

**Equilibrium Characterization for Resource
Allocation Games on Single-Path Serial Networks**

by

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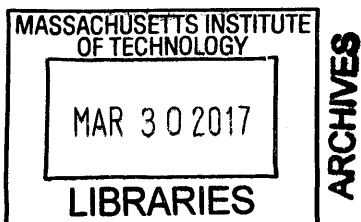
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Abstract

The Resource Allocation Game we examined in this work is a strategic interaction where a principal distribute an infinitely divisible good among different agents based on their specific valuations of said good. The distribution is done by a particular scheme first studied by Kelly [15] with no price-discrimination. In a further study by Johari and Tsitsiklis [13], they aim to distribute the link capacities of a network among different users. The authors prove existence of a unique Nash equilibrium (NE) for the base case of a single link, but for a general network only existence is proven, leaving open questions about uniqueness. In this study we characterize the NE for a distinct structure of networks, the single-path serial network.

The problem is tackled gradually. First, we give explicit solutions for the case with n players with affine utility functions on a single arc. Next for networks with different arc capacities and all players interested in the same path within the network, uniqueness of the NE is proved. Moreover the NE is characterized by a variational inequality that correspond to the first-order conditions of an optimization problem. Thereupon, for the case where players might have different origin-destination pairs without arcs in common, uniqueness of the NE in terms of flow is again proved.

Last but not least, we propose a sequential extension of the scheme. In this framework, players do not act simultaneously, but in a given order of precedence. For the base case of one arc and two players with linear utilities, we obtained an explicit Subgame Perfect Equilibrium. In addition, we get a price of anarchy better than the one obtained for the simultaneous case.

We propose some thoughts about the Transportation analysis application of this type of networks for liner shipping and highways, that is to say situations where there is a single-path of interest for every player.

Thesis Supervisor: Saurabh Amin

Title: Robert N. Noyce Career Development Assistant Professor

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Chapter 1

Resource Allocation Game

1.1 Conceptual framework

We framed the present work within Game Theory, which is the methodology of using applied mathematical tools to model and analyze situations of interactive decision making. Even though this discipline have connections with results from beforehand, e.g. Cournot competition from 19th century [7], its establishment as such occurs in 1944 by the publication of Theory of Games and Economic Behavior, authored by mathematician John von Neumann and economist Oskar Morgenstern [27]. For an introduction from basic to advanced game-theoretic concepts, see Fudenberg and Tirole [10]. A game, or a strategic interaction, is a situation where

- (a) There are several persons, in a broad sense. It means that these persons could be humans, firms, animals, softwares, et cetera. They are individuals, or groups of individuals, and are called *players*.
- (b) Each of these players has to do something. They have to choose their *strategy*, based on a set of possible actions.
- (c) The *utility* that each player receive does not depend only in their own choice, but may also depend on the choices of other players. This utility could be a payoff, happiness, money transfer, et cetera.

Maschler, Solan and Zamir [23] explain that these *games* are useful to model several situations in various fields, e.g.

- Theoretical economics: a market in which there are vendors that sell items to buyers, or an auction.
- Networks: the interactions between users and providers in Internet and cell phone networks, or the route choice for drivers in a congested network.
- Political science: the manner how political parties form a governing coalition and divide government ministries and other elected offices, or electoral systems.
- Military applications: a missile pursuing a fighter plane.
- Inspection: the enforcement of laws prohibiting drug smuggling, auditing of tax declarations by the tax authorities or ticket inspections on public transit.
- Biology: the insects and plants that by evolution inherit the properties which are the fittest for pollination and survival.

Even in the Philosophy, where it can contribute some insights into concepts related to morality and social justice, raising questions regarding human behavior in various situations that are of interest to psychology. Osborne and Rubinstein [31] highlight that game theory is sustained by the very basic assumption that decision-makers, or players, pursue well-defined exogenous objectives, i.e. they are *rational*, and take into account their knowledge and expectations of other decision-makers' behavior, i.e. they *reason strategically*. For a formal mathematical basis of these concepts, the reader is invited to visit Appendix A.

1.2 Game theoretic applications in Transportation

We have already mentioned in section 1.1 a couple of Transportation applications, e.g. the ticket inspection on public trains and buses and the congestion games on a network. The first direct application of Game Theory in Transportation goes back to,

| | |
|----------------------------------|---|
| Common models | Ordinary non-cooperative games Generalized Nash equilibrium game Stackelberg game Cournot game |
| Transportation analysis examples | <i>Road/parking tolls policy</i> Vehicle routing problems Transportation network reliability Urban traffic demand Drivers' response to guidance Transport modes competition Risk allocation Local competition's effects to overall situation |

Table 1.1: Applications in macro-policy analysis

at least, Wardrop's first principle [50], which is a Nash equilibrium characterization for noncooperative games in disguise. Since then, numerous other applications in games concerning both travelers and authorities have been studied. Zhang et al. [51] presents us a survey of the main applications in Transportation classifying them in two types: (i) Macro-policy analysis, which focuses on overall situations, where a large number of players take part in the game in a complex and large place; and (ii) Micro-policy simulation, which concentrates on confined situations, in which only few players are in the game in a very limited place. A summary of the common models and transportation analysis examples for macro-policy analysis (resp. micro-policy simulation) can be found in Table 1.1 (resp. Table 1.2)

Out of all these examples, we focus particularly in the road and parking tolls policy, because out of the typical applications is the one that relates with this Thesis' content. The quantity of works on any of these subjects is immense. Just to name a few, optimal tolls for multi-class traffic are studied by Holguin-Veras and Cetin [12], using a Stackelberg game. The toll agency is the leader and the equilibrating traffic are the followers. Steiner and Bristow [42] presents a case study for tolling in National Parks to shed light on non-urban context, which is the top of mind when we talk about road pricing. Teodorovic and Edara [43] using a combination of Dynamic Programming and Neural Networks make real-time changes to the road toll values.

| | |
|----------------------------------|---|
| Common models | Ordinary non-cooperative games Generalized Nash equilibrium game Stackelberg game Bounded rationality game Repeated game |
| Transportation analysis examples | Adjacent traffic signals strategy Emergency rescue Conflicts between two airplanes Collision avoidance between vehicles Conflict between pedestrians and vehicles |

Table 1.2: Applications in micro-policy simulation

Second best congestion pricing problems in the network are examined, among others, by Verhoef et al. [44–46]. Welfare considerations are also incorporated by Ferrari [9] or Arnott et al. [3]. They all try to find a proper toll-policy, which entails a road efficiency improvement by their implementation.

1.3 Basic model for one link

In this section we recapitulate the framework of the Resource Allocation problem based on the work of Johari and Tsitsiklis [13]. Traditionally this methodology has been more used in used in communication network, as in [32], [22], [16], [15]. Nonetheless, we can find capacity allocation in transportation networks in the survey of Johnston et al. [14] where different approaches to reducing congestion by capacity allocation are reviewed: *laissez-faire* allocation, allocation by passenger load, ramp metering, *road and parking pricing* or allocation by trip purpose. The decision made on the approach to be selected will be represented in the shape of our utility function.

1.3.1 Assignment scheme

Let \mathcal{I} be the set of players interested in using a single link with capacity $C > 0$. Each player $i \in \mathcal{I}$ has a utility $u^i(d^i)$, where $0 \leq d^i \leq C$ is the rate allocated to him. Utility function $u^i : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is strictly increasing, concave and continuously differentiable.

The assignment scheme, as defined in [16] , is as follows:

- Each player $i \in \mathcal{I}$ bids $w^i \geq 0$ for the link capacity.
- Given the vector $w = (w^i)_{i \in \mathcal{I}}$, the price μ is uniform for each player defined as

$$\mu = \frac{\sum_i w^i}{C} \quad (1.1)$$

- The allocation finally is given by the vector $d = (d^i)_{i \in \mathcal{I}}$, where

$$d^i = \frac{w^i}{\mu} = \frac{w^i C}{\sum_j w^j}$$

Remark 1.1. *This scheme assign the whole capacity of the arc. It is direct to calculate $\sum_i d^i = C$. In addition, this mechanism does not consider price discrimination among the different players, i.e. charges μ to everyone. There are other mechanisms that discriminate in price depending on the type of player interested, e.g. charging more to higher valuation, or less to players that buy more volume. An example of a mechanism that discriminates in price and is optimal in terms of efficiency was presented by Sanghavi and Hajek in [36].*

1.3.2 Game definition

Frank Kelly [16] in his pivotal work considered a game where players know that the abovesaid scheme will be consider, but they are price-takers. A central assumption in the definition of his competitive equilibrium is that each user does not anticipate the effect of their payment w^i on the price. Considering payoff defined as

$$P^i(w^i; \mu) = u^i \left(\frac{w^i C}{\mu} \right) - w^i \quad (1.2)$$

He showed that there exists a competitive equilibrium, and the resulting allocation solves SYSTEM to optimality. Johari and Tsitsiklis [13] consider a modification of this problem. In their model, players instead of merely taking the given price, as in equation 1.2; will foresee how their bids affect the price consistent with equation 1.1.

The players adjust their payoff accordingly, becoming price-anticipating users. We have already established the strategies of the players, corresponding to bids $(w_i)_{i \in \mathcal{I}}$ and the set of rules for the rate allocation, or the assignment mechanism. Now, it simply remains to define the payoffs of the players to complete the game definition. This last requirement is done below in equation 1.3.

$$Q^i(w^i, w^{-i}) = \begin{cases} u^i \left(\frac{w^i C}{\sum_j w^j} \right) - w^i & \text{if } w^i > 0 \\ 0 & \text{if } w^i = 0 \end{cases} \quad (1.3)$$

With these specific payoffs $\{Q^i\}_{i \in \mathcal{I}}$, we can describe the Nash equilibria as

Definition 1.2. *A Nash Equilibrium for the game defined by the payoffs $\{Q^i\}_{i \in \mathcal{I}}$, is a vector $w \geq 0$ such that for all $i \in \mathcal{I}$*

$$Q^i(w^i, w^{-i}) \geq Q^i(\underline{w}^i, w^{-i}), \quad \forall \underline{w}^i \geq 0$$

Remark 1.3. *The payoffs defined in equation 1.3 has a discontinuity in $w^i = 0$, when $\sum_{j \neq i} w^j = 0$. This fact is crucial, since it can lead to the non-existence of NE. Johari and Tsitsiklis [13] show an example of this case, but fortunately, the authors also show that said difficulty can be settle adding enough competence.*

1.3.3 Existence and uniqueness of Nash equilibrium

Hajek and Gopalakrishnan [11] prove both existence and uniqueness of a NE when multiple users share the link. In addition, they show a characterization of this Nash equilibrium by solving another optimization problem of a similar form to the problem SYSTEM, but with altered utility functions. It is very important to note that even though the Nash equilibrium is obtained by an optimization problem, this does not correspond to a potential. The formalization of this result is presented in an adapted form in [13]. Theorem 1.4 is the adapted version of the latter.

Theorem 1.4. (Johari and Tsitsiklis [13]) *Assume a set of players \mathcal{I} , where $|\mathcal{I}| > 1$, and that for each $i \in \mathcal{I}$, the utility function $u^i(\cdot)$ is concave, strictly increasing, and continuously differentiable. Then, there exists a unique Nash equilibrium $w \geq 0$ of the game defined by $\{Q^i\}_{i \in \mathcal{I}}$, and it satisfies $\sum_i w^i > 0$. In this case, the vector $d = (d^i)_{i \in \mathcal{I}}$ defined by*

$$d^i = \frac{w^i C}{\sum_j w^j}, \quad \forall i \in \mathcal{I}$$

is the unique solution to the following optimization problem GAME:

$$\begin{aligned} & \text{maximize} && \sum_{i \in \mathcal{I}} \hat{u}^i(d^i) \\ & \text{subject to} && \sum_{i \in \mathcal{I}} d^i \leq C \\ & && d^i \geq 0 \quad \forall i \in \mathcal{I} \end{aligned} \tag{1.4}$$

$$\text{where } \hat{u}^i(d^i) = \left(1 - \frac{d^i}{C}\right) u^i(d^i) + \frac{d^i}{C} \left(\frac{1}{d^i} \int_0^{d^i} u^i(z) dz\right).$$

For the proof of this theorem, please see [13]. In this article, it is shown as one of the steps of the proof that a variational inequality also characterizes the equilibrium. This result is presented in the following corollary

Corollary 1.4.1. (Johari and Tsitsiklis [13]) *The vector w is a Nash equilibrium if and only if at least two components of $(w)_{i \in \mathcal{I}}$ are positive, and for each player $i \in \mathcal{I}$, the following conditions hold:*

$$\begin{aligned} \frac{\partial u^i}{\partial d^i} \left(\frac{w^i C}{\sum_j w^j} \right) \left(1 - \frac{w^i}{\sum_j w^j} \right) &= \frac{\sum_j w^j}{C} && \text{if } w^i > 0 \\ \frac{\partial u^i}{\partial d^i}(0) &\leq \frac{\sum_j w^j}{C} && \text{if } w^i = 0 \end{aligned} \tag{1.5}$$

On the other hand, there exists a different optimization problem given by the SYSTEM. In this optimization problem, the aim is to find a social optimum for the aggregate utility given by

$$\begin{aligned}
& \text{maximize} && \sum_{i \in \mathcal{I}} u^i(d^i) \\
& \text{subject to} && \sum_{i \in \mathcal{I}} d^i \leq C \\
& && d^i \geq 0 \quad \forall i \in \mathcal{I}
\end{aligned}$$

Johari and Tsitsiklis also proved that the Price of Anarchy, check Definition A.5 that follows Papadimitriou [32], which is the loss of the aggregate utility given by the selfish strategies of the players, is a tight bound of $4/3$.

Theorem 1.5. (Johari and Tsitsiklis [13]) *Suppose that in addition to the assumption we already have on the utility function $u^i(\cdot)$, we also have that $u^i(0) \geq 0$, $\forall i \in \mathcal{I}$. Let d_S be a solution to SYSTEM, i.e. a social optimum, and d_G be the unique solution to GAME, then*

$$\sum_{i \in \mathcal{I}} u^i(d_G^i) \leq \frac{3}{4} \sum_{i \in \mathcal{I}} u^i(d_S^i)$$

The proof of this theorem is also found in [13]. Similar result for traffic routing games with affine link latency functions are found in [35]. A positive aspect of this proof is the fact that it is constructive, in the sense that calculate explicitly the form of the worst case that makes the bound tight, as shown in the following corollary.

Corollary 1.5.1. (Johari and Tsitsiklis [13]) *The worst case for the Price of Anarchy occurs when all players have linear utilities $u^i(d^i) = a_i d^i$ and $\max_i a_i = 1$, w.l.g. say player 1 is characterized as $a_1 = 1$.*

$$\begin{aligned}
d^1 &= 1/2 & a_1 &= 1 \\
d^i &= \frac{1/2}{|\mathcal{I}| - 1} & a_i &= \frac{1/2}{1 - d^i}, \quad \forall i \neq 1
\end{aligned}$$

This means that player 1 receives half the capacity, and the rest of players share equally the remaining half. More over, we have that $a_i \rightarrow 1/2$ as $|\mathcal{I}|$ increases.

1.4 General networks

First we need to define some notation. Let \mathcal{J} be the set with all links j in the network, where each link $j \in \mathcal{J}$ has a capacity given by C_j . Let \mathcal{P} be the set with all paths p that can be of interest for the players. We define each path $p \subset \mathcal{J}$ as a sequence of different links to get from an origin to a destination, and $\mathcal{P}_i \subset \mathcal{P}$ as the collection of paths that are interesting for player $i \in \mathcal{I}$.

1.4.1 Extended definitions and Assignment scheme

Since we are now studying the generalization of the previous case with only one link, we need to extend some of our definitions to fit the new model. The assignment scheme is defined analogously as follows:

- Each player $i \in \mathcal{I}$ bids $w_j^i \geq 0$ for each link $j \in \mathcal{J}$.
- Given the vector $w = (w_j^i)_{(i,j) \in \mathcal{I} \times \mathcal{J}}$, each link j receives a rate allocation of x_j^i which respond to

$$x_j^i = \begin{cases} \frac{w_j^i C_j}{\sum_k w_k^j} & \text{if } w_j^i > 0 \\ 0 & \text{if } w_j^i = 0 \end{cases}$$

Remark 1.6. *There is a uniform price for each link, as in the previous case. This means that even though the principal charges different prices to different players, it does not price discriminate. The difference is based on the different subset of paths of interest for each player $i \in \mathcal{I}$*

- For each player $i \in \mathcal{I}$, we have an allocation vector $x^i = (x_j^i)_{j \in \mathcal{J}}$, and the final allocation rate is given by $d^i(x^i(w))$, where the function $d^i(\cdot)$ solves the

Maximum flow problem presented below

$$\begin{aligned}
& \text{maximize} && \sum_{p \in \mathcal{P}_i} y_p \\
& \text{subject to} && \sum_{\substack{p \in \mathcal{P}_i \\ p \ni j}} y_p \leq x_j^i \quad \forall j \in \mathcal{J} \\
& && y_p \geq 0 \quad \forall p \in \mathcal{P}
\end{aligned}$$

We extend our previous definitions for payoffs and Nash equilibrium for this game.

Definition 1.7. *f From $d^i(x^i(w))$ we define the payoff for player $i \in \mathcal{I}$ as*

$$Q^i(w^i, w^{-i}) = u^i(d^i(x^i(w))) - \sum_{j \in \mathcal{J}} w_j^i$$

As a consequence, we define a Nash equilibrium for these new payoffs $\{Q^i\}_{i \in \mathcal{I}}$ as a vector $w \geq 0$ such that for all $i \in \mathcal{I}$

$$Q^i(w^i, w^{-i}) \geq Q^i(\underline{w}^i, w^{-i}), \quad \forall \underline{w}^i \geq 0$$

We realize that in reality the sum $\sum_j w_j^i$ is only on the arcs that are interesting for player i , since in the arcs that do not belong to any path relevant for him, he will naturally bid $w_j^i = 0$. A problem with this generalization is that the existence of a NE is not guaranteed. In fact, it could even be the case for a great number of players where the competition is assured. The reason for this is the presence of bottlenecks, then there could be cases where players are overbidding for an arc. This can be understood crystal clear with the example provided in [13].

1.4.2 Cases without Nash equilibrium

Let $|\mathcal{I}| > 1$ be the number of players that want to use the path of two links in figure 1-1 with capacities C_1 and C_2 , where $C_2 \gg C_1$. The player $i \in \mathcal{I}$ has a utility function $u^i(\cdot)$ strictly increasing, concave and continuously differentiable. First, we

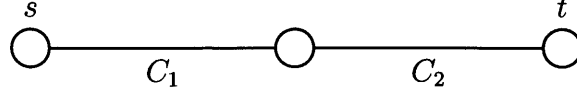


Figure 1-1: No NE even though there is enough competence

realize that the flow for each player is given by a maximum flow problem solution, that in this simple case is given by $d^i = \min\{x_1^i(w), x_2^i(w)\}$ (as will be formally shown in Lemma 2.3). Let's try to prove this using the *reductio ad absurdum* in two steps.

- (i) We notice that the players bid positively in both arcs, i.e. $\sum_i w_j^i > 0$, for $j \in \{1, 2\}$. This is clear since, other options are not viable. If nobody bids on any arc, i.e. $\sum_i w_1^i = \sum_i w_2^i = 0$ creates a situation where any player $i \in \mathcal{I}$ bidding $\epsilon > 0$ on each arcs gets $d^i = \min\{C_1, C_2\}$ and since the utility function is strictly increasing, player i sees his utility increased. Ergo there is a profitable deviation for him and we are not before a NE. The case where players bid in only one arc, say arc 1, it means a situation where $\sum_i w_1^i > 0$ and $\sum_i w_2^i = 0$. There exists a player $i \in \mathcal{I}$ that bids positively on the first arc and none on the second, i.e. his bids are $w_1^i > 0, w_2^i = 0$. But this player is getting $d^i = \min\{x_1^i(w), 0\} = 0$. Hence, it is clear that there exist profitable deviations for him, on account that he can still get the same rate but bidding less. Thus this situation cannot be a NE. The case is analogous if players choose to bid only in arc 2. Therefore if w is an equilibrium bidding profile, the condition $\sum_i w_j^i > 0$, for $j \in \{1, 2\}$ needs to be fulfilled.

- (ii) However, as noted in the scheme definition, this allocates the whole capacity of the arc, as consequence we have the following relations

$$\sum_{i \in \mathcal{I}} x_1^i = \sum_{i \in \mathcal{I}} \left(\frac{w_1^i C_1}{\sum_j w_1^j} \right) = C_1$$

$$\sum_{i \in \mathcal{I}} x_2^i = \sum_{i \in \mathcal{I}} \left(\frac{w_2^i C_2}{\sum_j w_2^j} \right) = C_2$$

Given that $C_1 \ll C_2$, it has to exist a player $\iota \in \mathcal{I}$ such that $x_1^\iota < x_2^\iota$, ergo $d^\iota = x_1^\iota$. This player $\iota \in \mathcal{I}$ has incentive to deviate and lower his bid w_2^ι , as he will be still getting the same d^ι , but paying less. Therefore, w is cannot be a Nash equilibrium.

Closing the argument, a Nash equilibrium must fulfill the condition that players bid positively on both arcs, as seen in (i), but on the other hand, this provokes that at least one player has incentive to deviate according to (ii). Finally, we conclude that it does not exist a Nash equilibrium for this configuration.

1.5 Modified games

Given the problem presented by cases similar to the example shown in section 1.4.2, where there are no Nash Equilibria, the authors consider necessary to extend the scope of action of the players. This extension is in the sense of allowing not only to bid to take capacity in any link, but also to ask for capacity when the sum of bids on a given link are zero. In this way, the bottleneck problem that can be found in configurations as simple as the one displayed in figure 1-1, does not occur. The idea is formalized as follows. The strategy of each player $i \in \mathcal{I}$ respect a link $j \in \mathcal{J}$ contains besides the bid a *rate request*. This strategy is a vector $\sigma^i = (w^i, \phi^i)$, where $\phi^i := (\phi_j^i)_{j \in \mathcal{J}}$ represent the request made by player i on every arc j of the network, and w^i is the same as before. This request only plays a roll when $\sum_i w_j^i = 0$, because in the case $\sum_i w_j^i > 0$, then the assignment is done an explained for the normal game in section 1.3.1.

Definition 1.8. *The scheme with rate request works as*

$$\sum_{i \in \mathcal{I}} w_j^i > 0 \implies x_j^i(w, \phi) = \frac{w_j^i C_j}{\sum_k w_k^j}$$

$$\sum_{i \in \mathcal{I}} w_j^i = 0 \implies x_j^i(w, \phi) = \begin{cases} \phi_j^i & \text{if } \sum_i \phi_j^i \leq C_j \\ 0 & \text{if } \sum_i \phi_j^i > C_j \end{cases}$$

Now the assignment for player $i \in \mathcal{I}$ is given by $x^i(\sigma) = (x_j^i(\sigma))_{j \in \mathcal{J}}$ and we redefine the payoffs and the Nash equilibrium as we did before in Definition 1.7

Definition 1.9. From $\sigma = (w, \phi)$ we define the payoff for player $i \in \mathcal{I}$ as

$$T^i(\sigma^i, \sigma^{-i}) = u^i(d^i(x^i(w, \phi))) - \sum_{j \in \mathcal{J}} w_j^i$$

As a consequence, we define a Nash equilibrium for these new payoffs $\{T^i\}_{i \in \mathcal{I}}$ as a vector $\sigma \geq 0$ such that for all $i \in \mathcal{I}$

$$T^i(\sigma^i, \sigma^{-i}) \geq T^i(\underline{\sigma}^i, \sigma^{-i}), \quad \forall \underline{\sigma}^i \geq 0$$

Theorem 1.10. Assume that for each player $i \in \mathcal{I}$, the utility function $u^i(\cdot)$ is concave, nondecreasing, and continuous. Suppose that w is a strategy vector for the game defined by payoffs $\{Q^i\}_{i \in \mathcal{I}}$. For each $i \in \mathcal{I}$, define

$$\phi_j^i = \begin{cases} \frac{w_j^i C}{\sum_k w_j^k} & \text{if } w_j^i > 0 \\ 0 & \text{if } w_j^i = 0 \end{cases}$$

For each player i , let $\sigma^i = (w^i, \phi^i)$. Then player i receives the same payoff in either game:

$$T^i(\sigma^i, \sigma^{-i}) = Q^i(w^i, w^{-i}).$$

Furthermore, if w is a Nash equilibrium of the game defined by $\{Q^i\}_{i \in \mathcal{I}}$, then σ is a Nash equilibrium of the game defined by $\{T^i\}_{i \in \mathcal{I}}$.

Once more, you can find the proof to this theorem in [13], on which we have based our framework. Analogously to the previous section 1.3, we define the social optimum as the optimal solution to the problem solved by the SYSTEM. Maximizing the aggregate utility of the players.

Definition 1.11. *The social optimum of a generalized network is a rate allocation that solves the SYSTEM optimization problem*

$$\begin{aligned}
& \text{maximize} && \sum_{i \in \mathcal{I}} u^i(d^i) \\
& \text{subject to} && \sum_{p \ni j} y_p \leq C_j \quad \forall j \in \mathcal{J} \\
& && \sum_{p \in \mathcal{P}} y_p = d^i \quad \forall i \in \mathcal{I} \\
& && y_p \geq 0 \quad \forall p \in \mathcal{P}
\end{aligned}$$

The authors show the existence of a Nash equilibrium and the Price of Anarchy is again $4/3$. Also it is noteworthy that in the article the hypothesis for the Theorem 1.10 are different than in the previous case in Theorem 1.4. Instead of *continuous differentiability*, now we have *continuity*. This means we are changing $u^i \in \mathcal{C}^1$ to $u^i \in \mathcal{C}^0 \supset \mathcal{C}^1$. Since we are relaxing the assumptions, the difficulty is increased. Notwithstanding the importance of the results presented so far, the question about uniqueness for other type of networks with complexity beyond the single arc remains open to further study.

1.6 Summary of Findings

- Our hypothesis was that there exists a special structure which can be exploited to characterize its equilibrium. In particular we get uniqueness results for the NE on single-path serial networks, considering the same assumptions on the utility functions as in Theorem 1.10. J. Ben Rosen [33] explains the diagonal strict convexity condition and proves that this, along with other simple-to-check requirements, are a sufficient condition for the existence, uniqueness, and stability of Nash equilibria in concave games. Unfortunately, in the problem posed in our hypothesis these requirements are not generally satisfied.
- We obtained the proof of uniqueness and a moreover a characterization of the NE as a solution of an optimization problem for a single-path serial network

with different arc capacities and where all users are interested in going across the whole network, i.e. every player has the same origin-destination pair. The importance of this characterization, in our opinion, is that it allows us to study how the equilibria change for different sets of parameters, and gives us the opportunity to study the dynamics of the equilibria when faced to a situation where parameters could change in time in regular basis or because of random events. For instance, the capacity on certain segments of a highway could change due to rush hours or accidents.

- We prove uniqueness of the Nash equilibrium in terms of flow for a more general case, where there is a single-path serial network where users may have different origin-destination pairs and not even share a common arc. An example of this situation is given in Figure 2-3.
- We present a sequential extension of the proportional assignment mechanism, where players do not act simultaneously, but in a given order of precedence. For the base case of one arc and two players with linear utilities, we obtained an explicit Subgame Perfect Equilibrium and the price of anarchy of $8/7$. This result coincides both in the threshold and the final allocation with the optimal mechanism for two players defined by Sanghavi and Hajek in [36].

Chapter 2

Equilibrium Characterization for Single-Path Serial Network

2.1 Paths where all links have the same capacity

In this section the case where there are equal capacities in all links, and also every player is interested only in one path of the network. They all have the same origin and destination in mind. In this section, we start tackling the problem from its most basic model for one link and affine utility functions. We get explicit results for the Nash equilibrium profiles for the case of two players, and then generalized for n players. Finally uniqueness in terms of flow is proven for the case with multiple links.

2.1.1 Base case for one link and affine utilities

We can first studying the case for two players to gain perspective. Let two players have affine utility functions i.e. $u^i(d^i) = a_i d^i + b_i$, for $i \in \{1, 2\}$. We can explicitly state the unique Nash equilibrium as

$$w^1 = \frac{Ca_1^2 a_2}{(a_1 + a_2)^2} \quad \& \quad w^2 = \frac{Ca_1 a_2^2}{(a_1 + a_2)^2}$$

To prove this we just have to calculate the rate allocation x_i , for $i = 1, 2$

$$x^1 = \frac{w^1 C}{w^1 + w^2} = \left(\frac{C^2 a_1^2 a_2}{(a_1 + a_2)^2} \right) \left(\frac{C a_1 a_2 (a_1 + a_2)}{(a_1 + a_2)^2} \right)^{-1} = \frac{C a_1}{a_1 + a_2}$$

Then we have to prove the variational inequality shown in equation 1.5. First we consider the left member

$$\begin{aligned} \frac{\partial u^1}{\partial d^1} \left(\frac{w^1 C}{w^1 + w^2} \right) \left(1 - \frac{w^1}{w^1 + w^2} \right) &= a_1 \left(1 - \frac{w^1}{w^1 + w^2} \right) \\ &= a_1 \left(1 - \frac{a_1}{a_1 + a_2} \right) \\ &= \frac{a_1 a_2}{a_1 + a_2} \end{aligned} \quad (2.1)$$

On the other hand, taking the right member of equation 1.5 we have

$$\begin{aligned} \frac{w^1 + w^2}{C} &= \frac{1}{C} \left(\frac{C a_1^2 a_2}{(a_1 + a_2)^2} + \frac{C a_1 a_2^2}{(a_1 + a_2)^2} \right) \\ &= \frac{a_1^2 a_2 + a_1 a_2^2}{(a_1 + a_2)^2} \\ &= \frac{a_1 a_2}{a_1 + a_2} \end{aligned} \quad (2.2)$$

Since we obtain the same result for both equation 2.1 and equation 2.2, we conclude the proof for two players. It can be generalized for a set of any cardinality $n \in \mathbb{N} + 2$.

Theorem 2.1. *Let $\mathcal{I} = \{1, 2, \dots, n\}$ be a set of players with affine utility functions $u^i(d^i) = a_i d^i + b_i$, for $i \in \mathcal{I}$. Then we have that the bids will be*

$$w^i = \frac{(n-1)C \left(\sum_{j \neq i} \frac{1}{a_j} - (n-2) \frac{1}{a_i} \right)}{\left(\sum_j \frac{1}{a_j} \right)^2}$$

and therefore the rate assignment

$$x^i = \frac{C \left(\sum_{j \neq i} \frac{1}{a_j} - (n-2) \frac{1}{a_i} \right)}{\sum_j \frac{1}{a_j}}$$

Proof. First let's calculate the value for $\sum_i w^i$

$$\begin{aligned}
\sum_{i=1}^n w^i &= \sum_{i=1}^n \left(\frac{(n-1)C \left(\sum_{j \neq i} \frac{1}{a_j} - (n-2) \frac{1}{a_i} \right)}{\left(\sum_j \frac{1}{a_j} \right)^2} \right) \\
&= \frac{(n-1)C \left(\sum_i \sum_{j \neq i} \frac{1}{a_j} - (n-2) \sum_i \frac{1}{a_i} \right)}{\left(\sum_j \frac{1}{a_j} \right)^2} \\
&= \frac{(n-1)C \left((n-1) \sum_j \frac{1}{a_j} - (n-2) \sum_i \frac{1}{a_i} \right)}{\left(\sum_j \frac{1}{a_j} \right)^2} \\
&= \frac{(n-1)C \left(\sum_i \frac{1}{a_i} \right)}{\left(\sum_j \frac{1}{a_j} \right)^2} \\
&= \frac{(n-1)C}{\sum_j \frac{1}{a_j}}
\end{aligned}$$

Once we have this result, we can calculate for any given $k \in \mathcal{I}$ that equation 1.5 holds

$$\begin{aligned}
\frac{\partial u^k}{\partial d^k} \left(\frac{w^k C}{\sum_j w^j} \right) \left(1 - \frac{w_k}{\sum_j w^j} \right) &= a_k \left(1 - \frac{\sum_{j \neq k} \frac{1}{a_j} - (n-2) \frac{1}{a_k}}{\sum_j \frac{1}{a_j}} \right) \\
&= a_k \left(\frac{\sum_j \frac{1}{a_j} - \sum_{j \neq k} \frac{1}{a_j} + (n-2) \frac{1}{a_k}}{\sum_j \frac{1}{a_j}} \right) \\
&= a_k \left(\frac{(n-1) \frac{1}{a_k}}{\sum_j \frac{1}{a_j}} \right) \\
&= \frac{(n-1)}{\sum_j \frac{1}{a_j}} \cdot \left(\frac{C}{C} \right) \\
&= \frac{\sum_i w^i}{C}
\end{aligned}$$

□

The degree of efficiency loss is known as the *Price of Anarchy*. First introduced by Koutsoupias and Papadimitriou [18] when evaluating the worst-case ratio of Nash equilibria to the social optimum in a general network. It was used in a traffic network by Roughgarden and Tardos [35] to optimize the performance under congestion. The formal definition can be found in Definition A.5

Theorem 2.2. *The Price of Anarchy for the single-link network where two players with linear utilities compete for the capacity $C > 0$ of the arc is given by $(1 + \sqrt{2})/2$*

Proof. We consider two players with linear utilities $u^i(d^i) = a_i d^i$, for $i \in \{1, 2\}$. First we need to calculate the social optimum of this problem.

$$\begin{aligned} & \text{maximize} && a_1 d^1 + a_2 d^2 \\ & \text{subject to} && d^1 + d^2 \leq C \\ & && 0 \leq d^1 \leq C \\ & && 0 \leq d^2 \leq C \end{aligned}$$

This linear program will have its solution in one of its vertices and the optimal value will be given by $W^* = C \max\{a_1, a_2\}$. On the other hand replacing the value for the allocations d^1 and d^2 according with Theorem 2.1, we have

$$u^1(d^1) = \frac{C a_1^2}{a_1 + a_2} \quad \& \quad u^2(d^2) = \frac{C a_2^2}{a_1 + a_2}$$

We define the welfare value of our unique NE as $W(e)$ and calculate the ratio with respect to the social optimum W^*

$$\begin{aligned} \frac{W^*}{W(e)} &= \frac{C \max\{a_1, a_2\}}{\frac{C a_1^2}{a_1 + a_2} + \frac{C a_2^2}{a_1 + a_2}} \\ &= \frac{(a_1 + a_2) \max\{a_1, a_2\}}{a_1^2 + a_2^2} \\ &= \frac{\left(1 + \frac{a_1}{a_2}\right) \max\left\{\frac{a_1}{a_2}, 1\right\}}{1 + \left(\frac{a_1}{a_2}\right)^2} \\ &= f\left(\frac{a_1}{a_2}\right) \end{aligned}$$

Thus, we can treat the Price of Anarchy as a function of the ratio of the slopes of the utilities, where function $f(\cdot)$ is represented in Figure 2-1. Solving for optimality:

$$\max_{x \geq 0} f(x) = f(\sqrt{2} - 1) = f(\sqrt{2} + 1) = \frac{1 + \sqrt{2}}{2}$$

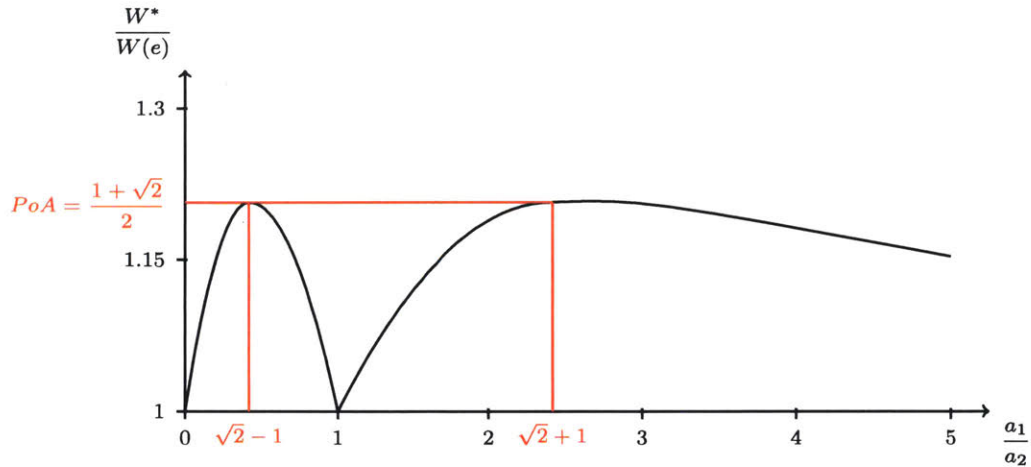


Figure 2-1: The PoA is reached when $a_1 = (\sqrt{2} - 1)a_2$

Therefore we can assure that the Price of Anarchy is reached when the ratio of the slopes of the utilities is $(\sqrt{2} - 1)$, or by symmetry its reciprocal $(\sqrt{2} - 1)^{-1} = \sqrt{2} + 1$. The value of the Price of anarchy is given by

$$PoA = \frac{1 + \sqrt{2}}{2}$$

□

2.1.2 Case with one path and multiple links

Lemma 2.3. *In the case of a single-path serial network, the maximum flow for a player $i \in \mathcal{I}$ is determined as*

$$d^i = \min_{j \in \mathcal{J}} \{x_j^i(\sigma)\}$$

where $x_j^i(\sigma)$ is the rate allocation for player $i \in \mathcal{I}$ on edge $j \in \mathcal{J}$.

Proof. We have to consider the maximum flow problem defined in section 1.4.1 at the page 22 for the specific topology of this type of network. Since there is only one path for every player, i.e. $|\mathcal{P}| = 1$, the optimization problem can be reduced to

$$\begin{aligned} & \text{maximize} && d^i \\ & \text{subject to} && d^i \leq x_j^i \quad \forall j \in \mathcal{J} \\ & && d^i \geq 0 \end{aligned}$$

The feasible region of this problem defined by the set of constraints is equivalent to

$$0 \leq d^i \leq \min_{j \in \mathcal{J}} \{x_j^i(\sigma)\}$$

Therefore it is clear that the solution of a linear problem over an interval occurs on one of the two vertices. In this case, it occurs in the upper bound of this interval. Henceforth $d^i = \min_{j \in \mathcal{J}} \{x_j^i(\sigma)\}$ □

Lemma 2.4. *In a Nash equilibrium, every player receives the same assignment on each arc where he bids positive, this means*

$$w_j^i, w_k^i > 0 \implies x_j^i(\sigma) = x_k^i(\sigma), \quad \forall i \in \mathcal{I}$$

Proof. Suppose for the sake of contradiction that there exist a player i that bids positively on two different edges j and k such that $x_j^i(\sigma) = x_k^i(\sigma)$. Without loss of generality we can assume that $x_j^i(\sigma) < x_k^i(\sigma)$, then by Lemma 2.3 we have that

$$d_k \leq x_j^i(\sigma) < x_k^i(\sigma)$$

Therefore, player i has incentive to decrease his bid on k and use that to increase his bid j getting a higher level of flow, ergo the point cannot be a NE as it says in the hypothesis and we have the contradiction. □

Theorem 2.5. *For a single-path serial network where all arc capacities are equal to $C > 0$ and every player at least one edge in common, then the NE is unique in terms of flow. Moreover, the flow is the same amount of rate assigned on that common arc.*

Proof. First we prove that every player needs to bid on the common arc q . The case where one players only request and other players bid is clearly not a NE, since the flow for that player would be null and he would have incentive to deviate. Suppose now that every player does not bid on the common arc and only request. Therefore there must exist at least two arcs j and k with positive bids.¹ The arcs where the bids are positive allocate the entire link capacity C , therefore we have that

$$\sum_{i \in \mathcal{I}^j} x_j^i(\sigma) = \sum_{i \in \mathcal{I}^k} x_k^i(\sigma) = C$$

where \mathcal{I}^m is the set of players that have arc $m \in \mathcal{J}$ on his path. On the other hand, it makes sense to request rate on the common arc q only if

$$\sum_{i \in \mathcal{I}^q} \phi_q^i \leq C$$

Since $q \in \mathcal{J}$ is the common arc, we necessarily have that $\mathcal{I}^j, \mathcal{I}^k \subsetneq \mathcal{I}^q = \mathcal{I}$. Therefore, without loss of generality, there must exist a player $\iota \in \mathcal{I}^j \setminus \mathcal{I}^k$ such that $x_j^\iota(\sigma) > \phi_q^\iota$ which means that player ι has incentives to bid $\epsilon > 0$ on the common arc to get the full capacity of this arc and increase his flow. This unilateral profitable deviation makes that such profile cannot be a NE. Finally knowing that on a NE all players have to bet on the common arc and by Lemma 2.4 we conclude the result of uniqueness for what we were looking. \square

Remark 2.6. *It is noteworthy that in the case there are multiple common arcs, then we may have different Nash equilibria where players bid on all the arcs, or they all bet in only one of them and request the same amount that would be assigned on said arc on the rest. If we define a subset $\mathcal{J}^q \subset \mathcal{J}$ with all common arcs, we know that*

¹The case where the NE is obtained by players that bid on no arc and request on all of them could only be a NE in the case where utilities are constant after a given quantity. This is a very specific case that we will not take into account in this framework.

in a Nash equilibrium profile $\exists q \in \mathcal{J}^a : w_q^i > 0$ and $\forall j \in \mathcal{J} \setminus \mathcal{J}^a : w_j^i = 0$ for all the players. These may be different equilibria in terms of bids, but they all have the same flows, therefore the uniqueness of the equilibrium in terms of flow is preserved.

2.2 Paths where links have different capacity

In this case we note that Lemma 2.3 and Lemma 2.4 are still valid, since they are based only in the topology of the network and not in the capacities. For the case where all capacities are different, i.e. there is no pair of arcs with the same capacity, we have the following result

Lemma 2.7. *In a Nash equilibrium there can be but one arc with positive bids*

Proof. Suppose there is a NE given by $\sigma = (w, \phi)$ such that there are two arcs j and k with positive bids. Without loss of generality, we assume that $C_j < C_k$. Since the scheme allocate the entire link capacity on arcs where there are bids, we have

$$\sum_{i \in \mathcal{I}} x_j^i = C_j < \sum_{i \in \mathcal{I}} x_k^i = C_k$$

Hence there exists a player $\iota \in \mathcal{I}$ such that $x_j^\iota < x_k^\iota$, ergo a profitable deviation exists for player ι , since he can deviate $-\epsilon < 0$ on his bid x_k^ι and the flow will remain the same given by Lemma 2.3

$$d^\iota = \min_{e \in \mathcal{J}} x_e^\iota \leq x_j^\iota < x_k^\iota - \epsilon$$

□

This lemma implies that the NE must bid in only one edge and request on the rest, which leads us to the next theorem.

Theorem 2.8. *The NE is unique in terms of flow. Moreover, the flow is the same amount of rate assigned on the arc m of minimum capacity such that $C_m = \min_{j \in \mathcal{J}} \{C_j\}$ and the equilibrium profile on the extended game is given by*

$$\sigma_i = \begin{cases} w_m^i > 0 \\ \phi_j^i = x_m^i & j \neq m \end{cases}$$

Proof. We know by Lemma 2.7 that a NE only bids in one arc and by contradiction we can show that it must bid on m . Suppose a NE where players bid on an edge $j \neq m$ and request on the rest of the arcs. Let $k \in \mathcal{J}$ be an arc with capacity $C_k < C_j$. The sum of the requests on arc k must be $\sum_i \phi_k^i \leq C_k < C_j$, because in other case the assignment according to Definition 1.8 would be zero and all players would have flow zero by Lemma 2.3. We have then that on arc k they receive $x_k^i = \phi_k^i$ and

$$\sum_{i \in \mathcal{I}} x_k^i \leq C_k < \sum_{i \in \mathcal{I}} x_j^i = C_j$$

Thus there must exist a player $\iota \in \mathcal{I}$ such that $x_k^\iota < x_j^\iota$, and with a reasoning analogous to the one used in Lemma 2.7, profitable deviations exist and this profile cannot be a NE. Therefore the arc on which the players bid must be the one of minimum capacity and the theorem is proved. \square

Remark 2.9. *Analogous to the Remark 2.6, there is some important note to point out. In the case that there are multiple arcs with minimum capacity, then we have different equilibria where players bid on all the arcs, or they all bet in only one of them and request the same amount that would be assigned on said arc on the rest. If we define a subset $\mathcal{J}^m := \{m \in \mathcal{J} : C_m = \min_{j \in \mathcal{J}} C_j\} \subset \mathcal{J}$ with all the arcs whose capacities are the minima, then we have in a Nash equilibrium profile $\exists m \in \mathcal{J}^m : w_m^i > 0$ and $\forall j \in \mathcal{J} \setminus \mathcal{J}^m : w_j^i = 0$ for all the players. These are different equilibria in terms of bids, but they all have the same flows, therefore the uniqueness of the equilibrium in terms of flow is preserved.*

Lemma 2.10. *A Nash Equilibrium for a single-path serial network does not depend on the request ϕ of players.*

Proof. What we want to prove in this Lemma is that in an equilibrium condition, the flow obtained can not be determined only by arcs where there are no bids but only requests. Suppose there is a NE $\sigma = (\phi, w)$ such that there is a player ι whose allocation d^ι is determined by an arc k , where he has no bid, and only a request. hence he has $d^\iota = \min_{j \in \mathcal{J}} x_j^\iota = x_k^\iota$. In this case, he has a profitable deviation if he bids $\epsilon > 0$ on arc k , gets a strictly increase in his allocation, and therefore an increase in his utility. Therefore, this profile cannot be a NE. \square

Proposition 2.11. *The unique Nash equilibrium is characterized by the following condition for all $i \in \mathcal{I}$*

$$\begin{aligned} \frac{\partial u^i}{\partial d^i} \left(\frac{w_m^i C}{\sum_j w_m^j} \right) \left(1 - \frac{w_m^i}{\sum_j w_m^j} \right) &\leq \frac{\sum_j w_m^j}{C} && \text{if } w_m^i > 0 \\ \frac{\partial u^i}{\partial d^i}(0) &\leq \frac{\sum_j w_m^j}{C} && \text{if } w_m^i = 0 \end{aligned}$$

Proof. We consider again the GAME problem of equation 1.4, which we know has a unique optimal solution because it is an optimization problem with a strictly concave objective function over a compact feasible set. Henceforth the unique optimum is characterized by the first order conditions

$$\begin{aligned} \frac{\partial u^i}{\partial d^i}(d^i) \left(1 - \frac{d^i}{C} \right) &= \sum_j \lambda^j \\ \frac{\partial u^i}{\partial d^i}(0) &\leq \sum_j \lambda^j \end{aligned}$$

According to Lemma 2.10 we know that the equilibrium does not depend on ϕ^i for any player i . Thus, the payoff function $Q^i(\cdot, w^{-i})$ is strictly concave and differentiable as function of $w^i = (w_j^i)_{j \in \mathcal{J}}$. So it reaches its optimum in the NE and meets the optimality condition

$$\nabla Q^i(w^i, w^{-i}) \leq 0 \tag{2.3}$$

This condition becomes interesting only in the bottleneck arcs, in the rest we have a trivial $0 \leq 1$. Based on Theorem 2.11 we know that this arc is the same where $w_j^i > 0$, since we know that the players bid positively only in the bottleneck. Assume now that σ meets the condition and does not depend on ϕ . Since the payoff function is strictly concave, it suffice to revert the argument, because the condition given by equation 2.3 characterized the optimum. The second inequality is a border condition.

Finally, to conclude our proof, we only have to identified

$$d^i = \frac{w_m^i C_m}{\sum_j w_m^j} \quad \& \quad \sum_{j \in \mathcal{J}} \lambda^j = \frac{\sum_j w_m^j}{C_m}$$

□

2.3 Different walks, but with one arc in common

A first intuition for this results comes from the fact that players will bid positively on an arc only when they face competition in that arc. In a theorem this means

Theorem 2.12. *If (w, ϕ) is a Nash equilibrium, then for all $(i, j) \in \mathcal{I} \times \mathcal{J}$ we have*

$$(i) \sum_{r \neq i} w_j^r = 0 \implies w_j^i = 0 \quad \& \quad (ii) \sum_{r \neq i} w_j^r > 0 \implies w_j^i > 0$$

Proof. Suppose there exist a player $i \in \mathcal{I}$ that in his walk passes through arc $j \in \mathcal{J}$.

(i) $\sum_{r \neq i} w_j^r = 0$ and $w_j^i > 0$ is not possible to occur in a Nash equilibrium. Because bidding $0 < w_j^i - \epsilon < w_j^i$ gives the same utility paying less, therefore he has profitable deviation.

(ii) $\sum_{r \neq i} w_j^r > 0$ and $w_j^i = 0$ is not possible to occur in a Nash equilibrium. By Lemma 2.3 player i is getting $d^i = 0$, so it is convenient for him to bid something and get $d^i > 0$ increasing unilaterally his utility.

□

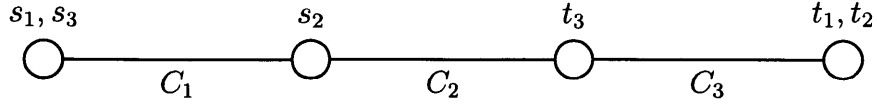


Figure 2-2: Different O-D pairs with arc 2 in common

An example to illustrate this case is shown in figure 2-2, where we are considering affine utility functions $u^i(d^i) = a_i d^i + b_i$. At first glance, as there is no additional information about the game, we have to study all possible options regarding the arcs in which players can bet. We only know that the option where players only request and do not bid cannot be a NE, unless the utilities involved in the game at some point become constant. Therefore, it ought to be the case that Player 1 must bid in at least one of the three arcs, Player 2 must bid on arc 2 or 3 and Player 3 must bet on arc 1 or 2.

Thus we have to study the four scenarios to reach a conclusion, or at least to sharpen the intuition of the result.

(i) Bidding only in arc 2. By Theorem 2.1 we have that every player gets

$$x_2^1 = \frac{C_2(a_1 a_2 + a_1 a_3 - a_2 a_3)}{a_1 a_2 + a_1 a_3 + a_2 a_3} \quad (2.4)$$

$$x_2^2 = \frac{C_2(a_1 a_2 + a_2 a_3 - a_1 a_3)}{a_1 a_2 + a_1 a_3 + a_2 a_3} \quad (2.5)$$

$$x_2^3 = \frac{C_2(a_1 a_3 + a_2 a_3 - a_1 a_2)}{a_1 a_2 + a_1 a_3 + a_2 a_3} \quad (2.6)$$

In addition to the conditions on the parameters for the nonnegativity of x_i , $i \in \{1, 2, 3\}$, we have to meet the condition that the request are less than the capacity of the arcs where there is no bidding, i.e.

$$x_1^1 + x_1^3 \leq C_1 \implies \frac{2C_2 a_1 a_3}{a_1 a_2 + a_1 a_3 + a_2 a_3} \leq C_1$$

$$x_3^1 + x_3^2 \leq C_3 \implies \frac{2C_2 a_1 a_2}{a_1 a_2 + a_1 a_3 + a_2 a_3} \leq C_3$$

(ii) Bidding in arc 1 and 2. On arc 1, the players involved are 1 and 3, that will get

$$x_1^1 = \frac{C_1 a_1}{a_1 + a_3}$$

$$x_1^3 = \frac{C_1 a_3}{a_1 + a_3}$$

On arc 2, the three players bid and we get the same results of equation 2.4. Again, besides the nonnegativity conditions, we need to meet other conditions.

$$x_1^1 = x_2^1$$

$$x_1^3 = x_2^3$$

$$C_3 \geq x_1^1 + x_2^2 = x_2^1 + x_2^2$$

The first two conditions are known because of Lemma 2.4. Combining these two, we obtain

$$\frac{C_2(a_1 a_2 + a_1 a_3 - a_2 a_3)}{C_1 a_1} = \frac{C_2(a_1 a_3 + a_2 a_3 - a_1 a_2)}{C_1 a_3}$$

$$a_3(a_1 a_2 + a_1 a_3 - a_2 a_3) = a_1(a_1 a_3 + a_2 a_3 - a_1 a_2)$$

$$a_3^2(a_1 - a_2) = a_1^2(a_3 - a_2)$$

And from the last condition we get

$$C_3 \geq \frac{2C_2 a_1 a_2}{a_1 a_2 + a_1 a_3 + a_2 a_3}$$

(iii) Bidding in arc 1 and 3 is totally symmetric to the case (ii).

(iv) Bidding on arcs 1, 2 and 3. Finally we consider the case where the players bid on every single arc. Since we have no request to consider, we only focus in the equality of the assignments.

$$\frac{C_1 a_1}{a_1 + a_3} = C_3 a_1 a_1 + a_2 \quad (2.7)$$

$$\frac{C_1 a_1}{a_1 + a_3} = \frac{C_2 (a_1 a_2 + a_1 a_3 - a_2 a_3)}{a_1 a_2 + a_1 a_3 + a_2 a_3} \quad (2.8)$$

$$\frac{C_1 a_3}{a_1 + a_3} = \frac{C_2 (a_1 a_3 + a_2 a_3 - a_1 a_2)}{a_1 a_2 + a_1 a_3 + a_2 a_3} \quad (2.9)$$

$$\frac{C_3 a_2}{a_1 + a_2} = \frac{C_2 (a_1 a_2 + a_2 a_3 - a_1 a_3)}{a_1 a_2 + a_1 a_3 + a_2 a_3} \quad (2.10)$$

Analogous to what we develop in (ii), combining 2.8 and 2.9 we get

$$a_3^2 (a_1 - a_2) = a_1^2 (a_3 - a_2) \quad (2.11)$$

Then, combining 2.7, 2.8 and 2.10

$$a_2^2 (a_1 - a_3) = a_1^2 (a_2 - a_3) \quad (2.12)$$

And associating these two equations 2.11 and 2.12 we get the last condition on the slope of the utilities of players 2 and 3

$$a_3^2 (a_1 - a_2) = a_2^2 (a_3 - a_1)$$

The bottom line for intuition is that, a priori, the cases studied in the example above do not fall in contradiction. There could be more than one profile that meets these conditions and is an equilibrium, but all of them will be the same in terms of flow, since by Lemma 2.4 the flows on every arc with positive bid must be the same.

Lemma 2.13. *In a single-path serial network, where players have walks with different origin-destination pairs, but they all share at least one single arc. Then the Nash equilibrium is unique.*

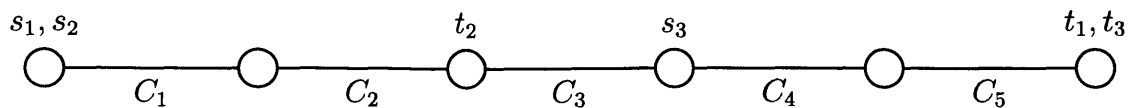


Figure 2-3: Different O-D pairs without common arc

Proof. Let j be the common edge and let s and t be their endpoints. Therefore every origin-destination path begins at most at s and finishes at least in t , otherwise the common edge would not be j . Besides for a single player always we have that, in a NE, $w_j^i > 0, w_k^i > 0$ for $j \neq k \implies x_j^i = x_k^i$. Because if we had that $x_j^i < x_k^i$, then it means that $d^i \leq x_k^i < x_j^i$, ergo we can deviate $-\epsilon$ in bid w_j^i and add it to w_k^i and we would be better off, ergo contradiction and the corollary is proved. \square

2.4 Players with different walks

Now we consider a case relaxing the assumption of the common arc that had before. An example to illustrate this case is shown in figure 2-3, where we are considering affine utility functions $u^i(d^i) = a_i d^i + b_i$. At first glance, as there is no additional information about the game, we have to study all possible options regarding the arcs in which players can bet. We only know, as before, that the option where players only request and do not bid cannot be a NE.

Thus we have to study the different cases to reach a conclusion, or at least to sharpen the intuition of the result. In this case there is no common arc, and we know that every player must bid in at least one arc, since the situation where there are only requests cannot be a NE. Ergo, the different options are reduced to four cases. Bid in arcs (1,4), (1,5), (2,5) or (2,5). We look to prove by contradiction the impossibility of multiple NE in this case. For this purpose, suppose there is two NE, one that bids on (1,4) and other in (1,5).

(i) Bidding in arc 1 we get on arc 1

$$x_1^1 = \frac{C_1 a_1}{a_1 + a_2} \quad \& \quad x_1^2 = \frac{C_1 a_2}{a_1 + a_2}$$

Besides the condition to make sense a request on arc 2, i.e. $C_2 \geq x_1^1 + x_1^2 = C_1$

In arc 4 we get

$$x_4^1 = \frac{C_4 a_1}{a_1 + a_2} \quad \& \quad x_4^2 = \frac{C_4 a_2}{a_1 + a_2}$$

Again with the condition that $C_4 \leq C_5$. Besides we have the condition given by Lemma 2.4 that states

$$x_1^1 = x_4^1 \implies \frac{C_1}{C_4} = \frac{a_1 + a_2}{a_1 + a_3}$$

(ii) Bidding in arc 1 and 5, analogously we arrive to the conclusion

$$x_1^1 = x_5^1 \implies \frac{C_1}{C_5} = \frac{a_1 + a_2}{a_1 + a_3}$$

However, a necessary condition for the existence of two equilibria of this form is that $C_4 = C_5$, since we know that make no sense bidding in an arc with slack of capacity. On the other hand, looking for two different equilibria we must have that

$$x_4^1 \neq x_5^1 \implies \frac{C_4 a_3}{a_1 + a_3} \neq \frac{C_5 a_3}{a_1 + a_3} \implies C_4 \neq C_5$$

Then we reach a contradiction. Therefore, it is not possible to have two different NE in a network as the one in Figure 2-3.

Theorem 2.14. *In a network with only one path and players with different O-D pairs, there exists a unique Nash equilibrium in terms of flow.*

Proof. We will prove this Theorem by contradiction. Suppose there exist two equilibria with different flow, d_1 and d_2 . Then the situation of figure 2-4 must occur at some point.

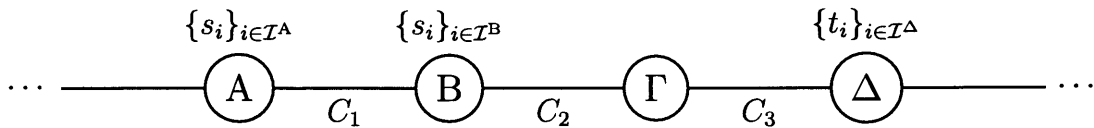


Figure 2-4: General one-path network with different O-D for players

- In node A we consider a nonempty proper subset of the players $\mathcal{I}^A \subsetneq \mathcal{I}$ whose origins are in or before this node. If there were no player that starts his walk in this node or before, then the node could be removed from the network without any incidence on the equilibria.
- In node B we consider a nonempty subset of players $\mathcal{I}^B \subset \mathcal{I} \setminus \mathcal{I}^A$ whose origins are exactly in this node. It could be the case that some of the players $i \in \mathcal{I}^A$ have their destination in this node also.
- In node Γ some players from \mathcal{I}^A could have their destination in this node, but **no player** from \mathcal{I}^B can finish their walk here. If this were the case, then both flows would compete and bid on arc 2.
- In node Δ , in addition to origin for new players and maybe destination for some player from \mathcal{I}^A , for the first time a player from \mathcal{I}^B can finish his walk started in node B.

Some important remarks to make before proceeding with the demonstration are

- (i) Both flows cannot have arcs in common with positive bid. If this were the case, then by Lemma 2.4 the flows would be the same.
- (ii) Both d_1 and d_2 must have a positive bid on at least one arc, since we have already cover the condition that not bidding on any arc, and only requesting cannot be a Nash equilibrium. Then, necessarily they must bid on one arc out of 2 or 3, and request on the other. The other player must do the symmetric bid and request interchanging the arcs.
- (iii) Suppose, without loss of generality, that NE flow d_1 (resp. d_2) is obtained by bidding on arc 3 (resp. 2). This mean we have the conditions

$$C_2 = \sum_{i \in \mathcal{I}} d_2^i \geq \sum_{i \in \mathcal{I}} d_1^i \quad (2.13)$$

$$C_3 = \sum_{i \in \mathcal{I}} d_1^i \geq \sum_{i \in \mathcal{I}} d_2^i \quad (2.14)$$

We realize that in the case where there is actual positive bidding, then the relation is a strict equality.

Lemma 2.15. *In this context, where we are assuming two different NE, let's suppose that an arc of capacity C has positive bidding under flow d_1 and only requests under flow d_2 . Then, it necessarily holds the condition*

$$C = \sum_{i \in \mathcal{I}} d_1^i < \sum_{i \in \mathcal{I}} d_2^i$$

Proof. This lemma can be proved by contradiction. Let's assume that the case where there were two different NE in the network such that

$$C = \sum_{i \in \mathcal{I}} d_1^i = \sum_{i \in \mathcal{I}} d_2^i$$

Then, there exist a player $\iota \in \mathcal{I}$ such that $d_1^\iota \neq d_2^\iota$. We have to remember that when in a NE, there is a request on one arc, it means that the throughput is given by the bottleneck which is *other* arc. Therefore, for player ι , and the rest of players that share at least a common with him, there exist two possible NE flows for a path with one arc in common, which is a contradiction with Lemma 2.13. \square

Thus, our condition 2.13 and 2.14 become

$$C_2 = \sum_{i \in \mathcal{I}} d_2^i > \sum_{i \in \mathcal{I}} d_1^i \tag{2.15}$$

$$C_3 = \sum_{i \in \mathcal{I}} d_1^i > \sum_{i \in \mathcal{I}} d_2^i \tag{2.16}$$

Lemma 2.16. *For a Nash equilibrium flow d and an arc with capacity C where there are only requests and no bids, the following condition holds. If $\sum_i d^i = K \leq C$, then the rate allocated to each player i is a proportional fraction of K , as if there had been bids on the arc.*

Proof. In fact, we know that an arc where there are only requests must verify the fact $\sum_i \phi^i \leq C$, and since by Lemma 2.3 we have that $d^i = \min_{j \in \mathcal{J}} x_j^i$, then it must hold

$$\sum_{i \in \mathcal{I}} d^i \leq \sum_{i \in \mathcal{I}} \phi^i \leq C$$

Therefore, in an arc where there is only requests, it is impossible to get through more flow than the one given by a bottleneck in *other* arc, in particular, one of the arcs where there were actual positive bids. In fact, the flow d^i that goes through this arc of capacity C is exactly the same that was allowed in the bottleneck. In view of the fact that in those bottleneck the assignment was given by the proportional assignment scheme, then the proportion on the arc of request must be the same. \square

Let's consider the case of a player $\beta \in \mathcal{I}^B$. It is important that this reasoning is the same for any player with the same destination than β . In arc 2, player β has $d_1^\beta = \rho_2^w C_2$ in the first NE flow d_1 and $d_1^\beta = \rho_3^\phi C_3$, where ρ_j^w (resp. ρ_j^ϕ) is the proportion of capacity C_j of arc j that player β receives by bidding w_j^β (resp. requesting ϕ_j^β).

Then, based on Lemma 2.16 we know that it must hold $d_1^\beta < d_2^\beta$ (resp. $d_1^\beta > d_2^\beta$) for arc 2 (resp. arc 3), i.e. we have the conditions

$$\rho_2^\phi C_2 < \rho_2^w C_2 \tag{2.17}$$

$$\rho_3^\phi C_3 < \rho_3^w C_3 \tag{2.18}$$

On the other hand, we know that when we bid positively on an arc, we obtain at least the rate of the flow given in the bottleneck. Where there is positive bid, then

$$\rho_2^\phi C_2 \geq \rho_3^w C_3 \tag{2.19}$$

$$\rho_3^\phi C_3 \geq \rho_2^w C_2 \tag{2.20}$$

Using equations 2.17, 2.18, 2.19 and 2.20 we get the following inequality

$$\rho_3^w C_3 \leq \rho_2^\phi C_2 \leq \rho_3^\phi C_3 < \rho_3^w C_3$$

Which is clearly a contradiction, and proves our result about the uniqueness of the Nash equilibrium we were looking to prove. \square

Remark 2.17. *The network on Figure 2-4 does not represent a loss of generality. Based on what has been previously explained, if it were possible to coexist two different NE flows, then they cannot have a common arc where players bid positively on it, besides in both cases players must bid positively in at least one arc. Thus, the only potential loss of generality could occur if there were more arcs between the arcs that find positive bids, in this case arcs 2 and 3. In case there exist more arcs in between them, where players can find their origin and players from node A or before can find their destination, in none of them could be a positive bid, since in that case the NE flow would be completely determined. This entails that Lemma 2.15 and 2.16 remain intact, and as a result the proof continues to be valid.*

2.5 Sequential Assignment Mechanism

In this section we present an extension of the standard assignment scheme presented in Chapter 2. As described by Leme et al. [20], typical game-theoretical models use simultaneous move games, nonetheless in real applications simultaneity is often hard to achieve. That is where sequential games play a crucial role, dropping the simultaneity assumption. The main difference consist in that now, instead of playing one-shot simultaneous games. Players face an interaction that happen in a sequence of rounds, where a single player acts in each round. In this particular case, we analyze the case of two players with linear functions in a single link with capacity $C = 1$.

We notice that, given a set of actions for the first $k < n$ players, it is induced naturally a *subgame* for the remaining players, i.e. players $k+1, \dots, n$. They take this information as a given and play a sequential game with $n - k$ players. The concept of

solution for a sequential game is given by the Subgame Perfect Equilibrium, originally formalized by Selten [39] extending the idea of backward induction

Analogously to our Definition A.5 for the Price of Anarchy, we need to define its extension for sequential games. The Sequential Price of Anarchy (SPoA) of the game, Definition A.9, is the ratio between the social optimal solution W^* and the quality of the worst subgame perfect equilibrium, $\min_{e \in SPE} W(e)$.

2.5.1 Base case for two players

Suppose there exist two players with linear utilities functions, i.e. $u^i(d^i) = a_i(d^i)$, for $i \in \{1, 2\}$, where $a_1, a_2 > 0$.

Theorem 2.18. *The unique SPE for a two-player sequential game, with linear utility functions and nonnegative bids occur when $a_1 \leq 2a_2$ and the flows are given by*

$$(d^1, d^2) = \left(\frac{a_1}{2a_2}, 1 - \frac{a_1}{2a_2} \right)$$

Proof. For this proof we analyze the game by steps.

1. Player 1 bids $w^1 \geq 0$.
2. Player 2 wants to maximize his payoff considering w^1 , then he solves

$$\begin{aligned} & \text{maximize} && a_2 \left(\frac{x}{w^1 + x} \right) - x \\ & \text{subject to} && x \geq 0 \end{aligned} \tag{2.21}$$

The result from equation 2.21 is equivalent to

$$w^2 = \max\{\sqrt{a_2 w^1} - w^1, 0\} \tag{2.22}$$

If we want that $w^2 \geq 0$, we need condition 2.23 to hold

$$w^1 \leq a_2 \tag{2.23}$$

We consider condition 2.23 to be true, since had not this condition hold, we could remove these players from the game and study a equivalent game with only positive bids. In all this analysis we consider only positive bids.

3. Player 1 knows that 2.24 will be the rational strategy of player 2. He anticipates to this answer and solves

$$\begin{aligned}
 \text{maximize } w^1 &= a_1 \left(\frac{x}{x + w^2} \right) - x \\
 \text{subject to } w^2 &= \max\{\sqrt{a_2 w^1} - w^1, 0\} \\
 w^1 &\leq a_2 \\
 x &\geq 0
 \end{aligned} \tag{2.24}$$

where we have considered condition 2.23 and equation 2.22 as constraint, we get that

$$w^1 = \min \left\{ \frac{a_1^2}{4a_2}, a_2 \right\}$$

Finally, we have to analyze two cases of interest

- (i) If $a_1 > 2a_2$, then $(w^1, w^2) = (a_2, 0)$. Ergo, as said after the condition 2.23, we disregard this case because we are only interested in games with positive bids.

- (ii) If $a_1 \leq 2a_2$, then $(w^1, w^2) = \left(\frac{a_1^2}{4a_2}, \frac{a_1}{2} \right)$. The NE flows are

$$\begin{aligned}
 d^1 &= \frac{w^1}{w^1 + w^2} = \frac{a_1^2}{4a_2} \left(\sqrt{\frac{a_1^2 a_2}{4a_2}} \right)^{-1} = \frac{a_1}{2a_2} \\
 d^2 &= \frac{w^1}{w^1 + w^2} = \left(\frac{a_1}{2} - \frac{a_1^2}{4a_2} \right) \frac{2}{a_1} = 1 - \frac{a_1}{2a_2}
 \end{aligned}$$

It is noteworthy that the SPE is unique given the concavity of the payoffs. □

2.5.2 Sequential Price of Anarchy

First we need to calculate the social optimum of this problem. Replacing the value for the allocations d^1 and d^2 according with Theorem 2.18, we have

$$u^1(d^1) = \frac{a_1^2}{2a_2} \quad \& \quad u^2(d^2) = a_2 - \frac{a_1}{2}$$

On the other hand the social optimum will be given by

$$\begin{aligned} & \text{maximize} && a_1 d^1 + a_2 d^2 \\ & \text{subject to} && d^1 + d^2 \leq 1 \\ & && 0 \leq d^1 \leq 1 \\ & && 0 \leq d^2 \leq 1 \end{aligned}$$

This is a Linear Program, therefore its optimum will be in one of the vertices. It is clear that in this case the the optimal value will be given by $\max\{a_1, a_2\}$

Theorem 2.19. *The SPoA for this game is equal to $8/7$.*

Proof. We define our unique SPE as $e = (w^1, w^2)$ and calculate the ratio with respect to $W^* = \max\{a_1, a_2\}$

$$\begin{aligned} \frac{W^*}{W(e)} &= \frac{\max\{a_1, a_2\}}{\frac{a_1^2}{2a_2} + a_2 - \frac{a_1}{2}} \\ &= \frac{2 \max\left\{\frac{a_1}{a_2}, 1\right\}}{\left(\frac{a_1}{a_2}\right)^2 + 2 - \frac{a_1}{a_2}} \\ &= f\left(\frac{a_1}{a_2}\right) \end{aligned}$$

Then, we can treat the ratio between welfare of the social optimum and the welfare of the SPE as a function of the ratio of the slopes of the utilities, where function $f(\cdot)$ is represented in Figure 2-5. Solving for optimality, we find

$$\max_{0 \leq x \leq 2} f(x) = f\left(\frac{1}{2}\right) = \frac{8}{7}$$

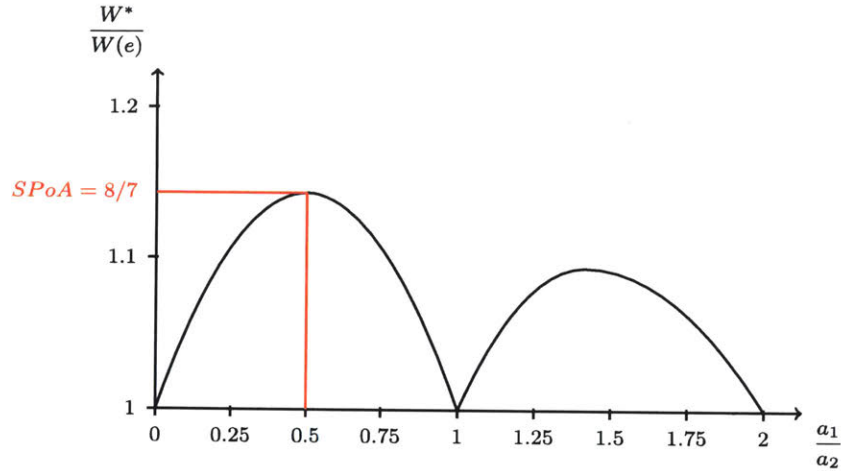


Figure 2-5: The SPoA is reached when $2a_1 = a_2$.

Therefore we can assure that the Sequential Price of Anarchy for this game is $8/7$. \square

The importance of the result of Theorem 2.19 is twofold. On the one hand, it is proved that the sequential mechanism improves the efficiency and gets a value for the welfare function of $7/8$ of the optimal value, which means that exceed the Price of Anarchy that we had before for the simultaneous game, i.e. $3/4$ that we had before. On the other hand, it meets the threshold given by Sanghavi and Hajek in [36]. In their work, the authors show that for two buyers, their allocation mechanism is found that guarantees that the aggregate value is always greater than $7/8$ of the maximum possible, and it is shown that no other mechanism achieves a larger ratio.

Chapter 3

Modeling Suggestions

We have studied and characterized the equilibrium for a game where all players are interested in only one path. This could model a situation where there is a single-path network, but also is useful to model a general network, where there is a specific path that every user must use. This could be a physical constraint, an agreement, a legal issue, et cetera. The latter is called Fixed Routing in connectivity networks and there is increasing interest in them in the internet and telecommunication literature. Nevertheless the application in Transportation is also vast and thought-provoking. In this section we present a little review of the results in communication network and then propose some ideas, suggestions, comments for modeling Transportation interactions.

3.1 Applications outside Transportation

Traditionally the study of allocation mechanism and evaluation of efficiency loss has been traditionally applied to communication networks. The study has focused for the most part in the difference between the aggregate welfare when noncooperative users choose their routes maximizing their own utility vs. a centralized planned routing that optimizes the welfare. Looking for a socially desirable outcome, researchers have proposed different methods to cope with selfish behavior. There are so many authors that we could cite here, that any list we could make would be insufficient. Just to

name a few, previous approaches include influencing the behavior of selfish agents via pricing policies [6], network switch protocols [40], routing a small portion of the traffic centrally [34], [17] generating a leader-follower situation, where after committing to this strategy, the rest of the players, or followers, make their decisions with knowledge of the leader's commitment, or algorithmic mechanism design [8], [29], [28].

Other authors have proposed different mechanisms to address the congestion and its undesired consequences. Shetty et al. [41] studied an Internet market. Assuming a certain form of demand function, they are able to derive the optimal capacity investments for Internet service providers under different regulation policies. Schwartz et al. [37] examine a low-cost regulatory tool. Establishing property rights over a small fraction of their capacity leads to a simplification of the capacity division and diminishes investment disincentives of Internet service providers. The social planner can reduce the harmful effects of transition to multiple service classes more easily using this tool. Application of market mechanisms to manage congestion in networks, and therefore improve quality of service, using different fixed rate pricing mechanisms, as done by Odlyzko [30] is another option. In his paper, the author explains that most of people are averse to varying prices. He gives examples where people are more inclined to accept variations in quality than in price. The author also gives examples to illustrate different *prima facie* irrational behaviors by the economic agents, but does not provide hard estimates. His goal is maximal simplicity for the user, rather than maximizing any given quantifiable measure.

3.2 Highways

In an inter-urban trip, there are multiple ways to reach and exit the highway. However, once in it, every car needs to go through the exact same road passing one by one every exit and entrance, i.e. all users go through a single-path network. Within urban context, it could also be seen for modeling expressways or arterial roads, which are the primary roads in urban land that are responsible for channeling the metropolitan long distance moves. Fulfilling the connection and distribution of vehicles in the

urban environment, therefore being a must for most drivers in the area. In this situation we could have an authority (governmental, municipal, regional organizations or companies that operate public transport services or toll roads) that decide to allocate capacity under some criteria and ask for different bids. However sometimes it is really hard to enforce the actual compliance of these capacities given a priori, for instance determining capacities on city street for a given type of vehicles, more over when we are dealing with populations of driver. This fact usually precludes the implementation of this type of measures. What is normally done is charging different toll prices, and maybe consider congestion prices as done originally by Vickery [47,48].

3.3 Maritime transportation

Before we start commenting how our single-path network model can fit with maritime transportation, in particular liner shipping, for the sake of clarity, we need to define some basic concepts necessary to comprehend this industry. Agarwal and Ergun [1] define *sea cargo* as the freight carried by ships, and it includes anything traveling by sea other than mail, people, and personal baggage. A *global sea carrier* is a private person, firm or organization that offers transportation services via the sea on a worldwide basis. A *shipper* is a person or company that is either the supplier or the owner of the cargo that is to be shipped.

With rates for sea cargo transportation at approximately one-tenth of air freight rates, fewer accidents and less pollution, maritime transportation is regarded as a cheap, safe, and clean transportation mode, compared with other modes of freight transportation. It is no surprise therefore that according to the United Nations Conference on Trade and Development [38] the international seaborne transportation volume in 2014 was about 9,842 millions of tons, or in the vicinity of 80% of total world merchandise traded. The breakdown for traded products is in Table 3.1. The five major bulks refer to iron ore, coal, grain, bauxite/alumina and phosphate rock.

The increase in container cargo is important to point out, since the development of the container is an example of *unintended consequences*, concept popularized by

| Year | 1980 | 1985 | 1990 | 1995 | 2000 | 2005 | 2010 | 2014 |
|------------------|-------|-------|-------|-------|-------|-------|-------|-------|
| Container | 102 | 152 | 234 | 371 | 598 | 969 | 1,280 | 1,631 |
| Other dry cargo | 1,123 | 819 | 1,031 | 1,125 | 1,928 | 2,009 | 2,022 | 2,272 |
| Five major bulks | 608 | 900 | 988 | 1,105 | 1,295 | 1,709 | 2,335 | 3,112 |
| Oil and gas | 1,871 | 1,459 | 1,755 | 2,050 | 2,163 | 2,422 | 2,772 | 2,862 |
| Total | 3,704 | 3,330 | 4,008 | 4,651 | 5,984 | 7,109 | 8,409 | 9,877 |

Table 3.1: International seaborne trade (millions of tons loaded)

Source: UNCTAD Review of Maritime Transportation 2015

the famous sociologist Robert K. Merton in 1936 [24]. Containerization began as a mean to cut costs of sending Malcom McLean's trucks between New York and North Carolina, but ended integrating East Asia into the world economy, earlier dominated by the North Atlantic, spearheading globalization and changing the whole landscape of maritime transportation industry as explained by Levinson [21]. As information flow was revolutionized by the computer, the container revolutionized the ocean trade, by cutting costs and enhancing reliability. Container-based shipping vastly increased the volume of international trade. Bernhofen et al. [4] gives us Figure 3-1 showing the growth of world trade in real terms from \$0.45 trillion in the early 1960s to \$3.4 trillion in 1990, by about a factor of 7. Other factors influenced as well, e.g. trade policy liberalization and other cost-reducing technological advancements.

Christiansen et al. [5] describe in detail the division of the global shipping industry into three different modes of operation:

- (i) Industrial shipping: The shipper owns the ship and looks for a minimization of the freight cost for a given origin-destination pair.
- (ii) Tramp shipping: A carrier accede to a contract with a shipper to carry bulk cargo between specified ports within a specific time window. It is possible that depending on availability of cargo and slack in capacity, the tramp shipping load other cargo looking for profit maximization, but it is not its core business.
- (iii) Liner shipping: A carrier decides on a fleet, a set of routes and a schedule of trips, and operates offering its services to the shippers.

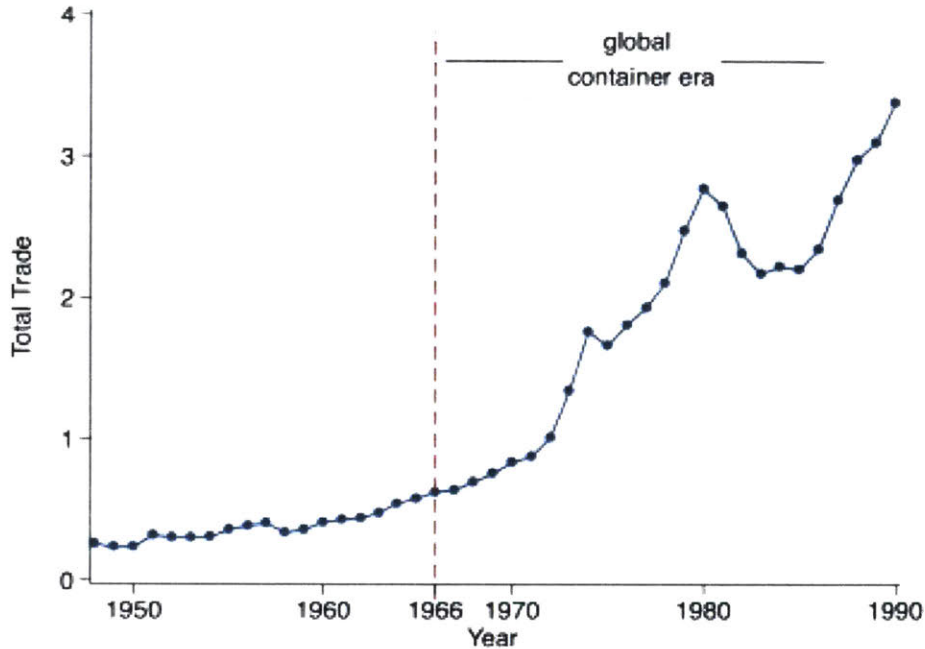


Figure 3-1: The growth of world trade (deflated): 1948-1990

Source: Estimating the effects of the container revolution on world trade [4]

In this manner, one can make the analogy with ground transportation, associating industrial shipping with a person driving his own car or a firm that manages its own trucks to deliver the products. Tramp shipping with a taxi service or some current application such as Uber or Cabify. Finally liner shipping with a bus service with certain and known schedules and itinerary. It is in this latter type of shipping, we mean liner shipping, that our model to allocate capacity between different users that are interested in the same route takes on greater significance. This is the domain where we think could get the utmost of this model. It is important to point out that in this situation a vessel has to go through a specific route, stopping in different ports in a determined order given by geography. This order cannot be change, but for small differences. The vessel has a finite capacity to offer to different players. These players can have different weights, willingness to pay, utility functions, et cetera.

To know more about this industry, please visit Agarwal and Ergun [2]. In this paper, the authors describe the liner shipping business emphasizing three factors

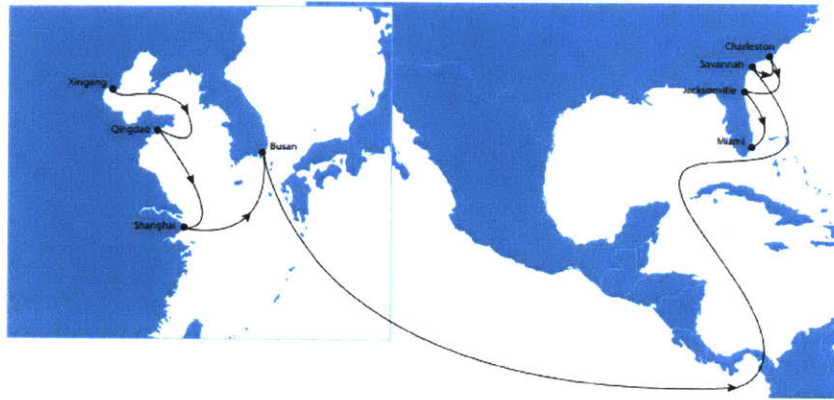


Figure 3-2: Route for Maersk Triple E vessels from America to Asia

that drive carriers to collaborate with their competitors and make alliances: (i) liner shipping is a capital-intensive industry; (ii) large containerships produce economies of scale, however, they require a longer period for container accumulation, resulting in a less frequent service; (iii) alliances help carriers to explore new markets and enhance their service scope.

Chapter 4

Concluding Remarks

4.1 Summary

In general terms, the main conclusion of this work is that the main objective defined at the beginning of the work is fulfilled. We were capable of made a little extension on the results already published by Johari and Tsitsiklis [13]. The authors had proved existence and uniqueness for a single link network, and only existence for the general network. Focusing in a particular class of networks, the one-path serial network, we exploited its distinct characteristics and were able to prove the uniqueness of Nash equilibrium, making a minuscule dent on the knowledge frontier. We gradually proved the uniqueness of the Nash equilibrium for this particular class of networks, relaxing assumptions one by one. Then we start with all links with equal capacity and at least one arc in common for every user;s path and proved the uniqueness in Theorem 2.5, more over we identified the NE profiles of bids and request in Theorem 2.8 as

$$\sigma_i = \begin{cases} w_m^i > 0 \\ \phi_j^i = x_m^i & j \neq m \end{cases}$$

where we could see that make sense to bid only in the bottleneck and secure a piece of capacity in that arc. Then we proved that this NE profile also solves an optimization problem and is characterized by the variational inequality given in Theorem 2.11. On

the other hand, for the sequential extension, we were also able to prove that for the base case of a single arc and two players with linear utilities, we obtained an explicit Subgame Perfect Equilibrium given by

$$(d^1, d^2) = \left(\frac{a_1}{2a_2}, 1 - \frac{a_1}{2a_2} \right)$$

and the price of anarchy of $8/7$. A noteworthy feature of this result, is that it coincides both in the threshold and the final allocation with the *optimal mechanism* for two players defined by Sanghavi and Hajek in [36], which is a mechanism that do *price-discriminate*.

4.2 Future work

There are several open question of interest to address yet. For starters, the main result of our study was made for a one-path serial network, but an interesting line for further research would be to have some kind of results for serial-parallel networks. The expansion that would mean this slight difference would be uncanny. For the general network it has not been proved the fact that there are multiple Nash equilibria, this could be got by a counterexample exploiting the characteristic that in a general network, the flow for different arcs is not necessarily the same, since the general network does not have the characteristic of only one possible path.

A possible interpretation of the result obtained in section 2.5 is that the order of precedence is in fact a discrimination, but could be interpreted and quantified with an equivalent price-discrimination. Increasing the number of players, even for three becomes the problem much more complicated and not possible to get explicit solutions. Try to have numerical solutions, and then work with heuristics would a solid choice of further work to characterize the SPE. Besides it is possible that for more than two players, the SPE may be not longer unique.

An hypothesis that would be very interesting to study is wether the efficiency of the mechanism is less when players less *economical* play first. Based on our result in

Theorem 2.19, and as we can see in Figure 2-5, the worst efficiency loss case when $a_1 < a_2$ is worse than the worst case when $a_2 < a_1$. This means that for the two-player case: when the more *economical* player, i.e. the one that has a higher utility for the same rate allocation, in this case a bigger slope, plays first the aggregate utility is higher than when he plays last. The extension to n players would be the next step.

Last but not least, the application of this results to real data and assess the efficiency of real firm mechanism would be the culmination of this effort. There are evidence of a strategic component on the pricing for capacity in liner shipping , as well as freight transportation by rail, therefore it would be of tripartite interest for (i) port authorities or operators, (ii) shippers or owners of the goods, and (iii) shipping companies; to have an in-depth understanding of this mechanism and its dynamics.

Appendix A

Preliminary Definitions

Formally, we can define the mathematical basis of Game Theory as done by Laraki, Renault and Sorin in [19]

Definition A.1. *A game in normal form G is defined by a triplet $(\mathcal{I}, \mathcal{S}, u)$ where*

- (a) *A set of players \mathcal{I} with cardinality $n \in \mathbb{N}$.*
- (b) *A set of strategies $\mathcal{S} := \prod_{i \in \mathcal{I}} \mathcal{S}^i$, where each \mathcal{S}^i is the non-empty set of strategies, or actions, for player $i \in \mathcal{I}$.*
- (c) *A mapping $u : \mathcal{S} \rightarrow \mathbb{R}^n$, where $u^i(s)$ is the utility, or payoff, for player $i \in \mathcal{I}$ given that the profile $s = (s^i)_{i \in \mathcal{I}} \in \mathcal{S}$ is played.*

The interaction is the following. Simultaneously, or at least at the moment when a player chooses his action, this player is not aware of the possible choices already made by the rest, each player $i \in \mathcal{I}$ chooses $s^i \in \mathcal{S}^i$. Then this player receives the payoff $u^i(s)$. With a little abuse of notation, we denote also $s = (s^i, s^{-i}) \in \mathcal{S}$, where $s^{-i} \in \mathcal{S}^{-i} := \prod_{j \neq i} \mathcal{S}^j$ is the vector of strategies s^j of all players but i . Finally each player wants to maximize his own utility. The framework of game G is common knowledge among players.

Definition A.2. *The set of all probability distributions over a finite set T is defined*

$$\Delta(T) = \left\{ x \in \mathbb{R}_+^T : \sum_{t \in T} x_t = 1 \right\}$$

Definition A.3. *The set-valued function, or correspondance, Best Response of player $i \in \mathcal{I}$, $BR^i : \mathcal{S}^{-i} \rightrightarrows \mathcal{S}^i$ is*

$$BR^i(s^{-i}) = \{s^i \in \mathcal{S}^i : u^i(s^i, s^{-i}) \geq u^i(t^i, s^{-i}), \quad \forall t^i \in \mathcal{S}^i\}$$

We also denote the Best Response global of the game G as $BR : \mathcal{S} \rightrightarrows \mathcal{S}$, where $BR : s \in \mathcal{S} \mapsto BR(s) = \prod_{i \in \mathcal{I}} BR^i(s^{-i}) \subset \mathcal{S}$. We define as well the concept of *Nash Equilibrium*, developed in the beginning of the 1950s by mathematician John Nash [25, 26] where no single player has an incentive to unilaterally deviate from his chosen strategy considering the choices of the rest of players and given that they do not deviate. Formally we define

Definition A.4. *The profile $s \in \mathcal{S}$ is a Nash Equilibrium of game G if and only if*

$$\forall i \in \mathcal{I}, \quad \forall t^i \in \mathcal{S}^i \quad u^i(s^i, s^{-i}) \geq u^i(t^i, s^{-i})$$

The concept of Nash Equilibrium help us to have an idea of a ‘*solution*’ for game G . A solution is a systematic description of the outcomes that may emerge in a family of games [31]. Since it is defined on the players’ selfish actions in a non-cooperative framework, it entails the question about the extent to which selfish behavior affects system efficiency. The Price of Anarchy, originally defined by Papadimitriou [32] and applied to Internet bandwidth allocation, but inspired in the work previously done with Koutsoupias [18] addresses this question as

Definition A.5. *The Price of Anarchy, usually shortened PoA, defined for a welfare function $W : \mathcal{S} \rightarrow \mathbb{R}$ is*

$$PoA = \frac{\max_{s \in \mathcal{S}} W(s)}{\min_{s \in \mathcal{E}} W(s)}$$

where $\mathcal{E} := \{s \in \mathcal{S} : s \in BR(s)\} \subset \mathcal{S}$ is the set of profiles that are Nash equilibria.

Other important concept is the *mechanism*, which in the context of this work, refers to assign one or more goods between interested players , who have different valuations of these assets. So far, the rules of the game were taken as given, the players

had complete information about the payoffs of others and the goal was predict the outcome of a game. Now, we deal with the inverse problem. As explained in [49], the distinguishing feature of mechanism design is that the game structure is *designed* by a game designer, called a *principal* who wants to choose a set of rules for his own goal or interest. Having incomplete information about the players, who are called the *agents*, and asking them to reveal their preferences. The information about the payoff of the players are not common knowledge, and the principal ask the agents to reveal their preferences thorough the agents' bids, or messages, and want that truth-telling is an equilibrium strategy.

Definition A.6. *A mechanism defines a message space \mathcal{W}^i for each agent $i \in \mathcal{I}$ and an allocation function $(d, t) : \mathcal{W} \rightarrow D \times \mathbb{R}^{\mathcal{I}}$. For a given vector of messages $w \in \mathcal{W} := \prod_{i \in \mathcal{I}} \mathcal{W}^i$, $d(w)$ is the decision while $t := (t^i)_{i \in \mathcal{I}}$ contains the transfer $t^i(w)$ of each agent $i \in \mathcal{I}$.*

Definition A.7. *A sequential game is defined by n players with action sets $\{A^i\}_{i=1}^n$, utility functions $u^i : \mathcal{A} := \prod_i A^i \rightarrow \mathbb{R}$ for each player i and an ordering of the players, say player $1, 2, \dots, n$. In each round i , player i observes the actions chosen by players $1, 2, \dots, i - 1$ and chooses an action $a^i \in A^i$. Therefore, the strategy of player i is a mapping $s^i : \prod_{j < i} A^j \rightarrow A^i$.*

Definition A.8. *A strategy profile s^* is a subgame perfect equilibrium (SPE) of the game if it is a Nash equilibrium of every subgame of it.*

Definition A.9. *Given a welfare function $W : \mathcal{A} \rightarrow \mathbb{R}_+$, . If $SPE \subset \mathcal{A}$ are the action profiles that can happen in a subgame perfect equilibrium and $W^* = \max_{a \in \mathcal{A}} W(a)$, then we define:*

$$SPoA = \max_{e \in SPE} \frac{W^*}{W(e)}$$

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