Largest Eigenvalue of the Laplacian Matrix: Its Eigenspace and Transitive Orientations

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LARGEST EIGENVALUE OF THE LAPLACIAN MATRIX: ITS EIGENSPACE AND TRANSITIVE ORIENTATIONS*

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Abstract. We study the eigenspace with largest eigenvalue of the Laplacian matrix of a simple graph. We find a surprising connection of this space with the theory of modular decomposition of Gallai, whereby eigenvectors can be used to discover *modules*. In the case of comparability graphs, eigenvectors are used to induce orientations of the graph, and the set of these induced orientations is shown to (recursively) correspond to the full set of transitive orientations.

Key words. graph spectrum, largest eigenvalue, Laplacian matrix, orientation, modular decomposition

AMS subject classifications. 05C85, 05C76, 05C50, 15B99

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1. Introduction.

Let $G = G([n], E)$ be a simple (undirected) graph, where $[n] = \{1, 2, \ldots, n\}$, $n \in \mathbb{P}$. The *adjacency matrix of* G is the $n \times n$ matrix $A = A(G)$ such that

$$(A)_{ij} = a_{ij} := \begin{cases} 1 & \text{if } \{i, j\} \in E, \\ 0 & \text{otherwise.} \end{cases}$$

The *Laplacian matrix of* G is the $n \times n$ matrix $L = L(G)$ such that

$$(L)_{ij} = l_{ij} := \begin{cases} d_i & \text{if } i = j, \\ -a_{ij} & \text{otherwise,} \end{cases}$$

where $d_i := d(G)_i$ is the degree of vertex i in G.

The spectral theory of these matrices, i.e., the theory about their eigenvalues and eigenspaces, has been the object of much study for the last 40 years. However, the roots of this line of research can arguably be traced back to *Kirchhoff's matrix-tree theorem*, whose first proof is often attributed to Borchardt (1860) even though at least one proof was already known by Sylvester (1857). A recollection of some interesting applications of the theory can be found in Spielman (2009), and more complete accounts of the mathematical backbone are Brouwer and Haemers (2011) and Chung (1997). We refer the reader to our references for further inquiries of the extensive literature on the subject.

One of the first observations that can be made about L is that it is positive-semidefinite, a consequence of it being decomposable as $L = QQ^T$, where $Q = Q(G, O)$ is the *incidence matrix* of an arbitrary *orientation* O of G. We will thus let

$$0 = \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n = \lambda_{\max} = \lambda_{\max}(G)$$

be the (real) eigenvalues of L and note that $\lambda_2 > 0$ if G is a connected graph with at least two vertices. Note that we have effectively dropped G from the notation for convenience. We will also let $\mathbf{E}_{\lambda_i}$ be the eigenspace corresponding to λ_i.

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Our work follows the spirit of Fiedler (2011), who pioneered the use of eigenvectors of the Laplacian matrix to learn about a graph's structure. In its most primitive form, *Fiedler's nodal domain theorem* (Fiedler (1975)) states that when G is connected then for all $x \in \mathbf{E}_{\lambda_2}$, the induced subgraph $G\left[\{i \in [n] : x_i \geqslant 0\}\right]$ is connected. Related work, also relevant to the present writing, might be found in Merris (1998).

We will go even further in the way in which we use eigenvectors of the Laplacian to learn properties of G. To explain this, let us first call a map,

$$O : E \to ([n] \times [n]) \cup E = [n]^2 \cup E,$$

such that $O(e) \in \{e, (i,j), (j,i)\}$ for all $e := \{i, j\} \in E$, a *partial orientation* of E or G, and say that furthermore, O is an *orientation* if $O(e) \neq e$ for all e and that O is *acyclic* if it is an orientation and the directed graph on vertex-set $[n]$ and edge-set $O(E)$ has no directed cycles. On numerous occasions, we will somewhat abusively also identify O with the set $O(E)$. With this setup, eigenvectors of the Laplacian and, more precisely, elements of $\mathbf{E}_{\lambda_{\max}}$ will be used to obtain orientations of certain (not necessarily induced) subgraphs of G. Henceforth, given G and for all $x \in \mathbb{R}^n := \mathbb{R}^{[n]}$, the reader should always automatically consider the orientation map $O_x = O_x(G)$ associated to x, $O_x : E \to [n]^2 \cup E$, such that for $e := \{i, j\} \in E$,

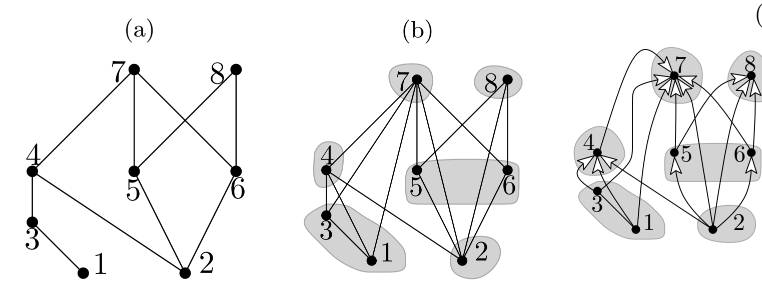
$$O_x(e) = \begin{cases} e & \text{if } x_i = x_j, \\ (i,j) & \text{if } x_i < x_j, \\ (j,i) & \text{if } x_i > x_j. \end{cases}$$

The orientation O_x will be said to be *induced* by x (e.g., Figure 1(c)). We will generally be interested in certain special vertex-subsets of G associated to x or O_x, called *fibers*.

DEFINITION 1. *For $x \in \mathbb{R}^n$, a fiber of x is a set $\xi \subseteq [n]$ in which x is constant, i.e., there exists some $\alpha \in \mathbb{R}$ such that $x_i = \alpha$ if and only if $i \in \xi$.*

Broadly, this article aims to fill one of the many gaps in our current knowledge of the spectra of graphs, namely, the lack of results about eigenvectors of the Laplacian with largest eigenvalue. More specifically, we will find that the eigenspace $\mathbf{E}_{\lambda_{\max}}$ is closely related to the theory of *modular decomposition* of Gallai (1967), and this relation will be established by exposing how orientations of G induced by elements of $\mathbf{E}_{\lambda_{\max}}$ lead naturally to the discovery of *modules*. In particular when $G = G([n], E)$ is a *comparability graph*, $n > 1$, we will show (I) that the collection of *fibers* of a generic $x \in \mathbf{E}_{\lambda_{\max}}$ is invariant with respect to the choice of x, (II) that these fibers are invariably either the connected components of G, or the connected components of $\overline{G}$, or disjoint unions of pairwise nonadjacent, maximal (by inclusion), proper modules of G and, furthermore, (III) that the orientations of G induced by all such generic elements iteratively correspond to and exhaust the *transitive orientations* of G. Arguably then, when G is a comparability graph, a recursive consideration of the space $\mathbf{E}_{\lambda_{\max}}$ is shown to solve the problem of finding the complete set of transitive orientations of G. It will be instructive to see Figure 1 at this point.

In section 2, we will introduce the background and definitions necessary to state the precise main contributions of this article. These punch line results will then be presented in section 3. The central theme of section 3 will be a stepwise proof of Theorem 15, our main result for comparability graphs, which summarily states that when G is a comparability graph, elements of $\mathbf{E}_{\lambda_{\max}}$ induce transitive orientations of the *copartition subgraph* of G. It will be along the natural course of this proof that



i	x_i
1	$a = -0.1515...$
2	$b = -0.2587...$
3	a
4	$c = -0.1021...$
5	$d = -0.1866...$
6	d
7	$e = 0.8855...$
8	$-a$

FIG. 1. (a) *Hasse diagram of a poset* P *on* $[8]$. (b) *Comparability graph* $G = G([8], E)$ *of the poset* P*, where closed regions are maximal proper modules of* G. (c) *Unit eigenvector* $x \in \mathbf{E}_{\lambda_{\max}}$ *of* G *fully calculated, where* $\dim\langle \mathbf{E}_{\lambda_{\max}} \rangle = 1$. *Arrows represent the induced orientation* O_x *of* G. *Notice the relation between* O_x*, the modules of* G*, and poset* P.

we present our three main results that apply to arbitrary simple graphs: Lemma 27 and Propositions 28 and 29.

Finally, in section 4, we will present a curious novel characterization of comparability graphs that results from the theory of section 3.

2. Background and definitions.

2.1. The graphical arrangement.

DEFINITION 2. *Let* $G = G([n], E)$ *be a simple (undirected) graph. The* graphical arrangement of G *is the union of hyperplanes in* $\mathbb{R}^n$*:*

$$\mathcal{A}_G := \{x \in \mathbb{R}^n : x_i - x_j = 0 \text{ for some } \{i, j\} \in E\}.$$

Basic properties of graphical arrangements and, more generally, of hyperplane arrangements are presented in Chapter 2 of Stanley (2004).

For $G = G([n], E)$, we will let $\mathcal{R}(\mathcal{A}_G)$ be the collection of all (open) connected components of the set $\mathbb{R}^n \backslash \mathcal{A}_G$. An element of $\mathcal{R}(\mathcal{A}_G)$ is called a *region* of $\mathcal{A}_G$, and every region of $\mathcal{A}_G$ is therefore an n-dimensional open convex cone in $\mathbb{R}^n$. Furthermore, the following is true about regions of the graphical arrangement.

PROPOSITION 3. *With* $G = G([n], E)$ *a simple graph, and for all* $R \in \mathcal{R}(\mathcal{A}_G)$ *and* $x, y \in R$*, we have that*

$$O_R := O_x = O_y.$$

Moreover, the map $R \mapsto O_R$ *from the set of regions of* $\mathcal{A}_G$ *to the set of orientations of* E *is a bijection between* $\mathcal{R}(\mathcal{A}_G)$ *and the set of* acyclic orientations *of* G.

Proof. Clearly our map is well-defined and has as codomain the acyclic orientations of G. Surjectivity can be established by considering, for any O an acyclic orientation of G, the partial order on $[n]$ whereby i is "less" than j if and only if there exists a directed path in O from i to j. A linear extension of this poset lives in a region R of $\mathcal{A}_G$, and $O_R = O$. Injectivity follows by considering, for any two disjoint regions of $\mathcal{A}_G$, a separating hyperplane in the arrangement between them, and then noting that the edge of G corresponding to this hyperplane is oriented differently in both regions under our map. ∎

Motivated by Proposition 3 and the comments before, we will introduce special notation for certain subsets of $\mathbb{R}^n$ obtained from $\mathcal{A}_G$.

Notation 4. Let $G = G([n], E)$ be a simple (undirected) graph. For an acyclic orientation O of E, we will let C_O denote the n-dimensional closed convex cone in $\mathbb{R}^n$ that is equal to the topological closure of the region of $\mathcal{A}_G$ corresponding to O in Proposition 3.

2.2. Modular decomposition. We need to concur on some standard terminology and notation from graph theory, so let $G = G([n], E)$ be a simple (undirected) graph and X a subset of $[n]$.

As customary, $\overline{G}$ denotes the *complement graph* of G. The notation $N(X)$ denotes the *open neighborhood* of X in G:

$$N(X) := \{j \in [n]\backslash X : \text{ there exists some } i \in X \text{ such that } \{i, j\} \in E\}.$$

The *induced subgraph* of G on X is denoted by $G[X]$, and the binary operation of graph *disjoint union* is represented by the plus sign $+$. Last, for $Y \subseteq [n]$, X and Y are said to be *completely adjacent (nonadjacent)* in G if

$$X \cap Y = \varnothing \text{ and}$$
$$\text{for all } i \in X \text{ and } j \in Y, \text{ we have that } \{i, j\} \in E \ (\{i, j\} \notin E).$$

The concepts of *module* and *modular decomposition* in graph theory were introduced by Gallai (1967) as a means to understand the structure of comparability graphs. The same work would eventually present a remarkable characterization of these graphs in terms of forbidden subgraphs. Section 3 of the present work will present an alternate and surprising route to modules.

DEFINITION 5. *Let $G = G([n], E)$ be a simple (undirected) graph. A* module *of G is a set $A \subseteq [n]$ such that for all $i, j \in A$,*

$$N(i)\backslash A = N(j)\backslash A = N(A).$$

Furthermore, A is said to be proper *if $A \neq [n]$,* nontrivial *if $|A| > 1$, and* connected *if $G[A]$ is connected.*

COROLLARY 6. *Two disjoint modules of G are either completely adjacent or nonadjacent.*

Let us now present some basic results about modules that we will need.

LEMMA 7 (Gallai (1967)). *Let $G = G([n], E)$ be a connected graph such that $\overline{G}$ is connected. If A and B are maximal (by inclusion) proper modules of G with $A \neq B$, then $A \cap B = \varnothing$.*

COROLLARY 8 (Gallai (1967)). *Let $G = G([n], E)$ be a connected graph such that $\overline{G}$ is connected, $n > 1$. Then, there exists a unique partition of $[n]$ into maximal proper modules of G, and this partition contains more than two blocks.*

From Corollary 8, it is therefore natural to consider the partition of the vertex-set of a graph into its maximal modules; the appropriate framework for doing this is presented in Definition 9. Hereafter, however, we will assume that our graphs are connected unless otherwise stated since (I) the results for disconnected graphs will follow immediately from the results for connected graphs, and (II) this will allow us to focus on the interesting parts of the theory.

DEFINITION 9 (Ramírez-Alfonsín and Reed (2001)). *Let $G = G([n], E)$ be a connected graph, $n > 1$. We will let the* canonical partition *of G be the set $\mathcal{P} = \mathcal{P}(G)$ such that*

- *if $\overline{G}$ is connected, $\mathcal{P}$ is the unique partition of $[n]$ into maximal proper modules;*
- *if $\overline{G}$ is disconnected, $\mathcal{P}$ is the partition of $[n]$ into the connected components of $\overline{G}$.*

Hence, in Definition 9, every element of the canonical partition is a module of the graph. Elements of the canonical partition of a graph on vertex-set [8] are shown in Figure 1(b).

Definition 10. *Let $G = G([n], E)$ be a connected graph with canonical partition $\mathcal{P}$. The* copartition subgraph *of G is the graph $G^{\mathcal{P}}$ on vertex-set $[n]$ and edge-set equal to*

$$E \setminus \{\{i,j\} \in E : i,j \in A \text{ for some } A \in \mathcal{P}\}.$$

Lemma 11. *We have that*

i. *$G^{\mathcal{P}}$ is connected,*

ii. *$\left(G^{\mathcal{P}}\right)^{\mathcal{P}} = G^{\mathcal{P}}$, i.e., the copartition subgraph of $G^{\mathcal{P}}$ is $G^{\mathcal{P}}$ itself.*

Proof. For claim i, note that for all $A \in \mathcal{P}$, $i,j \in A$, $\{i,j\} \in E$, there must exist some $k \in [n] \setminus A$ such that $\{i,k\}, \{j,k\} \in E$ since A is a proper module of G and G is connected. Hence, two vertices in G are connected if and only if they are connected in $G^{\mathcal{P}}$.

Claim ii is evident when $\overline{G}$ is disconnected from the definition of canonical partition. When $\overline{G}$ is connected, then clearly $\overline{G^{\mathcal{P}}}$ is connected and claim i shows that $G^{\mathcal{P}}$ partitions into maximal proper modules, its canonical partition. But $\mathcal{P}$ is also a partition into proper modules of $G^{\mathcal{P}}$, and therefore it must be a refinement of its canonical partition. However, then, from the edge-removal condition in the definition of copartition subgraph, every member of the canonical partition of $G^{\mathcal{P}}$ is also a module of G, and claim ii follows. ∎

2.3. Comparability graphs. We had anticipated the importance of comparability graphs in this work, so we need to define what they are.

Definition 12. *A* comparability graph *is a simple (undirected) graph $G = G(V, E)$ such that there exists a partial order on V under which two different vertices $u, v \in V$ are comparable if and only if $\{u, v\} \in E$.*

A comparability graph on vertex-set [8] is shown in Figure 1(b).

Comparability graphs are *perfectly orderable graphs* and, more generally, *perfect graphs*. These three families of graphs are all large hereditary classes of graphs.

Note that given a comparability graph $G = G(V, E)$ with $E \neq \varnothing$, we can find at least two partial orders on V whose comparability graphs agree precisely with G, and the number of such partial orders depends on the modular decomposition of G. Let us record this idea in a definition.

Definition 13. *Let $G = G(V, E)$ be a comparability graph, and let O be an acyclic orientation of E. Consider the partial order on V whereby for $u, v \in V$, u is "less" than v if and only if there is a directed path in O from u to v. If the comparability graph of this partial order agrees precisely with G, then we will say that O is a* transitive orientation *of G.*

2.4. Linear algebra. Some standard terminology of linear algebra and other related conventions that we adopt are presented here. First, we will always be working in Euclidean space $\mathbb{R}^n$, and all (Euclidean-normed real) vector spaces considered are assumed to live therein. Euclidean norm is denoted by $|| \cdot ||$. The standard basis of

$\mathbb{R}^n$ will be $\{e_i\}_{i\in[n]}$, as customary. Generalizing this notation, for all $I \subseteq [n]$, we will also let

$$e_I := \sum_{i\in I} e_i.$$

The orthogonal complement in $\mathbb{R}^n$ to $e_{[n]}$ will be of importance to us, so we will use special notation to denote it:

$$\mathbb{R}^{*[n]} := e_{[n]}^{\perp}.$$

For an arbitrary vector space $\mathcal{V}$ and a linear transformation $T : \mathcal{V} \to \mathcal{V}$, we will say that a set $U \subseteq \mathcal{V}$ is *invariant under* T, or that U is T*-invariant*, if $T(U) \subseteq U$. In the case when $\mathcal{V}$ is a linear subspace of $\mathbb{R}^n$ and $\dim\langle\mathcal{V}\rangle > 0$, often we will consider a *uniformly chosen unit vector* (u.c.u.v.) from $\mathcal{V}$, i.e., a vector chosen uniformly at random from the unit sphere $\{y \in \mathcal{V} : ||y|| = 1\}$.

2.5. Spectral theory of the Laplacian. We will need only a few background results on the spectral theory of the Laplacian matrix of a graph. We present these below in a single statement but refer the reader to Brouwer and Haemers (2011) for additional background and history.

LEMMA 14. *Let $G = G([n], E)$ be a simple (undirected) graph. Let $L = L(G)$ be the Laplacian matrix of G and $0 = \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n = \lambda_{\max} = \lambda_{\max}(G)$ be the eigenvalues of L:*

1. *The number of connected components of G is equal to the multiplicity of the eigenvalue 0 in L.*
2. *If $\overline{L}$ is the Laplacian matrix of $\overline{G}$, then $\overline{L} = nI - J - L$, where I is the $n \times n$ identity matrix and J is the $n \times n$ matrix of all-1's. Consequently, $\lambda_{\max} \leqslant n$ and the number of eigenvectors of L with eigenvalue n is equal to the number of connected components of $\overline{G}$ minus 1.*
3. *If H is a (not necessarily induced) subgraph of G on the same vertex-set $[n]$, and if $\mu_1 \leqslant \mu_2 \leqslant \cdots \leqslant \mu_n$ are the eigenvalues of the Laplacian of H, then $\lambda_i \geqslant \mu_i$ for all $i \in [n]$.*

The proof of part 1 is readily obtained from the decomposition $L = QQ^T$ presented in the introduction (section 1), where $Q = Q(G, O)$ is an incidence matrix of an arbitrary orientation O of G, so notably,

$$\operatorname{rank}(Q) = n - \#^{\text{connected}}_{\text{components}} \text{ of } G.$$

The equality of part 2 is a straightforward verification, and the second claim follows after noting that since $e_{[n]}$ is an eigenvector of L, we can always select a basis β for $\mathbb{R}^{*[n]}$ of eigenvectors of L. Then for $x \in \beta$ with eigenvalue λ, we have that $\overline{L}x = nx - 0 - Lx = (n - \lambda)x$, so x is an eigenvector of $\overline{L}$ with eigenvalue $n - \lambda$, and part 1 can be applied.

Part 3 is a more advanced result (Brouwer and Haemers (2011), Proposition 3.2.1).

3. Largest eigenvalue of a comparability graph. The main goal of this section is to prove the following theorem (see section 2 for the main definitions).

THEOREM 15. *Let $G = G([n], E)$ be a connected comparability graph with Laplacian matrix $L = L(G)$ and canonical partition $\mathcal{P} = \mathcal{P}(G)$. Let $\lambda_{\max} = \lambda_{\max}(G)$ be the largest eigenvalue of L and $\mathbf{E}_{\lambda_{\max}}$ its associated eigenspace. Then, the following are true:*

i. *If O is a transitive orientation of G, then*

$$\dim \langle C_O \cap \mathbf{E}_{\lambda_{\max}} \rangle = \dim \langle \mathbf{E}_{\lambda_{\max}} \rangle .$$

ii. $\mathbf{E}_{\lambda_{\max}} \subseteq \bigcup_O C_O$, *where the union is over all transitive orientations of G.*

iii. *Let $x \in \mathbf{E}_{\lambda_{\max}}$ be a* u.c.u.v. *Almost surely,*
   1. *if $A \in \mathcal{P}$, then A belongs to a fiber of x;*
   2. *if $A, A' \in \mathcal{P}$ are (completely) adjacent in G, then A and A' belong to different fibers of x;*
   3. *x induces a transitive orientation of $G^{\mathcal{P}}$; in particular, $G^{\mathcal{P}}$ is a comparability graph;*
   4. *all transitive orientations of $G^{\mathcal{P}}$ can be induced by x with positive probability;*
   5. *if ξ is a fiber of x, then*

$$G[\xi] = G[B_1] + \cdots + G[B_k],$$

   where for all $i \in [k]$, B_i is a connected module of G and $G[B_i]$ is a comparability graph;
   6. *G has exactly two transitive orientations if and only if* $\dim \langle \mathbf{E}_{\lambda_{\max}} \rangle = 1$ *and every fiber of x is an independent set of G.*

iv. *If $\overline{G}$ is connected, then* $\dim \langle \mathbf{E}_{\lambda_{\max}} \rangle = 1$. *If $\overline{G}$ is disconnected, then* $\dim \langle \mathbf{E}_{\lambda_{\max}} \rangle$ *is equal to the number of connected components of $\overline{G}$ minus one.*

Remark 16. In fact, as will be explained, all transitive orientations of G can be obtained with the following procedure: Select an arbitrary transitive orientation for $G^{\mathcal{P}}$, and select arbitrary transitive orientations for (the connected components of) each $G[A], A \in \mathcal{P}$. Therefore, claims i–iii imply an iterative algorithm that obtains every transitive orientation of G with positive probability. Even more remarkably, when G is selected uniformly at random from the set of comparability graphs on n vertices, $n \to \infty$, then only one calculation of $\mathbf{E}_{\lambda_{\max}}$ suffices to achieve this (Möhring (1984)).

The proof of Theorem 15 will be stepwise and its notation and conventions will carry over to the next results, unless otherwise stated. Let us begin with this work.

Using notation 4, we have the following.

PROPOSITION 17. *Let $G = G([n], E)$ be a connected comparability graph, $n > 1$, and let O be a transitive orientation of G. Then, C_O satisfies that*

$$\dim \langle C_O \cap \mathbf{E}_{\lambda_{\max}} \rangle = \dim \langle \mathbf{E}_{\lambda_{\max}} \rangle .$$

In particular, C_O contains a nonzero element of $\mathbf{E}_{\lambda_{\max}}$.

Proof. The proof consists of two main steps. First, we will prove that C_O is invariant under left-multiplication by L. Then, we will prove that $\dim \langle C_O \cap \mathbf{E}_{\lambda_{\max}} \rangle = \dim \langle \mathbf{E}_{\lambda_{\max}} \rangle$.

Step 1. $Lx \in C_O$ whenever $x \in C_O$.

To show that $Lx \in C_O$ whenever $x \in C_O$, we will show that for an arbitrary $\{i, j\} \in E$ with (i, j) in O, so that $x_i \leqslant x_j$, we also have that $(Lx)_i \leqslant (Lx)_j$.

To start,

$$
\begin{aligned}
(Lx)_j - (Lx)_i &= (x_j d_j - \sum_{k \in N(j)} x_k) - (x_i d_i - \sum_{\ell \in N(i)} x_\ell) \\
&= \sum_{k \in N(j)} (x_j - x_k) - \sum_{\ell \in N(i)} (x_i - x_\ell) \\
&= |N(i) \cap N(j)| (x_j - x_i) + \sum_{\ell \in N(j) \backslash N(i)} (x_j - x_\ell) \\
&- \sum_{m \in N(i) \backslash N(j)} (x_i - x_m).
\end{aligned}
$$

Now, since O is transitive and G is a comparability graph, if $\ell \in N(j)\backslash N(i)$, then we must have that (ℓ, j) is an edge in O, so that $x_\ell \leqslant x_j$. Otherwise, we would require that $\{i, \ell\} \in E$, which is false. Similarly, if $m \in N(i)\backslash N(j)$ we must have that (i, m) is an edge in O, so $x_m \geqslant x_i$. Since also $x_j \geqslant x_i$ then, we see that $(Lx)_j - (Lx)_i \geqslant 0$. This completes the proof.

Step 2. $\dim \langle C_O \cap \mathbf{E}_{\lambda_{\max}} \rangle = \dim \langle \mathbf{E}_{\lambda_{\max}} \rangle$.

Using Step 1, as C_O is a closed set, it is closed under

$$
\lim_{N \to \infty} \frac{L^N}{\lambda_{\max}^N} = \pi_{\mathbf{E}_{\lambda_{\max}}},
$$

the orthogonal projection onto $\mathbf{E}_{\lambda_{\max}}$. But since the relative interior of C_O is an open subset of $\mathbb{R}^n$, then $\pi_{\mathbf{E}_{\lambda_{\max}}}(C_O) = C_O \cap \mathbf{E}_{\lambda_{\max}}$ contains an open subset of $\mathbf{E}_{\lambda_{\max}}$, and hence the claim follows. ∎

Let us now recall Definition 5.

LEMMA 18. *Let $G = G([n], E)$ be a connected comparability graph, $n > 1$, and let O be a transitive orientation of G. Then for any $x \in C_O \cap \mathbf{E}_{\lambda_{\max}}$, $x \neq 0$, the connected components of fibers of x are proper modules of G.*

Proof. It is enough to prove that any choice of maximal (by inclusion) set $A \subseteq [n]$ such that (I) $G[A]$ is connected and (II) for some $\alpha \in \mathbb{R}$, we have that $x_k = \alpha$ for all $k \in A$ is also a proper module of G.

Primarily, for such an A we observe that $G[A]$ cannot be equal to G since that would imply that x is equal to $\alpha e_{[n]}$ and therefore that x is an eigenvector with eigenvalue $0 \neq \lambda_{\max}$, which contradicts our choice of x. Hence, $G[A]$ is a proper connected induced subgraph of G.

We will show that A is a (proper connected) module of G. Suppose on the contrary that A is not a module of G. Then, there must exist two vertices $i, j \in A$ such that $N(i)\backslash A \neq N(j)\backslash A$. Since $G[A]$ is connected, we can consider a path in $G[A]$ connecting i and j and observe that we may further assume that $\{i, j\} \in E$. Under this assumption, suppose now that (i, j) is an edge in O. As O is transitive, we must have that (i, k) is an edge in O whenever (j, k) is. Similarly, (k, j) must be an edge in O whenever (k, i) is. Hence, it must be the case that

$$
\begin{aligned}
x_i \leqslant x_k &\text{ for all } k \in N(i)\backslash(A \cup N(j)) \\
\text{and } x_k \leqslant x_j &\text{ for all } k \in N(j)\backslash(A \cup N(i)).
\end{aligned}
$$

Left-multiplying x by the Laplacian of G, we obtain

$$
\begin{aligned}
0 &= \lambda_{\max}\alpha - \lambda_{\max}\alpha = \lambda_{\max}x_j - \lambda_{\max}x_i \\
&= (Lx)_j - (Lx)_i = \sum_{k\in N(j)} (x_j - x_k) - \sum_{\ell\in N(i)} (x_i - x_\ell) \\
&= \sum_{k\in N(j)\setminus[A\cup N(i)]} (x_j - x_k) - \sum_{\ell\in N(i)\setminus[A\cup N(j)]} (x_i - x_\ell) \\
&= \sum_{k\in N(j)\setminus[A\cup N(i)]} |x_j - x_k| + \sum_{\ell\in N(i)\setminus[A\cup N(j)]} |x_i - x_\ell|.
\end{aligned}
$$

Since $N(i)\setminus A \neq N(j)\setminus A$ and A was chosen maximal, then at least one of the terms in the last summations must be nonzero and we obtain a contradiction. This proves that A is a module of G with the required properties. ∎

COROLLARY 19. *Let $G = G([n], E)$ be a connected comparability graph, $n > 1$, without proper nontrivial connected modules, and let O be a transitive orientation of G. If $x \in C_O \cap \mathbf{E}_{\lambda_{\max}}$, $x \neq 0$, then x is in the interior of C_O.*

Proof. This follows from Lemma 18 since for all $\{i, j\} \in E$ we must have that $x_i \neq x_j$, as otherwise a $\{i, j\} \in E$ such that $x_i = x_j$ is contained in some proper nontrivial connected module of G, contradicting our assumption about G. ∎

THEOREM 20. *Let $G = G([n], E)$ be a connected comparability graph, $n > 1$, without proper nontrivial connected modules. Then*

i. *any $x \in \mathbf{E}_{\lambda_{\max}}\setminus\{0\}$ induces a transitive orientation of G;*
ii. $\dim\langle\mathbf{E}_{\lambda_{\max}}\rangle = 1$*;*
iii. *G has exactly two transitive orientations.*

Proof. Fix a transitive orientation O of G and consider the cone C_O. Per Proposition 17, we have that $C_O \cap [\mathbf{E}_{\lambda_{\max}}\setminus\{0\}] \neq \varnothing$. By Corollary 19, $\mathbf{E}_{\lambda_{\max}}\setminus\{0\}$ does not intersect the boundary of C_O, so the set $C_O \cap [\mathbf{E}_{\lambda_{\max}}\setminus\{0\}]$ is a nonempty open and closed subset of $\mathbf{E}_{\lambda_{\max}}\setminus\{0\}$. However, since this set is also convex, then it cannot be equal to $\mathbf{E}_{\lambda_{\max}}\setminus\{0\}$, and hence $\mathbf{E}_{\lambda_{\max}}\setminus\{0\}$ is disconnected. It follows that $\dim\langle\mathbf{E}_{\lambda_{\max}}\rangle = 1$ and then easily that $\mathbf{E}_{\lambda_{\max}}\setminus\{0\}$ is covered by $\operatorname{int}(C_O)$ and $\operatorname{int}(C_{O_{dual}})$, where O_{dual} is obtained from O after a reversal of the orientation of all the edges. ∎

The remaining part of the theory will rely heavily on some standard results of the spectral theory of the Laplacian (subsection 2.5). These will be of central importance to establish Lemma 27 and Propositions 28 and 29, which deal with arbitrary simple graphs.

LEMMA 21. *Let $G = G([n], E)$ be a complete p-partite graph, $p > 1$, with partite sets $A_1, \ldots, A_p$. Then, $\lambda_{\max} = n$ and*

$$
\begin{aligned}
\mathbf{E}_{\lambda_{\max}} &= \{x \in \mathbb{R}^{*[n]} : x_i = x_j \textit{ for all } i, j \in A_q,\ q \in [p]\} \\
&= \langle e_{A_q} : q \in [p]\rangle \cap \mathbb{R}^{*[n]}.
\end{aligned}
$$

In particular, $\dim\langle\mathbf{E}_{\lambda_{\max}}\rangle = p - 1$.

Proof. The complement of G has p connected components, so following part 2 in Lemma 14 we observe that $\lambda_{\max} = n$ and $\dim\langle\mathbf{E}_{\lambda_{\max}}\rangle = p - 1$. Let $b_1, \ldots, b_p \in \mathbb{R}$ and let $x \in \mathbb{R}^{*[n]}$ be such that $x_i = b_q$ for all $i \in A_q$, $q \in [p]$. Hence, $\sum_{q\in[p]} |A_q|\, b_q = \sum_{i\in[n]} x_i = 0$ and for $i \in A_r$, $r \in [p]$, we have that

$$\begin{aligned}(Lx)_i &= (n - |A_r|)b_r - \sum_{q \in ([p] \setminus \{r\})} |A_q|\, b_q \\ &= (n - |A_r|)b_r - (0 - |A_r|\, b_r) \\ &= nb_r = nx_i.\end{aligned}$$

The set of all such x has dimension $p - 1$. ∎

LEMMA 22. *Let $G = G([n], E)$ be a connected bipartite graph with bipartition $\{X, Y\}$. Then, $\dim \langle \mathbf{E}_{\lambda_{\max}} \rangle = 1$. Furthermore, if $x \in \mathbf{E}_{\lambda_{\max}} \setminus \{0\}$, then either $x_i < 0$ for all $i \in X$ and $x_j > 0$ for all $j \in Y$ or vice versa.*

Proof. If G is complete 2-partite, this is a consequence of Lemma 21. Otherwise, as a connected bipartite graph, G is also a comparability graph and it has no proper connected nontrivial modules, so Theorem 20 shows that $\dim \langle \mathbf{E}_{\lambda_{\max}} \rangle = 1$ and that G has precisely two transitive orientations; in particular, these are the ones obtained by directing all the edges of G from one side of the bipartition to the other. The same theorem also shows that the elements of $\mathbf{E}_{\lambda_{\max}} \setminus \{0\}$ induce transitive orientations, so take $x \in \mathbf{E}_{\lambda_{\max}} \setminus \{0\}$ and suppose that x induces the transitive orientation of G where all edges are directed from X to Y. Then for all $i \in X$, $j \in N(i) \subseteq Y$, we have that $x_i < x_j$. Hence for $i \in X$,

$$\begin{aligned}\lambda_{\max} x_i &= (Lx)_i \\ &= \sum_{j \in N(i)} (x_i - x_j),\end{aligned}$$

which is a nonempty sum of negative terms, and so $x_i < 0$. Similarly, we obtain the reverse inequality for elements of Y. ∎

We have not found an agreed-upon notation in the literature for the following objects, so we will need to introduce it here.

DEFINITION 23. *Let $G = G([n], E)$ be a simple graph and let $\mathcal{Q} = \{X_1, \ldots, X_m\}$ be a partition of $[n]$ with nonempty blocks. Then, for all $k \in [m]$,*
- *G^{*X_k} denotes the graph on vertex-set $[n]$ and edge-set $\{\{i, j\} \in E : i, j \in X_k\}$;*
- *$R^{*X_k} := \{x \in \mathbb{R}^{*[n]} :$ for all $i \in ([n] \setminus X_k)$, $x_i = 0\}$.*

Also,
- *$R^{\mathcal{Q}} := \{x \in \mathbb{R}^{*[n]} : x$ is constant on each $X_k, k \in [m]\} = \langle e_{X_k} : k \in [m] \rangle \cap \mathbb{R}^{*[n]}$.*

Observation 24. We note that

$$\mathbb{R}^{*[n]} = R^{\mathcal{Q}} \oplus \left(\bigoplus_{k \in [m]} R^{*X_k} \right),$$

so $\mathbb{R}^{*[n]}$ decomposes as a direct sum of $R^{\mathcal{Q}}$ and the R^{*X_k}'s.

Last, we introduce a block decomposition of the Laplacian of a connected graph (with at least two vertices) based on its canonical partition. It will be convenient to recall Definitions 9 and 10 at this point.

DEFINITION 25. *Let $\mathcal{P} = \{A_1, \ldots, A_p\}$ be the canonical partition of a (connected) graph $G = G([n], E)$ with Laplacian L. We will let $L^{\mathcal{P}}$ denote the Laplacian matrix of the copartition subgraph $G^{\mathcal{P}}$ of G and let L^{*A_q} denote the Laplacian matrix of G^{*A_q} for $q \in [p]$.*

Remark 26 (block decomposition of the Laplacian L of G). Since

$$L = L^{\mathcal{P}} + \sum_{q=1}^{p} L^{*A_q},$$

we see that for any $x \in \mathbb{R}^{*[n]}$, if we let $x = y + x_1 + \cdots + x_p$ be the decomposition of Observation 24, so $y \in R^{\mathcal{P}}$ and $x_q \in R^{*A_q}$ for all $q \in [p]$, then

$$Lx = L^{\mathcal{P}} y + \sum_{q=1}^{p} (|N(A_q)|\, I + L^{*A_q}) x_q,$$

where I is the $n \times n$ identity matrix, and furthermore, this gives the unique analogous decomposition of Lx.

Proof. Since the A_q's are modules of G, then for all $q \in [p]$, R^{*A_q} is an eigenspace of $L^{\mathcal{P}}$ with eigenvalue $|N(A_q)|$, and also $R^{\mathcal{P}}$ is $L^{\mathcal{P}}$-invariant. More directly, for all $r \in [p]$ and $q \in [p]\backslash\{r\}$, both $R^{\mathcal{P}}$ and R^{*A_q} are in the null space of L^{*A_r}, and R^{*A_r} is L^{*A_r}-invariant. ∎

We are now ready to present the results about the space $\mathbf{E}_{\lambda_{\max}}$ for arbitrary simple graphs.

LEMMA 27. *Let $G = G([n], E)$ be a connected simple graph with canonical partition $\mathcal{P} = \{A_1, \ldots, A_p\}$, and L, $\mathbf{E}_{\lambda_{\max}}$ as usual. Then,*
 i. $\mathbf{E}_{\lambda_{\max}} \subseteq R^{\mathcal{P}}$*,*
 ii. $\mathbf{E}_{\lambda_{\max}}$ *coincides with the eigenspace of $L^{\mathcal{P}}$ with largest eigenvalue.*

Proof of Lemma 27. From Remark 26, it follows that there exists a basis for $\mathbb{R}^{*[n]}$ of eigenvectors of L such that its elements are either eigenvectors of $L^{\mathcal{P}}$ in $R^{\mathcal{P}}$ or eigenvectors of L^{*A_q} in R^{*A_q} for some $q \in [p]$. Hence, to prove our lemma, it suffices to prove that the largest eigenvalue of $L^{\mathcal{P}}$ in $R^{\mathcal{P}}$ is strictly greater than the eigenvalues of $|N(A_q)|\, I + L^{*A_q}$ for all $q \in [p]$. This will be enough since then $\mathbf{E}_{\lambda_{\max}}$ is the space of eigenvectors of $L^{\mathcal{P}}$ in $R^{\mathcal{P}}$ with largest eigenvalue, so $\mathbf{E}_{\lambda_{\max}} \subseteq R^{\mathcal{P}}$ and claim i follows; therefore, observing that the copartition subgraph of $G^{\mathcal{P}}$ is $G^{\mathcal{P}}$ itself (Lemma 11) and in particular that $\left(L^{\mathcal{P}}\right)^{\mathcal{P}} = L^{\mathcal{P}}$, we obtain that $\mathbf{E}_{\lambda_{\max}}$ is also the eigenspace of $L^{\mathcal{P}}$ with largest eigenvalue, proving claim ii.

The proof will be divided into two cases, depending on whether $\overline{G}$ is connected.

Case 1. $\overline{G}$ is connected. From part 2 in Lemma 14 we know that for all $q \in [p]$, the largest eigenvalue of L^{*A_q} is at most $|A_q|$, so the largest eigenvalue of $|N(A_q)|\, I + L^{*A_q}$ is at most $|N(A_q)| + |A_q|$. Consequently, in this case we will then prove that the largest eigenvalue of $L^{\mathcal{P}}$ in $R^{\mathcal{P}}$ is strictly greater than

$$\max_{q \in [p]} \{|N(A_q)| + |A_q|\}.$$

Our strategy will be to find, for an arbitrary $q \in [p]$, a (not necessarily induced) subgraph H of $G^{\mathcal{P}}$ whose largest eigenvalue is strictly greater than $|N(A_q)| + |A_q|$, and then to directly apply part 3 in Lemma 14.

To start, first note that both $G^{\mathcal{P}}$ and its complement are connected graphs, and that for $q \in [p]$, A_q is both a maximal proper module and an independent set of $G^{\mathcal{P}}$ (Lemma 11). So, for an arbitrary $q \in [p]$, consider the (not necessarily induced) subgraph H^q of $G^{\mathcal{P}}$ on vertex-set $A_q \cup N(A_q)$ and with edge-set

$$\{\{i, j\} \in E : i \in A_q \text{ and } j \in N(A_q)\}.$$

First, since $G^{\mathcal{P}}$ is connected, then $N(A_q) \neq \varnothing$ and so H^q is a complete 2-partite graph with largest eigenvalue $|N(A_q)| + |A_q|$, from Lemma 21. Second, since the complement of $G^{\mathcal{P}}$ is connected, we have that $A_q \cup N(A_q) \neq [n]$, and then again, since $G^{\mathcal{P}}$ is connected, it follows that $N(A_q \cup N(A_q)) \neq \varnothing$. Hence, there exists a (not necessarily induced) connected bipartite subgraph H of $G^{\mathcal{P}}$ such that H is obtained from H^q by adding a new vertex $j \in [n] \backslash (A_q \cup N(A_q))$ and connecting it to some vertex $i \in N(A_q)$. We claim that H is the subgraph that we are looking for. Indeed, consider a nonzero eigenvector x of the Laplacian of H^q with largest eigenvalue $|N(A_q)| + |A_q|$, and extend x to the vertex-set of H by placing a 0 in the new coordinate j. Let us call this new vector y, so $y_k = x_k$ for all $k \in A_q \cup N(A_q)$ and $y_j = 0$. If L_H denotes the Laplacian of H, then straightforward calculations show that

$$\begin{aligned}
(L_H y)_i &= (|N(A_q)| + |A_q| + 1) y_i, \\
(L_H y)_j &= -y_i, \text{ and} \\
(L_H y)_k &= (|N(A_q)| + |A_q|) y_k \text{ for all } k \in (A_q \cup N(A_q)) \backslash \{i\}.
\end{aligned}$$

But crucially, from Lemma 22 we know that $y_i = x_i \neq 0$, which implies that $||L_H y|| > (|N(A_q)| + |A_q|)\, ||y||$ and hence that the largest eigenvalue of L_H must be strictly greater than $|N(A_q)| + |A_q|$.

Case 2. $\overline{G}$ is not connected. In this case, for all $q \in [p]$ we have that $|N(A_q)| = n - |A_q|$, and we observe that $G^{\mathcal{P}}$ is complete p-partite so Lemma 21 applies. Hence, we must only verify that the largest eigenvalue of L^{*A_q} is strictly less than $|A_q|$. However, by the definition of canonical partition, for each $q \in [p]$ the complement of $G[A_q]$ is connected and then part 2 in Lemma 14 implies that the largest eigenvalue of L^{*A_q} is strictly less than $|A_q|$. ∎

Two immediate applications of Lemma 27 now follow.

PROPOSITION 28. *Let $G = G([n], E)$ be a connected simple graph such that $\overline{G}$ is connected, $n > 1$. For any fixed proper module A of G, the following is true: If $x \in \mathbf{E}_{\lambda_{\max}}$, then A belongs to a fiber of x.*

Proof. Since $\overline{G}$ is connected, G partitions nontrivially into maximal proper modules and this partition is $\mathcal{P}$. But then, A is contained in a member of $\mathcal{P}$ and claim i of the lemma applies. ∎

PROPOSITION 29. *Let $G = G([n], E)$ be a connected simple graph such that $\overline{G}$ is disconnected. Then, $\lambda_{\max} = n$ and*

$$\begin{aligned}
\mathbf{E}_{\lambda_{\max}} = \{&x \in \mathbb{R}^{*[n]} : x_i = x_j \\
&\textit{whenever } i \textit{ and } j \textit{ belong to the same connected component of } \overline{G}\}.
\end{aligned}$$

In particular, $\dim \langle \mathbf{E}_{\lambda_{\max}} \rangle$ *is equal to the number of connected components of $\overline{G}$ minus one, and $G^{\mathcal{P}}$ is a complete p-partite graph, where p is the number of connected components of $\overline{G}$.*

Proof. As it was previously observed, from the definition of canonical partition and of copartition subgraph it follows that $G^{\mathcal{P}}$ is a complete p-partite graph. Using claim ii of the lemma, the result is exactly Lemma 21. ∎

Let us now turn back our attention to comparability graphs and to the proof of Theorem 15. Comparability graphs are, as anticipated, specially amenable to apply the previous lemma and its two propositions. In fact, the following result already establishes most of Theorem 15.

Proposition 30. *Let $G = G([n], E)$ be a connected comparability graph with canonical partition $\mathcal{P}$.*

i. *For $x \in \mathbf{E}_{\lambda_{\max}}$ a u.c.u.v., the following hold true almost surely:*
   1. *If $A \in \mathcal{P}$, then A belongs to a fiber of x.*
   2. *If $A, A' \in \mathcal{P}$ are (completely) adjacent in G, then A and A' belong to different fibers of x.*
   3. *x induces a transitive orientation of $G^{\mathcal{P}}$. In particular, $G^{\mathcal{P}}$ is a comparability graph.*
   4. *The connected components of the fibers of x are comparability graphs, and their vertex-sets are modules of G.*

ii. *If $\overline{G}$ is connected, then*
   1. $\dim\langle \mathbf{E}_{\lambda_{\max}} \rangle = 1$*, and also, $G^{\mathcal{P}}$ has exactly two transitive orientations and each can be obtained with probability $\frac{1}{2}$ in claim* i*;*
   2. $\mathbf{E}_{\lambda_{\max}} \subseteq \bigcup_O C_O$*, where the union is over all transitive orientations of G.*

iii. *If $\overline{G}$ is disconnected, then* $\dim\langle \mathbf{E}_{\lambda_{\max}} \rangle = p - 1$*, where p is the number of connected components of $\overline{G}$. Also, $G^{\mathcal{P}}$ has exactly* $p!$ *transitive orientations and each can be obtained with positive probability in claim* i.

Proof. We will work on each case, whether $\overline{G}$ is connected or disconnected, separately.

Let O be a transitive orientation of G, and note that $C_O \cap (\mathbf{E}_{\lambda_{\max}} \backslash \{0\}) \neq \varnothing$ from Proposition 17.

Case 1. $\overline{G}$ is connected. Our strategy to prove claims i–ii will be to first prove claim i in the special case when $x \in C_O \cap (\mathbf{E}_{\lambda_{\max}} \backslash \{0\})$ and then to argue that $G^{\mathcal{P}}$ is a connected comparability graph without proper nontrivial connected modules, which from Theorem 20 and claim ii in Lemma 27 directly implies part 1 of claim ii. From $\dim\langle \mathbf{E}_{\lambda_{\max}} \rangle = 1$, it then follows that C_O and $C_{O_{dual}}$ cover $\mathbf{E}_{\lambda_{\max}} \backslash \{0\}$. Hence,

(I) for $x \in \mathbf{E}_{\lambda_{\max}}$ a u.c.u.v., x falls in C_O or $C_{O_{dual}}$ almost surely, so our proof of claim i in the special case also applies (almost surely) to x;

(II) indeed $\mathbf{E}_{\lambda_{\max}} \subseteq C_O \cup C_{O_{dual}}$, proving part 2 of claim ii.

So take any $x \in C_O \cap \mathbf{E}_{\lambda_{\max}}$, $x \neq 0$. From Proposition 28, we know that x is constant on each $A \in \mathcal{P}$, so part 1 of claim i holds. Moreover, since the elements of $\mathcal{P}$ are maximal proper modules of G, then Lemma 18 shows that (completely) adjacent $A, A' \in \mathcal{P}$ must belong to different fibers of x, so part 2 of claim i holds. But then x induces an orientation of $G^{\mathcal{P}}$ and this orientation coincides with the restriction of O to $G^{\mathcal{P}}$. As a consequence, if $(i,j), (j,k)$ are edges on this orientation of $G^{\mathcal{P}}$, they are also edges of O and so must be (i,k) since O is transitive, showing that $\{i,k\} \in E$. But as $x_i < x_j < x_k$ and $x_i \neq x_k$, then i and k cannot both belong to the same member of $\mathcal{P}$, and so $\{i,k\}$ is also an edge of $G^{\mathcal{P}}$, and moreover (i,k) is an edge of the orientation induced by x. This orientation of $G^{\mathcal{P}}$ is then transitive and it follows that $G^{\mathcal{P}}$ is a comparability graph, proving part 3 of claim i. Last, as the restriction of a transitive orientation to an induced subgraph is always transitive, part 4 of claim i follows directly from Lemma 18. Now, it is immediate from Lemma 11 that $G^{\mathcal{P}}$ is a connected graph without proper nontrivial connected modules, and we have already observed that $G^{\mathcal{P}}$ is a comparability graph, so part 1 of claim ii follows.

Case 2. $\overline{G}$ is disconnected. This is precisely the setting of Proposition 29, so parts 1–3 of claim i and claim iii follow after noting first that for $x \in \mathbf{E}_{\lambda_{\max}}$ a u.c.u.v., the fibers of x are the connected components of $\overline{G}$, second that p-partite graphs are comparability graphs, and last that their transitive orientations are exactly the acyclic orientations such that

- for every pair of maximal independent sets, all the edges between them (or having endpoints on both sets) are oriented in the same direction.

The proof of part 4 of claim i follows since the connected components of $\overline{G}$ are modules of G and since all connected components of induced subgraphs on modules of G are also modules. Again, restrictions of transitive orientations to induced subgraphs are transitive. ∎

COROLLARY 31. *Let $G = G([n], E)$ be a connected comparability graph with canonical partition $\mathcal{P}$, and let O be a transitive orientation of G. Then*

1. *the restriction of O to each of $G^{\mathcal{P}}$ and $G[A], A \in \mathcal{P}$, is transitive;*
2. *conversely, if we select arbitrary transitive orientations for each of $G^{\mathcal{P}}$ and $G[A], A \in \mathcal{P}$, and then take the union of these, we obtain a transitive orientation for G.*

Proof. To prove part 1, we note that the restriction of O to any induced subgraph of G is transitive and then argue that a u.c.u.v. of $\mathbf{E}_{\lambda_{\max}}$ belongs to C_O with positive probability, so part 3 of claim i in Proposition 30 applies for some $x \in C_O$. To prove this last claim, from Proposition 17, $\dim \langle C_O \cap \mathbf{E}_{\lambda_{\max}} \rangle = \dim \langle \mathbf{E}_{\lambda_{\max}} \rangle$. Hence, $C_O \cap \mathbf{E}_{\lambda_{\max}}$ is a full-dimensional, closed, convex cone of $\mathbf{E}_{\lambda_{\max}}$, and then its relative interior is a nonempty open cone of the same space. As a nonempty open cone, the later intersects with the unit sphere of $\mathbf{E}_{\lambda_{\max}}$ and this intersection is open in the sphere, so the claim follows.

For part 2, select transitive orientations for each of $G^{\mathcal{P}}$ and $G[A], A \in \mathcal{P}$, and let O be the orientation of E so obtained. Since each element of $\mathcal{P}$ is independent in $G^{\mathcal{P}}$ and since the restriction of O to $G^{\mathcal{P}}$ is transitive, then

($\star$) for $A, A' \in \mathcal{P}$ (completely) adjacent, the edges between A and A' must be oriented in O in the same direction.

This rules out the existence of directed cycles in O, so O is acyclic. Now, if O is not transitive, then there must exist $i, j, k \in [n]$ such that (i, j) and (j, k) are in O but not (i, k). By the choice of O, it must be the case that exactly two among i, j, k belong to the same $A \in \mathcal{P}$ and the other one to a different $A' \in \mathcal{P}$. The former cannot be i and k, per the argument above ($\star$). Hence, without loss of generality, we can assume that $i, j \in A$ and $k \in A'$. But then, A and A' must be (completely) adjacent and (i, k) must exist in O, so we obtain a contradiction.

Note. The argument for part 2 is essentially found in Ramírez-Alfonsín and Reed (2001), and the construction corresponds to a *partially ordered union of posets.* ∎

COROLLARY 32. *Let $G = G([n], E)$ be a connected comparability graph with at least one proper nontrivial connected module B and canonical partition $\mathcal{P}$. Then, G has more than two transitive orientations.*

Proof. Suppose, on the contrary, that G has only two transitive orientations. We will prove that, then, G cannot have proper nontrivial connected modules and so B does not exist.

From Corollary 31 and claims ii–iii in Proposition 30, a necessary condition for G to have no more than two transitive orientations is

($\star$) $G = G^{\mathcal{P}}$, and either $\overline{G}$ is connected or it has exactly two connected components.

Now, if $\overline{G}$ is connected, then $B \subseteq A$ for some $A \in \mathcal{P}$, so B is an independent set of G since A is independent. This contradicts the choice of B. Also, if $\overline{G}$ has two connected components, then G is a complete bipartite graph. However, it is clear that no such B can exist in a complete bipartite graph. ∎

Proof of Theorem 15. The different numerals of this result have, for the most part, already been proved.

- Claim i was proved in Proposition 17.
- Claim ii was proved in part 1 of claim ii in Proposition 30 when $\overline{G}$ is connected. When $\overline{G}$ is disconnected, let $x \in \mathbf{E}_{\lambda_{\max}}$ be a u.c.u.v., so part 3 of claim i in Proposition 30 shows that x induces a transitive orientation of $G^{\mathcal{P}}$, and this orientation can be extended to some transitive orientation O of G by part 2 of Corollary 31, hence $x \in C_O$. However, the union in claim ii is a closed set, so we are done.
- Parts 1–5 of claim iii and claim iv are precisely Proposition 30.
- For part 6 of claim iii, from Corollary 32 and claim iii of Theorem 20, G has exactly two transitive orientations if and only if G has no proper nontrivial connected modules. Now, if G has no proper nontrivial connected modules, then part 4 of claim i in Proposition 30 shows that the fibers of x are independent sets of G, and on the other hand, claim ii in Theorem 20 gives $\dim \langle \mathbf{E}_{\lambda_{\max}} \rangle = 1$. Conversely, assume that the fibers of x are independent sets of G and that $\dim \langle \mathbf{E}_{\lambda_{\max}} \rangle = 1$. From the first assumption, $G = G^{\mathcal{P}}$ by Part 1 of Claim i in Proposition 30 and the definition of copartition subgraph. From the second assumption, per Claims ii-iii in Proposition 30, then $\overline{G}$ has at most two connected components. Hence, $G = G^{\mathcal{P}}$ and $\overline{G}$ has at most two connected components. Consequently, G cannot have proper nontrivial connected modules, as observed before. ◻

4. A characterization of comparability graphs. This section offers a curious novel characterization of comparability graphs that results from our theory in section 3.

THEOREM 33. *Let $G = G([n], E)$ be a simple undirected graph with Laplacian matrix L, and let I be the $n \times n$ identity matrix.*

Then, G is a comparability graph if and only if there exists $\alpha \in \mathbb{R}$ and an acyclic orientation O of E such that C_O is invariant under left-multiplication by $\alpha I + L$.

If G is a comparability graph, the orientations that satisfy the condition are precisely the transitive orientations of G, and we can take $\alpha = 0$ for them.

Proof. If G is a comparability graph and O is a transitive orientation of G, then Step 1 of Proposition 17 shows that, indeed, $Lx \in C_O$ whenever $x \in C_O$.

Suppose now that G is an arbitrary simple graph, and let O be an acyclic orientation (of E) that is not a transitive orientation of G. Then, there exist $i, j, k \in [n]$ such that (i, j) and (j, k) are in O but not (i, k), and the following set is nonempty:

$$\begin{aligned} X := \{k \in [n] : \ & \text{there exist } i, j \in [n] \text{ and directed edges} \\ & (i, j), (j, k) \text{ in } O, \text{ but } (i, k) \text{ is not in } O\}. \end{aligned}$$

In the partial order on $[n]$ induced by O, take some $\ell \in X$ maximal, and consider the principal order filter $\ell^{\vee}$ generated by ℓ. The indicator vector of $\ell^{\vee}$ is $e_{\ell^{\vee}}$. Then, $e_{\ell^{\vee}} \in C_O$. Now, choose $i, j \in [n]$ so that (i, j) and (j, ℓ) are in O but not (i, ℓ). As ℓ was chosen maximal in X, for every $k \in \ell^{\vee}, k \neq \ell$, then both (i, k) and (j, k) are in O. Therefore, for any $\alpha \in \mathbb{R}$ we have

$$\begin{aligned} \left[(\alpha I + L) e_{\ell^{\vee}}\right]_i &= -\left|\ell^{\vee}\right| + 1 \text{ and} \\ \left[(\alpha I + L) e_{\ell^{\vee}}\right]_j &= -\left|\ell^{\vee}\right|. \end{aligned}$$

Hence, $\left[(\alpha I + L) e_{\ell^{\vee}}\right]_i > \left[(\alpha I + L) e_{\ell^{\vee}}\right]_j$ and $(\alpha I + L) e_{\ell^{\vee}} \notin C_O$ since (i, j) is in O. ◻

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