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Abstract

Using combinatorial techniques, we answer two questions about simple classical Lie groups. Define $N(G,m)$ to be the number of conjugacy classes of elements of finite order m in a Lie group G, and $N(G,m,s)$ to be the number of such classes whose elements have s distinct eigenvalues or conjugate pairs of eigenvalues. What is $N(G,m)$ for G a unitary, orthogonal, or symplectic group? What is $N(G,m,s)$ for these groups? For some cases, the first question was answered a few decades ago via group-theoretic techniques. It appears that the second question has not been asked before; here it is inspired by questions related to enumeration of vacua in string theory. Our combinatorial methods allow us to answer both questions.

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1 Introduction

Given a group G of linear transformations and integers m and s, let

$$E(G,m) = \{x \in G \mid x^m = 1\}. \tag{1}$$

Also let

$$E(G,m,s) = \{x \in E(G,m) \mid x \text{ has } s \text{ distinct eigenvalues}\} \tag{2}$$

for G a unitary group, and

$$E(G,m,s) = \{x \in E(G,m) \mid x \text{ has } s \text{ distinct conjugate pairs of eigenvalues}\} \tag{3}$$

for G a symplectic or orthogonal group, and let

$$\begin{aligned} N(G,m) &= \text{number of conjugacy classes of } G \text{ in } E(G,m), \\ N(G,m,s) &= \text{number of conjugacy classes of } G \text{ in } E(G,m,s). \end{aligned}$$

For Γ any finitely generated abelian group and G a Lie group, one can consider the space of homomorphisms $\mathrm{Hom}(\Gamma, G)$ and the space of representations of Γ in G, that is, consider

$$\mathrm{Rep}(\Gamma, G) \equiv \mathrm{Hom}(\Gamma, G)/G$$

(where G acts by conjugation); using this notation,

$$E(G,m) = \mathrm{Hom}(\mathbf{Z}/m\mathbf{Z}, G)$$

and

$$N(G,m) = |\mathrm{Rep}(\mathbf{Z}/m\mathbf{Z}, G)|.$$

For the case $\Gamma = \mathbf{Z}^n$, the spaces $\mathrm{Hom}(\mathbf{Z}^n, G)$ and $\mathrm{Rep}(\mathbf{Z}^n, G)$ have been studied for various Lie groups G in [4, 1, 2, 3] (and references therein), where there has been interest in their number of path-connected components and their cohomology groups.

It is the purpose of this paper to compute $N(G,m) = |\mathrm{Rep}(\mathbf{Z}/m\mathbf{Z}, G)|$ and $N(G,m,s)$ for G a unitary, orthogonal, or symplectic group. Unlike $\mathrm{Rep}(\Gamma, G)$ for $\Gamma = \mathbf{Z}^n$, the representation space $\mathrm{Rep}(\mathbf{Z}/m\mathbf{Z}, G)$ is a finite set, so we can count its number of elements. The results are summarized in Table 1.

The numbers $N(G,m,s)$ have never been studied before in the mathematical literature. What motivated their definition, as well as the definition of $N(G,m)$, was the need to find a formula for the number of certain vacua in the quantum moduli space of M-theory compactifications on manifolds of G_2 holonomy. In that context, the numbers $N(SU(p), q)$ and $N(SU(p), q, s)$, where q and p are relatively prime, were computed in [8]. These numbers are related to symmetry breaking patterns in grand unified theories, with the number $N(SU(p), q, s)$ being particularly significant as s is related to the number of massless fields in the gauge theory that remains after the symmetry breaking. The connections with symmetry breaking patterns arise from the fact that if M is a manifold and $\pi_1(M)$ is its fundamental group, then $\mathrm{Rep}(\pi_1(M), G)$ is the moduli space of isomorphism classes of flat connections on principal G-bundles over M; in grand unified theories arising from string or M-theory, these flat connections (called Wilson lines) serve as a symmetry breaking mechanism. For more on the physical applications and implications of these numbers, see [9].

As for $N(G,m)$, certain cases have been studied previously in the mathematical literature, using different techniques than ours. Two of the quantities we derive, Theorems 2.2 and 3.1, were obtained in [6, 7] using the full machinery of Lie structure theory with a generating function approach; in [16, 5], the case of certain prime power orders is computed; and in [11], Theorem 2.7 is obtained. Our methods are different; they are purely combinatorial and direct, and apply not only to simply connected or adjoint groups as in [6, 7], so we are able to derive formulas for $O(n)$, $SO(n)$, and $U(n)$ alongside those for $SU(n)$ and $Sp(n)$.

Other aspects of elements of finite order in Lie groups have been studied. See for example [10, 13, 12, 14, 15].

In addition to the quantities $N(G,m)$ and $N(G,m,s)$, which count conjugacy classes of elements of any order dividing m, we consider also conjugacy classes of elements of exact order m in G: let

$$F(G,m) = \{x \in G \mid x^m = 1, x^n \neq 1 \text{ for all } n < m\}.$$

Also let

$$F(G,m,s) = \{x \in F(G,m) \mid x \text{ has } s \text{ distinct eigenvalues}\}$$

for G a unitary group, and

$$F(G,m,s) = \{x \in F(G,m) \mid x \text{ has } s \text{ distinct conjugate pairs of eigenvalues}\}$$

for G a symplectic or orthogonal group, and let

$$\begin{aligned} K(G,m) &= \text{number of conjugacy classes of } G \text{ in } F(G,m), \\ K(G,m,s) &= \text{number of conjugacy classes of } G \text{ in } F(G,m,s). \end{aligned}$$

Since

$$\begin{aligned} N(G,m) &= \sum_{d|m} K(G,d), \\ N(G,m,s) &= \sum_{d|m} K(G,d,s), \end{aligned}$$

we have, by the Möbius inversion formula,

$$K(G,m) = \sum_{d|m} \mu(d) N(G, \frac{m}{d}), \tag{4}$$

$$K(G,m,s) = \sum_{d|m} \mu(d) N(G, \frac{m}{d}, s), \tag{5}$$

where $\mu(d)$ is the Möbius function.

The reader is invited to obtain $K(G,m)$ and $K(G,m,s)$ from Table 1 and equations (4) and (5) above.

Table 1: Number of conjugacy classes of elements of finite order in Lie groups

G	m	$N(G,m)$	$N(G,m,s)$
$U(n)$	any	$\binom{n+m-1}{m-1}$	$\frac{s}{n}\binom{n}{s}\binom{m}{s}$
$SU(n)$	$(n,m)=1$	$\frac{1}{m}\binom{n+m-1}{n}$	$\frac{s}{nm}\binom{n}{s}\binom{m}{s}$
	any	$\frac{1}{m}\sum_{d\mid(n,m)}\phi(d)\binom{(n+m-d)/d}{n/d}$	$\frac{1}{m}\sum_{d\mid(n,m)}\sum_{j\geq 0}\phi(d)\binom{(n+m-jd-d)/d}{(n-jd)/d}\binom{m/d}{j}\binom{jd}{s}(-1)^{j+s}$
$Sp(n)$	any	$\binom{n+[\frac{m}{2}]}{n}$	$\frac{s}{n}\binom{n}{s}\binom{[\frac{m}{2}]+1}{s}$
$SO(2n+1)$	any	$\binom{n+[\frac{m}{2}]}{n}$	$\frac{s}{n}\binom{n}{s}\binom{[\frac{m}{2}]+1}{s}$
$O(2n+1)$	$2k+1$	$\binom{n+[\frac{m}{2}]}{n}$	$\frac{s}{n}\binom{n}{s}\binom{[\frac{m}{2}]+1}{s}$
$O(2n)$	$2k+1$	$\binom{n+[\frac{m}{2}]}{n}$	$\frac{s}{n}\binom{n}{s}\binom{[\frac{m}{2}]+1}{s}$
$SO(2n)$	$2k+1$	$\binom{n+[\frac{m}{2}]-1}{n-1}\frac{n+m-1}{n}$	$\frac{s}{n}\binom{n}{s}\binom{[\frac{m}{2}]}{s}\frac{m+1-s}{[\frac{m}{2}]+1-s}$
$O(2n+1)$	$2k$	$2\binom{n+\frac{m}{2}}{n}$	$\frac{2s}{n}\binom{n}{s}\binom{\frac{m}{2}+1}{s}$
$O(2n)$	$2k$	$\binom{n+\frac{m}{2}-1}{n-1}\frac{4n+m}{2n}$	$\frac{2n-s-1}{n-s}\binom{n-2}{s-1}\binom{\frac{m}{2}+1}{s}$
$SO(2n)$	$2k$	$\binom{n+\frac{m}{2}}{n}+\binom{n+\frac{m}{2}-2}{n}$	$\frac{s}{n}\binom{n}{s}\left[2\binom{\frac{m}{2}}{s}+\binom{\frac{m}{2}-1}{s-2}\right]$

2 Counting conjugacy classes in unitary groups

We begin with $N(U(n), m)$, with no conditions on the integers m and n. Since every element of $U(n)$ is diagonalizable, every conjugacy class has diagonal elements. The diagonal entries are m^{th} roots of unity, $e^{2\pi i k_j/m}$, $k_j = 0, \ldots, m-1$, and $j = 1, \ldots, n$. In each conjugacy class there is a unique diagonal element for which the diagonal entries are ordered so that the k_j are nondecreasing with j. Therefore, $N(U(n), m)$ is the number of such diagonal matrices with nondecreasing k_j.

Let $\{n_k\} = (n_0, \ldots, n_{m-1})$, $\sum_{k=0}^{m-1} n_k = n$ with $n_k \geq 0$. Such a sequence is a weak m-composition of n, and it is well-known that there are $\binom{n+m-1}{m-1}$ such sequences [17]. There is a bijective map between such sequences and diagonal matrices in $U(n)$ with ordered entries: $\{n_k\}$ corresponds to the diagonal $U(n)$ matrix with n_k repetitions of the eigenvalue $e^{2\pi i k/m}$:

$$\mathrm{diag}(\underbrace{1, 1, \cdots, 1}_{n_0}, \underbrace{e^{2\pi i/m}, \cdots, e^{2\pi i/m}}_{n_1}, \cdots, \underbrace{e^{2(m-1)\pi i/m}, \cdots, e^{2(m-1)\pi i/m}}_{n_{m-1}}) \ . \tag{6}$$

Thus $N(U(n), m)$ is the number of weak m-compositions of n, so we obtain the following formula.

Theorem 2.1 *For any positive integers n and m,*

$$N(U(n), m) = \binom{n+m-1}{m-1} \tag{7}$$

Note that $N(U(n), m)$ is also the number of inequivalent unitary representations of $\mathbf{Z}/m\mathbf{Z}$ of dimension n.

Now we turn to the special unitary group $SU(p)$, and calculate $N(SU(p), q)$ where $(p, q) = 1$. Given a sequence $\{n_k\}$, $k = 0, \ldots q-1$ with $\sum_{k=0}^{q-1} n_k = p$, $n_k \geq 0$ (i.e. a weak q-composition of p), the determinant of the corresponding matrix x is $\exp \frac{2\pi i}{q} \left(\sum_{k=0}^{q-1} k n_k\right)$, so the condition $\det x = 1$ requires $\sum_k k n_k \equiv 0 \bmod q$. Thus for a weak q-composition of p to determine a matrix in $SU(p)$, we need $\sum_k k n_k \equiv 0 \bmod q$.

We now show the family of weak q-compositions of p are partitioned into sets of size q where in each such set there is exactly one such composition with $\sum k n_k \equiv 0$. Consider the q distinct sequences

$$\{n_k^{(j)}\} = \{n_{k+j}\} \qquad j = 0, 1, \ldots, q-1 \ , \text{ indices are understood mod q.} \tag{8}$$

(The only way for the sequences not to be distinct is if all n_k were equal, which would imply $q n_k = p$, impossible when $(p, q) = 1$). The determinant of the matrix x_j corresponding to the

j^{th} sequence is $\exp \frac{2\pi i}{q} \left(\sum_{k=0}^{q-1} kn_{k+j}\right)$. Since $(p,q)=1$ and

$$\sum_{k=0}^{q-1} kn_{k+j} - \sum_{k=0}^{q-1} kn_{k+j+1} \equiv p \bmod q \ , \tag{9}$$

exactly one of the q values of j gives the sum $\sum_k kn_{k+j} \equiv 0 \bmod q$, so $\det x_j = 1$ for that value of j. We therefore get the next result.

Theorem 2.2 *For* $(p,q)=1$,

$$N(SU(p),q) = \frac{1}{q}\binom{p+q-1}{q-1} = \frac{(p+q-1)!}{p!\,q!} \ . \tag{10}$$

Now we turn to counting conjugacy classes whose elements have a given number s of distinct eigenvalues. We begin with $N(U(n),m,s)$. A $U(n)$ matrix with s distinct eigenvalues (which has centralizer of the form $\Pi_{i=1}^{s} U(n_i)$) corresponds to a sequence $\{n_a\} = (n_1, \dots n_s)$, $\sum_{a=1}^{s} n_a = n$, $n_a \geq 1$. Such a sequence is an s-composition of n and there are $\binom{n-1}{s-1}$ such sequences [17]). There are also $\binom{m}{s}$ ways to choose the s eigenvalues themselves. We therefore obtain the following formula.

Theorem 2.3 *For any positive integers* n *and* m,

$$N(U(n),m,s) = \binom{n-1}{s-1}\binom{m}{s} = \frac{s}{n}\binom{n}{s}\binom{m}{s} \ .$$

For the special unitary group, again we impose $(p,q)=1$. Given an s-composition of p, $\{n_a\} = (n_1, \dots n_s)$, $\sum_{a=1}^{s} n_a = p$, $n_a > 0$, consider $\{\lambda_a\} = (\lambda_1, \dots, \lambda_s)$ where $\lambda_a \in \{0, \dots, q-1\}$ determine the eigenvalues $e^{\frac{2\pi i \lambda_a}{q}}$ with multiplicity n_a of the corresponding matrix. Arrange the $\binom{q}{s}s!$ possibilities for $\{\lambda_a\}$ in sets of size q given by

$$\{\lambda_a^{(j)}\} = (\lambda_1 + j, \dots, \lambda_s + j), \ \ j = 0, \dots, q-1 \quad \text{(all numbers are understood mod } q\text{)}. \tag{11}$$

The determinant of the matrix x_j corresponding to the j^{th} choice is

$$\exp \frac{2\pi i}{q} \left(\sum_{a=1}^{s} n_a(\lambda_a + j) \right) .$$

Since $(p,q)=1$ and

$$\sum_a n_a(\lambda_a + j) - \sum_a n_a(\lambda + j + 1) = p,$$

exactly one of the q matrices has determinant 1. Since so far neither the λ_a's nor the n_a's have been ordered, once we arrange the eigenvalues to have increasing λ_a's, each matrix would appear $s!$ times. Dividing by $s!q$, we obtain the following formula.

Theorem 2.4 *For* $(p,q)=1$,

$$N(SU(p),q,s)=\frac{1}{q}\binom{p-1}{s-1}\binom{q}{s}=\frac{s}{pq}\binom{p}{s}\binom{q}{s}\ . \tag{12}$$

From Theorems 2.2 and 2.4, we deduce an intriguing symmetry between p and q.

Corollary 2.5 *For* $(p,q)=1$,

$$\begin{aligned} N(SU(p),q) &= N(SU(q),p);\\ N(SU(p),q,s) &= N(SU(q),p,s). \end{aligned}$$

This symmetry has implications involving dualities of gauge theories; see [9].

It is clear that for any G and m, we must have

$$\sum_s N(G,m,s)=N(G,m). \tag{13}$$

Since $N(G,m,s)=0$ when $s>m$, the sum is finite. Applying equation (13) to $G=U(n)$ gives

$$\sum_s \binom{n-1}{s-1}\binom{m}{s}=\binom{n+m-1}{m-1}, \tag{14}$$

which is a special case of the Chu-Vandermonde identity [17].

We may also obtain both $N(SU(n),m)$ and $N(SU(n),m,s)$ without requiring $(n,m)=1$ via a generating function approach. Let

$$F(x,t,u)=\prod_{k=0}^{m-1}\left(1+u\sum_{a=1}^{\infty}(t^k x)^a\right).$$

A typical term in $F(x,t,u)$ is

$$x^{\sum n_k}\, t^{\sum k n_k}\, u^s,$$

where n_k, $k=0,\ldots,m-1$ are nonnegative integers and s is the number of k's for which $n_k\neq 0$. If $\sum n_k=n$ and $\sum kn_k\equiv 0 \bmod m$ then the sequence $\{n_k\}$ corresponds to a diagonal $SU(n)$ matrix of order m with s distinct eigenvalues. To pick out the terms in $F(x,t,u)$ for which $\sum kn_k\equiv 0 \bmod m$, let $\zeta=\exp 2\pi i/m$ and recall

$$\frac{1}{m}\sum_{j=0}^{m-1}\zeta^{jb}=\begin{cases} 1, & \text{if } m|b \\ 0, & \text{else}\end{cases}\ ,$$

so

$$G(x,u) = \frac{1}{m}\sum_{j=0}^{m-1} F(x,\zeta^j,u) = \sum_{n,s} N(SU(n),m,s)x^n u^s.$$

Rewriting

$$1 + u\sum_{a=1}^{\infty}(t^k x)^a = (1-u) + \frac{u}{1-t^k x} = \frac{1-t^k(1-u)x}{1-t^k x},$$

we have

$$G(x,u) = \frac{1}{m}\sum_{j=0}^{m-1}\prod_{k=0}^{m-1}\frac{1-\zeta^{kj}(1-u)x}{1-\zeta^{kj}x}.$$

For ζ^j a primitive d^{th} root of unity, we have the factorization $1-x^d = \prod_{l=0}^{d-1}(1-\zeta^{jl}x)$. Since ζ^j, $j = 0,\ldots,m-1$ is a primitive d^{th} root of unity $\phi(d)$ times, where $\phi(d)$ is Euler's function, we have

$$G(x,u) = \frac{1}{m}\sum_{d|m}\phi(d)\frac{\left[1-(1-u)^d x^d\right]^{m/d}}{(1-x^d)^{m/d}}.$$

Expanding in binomial series gives

$$G(x,u) = \frac{1}{m}\sum_{d|m}\phi(d)\sum_{k,j,l\geq 0}\binom{k+m/d-1}{k}\binom{m/d}{j}\binom{jd}{l}(-1)^{j+l}\,x^{d(k+j)}u^l.$$

Setting $d(k+j)=n$ and $l=s$ yields the next theorem.

Theorem 2.6 *For any positive integers $n,m,$ and s,*

$$N(SU(n),m,s) = \frac{1}{m}\sum_{d|(n,m)}\sum_{j\geq 0}\phi(d)\binom{n/d+m/d-j-1}{n/d-j}\binom{m/d}{j}\binom{jd}{s}(-1)^{j+s}.$$

We may deduce from Theorems 2.6 and 2.4 that for $(p,q)=1$,

$$\frac{1}{q}\sum_{j\geq 0}\binom{p+q-j-1}{p-j}\binom{q}{j}\binom{j}{s}(-1)^{j+s} = \frac{s}{pq}\binom{p}{s}\binom{q}{s}.$$

For $N(SU(n),m)$ we apply equation (13), or equivalently set $u=1$ in $G(x,u)$, and obtain (see also [11]) the next result.

Theorem 2.7 *For any positive integers n and m,*

$$N(SU(n),m) = \frac{1}{m}\sum_{d|(n,m)}\phi(d)\binom{n/d+m/d-1}{n/d}.$$

3 Counting conjugacy classes in symplectic groups

The diagonal elements of $U(n)$ and $SU(p)$ that we counted in the previous section belong to the maximal tori of those groups. For $\mathrm{Sp}(n) \equiv \mathrm{Sp}(n, \mathbf{C}) \cap U(2n)$, the maximal torus is

$$T_{\mathrm{Sp}(n)} = \left\{ (e^{2\pi i\theta_1}, \ldots, e^{2\pi i\theta_n}, e^{-2\pi i\theta_1}, \ldots, e^{-2\pi i\theta_n}) \right\}. \tag{15}$$

Since $\mathrm{Sp}(n)$ is compact and connected, we have $\mathrm{Sp}(n) = \bigcup_{x\in G} \; xT_{\mathrm{Sp}(n)}x^{-1}$. Hence, every element $x \in G$ can be conjugated into the torus, so every conjugacy class has elements in $T_{\mathrm{Sp}(n)}$. Any two elements x and x' of $T_{\mathrm{Sp}(n)}$ that differ only by $\theta_l' = -\theta_l$ for some l's are in the same conjugacy class; the symplectic matrix $E_{l,n+l} - E_{n+l,l}$, where $(E_{ab})_{cd} = \delta_{ac}\delta_{bd}$, conjugates them. So a conjugacy class is fully determined by n values of θ_l restricted to $[0, 1/2]$.

Conjugacy classes of elements of order m have a unique element in $T_{\mathrm{Sp}(n)}$ such that $\theta_l \in \frac{1}{m}(0, 1, \ldots, [\frac{m}{2}])$ and the θ_l are nondecreasing as i runs from 1 to n. Following the arguments leading to Theorem 2.1, and noting that here we have weak $([\frac{m}{2}]+1)$-compositions of n, rather than weak m-compositions of n, we obtain our next theorem.

Theorem 3.1 *For any positive integers n and m,*

$$N(\mathrm{Sp}(n), m) = \binom{n + [\frac{m}{2}]}{[\frac{m}{2}]}.$$

We now consider $N(\mathrm{Sp}(n), m, s)$ where s denotes the number of complex conjugate pairs of eigenvalues. Following the arguments leading to Theorem 2.3, but replacing m by $([\frac{m}{2}] + 1)$, we obtain the next result.

Theorem 3.2 *For any positive integers n, m, and s,*

$$N(\mathrm{Sp}(n), m, s) = \binom{n-1}{s-1}\binom{[\frac{m}{2}]+1}{s}.$$

4 Counting conjugacy classes in orthogonal groups

The maximal tori of the different orthogonal groups depend on the parity of l in $SO(l)$ or $O(l)$ and also on whether the orthogonal group is special or not:

$$T_{SO(2n)} = \{\mathrm{diag}(A(\theta_1), A(\theta_2), \ldots, A(\theta_n))\}, \tag{16}$$

$$T_{SO(2n+1)} = \{\mathrm{diag}(A(\theta_1), A(\theta_2), \ldots, A(\theta_n), 1)\}, \tag{17}$$

$$T_{O(2n)} = \left\{ \begin{array}{l} T_{1,even} = \mathrm{diag}(A(\theta_1), A(\theta_2), \ldots, A(\theta_n)) \\ T_{2,even} = \mathrm{diag}(A(\theta_1), A(\theta_2), \ldots, A(\theta_{n-1}), B) \end{array} \right\}, \tag{18}$$

$$T_{O(2n+1)} = \left\{ \begin{array}{l} T_{1,\mathrm{odd}} = \mathrm{diag}(A(\theta_1), A(\theta_2), \ldots, A(\theta_n), 1) \\ T_{2,\mathrm{odd}} = \mathrm{diag}(A(\theta_1), A(\theta_2), \ldots, A(\theta_n), -1) \end{array} \right\}, \tag{19}$$

where

$$A(\theta) = \begin{pmatrix} \cos 2\pi\theta & \sin 2\pi\theta \\ -\sin 2\pi\theta & \cos 2\pi\theta \end{pmatrix} \; ; \; B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{20}$$

In equations (18) and (19), the maximal torus is made of two parts. The first has elements of determinant 1 and is identical to the tori of equations (16) and (17), respectively; the second has elements of determinant -1.

The identity

$$BA(\theta)B^{-1} = A(-\theta) \tag{21}$$

will become useful below.

With the maximal tori defined as above, every element of the orthogonal group can be conjugated to the torus, so each conjugacy class has a nonempty intersection with the group's maximal torus.

The counting of conjugacy classes depends on the parity of the order m of the elements, so we treat the odd and even cases separately.

4.1 Odd m

We begin with $N(SO(2n+1), m)$. The block-diagonal matrix $\mathrm{diag}(B, I_{2n-2}, -1)$ is an element of $SO(2n+1)$ and equation (21) shows that conjugation by it takes $x \in T_{SO(2n+1)}$ to $x' \in T_{SO(2n+1)}$ where $\theta_1' = -\theta_1$ and the other θ_l remain the same. Similarly, two elements x and x' of $T_{SO(2n+1)}$ that differ by $\theta_l' = -\theta_l$ for any $l = 1, \ldots, n$ belong to the same conjugacy class. We therefore consider only elements of $T_{SO(2n+1)}$ with $\theta_l \in [0, 1/2]$ as we did for the symplectic case. As before, we order the θ_l to be nondecreasing with l.

For elements of order m, we have $\theta_l \in \frac{1}{m}(0, 1, \ldots, [\frac{m}{2}])$. So $N(SO(2n+1), m)$ is the number of weak $([\frac{m}{2}]+1)$-compositions of n.

Theorem 4.1 *For any positive integer n and any odd integer $m = 2k+1$,*

$$N(SO(2n+1), m) = \binom{n + \left[\frac{m}{2}\right]}{\left[\frac{m}{2}\right]}.$$

For $O(2n+1)$, there are two conjugacy classes of maximal tori, i.e. $T_{SO(2n+1)}$, and $T_{2,\mathrm{odd}}$ in equation (19). However, all elements of $T_{2,\mathrm{odd}}$ have even order, so none has order $m = 2k+1$. Therefore, the number of conjugacy classes of elements of odd order in $O(2n+1)$ is the same as that for $SO(2n+1)$, so we get the following result.

Theorem 4.2 *For any positive integer n and any odd integer $m = 2k+1$,*

$$N(O(2n+1), m) = \left(\begin{array}{c} n + \left[\frac{m}{2}\right] \\ \left[\frac{m}{2}\right] \end{array} \right).$$

For $O(2n)$, again $T_{2,even} \in T_{O(2n)}$ does not play a role when m is odd. Also, the block diagonal matrix $\mathrm{diag}(B, I_{2n-2})$ is an element of $O(2n)$, so the results for $O(2n+1)$ and $O(2n)$ are the same.

Theorem 4.3 *For any positive integer n and any odd integer $m = 2k+1$,*

$$N(O(2n), m) = \left(\begin{array}{c} n + \left[\frac{m}{2}\right] \\ \left[\frac{m}{2}\right] \end{array} \right).$$

Things become more subtle for $SO(2n)$: $\mathrm{diag}(B, I_{2n-2})$ has determinant -1 so it is not an element of $SO(2n)$. Therefore, it is no longer the case that if $x, x' \in T_{SO(2n)}$ differ only by $\theta'_i = -\theta_i$ for some i's then x and x' are necessarily in the same conjugacy class. However, the block diagonal matrix $\mathrm{diag}(B, B, I_{2n-4})$ *is* in $SO(2n)$, so if $\theta'_l = -\theta_l$ for an even number of l's, x and x' are in the same conjugacy class.

There are two cases to consider: $\theta'_1 = \theta_1 = 0$ and $\theta_l \neq 0$ for all l. In the first case, $A(\theta_1) = A(\theta'_1) = I_2$, and if $\theta'_l = -\theta_l$ for any additional $l \geq 2$ (not necessarily an even number of times), then x and x' are in the same conjugacy class. The number of conjugacy classes that are represented by elements of $T_{SO(2n)}$ with $\theta_1 = 0$ is the number of weak $\left(\left[\frac{m}{2}\right]+1\right)$-compositions of $n-1$. In the second case $\theta_l \neq 0$ for all l, the number of classes is the number of weak $\left[\frac{m}{2}\right]$-compositions of n; since here, flipping the sign of one θ_l, say $\theta'_1 = -\theta_1$ and leaving the others fixed lands in a different conjugacy class, we multiply the number by two to include all the classes. This leads to the following theorem.

Theorem 4.4 *For any positive integer n and any odd integer $m = 2k + 1$,*

$$N(SO(2n), m) = \binom{n + \left[\frac{m}{2}\right] - 1}{\left[\frac{m}{2}\right]} + 2\binom{n + \left[\frac{m}{2}\right] - 1}{\left[\frac{m}{2}\right] - 1} = \binom{n + \left[\frac{m}{2}\right] - 1}{\left[\frac{m}{2}\right]} \frac{n + m - 1}{n} .$$

We now turn to $N(SO(2n+1), m, s)$, where as for the symplectic groups, s denotes the number of distinct conjugate pairs of eigenvalues of the elements. For all the orthogonal groups, there are n θ_l's and $\binom{n-1}{s-1} = \frac{s}{n}\binom{n}{s}$ ways to partition them into s nonzero parts. There are $\left[\frac{m}{2}\right] + 1$ possible values for the θ_i. The same is true for $O(2n+1)$, and $O(2n)$, yielding the next result.

Theorem 4.5 *For any positive integers n and s, and any odd integer $m = 2k + 1$,*

$$N(SO(2n+1), m, s) = N(O(2n+1), m, s) = N(O(2n), m, s) = \frac{s}{n}\binom{n}{s}\binom{\left[\frac{m}{2}\right] + 1}{s}.$$

The above derivation does not apply to $SO(2n)$ because as before, some classes need to be counted twice due to the absence of (B, I_{2n-2}) in $SO(2n)$. First, we divide the n eigenvalue pairs into s nonzero parts (s-compositions of n). In choosing the s eigenvalues out of the $\left[\frac{m}{2}\right] + 1$ possibilities, we differentiate the cases where $\theta_1 = 0$, which we count once, from the cases where $\theta_1 \neq 0$, which we need to count twice to account for $\theta'_1 = -\theta_1$, $\theta'_l = \theta_l$, $l > 1$ which is in a distinct conjugacy class. We get the following formula.

Theorem 4.6 *For any positive integers n and s and any odd integer $m = 2k + 1$,*

$$\begin{aligned} N(SO(2n), m, s) &= \binom{n-1}{s-1}\left[\binom{\left[\frac{m}{2}\right]}{s-1} + 2\binom{\left[\frac{m}{2}\right]}{s}\right] \\ &= \frac{s}{n}\binom{n}{s}\binom{\left[\frac{m}{2}\right]}{s}\frac{m+1-s}{\left[\frac{m}{2}\right]+1-s} . \end{aligned}$$

4.2 Even m

Unlike the case for odd m, here we will have to consider T_2 in both $O(2n)$ and $O(2n+1)$. There will also be changes from the odd m case due to the fact that $\theta_l = 1/2$, corresponding to $A(\theta_l) = -I_2$, can appear.

For $SO(2n+1)$, we have essentially the same as we did for odd m, i.e. weak $\left(\frac{m}{2} + 1\right)$-compositions of n.

Theorem 4.7 *For any positive integer n and any even integer $m = 2k$,*

$$N(SO(2n+1), m) = \binom{n + \frac{m}{2}}{\frac{m}{2}} .$$

For $O(2n+1)$, we have to consider conjugacy classes with elements whose determinant is -1, that is, elements of the second part $T_{2,\text{odd}}$ of the torus $T_{O(2n+1)}$, not just the elements of determinant 1 as we did previously. But the counting is exactly the same as in $T_{1,\text{odd}}$, so the next theorem follows.

Theorem 4.8 *For any positive integer n and any even integer $m = 2k$,*

$$N(O(2n+1), m) = 2\begin{pmatrix} n + \frac{m}{2} \\ \frac{m}{2} \end{pmatrix}.$$

Turning to $O(2n)$, we note that elements in $T_{2,even}$ have only $n-1$ θ_l's. Other than that, the counting is the same as before, yielding the next result.

Theorem 4.9 *For any positive integers n and any even integer $m = 2k$,*

$$\begin{aligned} N(O(2n), m) &= \begin{pmatrix} n + \frac{m}{2} \\ \frac{m}{2} \end{pmatrix} + \begin{pmatrix} n + \frac{m}{2} - 1 \\ \frac{m}{2} \end{pmatrix} \\ &= \begin{pmatrix} n + \frac{m}{2} - 1 \\ \frac{m}{2} \end{pmatrix} \frac{4n+m}{2n}. \end{aligned}$$

For $SO(2n)$, again we need to be careful since $\theta_l' = \pm\theta_l$ does not always mean x and x' are in the same conjugacy class. Only when at least one of the θ_l is 0 or 1/2, so that $A(\theta_l) = \pm I_2$ for that l, which commutes with B, does $\theta_l' = \pm\theta_l$ mean x and x' are in the same conjugacy class. If no θ_l is 0 or 1/2 then if say $\theta_1' = -\theta_1$ and $\theta_l' = \theta_l$, $l > 1$, we have a different conjugacy class for x and x'. The number of conjugacy classes such that at least one θ_l is 0 or 1/2 is the number of weak $\left(\frac{m}{2}+1\right)$-compositions of $n-1$ (where we have fixed $\theta_1 = 0$) plus the number of weak $\left(\frac{m}{2}\right)$-compositions of $n-1$ (where we do not allow $\theta_l = 0$ and we require $\theta_l = 1/2$ for some l). The number of conjugacy classes where no θ_l is 0 or 1/2 is twice the number of weak $\left(\frac{m}{2}-1\right)$-compositions of n. After some algebra we obtain the next result.

Theorem 4.10 *For any positive integer n and any even integer $m = 2k$,*

$$N(SO(2n), m) = \begin{pmatrix} n + \frac{m}{2} \\ \frac{m}{2} \end{pmatrix} + \begin{pmatrix} n + \frac{m}{2} - 2 \\ \frac{m}{2} - 2 \end{pmatrix}.$$

For $N(SO(2n+1), m, s)$, we have the same calculation as for odd m, and for $N(O(2n+1), m, s)$, we simply double the result to account for the elements in $T_{2,\text{odd}}$, giving the following formulas.

Theorem 4.11 *For any positive integers n and s and any even integer $m = 2k$,*

$$N(SO(2n+1), m, s) \;=\; \frac{s}{n} \begin{pmatrix} n \\ s \end{pmatrix} \begin{pmatrix} \frac{m}{2}+1 \\ s \end{pmatrix};$$

$$N(O(2n+1), m, s) \;=\; \frac{2s}{n} \begin{pmatrix} n \\ s \end{pmatrix} \begin{pmatrix} \frac{m}{2}+1 \\ s \end{pmatrix}.$$

Next is $O(2n)$, where $T_{2,even}$ has only $n-1$ θ_l's, so the contribution from $T_{2,even}$ differs from that from $T_{1,even}$ by replacing n with $n-1$. After some algebra we get the following theorem.

Theorem 4.12 *For any positive integers n and s and any even integer $m = 2k$,*

$$N(O(2n), m, s) = \frac{2n-s-1}{n-s} \binom{n-2}{s-1} \binom{\frac{m}{2}+1}{s}.$$

For $SO(2n)$, for each s-composition of n, the number of conjugacy classes of $T_{SO(2n)}$ with $\theta_l \neq 0, 1/2$ for all l is $\binom{\frac{m}{2}-1}{s}$ and the number of conjugacy classes with at least one $\theta_l = 0, 1/2$ is the sum of $\binom{\frac{m}{2}}{s-1}$, which gives the number of conjugacy classes with $\theta_1 = 0$, and $\binom{\frac{m}{2}-1}{s-1}$ which gives the number of conjugacy classes with $\theta_l \neq 0$ $\forall l$ and $\theta_l = 1/2$ for some l. As before, we multiply the number for $\theta_l \neq 0, 1/2$ by 2, and add the rest. After some algebra, we have our final result.

Theorem 4.13 *For any positive integers n and s and any even integer $m = 2k$,*

$$N(SO(2n), m, s) = \begin{pmatrix} n-1 \\ s-1 \end{pmatrix} \left[\begin{pmatrix} \frac{m}{2}+1 \\ s \end{pmatrix} + \begin{pmatrix} \frac{m}{2}-1 \\ s \end{pmatrix} \right].$$

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