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MINIMAL ORDER DISCRETE WIENER FILTER IN THE PRESENCE OF COLORED MEASUREMENT NOISE

by

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ABSTRACT

The minimal order Wiener filter is constructively derived for a linear, time invariant, discrete system when the measurements are corrupted by both white and colored noise. It is shown that as all noise vanishes the steady-state error covariance associated with the filter converges to a null matrix. No Luenberger observer is used in combination with the filter.

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I. INTRODUCTION

Here we consider the discrete time system described by,

$$x_{k+1} = Fx_k + Bw_k, \quad k\geq 0 \tag{1.1}$$

$$z_k = Hx_k + v_k, \quad k\geq 0 \tag{1.2}$$

where the state vector x_k belongs to $\mathcal{R}^n$, the measurement vector z_k belongs to $\mathcal{R}^m$ and the noise vector w_k belongs to $\mathcal{R}^q$. The Gaussian random variable x_0 has mean $\bar{x}_0$ and the stationary white noise processes $\{w_k\}$ and $\{v_k\}$ are mutually independent and independent of x_0 and possess the statistics, $w_k = N(0,I_q)$, $v_k = N(0,R)$, where R is positive semidefinite and symmetric with rank $r<m$ and I_q is the identity in $\mathcal{R}^q$. The matrix H has rank q, the pair (H,F) is detectable and we seek the unbiased minimum mean squared error estimate $\hat{x}_{k+1}$ (Wiener filter) given the measurements $(z_k,z_{k-1},\ldots,z_{k-k_0})$ as $k_0 \to \infty$.

Systems of the form (1.1)-(1.2) having singular R arise for example when $\{v_k\}$ is not a white noise process, but a stionary Markov process. If the process noise can be described by

$$v_{k+1} = Av_k + \eta_k \ , \tag{1.3}$$

where $\{\eta_k\}$ is a stationary white noise process then the process noise $\{v_k\}$ is stationary Markov (colored). In this case the state vector may be augmented to x_k^a where

$$x_k^a = (x_k', \ v_k')' \ ,$$

and the augmented system has the form (1.1)-(1.2) with F replaced by

$$\begin{pmatrix} F & 0 \\ 0 & A \end{pmatrix},$$

H replaced by (H,I) and R replaced by a null matrix. If some measurements are corrupted by only white noise while other measurements are corrupted by colored noise then a non-zero, but singular measurment noise intensity occurs in the extended system. If the original pair (H,F) is detectable, then the augmented pair,

$$\left((H,I), \begin{pmatrix} F & 0 \\ 0 & A \end{pmatrix} \right)$$

is also detectable when the Markov process (1.3) is asymptotically stable.

Discrete Kalman filters for systems whose measurements contain colored noise were first discussed by Bryson and Johansen [1] and by Bryson and Henrikson [2]. In these papers only a restricted case was discussed, which was essentially the case when the restriction of the matrix HBB'H' to the kernel of R is nonsingular. More recently other authors (e.g. [3]-[7]) have considered the optimal, discrete state estimation problem in the presence of colored measurement noise, but they primarily concentrated upon obtaining a minimal order combined observer-estimator, where the minimal order was unknown but bounded below by n-m+r. References [1]-[7] all deal with finite horizon estimation. Here we obtain a steady-state (infinite horizon) state estimator whose order is exactly n-m+r, and our estimator is asymptotically stable on the disturbable subspace, i.e. on the subspace of modes which are disturbable by process noise. We obtain an optimal state estimator of lowest possible order without the addition of an observer, and we show that if R is null (all measurements are exact) then the covariance matrix of the state estimation error approaches a null matrix as the process noise intensity vanishes. This paper is a companion paper to [8], where the continuous version of this problem is discussed.

II. A FILTER WITH A FIRST ORDER SINGULARITY

Let $T_0 = (U_0', W_0')$ be a nonsingular coordinate transformation in the space of measurement variables, and suppose that

$$U_0RU_0' > 0, \quad W_0R = 0.$$

If $R = 0$ then W_0 is I_m and U_0 fails to exist. Define $D_0 = W_0H$ and assume that

$$D_0BB'D_0' > 0. \tag{2.1}$$

In section III we shall abandon assumption (2.1) and consider a more general case. From (1.2) we have

$$z_k^1 \triangleq U_0z_k = U_0(Hx_k + v_k) \tag{2.2}$$

and

$$W_0z_k = D_0x_k \,. \tag{2.3}$$

We shall consider (2.3) as an exact, or quiet, measurement and replace it by its first order forward difference,

$$z_k^2 \triangleq W_0\Delta z_k = D_0[(F-I)x_k + Bw_k]. \tag{2.4}$$

Note that equation (1.1) together with (2.2) and (2.4) define a new optimal filter problem. This problem has a nonsingular measurement noise intensity, but process and measurement noise are correlated. If we would allow one more measurement, namely z_{k+1}, in our original problem then the new optimal filter problem would be equivalent to the original. (Alternatively, we could replace (2.3) by its first order backward difference.)

Select an (n-m+r)xn matrix V_0 so that the matrix (D_0', V_0') has full rank and use this matrix to define a coordinate transformation in the state space. Let $(D_0', V_0')^{-1} = (C_0, E_0)'$. It was shown in [8] that when (H,F) is a detectable pair then V_0 may be found so that the pair $(U_0 H E_0, V_0 F E_0)$ is also detectable. Define a new state variable y_k, of dimension $n_0 = n-m+r$, by

$$y_k = V_0 x_k. \tag{2.5}$$

Then,

$$x_k = C_0 W_0 z_k + E_0 y_k . \tag{2.6}$$

Substitution of (2.5) and (2.6) into (1.1) yields

$$y_{k+1} = V_0 (F E_0 y_k + F C_0 W_0 z_k + B w_k). \tag{2.7}$$

Substitution of (2.6) into (2.2) and (2.4), defining new measurement variables ζ_k' and ζ_k^2, and noting that $D_0 E_0 = 0$ we find,

$$\zeta_k^1 \stackrel{\Delta}{=} U_0 (z_k - H C_0 W_0 z_k) = U_0 (H E_0 y_k + v_k) \tag{2.8}$$

and

$$\zeta_k^2 = D_0 (\Delta x_k - (F-I) C_0 W_0 z_k) = D_0 (F E_0 y_k + B w_k). \tag{2.9}$$

Equation (2.7)-(2.9) together with (2.3) define a new estimation problem in a reduced state space. This new problem is equivalent to the original problem provided $\hat{y}_0 = V_0 \bar{x}_0$. Note that if P_0 is the original initial time state error covariance matrix then $D_0 P_0 = 0$. This condition is merely an expression of the fact that the measurement $W_0 z_0$ is exact.

Furthermore, since $W_0 z_k$ is exact for all integers k then $D_0 P_k = 0$. Thus, the steady-state limit P of the optimal state error covariance matrix P_k must satisfy

$$D_0 P = 0. \tag{2.10}$$

Note that $V_0 P$ is to be determined by the estimator and that once this quantity is determined then so is P.

The filter problem defined by (2.7)-(2.9) with $\bar{y}_0 = V_0 \bar{x}_0$ is equivalent to the original problem, and in the next section we shall show that the optimal estimate $\hat{y}_{k+1}$ is given by

$$\begin{aligned}\hat{y}_{k+1} &= V_0 F E_0 \hat{y}_k + V_0 F C_0 W_0 z_k + K_1(\zeta_k^1 - U_0 H E_0 \hat{y}_k) \\ &\quad + K_2(\zeta_k^2 - D_0 F E_0 \hat{y}_k), \quad \hat{y}_0 = V_0 \bar{x}_0\end{aligned} \tag{2.11}$$

where K_1 and K_2 are found by solving a steady state Riccati equation in $\mathcal{R}^{n_0}$. We shall also show that under a certain nonsingular coordinate transformation, the optimal estimator splits into two parts. One part is defined on the "undisturbable subspace" of state coordinates that cannot be disturbed by process noise, and consists of a deterministic difference equation. The other part is an asymptotically stable estimator defined on the "disturbable subspace", the subspace of modes that are disturbable by process noise.

III. THE SEPARATION OF THE FILTER EQUATIONS

In this section we shall derive the formulas for the filter gains K_1 and K_2 appearing in equation (2.11). We begin by uncorrelating the process and measurement noise and transforming the system into disturbable (controllable) canonical form. To this end we define

$$M_0 = BB'D_0'(D_0BB'D_0')^{-1}$$

and add zero in the form

$$V_0M_0[\zeta_k^2 - D_0(FE_0y_k + Bw_k)]$$

to the right hand side of (2.7). We obtain,

$$y_{k+1} = V_0[(I-M_0D_0)FE_0y_k + FC_0W_0z_k + M_0\zeta_k^2 + (I-M_0D_0)Bw_k]. \qquad (3.1)$$

The process and measurement noise are now uncorrelated in the system described by (3.1) and (2.8)-(2.9). The measurements z_k and ζ_k^2 may be considered as known inputs for equation (3.1). Let j=0 and define

$$F_j = (I-M_jD_j)F, \quad B_j = (I-M_jD_j)B, \quad M_j = BB'D_j'(D_jBB'D_j')^{-1},$$

$$\mathcal{N} = \bigcap_{i=1}^{n_0} \mathrm{Ker}[(V_0B_j)'(E_0'F_j'V_0')^{i-1}],$$

and

$$\mathcal{C} = \langle V_0F_jE_0 | \mathrm{Im}(V_0B_j)\rangle \quad ,$$

where

$$\langle A|\mathrm{Im}\ B\rangle \triangleq \langle A|\mathcal{B}\rangle \triangleq \mathcal{B} + A\mathcal{B} + \ldots + A^{n-1}\mathcal{B} \quad .$$

Then $\mathcal{N}$ is the undisturbable subspace of the pair $(V_0F_jE_0, V_0B_j)$ and $\mathcal{C}$ is the orthogonal complement of $\mathcal{N}$ in $\mathcal{R}^{n_0}$. In a coordinate system compatible with the decomposition $\mathcal{R}^{n_0} = \mathcal{N} \oplus \mathcal{C}$ we have,

$$V_0 = \begin{pmatrix} V_{01} \\ \\ V_{02} \end{pmatrix}, \quad y_k = \begin{pmatrix} y_k^1 \\ \\ y_k^2 \end{pmatrix}, \; E_0 = (E_{01}, E_{02}),$$

$$V_0F_jE_0 = \begin{pmatrix} V_{01}F_jE_{01} & 0 \\ \\ V_{02}F_jE_{01} & V_{02}F_jE_{02} \end{pmatrix}, \quad V_0B_j = \begin{pmatrix} 0 \\ \\ V_{02}B_j \end{pmatrix} .$$

In this coordinate system equation (3.1) becomes,

$$y_{k+1}^1 = V_{01}(F_jE_{01}y_k^1 + FC_0W_0z_k + M_j\zeta_k^2) \tag{3.2}$$

and

$$y_{k+1}^2 = V_{02}(F_jE_{02}y_k^2 + F_jE_{01}y_k^1 + FC_0W_0z_k + M_j\zeta_k^2 + B_jw_k) . \tag{3.3}$$

Equations (2.8) and (2.9) become respectively,

$$\eta_k^1 \triangleq \zeta_k^1 - U_0HE_{01}y_k^1 = U_0(HE_{02}y_k^2 + v_k), \tag{3.4}$$

and

$$\eta_k^2 \triangleq \zeta_k^2 - D_0FE_{01}y_k^1 = D_0(FE_{02}y_k^2 + Bw_k). \tag{3.5}$$

Note that y_k^1 is found from a difference equation which is not corrupted by noise together with the best available estimate $\hat{y}_0^1$ for y_0^1, namely $\hat{y}_0^1 = V_{01}\bar{x}_0$. The pair $(V_{02}F_0E_{02}, V_{02}B_0)$ is controllable and we show in the appendix that the pair $(E_{02}'(H'U_0', F'D_0')', V_{02}F_jE_{02})$ is detectable.

Thus, by standard theory [9] the optimal estimator is described by,

$$\hat{y}^1_{k+1} = V_{01}(F_jE_{01}\hat{y}^2_k + FC_0W_0z_k + M_j\zeta^2_k), \quad \hat{y}^1_0 = V_{01}\bar{x}_0 \tag{3.6}$$

$$\hat{y}^2_{k+1} = V_{02}(F_jE_{02}\hat{y}^2_k + F_jE_{01}\hat{y}^1_k + FC_0W_0z_k + M_j\zeta^2_k$$

$$+ \bar{K}_1(\eta^1_k - U_0HE_{02}\hat{y}^2_k) + \bar{K}_2(\eta^2_k - D_jFE_{02}\hat{y}^2_k) \ ,$$

$$\hat{y}^2_0 = V_{02}\bar{x}_0 \ , \tag{3.7}$$

where

$$(\bar{K}_1,\bar{K}_2) = \bar{K} = V_{02}F_jE_{02}\Sigma E'_{02}(H'U'_0, \ F'D'_j)\bar{R}_j^{-1}, \tag{3.8}$$

$$\bar{R}_j = \begin{pmatrix} U_0(HE_{02}\Sigma E'_{02}H' + R)U'_0 & U_0HE_{02}\Sigma E'_{02}F'D'_j \\ D_jFE_{02}\Sigma E'_{02}H'U'_0 & D_j(FE_{02}\Sigma E'_{02}F' + BB')D'_j \end{pmatrix} , \tag{3.9}$$

Σ is the unique positive definite symmetric solution of the steady state Riccati equation,

$$\Sigma = V_{02}(F_jE_{02}\Sigma E'_{02}F'_j + B_jB'_j)V'_{02} - \bar{K}\bar{R}_j\bar{K}' , \tag{3.10}$$

and the matrix

$$V_{02}(F_j - \bar{K}_1U_0H - \bar{K}_2D_jF)E_{02} \text{ is stable.} \tag{3.11}$$

Note that equations (3.2)-(3.5) describe the same system as equations (2.7)-(2.9) in appropriate coordinates, and that the variables in (3.2)-(3.5) may be manipulated to compare K_i with $\bar{K}_i$, i=1,2. If fact, in our canonical coordinate system (3.2) and (3.3) together are exactly the same as (2.7) and the terms in (3.6)-(3.7) can be reassociated to yield

$$\hat{y}_{k+1} = V_0FE_0\hat{y}_k + V_0FW_0z_k + \begin{bmatrix} 0 \\ K_1 \end{bmatrix} (\zeta_k^1 - U_0HE_0\hat{y}_k)$$

$$+ \begin{bmatrix} V_{01}M_j \\ K_2+V_{02}M_j \end{bmatrix} (\zeta_k^2 - D_jFE_0\hat{y}_k) \ . \tag{3.12}$$

Comparing (2.11) with (3.12) we see that

$$K_1' = (0,\bar{K}_1'), \quad K_2' = (M_j'V_{01}', \bar{K}_2' + M_j'V_{02}'). \tag{3.13}$$

We also find by comparing (3.2) with (3.6) that $V_{01}P$ is null. Hence in our coordinate system we have

$$V_0PV_0' = \begin{pmatrix} 0 & 0 \\ 0 & V_{02}PV_{02}' \end{pmatrix} = E_0V_0PV_0'E_0'$$

$$= E_{02}V_{02}PV_{02}'E_{02}' \ ,$$

and

$$\Sigma = V_{02}PV_{02}' \ . \tag{3.14}$$

We summarize our results in the following theorem.

<u>Theorem 3.1</u>. Given the system described by (1.1)-(1.2) where H has full rank, the matrix pair (H,F) is detectable and where the random variable x_0 and the stationary white noise processes $\{w_k\}$ and $\{v_k\}$ possess the statistics indicated in the introduction. Then there is a state estimator that minimizes the limiting value of the trace of the state error covariance matrix as the number of available measurements increases. There exist matrices C_0, V_0

such that this estimator is described by (2.3) and (3.6) - (3.11) where $j=0$ in an appropriate coordinate system. The order of the estimator is $n_0 = n-m+r$, where n is the number of state variables, m is the number of measurement variables and r is the rank of the measurement noise intensity.

IV. A FILTER HAVING A HIGHER ORDER SINGULARITY

We shall now suppose that (2.1) does not hold. Then there is a nonsingular transformation $T_1 = (U_1', W_1')$ of the measurement space such that U_1D_0B has full rank and

$$W_1D_0B = 0 \quad . \tag{4.1}$$

If D_0B is null then W_1 is an identity matrix and U_1 is absent. Pre-multiplying equation (2.9) by U_1 and pre-multiplying (2.4) by W_1 we find respectively,

$$\zeta_k^{21} \triangleq U_1\zeta_k^1 = U_1D_0(FE_0y_k + Bw_k) \tag{4.2}$$

and

$$W_1W_0\Delta z_k = W_1D_0(F-I)x_k \quad . \tag{4.3}$$

Forming the first difference on both sides of (4.3) and noting (4.1) we find,

$$W_1W_0\Delta^2 z_k = W_1D_0[(F-I)^2 x_k + FBw_k] \quad . \tag{4.4}$$

Applying the coordinate transformation (2.6) to (4.4) we find,

$$\zeta_k^{22} \triangleq W_1(W_0\Delta^2 z_k - D_0(F-I)^2 C_0W_0z_k) = W_1D_0(F-I)[(F-I)E_0y_k + Bw_k] \quad . \tag{4.5}$$

Note that $W_1D_0(F-I)B = W_1D_0FB$. Equations (2.7), (2.8), (4.2) and (4.5) define a new optimal filtering problem. This problem is nonsingular if the measurement noise intensity matrix,

$$\begin{pmatrix} U_1 D_0 BB' D_0' U_1' & U_1 D_0 BB' F' D_0' W_1' \\ W_1 D_0 FBB' D_0' U_1' & W_1 D_0 FBB' F' D_0' W_1' \end{pmatrix} \tag{4.6}$$

is nonsingular. In this event we may proceed as before to find the optimal filter. We note that because we made use of the second difference of a measurement vector then the optimal filter is really a two-step smoother.

Define the matrix D_1 by

$$D_1' = (D_0' U_1', \ F' D_0' W_1') \ .$$

Then the matrix (4.6) is just $D_1 BB' D_1'$, and the condition $D_1 BB' D_1' > 0$ guarantees that we can find the optimal steady state filter for the system described by (2.7) (2.8), (4.2) and (4.5) by standard techniques.

If $D_1 BB' D_1'$ does not have full rank we define U_2, W_2 so that $U_2 W_1 D_0 FB$ has full rank and $W_2 W_1 D_0 F_1 B = 0$ and we difference equation (4.4) pre-multiplied by W_2.

In general, let $T_i = (U_i', W_i')$ be a sequence of nonsingular transformations satisfying,

$$U_i W_{i-1} \cdots W_2 W_1 D_0 F^{i-1} BB' (F')^{i-1} D_0' W_1' W_2' \cdots W_{i-1}' U_i' > 0,$$

$$W_i \cdots W_2 W_1 D_0 F^{i-1} B = 0, \quad i=1,\ldots,j-1$$

and define the matrix D_i by

$$D_i' = (D_0' U_1', F' D_0' W_1' U_2', \ldots, (F')^{i-1} D_0' W_1' \cdots W_{i-1}' U_i), \ i=1,\ldots,j,$$

where U_j is an identity matrix. Note that F may be replaced by (F-I) in the last three expressions. Suppose that

$$D_j BB'D_j' > 0 \; . \tag{4.7}$$

Then by procedures similar to those above, we can find an optimal steady-state filter. We find that the optimal filter is described by

$$\hat{y}_{k+1} = V_0 F E_0 \hat{y}_k + V_0 F C_0 W_0 z_k + K_1(\zeta_k^1 - U_0 H E_0 \hat{y}_k)$$
$$+ K_2 D_j(\zeta_k^2 - F E_0 \hat{y}_k)$$

$$\zeta_k^1 = U_0(H E_0 y_k + v_k) \; , \tag{4.8}$$

and

$$\zeta_k^2 = D_j((F-I)E_0 y_k + Bw_k) \; , \tag{4.9}$$

where K_1 and K_2 satisfy (3.13) and (3.8)-(3.11). The details of the calculations are similar to ones in [8]. If there is no nonnegative integer j for which (4.7) holds then an optimal Wiener filter does not exist. If j does exist then by the Cayley-Hamilton theorem, $j \leq n-1$.

We summarize our results in the following theorem.

Theorem 4.1. Given the same hypothesis as in Theorem 3.1 except that (2.1) is replaced by (4.7). Then $j \leq n-1$ and there exist matrices V_0, C_0 so that a steady-state optimal state estimator exists and can be represented in an appropriate coordinate system by (2.3) together with (3.6)-(3.11).

V. CONVERGENCE OF THE STATE ERROR COVARIANCE AS THE PROCESS AND MEASUREMENTS BECOME QUIET

Here we consider the steady-state optimal estimation problem defined by

$$x_{k+1} = Fx_k + \epsilon Bw_k, \quad k \geq 0 \tag{5.1}$$

$$z_k = Hx_k, \; k \geq 0 \tag{5.2}$$

where a small parameter ϵ multiplies the process noise. The statistics of x_0 and $\{w_k\}$ are the same as before and we again suppose that H has full rank m and that (H,F) is a detectable pair. We shall show that as $\epsilon \to 0$ the state error covariance matrix converges to a null matrix. For simplicity of exposition we shall assume

$$D_0BB'D_0' = HBB'H' > 0 \; .$$

If this is not true, but $D_kBB'D_k' > 0$ for some positive integer k then obvious modifications are to be made to the proof.

We need be concerned here only with $\Sigma = V_{02}PV_{02}'$, defined in (3.14). For the problem of this section, repeating the procedure of section 3 we find that the reduced order system is given by

$$y_{k+1}^1 = V_{01}(F_0E_{01}y_k^1 + FC_0z_k + M_0\zeta_k) \tag{5.3}$$

$$y_{k+1}^2 = V_{02}(F_0E_{02}y_k^2 + F_0E_{01}y_k^1 + FC_0z_k + M_0\zeta_k + \epsilon B_0w_k) \tag{5.4}$$

$$\eta_k = H(FE_{02}y_k^2 + \epsilon Bw_k), \tag{5.5}$$

where

$$\zeta_k = H(FE_{01}y_k^1 + FE_{02}y_k^2 + \epsilon Bw_k). \tag{5.6}$$

Equations (5.3) - (5.6) are the same as equations (3.2), (3.3), (3.5) and (2.9) with D_0 replaced by H, B replaced by ϵB and W_0 by I. Equations (2.8) and (3.4) are absent since all original measurements are quiet. The Wiener filter for the system is given by

$$\hat{y}^1_{k+1} = V_{01}(F_0E_{01}\hat{y}^1_k + FC_0z_k + M_0\zeta_k), \tag{5.7}$$

$$\hat{y}^2_{k+1} = V_{02}(F_0E_{02}\hat{y}^2_k + F_0E_{01}\hat{y}^1_k + FC_0z_k + M_0\zeta_k)$$

$$+ K(\eta_k - HFE_{02}\hat{y}^2_k) \quad , \tag{5.8}$$

where

$$K = V_{02}F_0E_{02}\bar{\Sigma}E'_{02}F'H'[H(FE_{02}\bar{\Sigma}E'_{02} + BB')H']^{-1}, \tag{5.9}$$

$$\bar{\Sigma} = \epsilon^{-2}\Sigma_\epsilon \; ,$$

Σ_ϵ is the steady state error covariance matrix and $\bar{\Sigma}$ is the unique symmetric, positive definite solution of the steady-state Riccati equation,

$$\bar{\Sigma} = V_{02}\{(F_0E_{02}\bar{\Sigma}E'_{02}F'_0 + B_0B'_0) \tag{5.10}$$

$$- F_0E_{02}\bar{\Sigma}E'_{02}F'H'[H(F\Sigma_{02}\bar{\Sigma}E'_{02}F' + BB')H']^{-1}HFE_{02}\bar{\Sigma}E'_{02}F'\}V'_{02} \quad .$$

Since none of the coefficients in (5.9) or in (5.10) depend on ϵ, both K and $\bar{\Sigma}$ are independent of ϵ. Hence,

$$\lim_{\epsilon\to 0} \Sigma_\epsilon = \lim_{\epsilon\to 0} \epsilon^2\bar{\Sigma} = 0 \quad . \tag{5.11}$$

We summarize our results in the following theorem.

<u>Theorem 5.1.</u> Given the steady-state optimal estimation problem defined by (1.1)-(1.2), where the independent Gaussian random variable x_0 and the

random processes $\{v_k\}$ and $\{w_k\}$ are mutually independent and stationary with the statistics described above. We assume that H has full rank and that the matrix pair (H,F) is detectable. If $D_k BB'D_k' > 0$ for some nonnegative integer k the steady-state error covariance matrix Σ described by (3.8)-(3.10) converges to a null matrix as all noise vanishes.

VI. CONCLUSIONS

We have derived a discrete Wiener filter for a stationary linear system when some measurements are corrupted by colored noise and others by white noise. We have shown that in the limit as all noise disappears the steady-state error covariance matrix resulting from the filter approaches a null matrix. Our treatment includes systems whose measurements are corrupted by asymptotically stable Markov noise.

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APPENDIX

Here we shall show that if the matrix pair (U_0HE_0, V_0FE_0) is detectable then so is the pair

$$((H'U_0', F'D_0')'E_{02}, \quad V_{02}F_0E_{02}) \tag{A.1}$$

where

$$E_0 = (E_{01}, E_{02}) \text{ and } V_0' = (V_{01}', \quad V_{02}').$$

Since (U_0HE_0, V_0FE_0) is a detectable pair there exists a matrix G such that $V_0FE_0 - GU_0HE_0$ is stable (i.e. all its eigenvalues are strictly interior to the unit circle). If $K_1=G$ and $K_2 = -V_0M_0$ then

$$V_0F_0E_0 - (K_1,K_2)(H'U_0', F'D_0')'E_0 = V_0FE_0 - GU_0HE_0$$

Thus, the matrix pair

$$((H'U_0', F'D_0')'E_0, V_0F_0E_0) \tag{A.2}$$

is detectable and the unstable modes of $E_0'F_0'V_0'$ lie in the controllable subspace of the pair, $(E_0'F_0'V_0', \quad E_0'(H'U_0', F'D_0'))$. Let $\mathcal{X}$ denote the vector space $\mathcal{R}^{n_0}$, where $n_0 = n-m+r$, and let $\bar{\mathcal{X}}$ denote the reduced space $\mathcal{X}$ (mod $\mathcal{N}$) where $\mathcal{N}$ is defined in section 3. Let T' denote the canonical projection of $\mathcal{X}$ on $\bar{\mathcal{X}}$, let $\bar{F}_0'$ denote the unique map induced in $\bar{\mathcal{X}}$ by $E_0'F_0'V_0'$ and let $\bar{B}_0'$ denote the unique map defined by $B_0'V_0' = \bar{B}_0'T'$. Then the matrix pair, $(\bar{F}_0', T'E_0'(H'U_0', F'D_0'))$ is stabilizable and the matrix pair $(\bar{B}_0', \bar{F}_0')$ is observable (see [10]). But $\bar{F}_0 = V_{02}F_0E_{02}$, $\bar{B}_0 = V_{02}B_0$, and $[T'E_0'(H'U_0', F'D_0')]' = (H'U_0', F'D_0')'E_{02}$. Thus, the pair (A.1) is detectable and in addition, the pair $(\bar{F}_0, \bar{B}_0)$ is controllable. These two properties allow us to apply standard theory to the system described by (3.2)-(3.5).