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MINIMUM VARIANCE CONTROL OF DISCRETE TIME MULTIVARIABLE ARMAX SYSTEMS***

by

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ABSTRACT

We consider multivariable ARMAX stochastic systems. These systems can incorporate the following complicating features: general delay structures, non-minimum phase transfer functions, different dimensions for input and output vectors. We obtain the control laws which minimize the variance of the output process while maintaining system stability.

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I. INTRODUCTION

We consider multivariable linear stochastic systems in an ARMAX format:

$$A(z)y(t) = z^d B(z)u(t) + C(z)w(t) \qquad (1)$$

Here z is the backward shift operator: $zy(t) := y(t-1)$. $y(t) \in \mathbb{R}^m$ is the output, $u(t) \in \mathbb{R}^\ell$ is the input and $w(t) \in \mathbb{R}^m$ is a white noise process with mean 0 and covariance $E\, w(t)w^T(t) = Q$.

(2.i) $A(z) = I + \sum_{i=1}^{n} A_i z^i$

(2.ii) $B(z) = B_0 + \sum_{i=1}^{n} B_i z^i$. $B_0 \neq 0$ and B(z) is of full rank. (2)

(2.iii) $C(z) = C_0 + \sum_{i=1}^{n} C_i z^i$. $C^{-1}(z)$ is analytic inside the closed unit disc.

(2.iv) d, the delay, is an integer with $d \geq 1$.

We shall define as "admissible", control laws which are of the form $u(t) = M(z)y(t)$ where

(3.i) M(z) is a matrix of rational functions

(3.ii) M(z) is analytic at z=0. (3)

The condition (3.ii) restricts us to the set of non-anticipative control laws, while (3.i) is imposed merely for convenience.

We shall further say that an admissible control law $u(t) = M(z)y(t)$ is "stabilizing" if the four transfer functions

$$M(z)[I-z^dA^{-1}(z)B(z)M(z)]^{-1},\ [I-z^dA^{-1}(z)B(z)M(z)]^{-1}$$
$$[I-z^dM(z)A^{-1}(z)B(z)]^{-1} \text{ and } z^dA^{-1}(z)B(z)[I-z^dM(z)A^{-1}(z)B(z)]^{-1} \quad (4)$$

are all analytic inside the closed unit disc.

Our goal in this paper is to find a control law, from among the set of all admissible stabilizing control laws, which minimizes the variance $Ey^T(t)y(t)$ of the output process.

For single-input, single-output (i.e., $m=\ell=1$) minimum phase systems, the problem has been solved by Astrom [1]. The minimum variance control law is shown to be

$$u(t) = -\frac{-G(z)}{B(z)F(z)}\,y(t) \quad (5.i)$$

where F(z), a polynomial of degree d-1, and G(z), a polynomial, satisfy

$$C(z) = A(z)F(z) + z^dG(z) \quad (5.ii)$$

If the system is of non-minimum phase, then while the above control law still minimizes the variance of the output process from among the set of all admissible control laws, it is <u>not</u> however stabilizing. To satisfy stability, one must "sacrifice" some variance. This constrained optimization problem of obtaining a control law which minimizes the output variance over the set of all admissible, stabilizing control laws, for single-input, single-output systems has been solved by Peterka [2]. It is shown to be

$$u(t) = -\frac{S(z)}{R(z)}\,y(t) \quad (6.i)$$

where R(z), a polynomial of degree (n+d-1), and S(z), a polynomial, satisfy

$$B^*(z)C(z) = A(z)R(z) + z^d B(z)S(z) \tag{6.ii}$$

Here, $B^*(z)$ is the minimum phase spectral factor of $B(z)B(z^{-1})$.

In the multi-input, multi-output case, Borison [3] has considered the situation where i) the number of inputs is equal to the number of outputs ii) B_0 is invertible and iii) $B(z)$ is of minimum phase, i.e. det $B(z) \neq 0$ for $0<|z|\leq 1$. Under these conditions, the optimal solution is given by a multivariable analog of (5.i,ii). This treatment is not fully general from several points of view. Firstly, conditions i) and iii) are restrictive. Secondly, the restriction that B_0 is invertible, condition ii), means that by defining a new control $\bar{u}(t) := B_0 u(t)$, we really have a system where for each output variable there is one special input variable which influences that output variable after other input variables have ceased to influence it. Moreover, the different output variables will be influenced by their special input variables with the same delay. This simplifies the problem considerably and in fact one outgrowth of this restriction is that the control law really minimizes, separately, the variance of each output variable, or equivalently, the same control law simultaneously minimizes $Ey^T(t)Ry(t)$ for all $R\geq 0$. We shall see that this situation is not true in general.

In another treatment of the multi-input, multi-output case, Goodwin, Ramadge and Caines [4] assume that $A(z) = (1+\alpha_1 z+\ldots+\alpha_n z^n)I$ where $\alpha_1,\ldots,\alpha_n$ are scalars. Stability of the solution is not considered, but use is made of the solution only when d=1, the number of inputs is equal to the number of outputs, B_0 is invertible, and the system is of minimum phase, i.e. det $B(z) \neq 0$ for $0<|z| \leq 1$, in which situation there are no problems.

We also refer the reader to Bayoumi and El Bagoury [5] for some errors in previous attempts to deal with the problem of minimum variance control of multi-variable systems.

In this paper, our goal is to treat all the complications caused by i) B_0 possibly singular, i.e. general delay structures, ii) non-minimum phase systems, i.e. det $B(z)$ possibly vanishing in $0<|z|<1$ and iii) rectangular systems where the number of inputs is different from the number of outputs. Throughout, we address the problem of minimizing $Ey^T(t)y(t)$ while maintaining system stability.

If one wishes to minimize $Ey^T(t)Ry(t)$ for some positive definite R, than this is easily accomplished by defining $\bar{y}(t) := R^{1/2}y(t)$, $\bar{A}(z) := R^{1/2}A(z)R^{-1/2}$, $\bar{B}(z) := R^{1/2}B(z)$, $\bar{C}(z) := R^{1/2}C(z)$ and considering the system $\bar{A}(z)\bar{y}(t) = z^d\bar{B}(z)u(t) + \bar{C}(z)w(t)$, which satisfies assumptions (2.i-iv).

Our treatment proceeds in the order of increasing generality. In Section II we treat systems with general delay structures, with the solution given by Theorems 2.1, 2.2 and 2.3. In Section III we treat non-minimum phase systems, with the solution given in Theorem 3.1 and finally in Section IV we treat rectangular systems, with the solution provided in Theorem 4.1.

II. NON-UNIFORM DELAY SYSTEMS

In this section we obtain the admissible, stabilizing, minimum variance control law for the multivariable ARMAX system (1), when it has a general delay structure. For this reason we allow det $B(z)$ to have zeroes at the origin, because such zeroes correspond to non-uniform transmission delays in different input-output channels.

Except for such zeroes at the origin, we assume that the system is of a minimum phase, i.e., det $B(z) \neq 0$ for $0<|z|<1$. The system is also assumed to have the same number of inputs and outputs, i.e. it is square.

The complete solution for this problem is furnished by the following three Theorems.

Theorem 2.1

Suppose there exist $F(z)$ and $G(z)$ which satisfy:

(7.i) $F(z) = \sum_{i=0}^{d+p-1} F_i z^i$ for some p, and F_0 is invertible.

(7.ii) $G(z)$ is a matrix of rational functions which are analytic at $z=0$. (7)

(7.iii) $\lim_{z \to 0} z^d F^T(z^{-1}) A^{-1}(z) B(z) = 0$

(7.iv) $C(z) = A(z)F(z) + z^d B(z)G(z)$

Then, the admissible, stabilizing control law which minimizes the variance $Ey^T(t)y(t)$ of the output, is

$$u(t) = -G(z)F^{-1}(z)y(t)$$

The resulting minimum variance is

$$Ey^T(t)y(t) = \text{tr} \sum_{i=0}^{d+p-1} F_i^T F_i Q$$

Theorem 2.2

Define the following:

(8.i) Let $\sum_{i=0}^{\infty} D_i z^i$ be a power series expansion of $A^{-1}(z)B(z)$.

(8.ii) Let p be the largest power of z^{-1} in $B^{-1}(z)A(z)$

(8.iii) Let $E_0, E_1, \ldots, E_p$ be matrices satisfying

$$B^{-1}(z)A(z) = E_p z^{-p} + E_{p-1} z^{-p+1} + \ldots + E_0 + o(1) \tag{8}$$

(8.iv) Let $\mathcal{W}_m^n := \begin{bmatrix} 0 & \cdot & \cdot & 0 & D_m \\ \cdot & & & & \\ \cdot & & & D_m & D_{m+1} \\ 0 & & \cdot & \cdot & \cdot \\ & \cdot & & \cdot & \cdot \\ D_m & \cdot & \cdot & D_{n-1} & D_n \end{bmatrix}$ and $\mathcal{E}_m^n := \begin{bmatrix} E_n & 0 & \cdot & \cdot & 0 \\ & & & & \cdot \\ E_{n-1} & E_n & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & & 0 \\ \cdot & & & \cdot & \\ E_m & E_{m+1} & \cdot & \cdot & E_n \end{bmatrix}$

Then, the matrix

$$\left[\mathcal{W}_0^{p-1}, \mathcal{E}_1^{pT}\right]$$

has full rank.

Theorem 2.3

Define the following:

(9.i) Define $F_0, \ldots, F_{d-1}$ recursively by $F_0 := C_0$ and

$$F_k := C_k - \sum_{i=1}^{k} A_i F_{k-i} \quad \text{for } k=1,\ldots,d-1 \tag{9}$$

(9.ii) Define $H_d, H_{d+1}, \ldots$ as the coefficient matrices in the power series expansion

$$A^{-1}(z)C(z) - \sum_{i=0}^{d-1} F_i z^i =: \sum_{i=d}^{\infty} H_i z^i$$

(9.iii) Let K and J be matrices which satisfy the linear system of equations

$$[\mathcal{W}_0^{p-1}, \mathcal{E}_1^{pT}][K^T, J^T]^T = [H_d^T, \ldots, H_{d+p-1}^T]^T$$

(9.iv) Define $F_d, \ldots, F_{d+p-1}$ by

$$[F_d^T, \ldots, F_{d+p-1}^T] := \mathcal{E}_1^{pT} J$$

(9.v) Define $F(z) := \sum_{i=0}^{d+p-1} F_i z^i$ and

$$G(z) := z^{-d} B^{-1}(z)[C(z) - A(z)F(z)]$$

Then, $F(z)$ and $G(z)$ satisfy (7.i-7.iv).

The significance of the three Theorems 2.1, 2.2 and 2.3 is the following. Theorem 2.1 gives sufficient conditions for the solution $u(t) = -G(z)F^{-1}(z)y(t)$ to be optimal. Theorem 2.2 asserts that a certain matrix is of full rank. Theorem 2.3 uses the solution of a system of linear equations, guaranteed to exist by Theorem 2.2, to construct $F(z)$ and $G(z)$ which satisfy the sufficient conditions of Theorem 2.1. Thus, we have a constructive procedure for obtaining an admissible, stabilizing, minimum variance control law.

One useful property of the minimum variance control law is that it does <u>not</u> depend on the noise covariance Q. Thus, the <u>same</u> control law is optimal irrespective of the noise covariance.

As we have mentioned earlier at the end of Section I, the above Theorems can be employed to solve the problem of minimizing $Ey^T(t)Ry(t)$ for any positive definite R. However, in general, the solution will depend on R. This means, in particular, that the control law of Theorems 2.1, 2.2 and 2.3 does not separately minimize the variance of each output variable. This differentiates the case det $B_0 \neq 0$, considered in [3], from the general delay structures considered here.

The minimum variance tr $\sum_{i=0}^{d+p-1} F_i^T F_i Q$ can be decomposed into two parts. tr $\sum_{i=d}^{d+p-1} F_i^T F_i Q$ can be regarded as the increase in variance resulting from the singularity of B_0, while the remaining part $\sum_{i=0}^{d-1} F_i^T F_i Q$ is the variance due to the delay of d time units. In the case considered in [3], only the latter part is present.

The proofs of Theorems 2.1, 2.2 and 2.3 follow immediately from Lemmas 2.4-2.10 below.

Lemma 2.4

Suppose F(z) is a matrix of polynomials, which, together with a certain G(z) satisfies (7.ii,iii and iv). Let $u(t) = M(z)y(t)$ be any admissible control law which is applied to the system (1). Then, the output y(t) of the closed loop system can be decomposed as

$$y(t) = F(z)w(t) + z^d A^{-1}(z)B(z)[G(z) + T(z)A^{-1}(z)C(z)]w(t)$$

where

$$T(z) := M(z)[I-z^d A^{-1}(z)B(z)M(z)]^{-1}$$

Furthermore, the two components

$$F(z)w(t) \text{ and } z^d A^{-1}(z)B(z)[G(z) + T(z)A^{-1}(z)C(z)]w(t)$$

are uncorrelated.

Proof

The closed loop system is clearly $Ay = z^d BMy + Cw$, and so $y = (I - z^d A^{-1}BM)^{-1}A^{-1}Cw$ and $u = TA^{-1}Cw$. Substituting for u, we therefore get $Ay = z^{d}BTA^{-1}Cw + Cw$. Using (7.iv) for C gives the required decomposition for the closed-loop output y. To see that the two components are uncorrelated, note first that

$$\mathrm{cor}(Fw,\ z^{d}A^{-1}B[G+TA^{-1}C]w) = \frac{\mathrm{tr}}{2\pi i}\oint F^{T}(z^{-1})z^{d}A^{-1}(z)B(z)[G(z)+T(z)A^{-1}(z)C(z)]Q\frac{dz}{z}$$

where, here and in the sequel, the contour is a circle centered at the origin and with radius so small that it does not encircle any singularities of the integrand other than those at the origin. Now G(z) is analytic at the origin, by assumption. Also, because M(z) is analytic at the origin, so is T(z), and therefore also $T(z)A^{-1}(z)C(z)$. Utilizing (7.iii) we see that the above integral vanishes.

□

Lemma 2.5

Suppose that F(z) and G(z) satisfy (7.i-iv). Then, the control law which minimizes $Ey^{T}(t)y(t)$ over the set of all admissible control laws is $u(t) = M(z)y(t)$, where

$$M(z) = -G(z)F^{-1}(z)$$

and the resulting minimum variance is

$$Ey^{T}(t)y(t) = \mathrm{var}(F(z)w(t)) = \mathrm{tr}\sum_{i=0}^{p+d-1} F_i^{T}F_iQ\ .$$

Proof

From Lemma 2.4 it follows that for an admissible choice of M,

$$Ey^{T}(t)y(t) = \mathrm{var}(F(z)w(t)) \tag{10}$$
$$+\frac{\mathrm{tr}}{2\pi i}\oint [G(z^{-1}) + T(z^{-1})A^{-1}(z^{-1})C(z^{-1})]^{T}B^{T}(z^{-1})A^{-T}(z^{-1})$$
$$A^{-1}(z)B(z)[G(z)+T(z)A^{-1}(z)C(z)]Q\frac{dz}{z}$$

Since $F(z)$ does not depend on the choice of $M(z)$, the best that one can hope to do, if one wishes to minimize the variance, is to choose $M(z)$ so that the integral on the right hand side above is zero. One way to do this is to choose $M(z)$ so as to make $G(z) + T(z)A^{-1}(z)C(z) = 0$, i.e. $T(z) = -G(z)C^{-1}(z)A(z)$. Since $T = M[I-z^dA^{-1}BM]^{-1}$, M would have to be chosen so that $T^{-1} = M^{-1} - z^dA^{-1}B$, i.e. $M = [(I+z^dA^{-1}BT)T^{-1}]^{-1} = T(I+z^dA^{-1}BT)^{-1} = -GC^{-1}A(I-z^dA^{-1}BGC^{-1}A)^{-1} = -GC^{-1}A[I-A^{-1}(C-AF)C^{-1}A]^{-1} = -GF^{-1}$. It remains to be seen whether this choice of M is admissible. Clearly it is a matrix of rational functions and so (3.i) is satisfied. So we need to only check that (3.ii), i.e. non-anticipativity, is satisfied. Now $G(z)$ is analytic at the origin, by assumption, and also $F^{-1}(0) = F_0^{-1} = C_0^{-1}$ exists by assumption, showing that $M(z)$ is analytic at the origin. □

<u>Lemma 2.6</u>

Suppose $F(z)$ and $G(z)$ satisfy (7.i,ii,iv). Then, the control law

$$u(t) = -G(z)F^{-1}(z)y(t)$$

is stabilizing.

<u>Proof</u>

To determine that the control law is stabilizing, we need to check that the four transfer functions in (4) are all analytic inside the closed unit disc, with M given by $M = -GF^{-1}$. Simple calculation using (7.iv) shows that

$$M[I-z^dA^{-1}BM]^{-1} = -GC^{-1}A = -z^{-d}B^{-1}(C-AF)C^{-1}A \qquad (11.i)$$

$$[I-z^dA^{-1}BM]^{-1} = FC^{-1}A \qquad (11.ii)$$

$$[I-z^d MA^{-1}B]^{-1} = B^{-1}AFC^{-1}B = I + [-z^{-d}B^{-1}(C-AF)C^{-1}A][z^d A^{-1}B] \qquad (11.iii)$$

$$z^d A^{-1}B[I-z^d MA^{-1}B]^{-1} = z^d FC^{-1}B \qquad (11.iv)$$

B^{-1} is analytic inside the closed unit disc, except possibly at the origin, by assumption. (C-AF) is a polynomial by (7.i). Also C^{-1} is analytic inside the closed unit disc by (2.iii). Hence $z^{-d}B^{-1}(C-AF)C^{-1}A$ is analytic inside the closed unit disc, except perhaps at the origin. However $z^{-d}B^{-1}(C-AF)C^{-1}A = GC^{-1}A$ and since G is analytic at the origin, so is $GC^{-1}A$. Hence (11.i) is analytic inside the closed unit disc. (11.ii) and (11.iv) are both analytic inside the closed unit disc since $C^{-1}(z)$ is so and A,B,F are all matrices of polynomials. Examining (11.iii), $B^{-1}AFC^{-1}B$ is analytic inside the closed unit disc, except perhaps at the origin. However $z^d A^{-1}B$ is analytic at the origin, and (11.i) has also been shown to be so. Hence $B^{-1}AFC^{-1}B = I + [z^{-d}B^{-1}(C-AF)C^{-1}A][z^d A^{-1}B]$ is also analytic at the origin, thus showing that (11.iii) is analytic inside the closed unit disc. □

It may be noted that at this stage Theorem 2.1 has been proved. Now we need to establish Theorems 2.2 and 2.3

<u>Lemma 2.7</u>

Let $\tilde{\mathcal{E}}_0^p = \begin{bmatrix} \tilde{E}_p & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & 0 \\ \tilde{E}_0 & \cdots & & \tilde{E}_p \end{bmatrix}$ for some matrices $\tilde{E}_0,\ldots,\tilde{E}_p$. If

$\tilde{\mathcal{E}}_0^p\, \mathcal{W}_0^p = \mathrm{diag}(0,\ldots,0,I)$ then there exists a square matrix N of the form

$N = \begin{bmatrix} I & 0 & 0 \\ \alpha_1 & \ddots & \vdots \\ \vdots & \ddots & 0 \\ \alpha_p & \cdots\ \alpha_1 & I \end{bmatrix}$ such that $\tilde{\mathcal{E}}_0^p = N\mathcal{E}_0^p$.

Proof:

Since $\tilde{\mathcal{E}}_0^p \mathcal{W}_0^p = \text{diag}(0,\dots,0,I)$ it follows that

$$(\tilde{E}_p z^{-p} + \tilde{E}_{p-1} z^{-p+1} + \dots + \tilde{E}_0)(D_0 + D_1 z + \dots) = I + 0(z)$$

where $0(z) = \alpha_1 z + \alpha_2 z^2 + \dots$ for some matrices $\alpha_1, \alpha_2, \dots$ Hence

$$(\tilde{E}_p z^{-p} + \dots + \tilde{E}_0) = (I + 0(z))(D_0 + D_1 z + \dots)^{-1}$$

$$= (I + 0(z))(E_p z^{-p} + E_{p-1} z^{-p+1} + \dots + E_0 + o(1))$$

Equating coefficients of identical powers of z^{-1} we get

$$\tilde{E}_p = E_p,\ \tilde{E}_{p-i} = E_{p-i} + \sum_{k=1}^{i} \alpha_k E_{p+k-i} \text{ for } i=1,\dots,p.$$

Hence the suggested N suffices. □

Lemma 2.8

$$N(\mathcal{W}_0^{(p-1)T}) = R(\mathcal{E}_1^{pT})$$

Here $N(\cdot)$ denotes the null space and $R(\cdot)$ the range space.

Proof

Consider $(\beta_1^T,\dots,\beta_p^T)^T \in N(\mathcal{W}_0^{(p-1)T})$. Suppose to the contrary that $(\beta_1^T,\dots,\beta_p^T)^T \notin R(\mathcal{E}_1^{pT})$. Since $D_0 \neq 0$ we can find a row vector β_0^T so that $(\beta_0^T,\dots,\beta_p^T)[D_0^T,\dots,D_p^T]^T \neq 0$. Since $(\beta_1^T,\dots,\beta_p^T)^T \notin R(\mathcal{E}_1^{pT})$, it follows that $(\beta_1^T,\dots,\beta_p^T)^T \notin R([E_1,\dots,E_p]^T)$. Hence $(\beta_0^T,\dots,\beta_p^T)^T \notin R([E_0,\dots,E_p]^T)$. Since $[E_0,\dots,E_p][D_0^T,\dots,D_p^T]^T = I$, as is easily checked, it follows that by choosing (m-1) rows from $[E_0,\dots,E_p]$ (if I above is mxm) and the row $(\beta_0^T,\dots,\beta_p^T)$ we can build a matrix $[\bar{E}_0,\bar{E}_1,\dots,\bar{E}_p]$ with m rows and such that $[\bar{E}_0,\ldots,\bar{E}_p][D_0^T,\ldots,D_p^T]^T$ is of full rank. Premultiplying by an appropriate nonsingular matrix we can obtain $[\tilde{E}_0,\ldots,\tilde{E}_p]$ such that $[\tilde{E}_0,\ldots,\tilde{E}_p][D_0^T,\ldots,D_p^T]^T = I$ and the rows of $[\tilde{E}_0,\ldots,\tilde{E}_p]$ span the row -space of $[\bar{E}_0,\ldots,\bar{E}_p]$. Now let $\tilde{\mathcal{E}}_0^p$ be defined from $\tilde{E}_0,\ldots,\tilde{E}_p$ as in the statement of Lemma 2.7. It is easily checked that $\tilde{\mathcal{E}}_0^p\mathcal{W}_0^p = \mathrm{diag}(0,\ldots,0,I)$. Lemma 2.7 now applies and shows that the rows of $[\tilde{E}_0,\ldots,\tilde{E}_p]$ are linear combinations of the rows of $\mathcal{E}_0^p$. But then the rows of $[\tilde{E}_1,\ldots,\tilde{E}_p]$ are linear combinations of the rows of $\mathcal{E}_1^p$, which contradicts our assumption that $(\beta_1^T,\ldots,\beta_p^T)^T \notin R(\mathcal{E}_1^{pT})$. This shows that $N(\mathcal{W}_0^{(p-1)T}) \subseteq R(\mathcal{E}_1^{pT})$. The reverse containment $R(\mathcal{E}_1^{pT}) \subseteq N(\mathcal{W}_0^{(p-1)T})$ follows trivially from the relationship $\mathcal{E}_0^p\mathcal{W}_0^p = \mathrm{diag}(0,\ldots,0,I)$. □

<u>Lemma 2.9</u>

$[\mathcal{W}_0^{p-1}, \mathcal{E}_1^{pT}]$ is a full rank matrix.

<u>Proof</u>

Suppose $\rho^T[\mathcal{W}_0^{p-1}, \mathcal{E}_1^{pT}] = 0$ for some vector ρ. Since $\rho \in N(\mathcal{W}_0^{(p-1)T})$ it follows by Lemma 2.8 that $\rho = \mathcal{E}_1^{pT}\gamma$ for some γ. But $\rho^T\mathcal{E}_1^{pT}\gamma = 0$, and so $\gamma^T\mathcal{E}_1^p\mathcal{E}_1^{pT}\gamma = 0$. Hence $\rho = \mathcal{E}_1^{pT}\gamma = 0$. □

Thus we have also proved Theorem 2.2. Now we complete the proof of Theorem 2.3.

<u>Lemma 2.10</u>

If F(z) and G(z) are defined as in Theorem 2.3, then (7.i-iv) of Theorem 2.1 are satisfied.

Proof

(7.i) is trivial since $F(0) = F_0 = C_0$ is invertible by assumption. (7.iv) follows from the definition of $G(z)$. So we need to check only (7.ii) and (7.iii). Now

$$(7.iii) \iff \lim_{z\to 0} z^d[F_0^T + F_1^T z^{-1} + \ldots + F_{d+p-1}^T z^{-d-p+1}][D_0 + D_1 z + \ldots + D_{p-1} z^{p-1}] = 0$$

$$\iff \lim_{z\to 0} [F_d^T + F_{d+1}^T z^{-1} + \ldots + F_{d+p-1}^T z^{-p+1}][D_0 + D_1 z + \ldots + D_{p-1} z^{p-1}] = 0$$

$\iff$ coefficients of nonpositive powers of z vanish in

$$[F_d^T + \ldots + F_{d+p-1}^T z^{-p+1}][D_0 + D_1 z + \ldots + D_{p-1} z^{p-1}]$$

$$\iff [F_d^T, \ldots, F_{d+p-1}^T]^T \in N(\mathcal{W}_0^{(p-1)T})$$

$$\iff [F_d^T, \ldots, F_{d+p-1}^T] \in R(\mathcal{E}_1^{pT})$$

$$\iff [F_d^T, \ldots, F_{d+p-1}^T] = \mathcal{E}_1^{pT} J \text{ for some matrix } J.$$

Similarly

$$(7.iv) \iff z^{-d} B^{-1}(z)[C(z) - A(z)F(z)] = O(1)$$

$$\iff z^{-d} B^{-1}(z) A(z)[A^{-1}(z) C(z) - F(z)] = O(1)$$

$$\iff z^{-d} B^{-1}(z) A(z)[A^{-1}(z) C(z) - F_0 - F_1 z - \ldots - F_{d+p-1} z^{d+p-1}] = O(1)$$

$$\iff z^{-d}B^{-1}(z)A(z)[H_d z^d + H_{d+1}z^{d+1} + \ldots - F_d z^d - F_{d+1}z^{d+1} \ldots - F_{d+p-1}z^{d+p-1}\} = O(1)$$

$$\iff B^{-1}(z)A(z)[(H_d-F_d)+\ldots+(H_{d+p-1}-F_{d+p-1})z^{p-1} + O(z^p)] = O(1)$$

$\iff$ coefficients of strictly negative powers of z vanish in

$$[E_p z^{-p}+\ldots+E_1 z^{-1}+O(1)][(H_d-F_d)+\ldots+(H_{d+p-1}-F_{d+p-1})z^{p-1}+O(z^p)]$$

$$\iff [(H_d-F_d)^T,\ldots,(H_{d+p-1}-F_{d+p-1})^T]^T \in N(\mathcal{E}_1^p)$$

$$\iff [(H_d-F_d)^T,\ldots,(H_{d+p-1}-F_{d+p-1})^T]^T \in R(\mathcal{W}_0^{p-1})$$

$$\iff [(H_d-F_d)^T,\ldots,(H_{d+p-1}-F_{d+p-1})^T]^T \in \mathcal{W}_0^{p-1}K \text{ for some matrix } K$$

Thus if $[\mathcal{W}_0^{p-1}, \mathcal{E}_1^{pT}][K^T,J^T]^T = [H_d^T,\ldots,H_{d+p-1}^T]^T$ and $\mathcal{E}_1^{pT}J = [F_d^T,\ldots,F_{d+p-1}^T]$, as we have assumed, then both (7.ii) and (7.iii) are satisfied. □

The proofs of Theorems 2.1, 2.2 and 2.3 are now complete.

III. SQUARE NON-MINIMUM PHASE SYSTEMS

We now turn to the problem of minimizing the variance over the set of admissible, stabilizing control laws for systems which have non-minimum phase transfer functions besides those caused by pure delays.

Thus we consider systems for which det B(z) may vanish in $\{z:0<|z|<1\}$ besides possible vanishing in $\{z:z=0 \text{ or } |z| > 1\}$. We do not allow det B(z) to vanish in $\{z:|z| = 1\}$ since we have imposed the requirement in (4) that our closed-loop systems should be <u>strictly</u> stable as opposed to just stable, i.e. we have required analyticity of the four transfer functions in (4) in the <u>closed</u> unit disc and not just the <u>open</u> unit disc. If we are willing to admit such a relaxation, then our solution is valid even for det B(z) vanishing on the unit circle $\{z:|z| = 1\}$.

In this section, we also assume that the number of inputs is equal to the number of outputs, i.e. the system is square with $m=\ell$ in (1).

By Lemma 2.5 we see that we have already solved the problem of obtaining the admissible control law which minimizes the variance of the output, and the control law which does this is just the control law of Theorems 2.1., 2.2 and 2.3. However, this control law is <u>not</u> stabilizing, i.e. it does not satisfy (4), when det B(z) vanishes in $\{z:0<|z|\leq 1\}$. The reasons is that Lemma 2.6 is no more valid, as can be seen from an examination of (11).

A graphic illustration of the loss of stability and its consequences, which result when we attempt to just minimize the output variance without constraining ourselves to the set of stabilizing control laws, is given by the following example.

<u>Example</u>

Consider the non-minimum phase system

$$y(t+1) = -y(t) + u(t) - 2u(t-1) + w(t+1) \tag{12}$$

The control law which minimizes the output variance $Ey^2(t)$ over the class of all admissible control laws is

$$u(t) = 2u(t-1) + y(t) \tag{13}$$

and the resulting variance is

$$Ey^2(t) = Ew^2(t)$$

However, the recursion for the control (13) shows that $\{u(t)\}$ is an exploding sequence. This is clearly undesirable from several points of view. Note that one of the transfer functions of (4), $M[1-z^d A^{-1}BM]^{-1} = \frac{1+z}{1-2z}$ is unstable, and therefore our formulation specifically excludes such control laws. □

For single-input, single-output systems, such as in the above example, Peterka [2] has solved the problem of obtaining the control law which minimizes the output variance over the class of all admissible, stabilizing control laws. We now solve this problem for the multivariable case.

We will obtain the solution by reducing the problem to the type considered in the previous section. Accordingly we will denote the $F(z)$ and $G(z)$ generated by Theorem 2.3 by $F(A(\cdot),B(\cdot),C(\cdot),d)(z)$ and $G(A(\cdot),B(\cdot),C(\cdot),d)(z)$ in order to explicity display the functional arguments on which they depend. We note here that the algorithms of Theorems 2.2 and 2.3 can be employed even when $d=0$ to generate F and G.

<u>Theorem 3.1</u>

We assume that $A^{-1}(z)$ and $B^{-1}(z)$ have no poles in common inside the closed unit disc, $A^{-1}(z)$ and $B^{-1}(z^{-1})$ have no poles in common inside the closed unit disc and $A^{-1}(z)$ and $A^{-1}(z^{-1})$ have no poles in common. In the above and what follows, by a zero of $X(z)$ we shall mean a singularity of $X^{-1}(z)$, and by a pole of $X(z)$ we mean a singularity of $X(z)$.

(14.i) Let $\Delta(z)$ be a spectral factor which satisfies

$$\Delta^T(z^{-1})\Delta(z) = B^T(z^{-1})A^{-T}(z^{-1})A^{-1}(z)B(z)$$

and is such that its poles are those of $A^{-1}(z)B(z)$, while its non-zero zeroes are the outside the closed unit disc images of the nonzero zeroes of $A^{-1}(z)B(z)$. By an "outside the closed unit disc image of z", we mean η such that $\eta=z$ if $|z|>1$ and $\eta=z^{-1}$ if $|z|<1$.

(14.ii) Let $\alpha(z)$ and $\beta(z)$ be matrices of polynomials such that

$$\alpha^{-1}(z)\beta(z) = \Delta(z)$$

is a left coprime representation of $\Delta(z)$, and such that the zeroes of $\beta(z)$ are the zeroes of $\Delta(z)$, while the poles of $\alpha^{-1}(z)$ are the poles of $\Delta(z)$. (14)

(14.iii) Let $\theta(z) := \alpha^{-1}(z)\beta(z)B^{-1}(z)[C(z) - A(z)F(z)]z^{-d}$, where $F(z) := F(A(\cdot),B(\cdot),C(\cdot),d)(z)$ and $G(z) :=$

$$G(A(\cdot), B(\cdot), C(\cdot),d)(z)$$

(14.iv) Let $\theta_+(z)$ and $\theta_-(z)$ satisfying $\theta(z) = \theta_+(z) + \theta_-(z)$ be such that $\theta_+(z)$ is the sum of all the partial fraction terms of $\theta(z)$ which have poles either outside the closed unit disc (including infinity) or coinciding with the poles of $A^{-1}(z)$ inside the closed unit disc, and constant terms, if any.

(14.v) Let $\gamma(z)$ be a polynomial matrix such that

$$\theta_+(z) = \alpha^{-1}(z)\gamma(z)$$

(The existence of such a polynomial matrix $\gamma(z)$ will be proved).

(14.vi) Let $\tilde{F}(z) := F(\alpha(\cdot), \beta(\cdot), \gamma(\cdot), 0)(z)$ and

$\tilde{G}(z) := G(\alpha(\cdot), \beta(\cdot), \gamma(\cdot), 0)(z)$

Then, the control law which minimizes $Ey^T(t)y(t)$ over the class of all admissible, stabilizing control laws is given by

$$u(t) = -\tilde{G}(z)[F(z) + z^d A^{-1}(z)B(z)(G(z)-\tilde{G}(z))]^{-1}y(t) \tag{15}$$

The resulting minimum variance is

$$Ey^T(t)y(t) = \operatorname{tr}\sum_j F_j^T F_j Q + \operatorname{tr}\sum_j \tilde{F}_j^T \tilde{F}_j Q$$
$$+ \frac{\operatorname{tr}}{2\pi i}\oint \{\alpha^{-1}(z)\beta(z)[G(z)-\tilde{G}(z)]-\tilde{F}(z)\} \tag{16}$$
$$\{\alpha^{-1}(z^{-1})\beta(z^{-1})[G(z^{-1})-\tilde{G}(z^{-1})]-\tilde{F}(z^{-1})\}^T Q \frac{dz}{z}$$

where $F(z) =: \sum_j F_j z^j$ and $\tilde{F}(z) =: \sum_j \tilde{F}_j z^j$.

<u>Proof</u>

Let $u(t) = M(z)y(t)$. From (10), it follows that

$$Ey^T(t)y(t) = \operatorname{tr}\sum_j F_j^T F_j Q + \frac{\operatorname{tr}}{2\pi i}\oint B^T(z^{-1})A^{-T}(z^{-1})A^{-1}(z)B(z)[G(z) + T(z)A^{-1}(z)C(z)]$$
$$Q[G(z^{-1}) + T(z^{-1})A^{-1}(z^{-1})C(z^{-1})]^T \frac{dz}{z}$$

Since $\beta^T(z^{-1})\alpha^{-T}(z^{-1})\alpha^{-1}(z)\beta(z) = B^T(z^{-1})A^{-T}(z^{-1})A^{-1}(z)B(z)$ from (14.i,ii), it follows that

$$Ey^T(t)y(t) = \operatorname{tr}\sum_j F_j^T F_j Q$$
$$+ \frac{\operatorname{tr}}{2\pi i}\oint \beta^T(z^{-1})\alpha^{-T}(z^{-1})[\alpha^{-1}(z)\beta(z)G(z) + \alpha^{-1}(z)\beta(z)T(z)A^{-1}(z)C(z)]$$
$$Q[G(z^{-1}) + T(z^{-1})A^{-1}(z^{-1})C(z^{-1})]^T \frac{dz}{z}$$

Substituting $G(z) = B^{-1}(z)[C(z)-A(z)F(z)]z^{-d}$, and using (14.iii,iv) gives

$Ey^T(t)y(t) =$

$$\mathrm{tr}\sum_j F_j^T F_j Q + \frac{\mathrm{tr}}{2\pi i}\oint [\theta_+(z)+\theta_-(z)+\alpha^{-1}(z)\beta(z)T(z)A^{-1}(z)C(z)]$$

$$Q[\theta_+(z^{-1}) + \theta_-(z^{-1}) + \alpha^{-1}(z^{-1})\beta(z^{-1})T(z^{-1})A^{-1}(z^{-1})C(z^{-1})]^T \frac{dz}{z}$$

By (14.iv), $\theta_+(z)$ has no poles inside the closed unit disc except those of $A^{-1}(z)$. However, by (2.i), $A^{-1}(z)$ has no pole at the origin. Hence $\theta_+(z)$ is analytic at the origin. By (14.iv), $\theta_-(z)$ is a matrix of strictly proper rational functions and so $\frac{1}{z}\theta_-(z^{-1})$ is analytic at the origin. Moreover, to satisfy our stabilizing assumption (4) it is necessary that $T(z) = M(z)[I-z^d A^{-1}(z)B(z)M(z)]^{-1}$, which is one of the four transfer functions in (4), is analytic inside the closed unit disc, and in particular is analytic at the origin. Hence, $\alpha^{-1}(z)\beta(z)T(z)A^{-1}(z)C(z)$ is also required to be analytic at the origin. Therefore the cross term $\frac{\mathrm{tr}}{2\pi i}\oint [\theta_+(z) + \alpha^{-1}(z)\beta(z)T(z)A^{-1}(z)C(z)]Q\theta_-^T(z^{-1})\frac{dz}{z}$ vanishes since the integrand is analytic at the origin. Hence

$$Ey^T(t)y(t) = \mathrm{tr}\sum_j F_j^T F_j Q + \frac{\mathrm{tr}}{2\pi i}\oint \theta_-(z)Q\theta_-^T(z^{-1})\frac{dz}{z}$$

$$+ \frac{\mathrm{tr}}{2\pi i}\oint [\theta_+(z) + \alpha^{-1}(z)\beta(z)T(z)A^{-1}(z)C(z)]$$

$$Q[\theta_+(z^{-1})+\alpha^{-1}(z^{-1})\beta(z^{-1})T(z^{-1})A^{-1}(z^{-1})C(z^{-1})]^T \frac{dz}{z} \tag{17}$$

The first two terms in the right hand side of (17) do not depend on the choice of $T(z)$ and, therefore, on the choice of $M(z)$. Hence to minimize $Ey^T(t)y(t)$, we need to only minimize

$$\frac{tr}{2\pi i}\oint [\theta_+(z) + \alpha^{-1}(z)\beta(z)T(z)A^{-1}(z)C(z)]$$

$$Q[\theta_+(z^{-1}) + \alpha^{-1}(z^{-1})\beta(z^{-1})T(z^{-1})A^{-1}(z^{-1})C(z^{-1})]^T \frac{dz}{z} \tag{18}$$

Now let us examine $\theta_+(z)$. From (14.i,ii) we see that

$$\alpha^{-1}(z)\beta(z)B^{-1}(z) = \alpha^T(z^{-1})\beta^{-T}(z^{-1})B^T(z^{-1})A^{-T}(z^{-1})A^{-1}(z) \tag{19}$$

An examination of the right hand side of (19) shows that the only poles of (19) which do not coincide with those of $A^{-1}(z)$ are either at the origin or coincide with the poles of $\beta^{-T}(z^{-1})$, and so all the poles of the left hand side of (19) which do not coincide with the poles of $A^{-1}(z)$ are inside the closed unit disc. Substituting (19) in the expression for $\theta(z)$ in (14.iii) we obtain

$$\theta(z) = \alpha^T(z^{-1})\beta^{-T}(z^{-1})B^T(z^{-1})A^{-T}(z^{-1})A^{-1}(z)[C(z)-A(z)F(z)]z^{-d}$$

Utilizing the definition of $\theta_+(z)$, we see therefore that

$$\theta_+(z) = \alpha^{-1}(z)\gamma(z)$$

for some polynomial matrix $\gamma(z)$. This proves the existence of $\gamma(z)$ claimed in (14.v). Now substituting for $\theta_+(z)$ in (18) shows that to minimize $Ey^T(t)y(t)$, we need to minimize

$$\frac{tr}{2\pi i}\oint [\alpha^{-1}(z)\gamma(z) + \alpha^{-1}(z)\beta(z)T(z)A^{-1}(z)C(z)]$$

$$Q[\alpha^{-1}(z^{-1})\gamma(z^{-1}) + \alpha^{-1}(z^{-1})\beta(z^{-1})T(z^{-1})A^{-1}(z^{-1})C(z^{-1})]^T \frac{dz}{z}$$

Define $S(z) := T(z)A^{-1}(z)C(z)$ and our problem now is how to choose $S(z)$, analytic at the origin, so as to minimize

$$\frac{tr}{2\pi i}\oint [\alpha^{-1}(z)\gamma(z) + \alpha^{-1}(z)\beta(z)S(z)]Q[\alpha^{-1}(z^{-1})\gamma(z^{-1})$$
$$+ \alpha^{-1}(z^{-1})\beta(z^{-1})S(z^{-1})]^T \frac{dz}{z} \qquad (20)$$

But this resembles the problem of Section II, where since we had

$$y(t) = [A^{-1}(z)C(z) + z^d A^{-1}(z)B(z)T(z)A^{-1}(z)C(z)]w(t)$$
$$= [A^{-1}(z)C(z) + z^d A^{-1}(z)B(z)S(z)]w(t)$$

we had to choose $S(z)$, analytic at the origin, so as to minimize

$$\frac{tr}{2\pi i}\oint [A^{-1}(z)C(z) + z^d A^{-1}(z)B(z)S(z)]$$
$$Q[A^{-1}(z^{-1})C(z^{-1}) + z^{-d}A^{-1}(z^{-1})B(z^{-1})S(z^{-1})]^T \frac{dz}{z} \qquad (21)$$

Making the obvious identifications between (20) and (21), we can apply the results of Section II and see that the optimal choice for $S(z)$ is

$$S(z) = -\tilde{G}(z) \qquad (22)$$

where $\tilde{G}(z)$ is as in (14.vi). Furthermore the minimum value of (20) is

$$\sum_j \tilde{F}_j^T \tilde{F}_j Q \qquad (23)$$

Since (23) is the minimum value of the third term in right hand side of (17), it follows by substituting in (17) that the resulting variance is

$$Ey^T(t)y(t) = tr\sum_j F_j^T F_j Q + tr\sum_j \tilde{F}_j^T \tilde{F}_j Q$$
$$+ \frac{tr}{2\pi i}\oint \theta_-(z)Q\theta_-^T(z^{-1})\frac{dz}{z} \qquad (24)$$

Since $C(z) = A(z)F(z) + z^d B(z)G(z)$ and $\gamma(z) = \alpha(z)\tilde{F}(z) + \beta(z)\tilde{G}(z)$, which follow from the definitions of F, G, $\tilde{F}$ and $\tilde{G}$ in (14.iii,vi) we obtain $G(z) = B^{-1}(z)[C(z)-A(z)F(z)]z^{-d}$ and $\alpha^{-1}(z)\gamma(z) = \tilde{F}(z) + \alpha^{-1}(z)\beta(z)\tilde{G}(z)$. Substituting these two expressions in the definitions of $\theta(z)$ and $\theta_+(z)$ in (14.iii,v), we get

$$\begin{aligned}\theta_-(z) &= \theta(z) - \theta_+(z)\\ &= \alpha^{-1}(z)\beta(z)G(z) - \tilde{F}(z) - \alpha^{-1}(z)\beta(z)\tilde{G}(z)\\ &= \alpha^{-1}(z)\beta(z)[G(z)-\tilde{G}(z)] - \tilde{F}(z)\end{aligned} \tag{25}$$

Substituting (25) in (24) gives the expression (16) claimed as the minimum variance. We still need to determine that the choice of $S(z)$ in (22) corresponds to (15) and also that it is stabilizing. Since $S(z) = T(z)A^{-1}(z)C(z)$, we obtain that the choice of $T(z)$ is $T(z)=-\tilde{G}(z)C^{-1}(z)A(z)$ and since $T = M[I-z^d A^{-1}BM]^{-1}$ it follows that

$$\begin{aligned}M(z) &= T(z)[I + z^d A^{-1}(z)B(z)T(z)]^{-1}\\ &= T(z)[I - z^d A^{-1}(z)B(z)\tilde{G}(z)C^{-1}(z)A(z)]^{-1}\\ &= -\tilde{G}(z)[A^{-1}(z)C(z) - z^d A^{-1}(z)B(z)\tilde{G}(z)]^{-1}\\ &= -\tilde{G}(z)[F(z) + z^d A^{-1}(z)B(z)(G(z) - \tilde{G}(z))]^{-1}\end{aligned}$$

which coincides with the control law of (15). It remains to be shown that this choice of $M(z)$ is stabilizing, i.e. it satisfies (4). Simple calculations show that two of the transfer functions in (4) are

$$\begin{aligned}&M[I-z^d A^{-1}BM]^{-1} = -\beta^{-1}(\gamma-\alpha\tilde{F})C^{-1}A\\ &[I - z^d MA^{-1}B]^{-1} = I - z^d\tilde{G}C^{-1}A = I - z^d\beta^{-1}(\gamma-\alpha\tilde{F})C^{-1}B\end{aligned} \tag{26}$$

which are both analytic inside the closed unit disc, since β^{-1} and C^{-1} are. The third transfer function in (4) is $[I-z^{d}A^{-1}(z)B(z)M(z)]^{-1}$ which can, by simple calculation, be seen to be to equal to $I-z^{d}A^{-1}(z)B(z)\tilde{G}(z)C^{-1}(z)A(z)$, which in turn is $I-z^{d}A^{-1}(z)B(z)\beta^{-1}(z)[\gamma(z)-\alpha(z)\tilde{F}(z)]C^{-1}(z)A(z)$. Except for the term $A^{-1}(z)$, all other quantities are analytic inside the closed unit disc, and so if this transfer function has any singularities inside the closed unit disc, they must coincide with those of $A^{-1}(z)$. However, we also have $[I-z^{d}A^{-1}(z)B(z)M(z)]^{-1} = [F(z) + z^{d}A^{-1}(z)B(z)(G(z)-\tilde{G}(z))]C^{-1}(z)A(z)$ $= \{F(z) + z^{d}A^{-1}(z)B(z)\beta^{-1}(z)\alpha(z)[\theta_{-}(z)-\tilde{F}(z)]\}C^{-1}(z)A(z)$ which, if it has any singularities inside the closed unit disc coinciding with those of $A^{-1}(z)$, can only be singularities of $A^{-1}(z)B(z)\beta^{-1}(z)\alpha(z)$ inside the closed unit disc coinciding with those of $A^{-1}(z)$. However, by (14.i,ii), we have $A^{-1}(z)B(z)\beta^{-1}(z)\alpha(z) = A^{T}(z^{-1})B^{-T}(z^{-1})\beta^{T}(z^{-1})\alpha^{-T}(z^{-1})$. The only poles of the right hand side inside the closed unit disc are either at the origin or coincident with the poles of $\alpha^{-1}(z^{-1})$ or $B^{-1}(z^{-1})$. By our assumptions, there can however be no poles of $A^{-1}(z)$ in any of these locations, showing that $[I-z^{d}A^{-1}BM]^{-1}$ is analytic inside the closed unit disc. The last transfer function of (14) we need to check is $z^{d}A^{-1}B[I-z^{d}MA^{-1}B]^{-1}$. Since (26), which is a factor, has no poles inside the closed unit disc, it follows that if there are poles of $z^{d}A^{-1}B[I-z^{d}MA^{-1}B]^{-1}$ inside the closed unit disc, they must be poles of A^{-1}. Simple calculation shows that $z^{d}A^{-1}B[I-z^{d}MA^{-1}B]^{-1} = z^{d}FC^{-1}B +$ $z^{d}A^{-1}B\beta^{-1}\alpha(\theta_{-}+\tilde{F})C^{-1}B$. The first term is analytic inside the closed unit disc, and so is $(\theta_{-}+\tilde{F})C^{-1}B$. Hence we only need to show that $A^{-1}(z)B(z)\beta^{-1}(z)\alpha(z)$ has no poles inside the closed unit disc which coincide with those of $A^{-1}(z)$. But we have already done this. □

If the system is of minimum phase, i.e. $B^{-1}(z)$ is analytic in $\{z:0<|z|\leq 1\}$, then $\alpha=A$, $\beta=B$ and so $\theta = A^{-1}[C-AF]z^{-d}$, thus showing that $\gamma = [C-AF]z^{-d}$. Hence $\tilde{F}=0$ and $\tilde{G}=G$. Thus the control law (15) above reduces to what it is in the minimum phase case of Theorem 2.1. Moreover the minimum variance (16) also reduces to what it is in Theorem 2.1.

The additional cost of stably controlling a non-minimum phase system is therefore

$$\mathrm{tr} \sum_j \tilde{F}_j^T \tilde{F}_j Q + \frac{\mathrm{tr}}{2\pi i} \oint \{\alpha^{-1}(z)\beta(z)[G(z)-\tilde{G}(z)] - \tilde{F}(z)\}$$

$$Q\{\alpha^{-1}(z^{-1})\beta(z^{-1})[G(z^{-1})-\tilde{G}(z^{-1})] - \tilde{F}(z^{-1})\}^T \frac{dz}{z}$$

This is the "sacrifice" in variance that must be made to obtain a stable system. If one just wants to minimize the variance without paying attention to stability, then this sacrifice need not be made.

One useful property of the control law (15) is that it does not depend on the noise covariance $Ew(t)w^T(t)$. Thus, the same control law is optimal irrespective of the covariance $Ew(t)w^T(t)$.

IV. RECTANGULAR SYSTEMS

Now we consider rectangular systems, i.e. systems where the number of inputs is not equal to the number of outputs.

If the system has more inputs than outputs, then the previous results can still be used if we replace $B^{-1}(z)$ by $B^{\#}(z)$, any right inverse of $B(z)$. The proofs proceed as before.

So we turn our attention to systems where the number of inputs is less than the number of outputs. Before describing the solution, we first discuss some pitfalls. One way of proceeding, it might appear, is to make the system "square" by adding ficititous inputs with small "gains" ϵ which are then driven to zero. This can however result in matrices M(z) and T(z) which become unbounded as $\epsilon \to 0$. Another way of making the system square is to add ficititous inputs which have delays which are then driven to infinity. However, the resulting solution for F(z) will be a power series, at best.

We therefore adopt the more fruitful approach of the following Theorem. As in previous sections, we assume that the system has no zeroes exactly on the unit circle $\{z:|z|=1\}$, or more precisely, $B^T(z^{-1})A^{-T}(z^{-1})A^{-1}(z)B(z)$ has no zeroes on the unit circle $\{z:|z|=1\}$.

Theorem 4.1

We assume that $A^{-1}(z)$ and $A^{-1}(z^{-1})$ have no poles in common and also that for every pole t_k of $A^{-1}(z)$ inside the closed unit disc, its residue R_k in the partial fraction expansion of $A^{-1}(z)B(z)$ satisfies the condition $\lim_{z \to t_k} B^T(z^{-1})A^{-T}(z^{-1})R_k \neq 0$.

(27.i) Let $\delta(z) := A^{-1}(z)B(z)$

(27.ii) Let $\Delta(z) = \alpha^{-1}(z)\beta(z)$ be a square minimum phase spectral factor satisfying $\Delta^T(z^{-1})\Delta(z) = \delta^T(z^{-1})\delta(z)$ and such that the non-zero zeroes of the polynomial matrix $\beta(z)$ are the outside the unit circle images of the nonzero zeroes of $\delta^T(z^{-1})\delta(z)$ while the poles of the polynomial matrix $\alpha^{-1}(z)$

are those of $\delta(z)$.

(27.iii) Define $\tilde{\theta}(z) := \Delta^{-T}(z^{-1})\delta^{T}(z^{-1})A^{-1}(z)C(z)$ and decompose $\tilde{\theta}(z)$ as $\tilde{\theta}(z) =: \tilde{\theta}_{+}(z) + \tilde{\theta}_{-}(z)$ where $\tilde{\theta}_{-}(z)$ consists of those partial fraction terms with poles which are inside the unit circle and not coinciding with those of $A^{-1}(z)$.

(27.iv) Let $\tilde{\theta}_{+}(z) = \tilde{\alpha}^{-1}(z)\tilde{\gamma}(z)$ where $\tilde{\alpha}(z)$ is a square polynomial matrix with zeroes corresponding to those of $A(z)$ and $\tilde{\gamma}(z)$ is a rectangular matrix of polynomials with more columns than rows.

(27.v) Let $\tilde{F}(z) := F(\tilde{\alpha}(\cdot), \beta(\cdot), \tilde{\gamma}(\cdot), d)(z)$ and $\tilde{G}(z) := G(\tilde{\alpha}(\cdot), \beta(\cdot), \tilde{\gamma}(\cdot), d)(z)$. Then, the control law which minimizes the output variance $Ey^{T}(t)y(t)$ over the set of all admissible stabilizing control laws is

$$u(t) = -\tilde{G}(z)[C(z) - z^{d}B(z)\tilde{G}(z)]^{-1}A(z)y(t)$$

<u>Proof</u>

Let $\tilde{\delta}(z)$ be a full rank left annihilator of $\delta(z)$. Clearly

$$\begin{bmatrix} \tilde{\delta}(z) \\ [\delta^{T}(z^{-1})\delta(z)]^{-1}\delta^{T}(z^{-1}) \end{bmatrix} \quad [\tilde{\delta}^{T}(z^{-1})[\tilde{\delta}(z)\tilde{\delta}^{T}(z^{-1})]^{-1}, \ \delta(z)] = I$$

and so each of the matrices on the left hand side of the above is the inverse of the other. Multiplying the two matrices above in the reverse order gives

$$\tilde{\delta}^{T}(z^{-1})[\tilde{\delta}(z)\tilde{\delta}^{T}(z^{-1})]^{-1}\tilde{\delta}(z) + \delta(z)[\delta^{T}(z^{-1})\delta(z)]^{-1}\delta^{T}(z^{-1}) = I$$

Hence, for any admissible $u(t) = M(z)y(t)$, we can decompose $y(t) = y_1(t) + y_2(t)$ where

$$y_1(t) = \tilde{\delta}^T(z^{-1})[\tilde{\delta}(z)\tilde{\delta}^T(z^{-1})]^{-1}\tilde{\delta}(z)A^{-1}(z)C(z)w(t), \quad \text{and} \tag{28}$$

$$y_2(t) = \delta(z)\{[\delta^T(z^{-1})\delta(z)]^{-1}\delta^T(z^{-1})A^{-1}(z)C(z) + z^d T(z)A^{-1}(z)C(z)\}w(t)$$

where $T(z)$ is as in Lemma 2.4. By integrating over a small circle around the origin, it can be seen that $Ey_1^T(t)y_2(t) = 0$. So $Ey^T(t)y(t) = Ey_1^T(t)y_1(t) + Ey_2^T(t)y_2(t)$. The first term on the right hand side does not depend on the choice of $M(z)$. Hence, to minimize $Ey^T(t)y(t)$ we need to minimize only $Ey_2^T(t)y_2(t)$. Now

$$Ey_2^T(t)y_2(t) = \frac{tr}{2\pi i}\oint \delta^T(z^{-1})\delta(z)\{[\delta^T(z^{-1})\delta(z)]^{-1}\delta^T(z^{-1}) + z^d T(z)\}\cdot$$

$$\cdot A^{-1}(z)C(z)QC^T(z^{-1})A^{-T}(z^{-1})\{[\delta^T(z)\delta(z^{-1})]^{-1}\delta^T(z) + z^{-d}T(z^{-1})\}^T\frac{dz}{z}$$

and so, using $\Delta^T(z^{-1})\Delta(z) = \delta^T(z^{-1})\delta(z)$ we get

$$Ey_2^T(t)y_2(t) =$$

$$\frac{tr}{2\pi i}\oint [\Delta^{-T}(z^{-1})\delta^T(z^{-1}) + z^d\Delta(z)T(z)]A^{-1}(z)C(z)QC^T(z^{-1})A^{-T}(z^{-1})$$

$$[\Delta^{-T}(z)\delta^T(z) + z^{-d}\Delta(z^{-1})T(z^{-1})]^T\frac{dz}{z}$$

$$= \frac{tr}{2\pi i}\oint [\tilde{\theta}_-(z)+\tilde{\theta}_+(z)+z^d\alpha^{-1}(z)\beta(z)T(z)A^{-1}(z)C(z)]$$

$$Q[\tilde{\theta}_-(z^{-1}) + \tilde{\theta}_+(z^{-1}) + z^{-d}\alpha^{-1}(z^{-1})\beta(z^{-1})T(z^{-1})A^{-1}(z^{-1})C(z^{-1})]^T\frac{dz}{z}$$

As in Theorem 3.1, the cross term

$$\frac{tr}{2\pi i} \oint [\tilde{\theta}_+(z) + z^d \alpha^{-1}(z)\beta(z)T(z)A^{-1}(z)C(z)]Q\tilde{\theta}_-^T(z^{-1})\frac{dz}{z}$$

vanishes because of the stabilizing requirement on M(z) and the consequence that T(z) is analytic at the origin. Also, the term $\frac{tr}{2\pi i} \oint \tilde{\theta}_-(z)Q\tilde{\theta}_-^T(z^{-1}) \frac{dz}{z}$ can be ignored since it does not depend on M(z). Hence, the problem becomes one of minimizing

$$\frac{tr}{2\pi i} \oint [\tilde{\theta}_+(z) + z^d \alpha^{-1}(z)\beta(z)T(z)A^{-1}(z)C(z)]$$

$$Q[\tilde{\theta}_+(z^{-1}) + z^{-d}\alpha^{-1}(z^{-1})\beta(z^{-1})T(z^{-1})A^{-1}(z^{-1})C(z^{-1})]^T \frac{dz}{z}$$

$$= \frac{tr}{2\pi i} \oint [\tilde{\alpha}^{-1}(z)\tilde{\gamma}(z) + z^d\alpha^{-1}(z)\beta(z)S(z)]Q[\tilde{\alpha}^{-1}(z^{-1})\tilde{\gamma}(z^{-1}) + + z^{-d}\alpha^{-1}(z^{-1})\beta(z^{-1})S(z^{-1})]^T\frac{dz}{z} \tag{29}$$

The slight difference, because $\alpha \neq \tilde{\alpha}$, between the problem of minimizing (29) and the problem of minimizing (20) is unimportant, and the rest of the proof proceeds as in the proof of Theorem 3.1. □

The variance of (28) represents an additional cost due to the non-squareness of the system. The cost term $\frac{tr}{2\pi i} \oint \tilde{\theta}_-(z)Q\tilde{\theta}_-^T(z^{-1})\frac{dz}{z}$ is also affected by the non-squareness and will be positive even if the system is of minimum phase.

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