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## Infinitesimal change of stable basis

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# INFINITESIMAL CHANGE OF STABLE BASIS

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ABSTRACT. The purpose of this note is to study the Maulik-Okounkov $K-$theoretic stable basis for the Hilbert scheme of points on the plane, which depends on a "slope" $m \in \mathbb{R}$. When $m = \frac{a}{b}$ is rational, we study the change of stable matrix from slope $m - \varepsilon$ to $m + \varepsilon$ for small $\varepsilon > 0$, and conjecture that it is related to the Leclerc-Thibon conjugation in the $q-$Fock space for $U_q\widehat{\mathfrak{gl}}_b$. This is part of a wide framework of connections involving derived categories of quantized Hilbert schemes, modules for rational Cherednik algebras and Hecke algebras at roots of unity.

## 1. INTRODUCTION

Maulik and Okounkov [25, 26] developed a new paradigm for constructing interesting bases in the equivariant cohomology and $K$-theory of certain algebraic varieties with torus actions. These are called **stable bases** and can be defined for any conical symplectic resolution in the sense of [6, 7], in particular, for Nakajima quiver varieties. In this paper, we present an explicit conjectural description of the $K-$theoretic stable bases for $\mathrm{Hilb}_n$, the Hilbert scheme of $n$ points on $\mathbb{C}^2$.

The definition of the stable basis involves a choice of a Hamiltonian one parameter subgroup, which is unique in the case of Hilbert schemes (strictly speaking, there are two possible choices since one can invert the parameter, but we fix it without loss of generality), and a choice of $\mathcal{L} \in \mathrm{Pic}(\mathrm{Hilb_n}) \otimes (\mathbb{R}\backslash\mathbb{Q})$. We abuse notation and refer to such $\mathcal{L}$ as "line bundles", though they are formal irrational multiples of actual line bundles. Since $\mathrm{Pic}(\mathrm{Hilb_n})$ has rank 1 with generator $\mathcal{O}(1)$, we write $\mathcal{L}_m$ for the line bundle associated to $m \in \mathbb{R}\backslash\mathbb{Q}$. The construction of [26] produces a basis:

$$\left\{ s_\lambda^m \right\}_{\lambda \vdash n} \in K_{\mathbb{C}^* \times \mathbb{C}^*}(\mathrm{Hilb}_n) \qquad \forall\, m \in \mathbb{R}\backslash\mathbb{Q} \qquad (1.1)$$

For $m = 0$ the basis $s^m$ is expected to match the (plethystically transformed) Schur polynomial basis, and for $m = \infty$ it coincides with the (modified) Macdonald polynomial basis. Therefore, the stable basis for general $m$ can be thought of as interpolating between the bases of Schur and Macdonald polynomials. We are interested in "walls", i.e. those:

$$m \in \mathbb{R} \qquad \text{such that} \qquad \left\{ s_\lambda^{m+\varepsilon} \right\}_{\lambda \vdash n} \neq \left\{ s_\lambda^{m-\varepsilon} \right\}_{\lambda \vdash n}$$

Throughout this paper, $\varepsilon$ denotes a very small positive real number. There are only discretely many walls for each fixed $n$, all expected to be of the form $m = \frac{a}{b}$ with $0 < b \leq n$. The following conjecture prescribes how the stable basis changes upon crossing these walls:

**Conjecture 1.2.** *(see Conjecture 4.18 for the precise formulation): For $m = \frac{a}{b}$ with $\gcd(a, b) = 1$:*

$$\textit{the matrix taking} \quad \left\{ s_\lambda^{m+\varepsilon} \right\}_{\lambda \vdash n} \quad \textit{to} \quad \left\{ s_\lambda^{m-\varepsilon} \right\}_{\lambda \vdash n}$$

*coincides with the Leclerc-Thibon involution [22, 23] for $U_q\widehat{\mathfrak{gl}}_b$, up to conjugation by the diagonal matrix that produces the renormalization (4.17).*

We prove the above conjecture for $b = 1$, where the Leclerc-Thibon involution is trivial:

**Proposition 1.3.** *We have $s_\lambda^\varepsilon = s_\lambda^{-\varepsilon}$ for all partitions $\lambda \vdash n$.*

The proof of Proposition 1.3, as well as an idea to tackle Conjecture 1.2 in general, is based on a principle that goes back to the work of Grojnowski and Nakajima, which says that one should work with all $\text{Hilb}_n$ together, for all $n \in \mathbb{N}$. Namely, define:

$$K = \bigoplus_{n=0}^{\infty} K_{\mathbb{C}^* \times \mathbb{C}^*}(\text{Hilb}_n) \tag{1.4}$$

Feigin-Tsymbaliuk [12] and Schiffmann-Vasserot [32] have constructed an action of the spherical double affine Hecke algebra (DAHA) $\mathcal{A}$ of type $GL_\infty$ on $K$, albeit each in a different language. The algebra $\mathcal{A}$ has numerous $q$–Heisenberg subalgebras $\mathcal{A}^{(m)}$, parametrized by rational numbers $m$. In previous work ([29, 30]) the second named author proved that the action of $\mathcal{A}^{(m)}$, written in the stable basis $s^m$, is given by ribbon tableau formulas akin to those studied by Lascoux, Leclerc and Thibon [20]. We conjecture that this is a special case of the following more general phenomenon.

**Conjecture 1.5.** *(see Conjecture 5.4 for the precise formulation): For $m = \frac{a}{b}$ with $\gcd(a,b) = 1$:*

$$there\ exists\ an\ action\ \ U_q\widehat{\mathfrak{gl}}_b \ \curvearrowright \ K \tag{1.6}$$

*such that:*

(1) *$K$ is a level 1 vacuum module for $U_q\widehat{\mathfrak{gl}}_b$, isomorphic to the Fock space*
(2) *The subalgebra $\mathcal{A}^{(m)}$ embeds into $U_q\widehat{\mathfrak{gl}}_b$ as the standard diagonal $q$–Heisenberg subalgebra, and this embedding intertwines its action on $K$ from [12, 29, 32] with the action (1.6)*
(3) *The bases $s^{m-\varepsilon}$ and $s^{m+\varepsilon}$ are, respectively, the standard and costandard bases for the action (1.6), up to renormalization.*

We expect that the above "slope $m$ action" of $U_q\widehat{\mathfrak{gl}}_b$ on Fock space has interesting algebraic, geometric and combinatorial meaning, generalizing recent results about the "slope $m$ action" $\mathcal{A}^{(m)} \curvearrowright K$ [3, 16, 27]. We support the conjectures with the following results.

**Theorem 1.7.** *Suppose that $\gcd(a,b) = \gcd(a',b) = 1$. Then the actions of $\mathcal{A}^{(\frac{a}{b})}$ and of $\mathcal{A}^{(\frac{a'}{b})}$ on $K$ are conjugate to each other by the transition matrix between the bases $s^{\frac{a}{b}}$ and $s^{\frac{a'}{b}}$.*

**Theorem 1.8.** *Conjectures 1.2 and 1.5 are equivalent.*

Conjecture 1.2 was verified for $n \le 6$ and all rational slopes $m = \frac{a}{b}$ by explicit computer calculations. Note that by (4.15), it is sufficient to check slopes $m \in [0,1)$ and by Proposition 4.20 one can assume $b \le n(n-1)$. Therefore, one has finitely many slopes to check for each $n$.

## 2. Symmetric functions and Hilbert schemes

**2.1.** Much of the present paper is concerned with the ring of symmetric functions in infinitely many variables $x_1, x_2, ...$:

$$\Lambda = \mathbb{Z}[x_1, x_2, ...]^{\text{Sym}} \tag{2.1}$$

There are a number of generating sets of (2.1), perhaps the most fundamental being the collection of monomial symmetric functions:

$$m_\lambda = \text{Sym}\left(x_1^{\lambda_1} x_2^{\lambda_2}...\right)$$

where $\lambda = (\lambda_1 \ge \lambda_2 \ge ...)$ goes over all partitions of natural numbers. Particular instances of monomial symmetric functions are the power sum functions:

$$p_k = m_{(k)} = x_1^k + x_2^k + ...$$

and the elementary symmetric functions:

$$e_k = m_{(1,1,...,1)} = \sum_{i_1 < ... < i_k} x_{i_1}...x_{i_k}$$

As a ring, $\Lambda$ is generated by the elementary symmetric functions:

$$\Lambda = \mathbb{Z}[e_1, e_2, ...]$$

and is generated by power sum functions upon tensoring with $\mathbb{Q}$:

$$\widetilde{\Lambda} := \Lambda \bigotimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}[p_1, p_2, ...]$$

Additive generators are always indexed by partitions $\lambda$:

$$\Lambda = \mathbb{Z}[e_\lambda]_{\lambda \text{ partition}} \qquad \text{where} \quad e_\lambda = e_{\lambda_1} e_{\lambda_2} ...$$

and:

$$\widetilde{\Lambda} = \mathbb{Q}[p_\lambda]_{\lambda \text{ partition}} \qquad \text{where} \quad p_\lambda = p_{\lambda_1} p_{\lambda_2} ...$$

A symmetric function is called **integral** if it lies in the image of $\Lambda \hookrightarrow \widetilde{\Lambda}$. A basis of $\widetilde{\Lambda}$ is called integral if it consists only of such functions.

**2.2.** There is a one-to-one correspondence between partitions and Young diagrams, the latter being stacks of $1 \times 1$ boxes placed in the corner of the first quadrant. For example, the Young diagram:
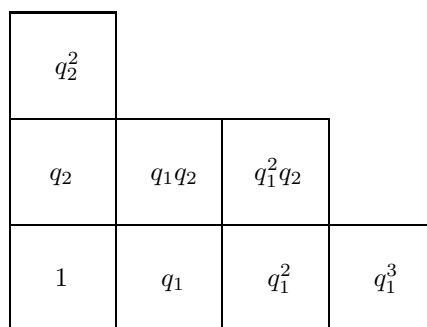


Figure 1

represents the partition $(4, 3, 1)$, because it has 4 boxes on the first row, 3 boxes on the second row, and 1 box on the third row. The monomials displayed in Figure 1 are called the **weights** of the boxes they are in, and are defined by the formula:

$$\chi_\square = q_1^x q_2^y \tag{2.2}$$

where $(x, y)$ are the coordinates of the southwest corner of the box in question. We call the integer:

$$c_\square = x - y \tag{2.3}$$

the **content** of the box, and note that $c_\square$ is constant across diagonals. Finally, to every box in a Young diagram we may associate its **arm–length** and **leg–length**:

$$a(\square) \text{ and } l(\square) \in \mathbb{Z}_{\geq 0}$$

These numbers count the distance between the box $\square$ and the right and top borders of the partition, respectively. For example, the box of weight $q_2$ in Figure 1 has $a(\square) = 2$ and $l(\square) = 1$. We will write:

$$c_\lambda = \sum_{\square \in \lambda} c_\square \qquad\qquad \chi_\lambda = \prod_{\square \in \lambda} \chi_\square \tag{2.4}$$

We write $\mu \leq \lambda$ if the Young diagram of $\mu$ is completely contained in that of $\lambda$, and call $\lambda \backslash \mu$ a **skew Young diagram**. If such a skew diagram is a connected set of $b$ boxes which contains no $2 \times 2$ squares, we call it a $b$–**ribbon**. Note that the contents of the boxes of a $b$–ribbon $R$ are consecutive integers. Set:

$$\mathbf{h}(\text{ribbon } R) = \max_{\square, \blacksquare \in R} y(\square) - y(\blacksquare)$$

A skew diagram $S$ is called a **horizontal** $k$–**strip** of $b$–ribbons if it can be tiled with $k$ such ribbons $R_1, ..., R_k$ in such a way that the the northwestern most box of $R_i$ does not lie below a box of $R_j$ for any $1 \leq j \neq i \leq k$. Note that such a tiling of a skew diagram $S$ is always unique. We set:

$$\mathbf{h}(\text{strip } S) = \mathbf{h}(R_1) + ... + \mathbf{h}(R_k)$$

The $b$–**core** of a partition $\lambda$ is defined as the minimal partition which can be obtained by removing $b$–ribbons from $\lambda$. It is well known that the $b$–core does not depend on which set of ribbons we choose to remove, as long as this set is maximal.

**2.3.** We will now extend our rings of constants, and work instead with:

$$\Lambda_{q_1,q_2} = \Lambda \bigotimes_{\mathbb{Z}} \mathbb{Z}[q_1^{\pm 1}, q_2^{\pm 1}] = \mathbb{Z}[q_1^{\pm 1}, q_2^{\pm 1}][x_1, x_2, ...]^{\text{Sym}}$$

$$\widetilde{\Lambda}_{q_1,q_2} = \widetilde{\Lambda} \bigotimes_{\mathbb{Q}} \mathbb{Q}(q_1, q_2) = \mathbb{Q}(q_1, q_2)[x_1, x_2, ...]^{\text{Sym}}$$

Since the Macdonald inner product respects the degree of symmetric polynomials and the Hopf algebra structure of $\widetilde{\Lambda}_{q_1,q_2}$, it is uniquely determined by the pairing of $p_k$ with itself:

$$\langle \cdot, \cdot \rangle_0 : \widetilde{\Lambda}_{q_1,q_2} \bigotimes_{\mathbb{Q}(q_1,q_2)} \widetilde{\Lambda}_{q_1,q_2} \longrightarrow \mathbb{Q}(q_1, q_2) \tag{2.5}$$

$$\langle p_k, p_k \rangle_0 = k \cdot \frac{1 - q_1^k}{1 - q_2^{-k}}$$

Macdonald polynomials $\{P_\lambda\}_{\lambda \text{ partition}}$ are the only orthogonal basis of $\widetilde{\Lambda}_{q_1,q_2}$:

$$\langle P_\lambda, P_\mu \rangle_0 = 0 \qquad \forall \lambda \neq \mu$$

which is unitriangular in the basis of monomial symmetric functions:

$$P_\lambda = m_\lambda + \sum_{\mu \vartriangleleft \lambda} m_\mu c_\lambda^\mu \tag{2.6}$$

for certain coefficients $c_\lambda^\mu \in \mathbb{Q}(q_1, q_2)$. In the above formula, recall that the **dominance ordering** on partitions of the same size $|\mu| = |\lambda|$ is:

$$\mu \trianglelefteq \lambda \qquad \text{if} \qquad \mu_1 + ... + \mu_i \leq \lambda_1 + ... + \lambda_i \quad \forall i \tag{2.7}$$

An element of $\widetilde{\Lambda}_{q_1,q_2}$ is called integral if it lies in the image of $\Lambda_{q_1,q_2} \hookrightarrow \widetilde{\Lambda}_{q_1,q_2}$. Because the coefficients $c_\lambda^\mu$ of (2.6) are rational functions in general, Macdonald polynomials are not integral. However, the following renormalization:

$$\widetilde{J}_\lambda = P_\lambda \cdot q_2^{-|\lambda|} \prod_{\square \in \lambda} \left( q_2^{l(\square)+1} - q_1^{a(\square)} \right) \tag{2.8}$$

is integral. It is well-known that the pairing of $\widetilde{J}_\lambda$ with itself is given by:

$$\langle \widetilde{J}_\lambda, \widetilde{J}_\mu \rangle_0 = \delta_\mu^\lambda \cdot q_2^{-|\lambda|} \prod_{\square \in \lambda} \left( q_2^{l(\square)+1} - q_1^{a(\square)} \right) \left( q_2^{l(\square)} - q_1^{a(\square)+1} \right) \tag{2.9}$$

## 3. Fock representation and global canonical bases

**3.1.** We recall the explicit construction of the action of the quantum affine algebra $U_q \widehat{\mathfrak{gl}}_b$ on the $q$–Fock space $\Lambda_q$, following Kashiwara-Miwa-Stern [19] and Leclerc-Thibon [21, 22, 23]. The **standard basis** in $\Lambda_q$ will be denoted by $|\lambda\rangle$, so we define:

$$\Lambda_q = \bigoplus_{\lambda \text{ partition}} \mathbb{Q}(q) \cdot |\lambda\rangle$$

Consider partitions $\lambda, \mu$ such that the former is obtained from the latter by adding an $i$–**node**, by which we mean a box $\blacksquare$ with content $\equiv i$ modulo $b$. We call this box a **removable** $i$–**node** for $\lambda$ and an **indent** $i$–**node** for $\mu$. Let $I_i(\mu)$ be the number of indent $i$–nodes of $\mu$, $R_i(\lambda)$ the number of removable $i$–nodes of $\lambda$, $I_i^l(\lambda, \mu)$ (resp. $R_i^l(\lambda, \mu)$) the number of indent $i$–nodes (resp. of removable $i$–nodes) situated to the

left of ■, and similarly, let $I_i^r(\lambda, \mu)$ and $R_i^r(\lambda, \mu)$ be the corresponding numbers of nodes located on the right of ■. Set:

$$N_i(\lambda) = I_i(\lambda) - R_i(\lambda)$$

for all partitions $\lambda$, as well as:

$$N_i^l(\lambda, \mu) = I_i^l(\lambda, \mu) - R_i^l(\lambda, \mu)$$

$$N_i^r(\lambda, \mu) = I_i^r(\lambda, \mu) - R_i^r(\lambda, \mu)$$

for all pairs $\lambda, \mu$ such that $\lambda \backslash \mu$ consists of an $i$–node ■. Then the following assignments:

$$e_i|\lambda\rangle = \sum_{\substack{\lambda/\mu \text{ is} \\ \text{an } i-\text{node}}} q^{N_i^l(\lambda,\mu)}|\mu\rangle, \qquad f_i|\mu\rangle = \sum_{\substack{\lambda/\mu \text{ is} \\ \text{an } i-\text{node}}} q^{N_i^r(\lambda,\mu)}|\lambda\rangle, \qquad (3.1)$$

$$q^{h_i}|\lambda\rangle = q^{N_i(\lambda)}|\lambda\rangle, \qquad q^D|\lambda\rangle = q^{N_0(\lambda)}|\lambda\rangle \qquad (3.2)$$

give rise to an action of $U_q\widehat{\mathfrak{sl}}_b$ on the Fock space $\Lambda_q$. One wishes to enhance (3.1)–(3.2) to an action of:

$$U_q\widehat{\mathfrak{gl}}_b = U_q\widehat{\mathfrak{sl}}_b \otimes U_q\widehat{\mathfrak{gl}}_1$$

on the Fock space, where the $q$−Heisenberg algebra is:

$$U_q\widehat{\mathfrak{gl}}_1 = \mathbb{Q}(q)\langle..., B_{-2}, B_{-1}, B_1, B_2, ...\rangle \Big/ [B_k, B_l] - k\delta_{k+l}^0[b]_{q^k}$$

where $[b]_x = 1 + x + ... + x^{b-1}$. In other words, we must define an action of the generators $B_k$ on Fock space which commutes with the one prescribed by formulas (3.1)–(3.2). To do so, let us consider the following alternative system of generators:

$$\sum_{k=0}^{\infty} V_{\pm k} z^k = \exp\left(\sum_{k=1}^{\infty} \frac{B_{\mp k} z^k}{k}\right)$$

In [20], the authors introduced the following action $U_q\widehat{\mathfrak{gl}}_1 \curvearrowright \Lambda_q$ and showed that it commutes with the action of $U_q\widehat{\mathfrak{sl}}_b$ defined in (3.1)–(3.2), thus giving rise to an action $U_q\widehat{\mathfrak{gl}}_b \curvearrowright \Lambda_q$:

$$V_k|\mu\rangle = \sum_\lambda (-q)^{-\mathbf{h}(\lambda/\mu)}|\lambda\rangle, \qquad V_{-k}|\lambda\rangle = \sum_\mu (-q)^{-\mathbf{h}(\lambda/\mu)}|\mu\rangle \qquad (3.3)$$

where the sums go over all horizontal $k$–strips of $b$–ribbons $\lambda/\mu$, as in Subsection 2.2.

**3.2.** As observed by Leclerc and Thibon, there is a unique involution of the Fock space $\Lambda_q$ satisfying:

(1) Semilinearity: $\overline{a(q)x + b(q)y} = a(q^{-1})\overline{x} + b(q^{-1})\overline{y}$
(2) Identity on vacuum: $\overline{|\emptyset\rangle} = |\emptyset\rangle$
(3) Invariance under the creation operators: $\overline{f_i v} = f_i\overline{v}$, $\overline{B_{-k}v} = B_{-k}\overline{v}$.

Indeed, products of $f_i$ and $B_{-k}$ applied to the vacuum span the Fock space, and this implies uniqueness. Note that $\overline{V_k v} = V_k\overline{v}$ for all $k > 0$, because the operators $V_k$ are monomials in the generators $B_{-k}$ with constant coefficients. Define the matrix $A_b(q) = (a_\lambda^\mu(q))$ by the equation

$$\overline{|\lambda\rangle} = \sum_\mu a_\lambda^\mu(q) \cdot |\mu\rangle. \qquad (3.4)$$

Clearly, $A_b(q)A_b(q^{-1}) = \text{Id}$ by the semilinearity property (1).

**Theorem 3.5.** ([22, 23]) *The matrix $A_b(q)$ has the following properties:*

a) $a_\lambda^\mu(q) \in \mathbb{Z}[q, q^{-1}]$
b) $a_\lambda^\mu(q) = 0$ *unless* $|\lambda| = |\mu|$, $\mu \trianglelefteq \lambda$ *and* $\lambda, \mu$ *have the same $b$–core*
c) $a_\lambda^\lambda(q) = 1$
d) $a_\lambda^\mu(q) = a_{\mu'}^{\lambda'}(q)$,

*where $\lambda'$ is the transpose of the Young diagram $\lambda$.*

*Example* 3.6. As an exercise, let us compute the matrix $A_2(q)$ in degree 2. We have $f_0|\emptyset\rangle = |(1)\rangle$ and:

$$f_1 f_0|\emptyset\rangle = f_1|(1)\rangle = |(2)\rangle + q|(1,1)\rangle, \qquad \text{while} \qquad V_1|\emptyset\rangle = |(2)\rangle - q^{-1}|(1,1)\rangle$$

By condition (3), the vectors $f_1 f_0|\emptyset\rangle$ and $V_1|\emptyset\rangle$ should be preserved by the bar-involution, so the matrix:

$$T = \begin{pmatrix} 1 & 1 \\ q & -q^{-1} \end{pmatrix}$$

satisfies $A_2(q)T(q^{-1}) = T(q)$. We conclude that:

$$A_2(q) = T(q)T(q^{-1})^{-1} = \begin{pmatrix} 1 & 0 \\ q - q^{-1} & 1 \end{pmatrix}.$$

*Remark* 3.7. A similar method can be used to compute the matrix $A_b(q)$ in general: using the matrices of $f_i$ and $V_i$ defined by (3.1)–(3.3), one can write a basis of bar-invariant vectors in $\Lambda_q$, write their coordinates in a matrix $T$ and obtain $A_b(q) = T(q)T(q^{-1})^{-1}$. See [21] for further details. Note that this approach does not explain the triangularity of $A_b(q)$, namely property b) of Theorem 3.5.

**3.3.** We will also encounter the **costandard basis** $\overline{|\lambda\rangle}$ of $\Lambda_q$. By definition, $A_b(q)$ is the transition matrix between the standard and the costandard bases. Furthermore, the action of the creation operators in the costandard basis is given by the following equations:

$$f_i\overline{|\mu\rangle} = \overline{f_i|\mu\rangle} = \overline{\sum_\lambda q^{N_i^r(\lambda,\mu)}|\lambda\rangle} = \sum_\lambda q^{-N_i^r(\lambda,\mu)}\overline{|\lambda\rangle}, \tag{3.8}$$

and similarly:

$$V_k\overline{|\mu\rangle} = \sum_\lambda (-q)^{\mathbf{h}(\lambda\backslash\mu)}\overline{|\lambda\rangle}, \tag{3.9}$$

where the sums over $\lambda$ and $\mu$ are the same as in (3.1) and (3.3).

*Remark* 3.10. Since the Fock space is an irreducible representation of $U_q\widehat{\mathfrak{gl}}_b$, the equations (3.1)–(3.3) and (3.8)–(3.9) define the standard and the costandard bases completely.

Furthermore, [22, 23] define yet another basis in the Fock space called the *global canonical basis*.

**Theorem 3.11.** ([22, 23]) *There exist unique bases* $G^\pm(\lambda)$ *in* $\Lambda_q$ *such that:*
  (1) $\overline{G^\pm(\lambda)} = G^\pm(\lambda)$.
  (2) $G^\pm(\lambda) \cong |\lambda\rangle \mod q^{\pm 1}\Lambda[q^{\pm 1}]$

Consider the matrix $(d_\lambda^\mu(q))$ defined by the equation:

$$G^+(\lambda) = \sum_\lambda d_\lambda^\mu(q) \cdot |\mu\rangle. \tag{3.12}$$

One can check that this matrix is lower-triangular, so $d_\lambda^\mu(q) = 0$ unless $\mu \trianglelefteq \lambda$.

*Example* 3.13. Let us compute the basis $G^+$ using Example 3.6. By triangularity,

$$G^+(1,1) = |(1,1)\rangle, \ G^+(2) = |(2)\rangle + \beta(q)|(1,1)\rangle.$$

The bar-invariance implies $\beta(q) - \beta(q^{-1}) = q - q^{-1}$ which, together with condition (2) in Theorem 3.11, uniquely determines $\beta(q) = q$. Therefore

$$(d_\lambda^\mu(q)) = \begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix}$$

## 4. Hilbert schemes and stable bases

**4.1.** We consider the Hilbert scheme $\text{Hilb}_n$ of $n$ points in the plane. This is a smooth quasi-projective variety of dimension $2n$. It is endowed with a torus action:

$$T = \mathbb{C}_q^* \times \mathbb{C}_t^* \curvearrowright \text{Hilb}_n \tag{4.1}$$

In the above formula, $q$ and $t$ are equivariant parameters, namely the standard coordinates on rank 1 tori. We will often denote $q_1 = qt$ and $q_2 = qt^{-1}$ and think of these monomials as the torus characters

acting on the coordinate lines of $\mathbb{C}^2$. Fixed points of the Hilbert scheme with respect to the torus action (4.1) are monomial ideals:

$$I_\lambda = (x^{\lambda_1-1}, x^{\lambda_2-1}y, x^{\lambda_3-2}y^2, \ldots) \in \text{Hilb}_n \tag{4.2}$$

for any partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots)$. The torus character in the tangent space to $\text{Hilb}_n$ at the fixed point $I_\lambda$ is given by the well-known formula:

$$T_\lambda \text{Hilb}_n = \sum_{\square \in \lambda} \left( q_1^{a(\square)} q_2^{-l(\square)-1} + q_1^{-a(\square)-1} q_2^{l(\square)} \right) \tag{4.3}$$

We will work with the equivariant $K-$theory group:

$$K = \bigoplus_{n=0}^{\infty} K_{q,t}(\text{Hilb}_n)$$

By definition, $K$ is the additive group generated by the classes of $\mathbb{C}_q^* \times \mathbb{C}_t^*-$equivariant vector bundles on Hilbert schemes $\text{Hilb}_n$, modulo relations imposed by exact sequences. Important elements of $K$ are the skyscraper sheaves at the torus fixed points (4.2), which we denote by the same letter as the fixed point itself:

$$[\widetilde{I_\lambda}] \in K$$

Recall the equivariant localization formula, which expresses any class $f \in K$ in terms of its restrictions to torus fixed points:

$$f = \sum_{\lambda \vdash n} \frac{f|_\lambda \cdot [\widetilde{I_\lambda}]}{[T_\lambda \text{Hilb}_n]} \tag{4.4}$$

where in the denominator we write $[x] = 1 - x^{-1}$ and extend this notation additively: $[x+y] = [x] \cdot [y]$. Because of the presence of denominators, the equality (4.4) holds in the localized $K$–theory group:

$$\widetilde{K} = K \bigotimes_{\mathbb{Z}[q_1^{\pm 1}, q_2^{\pm 1}]} \mathbb{Q}(q_1, q_2)$$

In this localization, we may renormalize the classes of fixed points:

$$[I_\lambda] = \frac{[\widetilde{I_\lambda}]}{[T_\lambda \text{Hilb}_n]} \in \widetilde{K}$$

The restriction of a class to a fixed point is precisely its coefficient when expanded in the basis $[I_\lambda]$:

$$f = \sum_{\lambda \vdash n} f|_\lambda \cdot [I_\lambda] \tag{4.5}$$

**4.2.** The well-known Bridgeland-King-Reid construction [8] is an equivalence between the derived category of coherent sheaves on $\text{Hilb}^n(\mathbb{C}^2)$ and the derived category of $S_n-$equivariant coherent sheaves on $\mathbb{C}^{2n}$, which in particular allows one to identify:

$$K \cong \Lambda_{q_1, q_2} \tag{4.6}$$

Haiman showed that the classes of fixed points correspond to **modified Macdonald polynomials** $\widetilde{H}_\lambda$:

$$[\widetilde{I_\lambda}] \leftrightarrow \widetilde{H}_\lambda$$

where $\widetilde{H}_\lambda[X] = \varphi(\widetilde{J}_\lambda)$ is the image of (2.8) under the algebra homomorphism:

$$\varphi : \widetilde{\Lambda}_{q_1, q_2} \to \widetilde{\Lambda}_{q_1, q_2} \qquad \varphi(p_k) = \frac{p_k}{1 - q_2^{-k}} \tag{4.7}$$

Because of thishomomorphism, it makes sense to study the following modification of the inner product (2.5):

$$\langle \cdot, \cdot \rangle : \widetilde{\Lambda}_{q_1, q_2} \bigotimes_{\mathbb{Q}(q_1, q_2)} \widetilde{\Lambda}_{q_1, q_2} \longrightarrow \mathbb{Q}(q_1, q_2) \tag{4.8}$$

$$\langle \varphi(f), \varphi(g) \rangle = \langle f, g \rangle_0$$

which explicitly is generated by the following formula for the pairing of $p_k$ with itself:

$$\langle p_k, p_k \rangle = k \cdot \left(1 - q_1^k\right) \left(1 - q_2^k\right)$$

With respect to this inner product, (2.9) implies on general grounds that:

$$\langle \widetilde{H}_\lambda, \widetilde{H}_\mu \rangle = \delta_\mu^\lambda \cdot (-1)^{|\lambda|} \prod_{\square \in \lambda} \left( q_2^{l(\square)+1} - q_1^{a(\square)} \right) \left( q_2^{l(\square)} - q_1^{a(\square)+1} \right) \tag{4.9}$$

Meanwhile, the natural Euler form is:

$$(\widetilde{H}_\lambda, \widetilde{H}_\mu) = \delta_\mu^\lambda \cdot [T_\lambda \text{Hilb}_n] = \delta_\mu^\lambda \prod_{\square \in \lambda} \left( 1 - q_1^{-a(\square)} q_2^{l(\square)+1} \right) \left( 1 - q_1^{a(\square)+1} q_2^{-l(\square)} \right) \tag{4.10}$$

Comparing (4.9) with (4.10), we conclude that $\langle f, g \rangle = (\nabla f, g)$, where the Bergeron–Garsia operator $\nabla$ is defined to be diagonal in the basis of modified Macdonald polynomials:

$$\nabla : \Lambda_{q_1, q_2} \longrightarrow \Lambda_{q_1, q_2}, \qquad \widetilde{H}_\lambda \mapsto \widetilde{H}_\lambda \cdot \chi_\lambda$$

where $\chi_\lambda$ was defined in (2.4). If we observe that $\chi_\lambda$ is the torus weight of the restriction of the line bundle $\mathcal{O}(1)$ to the fixed point $\lambda$, then the operator $\nabla$ corresponds to the operator of multiplication by $\mathcal{O}(1)$ under the isomorphism (4.6).

**4.3.** In [25], Maulik and Okounkov defined the **stable basis** for the cohomology of a wide class of symplectic resolutions $X$. The $K-$theoretic version of their construction has not yet been published, but the interested reader can read about it in [1, 30, 31]. We will review their particular construction in the case at hand $X = \text{Hilb}_n$:

$$\forall \ m \in \mathbb{R} \backslash \mathbb{Q} \quad \rightsquigarrow \quad \text{an integral basis } \{s_\lambda^m\}_{\lambda \vdash n} \in K_T(\text{Hilb}_n) \tag{4.11}$$

which is triangular in terms of renormalized fixed points:

$$s_\lambda^m = \sum_{\mu \trianglelefteq \lambda} \gamma_\lambda^\mu [I_\mu] \qquad \text{where} \qquad \gamma_\lambda^\lambda = \prod_{\square \in \lambda} \left( q_2^{l(\square)} - q_1^{a(\square)+1} \right) \tag{4.12}$$

and the coefficients $\gamma_\lambda^\mu \in \mathbb{Z}[q^{\pm 1}, t^{\pm 1}]$ have the property:

$$\min \deg \gamma_\lambda^\mu(q, t) \geq -n(\mu) + m \cdot (c_\mu - c_\lambda) \tag{4.13}$$

$$\max \deg \gamma_\lambda^\mu(q, t) \leq n(\mu') + |\mu| + m \cdot (c_\mu - c_\lambda) \tag{4.14}$$

Recall that $n(\lambda) = \sum_{\square \in \lambda} l(\square)$. Here and throughout this paper, "min deg" and "max deg" refer to the minimal and maximal degrees of a Laurent polynomial in the variable $t$. Formulas (4.13)–(4.14) are arranged so that when $\lambda = \mu$, the leading coefficient of (4.12) forces the two inequalities to be equalities. Maulik–Okounkov claim that for any $m \in \mathbb{R} \backslash \mathbb{Q}$, there is a unique integral basis with properties (4.12), (4.13), (4.14). Moreover, the basis is unchanged under small perturbations of $m$. Note that uniqueness implies:

$$s_\lambda^{m+1} = \frac{\nabla s_\lambda^m}{\chi_\lambda} \tag{4.15}$$

*Remark* 4.16. Geometrically, we may think of the flow given by the rank one torus $\mathbb{C}_t^*$ on $\text{Hilb}_n$. There is a flow line from $I_\mu$ to $I_\lambda$ only if $\mu \triangleleft \lambda$, and conversely, if $\mu \triangleleft \lambda$ then there exists a broken flow line:

$$I_\mu \to I_{\nu_1} \to \ldots \to I_{\nu_k} \to I_\lambda$$

At each fixed point, the flow divides torus fixed tangent directions into either attracting or repelling, and this is determined by whether the power of $t$ in that tangent direction is positive or negative. The $K-$theory class:

$$\prod_{\square \in \lambda} \left( q_2^{l(\square)} - q_1^{a(\square)+1} \right) \cdot |\lambda\rangle$$

coincides with the localized structure sheaf of the attracting submanifold at $\lambda$, up to a monomial multiple. Then (4.12) means that we define the stable basis vector $s_\lambda^m$ by correcting the attracting submanifold of $\lambda$ with contributions that come from "downstream" fixed points $\mu \triangleleft \lambda$.

**4.4.** The existence and uniqueness of (4.11) also holds for $m \in \mathbb{Q}$, but we must require either (4.13) or (4.14) to be a strict inequality. Fix a rational slope $m \in \mathbb{Q}$. Since the stable basis is locally constant on a small punctured neighborhood of $m$, we have the two different bases:

$$\{s_\lambda^{m-\varepsilon}\}_{\lambda \text{ partition}} \subset \Lambda_{q_1,q_2} \supset \{s_\lambda^{m+\varepsilon}\}_{\lambda \text{ partition}}$$

Our main object of study will be the transition matrix between the above stable bases:

$$A : \Lambda_{q_1,q_2} \longrightarrow \Lambda_{q_1,q_2}$$

$$A\left(s_\lambda^{m+\varepsilon}\right) = s_\lambda^{m-\varepsilon}$$

for all partitions $\lambda$. When $m = \frac{a}{b}$ with $\gcd(a,b) = 1$, we will relate the matrix $A$ with the representation theory of $U_q \widehat{\mathfrak{gl}}_b$, as in Section 3. Specifically, we consider the renormalized stable basis given by:

$$\tilde{s}_\lambda^{m\pm\varepsilon} = s_\lambda^{m\pm\varepsilon} \cdot o_\lambda^m \cdot \prod_{\substack{\lambda\backslash\text{core }\lambda = \\ = R_1 \sqcup \ldots \sqcup R_k}} \prod_{i=1}^{k} \prod_{j=1}^{b-1} q^{\#_j^i} \tag{4.17}$$

where $o_\lambda = t^{c_\lambda}$ (see (2.4)), the product is taken over any maximal set of $b-$ribbons contained in $\lambda$, and:

$$\#_j^i = \begin{cases} mj - \lfloor mj \rfloor & \text{if the } j - \text{th step in the ribbon } R_i \text{ is to the right} \\ \lceil mj \rceil - mj & \text{if the } j - \text{th step in the ribbon } R_i \text{ is down.} \end{cases}$$

**Conjecture 4.18.** *In the renormalized stable basis, we have:*

$$\tilde{s}_\lambda^{m-\varepsilon} = A\left(\tilde{s}_\lambda^{m+\varepsilon}\right) = \sum_\mu a_\lambda^\mu(q) \cdot \tilde{s}_\mu^{m+\varepsilon}$$

*where $(a_\lambda^\mu(q))$ is the matrix of the Leclerc-Thibon involution (3.4). In particular, we claim that the matrix $A$ depends only on $q$ and the denominator $b$ of the rational number $m$.*

*Remark* 4.19. The fact that the matrix $A$ depends only on $q$ (and not on $t$) is part of the general behavior of $K$–theoretic stable bases under small perturbations of $m$, although we will not use it in the present paper.

**4.5.** It is clear from the definition that the stable bases are locally constant in the parameter $m$. More precisely, we say that the stable basis for $\text{Hilb}_n$ has a **wall** at $m$ if $s^{m-\varepsilon} \neq s^{m+\varepsilon}$ for some small $\varepsilon > 0$.

**Proposition 4.20.** *If $m = \frac{a}{b}$ with $\gcd(a,b) = 1$ is a wall for $\text{Hilb}_n$, then the following statements hold:*
  a) *$b \leq n(n-1)$.*
  b) *The transition matrix between $s^{m+\varepsilon}$ and $s^{m-\varepsilon}$ is block-triangular. Two partitions $\lambda$ and $\mu$ belong to the same block if $m \cdot (c_\lambda - c_\mu) \in \mathbb{Z}$.*

**Proof** Since $|c_\lambda|, |c_\mu| \leq \frac{n(n-1)}{2}$, we conclude that:

$$b \leq c_\lambda - c_\mu \leq n(n-1).$$

which implies (a). Part (b) is immediate from equations (4.13) and (4.14). $\qquad \square$

Conjecture 4.18 implies stronger constraints on the set of walls than Proposition 4.20 does, and it also refines the blocks in the the wall-crossing matrices:

**Proposition 4.21.** *Assume that Conjecture 4.18 holds and $m = \frac{a}{b}$ is a wall for $\text{Hilb}_n$, $\gcd(a,b) = 1$. Then the following statements hold:*
  a) *$b \leq n$*
  b) *The transition matrix between $s^{m+\varepsilon}$ and $s^{m-\varepsilon}$ is block-triangular. Two partitions $\lambda$ and $\mu$ belong to the same block if they have the same $b$–core.*

**Proof** Part (b) follows from Theorem 3.5 (b). Suppose for the purpose of contadiction that $b > n$. Then every partition of $n$ is its own $b$–core, so all blocks are of size 1. Since the transition matrix should have 1's on the diagonal, it is an identity matrix, and therefore $m = \frac{a}{b}$ is not a wall. $\qquad \square$

## 5. Heisenberg actions

To prove Conjecture 4.18, for each $m = \frac{a}{b}$ one needs to present an action of $U_q \widehat{\mathfrak{gl}}_b$ on the Fock space such that the matrices of the generators in the renormalized stable bases $\tilde{s}^{m-\varepsilon}$ and $\tilde{s}^{m+\varepsilon}$ have particularly nice form. In this section, we present such an action of the diagonal Heisenberg subalgebra:

$$U_q \widehat{\mathfrak{gl}}_1 \subset U_q \widehat{\mathfrak{gl}}_b$$

following [29]. We will use a remarkable algebra $\mathcal{A}$ over the field $\mathbb{Q}(q, t)$, which is known by many names:

- The double shuffle algebra
- The Hall algebra of an elliptic curve
- The doubly-deformed $W_{1+\infty}$–algebra
- The spherical double affine Hecke algebra (DAHA) of type $GL_\infty$
- $U_{q,t}\left(\ddot{\mathfrak{gl}}_1\right)$

See [11, 28, 32] for various isomorphisms between different presentations of $\mathcal{A}$. It is known that the group $SL(2, \mathbb{Z})$ acts on $\mathcal{A}$ by automorphisms. Furhermore, there is a natural $q$–Heisenberg subalgebra of $\mathcal{A}$, which in the DAHA presentation is generated by symmetric polynomials in $X_i$ and their conjugates. By applying automorphisms $\gamma \in SL(2, \mathbb{Z})$ to this subalgebra, we get new $q$–Heisenberg subalgebras:

$$\mathcal{A} \supset \mathcal{A}^{(m)} = \mathbb{Q}(q, t) \left\langle ..., B_{-2}^{(m)}, B_{-1}^{(m)}, B_1^{(m)}, B_2^{(m)}, ... \right\rangle$$

labeled by rational numbers $m = a/b$, where $\gamma(1, 0) = (b, a)$. We will call $\mathcal{A}^{(m)}$ the **slope $m$ subalgebra** in $\mathcal{A}$. The following results relate $\mathcal{A}^{(m)}$ to slope $m$ stable bases.

**Theorem 5.1.** ([12, 32, 27]) *There is an action of $\mathcal{A}$ on $\Lambda_{q_1, q_2}$, where $q_1 = qt$ and $q_2 = qt^{-1}$.*

**Theorem 5.2.** ([29]) *The action of the slope $m$ subalgebra $\mathcal{A}^{(m)}$ in the renormalized stable basis $\tilde{s}^{m+\varepsilon}$ is given by equations (3.3).*

It turns out that the action of the slope $m$ subalgebra in the stable basis $\tilde{s}^{m-\varepsilon}$ can be described in similar terms, by analogy with the proof of *loc. cit.*

**Theorem 5.3.** *The action of the slope $m$ subalgebra $\mathcal{A}^{(m)}$ in the renormalized stable basis $\tilde{s}^{m-\varepsilon}$ is given by equations (3.9), i.e. replacing $q \leftrightarrow q^{-1}$ in Theorem 5.2.*

Conjecture 4.18 can be now reformulated in the following way, which is more interesting for geometric applications.

**Conjecture 5.4.** *Given $m = \frac{a}{b}$ with $\gcd(a, b) = 1$, there is an action of the quantum affine algebra $U_q \widehat{\mathfrak{sl}}_b$ on $\Lambda_{q_1, q_2}$, satisfying the following conditions:*

a) *It commutes with the action of the slope $m$ Heisenberg subalgebra $\mathcal{A}^{(m)}$*
b) *The action of the creation operators $f_i$ in the renormalized stable basis $\tilde{s}^{m+\varepsilon}$ is given by (3.8).*
c) *The action of the creation operators $f_i$ in the renormalized stable basis $\tilde{s}^{m-\varepsilon}$ is given by (3.1).*

**Theorem 5.5.** *Conjectures 4.18 and 5.4 are equivalent.*

**Proof** Assume that Conjecture 5.4 holds. By part a), $\mathcal{A}^{(m)}$ and $U_q \widehat{\mathfrak{sl}}_b$ generate an action of $U_q \widehat{\mathfrak{gl}}_b$ on the Fock space. By Theorems 5.2 and 5.3, the bases $\tilde{s}^{m+\varepsilon}$ and $\tilde{s}^{m-\varepsilon}$ are respectively standard and costandard for this action (see Remark 3.10), so the transition matrix between them coincides with $A_b(q)$.

Assume that Conjecture 4.18 holds. Define the action of $f_i$ by the matrices (3.1) in the basis $\tilde{s}^{m+\varepsilon}$. By Theorems 5.2–5.3 and the properties of the Leclerc-Thibon involution, the $U_q \widehat{\mathfrak{sl}}_b$ and $\mathcal{A}^{(m)}$ actions on Fock space commute. Altogether, one gets an action of:

$$U_q \widehat{\mathfrak{gl}}_b \curvearrowright K \cong \text{Fock space}$$

such that $\tilde{s}^{m+\varepsilon}$ is the corresponding standard basis. By Conjecture 4.18, $\tilde{s}^{m-\varepsilon}$ is the costandard basis for this action, so the matrices of $f_i$ in the basis $\tilde{s}^{m-\varepsilon}$ are given by (3.8).  □

**Proof of Proposition 1.3** Note that Conjecture 5.4 is vacuous when $b = 1$, hence Conjecture 4.18 for $b = 1$ follows. Although we will not prove it, the stable basis vectors $s_\lambda^\varepsilon = s_\lambda^{-\varepsilon}$ coincide with modified Schur functions. Therefore, properties (4.12)–(4.14) give an equivalent description of Schur functions in terms of the degrees of their coefficients when expanded in the basis of Macdonald polynomials.  □

## 6. Relation to rational Cherednik algebras

**6.1.** Let $V$ be a finite-dimensional vector space and let $G \subset GL(V)$ be a finite group generated by reflections. Let $S \subset G$ be the set of reflections, and let $c : S \to \mathbb{C}$ be a conjugation-invariant function.

**Definition 6.1.** The rational Cherednik algebra $H_c(G, V)$ attached to $(G, V)$ is the quotient of $\mathbb{C}[W] \ltimes T(V \oplus V^*)$ by the relations:

$$[x, x] = [y, y] = 0, \ [y, x] = (x, y) - \sum_{s \in S} c(s)(\alpha_s, y)(\alpha_s^*, x)s,$$

where $x, x' \in V^*, y, y' \in V$ and $\alpha_s$ is the equation of the reflecting hyperplane for $s$.

The category $\mathcal{O}_c(G, V)$ is defined in [14] as the category of $H_c(G, V)$–modules which are finitely generated over $\mathbb{C}[V]$ and locally nilpotent under $V$. For a representation $U$ of $G$, let $M_c(U)$ denote the Verma (or standard) module over $H_c(G, V)$ induced from $U$, i.e.:

$$M_c(U) = H_c(G, V) \otimes_{\mathbb{C}[G] \ltimes T(V)} U$$

For the remainder of the paper, we will work in type $A$, assuming $G = S_n$, $c(s) = m$ identically, and $V = \mathbb{C}^{n-1}$. To simplify notations, denote $H_m = H_c(G, V)$. Irreducible representations $V_\lambda$ of $S_n$ are labeled by partitions $\lambda \vdash n$, and we denote $M_m(\lambda) = M_m(V_\lambda)$. The Verma module $M_m(\lambda)$ has a unique irreducible quotient $L_m(\lambda)$. Clearly, $M_m(\lambda)$ and $L_m(\lambda)$ belong to the category $\mathcal{O}_c(S_n, \mathbb{C}^{n-1})$. The following results relate the representation theory of the rational Cherednik algebra to the constuctions of Leclerc and Thibon.

**Theorem 6.2.** *Fix $m = \frac{a}{b}$ with $\gcd(a, b) = 1$. Then the composition series of $M_m(\lambda)$ can be computed in terms of the global canonical basis for $U_q \widehat{\mathfrak{gl}}_b$:*

$$[M_m(\lambda)] = \sum_\mu d_\lambda^\mu(1) \cdot [L_m(\mu)], \tag{6.3}$$

*where the coefficients $d_\lambda^\mu$ are defined by (3.12).*

**Proof** By [35, 24], the category $\mathcal{O}_m(S_n, \mathbb{C}^{n-1})$ is equivalent to the category of modules over the $q$–Schur algebra $S_q(n)$, where $q = \exp(\pi i/b)$. Under this equivalence the Verma module $M_m(\lambda)$ goes to the Weyl module $W(\lambda)$ and simple modules go to simple modules. (6.3) follows from the main theorem of [36]. $\square$

**Theorem 6.4.** ([33, 34]) *Fix $m = \frac{a}{b}$ with $\gcd(a, b) = 1$. There exist commuting categorical actions of $\widehat{\mathfrak{sl}}_b$ and of the Heisenberg algebra on the category:*

$$\mathcal{O}_m = \bigoplus_n \mathcal{O}_m(S_n, \mathbb{C}^{n-1}).$$

*On the level of Grothendieck groups, these actions agree with the $U_q \widehat{\mathfrak{gl}}_b$ action (3.1) and (3.3) at $q = 1$.*

The actions of Theorem 6.4 were constructed using the Bezrukavnikov–Etingof parabolic induction and restriction functors [4]. For example, the class of the unique finite-dimensional representation ([2]) can be computed as:

$$[L_m] = [L_m(b)] = B_{-1}^{(m)}(\mathbf{1}).$$

Finally, we note that the rational Cherednik algebra and its representations are naturally graded in such a way that $x_i$ has degree 1, $y_i$ has degree $-1$ and $\mathbb{C}[S_n]$ has degree 0. The graded characters of standard modules can be computed (up to an overall factor) as

$$\mathrm{ch}_t M_m(\lambda) = t^{-mc_\lambda}(1 - t)\varphi(s_\lambda),$$

where, similarly to (4.7), $\varphi$ denotes the homomorphism which sends power sum $p_k$ to $p_k/(1 - t^k)$.

**6.2.** The algebra $H_m$ is naturally filtered: both $x_i$ and $y_i$ lie in filtration part 1, while $\mathbb{C}[S_n]$ lies in filtration part 0. One can easily see that $\mathrm{gr}\, H_m \simeq \mathbb{C}[S_n] \ltimes \mathbb{C}[x_1, \ldots x_n, y_1, \ldots y_n]$. If $M$ is an $H_m$–module with a compatible filtration, then $\mathrm{gr}\, M$ is a module over $\mathrm{gr}\, H_m$. By the work of Bridgeland-King-Reid [8] and Haiman [17, 18], such a module corresponds to a class in the derived category of the $\mathrm{Hilb}_n$. If, as in (4.6), we identify the $(\mathbb{C}^*)^2$–equivariant $K$-theory of $\mathrm{Hilb}_n$ with the space of degree $n$ symmetric polynomials, then the above chain of equivalences sends $M$ to the bigraded Frobenius character of $\mathrm{gr}\, M$. If $M$ is an object in the category $\mathcal{O}_m$, then $y_1, ..., y_n$ are nilpotent on $M$, so the corresponding complex of sheaves is supported on the subvariety:

$$\mathrm{Hilb}_n^{\{y=0\}} = \{p \in \mathrm{Hilb}_n : \lim_{t \to 0} t \cdot p \text{ exists}\}$$

In fact, this is an example of a more general construction [6, 7], which associates a generalization of category $\mathcal{O}$ to a *conical symplectic resolution* with a chosen line bundle. For the Hilbert scheme, the choice of a line bundle corresponds to the choice of the parameter $m$ in the rational Cherednik algebra. Okounkov and Bezrukavnikov conjectured that the stable basis in $K(\mathrm{Hilb}_n)$ with parameter $m$ consists of the images of associated graded spaces of Verma modules $M_m(\lambda)$.

**Conjecture 6.5.** ([5]) *Every representation in the category $\mathcal{O}_m$ admits a filtration such that the following statements hold:*

  a)  *The filtration is compatible with the filtration on $H_m$*
  b)  *The bigraded Frobenius character of $\mathrm{gr}\, M_m(\lambda)$ is equal to $s_\lambda^m$.*
  c)  *This filtration is compatible with the induction/restriction functors in [4] and the morphisms.*
  d)  *The categorical action of Theorem 6.4 admits a filtered lift, and it agrees with the $U_q \widehat{\mathfrak{gl}}_b$ action (3.1) and (3.3), where $q$ corresponds to the filtration shift.*
  e)  *The filtration on finite-dimensional simples $L_m$ agrees with the filtration constructed in [15].*

Parts d) and e) of Conjecture 6.5 supersede our previous conjecture [16, Conjecture 5.5].

**Corollary 6.6.** *If Conjecture 6.5 holds, then the bigraded Frobenius character of $L_m$ equals $B_{-1}^{(m)}(\mathbf{1})$.*

Based on the above discussion, let us formulate a generalization of the Macdonald positivity conjecture that Haiman proved in [17].

**Conjecture 6.7.** *For all positive slopes $m$, the stable basis is Schur-positive:*

$$s_\lambda^m = \sum_\mu k_{\lambda,\mu}^m s_\mu, \qquad k_{\lambda,\mu}^m \in \mathbb{N}[[q,t]].$$

Conjecture 6.7 would follow from Conjecture 6.5 b) since $s_\lambda^m$ would be a Frobenius character of a bigraded $S_n$–representation.

## APPENDIX A. STABLE BASES FOR $\mathrm{Hilb}_2$ AND $\mathrm{Hilb}_3$

We list the stable bases for the Hilbert schemes of $n = 2$ and $3$ points and certain values of the slope $m$. We note that there is no wall–crossing at integers, so $s^{m+\varepsilon} = s^{m-\varepsilon}$ if $m \in \mathbb{Z}$. The following are the matrices that go from the stable basis $s_\lambda^{m+\varepsilon}$ to the plethystically modified Schur basis $s_\lambda^0$. Specifically, the number indicated in front of each matrix is $m$, and the $\lambda$–th column of each matrix denotes the expansion of the plethystically modified Schur functions $s_\lambda^0$ in the stable basis at slope $m + \varepsilon$. We also factor these transition matrices as products of "wall-crossing" matrices, writing the coordinate of the wall as a subscript. We start with $n = 2$:

$$\frac{1}{2} \mapsto \begin{pmatrix} 1 & 0 \\ q_2 - \frac{1}{q_1} & 1 \end{pmatrix} \qquad \frac{3}{2} \mapsto \begin{pmatrix} 1 & 0 \\ q_2 - \frac{1}{q_1} + \frac{q_2^2}{q_1} - \frac{q_2}{q_1^2} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ q_2 - \frac{1}{q_1} & 1 \end{pmatrix}_{\frac{1}{2}} \begin{pmatrix} 1 & 0 \\ \frac{q_2^2}{q_1} - \frac{q_2}{q_1^2} & 1 \end{pmatrix}_{\frac{3}{2}}$$

The expansion of the stable bases into usual Schur functions has the form:

$$s_2^0 = \frac{s_2}{1 - q_2^2} + \frac{s_{1,1} q_2}{1 - q_2^2} \qquad\qquad s_{1,1}^0 = \frac{s_2 q_2}{1 - q_2^2} + \frac{s_{1,1}}{1 - q_2^2}$$

$$s_2^{1/2+\varepsilon} = \left[1 + \frac{q_2}{q_1(1 - q_2^2)}\right] s_2 + \frac{s_{1,1}}{q_1(1 - q_2^2)} \qquad\qquad s_{1,1}^{1/2+\varepsilon} = \frac{s_2 q_2}{1 - q_2^2} + \frac{s_{1,1}}{1 - q_2^2}$$

$$s_2^{3/2+\varepsilon} = \left[1 + \frac{q_2}{q_1} + \frac{q_2^2}{q_1^2(1 - q_2^2)}\right] s_2 + \left[\frac{1}{q_1} + \frac{q_2}{q_1^2(1 - q_2^2)}\right] s_{1,1} \qquad\qquad s_{1,1}^{3/2+\varepsilon} = \frac{s_2 q_2}{1 - q_2^2} + \frac{s_{1,1}}{1 - q_2^2}$$

Indeed, all the coefficients in the above expressions are nonnegative when expanded in $|q_2| < 1$. Finally, note that the characters of the simple representations of rational Cherednik algebras at $m = 1/2$ and at $m = 3/2$ can be expressed both in standard and in costandard bases near the corresponding wall:

$$\operatorname{ch} L_{1/2} \ = \ s_2 \ = \ s_2^0 - s_{1,1}^0 q_2 \ = \ s_2^{1/2+\varepsilon} - \frac{s_{1,1}^{1/2+\varepsilon}}{q_1}$$

$$\operatorname{ch} L_{3/2} = (q_1 + q_2)s_2 + s_{1,1} = s_2^{1/2+\varepsilon} q_1 - s_{1,1}^{1/2+\varepsilon} q_2^2 = s_2^{3/2+\varepsilon} q_1 - \frac{s_{1,1}^{3/2+\varepsilon} q_2}{q_1}$$

For $n = 3$ we just list the transition matrices between slope $0$ and slope $m+\varepsilon$, and also their decomposition into simpler "wall-crossing" matrices:

$$\frac{1}{3} \mapsto \begin{pmatrix} 1 & 0 & 0 \\ q_2 - \frac{1}{q_1} & 1 & 0 \\ \frac{1}{q_1^2} - \frac{q_2}{q_1} & q_2 - \frac{1}{q_1} & 1 \end{pmatrix}$$

$$\frac{1}{2} \mapsto \begin{pmatrix} 1 & 0 & 0 \\ q_2 - \frac{1}{q_1} & 1 & 0 \\ \frac{1}{q_1^2} - \frac{q_2}{q_1^2} + \frac{q_2^2}{q_1} - \frac{q_2}{q_1} & q_2 - \frac{1}{q_1} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{q_2^2}{q_1} - \frac{q_2}{q_1^2} & 0 & 1 \end{pmatrix}_{\frac{1}{2}} \begin{pmatrix} 1 & 0 & 0 \\ q_2 - \frac{1}{q_1} & 1 & 0 \\ \frac{1}{q_1^2} - \frac{q_2}{q_1} & q_2 - \frac{1}{q_1} & 1 \end{pmatrix}_{\frac{1}{3}}$$

$$\frac{2}{3} \mapsto \begin{pmatrix} 1 & 0 & 0 \\ q_2 - \frac{1}{q_1} + \frac{q_2}{q_1} - \frac{1}{q_1^2} & 1 & 0 \\ q_2^3 - \frac{q_2^2}{q_1} + \frac{q_2}{q_1^3} - \frac{q_2^2}{q_1^2} + \frac{1}{q_1^2} - \frac{q_2}{q_1} & q_2^2 - \frac{q_2}{q_1} + q_2 - \frac{1}{q_1} & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ \frac{q_2}{q_1} - \frac{1}{q_1^2} & 1 & 0 \\ \frac{q_2}{q_1^3} - \frac{q_2^2}{q_1^2} & q_2^2 - \frac{q_2}{q_1} & 1 \end{pmatrix}_{\frac{2}{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{q_2^2}{q_1} - \frac{q_2}{q_1^2} & 0 & 1 \end{pmatrix}_{\frac{1}{2}} \begin{pmatrix} 1 & 0 & 0 \\ q_2 - \frac{1}{q_1} & 1 & 0 \\ \frac{1}{q_1^2} - \frac{q_2}{q_1} & q_2 - \frac{1}{q_1} & 1 \end{pmatrix}_{\frac{1}{3}}$$

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