+y)Z$

then $\det R' \equiv \det \bar{R} \mod 8$.

Thus $R' \simeq \bar{R}$ and $v \in R'$

We may write $L = R' \oplus S'$ with $S' \simeq S$ by Theorem 1.10.

<u>Case 3</u>. ord $v^2 \geq 1$ v saturated.

Write $L = R \oplus S = \bar{R} \oplus \bar{S}$ $v \in R$ $\bar{v} \in \bar{R}$

(we may do this by Proposition 2.2.) $R, \overline{R}$ proper
$S, \overline{S}$ improper

Then v is saturated in R, $\overline{v}$ saturated in $\overline{R}$.
Assume we can show $R \simeq \overline{R}$ then $S \simeq \overline{S}$ by Witt's Theorem and Theorem 1.10.

Now let $x \in R$. and $\varphi(R) = \overline{R}$.
We have $v \cdot x \equiv \overline{v} \cdot x$ since $v, \overline{v}$ have the same type.
Now $\overline{v} \cdot \varphi(x) = \varphi^{-1}(\overline{v}) \cdot x \equiv \overline{v} \cdot x \bmod 2$ by Lemma 2.1 (4).

$$\equiv v \cdot x.$$

Thus $\overline{v}$ has the same type in R as v in R hence by Theorem 2.2 there exists an isometry ψ_0 with $\psi_0(R) = \overline{R}$ $\psi_0(v) = \overline{v}$
Let $\psi_1(s) = \overline{S}$ be an isometry. Then the desired isometry is $\psi_0 \oplus \psi_1$ on $R \oplus S$. If $S \simeq \overline{S}$ then $R \simeq \overline{R}$. Hence we may assume $S \not\simeq \overline{S}$. We may also assume, without loss of generality, that S contains a hyperbolic plane. Then $S \simeq H(0) \oplus T$, $\overline{S} \simeq H(\rho) \oplus T$ where T is a direct sum of hyperbolic planes, or empty. Then $R \not\simeq B(0)$, $R \not\simeq B(\rho)$

$$\overline{R} \simeq B(0), \; \overline{R} \not\simeq B(\rho)$$

by Proposition 2.3 (1,2).

<u>Thus</u> $R \simeq E(\epsilon,\delta)$, $\overline{R} \simeq E(\overline{\epsilon},\overline{\delta})$. Therefore ord $v^2 = 1$.
<u>Let</u> $R = vZ + wZ$, $\overline{R} = \overline{v}Z + \overline{w}Z$, where $v^2 = 2\delta$, $v \cdot w = 1$, $w^2 = \epsilon$

$$\overline{v}^2 = 2\overline{\delta}, \; \overline{v} \cdot \overline{w} = 1, \; \overline{w}^2 = \overline{\epsilon}.$$

Let $x \in S$ with $2\delta x^2 = 2(\overline{\epsilon}\overline{\delta} - \epsilon\delta) = \det \overline{R} - \det R$.
<u>Let</u> $R' = (v+x)Z + wZ$. <u>Note</u>: $\det R' = \det \overline{R}$ and $v \in R'$.

Write $L = R' \oplus S'$. Then $S' \simeq \overline{S}$. Therefore $R' \simeq \overline{R}$.

Chapter Three - Vectors with One Critical Index.

Definition: Let L be any lattice over a local field.

v a vector in L.

Let $L^{(i)} = \{v \in FL : v \cdot y \in \pi^i Z, \quad y \in L\}$ where

π is a prime element of F.

$$e_i = \min_{y \in L^{(i)}} \operatorname{ord} v \cdot y$$

We make the following definitions

1. If $e_{i-1} = e_i = (e_{i+1} - 1)$ then i is called a critical index of v.
2. If i is a critical index of v, then $(e_i - i)$ is called the critical exponent of v corresponding to the critical index i.

We shall always use the following notation

1. $\lambda_1, \lambda_2 \ldots$ are the critical indices of v in increasing order.
2. $f_1, f_2 \ldots$ are the corresponding critical exponents.
3. $s_i = \lambda_{i+1} + f_{i+1} - \lambda_i = f_i$

The following Theorem gives a better insight into the meaning and importance of the critical indices and exponents.

Theorem 3.1: Let v have critical indices λi, exponents fi.

Then there is a Jordan Splitting $L = \sum_{-m}^{n} \oplus Li$ with Li empty or 2^i-modular, such that v has representation $v = \Sigma \oplus 2^{h_i} v_{k_i}$ with v_{k_i} maximal in L_{k_i} and the following conditions holding:

1. $h_1 > h_2 > \dots$.

2. $h_1 + k_1 < h_2 + k_2 < \dots$.

Furthermore, for any Jordan Decomposition satisfying the above two conditions, we have that

$$h_i = f_i$$
$$k_i = \lambda_i$$

The previous Definition and Theorem are due to Rosenzweig [9], as are the following important facts about critical indices.

Let $L = \sum_{-m}^{n} \oplus L_i$ where L_i is 2^i-modular as empty.

$v = \Sigma \oplus 2^{h_i} v_i$ with v_i maximal in L_i (or possibly zero)

Suppose k is not a critical index of v. Then:

1. If $k < \lambda_1$ then $h_k \geq f_1 + \lambda_1 - k$.
2. If $\lambda_j < k < \lambda_{j+1}$ then $h_k \geq f_j$ when $f_j < k \leq f_j + s_j$.

   $h_k + k \geq f_{j+1} + \lambda_{j+1}$ when $f_j + s_j \leq k < \lambda_{j+1}$
3. If $k < \lambda_j$ then $h_k > f_j$
4. If $k > \lambda_j$ then $h_k + k > \lambda_j + f_j$
5. $\sum_{-m}^{\lambda_i + s_i} \oplus 2^{h_j} v_j$ has critical indices $\lambda_1 \dots \lambda_i$.

<u>Lemma</u> 3.1: $L = L_o \oplus \dots \oplus L_n$ L_i is a 2^i-modular.

v, w have critical index λ. $v \sim w$

$\rightarrow v_\lambda$ is of the same type as w_λ in L_λ.

<u>Proof</u>: $v, w \in L^{2^\lambda Z} = \{y: y \in L$ and $y \cdot L \in 2^\lambda Z\}$

$$= L_\lambda \oplus \{L_{\lambda+1} \oplus 2\, L_{\lambda-1}\} \oplus \dots$$

<u>Then</u> $n(L^{2^\lambda Z}) = 2^\lambda Z$.

<u>But</u> ord $y^2 \geq \lambda+1$ for $\forall\, y \in L^{2^\lambda Z}$ such that $y \cdot v = o$ iff $v\lambda$ is saturated in $L\lambda$.

<u>Hence</u>: v_λ is saturated in L_λ iff w_λ is saturated in L_λ, Since the above property is preserved under isometries on L. Now a method virtually the same as that used in Lemma 2.1 (4) shows that $v \cdot y \equiv w \cdot y \bmod 2$ $\forall\, y \in L^{2^\lambda Z}$.

<u>Lemma</u> 3.2: Let L be unimodular, v saturated in L. Let $y_i \in L$ have $|y_i^2| = 1$ $i = 1,2$. Then $\dfrac{(v \cdot y_1)^2}{y_1^2} \equiv \dfrac{(v \cdot y_2)^2}{y_2^2} \bmod 2$.

<u>Proof</u>: Let $v = \Sigma \alpha_i\, x_i$ $L = \oplus \langle x_i \rangle$. $x_i^2 = \Delta_i$ assume v is maximal where $\alpha_i^2\, \Delta_i \equiv \alpha_j^2 \Delta_j \bmod 2$.

$$y_1 = \Sigma\, \beta_i x_i \qquad y_2 = \Sigma\, \overline{\beta}_i x_i .$$

Then
$$\frac{(v \cdot y_1)^2}{y_1^2} = \frac{(\Sigma\, \alpha_i \beta_i \Delta_i)^2}{\Sigma\, \beta_i^2 \Delta_i} = \frac{\Sigma\, \alpha_i^2 \beta_i^2 \Delta_i^2}{\Sigma\, \beta_i^2 \Delta_i}$$

$$\equiv \alpha_1^2 \Delta_1 \frac{\Sigma\, \beta_i^2 \Delta_i}{\Sigma\, \beta_i^2 \Delta_i} = \alpha_1^2 \Delta_1 \equiv \frac{(v \cdot y_2)^2}{y_2^2} \bmod 2.$$

This Lemma indicates the existence of a very important invariant needed to show the equivalence of two vectors.

<u>Definition</u>: Let L be 2^k-modular. Let $x \in L$ $|x^2| = 1$ if L is proper. Then let $T(v) \equiv o \bmod 2$ if v is unsaturated.

$$T(v) \equiv \frac{(v \cdot x)^2}{x^2} \bmod 2 \text{ if } v \text{ is saturated.}$$

$T(v)$ is called the <u>type</u> of v. Clearly $T(v) = T(w)$ iff v and w are of the same type. Note that T is a mapping of L into the residue class field of F.

<u>Proposition</u> 3.1. Let $L = L_{-m} \oplus .. \oplus L_n$ be a J.D. for L where L_k is 2^k-modular. Let $v = v_o \oplus .. \oplus v_n$ be the representation of some vector v with respect to the above decomposition. Assume v has o as its only critical index and that $T(v_o) \neq o$ in L_o. Furthermore, assume that $1 + \frac{v_1^2 + .. + v_n^2}{T(v_o)}$ is a square.

Then $v \sim w$ where $w \in L_o$.

<u>Proof</u>: We may write $L_o = \Sigma \oplus \langle x_i \rangle$ where $v_o = \Sigma \oplus x_i$. Then if we let $\bar{L}_o = \langle x_1 \oplus v_2 \oplus .. \oplus v_n \rangle \oplus \langle x_2 \rangle \oplus \ldots$ then clearly $\bar{L}_o \simeq L_o$. If we write $L_o \oplus .. \oplus L_n = \bar{L}_o \oplus K$, then a simple application of Theorem 1.10 gives us that $L_1 \oplus .. \oplus L_n \simeq K$. Thus there is a J.D. of L given by $L = K_{-m} \oplus .. \oplus K_n$ such that $L_i \simeq K_i$, $v \in k_o$. Let φ be an isometry on L which maps k_i onto L_i. Then $w = \varphi(v)$ is the desired vector.

We now wish to find when two vectors having the same critical index λ are isometric. We first do the more difficult case where v_λ and w_λ are saturated in L_λ. We may assume, by scaling that $\lambda = o$.

Proposition 3.2. Let $L = L_{-m} \oplus \dots \oplus L_n$ with L_k 2^k-modular or empty. $v = v_{-m} \oplus \dots \oplus v_n$ with $v_k \in L_k$. Let $w = w_o$ $w_o \in L_o$. v, w maximal with critical index o. $v^2 = w^2$. Furthermore let w_o be saturated in L_o. v_o be saturated in L_o. $T(v_o) = T(w_o)$ in L_o. Then $v \sim w$ iff $1 + \frac{v_{-m}^2 + \dots + v_{-1}^2}{T(v_o)}$ is a square.

Proof: First we remark that the statement $1 + \frac{v_{-m}^2 + \dots + v_{-1}^2}{T(v_o)} \in F^2$ makes good sense for if $\eta^2 = 1 + \frac{2\alpha}{T}$ where $\alpha \in Z$, T a unit.

Then if $\lambda \in Z$, $1 + \frac{2\alpha}{T+2\lambda} \equiv 1 + \frac{2\alpha}{T} \mod 4\alpha$. But $\alpha \equiv o \mod 2$ by Theorem 1.1 (c). Therefore $\frac{2\alpha}{T} \equiv o \mod 8$. Thus $1 + \frac{2\alpha}{T+2\lambda}$ is a square by Hensel's Lemma.

Necessity: By Proposition 3.1 we may assume $v_{-3-k} = o$, $v_{3+k} = o$, $k \geq o$, since $\frac{T + \alpha + 8\eta}{T}$ is a square iff $\frac{T+\alpha}{T}$ is when $|T| = 1$. Assume $\frac{v_{-2}^2 + v_{-1}^2 + T(v_o)}{T}$ is not a square.

Let us also assume $v^2 \neq o$ for the present time. The procedure which will be used is to examine the Jordan Decompositions of $<v>^{\perp}$ and $<w>^{\perp}$ and show that one of the conditions 2 or 3 fails to hold. There are three case to be examined

1. $v_{-1} = o$, ord $v_{-2}^2 = +2$
2. ord $v_{-1} = +1$
3. ord $v_{-1} = +2$.

Case 1. $v_{-1} = o$ ord $v_{-2}^2 = +2$.

Let $L_o = <x_1> \oplus \ldots \oplus <x_s>$ $v_o = x_1 + \ldots + x_s$.

$T(v_o) = x_1^2 = \Delta_1$.

We obtain a new Jordan Decomposition for L in the following way:

a) Let $\bar{L}_i = L_i$ when $i \neq -2, o$.

b) Replace the vector x_1 in L_o by $\bar{x}_1 = x_1 + v_{-2}$ to obtain $\bar{L}_o$.

c) Find a new lattice $\bar{L}_{-2}$ such that $\bar{L}_{-2} \oplus \bar{L}_o = L_{-2} \oplus L_o$. L_{-2} is proper and $L_{-2} = <y_1> \oplus \ldots \oplus <y_t>$ where $4y_1 = v_{-2}$. Thus $\bar{L}_{-2}$ may be obtained by replacing y_1 by some linear combination of y_1 and x_1, call it $\bar{y}_1$. Note:

$$\det \bar{L}_{-2}/\det L_{-2} = \det \bar{L}_0/\det L_0 = 1 + 4\rho.$$
$$= \frac{T(v_0) + v_{-2}^{\ 2}}{T(v_0)} .$$

<u>The Jordan Decompositions for</u> $\langle v\rangle^{\perp}$ <u>and</u> $\langle w\rangle^{\perp}$ <u>have the following forms:</u>

$$\langle v\rangle^{\perp} = L_{-m} \oplus \dots \oplus L_{-3} \oplus \bar{L}_{-2} \oplus L_{-1} \oplus M \oplus \dots$$
$$\langle w\rangle^{\perp} = L_{-m} \oplus \dots \oplus L_{-3} \oplus L_{-2} \oplus L_{-1} \oplus N \oplus \dots$$

where $n(M), n(N) \subseteq 2Z$. We wish to show $\langle v\rangle^{\perp} \not\cong \langle w\rangle^{\perp}$

<u>Sub case</u> a) L_{-1} <u>is improper non-zero</u>.

Then $\dfrac{n(L_{-1})\cdot n(M)}{s(L_{-1})^2} \subseteq 8Z.$

But $\dfrac{d(\bar{L}_{-m}).\ \dots\ .\ d(\bar{L}_{-1})}{d(L_{-m}).\ \dots\ .\ d(L_{-1})} = 1 + 4\rho.$

Thus condition (2) of Theorem 1.10 is violated.

<u>Sub case</u> b) L_{-1} <u>is empty</u>. $\dfrac{n(L_{-2})\cdot n(M)}{x(\bar{L}_{-2})^2} \subseteq 8Z.$

<u>Similarly</u>: $\dfrac{d(\bar{L}_{-m})\ .\ \dots\ .\ d(\bar{L}_{-2})}{d(L_{-m})\ .\ \dots\ .\ d(L_{-2})} = 1 + 4\rho.$

<u>Sub case</u> c) L_{-1} <u>is proper</u> Then $n(M) \subseteq 4\, n(L_{-1})$.

For condition 3 of Theorem 1.10 to hold we must have $f(L_{-m} \oplus \dots \oplus L_{-1}) \to F(\bar{L}_{-m} \oplus \dots \oplus \bar{L}_{-1}) \oplus \langle \tfrac{1}{2}\rangle$.

<u>By Witt's Theorem</u> $F\, L_{-2} \to F\, \bar{L}_{-2} \oplus \langle \tfrac{1}{2}\rangle$, or

$\langle y_1\rangle \cdot \to \langle \bar{y}_1\rangle \oplus \langle \tfrac{1}{2}\rangle$.

Now: Let $y_1^2 = \frac{\epsilon}{4}$. Then $\bar{y}_1^2 = \frac{\epsilon}{4}(1 + 4\rho)$

$(1 + 4\rho$ has defect $= 4Z)$. Therefore $<\frac{\epsilon}{4}>$
$\to <\frac{\epsilon}{4}> \to <\frac{\epsilon}{4}(1 + 4\rho)> \oplus <\frac{1}{2}>$.

Applying Theorem 1.4 we have:

$$<\frac{\epsilon}{4}> \oplus <\frac{1 + 4\rho}{2}> \simeq <\frac{\epsilon}{4}(1 + 4\rho)> \oplus <\frac{1}{2}>.$$

or $<\epsilon> \oplus <2(1 + 4\rho)> \simeq <\epsilon(1 + 4\rho)> \oplus <2>$.

Applying Theorem 1.3 and using the simple facts about Hasse Symbols given in Chapter I we can easily see that the above Quadratic Spaces are not isometric. Hence $<v>^{\perp} \not\simeq <w>^{\perp}$.

Case 2. ord $v_{-1}^2 = 1$

Using a similar procedure as before we may write.

$$<v>^{\perp} = L_{-m} \oplus \ldots \oplus \overline{L_{-2}} \oplus \overline{L}_{-1} \oplus M \oplus \ldots$$

$$<w>^{\perp} = L_{-m} \oplus \ldots \oplus L_{-2} \oplus L_{-1} \oplus N \oplus \ldots$$

$n(M), n(N) \subseteq 2$.

$$\frac{d(L_{-m}) \cdot \ldots \cdot d(\overline{L}_{-2}) \cdot d(\overline{L}_{-1})}{d(L_{-m}) \cdot \ldots \cdot d(L_{-2}) \cdot d(L_{-1})} = 1 + 2\epsilon \quad \epsilon \text{ a unit.}$$

But $\frac{n(L_{-1})\, n(N)}{s(L_{-1})^2} \subseteq 4Z$. Thus condition 2 of Theorem 1.10 is violated.

Case 3. ord $v_{-1}^2 = 2$.

determinantal arguments like those used above work when L_{-1} is improper.

<u>Assume</u> L_{-1} <u>is proper</u>. In this case we may assume

$v_{-2} = o$ for $\frac{v_{-1}^2 + v_{-2}^2}{T(v_o)} + 1$ is a non-square.

Therefore $1 + \frac{v_{-2}^2}{T(v_o)}$ is a square or $1 + \frac{v_{-1}^2}{T(v_o)}$ is a square. In the first case we may assume $v_{-2} = o$ and the second case reduces to Case 1 above.

<u>Now write</u>: $L_{-1} = \langle y_1 \rangle \oplus \langle y_2 \rangle \oplus R$ with $y_1^2 = \frac{\varepsilon_1}{2}$, $y^2 = \frac{\varepsilon_2}{2}$ with $v_{-1} = y_1 + y_2$.

We already have $L_o = \langle x_1 \rangle \oplus .. \oplus \langle x_s \rangle$ with $T(v_o) = x_1^2 = \Delta_1$ and $x_i^2 = \Delta_i$.

We make an alteration in the Jordan Decomposition in the following manner: We leave all modular components fixed except for L_{-1} and L_o. We also leave R fixed. Now replace x_1 by $x_1 = 2y_1$ which we call $\bar{x}_1$. Replace y_1 by some vector $\bar{y}_1 = \alpha y_1 + \beta x_1$ with $\alpha, \beta \subset Z$, $|\alpha| = 1$ where $\bar{y}_1 \cdot \bar{x}_1 = o$. Now $\bar{y}_1^2 = C\, y_1^2$ where a determinental argument shows $\partial(C) = 2Z$.

Note that $C = \frac{x_1^2 + 2\varepsilon_1}{x_1^2}$

Now, having done this, we repeat the procedure with y_2 and $\bar{x}_1$ replacing $\bar{x}_1$ be a vector $\bar{\bar{x}}_1 = \bar{x}_1 + 2y_2$ and y_2 be a vector $\bar{y}_2 = \bar{\alpha}\, y_2 + \bar{\beta}\, \bar{x}_1$ with $|\bar{\alpha}| = 1$ and $\bar{\beta} \in Z$ such that $\bar{y}_2 \cdot \bar{\bar{x}}_1 = o$.

Note: a) $\bar{y}_2 \cdot \bar{y}_1 = o$

b) $\bar{y}_2^{\,2} = D\, y_2^{\,2}$ where $D = \dfrac{x_1^{\,2} + 2\epsilon_1 + 2\epsilon_2}{x_1^{\,2} + 2\epsilon_1}$

c) $\partial(CD) = 4$ by hypothesis.

So we now have a new Jordan Decomposition for L

$$L = L_{-m} \oplus \ldots \oplus L_{-2} \oplus \bar{L}_{-1} \oplus \bar{L}_o \oplus L_1 \oplus \ldots \quad \text{with}$$

$\bar{L}_{-1} = \langle \bar{y}_1 \rangle \oplus \langle \bar{y}_2 \rangle \oplus R$ and $v \cdot \bar{L}_{-1} = o$.

Hence $\langle v \rangle^{\perp} = L_{-m} \oplus \ldots \oplus \bar{L}_{-1} \oplus M \oplus \ldots$

$\langle w \rangle^{\perp} = L_{-m} \oplus \ldots \oplus L_{-1} \oplus \bar{M} \oplus \ldots$ with $n(M)$,

$n(\bar{M}) \subseteq 2Z$. Since w is saturated. We wish to show $\langle v \rangle^{\perp} \not\cong \langle w \rangle^{\perp}$. Note: $n(M) \subseteq 4n(\bar{L}_{-1})$. So if we can show $F(L_{-m} \oplus \ldots \oplus \bar{L}_{-1}) \not\rightarrow F(L_{-m} \oplus \ldots \oplus L_{-1}) \oplus \langle \tfrac{1}{2} \rangle$

That is we wish to show:

$$\langle \bar{y}_1 \rangle \oplus \langle \bar{y}_2 \rangle \not\rightarrow \langle y_1 \rangle \oplus \langle y_2 \rangle \oplus \langle \tfrac{1}{2} \rangle$$

By Theorem 1.4, we wish to show:

$$\langle C\, \epsilon_1 \rangle \oplus \langle D\, \epsilon_2 \rangle \oplus \langle CD \rangle \not\cong \langle \epsilon_1 \rangle \oplus \langle \epsilon_2 \rangle \oplus \langle 1 \rangle$$

A calculation of hasse symbols, using the fact that $\partial(CD) \subseteq 4Z$ shows that the above statement is equivalent to the statement: $(C, C\epsilon_1\epsilon_2) = -1$ or equivalently, the lattice $\langle C \rangle \oplus \langle C\epsilon_1\epsilon_2 \rangle \oplus \langle -1 \rangle = k$. is anisotropic.

We may assume, by scaling that $x_1^{\,2} = 1$. In this case,

$C = 1 + 2\epsilon_1$

$\partial(1+2\epsilon_1+2\epsilon_2) = 4Z$

By Proposition 2.3 (3) we need only show that k contains a lattice isometric to $H(p)$.

Let $x^2 = C \qquad y^2 = C\epsilon_1\epsilon_2 \qquad Z^2 = -1$

Then $k \simeq \langle x\rangle \oplus \langle y\rangle \oplus \langle Z\rangle$

Let $J = (\epsilon_1 x+y)Z + (\frac{x+Z}{C\epsilon_1})Z$.

$$\text{Then } J \simeq \begin{vmatrix} \epsilon_1^2 C + C\epsilon_1\epsilon_2 & 1 \\ 1 & \frac{C-1}{C^2\epsilon_1^2} \end{vmatrix}$$

$$\simeq \begin{vmatrix} C\epsilon_1(\epsilon_1+\epsilon_2) & 1 \\ 1 & \frac{2}{C^2\epsilon_1} \end{vmatrix}$$

Since $\epsilon_1 + \epsilon_2 \equiv o \mod 2$ $\quad J$ is improper.

We need only show $\det J \equiv -1 + 2(\epsilon_1+\epsilon_2) \mod 8$.

But $\det J = -1 + \frac{2(\epsilon_1+\epsilon_2)}{C}$

$$= -1 + 2(\epsilon_1+\epsilon_2) + (2\epsilon_1+4\epsilon_1^2+\ldots)(2(\epsilon_1+\epsilon_2))$$

$$\equiv -1 + 2(\epsilon_1+\epsilon_2) \mod 8.$$

Thus we have proven necessity of the conditions when $v^2 \neq o$. But when $v^2 = o$ we have $\langle v\rangle^{\perp} = S \oplus \text{rad } \langle v\rangle^{\perp}$

$\langle w\rangle^{\perp} = \overline{S} \oplus \text{rad } \langle w\rangle^{\perp}$

now we show $S \cong \overline{S}$ by exactly the same method.

Sufficiency: In this case we show there is a second Jordan decomposition $L = \overline{L}_{-m} \oplus \ldots \oplus \overline{L}_n$ in which the representation of the vector $v = \overline{v}_{-m} \oplus \ldots \oplus \overline{v}_n$ has the property

$\bar{v}_{-m} = .. = \bar{v}_{-1} = \bar{v}_{+1} = .. = \bar{v}_n = o$, and $L_i \simeq \bar{L}_i$.

This implies that $v \sim \bar{v}$ where $\bar{v} \in L_o$. We then apply Theorem 1.3.

So let $L_o = \langle x_1 \rangle \oplus \ldots \oplus \langle x_s \rangle$

with $v_o = x_1 \oplus .. \oplus x_n$

and $T(v_o) = x_1^2$.

We obtain the new Jordan decomposition of L as follows:

First we obtain an intermediate decomposition by replacing x_1 by $v_{-m} \oplus .. \oplus v_{-1} \oplus x_1$ to obtain L_o'.

Now $L_o' \simeq L_o$ since $\dfrac{v_{-m}^2 + .. + x_1^2}{x_1^2}$ is a square.

So now we may find a new decomposition of the lattice

$$L_{-m} \oplus \ldots \oplus L_o$$

which may be written $\bar{L}_{-m} \oplus \ldots \oplus \bar{L}_{-1} \oplus L_o'$ such that $\bar{L}_{-n} \simeq L_{-n}$.

Now we can do the same with the lattice $L_o' \oplus L_1 \oplus .. \oplus L_n$ provided we can show $\dfrac{T(v_o) + v_1^2 + .. + v_n^2}{T(v_o)}$ is also a square.

Now $1 + \alpha + \beta$ is a square if $1 + \alpha + \beta$ and $1 + \beta$ are squares with $|\alpha|, |\beta| < 1$. So we need only show, from the hypotheses that $\dfrac{v_{-m}^2 + .. + v_{-1}^2 + T(v_o) + v_1^2 + .. + v_n^2}{T(v_o)}$ is a square. That is $\dfrac{w^2 - v_o^2 + T(v_o)}{T(v_o)}$ is a square.

Now $v_o = x_1 \oplus \ldots \oplus x_s$.

But w has the same type as v_o. Thus

$w = (1+2\eta_1)x_1 \oplus .. \oplus (1+2\eta_s)x_s$ where $\eta_i \in Z$.

Therefore now $(1+2\eta_i)^2$ is, of course, a square

Hence $\dfrac{x_i^2(4\eta_i+4\eta_i^2) + x_1^2}{x_1^2}$ is a square since

$x_i^2 \equiv x_1^2(2)$.

Now $\dfrac{w^2 - v_o^2 + T(v_o)}{x_1^2} = \dfrac{x_1^2 + \Sigma(4\eta_i+4\eta_i^2)\, x_i^2}{x_1^2}$

But $1 + \dfrac{(4\eta_i+4\eta_i^2)x_i^2}{x_1^2}$ is a square $\forall_i$

Therefore $1 + \sum_i \dfrac{(4\eta_i+4\eta_i^2)x_i^2}{x_1^2}$ is a square Q.E.D.

So finally we have our result for saturated vectors:

<u>Proposition</u> 3.3. $L = L_{-m} \oplus \ldots \oplus L_n$ The Jordan Decomposition. L_i is 2^i-modular or empty. $v, w \in L$ are maximal, with critical index o. $v^2 = w^2$. $T(v_o) = T(w_o) \neq o$.

Then $v \sim w$ iff $\lambda = \dfrac{(v_{-m}^2+..+v_o^2) - (w_{-m}^2+..+w_o^2)}{T(v_o)} + 1$

is a square.

<u>Proof</u>: I. <u>Necessity</u>: Suppose the above expression is not a square. First note λ is a square iff

$\dfrac{(v_{-m}^2+\ldots+v_{-1}^2) - (w_{-m}^2+..+w_{-1}^2)}{T(v_o)} + 1$ is a square.

By the method used in the last part of the previous proposition we can find a new Jordan Decomposition

$$L = \bar{L}_{-m} \oplus \ldots \oplus \bar{L}_o \oplus L_1 \oplus L_2 \ldots$$

such that if $v = \bar{v}_{-m} \oplus \ldots\ldots \oplus \bar{v}_n$

$$w = \bar{w}_{-n} \oplus \ldots\ldots \oplus \bar{w}_n$$

with respect to this decomposition then

$$\bar{v}_{-n_1} \ldots \bar{v}_{-1} = o.$$

But $(\bar{v}_{-m}{}^2 + \ldots + \bar{v}_o{}^2) - (\bar{w}_{-m}{}^2 + \ldots + \bar{w}_o{}^2) = (v_{-m}{}^2 + \ldots + v_o{}^2)$

$$- (w_{-m}{}^2 + \ldots + w_o{}^2)$$

So now we may apply proposition 3.2

II. Sufficiency: The facts that $v^2 = w^2$ and $\frac{(v_{-m}{}^2 + .. + v_o{}^2) - (w_{-m}{}^2 + .. + w_o{}^2)}{T(v_o)} + 1$ is a square imply that $\frac{(v_o{}^2 + .. + v_n{}^2) - (w_o{}^2 + \ldots + w_n{}^2)}{T(v_o)} + 1$ is a sqyare.

Choose a Jordan Decomposition $L = \bar{L}_{-m} \oplus \ldots \oplus \bar{L}_n$ in which the above elements are still squares and in which $v \in L_o$. (This is done by "absorbing" the "left hand side" of v into $\bar{L}_o$, altering $L_{-m} \ldots\ldots L_{-1}$, then "absorbing" the right hand side and altering $L_1, \ldots, L_n$).

Now apply proposition 3.2.

Proposition 3.4: Let $L = J \oplus k = \bar{J} \oplus \bar{k}$ where $J, \bar{J}$ are modulars, improper. Then if $J \simeq \bar{J}$ we have $k \simeq \bar{k}$.

Proof: Assume J is unimodular: Write $J \cong H(o) \oplus \ldots \oplus H(o) \oplus H(\lambda)$

$$\bar{J} \cong H(o) \oplus \ldots \oplus H(o) \oplus H(\lambda)$$

where $\lambda = o$ or ρ.

Then $k \oplus H(\lambda) \simeq \bar{k} \oplus H(\lambda)$ by Theorem 1.9(a).

Thus $k \oplus H(\lambda) \oplus H(\lambda) \simeq \bar{k} \oplus H(\lambda) \oplus H(\lambda)$

But $H(\lambda) \oplus H(\lambda) \simeq H(o) \oplus H(o)$.

Thus $k \simeq \bar{k}$ by Theorem 1.9(a).

Proposition 3.5: $v, w \in L$. $v^2 = w^2$, v, w have one critical index λ, and are maximal. $T(v_\lambda) = T(w_\lambda) = o$. Then $v \sim w$. We may assume $\lambda = o$.

Proof: Choose a Jordan decomposition $L = L_{-m} \oplus \dots \oplus L_n$. with $v \in L_o$.

There are two cases:

Case 1: $\text{ord } v^2 = o$.

Then $\langle v\rangle^\perp$ and $\langle w\rangle^\perp$ have proper unimodular components since v_o is unsaturated. Furthermore $L = \langle v\rangle \oplus \langle v\rangle^\perp = \langle w\rangle \oplus \langle w\rangle^\perp$. We are now able to apply Theorem 1.9(b) to show that $\langle v\rangle^\perp \simeq \langle w\rangle^\perp$.

Case 2: $\text{ord } v^2 \geq 1$

We have $v \in L_o$. If L_o contains a hyperplane H we may map w_o onto a vector $\bar{w}_o \in H$ where $H = \bar{w}_o Z + xZ$ with $\bar{w}_o^{\,2} = \Delta \quad \bar{w}_o \cdot x = 1 \quad x^2 = o \qquad (1)$

by an integral isometry which leaves every element in L_i fixed when $i \neq o$. This follows from Theorem 2.3. Thus we may assume w_o satisfies conditions (1) for some vector $x \in L_o$. Now it is clear that $k = wZ + xZ$ is a hyperplane which splits L (Theorem 1.6) and contains w. Thus $L = H \oplus J = k \oplus \bar{J} \quad H \simeq k \simeq H(o)$

$v \in H$ $w \in k$ $J \simeq \overline{J}$ by Theorem 1.9(a). Now let $\psi(J) = \overline{J}$ be an isometry. We have an isometry $\varphi(H) = k$ with $\varphi(v) = w$ by Theorem 2.3. Thus $\varphi \oplus \psi$ on $H \oplus J$ is an isometry on L which maps v onto w.

Now suppose L_o contains no hyperplanes $H(o)$.

We may imbed w_o in an improper lattice H which splits L by Proposition 2.2. Thus $H \simeq H(\rho)$. Similarly imbed v in an improper lattice $\overline{H} \simeq H(\rho)$ which splits L.

Write: $H = w_oZ + xZ$ with multiplication table $\begin{vmatrix} \Delta & 1 \\ 1 & E \end{vmatrix}$

$\overline{H} = vZ + \overline{x}Z$ with multiplication table $\begin{vmatrix} \overline{\Delta} & 1 \\ 1 & \overline{E} \end{vmatrix}$

$\Delta E \equiv \overline{\Delta E} \equiv 4\rho \mod 8$.

Now let $H' = wZ + xZ$, if $H' \simeq H$ we obtain our result by applying Proposition 3.4 and Theorem 2.3. If $H' \not\simeq H$, then $H' \simeq H(o)$ and clearly $\operatorname{ord} {w_{-1}}^2 = 1$ or $\operatorname{ord} {w_1}^2 = 1$ where $w = \Sigma \oplus w_i$ $w_i \in L_i$.

Assume ${w_1}^2 = 2\eta$ $|\eta| = 1$. Since the residue class field is perfect (every element is a square) there is a unit ϵ such that $\overline{E} + \epsilon^2 {w_1}^2 \equiv o \mod 4$.

Let $H^{(2)} = vZ + (\overline{x} + \epsilon w_1)Z$ $H^{(2)} \simeq H(o) \simeq H'$.

Then $H^{(2)}$ and H' split L and $v \in H^{(2)}$, $w \in H'$.

Apply Proposition 3.4 and Theorem 2.3 to get that $v \sim w$.

Collecting the results of this Chapter we have the following Theorem:

Theorem 3.1: Let $L = L_o \oplus .. \oplus L_n$ be the Jordan Decomposition of a lattice.

Let $v, w \in L$ be maximal vectors with $v^2 = w^2$.

v, w have the same critical index λ.

$v = \Sigma \oplus v_i$ with the above decomposition

$w = \Sigma \oplus w_i$

Then $v \sim w$ iff 1. $T(v_\lambda) \equiv T(w_\lambda) \bmod 2$ (in L_λ).

2. if $T(v_\lambda) \neq 0$ then

$$1 + \frac{(v_o^2 + \ldots + v_\lambda^2) - (w_o^2 + \ldots + w_\lambda^2)}{T(v_\lambda) \cdot 2^\lambda}$$

is a square in F.

We have also proven the following Theorem which is interesting but difficult to apply:

Theorem 3.2: $v, w \in L$ are maximal $v^2 = w^2$ v, w have the same critical index λ. Then $v \sim w$ iff

1. $\langle v \rangle^\perp \simeq \langle w \rangle^\perp$
2. $T(v_\lambda) \equiv T(w_\lambda) \bmod 2$

Chapter Four - The General One-Dimensional Case.

Notation: When we have more than one Jordan Decomposition, will number them + number the components of each vector to indicate with which Jordan Form they are decomposed.

Example: $\mathcal{L}(1) = \Sigma \oplus L_i$ and $\Sigma \oplus \overline{L}_i = \mathcal{L}(2)$

we write $v = \oplus v_{(i)}^{(1)} = \oplus v_{(i)}^{(2)}$, $v_{(i)}^{(1)} \in L_i$

$v_i^{(2)} \in \overline{L}_i$.

We are also going to assume $s(L) = Z$. This of course will not lose us any generality. We will also write our Jordan Decompositions in the form; $L = \Sigma \oplus L_i$ where L_i is 2^i-modular or empty.

Definition: 1. $s_i = \lambda_{i+1} + f_{i+1} - \lambda_i - f_i > o$.

2. $v_{(i)} = v_o \oplus \dots \oplus v_{\lambda_i + s_i}$

3. $v_{[i]} = v_{(i)} - v_{(i-1)}$

4. $v_i = 2^{g_i} \hat{v}_i$ where $\hat{v}_i$ is maximal in L_i.

Of course the last three definitions depend on the decomposition chosen + must be numbered accordingly if there is more than one decomposition.

Lemma 4.1: Let $v \sim w$. Then 1. v, w have the same critical indices and exponents $\{\lambda_i, f_i\}$

2. $T(v_{\lambda_i}) = T(w_{\lambda_i})$ in L_{λ_i}.

Proof: 1. See [9].

2. a) First we show v_{λ_i} is saturated iff w_{λ_i} is saturated

2. If L_{λ_i} is improper, the result is trivial

If L_{λ_i} is proper let $L^{2^{\lambda_i}} = \{x \in L: x \cdot L \in 2^{\lambda_i} Z\}$

$$= L_{\lambda_i} \oplus \{2\, L_{\lambda_i - 1} + L_{\lambda_i + 1}\} \oplus \ldots$$

Define: $M = \{y \in L^{2^{\lambda_i}Z} : v \cdot y = o\}$.

We shall show that $\{\forall\, x \in M,\ \text{ord}\ x^2 > \lambda_i\}$ iff v_{λ_i} is saturated. This proves the theorem since an isometry on L induces an isometry on M, so if $v \sim \bar{v}$, $M \simeq \bar{M}$.

First, it is easily seen that there is a Jordan Decomposition $\mathcal{L}(2)$ such that $v = \oplus\, v_{\lambda_j}^{(2)}$ and $v_{\lambda_i}^{(2)}$ is saturated in $\bar{L}_{\lambda_i}$ off $v_{\lambda_i}^{(1)}$ is saturated in L_{λ_i}.

If $v_{\lambda_i}^{(2)}$ is unsaturated, there is, by definition, a vector $x \in L_{\lambda_i}$ with $v_{\lambda_i}^{(2)} \cdot x = o$ and $|x^2| = |2^{\lambda_i}|$.

If $v_{\lambda_i}^{(2)}$ is saturated, ord $x^2 = \lambda_i$, $x \in L^{2^{\lambda_i}Z}$ then

$$x = 2^{\lambda_i} x_o \oplus 2^{\lambda_i - 1} x_1 \oplus \ldots \oplus x_{\lambda_i} \oplus x_{\lambda_i + 1} \oplus$$

Now if ord $x^2 = \lambda_i$, then ord $x_{\lambda_i}{}^2 = \lambda_i$, therefore ord $x_{\lambda_i} \cdot v_{\lambda_i} = \lambda_i + f_i$ thus there must be a vector $2^{\lambda_i - \lambda_j} x_j$ or $x_{\lambda_i + k}$ with ord $(2^{\lambda_i - \lambda_j} x_{\lambda_j} \cdot v_{\lambda_j}) \leq \lambda_i + f_i \ldots$ (a)

or ord $(x_{\lambda_i + k} \cdot v_{\lambda_i + k}) \leq \lambda_i + f_i \ldots$ (b)

But a) is false since $f_j > f_i$

b) is false since $\lambda_{i+k} + f_{i+k} > \lambda_i + f_i$.

b) We now show that if $y \in L^{2^{\lambda_i}Z}$ ord $v^2 = \lambda_i$ then $v \cdot y \equiv T(v_{\lambda_i}) \cdot 2^{\lambda_i + f_i} \bmod 2^{\lambda_i + f_i + 1}$

Once again assume $v = \Sigma \oplus v_{\lambda_i}$ $y = \Sigma \oplus y_i$.

Then $v \cdot y = \Sigma\, v_{\lambda_j} \cdot y_{\lambda_j}$.

Now if $j < i$, then $\operatorname{ord} v_j \cdot y_{\lambda_j} \geq \lambda_j + f_j + (\lambda_i - \lambda_j) \geq \lambda_i + f_i + 1$

$j > i$, then $\operatorname{ord} v_j \cdot y_{\lambda_j} \geq \lambda_j + f_j \geq \lambda_i + f_i + 1$

Thus $v \cdot y \equiv v_{\lambda_i} \cdot y_{\lambda_i} \bmod 2^{\lambda_i + f_i + 1} \equiv T(v_{\lambda_i}) \cdot 2^{\lambda_i + f_i} \bmod 2^{\lambda_i + f_i + 1}$

<u>Theorem</u> 4.1: Let v have principal indices λ_i, exponents f_i and $\bar{v} \sim v$.

Then: $\bar{v}_{(i)}{}^2 \equiv v_{(i)}{}^2 \bmod 2^{\lambda_{i+1} + f_{i+1} + f_i}$ when $L_{\lambda_i + s_i}$ is proper.

$\bar{v}_{(i)}{}^2 \equiv v_{(i)}{}^2 \bmod 2^{\lambda_{i+1} + f_{i+1} + f_i + 1}$ when $L_{\lambda_i + s_i}$ is improper.

(Empty. Lattices are defined to be improper).

<u>Proof</u>: This Theorem is an analogue of a theorem for the non-dyadic case. The proof is similar. See Lemma 2.5 of [9].

We let $L = M \oplus N$ $M = L_0 \oplus \ldots \oplus L_{\lambda_i + s_i}$

$N = L_{\lambda_i + s_{i+1}} \oplus \ldots \oplus L_n$.

Write: $v = r \oplus \bar{r}$ $r \in M$ $\bar{r} \in N$

$\bar{v} = s \oplus \bar{s}$ $s \in M$ $\bar{s} \in N$

Let $\varphi(v) = \bar{v}$

$\varphi(r) = t \oplus \bar{t}$ $t \in M$ $\bar{t} \in N$

$\varphi(\bar{r}) = u \oplus \bar{u}$ $u \in M$ $\bar{u} \in N$

We have: $\varphi(v) = (s-u) \oplus (\bar{s}-\bar{u})$.

Now: $2^{f_i} \mid r, s, u$. Thus $2^{f_i} \mid (\bar{s}-\bar{u})$.

Thus: $r^2 \equiv (s-u)^2 \bmod 2^{2f_i+\lambda_i+s_{i+1}}$

$$r^2 \equiv (s-u)^2 \bmod 2^{f_i+f_{i+1}+\lambda_{i+1}+1}$$

Now: $\operatorname{ord} u\cdot L \geq \operatorname{ord} \bar{r}\cdot L \geq \lambda_{i+1} + f_{i+1}$..

$$2^{f_i} \mid s..$$

Thus: $2\, s\cdot u \equiv o \bmod 2^{f_i+f_{i+1}+\lambda_{i+1}+1}$.

Hence: $r^2 \equiv s^2 + u^2 \bmod 2^{f_i+f_{i+1}+\lambda_{i+1}+1}$.

Notice that: $\varphi(v) = u \oplus \bar{u}$ has critical indices

$\lambda_{i+1}, \lambda_{i+2}\ldots$ exponents $f_{i+1}, f_{i+2} \ldots$

Thus: $u = \overset{\lambda_i+s_i}{\underset{o}{\oplus}} 2^{h_j} \mu_j \qquad \mu_j$ maximal in L_j.

$$h_j \geq f_{i+1} + (\lambda_{i+1}-j).$$

Therefore: $\operatorname{ord} (2^{2h_j}\mu_j^2) \geq h_j + \lambda_{i+1} + f_{i+1}+1$ when L_j is improper.

$\geq f_i + f_{i+1} + \lambda_{i+1}$ when L_j is proper.

Combining these two conditions, we have

$$r^2 \equiv s^2 \bmod 2^{f_i+f_{i+1}+\lambda_{i+1}} \text{ when } L_{\lambda_i+s_i} \text{ is proper.}$$

$$r^2 \equiv s^2 \bmod 2^{f_i+f_{i+1}+\lambda_{i+1}+1} \text{ when } L_{\lambda_i+s_i} \text{ is improper,}$$

which is what we wished to prove.

The following proposition is simple but important in its implications. It will be used many times over:

Proposition 4.2: $v, w \in L$. $\{k_i\}$ a partition of the numbers $o, 1 \ldots n$. $k_o = o$; $k_{\ell} = n+1$

Let $\oplus L_i$ be any Jordan Decomposition of L.

Let $L_{[i]} = L_{k_i} \oplus \ldots \oplus L_{k_{i+1}-1}$

$$v_{[i]} = v_{k_i} \oplus \ldots \oplus v_{k_{i+1}-1}$$

Then $v \sim w$ iff there exist Jordan Decompositions

$$\mathcal{L}_{(1)} = \oplus M_i \qquad \mathcal{L}_{(2)} = \oplus N_i$$

such that $M_{[i]} \simeq N_{[i]}$

and $v_{[i]}^{(1)} \simeq w_{[i]}^{(2)}$ between these isometric lattices.

In our application of this proposition, the Lattices $L_{[i]}$ will be chosen such that the component $v_{[i]}$ of v in $L_{[i]}$ has only one critical index.

Proposition 4.3: The unimodular lattice L represents η by an unsaturated maximal vector x. Then L represents $\eta + 4\lambda$ by an unsaturated maximal vector if $\lambda \in Z$.

Proof: Case I. $|\eta| = 1$ a) If ord $\lambda \geq 1$ we have $\langle\eta\rangle \simeq \langle\eta + 4\lambda\rangle$ and thus L clearly represents $\eta + 4\lambda$.

b) If ord $\lambda = o$. We may write $L = \langle x\rangle \oplus \langle y\rangle \oplus \overline{L}$ since x is unsaturated. Choose $\epsilon \in Z$ with $y^2\epsilon^2 \equiv \lambda \bmod 2$. Let $\overline{x} = x + 2\epsilon y$. Then $\overline{x}^2 \equiv \eta + 4\lambda \bmod 8$. Now use part a).

Case II. ord $\eta \geq 1$. Then x may be imbedded in a two-dimensional improper lattice k which splits L.

$$k \simeq \begin{vmatrix} \eta & 1 \\ 1 & \mu \end{vmatrix} \simeq \begin{vmatrix} \eta + 4\lambda & 1 \\ 1 & \mu \end{vmatrix} \quad \text{by Theorem 2.1.}$$

Proposition 4.3 a) Let L be unimodular, then assume L represents η by a saturated maximal vector s. Then L represents $\eta + 4\lambda$ where $1 + \dfrac{4\lambda}{T(x)}$ is a square, by a saturated maximal vector of the same type.

Proposition 4.4: Let v, w have the same critical indices and exponents, and satisfy congruence relations (1) with respect to $\mathcal{L}_{(1)} = \oplus L_i$. Then if v_{λ_1} is unsaturated in L_{λ_1} there exists a second canonical form $\mathcal{L}_{(2)} \sim \Sigma \oplus \overline{L}_i$ with $\overline{L}_i \simeq L_i$ and such that one of the following two congruence relations holds:

$$1. \quad \{w_{(1)}^{(2)}\}^2 \equiv \{v_{(1)}^{(1)}\}^2 \mod 2^{f_1+f_2+\lambda_2+1}$$

$$2. \quad \{w_{(1)}^{(2)} - w_{\lambda_1+s_1}^{(2)}\}^2 \equiv \{v_{(1)}^{(1)} - v_{\lambda_1+s_1}^{(1)}\}^2 \mod 2^{f_1+f_2+\lambda_2+1}$$

Remark: The importance of this Proposition lies in the fact that $v - v_{(1)}^{(1)}$ and $v - (v_{(1)}^{(1)} - v_{\lambda_1+s_1}^{(1)})$ both have one less critical index than v.

Proof: The result follows from relations (1) if $L_{\lambda_1+s_1}$ is improper. So assume $L_{\lambda_1+s_1}$ is proper.

Define: $$x = v_0^{(1)} \oplus \ldots \oplus v_{\lambda_1+s_1-1} = v_{(1)}^{(1)} - v_{\lambda_1+s_1}^{(1)} .$$

$$y = w_0^{(1)} \oplus \ldots \oplus w_{\lambda_1+s_1-1}^{(1)} = w_{(1)}^{(1)} - w_{\lambda_1+s_1}$$

Since $L_{\lambda_1+s_1}$ is proper it contains a vector z with $z^2 \equiv \{v_{(1)}{}^{(1)}\}^2 - y^2 \bmod 2^{\lambda_2+f_2+f_1+1}$ (Here we use the perfectness of the residue class field).

By Proposition 4.3, there is an unsaturated vector $\bar{w}_{\lambda_1}{}^{(1)} \in L_{\lambda_1}$ with critical exponent f_1 such that

$$\{\bar{w}_{\lambda_1}{}^{(1)}\}^2 = \{w_{\lambda_1}{}^{(1)}\}^2 + \{v_{(1)}{}^{(1)}\}^2 - y^2 - z^2$$

Let $\bar{v}_{(1)}^{(1)} = w_o^{(1)} \oplus \dots \oplus w_{\lambda_1-1}^{(1)} \oplus \bar{w}_{\lambda_1}^{(1)} \oplus w_{\lambda_1+1}^{(1)} \oplus \dots \oplus w_{\lambda_1+s_1-1}^{(1)} \oplus z$

Then $\{\bar{v}_{(1)}^{(1)}\}^2 = \{v_{(1)}^{(1)}\}^2$ hence by Theorem 3.1 there is an isometry φ on $L_o \oplus \dots \oplus L_{\lambda_1+s_1}$ with $\varphi(v_{(1)}^{(1)}) = \bar{v}_{(1)}^{(1)}$.

Let $\bar{v} = \bar{v}_{(1)}^{(1)} \oplus v_{\lambda_1+s_1+1}^{(1)} \oplus \dots \oplus v_n^{(1)}$. Then $\bar{v} \sim v$ and furthermore $\{\bar{v}_{(1)}^{(1)} - \bar{v}_{\lambda_1+s_1}^{(1)}\}^2 \equiv \{w_{(1)}^{(1)} - w_{\lambda_1+s_1}^{(1)}\}^2 \bmod 2^{\lambda_2+f_2+f_1+1}$

<u>Proposition</u> 4.4 a) Let v, w have the same critical indices and exponents and satisfy the congruence relations (1) wrt. $\mathcal{L}_{(1)} = \Sigma \oplus L_i$. Furthermore let $s_1 \geq 2$ and v_{λ_1} be saturated in L_{λ_1}. Then there exists a second Jordan Splitting $\mathcal{L}_{(2)} = \Sigma \oplus L_i$ with $L_i \simeq \bar{L}_i$ and in which one of the congruence relations (2) holds.

<u>Proof</u>: Use the same method as before, only apply Proposition 4.3 a).

Definition: Let v be any vector space of finite dimension over a local field. We define a topology on v given by the norm

$$|\alpha_1, x_1 + \dots + \alpha_n, x_n| = \sup \{|\alpha_i|\}$$

where $\{x_i\}$ is a basis for v. It is a well known fact that the unit sphere in this topology is compact.

Proposition 4.5: The set of vectors equivalent to a given vector v is compact.

Proof: Let $x_1 \sim x_2 \sim x_3 \dots$ be any sequence of equivalent vectors. Let A be the matrix representing L. Then there exist matrices B_i with integral entries such that $x_1 B_i = x_i$ and $B_i A B_i = A$. If we consider the matrices B_i to be an n x n dimensional vector space over F, we have, by the compactness of the unit sphere, a subsequence $\{B_{k_i}\}$ of $\{B_i\}$ which converges to a matrix B with $B A B^T = A$ and B integral. Thus the subsequence $\{x_{k_i}\}$ converges to a vector x with $x = x_1 B$.

Proposition 4.6: Let $L = L_o \oplus L_n$ with L_o unimodular, improper. L_n is 2^n-modular.

$v = v_o^{(1)} \oplus v_n^{(1)}$ with critical indices o and n

$|\eta| = |(v_o^{(1)})^2|$. exponents f and o.

Then there is a Jordan Decomposition $\mathfrak{L}_{(2)} = \overline{L}_o \oplus \overline{L}_n$ such that $\{v_o^{(2)}\}^2 = \eta$ provided that:

1. $1 + 2^{-n}\{\{v_o^{(1)}\}^2 - \eta\} + T(v_n^{(1)})$ is a square when

$T(v_n^{(1)}) \neq o$ and $f = 1$.

2. $\{v_o^{(1)}\}^2 - \eta \equiv o \mod 2^{f+n+1}$ otherwise

Proof: We will show that, given ord $[\{v_o^{(1)}\}^2 - \eta]$ $= f+n+k+1$ with $k \geq o$ and provided (1) holds when $T(v_n^{(1)}) \neq o$ and $f = 1$. Then there is a Jordan Splitting $\mathfrak{L}_{(3)} = k_o \oplus k_n$ such that

$$\text{ord } [\{v_o^{(3)}\}^2 - \eta] \geq f+n+k+2.$$

This implies that there is a sequence of vectors

$$v \sim v_{(1)} \sim v_{(2)} \sim \cdots \text{ with } \{v_{(i),o}^{(1)}\}^2 \to \eta$$

Thus by Proposition 4.5 there is a vector w with $w \sim v$ and $\{w_o^{(1)}\}^2 = \eta$.

First we imbed $v_o^{(1)}$ in a two-dimensional, unimodular sublattice of L_o. So now we assume L_o is two-dimensional. Let $L_o = xZ + yz$ where $x^2 = y^2 = \Delta$ $x \cdot y = 1$

where $\Delta^2 = o$ or $\Delta^2 \equiv 4p \mod 8$.

Now let $v_o^{(1)} = 2^f(\epsilon x + 2^m \bar{\epsilon} y)$

Furthermore, let $\bar{v}_n$ be a vector in L_n with $v_n^{(1)} \cdot \bar{v}_n$ $= \alpha\, 2^n$ where α is a unit yet to be determined.

Let $\bar{v}_n^{\,2} = 2^{n+i}\, \bar{\partial}$ where ∂ is a unit. Furthermore, if v_n is unsaturated choose $\bar{v}_n$ such that $i \geq 1$.

This is easily done.

We are given that ord $[\{v_o^{(1)}\}^2 - \eta] = f+n+k+1$ with $k \geq o$.

We define the new lattice k_o to be $\bar{x}Z + \bar{y}Z$ where

$\bar{x} = x + 2^k \bar{v}_n$

$\bar{y} = y$

$k_o \simeq L_o$ and has the multiplication table:

$$\begin{vmatrix} \Delta + 2^{2k+n+i}\partial & 1 \\ 1 & \Delta \end{vmatrix}$$

and determinant $D = -1(1-\Delta^2-2\Delta 2^{2k+n+i}\partial)$

We may now write: $L = k_o \oplus k_n \sim \mathfrak{L}_{(3)}$ where $k_n \simeq L_n$.

Now let $v_o^{(2)} = \mu\bar{x} + \nu\bar{y}$

Then $v_o^{(3)}\cdot\bar{x} = \mu(\Delta + 2^{2k+n+i}\partial) + \nu = 2^f \epsilon \Delta + 2^{f+m}\bar{\epsilon} + 2^{k+n}\alpha$

$v_o^{(3)}\cdot\bar{y} = \mu + \Delta\nu = 2^f \epsilon + 2^{f+m}\bar{\epsilon}\Delta$.

Solving these two equations, we have

$$\mu = \{2^f \epsilon(\Delta^2-1) + \Delta 2^{k+n}\alpha\} \div D$$

$$\nu = \{2^{f+m}\bar{\epsilon}(\Delta^2-1) + 2^{f+2k+n+i}\epsilon + 2^{f+m+2k+n+i}\bar{\epsilon}\Delta - 2^{k+n}\alpha\} \div D$$

Let $\bar{u} = 2^f \epsilon(\Delta^2-1) \div \bar{D} = 2^f\epsilon$ where $\bar{D} = \Delta^2-1$

$\bar{\nu} = 2^{f+m}\bar{\epsilon}(\Delta^2-1) - 2^{k+n}\alpha\} \div \bar{D}$, ϵ_o is some unit

$= 2^{f+m}\bar{\epsilon} - 2^{k+n}\alpha + \epsilon_o 2^{k+n+1+s}$, s some positive integer.

It is clear that $\{\bar{u}\,\bar{x} + \bar{\nu}\,\bar{y}\}^2 \equiv \{\mu\bar{x} + \nu\bar{y}\}^2 \bmod 2^{n+f+k+2}$.

$= \{v_o^{(3)}\}^2$

Now $\{\bar{u}\,\bar{x} + \bar{\nu}\,\bar{y}\}^2 - \{2^f\epsilon x + 2^{f+m}\bar{\epsilon} y\}^2 =$

$= \epsilon^2 2^{2f+2k+n+i}\partial + \Delta 2^{2k+2n}\epsilon_o^2 2^{2k+2n}$

$$- 2^{f+m+k+n+1}\,\bar{\epsilon}_o\,\Delta - 2^{f+k+n+1}\,\alpha\epsilon + \epsilon_o^1\,2^{f+k+n+1+r}$$

ϵ_o^1, $\bar{\epsilon}_o$ are some units, r an integer.

Now, using the facts that $f > o$, $n > f$ and ord $\Delta \geq 1$ we can obtain $\{v_o^{(1)}\}^2 - \{v_o^{(3)}\}^2 \equiv \epsilon^2\,2^{2f+2k+n+i}$

$$- 2^{f+n+k+1}\alpha\epsilon \text{ or } \epsilon \bmod 2^{f+n+k+2}$$

Now remember that we chose $\bar{v}_n$ such that $i > o$ when $v_n^{(1)}$ was unsaturated. Thus we have $2f+2k+n+i > f+n+k+i$ provided $f \neq 1$ or $k \neq o$ or $v_n^{(o)}$ unsaturated.

In this case: $\{v_o^{(1)}\}^2 - \{v_o^{(3)}\}^2 \equiv 2^{f+n+k+1}\,\epsilon\,\alpha$

$$\bmod 2^{f+n+k+2}.$$

Here we need only choose $\bar{v}_n$ such that

$$\{v_o^{(1)}\}^2 - \eta \equiv \epsilon\,\alpha\,2^{f+n+k+1} \bmod 2^{f+n+k+2}$$

We have, by hypothesis, that $2^{-k}[\{v_o^{(1)}\}^2-\eta] = 4\mu + 4\mu^2$ where μ is an integer. So let $\bar{v}_n = \frac{\mu}{\epsilon}x_1$. Then $\epsilon\alpha = \mu$ $\quad \epsilon^2\partial = \mu^2$. Thus $2^{n+2}(\epsilon\alpha-\epsilon^2\partial) \equiv 2^n(4\mu+4\mu^2) \bmod 2^{n+3}$

$$\equiv [\{v_o^{(1)}\}^2-\eta] \bmod 2^{n+3}$$

<u>Proposition</u> 4.7: $L = L_o \oplus L_n \sim \mathfrak{L}_{(1)}$, where L_o is unimodular. L_n is 2^n-modular.

Let $v \in L$ have critical indices o, n, exponents f, o, Then there is a Jordan Splitting $\mathfrak{L}_{(2)}$ with $\{v_o^{(2)}\}^2 = \eta$ provided that:

1. $\{v_o^{(2)}\}^2 \equiv \eta \bmod 2$.

2. $1 + 2^{-n} \{(v_o^{(1)})^2 - \eta\} / T(v_n^{(2)})$ is a square when $T(v_n^{(1)}) \neq o$ and $f = 1$.

3. $1 + 2^{-n} \{(v_o^{(1)})^2 - \eta\} / T(v_o^{(2)})$ is a square when $T(v_o^{(1)}) \neq o$ and $f + 1 = n$.

4. $\{v_o^{(1)}\}^2 - \eta \equiv o \bmod 2^{f+n+1}$ otherwise.

Proof: By Proposition 4.6 we may assume L_n is proper, otherwise the result follows from applying the previous proposition to the dual lattice $L^{\#}$.

Also we may assume L_o is proper by Proposition 4.6.

Once again we use the method of "successive approximations" which was applied in Proposition 4.6.

Assume for the time being that $n \neq 2$ or $k \neq o$ where $\{v_o^{(1)}\}^2 - \eta = 2^{n+f+k+1}\lambda$. λ is some unit.

Write $L_o = \sum_{i=1}^{s} \oplus \langle y_i \rangle \oplus M$ where $s = 1$ or 2

$L_n = \sum \oplus \langle x_i \rangle$.

where $v_o^{(1)} = 2^f(y_1 + \Delta\, y_2)$ $\Delta = o$ or 1.

Let $y_1^2 = \epsilon$ $y_2^2 = \bar{\epsilon}$.

Choose $\bar{v}_n \in L_n$ such that $\bar{v}_n \cdot v_n^{(1)} = 2^n \alpha$ α a unit

$$\bar{v}_n^2 = 2^{n+i}\,\partial.$$

Note: we have free choice in α, and after ∂ is chosen we may still have $i > o$ if $v_n^{(1)}$ is unsaturated.

Now let $\bar{L}_o = \langle \bar{y}_1 \rangle \oplus \langle y_2 \rangle \oplus M$. $\bar{L}_o \simeq L_o$ where $\bar{y}_1 = y_1 + 2^k \bar{v}_n$. Let $\mathfrak{L}_{(2)_2} \sim \bar{L}_o \oplus \bar{L}_n$.

we have $$\{v_o^{(2)}\}^2 = \frac{(v \cdot \bar{y}_1)^2}{\bar{y}_1^{\,2}} + 2^{2f} \Delta \bar{\varepsilon}$$

A simple calculation shows:

$$\{v_o^{(2)}\}^2 - \{v_o^{(1)}\}^2 \equiv \partial\, 2^{2f+2k+1} + 2^{f+k+n+1} \alpha + 2^{2k+2n} \alpha^2/\varepsilon \mod 2^{f+k+n+2}$$

Now with the exception of the cases

a) $f = 1$, $k = o$, $v_n^{(1)}$ saturated

b) $f + 1 = n$, $k = o$

The above expression is congruent to

$$2^{f+k+n+1} \alpha \mod 2^{f+k+n+2}$$

so here we need only let $\alpha = \lambda$ to obtain our next approximation.

Now <u>Case a)</u> Since we assumed $n \neq 2$, we have $2k+2n > f+k+n+1$ so the above expression is congruent to

$$(\partial+\alpha)\, 2^{f+k+n+1} \mod 2^{f+k+n+2}$$

This is the same congruence we arrived at in Proposition 4.6 when $v_n^{(1)}$ was saturated and is handled in the same manner.

<u>Case b)</u> Here we examine $L^{\#}$. And in this lattice $f = 1$, $f \neq n-1$ since $n \neq 2$. But we have proven the result for these circumstances in the first part of the proposition.

Now let $n = 2$, $k = o$.

<u>Case 1</u>: $v_n^{(1)}$, $v_o^{(1)}$ both saturated.

Now $\{v_o^{(1)}\}^2 - \eta = 2^{n+2}\lambda$ λ a unit.

We use the notations as of before $L_o = \Sigma \oplus \langle y_1 \rangle$

$$L_n = \Sigma \oplus \langle x_1 \rangle$$

$$v_n^{(1)} = \Sigma \oplus x_1 \qquad v_o^{(1)} = 2\Sigma \oplus y_1$$

By scaling we may assume $x_1{}^2 = 2^n$. Let $\bar{v}_n = \alpha\, x_1$.

Then $i = o$. $\partial = \alpha^2$. By hypothesis $1 + 4\lambda$ and

$1 + \frac{4\lambda}{\epsilon}$ are both squares.

Now use the same method as in the previous section to let $\bar{y}_1 = y_1 + \alpha\, \bar{v}_n$ $\qquad \bar{L}_o = \langle \bar{y}_1 \rangle \oplus \langle y_2 \rangle \oplus \ldots$

to arrive at:

$$16\,((1+\tfrac{1}{\epsilon})\,\alpha^2 + \alpha) \equiv \{v_o^{(1)}\}^2 - \{v_o^{(2)}\}^2 \bmod 2^{n+3}.$$

Now if we can find an integer α such that

$$((\tfrac{\epsilon+1}{\epsilon})\,\alpha^2 + \alpha) \equiv 2^{-4}\,\{(v_o^{(1)})^2 - \eta\} \bmod 2$$

and such that $\bar{L}_o \cong L_o$, we shall be through.

But if α satisfies the above equation then

$$\alpha = \frac{\epsilon}{2(\epsilon+1)}\,\{-1 \pm \sqrt{1 - 4(1+\tfrac{1}{\epsilon})\lambda}\}.$$

Now $1 + 4\lambda$ is a square.

$1 + 4\lambda/\epsilon$ is a square.

Hence $1 - 4(1+\frac{1}{\epsilon})\,\lambda$ is a square. This solution exists to the above equation.

Since $|\frac{\epsilon+1}{\epsilon}| \leq 1$. $|\lambda| = 1$ it is clear that we may choose α with $\alpha \in Z$.

Now $\overline{L}_o \simeq L_o$ since if we let

$$v_o^{(2)} = 2(\beta\, \overline{y}_1 + y_2 + .. + y_n)$$

Then $y_1^{\,2} - \beta^2\, \overline{y}_1^{\,2} \equiv 4\lambda \mod 8.$

Therefore $\beta^2\, \overline{y}_1^{\,2} \equiv \epsilon + 4\lambda \mod 8$

Therefore $\overline{y}_1^{\,2}/y_1^{\,2}$ is a square.

Case 2: $v_o^{(1)}$ unsaturated, $v_n^{(1)}$ saturated.

Let $L_o = M \oplus N$ with $v_o^{(1)} \in M$. N proper

$N = \Sigma \oplus \langle y_i \rangle \qquad L_n = \Sigma \oplus \langle x_i \rangle \qquad x_1^{\,2} = 1 \qquad v_n^{(1)} = \Sigma \oplus x_i$

by scaling.

Let $\{v_o^{(1)}\}^2 - \eta = 2^4\lambda$ λ a unit and $1 + 4\lambda$ a square.

Choose ϵ such that $-(4\,\epsilon\, y_1)^2 \equiv 16\lambda \mod 32$ then

$1 + (2\,\epsilon\, y_1)^2$ is a square $= \mu^2$.

Let $\overline{v} = \{v_o^{(1)}\} \oplus 4\,\epsilon\, y_1 \oplus u\, x_1 \oplus x_2 \oplus x_3 \oplus .. \oplus x_5$.

Then $\overline{v}^2 = v^2 \quad \{\overline{v}_o^{(1)}\}^2 \equiv \{v_o^{(1)}\}^2 \mod 32$ and by Theorem 3.1, applied to $N \oplus \langle x_1 \rangle$ we have $\overline{v} \sim v$. $\overline{v}$ is our next approximation.

Case 3: $v_o^{(1)}$ saturated, $v_n^{(1)}$ saturated. Examine $L^\#$ and apply Case 2.

Case 4: $v_o^{(1)}$, $v_n^{(1)}$ unsaturated. We leave for the reader.

Proposition 4.8: Let $L = L_o \oplus .. \oplus L_n \sim \mathfrak{L}$ L_i is 2^i-modular or empty

$$= k_o \oplus .. \oplus k_n \sim \mathfrak{L}' \quad \text{with} \quad L_i \simeq k_i$$

Then there is a chain of Jordan Decompositions $\mathcal{L}_{(1)}, \mathcal{L}_{(2)}, \ldots, \mathcal{L}_{(m)}$ with $\mathcal{L}_1 = \mathcal{L}$, $\mathcal{L}_m = \mathcal{L}'$ such that $L_i^{(j)} \simeq L_i^{(k)}$ and $\mathcal{L}_{i+1}$ is obtained from $\mathcal{L}_i$ by altering either three consecutive components of $\mathcal{L}_i$ or else by altering some two components of $\mathcal{L}_i$.

<u>Proof</u>: We shall show that there is a chain (satisfying the conditions of the Theorem) $\mathcal{L}_1, \ldots, \mathcal{L}_t$ with $\mathcal{L}_t \sim k_o \oplus \ldots$

We then proceed by induction.

So let $k_o = x_1 Z + x_2 Z + .. + x_r Z$.

where $x_i = \bigoplus_{j=o}^{n} \lambda_{i_j} v_{i_j}$ $\qquad v_{i_j} \in L_j$.

Let $\bar{x}_i = \bigoplus_{j=o}^{2} \lambda_{i_j} v_{i_j}$ and $\bar{L}_o = \bar{x}_1 Z + \ldots + \bar{x}_r Z$.

Clearly $\bar{L}_o \simeq L_o \simeq k_o$. Since $\bar{x}_i \cdot x_j \equiv x_i \cdot x_j \bmod 8$.

Write $L_o \oplus L_1 \oplus L_2 = \bar{L}_o \oplus \bar{L}_1 \oplus \bar{L}_2$ $\quad \bar{L}_1 \simeq L_1$. $\bar{L}_2 \simeq L_2$.

Now let: $L = \bar{L}_o \oplus \bar{L}_1 \oplus \bar{L}_2 \oplus L_3 \oplus \ldots \qquad \sim \mathcal{L}_{(2)}$.

$\mathcal{L}_{(2)}$ satisfies the conditions of the Theorem since we have altered 3 consecutive lattices.

We now have: $x_i = v_{i_o} + \bigoplus_{j=3}^{n} \lambda_{i_j} v_{i_j}$, $v_{i_j} \in L_j$.

By induction, assume $x_i = v_{i_o} \bigoplus_{j=k}^{n} \lambda_{i_j} v_{i_j}$ with $k \geq 3$, and $v_{i_j} \in L_j$.

Form lattice $\bar{L}_o$ by replacing x_i by $x_i + \lambda_{i_k} v_{i_k}$.

Clearly $\bar{L}_o \simeq L_o$.

Choose $\overline{L}_k$ such that $\overline{L}_o \oplus \overline{L}_k = L_o \oplus L_k$ and $L_k \simeq \overline{L}_k$.

Then $\overline{L}_o \oplus L_1 \oplus L_2 \oplus \dots \oplus L_{k-1} \oplus \overline{L}_k \oplus L_{k+1} \oplus \dots$

is the next lattice in our chain and $x_i = \overline{v}_{i_o} + \bigoplus_{j=k+1}^{n} \lambda_{i_j} v_{i_j}$

with $\overline{v}_{i_o} \in \overline{L}_o$ $v_{i_j} \in L_j$.

After a finite number of steps we have a second Jordan Decomposition

$M_o \oplus \dots \oplus M_n$ with $M_i \simeq \overline{L}_i$

with $x_i \in M_o$. Clearly $M_o = k_o$

Corollary: Let φ be an isometry on L. Then there is a chain of Jordan Decompositions. $\mathcal{L}_i = L_o^{(i)} \oplus \dots \oplus L_n^{(i)}$ with $L_j^{(i)} \simeq L_j^{(k)}$ and a chain of isometries φ_i, such that φ_i leave all the $L_k^{(i)}$ fixed except for two of them or three consecutive ones, and $\varphi = \varphi_1 \dots \varphi_m$.

Proposition 4.4: $L_o \oplus \dots \oplus L_n \sim \mathcal{L}(1)$

$k_o \oplus \dots \oplus k_n \sim \mathcal{L}(2)$

are two different Jordan Decompositions of L with $L_i \simeq k_i$. $v = \Sigma \oplus v_i^{(1)} = \Sigma \oplus v_i^{(2)}$. $v_{\lambda_i}^{(1)}$ is saturated in L_{λ_i}. Then

$$\frac{\{v_{(1)}^{(1)} \oplus .. \oplus v_{\lambda_{i-1}}\}^2 - \{v_{(1)}^{(2)} \oplus .. \oplus v_{\lambda_{i-1}}^{(1)}\}^2 + 1}{T(v_{\lambda_i}) \cdot 2^{2f_i+\lambda_i}} \text{ is a square.}$$

Proof: By Proposition 4.8 we may assume $L_j = k_j$ when

a) $J \neq s, t$ or b) $j \neq r, r+1, r+2$.

Case a) $s < t$. Let $s = \lambda_i$. Then write

$$L_{\lambda_i} = \Sigma \oplus \langle x_i \rangle \quad \text{with} \quad v_{\lambda_i}^{(1)} = \Sigma \oplus x_i$$

$$\overline{L}_{\lambda_i} = \Sigma \oplus \langle y_i \rangle \quad \text{with} \quad y_i^2 = x_i^2$$

Now since $T(v_{\lambda_i}^{(1)}) = T(v_{\lambda_i}^{(2)})$ we have $v_{\lambda_i}^{(2)} = \Sigma \oplus (1+2\eta_i)y_i$

Therefore $\{v_{\lambda_i}^{(1)}\}^2 - \{v_{\lambda_i}^{(2)}\}^2 = \sum_i (4\eta_i + 4\eta_i^2)y_i^2$

Thus $$\frac{\{v_{\lambda_i}^{(1)}\}^2 - \{v_{\lambda_i}^{(2)}\}^2}{T(v_{\lambda_i})} \cdot 2^{-\lambda_i - 2f_i} \equiv \sum_i (4\eta_i + 4\eta_i^2) \bmod 8.$$

and $1 + \Sigma(4\eta_i + 4\eta_i^2) \equiv (1 + \Sigma\, 2\eta_i)^2 \bmod 8$.

Therefore $$\frac{\{v_s^{(1)}\}^2 - \{v_s^{(2)}\}^2}{2^{\lambda_i} T(v_{\lambda_i})} + 1 \text{ is a square.}$$

Of course the proof is the same if $t = \lambda_i$. So now we may assume $s < \lambda_i < t$. There are three subcases here:

$w = v_s^{(1)} \oplus v_t^{(1)}$ has critical incides 1. s and t

2. s

3. t

Case b) The result is trivial when $r \neq \lambda_i - 2, \lambda_i - 1, \lambda_i$

when $r = \lambda_i - 1$, the result follows from Theorem 3.1. Since in this case $v_r^{(1)} \oplus v_{r+1}^{(1)} \oplus v_{r+2}^{(1)}$ has only one critical index. The other two cases are similar to case a) when $s = \lambda_i$.

Subcase 1. Let the critical exponents of w be $\overline{f}_s$ and $\overline{f}_t$.

By Theorem 4.1 $\{v_s^{(1)}\}^2 \equiv \{v_s^{(2)}\}^2 \mod 2^{\bar{f}_s+\bar{f}_t+t+1}$

Now $\bar{f}_t + t \geq f_i + \lambda_i + 1$ by Proposition 4.0.

$\bar{f}_s \geq f_i + 1$

Thus $$\frac{\{v_s^{(1)}\}^2 - \{v_s^{(2)}\}^2}{2^{2f_i+\lambda_i}\cdot T(v_{\lambda_i})} \equiv 0 \mod 8$$

But $1 + 8\eta$ is always a square if $\eta \in Z$.

Subcase 2. Let $v_t^{(1)} = 2^{h_1}\hat{v}_t^{(1)}$ $v_t^{(2)} = 2^{h_2}\hat{v}_t^{(2)}$

$v_s^{(1)} = 2^{g_1}\hat{v}_s^{(1)}$ $v_s^{(2)} = 2^{g_2}\hat{v}_s^{(2)}$

where $\hat{v}_t^{(1)}, \hat{v}_t^{(2)}, \hat{v}_s^{(1)}, \hat{v}_s^{(2)}$ are all maximal.

Then $h_1 \geq g_1 \geq f_i+1$ $t \geq \lambda_i+1$

$\{v_t^{(1)}\}^2 \equiv o \mod 2^{2(f_i+1)+\lambda_i+1}$

$\{v_t^{(2)}\}^2 \equiv o \mod 2^{2(f_i+1)+\lambda_i+1}$

Thus $$\frac{\{v_t^{(1)}\}^2 - \{v_t^{(2)}\}}{2^{2f_i+\lambda_i}} \equiv o \mod 8.$$

Subcase 3. Proof is similar to subcase 2.

So we have finally collected enough information to prove our main result.

Theorem 4.2: Let $v, w \in L$. $v^2 = w^2$. $L = L_{-n} \oplus \dots \oplus L_m$ is any Jordan decomposition of L where L_i is empty or 2^i-modular. Then $v \sim w$ if and only if the following conditions hold.

1. v, w have the same critical indices and exponents $\{\lambda_i, f_i\}$.

2. $T(v_{\lambda_i}) \equiv T(w_{\lambda_i}) \bmod 2$.

3. $v_{-n}^2 + v_{-n+1}^2 + \ldots + v_{\lambda_i+s_i}^2 \equiv w_{-n}^2 + \ldots + w_{\lambda_i+s_i}^2 \bmod 2^{f_i+f_{i+1}+\lambda_{i+1}+\Delta_i}$

4. $\dfrac{\{v_{-n}^2 + \ldots + v_{\lambda_i}^2\} - \{w_{-n}^2 + \ldots + w_{\lambda_i}^2\}}{T_i} + 1$ is a square when $T(v_{\lambda_i}) \neq o$

where $\Delta_i = 1$ if $L_{\lambda_i+s_i}$ is improper

$\Delta_i = o$ of $L_{\lambda_i+s_i}$ is proper

$T_i = T(v_{\lambda_i})/2^{2f_i+\lambda_i}$.

Proof: Necessity

This follows from Proposition 4.9, Lemma 4.1 and Theorem 4.1.

Sufficiency

We do an induction on the number of critical indices. To do this, we need only show there is a Jordan Decomposition

$$\mathcal{L}_{(2)} \sim \overline{L}_{-n} \oplus \ldots \oplus \overline{L}_m \quad \text{with} \quad \overline{L}_i \simeq L_i$$

and such that $= \{w_{-n}^{(2)} + \ldots + w_{\lambda_1+s_1-\partial}^{(2)}\}^2$

$= \{v_{-n}^{(1)} + \ldots + v_{\lambda_1+s_1-\partial}^{(1)}\}^2$

where $\partial = o$, or 1. Then Proposition 4.9 will imply the validity of the hypotheses of Theorem 3.1 for the above two vectors over the isometric lattices

$$L_{-n} \oplus .. \oplus L_{\lambda_1+s_1-\partial} \quad \text{and} \quad \overline{L}_{-n} \oplus \ldots \oplus \overline{L}_{\lambda_1+s_1-\partial}.$$

Hence there will exist an isometry φ: $L_{-n} \oplus \ldots \oplus L_{\lambda_1+s_1-\partial}$ onto $\overline{L}_{-n} \oplus \ldots \oplus \overline{L}_{\lambda_1+s_1-\partial}$ which maps

$\overline{v} = v_{-n}^{(1)} \oplus \ldots \oplus v_{\lambda_1+s_1-\partial}^{(1)}$ onto $\overline{w} = w_{-n}^{(2)} \oplus .. \oplus w_{\lambda_1+s_1-\partial}^{(2)}$

Furthermore, $v - \overline{v}$, $w - \overline{w}$ have one less critical index than v and w and they both satisfy the hypotheses of Theorem 4.2 over the isometric lattices $L_{\lambda_1+s_1-\partial+1} \oplus \ldots \oplus L_m$ and $\overline{L}_{\lambda_1+s_1-\partial+1} \oplus .. \oplus \overline{L}_m$ and so we are able to carry through the induction step.

So we must find a decomposition $\mathcal{L}_{(2)} \sim \overline{L}_{-n} \oplus \ldots \oplus \overline{L}_m$ with $L_i \simeq \overline{L}_i$ such that $\{w_{-n}^{(2)} \oplus .. \oplus w_{\lambda_1+s_1-\partial}^{(2)}\}^2$ $= \{v_n^{(1)} \oplus \ldots \oplus v_{\lambda_1+s_1-\partial}^{(1)}\}^2$.

We break the proof up into several subcases.

<u>Case 1</u>: v_{λ_1} unsaturated, and $f_1-f_2 \neq 1$ when v_{λ_2} saturated; or v_{λ_1} saturated, $s_1 \neq 1$ and $f_1-f_2 \neq 1$ when v_{λ_2} saturated. By Proposition 4.4 when v_{λ_1} is unsaturated and by Proposition 4.4 a) when v_{λ_1} is saturated there is

a Jordan Splitting $\mathfrak{L}_{(3)} \sim L'_{-n} \oplus \ldots \oplus L'_{m}$ with $L_k^1 \simeq L_k$ and such that

$$\{w_{-n}^{(3)} \oplus \ldots \oplus w^{(3)}_{\lambda_1+s_1-\partial}\}^2 \equiv \{v_{-n}^{(1)} \oplus \ldots \oplus v^{(1)}_{\lambda_1+s_1-\partial}\}^2$$

mod $2^{f_1+f_2+\lambda_2+1}$ where $\partial = o$ or 1.

Let $\mu = \{w_{-n}^{(3)} \oplus \ldots \oplus w^{(3)}_{\lambda_1+s_1-\partial}\}^2$

$- \{v_{-n}^{(1)} \oplus \ldots \oplus v^{(1)}_{\lambda_1+s_1-\partial}\}^2$

$\mu \equiv o \bmod 2^{f_1+\lambda_2+f_2+1}$ when v_{λ_1}, v_{λ_2} are unsaturated

$\mu \equiv o \bmod 2^{2f_1+\lambda_1+3}$ when v_{λ_1} is saturated.

$\mu \equiv o \bmod 2^{2f_2+\lambda_2+3}$ when v_{λ_2} is saturated.

Hence we are able, in virtue of Proposition 4.6, applied to $L'_{\lambda_1} \oplus L'_{\lambda_2}$ to find a J. D. $\mathfrak{L}_{(2)} = \bar{L}_{-n} \oplus \ldots \oplus \bar{L}_m$ with $\bar{L}_i \simeq L_i$ such that $\{w_{-n}^{(2)} \oplus \ldots \oplus w^{(2)}_{\lambda_1+s_1-\partial}\}^2$ $= v_{-n}^{(1)} \oplus \ldots \oplus v^{(1)}_{\lambda_1+s_1-\partial}\}^2$.

<u>Case 2</u>. $f_1 = f_2 + 1$. v_{λ_2} saturated, $\lambda_2 - \lambda_1 \neq 2$ when v_{λ_1} is saturated.

We may assume by scaling that L_{λ_2} is unimodular, $T(v_{\lambda_2}) = 1$. Then $1 + (v_{-n}^{(1)} + \ldots + v_{\lambda_1}^{(1)})^2$ $- (w_{-n}^{(1)} + \ldots + w_{\lambda_2}^{(1)})^2$ is a square by hypothesis.

Now apply Proposition 4.7 (2) to $L_{\lambda_1} \oplus L_{\lambda_2}$ to obtain a J.D. $\mathcal{L}_{(2)} = L_{-n} \oplus .. \oplus L_{\lambda_1-1} \oplus \overline{L}_{\lambda_1} \oplus L_{\lambda_1+1} \oplus ..$

$$\oplus L_{\lambda_2-1} \oplus \overline{L}_{\lambda_2} \oplus \ldots$$

in which $\overline{L}_{\lambda_1} \simeq L_{\lambda_1}$ and $(v_{-n}^{(1)} \oplus .. \oplus v_{\lambda_2-1}^{(1)})^2$ $= (w_{-n}^{(2)} \oplus .. \oplus w_{\lambda_2-1}^{(2)})^2$ and now note that $\lambda_2 - 1 = \lambda_1 + s_1$.

Case 3. $s_1 = 1$. v_{λ_1} saturated, $\lambda_2 - \lambda_1 \neq 2$ when v_{λ_2} is saturated.

Apply Proposition 4.7 (3) to obtain a Jordan Decomposition $\mathcal{L}_{(2)} = \Sigma \oplus \overline{L}_i$ with $\overline{L}_i \simeq L_i$ and $(v_{-n}^{(1)} \oplus .. \oplus v_{\lambda_1}^{(1)})^2 = (w_{-n}^{(2)} \oplus .. \oplus w_{\lambda_1}^{(2)})^2$ and note that $\lambda_1 = \lambda_1 + s_1 - 1$.

Case 4. v_{λ_1}, v_{λ_2} saturated $\lambda_2 - \lambda_1 = 2$.

We may assume $\lambda_2 = o$, $T(v_{\lambda_1}) = 1$.

If L_{λ_1+1} is empty, the result follows from Proposition 4.3 (2,3). Therefore we may assume L_{λ_1+1} is not empty, also by changing the basis if necessary that $w_{\lambda_1+1} = o$.

Now $v_{-1}^2 = \{v_{-n} + .. + v_{-1}^2\} - (v_{-n}^2 + .. + v_{-2}^2\}$

$$= +\alpha - \beta$$

$$\text{where} \quad 1 + \frac{\alpha}{T(v_o)} \quad \text{is a square}$$

$$1 + \frac{\beta}{T(v_{-2})} \quad \text{is a square.}$$

<u>Note</u>: ord $(\alpha-\beta) \geq 2$

Now suppose ord $(\alpha-\beta) \geq 3$. Then

$$\frac{\{v_{-n}{}^2 + .. + v_{-1}{}^2\} - \{w_{-n}{}^2 + .. + w_{-1}{}^2\}}{R} + 1 \quad \text{is a square}$$

where $R = T(v_o)$ or $T(v_{-2})$. Hence we may apply Proposition 4.7 (2,3) to $L_{-2} \oplus L_o$, and we are finished.

Now suppose ord $(\alpha-\beta) \equiv 2$.

Then there is a vector $x \in L_{-1}$ with $x^2 \equiv \alpha \bmod 8$, and a vector $\overline{w}_o \in L_o$ with $T(\overline{w}_o) = T(w_o)$ and such that $x^2 + \overline{w}_o{}^2 = w^2$.

Now let $\overline{w} = w_{-n}^{(1)} \oplus .. \oplus w_{-2}^{(1)} \oplus x \oplus \overline{w}_o \oplus w_1^{(1)} \oplus ..$.

Then $\overline{w} \sim w$.

Also $$\frac{\{\overline{w}_{-n}^{(1)} \oplus .. \oplus \overline{w}_{-1}^{(1)}\}^2 - \{v_{-n}^{(1)} \oplus .. \oplus v_{-1}^{(1)}\}^2}{T(v_o)} + 1 \quad \text{is a square.}$$

$$\frac{\{\overline{w}_{-n}^{(1)} \oplus .. \oplus w_{-1}^{(1)}\}^2 - \{v_{-n}^{(1)} \oplus .. \oplus v_{-1}^{(1)}\}^2}{T(v_{-2})} + 1$$

$$= \frac{\{w_{-n}^{(1)} \oplus .. \oplus w_{-2}^{(1)}\}^2 - \{v_{-n}^{(1)} \oplus .. \oplus v_{-2}^{(1)}\}^2}{T(v_{-2})} + 1$$

$$+ \frac{\{\overline{w}_{-1}^{(1)}\}^2 - \{v_{-1}^{(1)}\}^2}{T(v_{-2})}$$

$$= 1 + 4u + 4u^2 + \frac{\alpha - (\alpha-\beta) + 8\nu}{T(v_{-2})} \quad \text{where} \quad u, \nu \in Z.$$

which is a square since $1 + \frac{\beta}{T(v_{-2})}$ is a square.

Now apply Proposition 4.7 (2,3) and we are finished.

We now apply Theorem 4.2 to the special case where F is the 2-adic completion of the rational numbers. Using the facts that any two saturated vectors have the same type and that $1 + 2\eta$ is a square iff $\eta \equiv o \bmod 4$ we have

<u>Theorem</u> 4.3: $v, w \in L \cdot v^2 = w^2$ $L = \Sigma \oplus L_i$ the Jordan Decomposition. Then $v \sim w$ if and only if the following conditions hold.

1. v, w have the same critical indices and exponents $\{\lambda_i, f_i\}$.

2. v_{λ_i} saturated iff w_{λ_i} saturated (in L_{λ_i}).

3. $v_{-n}^2 + .. + v_{\lambda_i+s_i}^2 \equiv w_{-n}^2 + .. + w_{\lambda_i+s_i}^2 \bmod 2^{f_i+f_{i+1}+\lambda_{i+1}+\Delta_i}$

4. $v_{-n}^2 + .. + v_{\lambda_i}^2 \equiv w_{-n}^2 + .. + w_{\lambda_i}^2 \bmod 2^{\lambda_i+2f_i+3}$ when v_{λ_i} is saturated

where $\Delta_i = 1$ if $L_{\lambda_i+s_i}$ is improper.

$\Delta_i = o$ if $L_{\lambda_i+s_i}$ is proper.

The solution of this problem for dyadic local fields in which 2 is ramified would certainly be much more difficult, for none of the theorems here proven generalize to that case. The main reason for this is that vectors in a modular lattice cannot be divided into merely two catagories (saturated and unsaturated). This results from the fact that when 2 is a prime, there are at most two lattices over a given quadratic space - a proper lattice and an improper lattice - whereas in the ramified case, there can be several lattices over a given quadratic space. Hence any attempt to generalize the method to the ramified case would have to begin by somehow generalizing the concept of saturization of a vector in a modular lattice.

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BIOGRAPHICAL NOTE

Allan Trojan was born in 1940 in Vancouver, British Columbia. He attended school in British Columbia, graduating from North Vancouver High School in 1957. The same year he entered the University of British Columbia where he enrolled in the Honors programme in Physics and Mathematics. He received a B.Sc. degree in 1961 and since that time has studied at the Massachusetts Institute of Technology under a Woodrow Wilson Fellowship and an Institute Research Assistantship.