Jet Processing of Direct Chill Cast Aluminum Ingots

by

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Abstract

Macrosegregation of solute elements during casting operations has been a subject of observation, discussion, and research for several hundred years. The unique casting conditions found in the Direct-Chill aluminum casting process can lead to significant accumulation and depletion of solute elements throughout the cast product. In particular, the center of ingots are generally characterized by a depletion of up to 20% of solute elements. In this investigation we confirm the dominant role mobile, solute-depleted grains play on macrosegregation along ingot centerlines. We propose that avalanche events drive the grains from the solidification front to the ingot center. In order to demonstrate and control this preferential settling, we propose the use of a turbulent jet to re-suspend the avalanching grains. An analytical expression is derived in order to optimize the dimension of the jet with the goal of minimizing centerline segregation. We then performed a series of experiments in order to validate the analytical expression across ingot dimensions and alloy families. Our experiments indicate that the use of a properly sized turbulent jet can reduce the degree of centerline segregation by up to 70% from standard conditions.

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Chapter 1

Introduction

Aluminum is arguably the second most important metal economically after iron. Of the approximately 80 million tonnes of aluminum cast each year, approximately half is produced using the Direct-Chill (DC) method at hundreds of different casting centers worldwide, meaning that this singular process is representative of a multibillion dollar activity.

The technology behind DC casting is rather well developed with a lengthy history stretching back to the 1940’s. A review of the theories, hypotheses, and factual data available through extensive research of the DC process illustrate, that even after over 70 years of research and practice, structure and defect formation are still not completely understood, let alone controlled. Perhaps one of the most pernicious of these defects is macrosegregation, i.e. large scale compositional inhomogeneity throughout the dimensions of the casting. While the overall composition may match the target product, local compositional variances can fluctuate up to 20% from nominal. Such variances can cause drastic changes in physical properties throughout the ingot [1].

While research into the causes of macrosegregation has been steady since the development of DC casting, there has been very little progress in treating its fundamental causes. Instead the focus has been on minimizing the effects of macrosegregation while maintaining productivity. The present research has been undertaken in an effort to leverage the knowledge gained over the past 70 years in order to eliminate macrosegregation from DC cast rolling slab ingots without sacrificing production.
1.1 Background

1.1.1 Macrosegregation

Modern casting techniques used for the production of wrought aluminum alloys exhibit high productivities and produce high quality as-cast product. Specifically referencing quality, segregation, the inhomogenous distribution of alloying elements on a variety of length scales, is characteristic of all cast products. It is well established that the solidification of alloys is accompanied by varying degrees of microsegregation of alloying elements within individual crystals of polycrystalline systems. This form of segregation is due to the difference in diffusivity between the solidified and liquid phases. In the case of hypoeutectic alloys, used for most wrought products, this leads to an enrichment of solute elements in the liquid phase as solidification progresses. If, subsequently, an overall net movement between the liquid and solid occurs, microsegregation translates to macro-scale composition differences, known as macrosegregation. Thus one can define macrosegregation as the spatial non-uniformity in the chemical composition whose length scale is greater than the diameter of an individual crystal. The concentrations of alloying elements may vary substantially throughout a cross-section of a cast. In the most extreme cases, the composition in certain regions is outside of the allowed limits established for the bulk alloy. While microsegregation (where diffusion distances are on the order of one dendrite arm spacing or cell size, usually between 10 and $100\mu m$) may be minimized or eliminated by subsequent heat treatments (homogenization), the length scales associated with macrosegregation (centimeter to meter) make it unaffected by a homogenizing heat treatment. Hence macrosegregation constitutes an irreversible defect from a manufacturing perspective, influencing the efficiency of subsequent precipitation heat treatments, leading to property variations, and ultimately affecting the quality of the product. To date, the occurrence of macrosegregation limits the size, alloy composition, and productivity of direct chill (DC) casting centers worldwide.

The degree of macrosegregation of a given alloy is dependent on the ingot dimensions, alloy, and the process used for casting. Continuous, or semi-continuous casting
routes are frequently used to produce ingots of wrought-alloy compositions, destined for subsequent processing [2–7]. Moreover, the semi-continuous direct-chill casting technique is the most efficient technology for the production of large ingots destined for rolling or forging. The presence of macrosegregation in these products sets limitations on the size and composition that can be cast in a productive and efficient manner. Thus the importance of macrosegregation during the production of cast products cannot be understated. Considered a major defect in DC cast aluminum alloys, the occurrence of this metallurgical flaw is determined by solidification conditions. In general, it is known that macrosegregation is influenced by solid-liquid flow conditions in the semi-solid region of the casting [4,8–12]. A complete review of the various mechanisms and their role specifically in DC cast products has been completed elsewhere [13]. We propose a review of the most relevant parameters in order to prepare the reader for our subsequent modifications to current theory and practice.

1.1.2 Direct-chill Casting Process-a brief introduction

The modern Direct-Chill (DC) process was invented in 1936-38 nearly simultaneously in Germany (W. Roth, VAW) and the USA (W.T. Ennor, ALCOA). Their process was based on prior methods (for copper and aluminum alloy casting) proposed by B. Zunkel (1935) and S. Junghans (1933). The rapid development and deployment of DC casting was facilitated extensively by requirements from the aviation industry for large ingots (both round and flat). At first, the demand was driven by the booming passenger airline industry, and later by military requirements during World War II. By 1945, nearly all wrought aluminum was produced by DC casting in the United States of America, Germany, and the Soviet Union. Excellent reviews are available on the key process developments along with insights into the process mechanics [2, 3, 5, 14]. Typical products formed through the DC process include large rectangular sections known as ingots (500-700mm X 1500-2000mm which are rolled into plate, sheet, or foil) and circular sections known as billets (up to 1100mm in diameter, which are forged or extruded into rods, bars, tubes, or wires). While these two terms may be used interchangeably when referring to general processes, the present work is dedicated to rolling-slab ingots, though the methodology
and models may readily be modified for use in billets if so desired.

Figure 1-1 is a schematic representation of the modern DC casting process. During DC casting, liquid metal is poured into a mold cooled by water. The mold is initially sealed at the bottom by a starting block made of steel or aluminum. The liquid metal fills the cavity and is allowed to freeze on the starting block and form a solid shell adjacent to the mold walls. Once this shell reaches the desired thickness, the starting block is lowered into a pit at a controlled casting speed (refer to Table of Symbols), while maintaining a specific metal level in the mold. A solid shell is continuously formed due to heat extraction through the water-cooled mold (primary cooling). The solid shell is used to retain the still liquid/semi-solid core of the ingot. Further solidification of the ingot is accomplished by 'chilling' (cooling or quenching are alternative terms) the solid shell directly by an array of impinging water jets as the ingot descends past the lower edge of the mold (secondary cooling, see Figure 1-1). This secondary cooling process extracts approximately 95% of the heat of solidification, and thus governs the final microstructure of the ingot. In a vertical process variant, the casting ceases when the starting block reaches the bottom of the pit. Horizontal versions of the process are truly continuous, equipped with flying saws to section the solidified product, thereby allowing casting to continue.

The main DC casting parameters are:

- casting speed (the speed at which the solid is withdrawn from the mold)
- water flow rate (which influences the cooling rate)
- liquid temperature (degree of superheat).

The optimum casting speed, usually between 30-200 mm/min, depends on the alloy composition and the size of the ingot being cast. The water flow rate ranges from 2 to 4 L/s per mm of mold circumference [14]. Typical melt temperatures are in the range of 690-725°C for commercial aluminum alloys.
Figure 1-1: Schematic of a modern DC casting apparatus illustrating the key elements of the process. Note the approximate shape and position of the liquid sump relative to the mold. Under steady state casting conditions, this shape and position is constant.
1.1.3 Solidification Front Characteristics

The process variables described above determine the thermal boundary conditions of casting, and therefore set the temperature distribution within the ingot. In conjunction with the alloy composition, this determines the geometry and dimension of the sump of the ingot. The ingot sump consists of the liquid pool and the semi-solid transition region which evolves from 0% solid to 100% solid. A representation of the transition region has been produced in Figure 1-2. Once the temperature of the liquid metal descends below the liquidus temperature (654°C for Al4.5Cu), grain nucleation occurs heterogeneously through the addition of grain refiner (TiB or AlC). As the temperature continues to drop, these grains grow in size but continue to be mobile and can move from their unique nucleation site. This portion of the transition region is known as the slurry region. Once the grains have grown sufficiently in size (occupying 20 vol% for Al4.5Cu [15]) the grains impinge on one another, forming a solid cohesive network [15]. This point is known as the coherency isotherm and defines the boundary between the upper slurry region and the lower mushy zone. Since this point is one that can be physically probed, to practitioners the coherency isotherm generally marks the solidification front. As the temperature continues to drop solidification continues, until the remaining interstitial liquid is transformed to solid at the solidus temperature. In some texts, the entire transition region is labeled as the mushy zone, but in this study we have made the distinction between the two regions for ease of comprehension. During steady state conditions, the positions of these isotherms (liquidus, coherency, and solidus) are stable in the reference frame of the mold, and thus the various isotherms can be considered stable reference points. Different mechanisms of macrosegregation are dominant in the slurry and mushy zones, and are also dependent on the specific location within the ingot [13].

The depth of the sump (defined at the center of the ingot, from the liquid surface to the coherency isotherm) is one of the defining features of the solidification profile in DC casting (see [13] for example). The sump depth primarily depends on casting speed, alloy type, and dimension of the casting. For ingots, the sump depth ($H$) increases with the square of the ingot width ($w$), linearly with casting speed ($U_c$), and is inversely pro-
Figure 1-2: (Left) Phase diagram of the Aluminum end of the Al-Cu system with an arrow marking the equilibrium solidification path for Al4.5Cu. (Right) A schematic representation of the microstructural evolution of the two-phase region illustrating the nucleation of mobile grains in the slurry region, their attachment at the coherency point, and their transformation to fully solid.

Portional to the alloy thermal conductivity as described by the following experimentally determined relationship [2,16]

\[ H = \frac{A U_w w^2}{4 \lambda_s (T_m - T_{surf})} \]  

Here, \( T_m \) is the melting temperature, \( T_{surf} \) is the temperature of the ingot surface (not to be confused with \( T_s \)), \( \lambda_s \) is the thermal conductivity of the solid, and \( A \) is a parameter that is alloy dependent determined by:

\[ A = L_f \rho_s + 1/2 c_p \rho_s (T_m - T_{surf}) \]  

Where \( L_f \) is the enthalpy of fusion, \( \rho_s \) is the density of the solid, and \( c_p \) is the specific heat capacity of the solid. These equations were originally determined for billets, but the radius has been replaced by the width of the ingot. An analytical solution has also been derived to analyze the complete 3-dimensional shape of the interface formed during ingot casting. Its derivation, and results are detailed in Appendix A.
1.1.4 Flow Patterns

The fundamental cause behind macrosegregation is the relative movement between solid and liquid phases. In the case of hypoeutectic aluminum alloys, this represents the movement of solute depleted solid relative to solute enriched liquid. In order to understand the driving mechanisms, we must identify how this two-phase flow problem evolves through the solidification process. As discussed previously, it is convenient to divide the transition (two-phase) region into two distinct parts by the coherency isotherm:

- a slurry zone in which the solid phase is suspended in the liquid phase
- a mushy zone where the solid phase forms a coherent network and moves downward at a uniform speed (casting speed).

The flow patterns in these two regions are distinct from one another. In the slurry zone, the volume fraction of grains is in the range [0 - 20%] (for Al4.5Cu). The drag force imposed on the solid phase by the liquid phase is relatively small, and thus the flow characteristics are primarily driven by forced and natural convection. The forced convection can be a direct result of filling conditions, or an intrinsic feature of a casting technology, e.g. electro-magnetic casting. Natural convection driven by temperature and composition gradients has been extensively reviewed and simple equations used in the implementation of a finite element model can be found in reference [17].

Another essential element of the flow in the slurry zone is the settling of suspended grains during their movement with the liquid, due to the higher density of the solid phase as compared to the liquid. Perhaps the simplest eulerian expression to illustrate these phenomena has been expressed by Ni and Incropera [18] as:

\[ U_s - U_l = \frac{1 - c_v}{18\mu_m} (\rho_s - \rho_l) d_s^2 g \]  

(1.3)

where \( U_s \) and \( U_l \) are the velocities of the solid and liquid phases respectively; \( c_v \) is the fraction of solid; \( \rho_s \) and \( \rho_l \) are the densities of the solid and liquid, \( d_s \) is the grain size; \( g \) is the gravitational acceleration, and \( \mu_m \) is the apparent viscosity of the solid-liquid mixture.
In the mushy zone, the solid fraction increases to 100% as solidification progresses; and the solid phase progressively forms a rigid, cohesive network, which effectively limits the degree of liquid flow.

1.2 Macrosegregation in DC casting of aluminum alloys

Macrosegregation is an irreparable defect, occurring most prominently in large castings. Full reviews of all of the mechanisms found in large aluminum castings have been previously reported by the author [13,22]. However, the question persists: to what extent can we minimize macrosegregation by exerting control over its driving mechanisms? For any alloy, macrosegregation is linked to a set of structural parameters such as the morphology of the evolving solid phase, magnitude of solidification shrinkage, level of solute rejection into the melt, and the movement of the solid phase in the liquid and slurry regions. Any and all of these parameters can play dominant roles in macrosegregation at specific locations during the solidification process.

Modelling of macrosegregation is typically aimed at quantitative prediction of the occurrence and severity of macrosegregation by considering the basic mechanisms involved. While the fundamental mechanisms have been well described and reported individually, the first challenge is to determine the relative importance of each.

1.2.1 Segregation patterns in DC cast alloys

As already elucidated, the reason why macrosegregation occurs in castings is the relative movement of solid and liquid phases. Microsegregation is the partitioning of solute elements between liquid and solid phases during solidification of a single grain (100μm in diameter). The bulk movement of these partitioned elements (e.g. many grains together or an entire liquid volume) can then drive macroscopic solute partitioning. Thus the observed composition profiles in an ingot cross-section, depend on the partition coefficient, $K$, and its magnitude. The partition coefficient is defined as the slope of the liquidus divided by the slope of the solidus on an equilibrium phase diagram. The majority of the elements found in wrought alloys (e.g. Cu, Mg, Zn, Li, Mn, Si, Fe) are hypoeutectic
Figure 1-3: 1-Dimensional schematic representation of the general segregation pattern formed by elements with partition coefficient, \( K < 1 \).

Elements with partition coefficients, \( K < 1 \), exhibit a negative (solute depleted) segregation in the center and subsurface, while there is positive (solute enriched) segregation at the surface and quarter thickness. In the hypothetical condition of no macrosegregation, this plot would simply be a straight line at zero compositional variation.

It has additionally been observed that the magnitude of the compositional variations follows closely the magnitude of the partition coefficient [19]. If \( K \) is close to 1, there is little difference between the slopes of the liquidus and solidus lines, and little tendency for segregation to occur. Conversely, elements with \( K \) much smaller than 1 have a tendency to strongly segregate. Table 1-1 is a list of the most common solute elements in aluminum alloys and their respective partition coefficients [19, 20].

The compositional deviation in commercial aluminum alloys is generally inversely proportional to the partition coefficients of the respective alloy element [10, 19, 21]. For example, it has been demonstrated in an AA2024 alloy that Fe exhibits the highest degree
Table 1.1: Partition coefficient, K, for the most common solute elements in aluminum alloys

<table>
<thead>
<tr>
<th>Element</th>
<th>Fe</th>
<th>Si</th>
<th>Cu</th>
<th>Mg</th>
<th>Zn</th>
<th>Mn</th>
<th>Cr</th>
<th>Ti</th>
</tr>
</thead>
<tbody>
<tr>
<td>K</td>
<td>0.03</td>
<td>0.13</td>
<td>0.17</td>
<td>0.43</td>
<td>0.45</td>
<td>0.90</td>
<td>2.0</td>
<td>9.0</td>
</tr>
</tbody>
</table>

Figure 1-4: Surface contour representing deviation from furnace composition as observed in one quadrant of a horizontal section taken from a 600x1750mm Al4.5Cu ingot. The color code is based on relative deviation from furnace composition (in %), positive values corresponding to solute enrichment, while negative values represent solute depletion. All X and Y axes are identical with (0,0) representing the center of the cross-section. The macrosegregation index (Equation 1.4) for this slice is 0.1034.

of segregation, while Mn exhibits the least [19, 21].

The transition regions of billets are axisymmetric structures so the representation in Figure 1-3 is a fair schematic of the segregation found in such castings. The sump of an ingot is however a 2D structure [13] and thus such a simple representation is not appropriate. A 2-D segregation profile for a DC cast ingot is presented in Figure 1-4.

A full analysis of this ingot including segregation and microstructure has been published previously [13, 22], readers are directed to these references for data and analysis of this ingot.

Perhaps the most striking feature of Figure 1-4 is the large depleted region along the centerline of the ingot (represented in shades of blue). At the center of the ingot, this region is 15% lower than the target product composition. For an Al4.5Cu alloy, this corresponds to a local composition of 3.825 Cu. It is well recognized that this region
is responsible for the drastic changes in mechanical properties across the thickness of plate products [1].

We propose a metric called the Macrosegregation Index (MI) in order to quantify the degree of centerline segregation. Equation 1.4 is a modified second-area moment equation that assigns quantitative values to the concentration measured at each position, based on its deviation from the target alloy composition and its distance from the center.

\[
MI = \frac{1}{C_0} \left[ \frac{Y}{A} \int_A (C - C_0)^2 \frac{1}{Y} dA \right]^{1/2}
\]

(1.4)

This allows a quantifiable comparison of macrosegregation profiles where modifications to composition profiles can be drastic and difficult to compare. For example, if the depleted centerline became enriched, Figure 1-4 would appear differently but without a means of quantification, any degree of comparison between the two cases would be impossible. For the ingot displayed above in Figure 1-4, the macrosegregation index is 0.1034. This will be the base for comparison in any attempt to compare degrees of macrosegregation.

1.3 Centerline Depletion in DC cast ingots

As mentioned above, one of the most striking features of macrosegregation patterns in large ingots is the severely depleted centerline region. Excellent reviews of all of the mechanisms driving segregation in large castings are available elsewhere [23–26], and in this section we will focus on the potential mechanisms driving the depleted centerline of these ingots.

1.3.1 Shrinkage flow

It has been well documented that one of the primary reasons for negative centerline segregation is the flow driven by shrinkage of the alloy as it solidifies (6-10 vol % in aluminum alloys). This shrinkage driven flow is only significant (as mentioned previously)
in the mushy zone of the transition region. While these flow velocities are small in magnitude \(10^{-4} \text{ m/s}\), their effects can be significant because they involve the transport of highly enriched material. This material is drawn from the hotter part of the transition region and directed normal to the coherency isotherm, deeper into the mushy zone. It then contributes to negative centerline segregation by "robbing" the centerline of rejected solute \([11, 27, 28]\). Eskin et al. \([29]\) have proposed a model to estimate the magnitude of the macrosegregation caused by shrinkage flow. The shrinkage flow, which is always directed normal to the solidification isotherms can be decomposed into horizontal and vertical components (see Figure 1-5). Negative centerline segregation occurs due to the horizontal component only \([8, 30]\). The solute transport from the centerline to the surface occurs very slowly, gradually a bulk solute transfer is generated. The centerline becomes depleted because there is no horizontal influx of additional enriched material. Eskin et al. proposed the following equation to calculate the horizontal solute transfer distance:

\[
L_H = A C_0 L m \beta (\sin 2\alpha) / 2
\]

Where the slope of the local isotherm is \(\alpha\), the mushy zone thickness is \(L_m\), \(\beta\) is the shrinkage ratio (0.1 for Al4.5Cu), \(C_0\) is the nominal bulk composition, and \(A\) is a parameter that takes into account the solidification path (0.78 using the Scheil Equation). The first derivative of this relationship with respect to the ingot width \((dL_H / dw)\) gives a net solute efflux. Normalizing this solute efflux by the bulk composition \((dL_H / dw) / C_0\) then reflects the local relative segregation. The results of this analytical solution have been applied to simple finite element simulations, incorporating uniquely heat transfer conditions, and ignoring convective conditions. These results indicate the potential to produce centerline segregation in castings absent of other effects. In reality, such a model would be difficult to apply using experimental data because it assumes a uniform composition at the solidification front, and requires the careful measurement of the mushy zone thickness. Thus, while shrinkage flow can explain some of the negative centerline segregation observed in industrial ingots, it is not the only cause of such segregation.
Figure 1-5: Schematic representation of solidifying interface formed during DC casting illustrating the direction and magnitude of shrinkage induced flow in a solidifying ingot ($V_{SHR}$). Normally only the horizontal component ($V_H$) is responsible for macrosegregation, meaning that the steepest part of the sump should exhibit the greatest degree of shrinkage induced flow. Adapted from [29]
1.3.2 Sedimenting grains

The movement of free moving grains from the slurry region, with sedimenting or free-floating grains, has long been identified as one of the dominant mechanisms behind centerline segregation in DC cast ingots. The original theory was proposed in 1929 by Voronov [31]. He proposed that solute-depleted crystals detached from the solidified shell and were then carried to the center of the ingot by the advancing solidification front or particle pushing. This theory was largely proposed due to the appearance of a duplex structure made of fine and coarse dendrites in the center of these castings. However, the validity of the theory was questioned in the 1940's due to experimental evidence illustrating the presence of centerline segregation in the absence of a duplex structure. Of course now the role of shrinkage flow illustrates how this latter condition could exist. The theory was then re-instated by Yu and Granger [9] and Chu and Jacoby [10] in the 1980’s due to the persistent observation of the duplex structure in some castings [21, 32–36]. There has been some debate on the source of depletion originating in the duplex structure, but it has been determined that coarse dendrites are more depleted in solute compared to the fine dendrites [26, 37]. Researchers have utilized numerical modeling approaches to investigate how the movement of grains can influence macrosegregation [38, 39], whose results indicated that the gravity-driven sedimentation [18] of grains to the center of ingots can result in centerline depletion.

Today there is little debate on the presence of sedimenting grains in large castings. The primary open question however is why a duplex structure, and evidence of solid phase transport is present in only some castings. It has been well documented from early works [2, 9, 19, 21, 32–34, 40], to modern investigations, that the severity of macrosegregation tends to increase with casting speed. While this is easily explained by the deepening sump as seen in Equation 1.1, it has also been noted that the increase in casting speed leads to an increase in the volume fraction in sedimenting grains [41]. This effect has led us to hypothesize that there is an additional effect beyond normal convective flow driving the transport of sedimenting grains. This effect and its consequences are presented below.
1.3.3 Avalanche effect

It is generally assumed that the transport of sedimenting grains is driven by convective flow conditions within the slurry region. However, convective flow conditions are present in all castings, and thus the transport of solid phase should always be an observable phenomenon. It is however, not consistently observed. We have previously noted that an increase in casting speed not only increases the degree of centerline segregation, but also the amount of sedimented grains along the centerline. We propose that the deepening of the sump causes the angle of the walls of the sump to exceed the angle of repose of the growing grains within the slurry region. Due to avalanche dynamics (see Appendix B), exceeding the angle of repose of any granular material causes a catastrophic movement of particles until the angle of inclination is once again below the static angle of repose. Using the relations presented in Appendix A, and property values for the Al4.5Cu alloy, we have plotted the angle of inclination of the sump as a function of ingot width and casting speed. This relationship is displayed in Figure 1-6.

In this figure, we can see that as casting speed and ingot width increase, the angle of inclination also increases. In the 1970's Livanov et al [42] performed a series of tests on a variety of molds at different casting speeds in order to determine the optimum casting speed for zero centerline segregation. They found that operating at speeds lower than this (for a given width) generated positive centerline segregation, while operating faster generated negative centerline segregation (white dashed line). While this bears resemblance to the segregation conditions proposed by Flemings et al [43] involving flow conditions, we propose that this shift is due to the increase in sedimenting grains once the angle of inclination is exceeded. As previously mentioned, it has been determined that fine-cell dendrites are richer in solute than coarse-cell dendrites. It is important to realize that fine-celled dendrites represent coarser particles while the opposite is true for coarse-celled dendrites. As the sump depth increases, it is the coarser-dendrites (smoother particles) that will begin to avalanche first. Since these grains are depleted in solute, there is a preferential accumulation of solute-lean grains in the center of the ingot leading to increased solute depletion. As the sump becomes deeper, more of these
Figure 1-6: Representation of the angle of inclination of the sump as a function of both casting speed and ingot width. The color scale is such that higher angles of inclination are red, while lower ones are blue. The colorbar at right represents this scale from 0 to 90° of inclination, with labels for the angle of repose of smooth and coarse media for reference. The dashed line and labels in the lower plot represent the findings of Livanov in reference [42]
grains sediment thereby increasing this effect.

The non-generalized observation of the duplex microstructure could be due to several effects. Firstly, it is necessary to have a sufficient thickness of the slurry zone in order to generate sufficient sedimenting grains to have an observable impact on structure even though it may still be occurring. Secondly, depending on the alloy in production, different speeds may be used for similar ingot dimensions. Thus in addition to the slurry zone thickness difference (through alloy change), the change in speed will result in a sump geometry change either promoting or inhibiting the avalanche effect. This makes comparison between casting conditions very difficult, and potentially explains why duplex structure was not always observed even though centerline depletion was present.

In an attempt to visualize the avalanche of grains within the sump, we slowly poured molten zinc into the head of an ingot during casting. This allowed us to preferentially mark the transition region at a series of "snapshots" and observe any transient behavior. This was similar to the procedure used to mark the sump by the author in a previous study [13]. Once casting was completed, the ingot was sliced at the midway along the long face, polished, and etched with sulfuric acid. The results are presented below in Figure 1-7. The small red arrows indicate incoming periodic instabilities which are poor in zinc. Since the very bottom of the sump was continually filled with molten zinc, it is proposed that these instabilities represent "avalanches" of grains (50μm in diameter) as they fall to the bottom of the sump. Since they displace the zinc and do not discolor when etched (like zinc) their effects are preserved beyond casting.
Figure 1-7: Zinc poisoned ingot sliced parallel to the short axis and etched with a Tri-acid etchant. This ingot displays the periodic avalanche events which drive the formation of centerline depletion in DC cast ingots. The less alloyed grains appear brighter because they are etched to a lesser extent, and which allows us to visualize them in a heavily alloyed (Zn) environment.
References


Chapter 2

Hypothesis

2.1 Theory

It is undeniable that shrinkage flow and grain sedimentation both contribute to the depleted centerline in DC cast products. Typical investigations of macrosegregation have focused on the eutectic forming elements elements (K<1) which are added for enhanced material properties in final products. Grain refiners are added to aluminum castings in order to artificially refine grain structures, and inhibit crack generation and propagation in the mechanical steps of the casting process. Grain refiners are peritectic formers (K>1) and are thus the first phase to solidify during the solidification process. Practitioners use this property to generate peritectic nuclei on which the aluminum phase then nucleates and grows. As a consequence of this solidification path, peritectic elements are not rejected as eutectic formers are, and they should exhibit no segregation due to shrinkage-induced flow. Since all of the peritectic material is connected to the solid phase, the only possible mechanism by which segregation of peritectic elements could occur is via the bulk movement of grains. Therefore, regions affected by the accumulation of these mobile grains should not only exhibit depletion of eutectic forming elements; but must also exhibit a corresponding enrichment of peritectic formers.

Figure 2-1 is data reproduced from Gariepy and Caron [1] displaying the variation in concentration of a peritectic former (Ti) alongside a eutectic former (Mg). We can see that while the magnesium distribution matches the typical depletion pattern we would
expect (see Figure 1-3), the titanium segregation pattern is inverted exhibiting enrichment instead of depletion. We can also observe that the enriched titanium region corresponds exactly to the depleted magnesium region. Gariepy and Caron [1] concluded that the only way for such agreement to occur is if the fundamental mechanism underlying both observations is the same. This conclusion re-established the importance of grain sedimentation in centerline depletion theory.

Figure 2-2 is a reproduction of data collected from Yu and Granger [2], who performed experiments on an AA2024 ingot cast using a peritectic forming grain refiner (Ti). They examined the effect of varying the casting speed on the distribution of titanium across the slab thickness. They found that along the ingot centerline, there was a marked increase in titanium composition (similar to Gariepy and Caron), a trait that was exacerbated by an increase in casting speed.

Since we know that enriched peritectic centerlines occur only through grain sedimentation, an effect that can create depleted eutectic centerlines, we can conclude that grain sedimentation is a driving mechanism for centerline macrosegregation. However, the observations of Yu and Granger also illustrate how an increase in casting speed also increases the concentration of peritectic formers (Ti), which could only occur via a corresponding increase in grain sedimentation. The only mechanism which could drive the added sedimentation of mobile grains is the avalanche effect due to the increased angle of inclination of the sump walls caused by the increase in casting speed.

### 2.2 Hypothesis

If the dominant mechanism of centerline segregation in DC cast aluminum ingots is the sedimentation of grains during the casting process; we can infer that the removal of these superfluous grains would lead to a corresponding reduction in centerline segregation. We therefore hypothesize that a controlled removal of the grains which sediment during the casting process, would minimize the degree of centerline segregation in DC cast aluminum ingots.

The chemical engineering sector frequently requires the re-suspension of solids in
Figure 2-1: Data reproduced from reference [1] displaying the variation in composition for a peritectic former (Ti) and a eutectic former (Mg) across the thickness of a rolling slab ingot. The depletion of eutectic elements along the centerline corresponds to the enrichment of peritectic elements, a situation which could uniquely be caused by the sedimentation of grains regardless of the presence of shrinkage induced flow.
Figure 2-2: Data reproduced from reference [2] displaying the segregation of a peritectic forming element (Ti) throughout the width of a rolling slab ingot cast at two different speeds. The increase in casting speed leads to an increase in sump depth and a greater degree of sedimentation, which accounts for the larger amount of peritectic titanium found in the center of the ingot.
reacting vessels. One of the oldest, best documented, and most reliable methods practitioners utilize is a turbulent mixing jet. By removing a volume of fluid from the reacting vessel and re-introducing it via a turbulent jet, chemical engineers are able to prevent the accumulation of solids. If we treat the ingot sump as a reaction vessel (solidifier or crystalizer) we can see how a similar scheme could function to prevent the sedimentation of grains. While the removal, transport and re-introduction of molten metal may be impractical, perhaps the simplest procedure to employ is to use the existing metal entrance tube or downspout, (Figure 1-1 tube linking the tundish to the molten sump) as a downwardly impinging jet. This configuration is represented below in Figure 2-3. The large arrow represents the jet directed in the Z direction into the molten pool. The boundary of the sump is represented by the multi-colored surface, which has been colored by depth. When the jet impacts the bottom of the sump, the flow is re-directed into the positive and negative X directions by the sump walls. The re-directed flow continues along the centerline (blue region in Figure 2-3) re-suspending any sedimented grains and dissipating its kinetic energy.

Typical DC casting operations utilize a mesh distribution bag to distribute the in-
coming superheated metal more evenly over the surface of the ingot. By removing this mesh bag, and modifying the diameter of the downspout, we can modulate the stirring motion directed into the center of the ingot. While there is little doubt if a turbulent jet can remove a quantity of grains, we anticipate that it will be possible to re-suspend all of the superfluous grains found along the ingot centerline by properly sizing the diameter of the jet according to the alloy, mold dimensions, and casting parameters. In removing only the superfluous grains from the centerline we should be able to minimize the degree of centerline segregation for a given product.

2.3 Experiment Specifications

In order to control the degree of sedimentation of mobile grains into the center of DC cast aluminum ingots it is necessary to facilitate the formation of a sufficient number of grains along the ingot centerline. This has two strong practical implications for the experimental setup regarding the appropriate alloy and size/scale of the casting.

The unique chemistry of each alloy determines the thickness of the slurry region and thus the number of mobile grains. Choosing an alloy with insufficient slurry zone thickness, as is the case with many relatively pure alloys, means that there are very few mobile grains to potentially move and thus very little centerline segregation generated by their sedimentation. The alloy we have chosen to investigate is Al4.5Cu, which is a simple alloy to cast and manufacture and reaches coherency at approximately 20% solid [3]. This degree of free-moving solid is expected to effectively generate centerline segregation via sedimenting grains, while maintaining castability. The added benefit of this alloy choice is that it was the precursor to the current aerospace grade alloys (6xxx and 7xxx), and thus was the subject of substantial macrosegregation investigations due to the presence of severe macrosegregation in cast products. The availability of these results makes our own studies easily comparable to those of previous researchers.

Alloy composition and thermodynamics certainly provides the mechanism of formation of a number of free-moving grains. During standard casting conditions, each grain may only be 50 \( \mu m \) in diameter (see Appendix C for experimental results deter-
mining this). It is important to ensure that the size and scale of the casting is correct to ensure that the bulk sedimentation of grains can generate observable centerline segregation. Even with an appropriate alloy composition, if the angle of inclination is insufficient then centerline segregation may not be observed at all. According to Figure 1-6, for small ingot widths (<10cm), casting speeds in excess of 2 mm/s would be required to generate zero centerline segregation. In order to generate negative centerline segregation, casting speeds of at least 4 mm/s would be required. While mechanically these conditions may be possible, we must take into account the relative velocity of the settling grains as well. For a 50μm solid aluminum grain falling in liquid aluminum, its terminal velocity is on the order of 2 mm/s (calculated from the stokes settling velocity [4]). Thus while the angle of inclination of the sump may be sufficient to generate avalanche events, the solidification front itself would be advancing twice as fast as the grains could sediment. This would make their rate of attachment much higher, and would decrease the degree of observed centerline segregation. In order to operate at casting speeds below the sedimentation velocity of the grains and still promote centerline segregation, we would need to use a significantly larger ingot width. To satisfy this requirement, we opted to use a 0.6 m x 1.75 m ingot cast at 1mm/s. This dimension and casting speed is well within the negative centerline region presented in Figure 1-6, and is sufficiently slow to allow grains to sediment before they are attached.

Due to the large size of ingots required to investigate sedimenting grains and centerline macrosegregation, certain previous methodologies to promote stirring, such as electromagnetics [5–7] are impractical. The magnetic fields requisite to generate motion in the center of a 0.6m thick ingot would have to be generated by complex electromagnet configurations.

### 2.4 Summation of Hypothesis

In this section we have proposed the hypothesis which we will investigate in the current study. This hypothesis can be summarized in three distinct steps:
• Sedimenting grains are the driving mechanism behind centerline segregation. These grains are depleted in solute and their bulk accumulation creates depleted centerlines in DC cast ingots. The steep angle of inclination of the sump walls during casting causes avalanche events during which, the grains migrate to the ingot centerline.

• These grains are mobile upon initial sedimentation. By utilizing a turbulent mixing jet, they can be re-suspended and removed from the ingot centerline.

• There is an optimum jet configuration which removes all of the superfluous grains from the ingot centerline thereby minimizing the centerline segregation for a given DC cast product.
References


Chapter 3

Analytical Framework

3.1 Theory

An eulerian fluid dynamic approach to two-phase flow and in particular grain erosion from granular beds is a topic that has received significant prior attention, for example with the goal of understanding sediment removal and transport or designing pipe or reactor flow conditions for suspensions. The relevant governing parameters have all been experimentally determined for horizontal flow over a horizontal bed, and a pertinent review is presented in Appendix D. We can redefine the corresponding parameters for a jet impinging perpendicularly onto a granular bed. Figure 3-1 is a schematic representation of this condition, specifically referenced to the case of solidifying metal, though valid for other fluid/grain relationships.

The primary goal of the impinging jet is to remove the excess grains which may be found along the ingot centerline due to avalanche events. This requires a fundamental understanding of the rate of grain transport as a function of jet power. We can visualize as in Figure 3-1, that an arbitrary jet impinging on a granular bed will promote the re-suspension of grains within a certain radius of the impingement point. As these grains are transported from the center due to the influence of the jet they form a crater, whose radius and deepening rate are functions of the grains and the jet itself. In a previous study [1] we have derived an equation for the deepening rate of the formed crater as a function of these aforementioned parameters:
Figure 3-1: Schematic of a liquid metal jet with volume flux $Q_0$, impinging on a granular bed through a nozzle of radius $b_0$. The velocity of the jet exiting the nozzle is $U_0$. The jet is situated a height, $H_0$ above the coherency isotherm. Above this is a granular bed (grain diameter $d$) of material forming the slurry region of height $h_0$.

\[ U_c \approx C_1 U_{th} Re_g^{3/2} \left(1 - C_2 \frac{U_{th}/U_j}{Re_g} \right) \tag{3.1} \]

Where $U_c$ is the descent velocity of the bottom of the crater, $U_{th}$ is the terminal settling velocity of the grains, $Re_g$ is the granular Reynolds number, and $U_j$ is the velocity of the jet at the surface of the granular bed (slurry region). The derivation of this equation, and the definitions of relevant parameters are provided in Appendix E. While this equation is valid for any jet impacting a non-cohesive granular bed, it is important to recast these parameters into the DC casting space and appropriate terminology. The granular Reynolds number and terminal settling velocity of the grains are related to the shape, size, and relative density of the solid aluminum grains in the liquid aluminum. We have performed a series of experiments to determine these parameters for the sedimenting grains. A review of these experiments and their results are presented in Appendix C. The velocity of the jet at the surface of the slurry region can be determined from the theory of turbulent jets [2]. Explicitly, $U_j$ is described as:
\[ U_j = U_0 \frac{b_0}{a} \left( \frac{H_0 - h_0 + \frac{b_0}{2a}}{2a} \right)^{-1} \]  

(3.2)

where \( b_0 \) is the nozzle radius and \( U_0 \) is the mean velocity of the fluid at the nozzle outlet in the bulk fluid, expressed as a function of the volumetric flow rate \( Q_0 \).

\[ U_0 = \frac{Q_0}{(\pi b_0^2)} \]  

(3.3)

\( H_0 \) and \( h_0 \) represent overall depth of the fluid and the granular bed respectively (See Figure 3-1). For a turbulent jet, the entrainment constant \( a \) can be taken to be \( a = 0.08 \) [3, 4]. In the case of DC casting, \( H_0 \) is typically taken to be the sump depth measured at the coherency isotherm, because the slurry zone is difficult to probe. Numerous relationships exist for the depth of the sump as a function of casting parameters, but the authors have found good agreement between their own derivation of the sump shape and experimental results (See Appendix A). It is important to realize that the fundamental governing parameters of the sump depth (and thus the jet velocity at the bottom of the sump) are the physical dimensions of the mold, the alloy, and the casting speed. Having fixed these parameters in §2.3, we can treat the sump depth as constant. Correspondingly, having fixed the casting speed and the dimensions of the mold, we have also fixed the volumetric flow rate of the jet. Upon careful consideration of Equations (3.2) and (3.3) we can see that the only remaining variable we can use to specify the velocity of the jet is its radius, \( b_0 \). This means that when seeking to optimize a jet velocity to minimize the effects of sedimenting grains, the diameter of the nozzle is the key variable to change the jet velocity, while maintaining all other parameters constant.

Returning to the thought of an optimized jet to minimize centerline segregation, we can visualize the optimum result. Given steady state casting conditions, viewing the sump of the ingot in the reference frame of the mold, we will have a stable sump of a given shape. The jet impinges downward on the slurry region to maintain a certain volume fraction of grains. Avalanche events along the inclined walls of the sump draw volumes of grains to the center of the ingot. As these grains reach the bottom, they are re-suspended by the jet and forced away from the ingot centerline. A jet that is too weak
will allow the avalanching grains to remain in the center, while a jet that is too strong will actively erode the bottom of the sump, and cause it to be deeper than normal conditions. In order to actively balance these two cases we need to provide a jet whose crater descent velocity \( U_c \) in Equation 3.1) precisely matches the velocity of our reference frame, otherwise known as the casting speed. By specifying the descent velocity to be the casting speed, we can guarantee no accumulation of grains in the center of the ingot, and that the power of the jet is then dissipated in the re-suspension of grains.

### 3.2 Application of the Model

With fixed alloy composition and mold dimensions, we are most interested in seeing how the required jet power (diameter) varies as a function of casting speed. In the previous section we have described how we wish to equate the crater descent velocity \( U_c \) in Equation 3.1 with the casting speed. Using our mold dimensions and alloy properties we can calculate the sump depth \( (H_0 - h_0) \) as a function of this casting speed according to the relationship derived in Appendix A. Since the granular Reynolds number \( Re_g \), and hindered terminal settling velocity \( U_{th} \), are functions of the fluid and grains themselves (See Appendix D), this means the final variable left to specify in Equation 3.1 is the jet velocity at the bed surface (bottom of the sump) specified by \( U_j \). Normally we could simply pick any arbitrary jet which fits this parameter \( U_j \), but by fixing the casting speed, we have also inadvertently fixed the volumetric flow rate, \( Q_0 \). This means by implementing Equations 3.2 and 3.3, there is a unique jet diameter which generates the appropriate jet velocity at the surface of the bed in order to generate a crater descending at the same velocity as the casting speed. Using appropriate material parameters from [2] it is possible to provide "boundary" curves representing the effective processing parameters for minimum centerline segregation for a range of typical aluminium alloys, as shown in Figure 3-2. This figure represents the range of predicted jet Reynolds numbers as a function of mold Reynolds number, respectively defined as:

\[
Re_j = \frac{2M_1M_\mu U_c}{\pi b_0 \nu} \quad Re_m = \frac{2M_1M_\mu U_c}{\nu(M_l + M_\mu)}
\]  

(3.4)
Figure 3-2: Predictive plot for jet processing parameters. The mold Reynolds number is based on an equivalent hydraulic radius, and casting speed. The jet Reynolds number is based on the jet velocity and diameter. Shaded region represents a range of values dependent on alloy properties. The dashed line represents the prediction for Al4.5Cu. The squares represent the conditions tested on a commercial DC caster.

where $M_l$ and $M_w$ represent the mold length and width respectively. The upper and lower bounds of the shaded region has been defined by two alloys which have been identified as limiting cases; nearly all other alloys will fall between these boundaries.

### 3.2.1 Crater Shape

In order to better understand the re-suspension potential of the proposed jet arrangement it becomes necessary to discuss the erosion boundary formed by a jet impinging on a granular bed. We will use the same schematic representation in Figure 3-1 to discuss this condition, which have been discussed in-depth by the author in previous studies [1, 6].
3.2.2 Influence of casting parameters on force distributions

Appendix F contains a review of the pressure and shear forces generated by a downwardly impinging circular jet on a flat, non-permeable bed. The analysis presented in Appendix G provides the conditions of stability of the crater walls formed by the impinging jet as a function of the forces \( F_r \) and \( F_y \), generated within the bed. These forces are evaluated for a fixed nozzle height above the bed \( H \). In the case of DC casting, the height above the transition region (sump depth) is a function of the physical mold dimensions, as well as the casting speed and alloy composition. This means that \( H \) is a functional variable, and must be determined for each specific casting condition. Figure 3-3 is a representation of the distribution of shear and pressure forces as a function of casting speed. These calculations were performed using the sump depth relationship proposed in Appendix A as a proxy for the height \( H \) above the granular bed, for an ingot of Al4.5Cu cast in a 600x1750mm mold with a 20mm diameter jet. Since the diameter and velocity of the expanding jet (plume) is a function of the height above the bed, we can expect the distribution of fluid forces to also vary as a function of casting conditions.

3.2.3 Influence of casting parameters on crater shape

Given the distribution of forces described in Appendix F, it is possible to calculate the fluid flow within the granular bed. The percolation of shear and pressure forces through the bed allows the suspension of grains located deep within the bed, thus generating an erosion crater. The calculations of the fluid dynamic forces through a permeable bed are presented in Appendix H. With the distribution of forces within the bed, the equilibrium shape of the crater is obtained by enforcing the equality in Equation G.3. A description of the numerical method is provided in Appendix I. While these calculations presented in appendices are performed for a jet of arbitrary velocity and dimension, it is possible to couple the casting parameters to find the equilibrium shape of the crater in the case of DC casting of aluminum alloys. By specifying the casting speed as well as the mold and jet sizes we can determine the velocity of the jet as it enters the molten pool. Using the sump-depth relation described in Appendix A, the degree of expansion of the
Figure 3-3: Pressure ratio (top) and shear stress (bottom) distributions for a radially impinging turbulent jet, as a function of casting speed for an Al4.5Cu ingot cast in a 600mmx1750mm mold using a 20mm diameter jet. The height ($H$) was calculated from the sump depth relation proposed in Appendix A.
Figure 3-4: 3-dimensional shape of a crater formed by casting an Al4.5Cu ingot in a 600x1750mm mold at 60mm/min with a 20mm diameter spout. Point (0,0) represents the location of the jet impingement on the granular bed (slurry region).

jet is obtained. Coupling these parameters to the force parameters described above allows for the full computation of the crater shape for a given casting condition. Figure 3-4. is a 3-dimensional representation of the crater formed in the slurry region of a 600mmx1750mm Al4.5Cu ingot cast at 60 mm/min with a 20mm jet.

Figure 3-5 is a representation of the various crater profiles formed by varying the casting speed of a 600mmx1750mm Al4.5Cu ingot cast with a 20mm diameter jet. As mentioned above, casting parameters will have an influence on the distribution of forces transmitted to the bed. Thus, since the combination of these forces determines the final crater shape, we can expect casting parameters to also have an influence on the crater shape.
Figure 3-5: Crater profile as a function of casting speed for a 600x1750mm Al4.5Cu ingot cast at 60mm/min with a 20mm diameter jet. The radial position (0) corresponds to the point of jet impingement.


References


Chapter 4

Experimental validation via large-scale trials

4.1 Setup

In order to validate the proposed analytical model, and experimentally verify our hypothesis, a series of experiments were designed to compare existing magrosegregation data against magrosegregation data from ingots cast using a range of jet powers. Using our previously determined mold size (0.6m x 1.75m) and alloy (Al4.5Cu), we performed a series of experiments whose conditions were identical to those described in [1, 2], the only difference being that each ingot implemented a distinct jet diameter. Figure 3-2 displayed the predicted jet parameters based on the model predictions and the experimental casting conditions (mold dimensions, casting speed). Based on the predictive plot, the minimum macrosegregation should be observed for Al4.5Cu with a jet characterized by a Reynolds number of approximately 97,000 for a mold Reynolds number of approximately 1600. Following the casting, cross sections were taken from each of the ingots at a cast length of 1800mm, and then analyzed using an Olympus Alloy Plus XRF Analyzer according to the procedure outlined in [3].

The majority of experimental studies have focused on the steady state cross sectional profiles [1-4]. However, these profiles only represent single snapshots in the entire macrosegregation history of the ingot. If we fix the ingot width in Figure 1-6, but vary
the casting speed, the frequency of avalanche events is predicted to increase or decrease, as the sump gets respectively deeper or shallower. During the beginning (start-up) of a casting practice, the sump depth gradually increases to steady state values. At the end of the cast (shut-down), the sump depth gradually decreases from steady state values. In order to investigate the performance of the mixing jet during these transient regimes, as well as any fluctuations during the length of the cast, the aforementioned ingots were also sectioned lengthwise and analyzed along the centerline according to the scheme described in reference [5].

4.2 Steady State Results

Figure 4-1:A-F are surface plots of the top left quadrant of a horizontal section of a slab (seen from the top), showing the relative deviations from furnace composition for the five casting conditions shown in Figure 3-2. In our previous study involving the use of a mixing nozzle (see [3]) we noticed that the jet has the ability to modify the macrosegregation profile of an Al4.5Cu ingot. Using a range of jet diameters, we can see that there are patterns depending on the power of the jet used during casting. Specifically, ingots cast with a jet Reynolds number below 97000 exhibit positive (enriched) centerline segregation, as opposed to the negative segregation observed for ingots cast without a jet (Figure 4-1A). In contrast, ingots cast with a jet Reynolds numbers of 97000 (Figure 4-1E) or above hardly exhibit centerline segregation, and if any a negative (depleted) segregation. In addition, the extent of the centerline region is significantly narrower with respect to the short axis when using a jet, with a few centimeters in thickness compared to tens of centimeters in absence of the jet.

4.3 Transient Results

Figure 4-2 is a surface plot representing the experimentally determined centerline segregation values for the five turbulent jets evaluated through the entire cast length.

It is noticed that for all of the jets except the most turbulent (Re=121 000), the de-
Figure 4-1: A-F Surface contours representing deviation from furnace composition as observed in one quadrant of horizontal sections taken at 1800mm of cast length: A-Standard casting procedure B-F using impinging jets of Rej = 64000, 69000, 81000, 97000, 121000. The color code is based on relative deviation from furnace composition (in %), positive values corresponding to solute enrichment, while negative values represent solute depletion. All X and Y axes are identical with (0,0) representing the center of the cross-section.
Figure 4-2: Surface plot representing longitudinal centerline segregation for each of the analyzed jet conditions. Segregation is determined as a percentage deviation from furnace composition. Length position has been normalized by the overall cast length. The horizontal grid represents the 0% deviation plane (for reference).
gree of segregation gradually decreases to a steady state value of approximately -15% from furnace composition at approximately 20% of cast length. This behavior is similar to that observed in [2]. For these four ingots the centerline composition remains relatively constant at this value until approximately 80% of cast length. In the case of the most turbulent jet, a nearly opposite behavior is observed, with the composition rapidly descending to -25% before rising to -5% at approximately 20% of cast length. For the remaining 60% of the cast, the trend is fairly non-uniform characterized by fluctuations between -5% and -18% of furnace composition. In all of the ingots, a sudden drop in centerline composition to approximately -20% is observed at approximately 80% of cast length. Immediately follows a linear increase in composition until the end of the cast.
References


Chapter 5

Discussion

5.1 Steady State Results

The qualitative analysis of the plots in Figure 4-1 illustrates the potential for jets to modify centerline segregation in rolling slab ingots. The fact that the centerline segregation zone itself is reduced is a macroscopic confirmation of our hypothesis. However, in order to perform a more quantitative analysis of the process performance, we have calculated the Macrosegregation Index (Eq 1.4) of each of the cross-sections investigated during this series of trials. Figure 5-1 is a plot of the MI for each of the jet tests reported above, identified by their jet Reynolds number. The standard DC profile analyzed in [1] and reproduced in Figure 4-1A has been plotted using an equivalent Reynolds number of 15,000. For the range of jet diameter tested, the macrosegregation index shows at least a 30% reduction from the standard casting method. The best performing jet, (Re=97,000) allows a 60% reduction in centerline segregation. The original derivation of Equation 3.1 relied on the presence of an optimum jet whose crater erosion rate matched the casting speed. Based on the improved performance of the best performing jet we can simultaneously confirm the validity of the analytical model and the presence of an optimum jet.
Figure 5-1: Macrosegregation index (MI) for each of the experimental jet conditions. The standard DC condition has been converted to an equivalent Reynolds number (15000) for ease of comparison. The predicted optimum corresponds to the jet with a Reynolds number of 97000 per the analytical calculation.

5.2 Transient

The initial transient behavior of the jets characterized by Reynolds numbers smaller than 121,000 is in agreement with the model and analysis inherited from traditional DC cast results. As the sump deepens during the start-up phase the angle of inclination of the solidification front gradually increases. This increase in angle of inclination causes a corresponding increase in the frequency of avalanche events. This generates the downward slope seen at the "ingot butt" side of Figure 4-2 as the increasing degree of sedimenting grains drives an increase in centerline depletion. Once reaching steady state, the degree of centerline sedimentation of grains remains constant thereby generating a uniform deviation from furnace composition. At the end of the cast, a decrease in sedimentation can explain the linear increase in composition reported in Figure 4-2. As the bottom of the sump rises and exhibits a smaller average angle of inclination, avalanche events will become less frequent and the degree of grain sedimentation is expected to decrease. This decrease in avalanche events and grain sedimentation will
correspondingly decrease the amount of centerline segregation.

All of the plots show a sudden decrease in composition at approximately 80% of cast length. Since our measurements are reported along the ingot centerline, this position would normally correspond to the bottom of the sump when the metal flow into the mold was shut off and casting ceased. Assuming grains were suspended by the impinging jet, such reduction in turbulent kinetic energy caused a significant fraction of the larger grains to suddenly fall out of suspension. This fallout would increase the amount of sedimented grains at the bottom of the sump, thus locally increasing the compositional deviation. This sudden change in composition, not normally observed in a traditional DC cast, confirms the ability of the jet to remove a certain fraction of the sedimented grains as previously described [2].

Figure 5-2 is a plot of the data from the geometric center and data taken from the area 50mm adjacent to the geometric center. This illustrates that the area directly underneath the jet (in the geometric center) undergoes less erosion than the areas surrounding the point of impingement. The lesser degree of erosion allows for sedimenting grains to accumulate in spite of the jet and create significant centerline depletion. This result is seemingly non-intuitive as we would expect the area directly under the jet to undergo the most erosion. In order to better understand the position dependent erosion potential of an impinging jet, it is necessary to investigate the interplay of forces driving the re-suspension of grains. A discussion relating the crater shape to the observed segregation is presented below.

### 5.2.1 Discussion of crater shape

As observed in Figure 3-3, the functional form of the distribution of forces remains consistent through a relatively large range of casting speeds. However, the increased depth ($H$) obtained at higher casting speeds lead to a larger jet expansion. This leads to a broader radial distribution of the forces, but does not modify the trend. This broader distribution of forces corresponds to an effective widening of the crater as can be seen in Figure 3-5. Perhaps contrary to expectation, increased casting speed also causes a deepening of the crater. We may expect the depth of the crater to decrease with in-
Figure 5-2: Centerline segregation data taken from the $Re_J = 97,000$ cast. Adjacent segregation points were taken from [2], and represent the average segregation 50mm from the geometric center of the ingot (impingement point). Trend lines have been added for reference only.
creased sump depth. However, it is important to recognize that an increase in casting speed corresponds to a higher volumetric flow rate, and thus a higher jet velocity. This means that there is a higher centerline jet velocity at the bed despite a larger sump depth (and increased degree of jet expansion).

As can be noticed in Figure 3-4, the impingement region directly underneath the jet actually undergoes significantly less erosion than the region immediately adjacent to the zone of impingement. This means that even though the bulk erosion of grains has an impact on the slurry zone in the center of the ingot, the region directly underneath the jet will not experience the same degree of re-suspension due to the negligible shear stresses at the impingement point. Thus while the overall cross sectional macrosegregation profile exhibits significant improvement over the standard as demonstrated in [2] and can be seen in Figures 4-1 and 5-1, the longitudinal segregation; measured exactly at the impingement point; does not exhibit as dramatic effects. The longitudinal centerline concentrations reported for all of the jet diameters are on par with the one observed in standard DC cast ingot. The sample size for those reports is a few cm2 of area, as compared to the 0.6cm2 analyzed in the standard case [1]. Thus, while the experimental sample taken at the impingement point exhibits little difference from the standard case, it should not be taken as a representation of the jet behavior immediately adjacent to the impingement point. Figure 5-2 is a plot of the centerline data for the Re=97,000 cast, along with the average composition 50mm from the exact center of the ingot (taken from [2]). Here we can see that the area in the geometric center of the ingot displays macrosegregation patterns similar to the standard cast. However, the area adjacent to the geometric center displays significantly less macrosegregation, on the order of a few percent. The region with maximum erosion occurs at $r = 5cm$.

Given that the turbulent jet is designed to remove only the mobile grains in the slurry region, once these grains are removed, the remaining energy of the jet is directed into the cohesive mass of interconnected grains known as the mushy zone. A jet impinging on such a surface no longer behaves in the same way as a jet impinging on a cohesionless granular bed. Mazurek and Hossain [6] noted that under such conditions the bed removal of cohesive soils is much more random, as large masses of material can break
off at once. In comparison with the uniform removal of loose granular material, a jet of sufficient power could cause the bulk erosion of the coherent network of grains (mushy zone) instead of uniquely suspending mobile grains (slurry region). The periodic removal of large masses of coherent grains could contribute to the random fluctuations in composition found in the most turbulent jet conditions seen in Figure 4-2.

The experimental observations of other researchers [6] match our prediction of crater formation in axisymmetric castings. However, experimental results do not seem to indicate the lack of erosion present along the centerline. The influence of turbulent interactions in those experiments may be the cause of this discrepancy. Our work indeed demonstrates that the region directly underneath the jet has a significantly reduced potential for erosion. In the case of planar, axisymmetric beds, turbulent eddies would most likely supplement this deficiency. However, in the case of rolling slab ingots, the shape of the sump is notably V-shaped when viewed from the short face (See Appendix A). The corresponding walls can act as baffles, guiding the flow out and away from the impingement point, towards the short face of the ingot. This flow redirection would serve the additional purpose of re-suspending additional grains found along the centerline of the ingot, while inhibiting the deep erosion found by Zhang et al [6]. In the case of rolling slabs, the re-direction of the majority of the jet flow could inhibit turbulent interactions and allow the grains directly under the jet to remain undisturbed. This would then account for the depleted region found in the center of the ingot in [5] and can be seen in Figure 4-2.
References


Chapter 6

Validation of the Analytical Approach and Conclusion

The experimental results presented in Chapter 4 were completed using an Al4.5Cu alloy and constant mold dimensions. The analytical framework presented in Chapter 3 however, was derived independent of composition and dimension. In order to further validate our hypothesis, the framework was tested against a different alloy system and mold size.

6.1 AA3104 Trials

Aluminum Alloy 3104 (AA3104) is a wrought alloy primarily used in the production of beverage cans. The primary solute element additions to this product are manganese (0.8-1.4%) and magnesium (0.8-1.3%), other elements in appreciable quantities include iron (0.8%), silicon (0.6%), zinc (0.25%) and copper (0.05-0.25%) [2]. Due to the high levels of solute additions (approximately 4.6% of total) this alloy also has a large freezing range and reaches the coherency temperature at approximately 0.17% of fraction solid [1]. This makes it an ideal candidate for an additional trial due to the high volume fraction of mobile grains and large fraction of solute elements to exhibit macrosegregation. Figure 6-1 is a plot of the analytical framework equations (Chapter 3) re-cast using the property values for AA3104 [2]. Using this prediction we designed an experiment
which changes both composition and physical dimension in order to test the robustness of our analytical calculation. We were able to obtain a mold with bore dimensions of 698.5mmx1587.5mm. Based on the desired casting speed of 0.001 m/s, our analytical model predicts a jet diameter of 22.5mm. This experimental condition is displayed by the square in Figure 6-1.

Two ingots were produced, one using the standard technique and one utilizing the optimized turbulent jet. Following casting, the ingots were sectioned identically to those from Chapter 4 and analyzed using the same techniques. The resulting macrosegregation profiles are presented in Figure 6-2 and 6-3. Immediately observable is the lack of centerline segregation in the stirred cast as compared to the standard, depleted or otherwise (MI is 0.036 for stirred, 0.132 for standard). The colors displayed do indicate some degree of isolated segregation pockets throughout the casting, but these variations are within the error range of the XRF analyzer. There is a wider variation in color pattern as compared to the Al4.5Cu casts, likely due to the difficulty of detection of magnesium.
Macrosegregation Surface for Standard AA3104

Figure 6-2: Macrosegregation surface of the standard AA3104 trial. It exhibits significant centerline segregation and is exemplified by a Macrosegregation Index of 0.132.

Macrosegregation Surface for Jet Stirred AA3104

Figure 6-3: Macrosegregation surface of the AA3104 trial performed using the optimum jet diameter. It exhibits nearly no observable centerline segregation and is exemplified by a Macrosegregation Index of 0.036.
6.2 Review and Conclusion

In Chapter 2 we presented a hypothesis in three parts regarding centerline segregation in DC cast aluminum ingots. This hypothesis can be broken down into the following components:

- Centerline segregation is predominantly driven by sedimenting grains which are depleted in solute. They are driven to the ingot centerline through avalanche events due to the steep angle of inclination of the sump walls.

- These grains are mobile, and can be removed from the ingot centerline through the use of a turbulent jet.

- By carefully matching the erosion speed of the jet to the casting speed of the ingot we can optimize the jet power to minimize the presence of centerline segregation.

The experimental results of Livanov et al. [3] illustrated the presence of an optimum casting speed for a specific mold dimension in order to generate zero centerline segregation. We postulated that these results were due to the static angle of repose of the grains (see Appendix B) and by increasing the casting speed beyond a certain threshold, the sump walls became sufficiently steep to promote avalanche events driving the sedimentation of coarse grains. The presence of peritectic titanium presented by Gariepy [4] confirmed the movement of grains from the sidewalls of the sump to the centerline. Furthermore Yu and Granger's results [5] further indicated that an increase in casting speed increased the concentration of titanium found at the centerline. We indicated that this was due to the increased frequency of avalanche events caused by the steeper sump walls formed at a higher casting speed (see Appendix A). We believe that this collection of experimental evidence is sufficient to justify our claims regarding the source and motion of sedimenting grains.

If the sedimentation of mobile grains through avalanche dynamics generates the intense centerline segregation observed in Figure 1-4, then the removal of these grains should significantly impact the size and severity of the centerline segregation found in DC cast ingots. We have proposed the use of a downwardly impinging circular jet to
re-suspend these sedimented grains and prevent their bulk accumulation along the ingot centerline. By performing a series of tests with jets of different power, we have been able to demonstrate that downwardly impinging turbulent jets do significantly reduce the size and magnitude of the depleted region along ingot centerlines (see Figure 4-1). Furthermore, we have been able to observe and sample mobile grains impacting the meniscus at the surface of the molten pool (see Appendix C), thereby proving the potential of the jet to suspend mobile grains and transport them from the ingot center.

Given the potential of a turbulent jet to mobilize grains, we postulated the existence of an optimum jet which would minimize centerline segregation in DC cast ingots. By carefully analyzing crater formation due to impinging jets on beds of granular media (see Chapter 3 and corresponding Appendices), we have specified a condition which analytically predicts the optimum jet for any DC casting condition. Utilizing the casting speed as the crater descent speed \( U_c \) in Equation 3.1 we create a careful balance where the bottom of the sump is washed by the impinging jet. Any excess grains are re-suspended while a certain allowed fraction remain thereby preventing the severe crater formation predicted by other researchers [6]. We have validated this prediction in an Al4.5Cu alloy (see Figure 4-1E) which exhibited over 60% less segregation than the standard case. We further tested the robustness of our hypothesis and analytical prediction by changing both alloy system and mold dimension and once again observing zero centerline segregation (see Figure 6-2). We believe that this validation of our analytical model confirms our hypothesis regarding the existence of an optimized jet to minimize segregation.

6.2.1 Further Work

It is clear to the discerning reader that this work is just beginning. While we postulated the existence of the avalanche effect and were able to observe its transient behavior (Figure 1-7), more robust experiments are required to analyze the angle of stability of grains along an inclined sump wall and the frequency of avalanche events. A more detailed sampling of the mobile grains (re-suspended or sedimenting) is also necessary to definitively prove the existence of sedimenting grains in situ and better understand their
suspension dynamics.

Our analytical model relied heavily on the volume of sedimented grains as calculated from post-mortem composition analyses. A superior technique to more accurately represent the volume of suspended grains which will increase the precision of our predictive ability. Key to our hypothesis is the idea that an increase in sump depth increases the frequency of avalanche events. This correspondingly increases the volume fraction of grains found at the centerline. We are currently limited in our ability to increase this volume fraction of grains (and push the validity of our model) by existing casting technology. Without the ability to cast larger ingots, and cast them faster (and thereby increase sump surface area) we cannot push the limit of performance of our turbulent jet. Research should be undertaken to increase the performance of DC casting technology and observe the limits of macrosegregation and what we can do to impact it.

All of our calculations have been the result of a zero-dimensional model we have derived. Without an appropriate model incorporating grain nucleation, migration, and attachment a fully computational model will not accurately reflect our experimental results. Additional work is needed to empirically and computationally model the lifecycle of these grains within the molten pool.
References


Appendix A

An Analysis of the Sump Shape During Continuous Casting

A rolling slab ingot is formed by the typical DC process as schematically represented in Figure 1. Normally a thin shell of solidified material is formed within the mold and some heat extraction does occur. However, upon solidification the thin shell contracts creating an air gap between the shell and the mold. This poor thermal contact leads us to assume that the walls of the mold are effectively insulating. Consequently, the majority of heat flow required for the phase change from liquid to solid at the interface is transferred down and out (following black arrows) through the cooled sides of the ingot. We make the assumption that these faces are cooled by an engineered spray such that they are essentially at a uniform temperature $T_c$.

The ingot is withdrawn from the mold at a uniform casting speed, $u$; thus the rate at which material solidifies for a time $dt$ is $udt$ (per unit area of mold). This means that the latent heat of fusion that must be removed from the interface per unit area and time is: $(\rho \lambda u) \cos \theta$ where $\theta$ is the angle between the $y$-axis of the ingot and the vector normal to the surface (Figure 2). The temperature above the molten metal we will assume to have very little superheat (enthalpy of fusion $>>$ specific heat capacity). This means that heat flowing across the interface is only due to solidification. At the solidification interface, the metal is isothermal $T_f$, meaning that the heat flow is normal to the interface. This yields the boundary condition at each position along the interface:
Figure A-1: Rolling Slab ingot formed by the Semi-Continuous DC Process being withdrawn from a water cooled mold

\[ \frac{k}{\partial n} = (\rho \lambda u) \cos \theta \]  

(A.1)

If we then use the relations found in [1], that \( \partial T/\partial x = (\partial T/\partial n) \sin \theta \) and \( \partial T/\partial y = (\partial T/\partial n) \cos \theta \), we can resolve equation (1) into two different components:

\[ \frac{\partial T}{\partial x} |_s = \frac{\partial T}{\partial n} |_s \sin \theta = \left( \frac{\rho \lambda}{k} u \right) \sin \theta \cos \theta \]  

(A.2)

\[ \frac{\partial T}{\partial y} |_s = \frac{\partial T}{\partial n} |_s \cos \theta = \left( \frac{\rho \lambda}{k} u \right) \cos^2 \theta \]  

(A.3)

Along the cooled boundaries of the ingot we can assume the temperature to be con-
Due to the air gap mentioned above, the portion of the ingot within the mold is assumed to have no heat flux normal to the surface:

\[
\frac{\partial T}{\partial x} = 0 \quad (x = \pm a, \ y > b) \tag{A.5}
\]

At the solidification interface, the rate at which latent heat must be removed (per unit width) is \( \rho u \lambda \). The maximum rate at which thermal energy can be transported as heat capacity as the ingot is withdrawn from the mold is bounded by the highest temperature above the coolant temperature a solid ingot can exist: \( \rho u c_p (T_f - T_c) \). Thus we can write the relative magnitude of heat removal due to latent heat to heat capacity transport as: \( \rho u \lambda / \rho u c_p (T_f - T_c) = \lambda / c_p (T_f - T_c) \). Note that this is independent of casting speed, meaning that it is reasonable to assume that the dominant mode of heat transport is via
conduction. Given this, the temperature distribution within the ingot is governed by the heat conduction equation:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$  \hspace{1cm} (A.6)

Normally in the solution of Laplace's equation, either the temperature or derivative of temperature is specified at each given boundary. For the present case however, we have both a temperature $T_f$ and a normal derivative $(\rho \lambda u) \cos \theta / k$ and instead we are searching for the unknown shape of the solidifying interface.

**Potential Plane**

We can define a potential function as $\Phi \equiv (T_f - T)/(T_f - T_c)$, which then yields through equation (6) within the ingot that:

$$V^2 \Phi = 0$$  \hspace{1cm} (A.7)

Along the solidifying interface, defined to be at $T_f$:

$$\Phi = 0$$  \hspace{1cm} (A.8)

and at the cooled ingot walls, where $T = T_c$:

$$\Phi = 1$$  \hspace{1cm} (A.9)

Equation (1) then along the undetermined interface becomes:

$$-\frac{\partial \Phi}{\partial x} = \cos \theta$$  \hspace{1cm} (A.10)

and then along the insulated mold walls:

$$\frac{\partial \Phi}{\partial x} = 0$$  \hspace{1cm} (A.11)

We can nondimensionalize all lengths such as $x$ by dividing by $\gamma = k(T_f - T_c) / (a\rho \lambda)$. This generates a nondimensional width, $A = a / \gamma = a u \rho \lambda / k(T_f - T_c)$ which becomes a
parameter in the solution.

Lines of constant $\Phi$ correspond to constant temperature lines, thus a heat flow function, $\Psi$, can be defined normal to these isotherms. Using the Cauchy-Riemann equations, at the interface $\partial \Phi / \partial N \big|_s = -\partial \Psi / \partial S \big|_s$. Then, using Equation (10):

$$\frac{\partial \Psi}{\partial X} \bigg|_s = \frac{\partial \Psi}{\partial S} \bigg|_s \frac{\partial S}{\partial X} = -\frac{\partial \Phi}{\partial N} \bigg|_s \left( -\frac{1}{\cos \theta} \right) = -1$$  \hspace{1cm} (A.12)

so that at the solidification interface, if we specify $\Psi = 0$ along the centerline ($X = 0$) we can see:

$$\Psi = -X \text{ along } \overline{12} \text{ and } \overline{89}$$  \hspace{1cm} (A.13)

Then along $\overline{23}$ and $\overline{78}$, which are lines of constant heat flow (constant $\Psi$) that join the interface at $X = -A$ and $A$,

$$\Psi = A \text{ along } \overline{23}, \Psi = -A \text{ along } \overline{78}$$  \hspace{1cm} (A.14)

The boundary conditions, nondimensionalized in terms of $\Phi$ and $\Psi$ are summarized in Figure 2.

From Figure 2, we can see that each portion of the ingot boundary is either a line of constant $\Phi$ or $\Psi$. This means that the ingot region forms a rectangle in the complex potential plane $W = \Psi + i\Phi$, which has been represented in Figure 3. In the $W$ plane, the $X$ and $Y$ are dependent variables of $\Psi$ and $\Phi$. Since $\Psi$ and $\Phi$ are functions of $X$ and $Y$, then $X$ and $Y$ are functions of $\Psi$ and $\Phi$ as well and thus satisfy:

$$\frac{\partial^2 X}{\partial \Psi^2} + \frac{\partial^2 X}{\partial \Phi^2} = 0$$  \hspace{1cm} (A.15)

$$\frac{\partial^2 Y}{\partial \Psi^2} + \frac{\partial^2 Y}{\partial \Phi^2} = 0$$  \hspace{1cm} (A.16)

in the region bounded by $\Phi = 0$ and 1.

We can consider the insulated boundaries $\overline{23}$ and $\overline{78}$ in Figure 3 to be symmetry lines, with the rectangular region repeated in a periodic fashion to extend along the entire $\Psi$
axis. Then from the results from [4] for a Cauchy boundary value problem that:

\[ Z(\Psi, \Phi) = X(\Psi, \Phi) + iY(\Psi, \Phi) = Re[X_0(\Psi + i\Phi)] + \frac{1}{2i} \int_{\Psi-i\Phi}^{\Psi+i\Phi} \left[ \left( \frac{\partial X}{\partial \Psi} \right)_0(\zeta) + i \left( \frac{\partial Y}{\partial \Psi} \right)_0(\zeta) \right] d\zeta \] 

(A.17)

where a zero subscript denotes a value along \( q = 0 \). Using the Cauchy-Riemann equations, equation (17) is equal to:

\[ X(\Psi, \Phi) + iY(\Psi, \Phi) = Re[X_0(\Psi + i\Phi)] + \frac{1}{2i} \int_{\Psi-i\Phi}^{\Psi+i\Phi} \left[ \left( \frac{\partial Y}{\partial \Psi} \right)_0(\zeta) + i \left( \frac{\partial X}{\partial \Psi} \right)_0(\zeta) \right] d\zeta \] 

(A.18)

If the functions \( X_0(\Psi) \) and \( Y_0(\Psi) \) are known along the \( \Phi = 0 \) boundary from Figure 3,
then by integrating Equation (18) we get:

\[
X(\Psi, \Phi) + i Y(\Psi, \Phi) = \text{Re}[X_0(\Psi + i\Phi)] + i \text{Re}[Y_0(\Psi + i\Phi)] \\
+ \frac{1}{2i} \left[ -Y_0(\zeta) + i X_0(\zeta) \right]^{\Psi + i\Phi}_{\Psi - i\Phi} = X_0(\Psi + i\Phi) + i Y_0(\Psi + i\Phi) \tag{A.19}
\]

If we then separate the real and imaginary parts

\[
X(\Psi, \Phi) = \text{Re}X_0(\Psi + i\Phi) - \text{Im}Y_0(\Psi + i\Phi) \tag{A.20}
\]

\[
Y(\Psi, \Phi) = \text{Im}X_0(\Psi + i\Phi) - \text{Re}Y_0(\Psi + i\Phi) \tag{A.21}
\]

From equation (13), \(X_0(\Psi) = -\Psi\), thus:

\[
X_0(\Psi + i\Phi) = -\Psi - i\Phi \tag{A.22}
\]

which then lets equations (20) and (21) become:

\[
X(\Psi, \Phi) = -\Psi - \text{Im}Y_0(\Psi + i\Phi) \tag{A.23}
\]

\[
Y(\Psi, \Phi) = -\Phi + \text{Re}Y_0(\Psi + i\Phi) \tag{A.24}
\]

If we express the unknown shape of the solidification interface as a Fourier Cosine series where \(B_0\) and \(B_n\) are the unknown coefficients:

\[
Y_0(X) = B_0 + \sum_{n=1}^{\infty} B_n \cos n\pi \frac{X}{A} \tag{A.25}
\]

Using \(X = -\Psi\) along the solidifying interface, we can insert this into equations (23) and (24) to obtain:

\[
X(\Psi, \Phi) = -\Psi + \sum_{n=1}^{\infty} B_n \sin n\pi \frac{\Psi}{A} \sinh n\pi \frac{\Phi}{A} \tag{A.26}
\]

\[
Y(\Psi, \Phi) = -\Phi + B_0 + \sum_{n=1}^{\infty} B_n \cos n\pi \frac{\Psi}{A} \cosh n\pi \frac{\Phi}{A} \tag{A.27}
\]

In order to solve for \(B_n\), we can use the fact that along \(\delta 4\), \(\Phi = 1\), \(X = -A\) and \(\Psi\) varies

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from \( A \) to 0. Using these values in Equation (26) yields:

\[
\Psi - A = \sum_{n=1}^{\infty} B_n \sin n\pi A \frac{\Psi}{A} \sinh \frac{n\pi}{A} \tag{A.28}
\]

The \( B_n \) are found as Fourier coefficients from

\[
B_n = \frac{2}{\pi} \frac{1}{\sinh \frac{n\pi}{A}} \int_{0}^{A} (\Psi - A) \sin n\pi A d\Psi = -\frac{2A}{n\pi \sinh \frac{n\pi}{A}} \tag{A.29}
\]

At the origin, \( \Phi = \Psi = 0 \) and \( Y = 0 \), so from equation (27), \( B_0 \) is:

\[
B_0 = -\sum_{n=1}^{\infty} B_n = \frac{2A}{\pi} \sum_{n=1}^{\infty} \frac{1}{n \sinh \frac{n\pi}{A}} \tag{A.30}
\]

The \( B \) values are then substituted into equations (26) and (27) to yield the final expressions that map the rectangle of Figure 3 into the geometry of Figure 2.

\[
X(\Psi, \Phi) = -\Psi - \frac{2A}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin n\pi A \frac{\Psi}{A} \sinh(n\pi A) \tag{A.31}
\]

\[
Y(\Psi, \Phi) = -\Phi - \frac{2A}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sinh \frac{n\pi}{A} (\cos n\pi A \frac{\Psi}{A} \cosh n\pi A - 1) \tag{A.32}
\]

The heat flow condition at the interface was properly accounted for by using equation (13).

The shape of the interface is found by letting \( \Phi = 0 \) and \( \Psi = -X \) in equation (32) to yield:

\[
\frac{Y_2(X)}{A} = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sinh \frac{n\pi}{A} (1 - \cos n\pi A X) - A \leq X \leq A \tag{A.33}
\]

While equation (33) gives the overall shape of the interface, the position of the interface must be positioned vertically in the mold. This represents the Advanced Cooling Distance (ACD) in casting practices today. We can do this by obtaining from equation (32) the height \( H \), in Figure 2 by using \( H = Y_2 - Y_3 = Y(\Psi = A, \Phi = 0) - Y(\Psi = A, \Phi = 1) \).

The resulting expression can be simplified by adding and subtracting:

\[
\frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -\frac{2}{\pi} \ln(2) \tag{A.34}
\]
which yields the final relation:

\[
\frac{H}{A} = \frac{1}{\pi} - \frac{2 \ln(2)}{\pi} + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \frac{e^{-\frac{nn}{\alpha}} - e^{-\frac{2nn}{\alpha}}}{1 - e^{-\frac{2nn}{\alpha}}}
\]  
(A.35)

This final set of equations can be evaluated to determine the sump (interface) shape of a continuously cast ingot of any given chemical composition and set of casting parameters. For the ingots investigated in the current document whose composition is Al4.5Cu and whose minor dimension (the dimension driving the interface depth) is 0.60m a set of curves has been generated below in Figure 4, viewed from the short face, which illustrates the effect of casting speed on the interface depth and shape.

Using these same computations, we can calculate the 3-dimensional form of the interface formed during the experimental conditions under investigation. Figure 5 is a representation of the interface formed during the casting of an Al4.5Cu ingot at 1.00 mm/s in a 1.75mx0.6m mold.

While these computed sump shape profiles match the shape of those presented in [3], we also performed experiments to validate the depth of the sump as this parameter is of critical importance to the analytical framework of this investigation. Using a numerically controlled linear slide, we inserted an alumina rod into the molten pool of the ingot until the rod contacted the coherecy isotherm. The rigid contact caused the rod to stop motion relative to the slide, which triggered an inductive proximity sensor. This setup has been produced below in Figure A-6. Using an AA3104 alloy we varied the casting speed within the limits of the operational range and measured the sump depth once conditions had equilibrated. The results of these trials alongside the predicted results are found in Figure A-7.
Figure A-4: Interface shape of the Al4.5Cu ingots investigated in the present study. Viewed from the short axis of the 1.75mx0.6m ingot cross-section, the multiple lines illustrate how casting speed can impact the depth and shape of the interface.
Figure A-5: 3-Dimensional computation of the interface formed during the casting of an Al4.5Cu ingot in a 1.75mx0.6m mold at 1.00 mm/sec. This shape can be compared directly to the experimentally determined interface in [3].

Figure A-6: Schematic of the actuator mechanism used to measure the sump depth as a function of the casting speed.
Figure A-7: Plot of the predicted and experimental results of the sump depth of an AA3104 ingot cast at a variety of casting speeds. These results indicate that the analytical framework can produce results within approximately 1 cm of experiment. These results are important in establishing a reliable height $H$ in the analytical framework for the jet.
References


Avalanche Dynamics and the Angle of Stability of a Pile of Grains

Granular materials display behaviors that distinguish them from other forms of matter. Unlike solids, granular media conform to the shape of a container and will flow if this container is sufficiently tilted. Unlike liquids, however, a granular media is stable when its container is tilted slightly as long as the top surface is at a slope less than the angle of maximum stability $\theta_m$. When the slope is increased above $\theta_m$, grains begin to flow and an avalanche of particles occurs, the angle of the pile decreasing to the angle of repose $\theta_r$. However, instead of uniform motion throughout the sample, all of the motion occurs in a relatively thin boundary layer (10 grains) at the surface [1].

Experimental measurements of the angle of repose [2] reveal that $\theta_r$ depends strongly on the shape and surface roughness of the grains. The typical measured value for $\theta_r$ is $\approx 22^\circ$ for smooth spheres, but $\theta_r$ can attain $64^\circ$ for materials containing rough, irregular particles. Cohesion between grains can also dramatically change the physical properties of a granular material, including $\theta_r$ and $\theta_m$ [3–8]. Such cohesion is commonly caused by the presence of a liquid in the material that forms interstitial bridges resulting in attractive forces between grains.

Typically $\theta_r$ and $\theta_m$ are measured experimentally for different materials. The most detailed theoretical predictions are provided from molecular dynamics studies, or statistical mechanics [9]. While these investigations have profoundly improved our under-
standing of \( \theta_r \), they have not provided a simple way to calculate \( \theta_r \) or \( \theta_m \). In this section we will not attempt such an involved prediction, but instead will use a simple geometric approach to predict a certain critical angle \( \theta_c \) which is perhaps best described as a mean field between \( \theta_c \) and \( \theta_m \). Since this prediction will be used later to describe the time averaged continuous precipitation of grains, this description offers a simple mathematical approach.

**Stability criteria.** The basic scheme of this approach is best illustrated first in 2-D. Consider a randomly packed sandpile of disks with equal radii, as shown in Figure 1. If we add an additional grain to a local surface minimum, its stability will depend entirely on the configuration of the two disks supporting it [10]. In order to quantify this, we define the local slope, \( \theta \), to be the tangent of the two supporting spheres. If \( \theta \) is small, the newly added grain is stable, while if \( \theta \) is larger than the critical value \( \theta_c \), it is unstable and will roll down the slope, forming an avalanche. For a pile of disks with equivalent radii, simple geometric considerations defines \( \theta_c = 30^\circ \).

This same argument can be extended to the three-dimensional case, where the geometry is a bit more complicated. We now have to study the arrangement of three spheres supporting a fourth. For simplicity we assume that all of the spheres are identical in size and roughness. We again define the local slope of the sandpile, \( \theta \) to be the angle between the plane tangent to the spheres and the horizontal plane (see Figure 2).
Figure B-2: (a) in three dimensions a newly added particle (top sphere) is supported by three surface particles. The local slope, $\theta$, is the angle between the plane tangent to the three supporting spheres and the horizontal plane. The angle is increased by rotating the inclined plane around the axis $\hat{x}$. (b) Top view of the four spheres. The orientation of the triangle defined by the three supporting particles is identified by angle $\phi$, which is related to the angle in the figure as $\phi = \pi/3 - \theta'$, where we limit $0 < \phi < \pi/3$.

For the case of $\theta = 0$ the top sphere is stable, being supported by three base spheres. Increasing $\theta$, the sphere remains in equilibrium for $\theta < \theta_c(\phi)$, where $\theta_c(\phi)$ depends on the relative orientation of the base spheres, quantified by the angle $\phi$.

Without the addition of any cohesive or frictional forces, the top sphere is stable only when the gravitational force vector points within the base triangle projected onto the horizontal plane. The addition of this criterion defines the maximum angle of stability as a function of $\phi$:

$$\theta_c(\phi) = \arctan \frac{1}{2\sqrt{2} \cos(\pi/3 - \phi)}$$  \hspace{1cm} (B.1)

The orientation of the base triangle, described by $\phi$, will be random for any real pile of grains. Thus, one can expect that a randomly packed pile will be characterized by the average $\theta_c(\phi)$, i.e.,

$$\theta_c = (3/\pi) \int_0^{\pi/3} \theta_c(\phi) d\phi$$  \hspace{1cm} (B.2)

Evaluation of this integral yields the result of $\theta_c = 23.4^\circ$. 

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References


Appendix C

Mobile grain size analysis

The size and composition of the mobile grains within the slurry region are of key importance to the accuracy of the analytical framework. In order to determine their size and composition we have implemented a strategy initially conceived by Martinez and Flemings [1]. Figure C-1 is a schematic of the vacuum quench probe utilized to rapidly quench samples of multi-phase solutions. The solidification is so rapid (200°C/sec) that there is no evolution of the solid phase from the moment the sample is taken.

A set of quenched samples were taken at regular intervals during each of the jet casts, which were then sectioned, polished, and etched with a Keller's etch. An example micrograph of the observed grain samples is produced in Figure 1-2.

Each of the samples were analyzed for size and composition of the primary phase particles using an SEM EDS linescan technique. Plotting the composition and size of these grains as a function of $Re_j$ in Figure 1-3 illustrates that the mobile grains vary little in size or composition as a function of jet power.
Figure C-1: Schematic of the quench mold employed by [1] and used in the present study to determine the size and composition of grains.
Figure C-2: Sample of a quenched grain taken by the vacuum quench mold, which was then analyzed for size and composition.
Figure C-3: Size and composition of the grains quenched from each ingot, plotted as a function of $Re_f$. 
References

Appendix D

A Review of Suspension Flow Parameters

In studies of uniform, statistically steady turbulent flow over a granular bed, the suspension and transport of particles is generally characterized by the Shields parameter, \( Sh \), representing the ratio of shear stress due to fluid flow relative to the weight per area of individual grains inside the bed. This has been displayed below in D.1, where \( U \) is the characteristic flow velocity, \( d_g \), grain diameter, and \( \rho_f \) and \( \rho_g \) the fluid and grain densities respectively.

\[
Sh = \frac{\rho_f U^2}{g(\rho_g - \rho_f)d_g} \tag{D.1}
\]

Transport of grains occurs if the Shields parameter exceeds a critical value, which depends on grain size, shape, cohesion and buoyancy [1–3]. This critical Shields parameter can be difficult to determine experimentally [4] [5], partially because the physical mechanism for re-suspension occurs transiently due to turbulent fluctuations.

An alternative classification of granular resuspension and sedimentation is expressed by the Rouse number, \( Rs \), which is proportional to the ratio of the settling speed of the grains, and the turbulent shear velocity of the bed. This relation is expressed below in D.2

\[
Rs = \frac{U_s}{\kappa u_*} \tag{D.2}
\]

where \( u_* \) is the shear velocity, \( \kappa = 0.41 \) is the von Kármán constant, and \( U_s \) is the
terminal settling velocity of the grains.

Below a critical value of $Rs$, the flow is capable of maintaining grains in suspension because turbulent velocity fluctuations are larger than the terminal velocity of each grain. In unidirectional, steady flow, full bed transport is anticipated for $Rs \leq 2.5$, and significant resuspension occurs if $Rs \leq 1$. Unlike the Shields number, the Rouse number accounts for the influence of viscosity upon each particle through the value of its respective settling speed, $U_s$. For very small grains ($< 70 \mu m$ in the aluminum system) $U_s$ is given by the Stokes settling velocity.

$$U_s = \frac{g(\rho_g - \rho_f)d^2}{18v\rho_f} \quad \text{(D.3)}$$

The granular Reynolds number $Re_g$ is most usefully defined using $U_s$ as the characteristic velocity, thereby forming:

$$Re_g = \frac{U_s d_g}{v} \quad \text{(D.4)}$$

where $d_g$ is the grain diameter, and $v$ is the kinematic viscosity (generally assumed to be approximately $5.5 \times 10^{-7} m^2/s$ for molten aluminum).
References


Appendix E

Derivation of Crater Descent Relationship

We propose to define the Rouse number for a turbulent jet impacting a bed of particles as:

\[ Rs = \frac{U_s}{\kappa U_j} \]  

(E.1)

where \( U_j \) is the velocity of the jet at the surface of the granular bed (distance \( H_0 - h_0 \) from the nozzle). For perfectly spherical particles obeying Stokes law (i.e. grain Reynolds number, \( Re_g < 0.1 \), \( Sh \) (Shield), \( Rs \), and \( Re_g \) can be related by:

\[ Sh = \frac{1}{18 \kappa^2} \frac{Re_g}{Rs^2} \approx 0.33 \frac{Re_g}{Rs^2} \]  

(E.2)

Given the fact that the critical Shields number, \( Sh_c \) is related to the granular Reynolds number according to \( Sh_c = Re_g^{-1/2} \) [1], we can determine that the critical Rouse number scales with the granular Reynolds number following:

\[ Rs_c \sim Re_g^{3/4} \]  

(E.3)

The presence of several grains falling together leads to a swarm velocity [2], \( U_{th} \), provided by
\[ U_{th} = U_s(1 - C_v)^m \]  
(E.4)

where \( C_v \) is the volume fraction of solid particles. Assuming Stokes law for grains of less than 70 microns, \( m \) is a constant function of the grain Reynolds numbers:

\[
m = \frac{4.7(1 + 0.15Re_g^{0.687})}{1 + 0.253Re_g^{0.687}}
\]  
(E.5)

Using Equation 5 from Chu and Jacoby [3], the volume fraction of sedimented grains can be calculated using the observed solute depletion at the centerline reported by the author [4]. \( C_v \) is then determined to be of the order of 0.2. Using the modified settling velocity (Equations E.4 and E.5) in Equation E.3 leads to:

\[ R_s \sim Re_g^{1/2} \]  
(E.6)

For horizontal channel flow, bedload transport is defined by the volume flux of grains per unit width of flow \( Q \). This is then normalized by the grain size and settling speed to give a non-dimensional flux per unit width [5], \( \bar{Q} \). Through extensive experiments examining bedload transport of uniform horizontal flow over granular beds, empirical relationships have been proposed to relate \( \bar{Q} \) to the difference, \( Sh - Sh_c \), following:

\[ \bar{Q} = C_s(Sh - Sh_c)^P \]  
(E.7)

\( P \) and \( C_s \) are constants dependent upon grain size, density, and the stress imposed by the flow over the bed.

In a study of jet scour, Mazurek and Hossain [6] showed that the radius of the crater generated by an impinging jet does not vary significantly with increasing jet power. Extrapolating from this, it is proposed that the crater deepens while its radius \( r_0 \) is almost constant with increasing jet velocity at the base of the crater. This assumption and extrapolation is likely only valid for cohesion-less grains forming a permeable bed, as found in the slurry region of the sump. In addition, the act of impingement on such bed is assumed to lead to a surface pressure distribution and a seepage flow within the
bed [7]. This seepage flow allows shear stresses to act deep into the bed. Such assumptions likely do not apply for cohesive granular beds, such as welded grains located in the mushy zone of the sump. For such cohesive bed, one would rely on high-temperature creep effects for the applied shear stress to enable re-suspension. This type of bed tends to "reflect" the impinging jet and thus leads to non-uniform and less predictable grain re-suspension.

With the assumption of a cohesion-less bed and a constant crater radius, the volume flux of grains suspended from the crater due to the jet can be represented as:

\[ Q \propto r_c^2 U_c \]  
(E.8)

where the crater descent velocity \( U_c \) is assumed to be constant, i.e. the volume flux is constant. Similar to statistically steady horizontal flow, a non-dimensionalizing constant can be defined by the granular settling flux divided by the across-flow granular density:

\[ \frac{d_g^2 U_{th}}{(d_g/r_0)^2} \]  
(E.9)

In this definition we have used the hindered settling velocity to account for inter-granular interactions within the crater. Equation E.8 then allows defining the non-dimensional volume flux as:

\[ \tilde{Q} = \frac{U_c}{U_{th}} \]  
(E.10)

This relationship defines the "relative crater descent velocity", suggesting that the crater descends independently of the properties of the grains themselves.

Due to our definition of Rs (Equation E.1) explicitly invokes the settling velocity of the grains and thus can account for the influence of turbulent fluctuations, we propose to replace Equation E.7 with a non-dimensional flux of grains of the form:

\[ \tilde{Q} = C_r Re_g (R_{sc} - Rs) \]  
(E.11)
Using Equations E.1, E.10 and the relationship for the critical Rouse number (Equation E.6), Equation E.11 provides an explicit expression for the crater descent speed $U_c$ as a function of the impacting jet velocity on the bed:

$$U_c \approx C_1 U_{th} Re_g^{3/2} \left(1 - c_2 \frac{U_{th}/U_j}{Re_g}\right)$$

(E.12)
References


Appendix F

Impinging Jet Flow Force Distribution

The impinging jet flow generates the pressure and shear stress distributions within the bed discussed above. Beltaos and Rajaratnam [1] and Rajaratnam [2] performed experiments of jets impinging on a rigid plane. The source of the jet had velocity $U_0$, diameter $2b_0$, and was at a height $H$ above the plane. The continual impingement of a jet onto an erodible bed generates substantially different flow patterns from the impingement on a rigid boundary because the bed topography is actively modified and thereby creates a dynamic flow regime. In our analysis we consider the rigid plane analysis because the individual period of consideration is small enough to ignore large scale flow pattern changes due to erosion. Beltaos and Rajaratnam [1] found experimentally that the pressure at the stagnation point is given by:

$$p_0 = 25\rho U_0^2 / \left( \frac{H}{2b_0} \right)^2 \quad (E.1)$$

and the distribution of surface pressure is given by:

$$\frac{p_s(r)}{p_0} = exp(-114 \left( \frac{r}{H} \right)^2) \quad (E.2)$$

where $r$ is the radial distance from the point of stagnation. Additionally, they found that the maximum shear stress exerted on the boundary is:

$$\sigma_0 = 0.16\rho U_0^2 / \left( \frac{H}{2b_0} \right)^2 \quad (E.3)$$
Far from the stagnation point, the shear stress decreases with the inverse-square of the radial distance $r$. One can then propose the following equation for the shear stress distribution:

$$\frac{\sigma(r)}{\sigma_0} = 0.033 \frac{1 - exp(-114\left(\frac{r}{H}\right)^2)(1 + 114\left(\frac{r}{H}\right)^2 + 11.09\left(\frac{r}{H}\right)^3)}{\left(\frac{r}{H}\right)^2}$$  \hspace{1cm} (F4)

Figures F-1 and F-22 plot the pressure and shear stress distributions as a function of radial distance from the impingement point of such a jet.

The experimental data of Beltaos and Rajaratnam fixes the ratio of the maximum stagnation pressure $p_0$ to the maximum shear stress $\sigma_0$, given as $p_0/\sigma_0 = 150$ Johanson et al. [3] studied the impingement of turbulent events and found this value to be an order of magnitude less than Beltaos and Rajaratnam. Nevertheless, the remaining parts of this analysis may be followed regardless of the actual value of $p_0/\sigma_0$ since the functional form of the relationship between volume of entrained particles and hydrodynamic forces is independent of this ratio.
Figure F-2: Shear Stress distribution for a radially impinging turbulent jet, set at height $H=0.6m$ above the bed.
References


Appendix G

Stability criterion and zone of erosion

This section considers the forces exerted on the grains comprising the granular bed. This enables the determination of the zone of erosion and quantifies the movement of grains as a function of the fluid dynamic characteristics of the jet.

Three forces act on a single grain reposing on a planar bed at angle \( \beta \) from horizontal, the force driving motion in the vertical and radial directions \( F_r \) and \( F_y \), and the weight of the grain \( W \). Motion is resisted by a frictional force arising from the contact of the grain with the surface. The static angle of repose is defined as \( \alpha \) which is the minimum angle of inclination of the bed for grains to initiate their movement without any fluid flow (for spherical grains, the static angle of repose is approximately \( \alpha = 23^\circ \)). This configuration is illustrated in Figure G-1.

An individual grain will begin to move if the rotational moment about its point of contact with a neighboring grain exceeds the moment due to the weight of the grain. This condition can be expressed as:

\[
F_r \cos(\alpha - \beta) + F_y \sin(\alpha - \beta) - W \sin(\alpha - \beta) > 0 \tag{G.1}
\]

thus we can see that the condition for the initiation of motion is:

\[
\frac{F_r}{-F_y + W} > \tan(\alpha - \beta) \tag{G.2}
\]

Motion will inevitably occur if the planar inclination exceeds the angle of static re-
Figure G-1: Forces acting on a grain located on an inclined bed. The horizontal and vertical components of the driving force are $F_r$ and $F_y$ respectively. The weight of the grain is denoted $W$ while the static angle of repose is $\alpha$.

pose ($\beta > \alpha$). The forces $F_r$ and $F_y$ are the forces induced within the bed by the impinging jet. The submerged weight per unit volume (accounting for buoyancy) is approximately $470 \, N/m^3$, as calculated by $W = C_b \rho (s - 1) g$. Where $C_b$ is the concentration of grains within the bed (20 vol%). This sets the weight required to complete the inequality in Equation G.2, evaluated here at $\eta = F(\xi)$ resulting in the following relationship:

$$\left| \frac{F_r}{F_y + \rho (s - 1) C_b g H / p_0} \right|_{\eta=F(\xi)} \leq \tan(\alpha - \beta)$$  \hspace{1cm} (G.3)

$$\text{where} \quad \frac{dF}{d\xi} = -\tan(\beta)$$  \hspace{1cm} (G.4)

The above two equations allow us to introduce a new dimensionless parameter, defined as: $\Theta = p_0 / \rho (s - 1) C_b g h$. Which is the ratio of the pressure gradient to the gravitational force. Similar to the Shields parameter, $\Theta$ represents the relationship between the driving and stabilizing forces. We will call this the 'effective' erosion parameter, similar to that described by Rajaratnam [1].
This analysis defines the condition for a downwardly impinging turbulent jet to move grains. The rotational moment about the contact point derived from the radial force $F_r$, must be sufficient to overcome the resistive moment derived from the weight of the grain, even when the downward force of the impinging jet may augment the weight of the grain.
References

Appendix H

Flow Within the Loose Permeable Bed

We will treat the flow within the bed by the Brinkman equation for filter flow through a porous media [1]. This is a simple modification of Darcy's law, which permits the modeling of flow when the porous region is bordered by a pure fluid region [2]. The bed is assumed to be semi-infinite and homogenous in structure. We will denote the difference between the fluid pressure and the hydrostatic pressure by $p$, the permeability of the porous bed by $\Lambda$, and the dynamic viscosity by $\mu$. This means that the equations governing the flow within the bed are given by:

\[ \mu \nabla^2 \mathbf{u} - \frac{\mu}{\Lambda} \mathbf{u} = \nabla p \]  
(H.1)

\[ \nabla \cdot \mathbf{u} = 0 \]  
(H.2)

\[ p = 0, \mathbf{u} \rightarrow 0, y \rightarrow -\infty \]  
(H.3)

\[ p = p_s(r), \sigma = \sigma_s(r), v = 0, y = 0 \]  
(H.4)

where $\mathbf{u} = (u, v)$ are the radial and vertical velocity components. The permeability of the bed is related to the grain size making up the bed. Bear [3] proposed the following experimentally derived relation for permeability in $m^2$.
$\Lambda = 0.617 d^2 \times 10^{-3}$  \hspace{1cm} (H.5)

We then introduce the following non-dimensional variables for vertical and radial coordinates and the pressure and velocity fields

$$Q = \frac{p}{p_0}, ~ U = \frac{u\mu H}{\Lambda p_0}, \xi = \frac{r}{H}, \eta = \frac{y}{H}$$  \hspace{1cm} (H.6)

This pressure field satisfies Laplace's equation and if one assumes that the surface pressure distribution is localized to the point of impingement, we can derive the axisymmetric pressure distribution by solving equations H.1-H.4, which reduce to:

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial p}{\partial r} \right) + \frac{\partial^2 p}{\partial y^2} = 0$$  \hspace{1cm} (H.7)

$$p \to 0, y \to -\infty$$  \hspace{1cm} (H.8)

$$p = p_s(r), z = 0$$  \hspace{1cm} (H.9)

We can take the Fourier-Bessel transform of the pressure field with respect to $r$, if we assume that the pressure field decays rapidly away from the stagnation point (See Appendix E). We denote the transformed variables by $\tilde{p} = (k, y)$ and thus:

$$\tilde{p}(k, y) = \tilde{p}_s(k) \exp(\gamma y)$$  \hspace{1cm} (H.10)

If we invert the transform, we can find the standard result [4]:

$$p(r, y) = \int_0^\infty k \exp(ky) J_0(kr) \int_0^\infty r' p_s(r') J_0(kr') dr'dk$$  \hspace{1cm} (H.11)

Here we have denoted $J_0(r)$ as the zeroth-order Bessel function of the first kind. If we substitute the surface pressure distribution $p_s = p_0 \exp(-\Lambda q(-\gamma r^2))$ into this expression yields:
\[ p(r, y) = \int_0^\infty k \exp(ky) f_0(kr) \exp(-k^2/4y) / 2y \, dk \]  \hspace{1cm} (H.12)

Using a numerical integration scheme, we can derive the pressure distribution within the bed:

\[ Q(\xi, \eta) = \int_0^\infty \frac{k}{228} \exp \left( k\eta - \frac{k^2}{456} \right) f_0(k\xi) \, dk \]  \hspace{1cm} (H.13)

Since the non-dimensional permeability \( \lambda^{-2} \equiv \Lambda/H^2 << 1 \), an "effective" boundary layer exists within the flow through the porous media next to \( y = 0 \). Within this boundary layer, viscous dissipation cannot be ignored. Outside of this layer however, Darcy stresses are dominant and the velocity field is linearly related to the gradient of the pressure field (i.e. we can ignore the first term of H.1). Within the boundary layer, variations in vertical direction become more significant than those in the horizontal direction, and we can determine an approximate velocity field given by:

\[ U = -\frac{\partial Q}{\partial \xi} + \left( \frac{\sigma_4(\xi)\lambda}{p_0} + \frac{1}{\lambda} \frac{\partial^2 Q}{\partial \xi \partial \eta} \right) \exp(\eta \lambda) \]  \hspace{1cm} (H.14)

\[ V = -\frac{\partial Q}{\partial \eta} + \frac{\partial Q}{\partial \eta} \bigg|_{\eta=0} \exp(\eta \lambda) \]  \hspace{1cm} (H.15)

Using these relations, we can approximate the dimensionless force per unit volume acting on the grains, averaged over a grain diameter, consisting of radial and vertical components decomposed into:

\[ F_r = -\frac{\partial Q}{\partial \eta} + \frac{H}{d} \exp(\eta \lambda) \left( \frac{\sigma_4(\xi)\lambda}{p_0} + \frac{1}{\lambda^2} \frac{\partial^2 Q}{\partial \xi \partial \eta} \bigg|_{\eta=0} \right) \]  \hspace{1cm} (H.16)

\[ F_y = -\frac{\partial Q}{\partial \eta} + \frac{H}{d} \exp(\eta \lambda) \frac{1}{\lambda} \frac{\partial Q}{\partial \eta} \bigg|_{\eta=0} \]  \hspace{1cm} (H.17)

The boundary layer thickness is generally on the order of \( \frac{1}{\lambda} \), representing the depth through which the surface stress is transmitted. Thus, we can conclude that the shear stress is absorbed by the granular bed over a short vertical distance which is on the order
of a single grain diameter, whereas pressure distributions are transmitted over much greater length scales. Similar experimental conclusions were made by Fernandez-Luque [5] and Wiegel [1].
References


Appendix I

Erosion Boundary Calculation

The erosion boundary can be calculated from Equations G.3, G.4, H.16 and H.17. Far from the impaction point, there is no excess pressure and so the boundary returns to the bed surface. We can solve for the point where this occurs by enforcing the equality from Equation G.3 and solving for $\tan(\beta) = 0$. The distance between the impaction (stagnation) point and this far boundary condition is the zone of influence of the jet.

Erosion boundary profiles are calculated for various values of the erosion parameter and are shown below in Figure I-1.

For these profiles, $\lambda$ has been set to 1000. Since this permeability coefficient is generally a constant for a granular bed, the form of these erosion profiles will remain consistent. Two distinct regions can be seen through these erosion profiles. Close to the surface of the bed, a thin shear layer mobilizes particles. However these effects dissipate rapidly, and further from the surface, the induced pressure field causes grain migration.
Figure 1-1: Theoretical predictions of erosion profiles for various values of the effective erosion parameter $\Theta$. 