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STÄCKEL SYSTEMS

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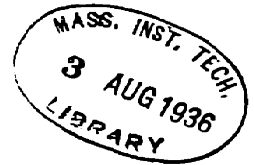
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STÄCKEL SYSTEMS

## STÄCKEL SYSTEMS

The following three topics are closely connected by Stäckel systems, coordinate systems in which the fundamental form is

$$ds^2 = \sum_{\kappa=1}^{n-1} \frac{\phi}{\phi_{\kappa 1}} (dq_{\kappa})^2$$

where  $\phi$  is the determinant of the  $n^2$  functions  $\varphi_{\kappa\lambda}(q)$ ;  $\phi \neq 0$ ;  $\kappa, \lambda = 1, 2, 3, \dots, n$ , and  $\phi_{\kappa 1}$  is the minor of the element  $\varphi_{\kappa 1}(q_{\kappa})$ : first, the separation of the variables in the Hamilton-Jacobi equation which leads to Stäckel systems, second, the finding of geodesic lines by quadratures and quadratic first integrals, and third, confocal quadrics and the generalization of certain well known properties which they possess to Riemannian 3-space. This paper is a historical resume of what has been accomplished in these three fields insofar as the work done has any direct connection with Stäckel systems.

In 1839 Jacobi(1) used the following substitutions for finding the equation of the geodesic lines on the ellipsoid:

$$\frac{x_1^2}{a_1} + \frac{x_2^2}{a_2} + \frac{x_3^2}{a_3} = 1$$

where  $a_1, a_2, a_3$  are real, positive constants and

$$a_1 < a_2 < a_3$$

$$x_1 = \sqrt{a_1 / (a_3 - a_1)} \sin \varphi \sqrt{a_2 \cos^2 \psi + a_3 \sin^2 \psi - a_1}$$

$$x_2 = \sqrt{a_2} \cos \varphi \sin \psi$$

$$x_3 = \sqrt{a_3 / (a_3 - a_1)} \cos \psi \sqrt{a_3 - a_1 \cos^2 \varphi - a_2 \sin^2 \varphi}$$

$\varphi$  and  $\psi$  are the coordinates of any point on the ellipsoid, which may be shown to be the same as giving a point on the surface as the intersection of the two lines of curvature on which it lies (the lines of curvature being the intersection of the ellipsoid with all confocal hyperboloids).

When the above substitutions are made the geodesic lines are given by the equation

$$\alpha = \int \frac{\sqrt{a_1 \cos^2 \varphi - a_2 \sin^2 \varphi} d \varphi}{\sqrt{a_3 - a_1 \cos^2 \varphi - a_2 \sin^2 \varphi} \sqrt{(a_2 - a_1) \cos^2 \varphi - \beta}}$$

$$- \int \frac{\sqrt{a_2 \cos^2 \psi + a_3 \sin^2 \psi} d \psi}{\sqrt{a_2 \cos^2 \psi + a_3 \sin^2 \psi - a_1} \sqrt{(a_3 - a_2) \sin^2 \psi + \beta}}$$

where  $\alpha$  and  $\beta$  are the two arbitrary constants of integration.

Later, in 1842, Jacobi(2) found the equation of the geodesic lines on certain n-dimensional surfaces (a generalization of ellipsoids) by means of general elliptical coordinates.

The surface is given by:

$$\frac{x_1^2}{a_1 + \lambda} + \frac{x_2^2}{a_2 + \lambda} + \dots + \frac{x_n^2}{a_n + \lambda} = 1$$

where  $a_1, a_2, \dots, a_n$  are positive, real constants, and  $a_1 < a_2 < \dots < a_n$ .

For any fixed point on the surface, this is an equation of the  $n^{\text{th}}$  degree in  $\lambda$  which, it may be proved, has  $n$  real distinct roots. If we let the roots be designated by  $\lambda_1, \lambda_2, \dots, \lambda_n$  where  $\lambda_1 < \lambda_2 < \lambda_3 < \dots < \lambda_n$ , then from the set of equations

$$\frac{x_1^2}{a_1 + \lambda_i} + \frac{x_2^2}{a_2 + \lambda_i} + \dots + \frac{x_n^2}{a_n + \lambda_i} = 1$$

$i = 1, 2, \dots, n.$

by the successive elimination of  $x^2$  terms we may obtain the following expressions

$$x_i^2 = \frac{(a_i + \lambda_1)(a_i + \lambda_2)(a_i + \lambda_3) \dots (a_i + \lambda_n)}{(a_i - a_2)(a_i - a_3) \dots (a_i - a_{i-1})(a_i - a_{i+1}) \dots (a_i - a_n)}$$

Differentiating this expression and dividing through by  $x_i^2$  we find

$$\frac{2dx_i}{x_i} = \sum_{m=1}^{m=n} \frac{dx_m}{a_i + \lambda_m} \quad i = 1, 2, 3 \dots n$$

$$\begin{aligned} \therefore 4 \sum_{m=1}^{m=N} dx_m^2 &= \sum_{m=1}^{m=N} \sum_{m=1}^{m=N} \frac{x_m^2}{(a_m + \lambda_s)} 2d\lambda_s \\ &+ 2 \sum_{r,s=1}^{r,s=N} \sum_{m=1}^{m=N} \frac{x_m^2}{(a_m + \lambda_s)(a_m + \lambda_r)} d\lambda_r d\lambda_s \end{aligned}$$

By a theorem of Jacobi's

$$\begin{aligned} M_i &= \sum_{m=1}^{m=N} \frac{x_m^2}{(a_m + \lambda_s)^2} \\ &= \frac{(\lambda_s - \lambda_1)(\lambda_s - \lambda_2) \dots (\lambda_s - \lambda_{s-1})(\lambda_s - \lambda_{s+1}) \dots (\lambda_s - \lambda_n)}{(a_1 + \lambda_s)(a_2 + \lambda_s) \dots (a_n + \lambda_s)} \end{aligned}$$

and

$$\sum_{m=1}^{m=N} \frac{x_m^2}{(a_m + \lambda_s)(a_m + \lambda_r)} = 0 \quad r \neq s$$

$$\therefore 4 \sum_{m=1}^{m=N} dx_m^2 = \sum_{m=1}^{m=N} M_m d\lambda_m^2$$

or

$$8T = 4 \sum_{m=1}^{m=N} \left( \frac{dx_m}{dt} \right)^2 = \sum_{m=1}^{m=N} M_m \left( \frac{d\lambda_m}{dt} \right)^2$$

But

$$\frac{\partial T}{\partial \dot{\lambda}_i} = \frac{\partial W}{\partial \lambda_i} \quad \text{where} \quad \dot{\lambda}_i = \frac{d\lambda_i}{dt}$$

$$\therefore \dot{\lambda}_i = \frac{4}{M_i} \frac{\partial W}{\partial \lambda_i}$$

and

$$8T = \sum_{m=1}^{m=N} \left(\frac{W}{x_m}\right)^2 = 4 \sum_{m=1}^{m=N} \frac{1}{M_m} \left(\frac{W}{\lambda_m}\right)^2$$

$$= \frac{(a_1 + \lambda_m)(a_2 + \lambda_m) \dots \dots \dots (a_n + \lambda_m)}{(\lambda_m - \lambda_1)(\lambda_m - \lambda_2) \dots (\lambda_m - \lambda_{m-1})(\lambda_m - \lambda_{m+1}) \dots (\lambda_m - \lambda_n)} \left(\frac{W}{\lambda_m}\right)^2$$

which is an equation in which the variables separate.

If we consider the surface under consideration as the surface  $\lambda_1 = \text{constant}$ , then the general solution of such an equation is

$$W(\lambda_2, \lambda_3, \dots, \lambda_n, \alpha_1, \alpha_2, \dots, \alpha_{n-1}) = \sum_{i=2}^n \int \frac{\partial W}{\partial \lambda_i} dx_i,$$

where  $\alpha_1, \alpha_2, \dots, \alpha_{n-1}$  are arbitrary constants.

The equations of the geodesics on this surface are

$$\frac{\partial W}{\partial \alpha_i} = \beta_i \quad i = 2, \dots, n-1$$

where the  $\beta_i$  are arbitrary constants.

In 1846 Liouville(3) found the equations of the trajectories of motion and the time on a surface which has a line element of the form

$$ds^2 = \{K(q_1) - \lambda(q_2)\} \{dq_1^2 + dq_2^2\}$$

and the force function is of the form

$$2U = \frac{f(q_1) - g(q_2)}{K(q_1) - \lambda(q_2)} - C$$

In this case the equations of motion may be



integrated, and the trajectories are given by

$$\frac{dq_1}{\sqrt{2f(q_1) + C(q_1) - A}} = \frac{dq_2}{\sqrt{A - 2g(q_2) - C\lambda(q_2)}}$$

where A is an arbitrary constant.

Then the geodesics are found by setting  $U = 0$ , that is by setting  $f(q_1)$  and  $g(q_2) = 0$ .

Liouville then takes up some applications of this method, namely the cases when a mass point moves in a plane, on a sphere, on an ellipsoid (which leads to Jacobi's elliptical coordinates) and on a surface of rotation.

In a second paper the same year Liouville gives a method for solving the problem of the free motion of a mass point in 3-space, where the force function is of the form  $U = U(x_1, x_2, x_3)$ .

Jacobi has shown (Liouville's Journal, Vol. III p. 81) that a sufficient condition for the finding of the trajectories of motion is the existence of a function  $\mathcal{H} = \mathcal{H}(x_1, x_2, x_3)$  containing three arbitrary constants A, B, C, distinct from those which can be formed by addition, which satisfies the equation

$$\left(\frac{d\mathcal{H}}{dx}\right)^2 + \left(\frac{d\mathcal{H}}{dy}\right)^2 + \left(\frac{d\mathcal{H}}{dz}\right)^2 = 2(U + C) .$$

Then the equations of the trajectories and time

are

$$\frac{d\psi}{dA} = A', \quad \frac{d\psi}{dB} = B', \quad \frac{d\psi}{dC} = t + C'$$

where  $A'$ ,  $B'$  and  $C'$  are constants.

Liouville shows that by the introduction of elliptical coordinates such a function  $\psi$  can be found, when  $U$  has the form

$$U = \frac{(\mu^2 - \nu^2) f(\rho) - (\rho^2 - \nu^2) g(\mu) - (\rho^2 - \mu^2) h(\nu)}{(\rho^2 - \mu^2)(\rho^2 - \nu^2)(\mu^2 - \nu^2)}$$

where  $\rho, \mu, \nu$  are the elliptical coordinates given by

$$\frac{x_1^2}{\rho^2} + \frac{x_2^2}{\rho^2 - b^2} + \frac{x_3^2}{\rho^2 - c^2} = 1$$

$$\frac{x_1^2}{\mu^2} + \frac{x_2^2}{\mu^2 - b^2} - \frac{x_3^2}{c^2 - \mu^2} = 1$$

$$\frac{x_1^2}{\nu^2} - \frac{x_2^2}{b^2 - \nu^2} - \frac{x_3^2}{c^2 - \nu^2} = 1$$

Then

$$\begin{aligned} \psi &= \int d\rho \sqrt{\frac{2f(\rho) + A + B^2 + 2C^4}{(\rho^2 - b^2)(\rho^2 - c^2)}} \\ &+ \int d\mu \sqrt{\frac{2g(\mu) - A - B^2 - 2C^4}{(\mu^2 - b^2)(c^2 - \mu^2)}} \\ &+ \int d\nu \sqrt{\frac{2h(\nu) + A + B^2 + 2C^4}{(b^2 - \nu^2)(c^2 - \nu^2)}} \end{aligned}$$

If we set  $f(\rho)$ ,  $g(\mu)$ , and  $h(\nu) = 0$  in the above integral, then the equations of the geodesics are given by

$$\frac{d\textcircled{A}}{dA} = A', \quad \frac{d\textcircled{B}}{dB} = B'.$$

In 1887 Roschatus(5) found out what the force function must be in order that the Hamilton-Jacobi equation expressed in generalized elliptical coordinates permit the separation of variables.

In 1880 Morera(6) proved that the Hamilton-Jacobi equation (for  $n = 2$ ) permits a separation of variables if it is of any one of the following forms:

$$\text{I} \quad \frac{W_1^2 + W_2^2}{\kappa(q_1) + \lambda(q_2)} - 2 \left\{ \frac{\mu(q_1) + \nu(q_2)}{\kappa(q_1) + \lambda(q_2)} + \alpha \right\} = 0$$

$$\text{II a) } E(q_1)W_1^2 + 2F(q_1)W_1W_2 + G(q_1)W_2^2 - 2\{\Pi(q_1) - \alpha\} = 0$$

$$\text{b) } E(q_2)W_1^2 + 2F(q_2)W_1W_2 + G(q_2)W_2^2 - 2\{\Pi(q_1) + \alpha\} = 0$$

where  $W_1 = W_1(q_1; \alpha, \beta)$

$$W_2 = W_2(q_2; \alpha, \beta)$$

and  $\alpha$  and  $\beta$  are two arbitrary constants.

Just about this time there appeared a number of discussions of the integral equations resulting from the separation of the variables in the Hamilton-Jacobi equation, namely, the discussions of Weierstrass(4) (1866), Stäckel(7)

(1885) and Staude(8) (1877).

In 1887 Morera(9) showed that if an equation of the general form  $F(W_1, W_2; q_1, q_2) = 0$  permits a separation of variables, then it is necessary and sufficient that there exist a functional equation of the form

$$\Psi\{\Psi(F, W_1, q_1), \chi(F, W_2, q_2)\} = 0.$$

This does not appear to be a very useful fact, however, since the important question is whether or not the variables in a given equation separate.

In 1890 Stäckel(10) proved that for Liouville surfaces, if the force function is of the form considered by Liouville in 1846(3), the variables in the Hamilton-Jacobi equation separate. Conversely, if the Hamilton-Jacobi equation ( $n = 2$ ) allows a separation of variables, then the line element can be put in Liouville's form.

The first paper which deals with Stäckel systems as such (Liouville systems, systems in which  $ds^2 = \{\chi(q_1) + \lambda(q_2)\}\{dq_1^2 + dq_2^2\}$ , and elliptical coordinate systems being special cases of Stäckel systems) is Stäckel's paper of 1891(11). Here he considers the possibility of separating variables in the Hamilton-Jacobi equation

$$H^* = \frac{1}{2} \sum_{\kappa=1}^{\kappa=N} A_{\kappa} \left( \frac{\partial W}{\partial p_{\kappa}} \right)^2 - (\Pi + \alpha_1) = 0,$$

which has associated with it the line element

$$ds^2 = \sum_{\kappa=1}^{\kappa=N} \frac{(dp_{\kappa})^2}{A_{\kappa}}$$

$A_K$  is a function of  $p_1, p_2, \dots, p_n$ ,  
 is the force function, also a function of  $p_1, p_2, \dots, p_n$ , and  $\alpha_1$  is an arbitrary constant.

A general solution of the Hamilton-Jacobi equation is

$$W = \sum_{K=1}^{K=N} \int \frac{\partial W}{\partial p_K} dp_K = \sum_{K=1}^{K=N} \int W_K dp_K$$

If the variables in  $H^*$  separate, then

$$\frac{\partial W}{\partial p_K} = W_K(p_K, \alpha_1, \alpha_2, \dots, \alpha_n)$$

and there must exist  $n(n+1)$  variables  $\varphi_{K\lambda}(p)$  where  $K = 1, 2, 3, \dots, n$ , and  $\lambda = 0, 1, 2, 3, \dots, n$ , such that

$$A_1 = \frac{\phi_1}{\phi}, \quad A_2 = \frac{\phi_2}{\phi} \quad \dots \quad A_n = \frac{\phi_n}{\phi}, \quad \Pi = \frac{\phi'}{\phi}$$

where  $\phi = |\varphi_{K\lambda}| = \sum_{K=1}^{K=N} \varphi_{K1}(p_K) \phi_K$

$$\phi' = \sum_{K=1}^{K=N} \varphi_{K\alpha}(p_K) \phi_K.$$

Stäckel reaches this result by substituting in  $H^*$  for  $\frac{\partial W}{\partial p_K}$  its value  $W_K(p_K, \alpha_1, \alpha_2, \dots, \alpha_n)$ , then differentiating with respect to the  $\alpha$ 's.

$$\sum_{K=1}^{K=N} A_K \frac{\partial W_K^2}{\partial \alpha_\mu} = 2 \begin{cases} 1 & \mu = 1 \\ 0 & \mu \neq 1 \end{cases}$$

$$\text{setting } \left| \frac{\partial W_k^2}{\partial \alpha_\mu} \right| = Q = \sum_{k=1}^{k=N} \frac{\partial W_k^2}{\partial \alpha_1} Q_k$$

$$A_k = 2 \frac{Q_k}{Q}$$

Substituting these values of  $A_k$  back into  $H^*$  we find that if the variables are assumed separable then

$$\Pi = \frac{\sum_{k=1}^{k=N} (W_k^2 - \alpha_1 \frac{\partial W_k^2}{\partial \alpha_1}) Q_k}{\sum_{k=1}^{k=N} \frac{\partial W_k^2}{\partial \alpha_1} Q_k}$$

and

$$\frac{\partial (W_k^2)}{\partial \alpha_\mu} = 2 \psi_{k\mu}(p_k)$$

from which the above results follow.

The remainder of the paper is taken up by a discussion of the integral equations resulting from such a separation of variables.

Under certain assumptions they are given as  $n$ -fold periodic functions of the time.

The geodesics are found by setting  $\Pi = 0$ , then  $\frac{\partial W}{\partial \alpha_\mu} = \beta_\mu$  (where  $\mu = 2, 3, \dots, n$ ) are the required equations.

In 1893 Stäckel(13) generalized the fact that surfaces on which the line element is reducible to the form of Liouville admit (besides the integral of the fundamental form) one homogeneous integral of the second degree in the velocities.

If the line element on a surface is reducible to the form

$$ds^2 = \sum_{\kappa=1}^{\kappa=N} \frac{\phi}{\phi_{\kappa 1}} (dq_{\kappa})^2,$$

and  $\Pi = \text{constant}$ , the Hamilton-Jacobi equation associated with this line element is

$$H^* = \frac{1}{2} \sum_{\kappa=1}^{\kappa=N} \frac{\phi_{\kappa 1}}{\phi} \left( \frac{\partial W}{\partial q_{\kappa}} \right)^2 - (\Pi + \alpha_1) = 0$$

where

$$\phi = |\varphi_{\kappa \lambda}(q_{\kappa})| = \sum_{\kappa=1}^{\kappa=N} \varphi_{\kappa \lambda} \phi_{\kappa \lambda},$$

as defined in the previous paper.

If we assume that

$$\frac{1}{2} \left( \frac{\partial W}{\partial q_{\kappa}} \right)^2 = \varphi_{\kappa 1} \alpha_1 + \varphi_{\kappa 2} \alpha_2 + \dots + \varphi_{\kappa n} \alpha_n$$

where  $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n$  are arbitrary constants, then a general solution of  $H^*$  is

$$W = \sum_{\kappa=1}^{\kappa=N} \int \sqrt{\varphi_{\kappa 1} \alpha_1 + \varphi_{\kappa 2} \alpha_2 + \dots + \varphi_{\kappa n} \alpha_n} dq_{\kappa}$$

$$\text{But } \frac{\partial T}{\partial \dot{q}_{\kappa}} = \frac{\partial W}{\partial q_{\kappa}} = \frac{\phi}{\phi_{\kappa 1}} \dot{q}_{\kappa}, \quad \text{and}$$

$$\sum_{\kappa=1}^{\kappa=N} \frac{\phi_{\kappa \lambda}}{\phi} \left( \frac{\partial W}{\partial q_{\kappa}} \right)^2 = \alpha_{\lambda} \quad \lambda = 1, 2, 3, \dots, n.$$

Therefore we have of necessity  $(n - 1)$  quadratic first integrals

$$\sum_{\kappa=1}^{\kappa=n} \frac{\phi_{\kappa\lambda}}{\phi_{\kappa 1}} \dot{q}_{\kappa}^2 = \alpha_{\lambda} \quad \lambda = 2, 3, \dots, n.$$

The term  $\lambda = 1$  is omitted since this gives us only the integral of the energy.

Then the problem may be reduced by quadratures and the equations of motion are

$$\frac{\partial W}{\partial \alpha_{\lambda}} = T - t$$

$$\frac{\partial W}{\partial \alpha_{\mu}} = \beta_{\mu}$$

where  $\mu = 2, 3, \dots, n$  and  $T$  and  $\beta_{\mu}$  are arbitrary constants. If in our original equation  $H^*$  we take  $\Pi = 0$ , then we get the equations of the geodesics.

In 1893 Stäckel(14) connected together the idea of "analytical equivalents" as given in an earlier paper(12), a special class of motion of a point in 3-dimensions, the integration of the Hamilton-Jacobi equation through separation of variables, the solution of the integral equations thus obtained by means of  $n$ -fold periodic functions, and a class of motions of a point on an  $n$ -dimensional surface which corresponds to the Jacobian motion on a surface of



rotation. By the same line of reasoning which Stäckel used in 1893 to show the existence of  $(n - 1)$  quadratic first integrals belonging to the equation

$$H^* = \frac{1}{2} \sum_{\kappa=1}^{\kappa=N} \frac{\phi_{\kappa 1}}{\phi} \left( \frac{\partial W}{\partial q_{\kappa}} \right)^2 - (\Pi + \alpha_1) = 0$$

when  $\Pi = \text{constant}$ , he used again in 1895(15) to show that when

$$\Pi = \sum_{\kappa=1}^{\kappa=N} \frac{\varphi_{\kappa 0} \phi_{\kappa 1}}{\phi}$$

we still have  $(n-1)$  quadratic first integrals, this time of the form

$$\frac{\phi \phi_{\kappa \lambda}}{\phi_{\kappa 1}} \dot{q}_{\kappa}^2 = \frac{\psi_{\lambda}}{\phi} + \alpha_{\lambda} \quad \lambda = 2, 3, \dots, n,$$

where 
$$\psi_{\lambda} = \sum_{\kappa=1}^{\kappa=N} \varphi_{\kappa 0}(q_{\kappa}) \phi_{\kappa \lambda} .$$

He then generalizes this result still farther, using the same type of proof given in 1893, to include the case when the position of the mass point is given by the  $r$  quantities

$$q_{11}, q_{12}, \dots, q_{1h1}$$

$$q_{21}, q_{22}, \dots, q_{2h2}$$

⋮  
⋮  
⋮

$$q_{n1}, q_{n2}, \dots, q_{nhn}$$

In this case we have

$$\frac{1}{2} \sum_{\rho_k=1}^{\rho_k=h_k} \sum_{\sigma_k=1}^{\sigma_k=h_k} A_{\rho_k \sigma_k} \frac{\partial W}{\partial q_{\rho_k}} \frac{\partial W}{\partial q_{\sigma_k}}$$

$$= \varphi_{k0} + \varphi_{k1} \alpha_1 + \dots + \varphi_{kn} \alpha_n$$

and

$$H^* = \frac{1}{2} \sum_{k=1}^{k=N} \frac{\phi_{k1}}{\phi} \sum_{\rho_k=1}^{\rho_k=h_k} \sum_{\sigma_k=1}^{\sigma_k=h_k} A_{\rho_k \sigma_k} \frac{\partial W}{\partial q_{\rho_k}} \frac{\partial W}{\partial q_{\sigma_k}}$$

$$= \frac{\psi_1}{\phi} + \alpha_1$$

There must then exist (n-1) quadratic first integrals of the form

$$\frac{1}{2} \sum_{k=1}^{k=N} \frac{\phi \cdot \phi_{k\lambda}}{\phi_{k1}^2} \sum_{\rho_k=1}^{\rho_k=h_k} \sum_{\sigma_k=1}^{\sigma_k=h_k} B_{\rho_k \sigma_k} \dot{q}_{\rho_k} \dot{q}_{\sigma_k}$$

$$= \frac{\psi_\lambda}{\phi} + \lambda$$

$$\lambda = 2, 3, \dots, n.$$

$B_{\rho_k \sigma_k}$  is the ~~minor~~ <sup>cotactor</sup> of  $A_{\rho_k \sigma_k}$  <sup>the MINOR</sup> divided by the determinant of the  $A_{\rho_k \sigma_k}$  terms, where  $\rho_k, \sigma_k = 1, 2, \dots, h_k$  } and the other symbols are those defined in the earlier papers discussed in this resume.

This enables one to find new line elements which permit the finding of the geodesic lines by quadratic

first integrals.

Stäckel generalized the above results still farther in a paper published in 1897.

In 1897 Stäckel also showed that the results stated in 1890 are valid for complex as well as real quantities. The same year Painlevé(19) reached independently the same conclusions Stäckel arrived at in 1890.

In 1897(20) Levi Civita discussed the dynamical problem which admits a quadratic first integral of the form

$$H = c_{rs} \dot{x}_r \dot{x}_s$$

A necessary and sufficient condition for the existence of this integral is

$$\nabla_{(r} c_{st)} = 0.$$

Using this and the fact that the principal directions are given by

$$|c_{st} - P a_{st}| = 0$$

we ~~may find~~ <sup>find</sup> the equations

$$I \quad (P_h - P_i) \delta_{hij} + (P_i - P_j) \delta_{ijh} + (P_j - P_h) \delta_{jhi} = 0$$

$$h, i, j = 1, 2, \dots, n; \quad h \neq i \neq j$$

$$II \quad \frac{\partial P_h}{\partial s_i} = 2(P_h - P_i) \delta_{ihh}$$

$$h, i = 1, 2, 3, \dots, n$$

where the  $\rho_i$  ( $i = 1, 2, \dots, n$ ) are the directions given by setting the determinant  $\begin{vmatrix} c_{st} - a_{st} \end{vmatrix} = 0$ .

If all these roots are distinct, then there are  $(n-1)$  quadratic first integrals, the solution corresponding to the Stäckel case, which may be found by making the assumption that the congruence of reference is normal. In this case I is fulfilled identically and II gives the Stäckel results.

An investigation as to the form of the geodesics on surfaces whose line elements can be put in the Liouville form was given by Stäckel in 1905(21).

In 1911 J. Hadamard(22) considered the converse problem of finding the equations of motion, that of determining the functions in a given set of equations of motion. Here he uses a theorem of function theory which he had set up prior to this.

In 1916 Arwin(23) gave a new proof of the Hamilton-Jacobi, and a discussion of the integration of the equation of the geodesics on a surface.

In 1923 E. Turriere(24) confirmed the fact that the equations of motion, in Liouville and Stäckel cases, may actually be solved by quadratures.

In 1923, a paper by Weinacht(25) showed the connection between Stäckel systems and confocal quadrics.

Weinacht proved that each separable mechanical problem of the motion of a mass point in 3-space must come under Stäckel's orthogonal case, and must be separable through the use of elliptical coordinates or their degenerates. In other words the only orthogonal systems of Stäckel in euclidean 3-space are the confocal quadrics and their degenerates.

In 1927 Drach(26) discussed Liouville elements and algebraic integrals of the equation of the geodesics. Later the same year he considered the case of Liouville surfaces which admit at least two first integrals.

A paper by Blaschke(27) generalizing certain well known properties of confocal quadrics to Riemannian space which leads to Stäckel systems was published in 1927. This work is included in a later paper which will be considered farther on.

At this point a paper by Robertson(28) appeared, connecting Stäckel systems with quantum mechanics. Given the Schrödinger wave equation

$$\sum_{i=1}^{i=n} H \frac{\partial}{\partial x_i} \left( \frac{H}{H_i^2} \frac{\partial \Psi}{\partial x_i} \right) + k^2 (E - V) \Psi = 0$$

where  $H = H_1 \dots H_n$ , Robertson proves that if this equation can be solved by the separation of variables,

that is if a solution of the form  $\prod X_i(x_i)$  exists then the following conditions must be fulfilled.

(1) The functions  $H_i^2$  must be of the Stäckel form.

$$(2) \quad v = \sum_{i=1}^{i=\mathcal{N}} \frac{f(x_i)}{H_i^2}$$

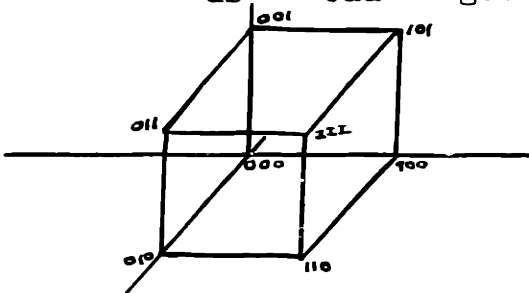
(3)  $\varphi = \prod \frac{H_i}{\varphi(x_i)}$ , where  $\varphi$  is the determinant of

the Stäckel functions.

In 1928 Blaschke(29) continued his earlier work(28) of generalizing certain properties of confocal quadrics to the coordinate surfaces  $u, v, w$  in Riemannian space.

If we assume that Ivory's diagonal property holds for the coordinates  $u, v, w$ , that is, if the distance between the points  $(u_0, v_0, w_0)$  and  $(u_1, v_1, w_1)$  equals the distance between the two points  $(u_0, v_0, w_1)$  and  $(u_1, v_1, w_0)$ , then the line element must have Stäckel's form. The proof of this theorem is as follows:

$$ds^2 = edu^2 + gdv^2 + hdw^2 .$$



If  $D$  is the geodesic diagonal between  $(0,0,0)$  and  $(1,1,1)$ , assuming the diagonal property stated above to hold

$$\delta D = e \frac{du}{ds} u + g \frac{dv}{ds} v + h \frac{dw}{ds} w$$

$$\frac{\partial D}{\partial u_1} = e \left[ \frac{du}{ds} \right]_{111} = e \left[ \frac{du}{ds} \right]_{110}$$

$$\frac{D}{v_1} = g \left[ \frac{dv}{ds} \right]_{111} = g \left[ \frac{dv}{ds} \right]_{110}$$

Setting  $e \frac{du}{ds} = \varphi$ ,  $f \frac{dv}{ds} = \psi$ ,  $g \frac{dw}{ds} = \omega$ , it may be proved that  $\varphi = \varphi(u)$ ,  $\psi = \psi(v)$ ,  $\omega = \omega(w)$ .

But if the point (0,0,0) has no special position there must exist a two parameter family of geodesics obeying these equations. That is

$$U = U(u; c_1, c_2) = \varphi^2$$

$$V = V(v; c_1, c_2) = \psi^2$$

$$W = W(w; c_1, c_2) = \omega^2$$

But

$$\frac{U}{e} + \frac{V}{f} + \frac{W}{g} = 1$$

$$\frac{U_1}{e} + \frac{V_1}{f} + \frac{W_1}{g} = 0$$

where  $U_1 = \frac{\partial U}{\partial c_1}$ ,  $V_1 = \frac{\partial V}{\partial c_1}$ ,  $W_1 = \frac{\partial W}{\partial c_1}$

$$\frac{U_2}{e} + \frac{V_2}{f} + \frac{W_3}{g} = 0$$

where  $U_2 = \frac{\partial U}{\partial c_2}$ , etc.

Solving these two equations for the reciprocals of  $e$ ,  $f$ ,  $g$ , we find that the line element under consideration must have the Stäckel form which is in this case

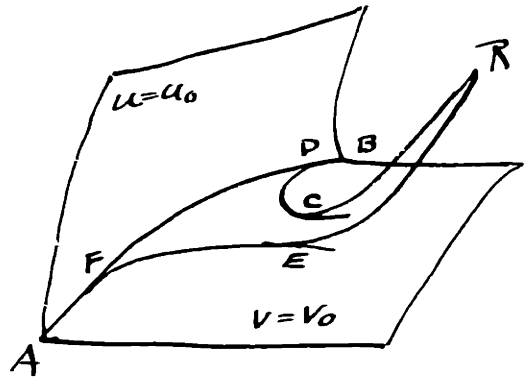
$$ds^2 = \begin{vmatrix} U & V & W \\ U_1 & V_1 & W_1 \\ U_2 & V_2 & W_2 \end{vmatrix} \left\{ \frac{du^2}{(W_2 V_1 - V_2 W_1)} + \frac{dv^2}{(W_1 U_2 - W_2 U_1)} + \frac{dw^2}{(U_1 V_2 - U_2 V_1)} \right\}$$

Blaschke also proves the converse of this theorem, which is that Stäckel's line element implies Ivory's diagonal property.

In the second part of the paper he shows that on each surface  $W = \text{constant}$  the parameters  $u$  and  $v$  are Liouville parameters.

The third theorem proved is that the common geodesic tangents to any two of the given coordinate surfaces form a system which a family of parallel surfaces intersect orthogonally.

Blaschke also generalizes Staude's principle for confocal ellipsoids. If  $u = u_0$ ,  $v = v_0$  are two of the coordinate surfaces under consideration, and  $AB$  their line of intersection;  $CD$  and





EF are two surface geodesics (tangent to AB), CR and ER are two space geodesics tangent to the surface geodesics. Then, if we consider an inelastic string passed through a loop at R along the space tangent to E, along the surface tangent to F, then along the line of intersection to R and back to R passing through C, this loop at P can be moved only on the surface  $V = \text{constant}$ , without spoiling the tangent contacts or breaking the string.

The last proof in the paper is a geometrical proof of Weinacht's theorem on Stäckel systems in euclidean 3-space.

In 1932 W. Vogelsang(30) investigated the case in a Riemannian space ( $n = 3$ ) when the common geodesic tangents to two orthogonal surfaces form a normal congruence and proved that it is sufficient that the space have Stäckel's line element. In this case the surfaces considered are used as coordinate surfaces. Secondly he proved that if all components of the curvature tensor with three different indices vanish, then the line element must be Liouville's. In the third part of his paper he proves that the only solutions to the first problem in euclidean 3-space are the confocal quadrics.

In 1934(31) Eisenhart continued on with the connection between Stäckel systems and quantum mechanics. First he proves that the  $\varphi$  condition of Robertson is

equivalent to  $R_{ij} = 0$ , where  $R_{ij}$  is the Ricci tensor used by Einstein. His second theorem, which is really Weinacht's theorem, is that the only orthogonal coordinate systems in which the 3-dimensional Schrödinger wave equation can be solved by separation of variables are the confocal quadrics and their degenerates (one or more families consisting of planes). The last part of this paper shows that both these results can be generalized to include spaces of higher dimensions.

Later, in 1935. Eisenhart(32) considered Stäckel systems in connection with conformal spaces. Given a Riemannian n-space with the fundamental form

$$ds^2 = \sum_{i=1}^{i=n} H_i^2 (dx_i)^2,$$

he investigates the case when

$$\bar{H}_i^2 = e^{2\sigma} H_i^2 \quad \begin{array}{l} \sigma = (x_1, x_2, \dots, x_n) \\ i = 1, 2, \dots, n \end{array}$$

are the coefficients of the fundamental form of an euclidean n-space, in which case the Riemannian space is conformally flat. By using the results of his previous paper(31) and the facts that  $\bar{g}_{ij} = e^{2\sigma} g_{ij}$  and  $\bar{R}_{ij} = R_{ij} + (n-2)(\sigma, ij - \sigma, i \sigma, j) + g_{ij}(\Delta_2^\sigma + (n-2) \Delta_1^\sigma)$  where  $g_{ij}$  is the fundamental tensor of the Riemannian space;  $\sigma, i = \frac{\partial \sigma}{\partial x_i}$ ;  $\sigma, ij$  is the second covariant derivative

of  $\sigma$  with respect to the  $g$ 's,

$$\Delta_1 \sigma = g^{ij} \frac{\partial \sigma}{\partial x^i} \frac{\partial \sigma}{\partial x^j}, \quad \text{and} \quad \Delta_2 \sigma = g^{ij} \sigma_{,ij},$$

he finds three necessary conditions. Among the solutions of the equations giving these three conditions are Stäckel systems for spaces of constant Riemannian curvature.

The papers listed here all show how Stäckel systems connect together various problems in ordinary mechanics, quantum mechanics and differential geometry. Of particular interest from the point of view of geometry are the following facts: first, the generalizations of many well known properties of Liouville surfaces to Stäckel spaces; second, the finding of geodesic lines by quadratures and quadratic first integrals; and third, the fact that Stäckel systems in euclidean 3-space are the confocal quadrics and their degenerates. From the point of view of ordinary mechanics the separation of the Hamilton-Jacobi equation and its connection with Stäckel systems is the interesting piece of information, while from the point of view of quantum mechanics solving the Schrödinger wave equation in three dimensions and the proof that the confocal quadrics and their degenerates are the only coordinate systems in euclidean 3-space in which the Schrödinger wave equation can be solved is important.

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