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# Classification of (2+1)-dimensional topological order and symmetry-protected topological order for bosonic and fermionic systems with on-site symmetries

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In 2+1-dimensional space-time, gapped quantum states are always gapped quantum liquids (GQL) which include both topologically ordered states (with long range entanglement) and symmetry protected topological (SPT) states (with short range entanglement). In this paper, we propose a classification of 2+1D GQLs for both bosonic and fermionic systems: 2+1D bosonic/fermionic GQLs with finite on-site symmetry are classified by nondegenerate unitary braided fusion categories over a symmetric fusion category (SFC) $\mathcal{E}$, abbreviated as UMTC$_{/\mathcal{E}}$, together with their modular extensions and total chiral central charges. In our classification, SFC $\mathcal{E}$ describes the symmetry, which is Rep($G$) for bosonic symmetry $G$, or sRep($G^f$) for fermionic symmetry $G^f$. As a special case of the above result, we find that the modular extensions of Rep($G$) classify the 2+1D bosonic SPT states of symmetry $G$, while the $c = 0$ modular extensions of sRep($G^f$) classify the 2+1D fermionic SPT states of symmetry $G^f$. Many fermionic SPT states are studied based on the constructions from free-fermion models. But free-fermion constructions cannot produce all fermionic SPT states. Our classification does not have such a drawback. We show that, for interacting 2+1D fermionic systems, there are exactly 16 superconducting phases with no symmetry and no fractional excitations (up to $E_8$ bosonic quantum Hall states). Also, there are exactly 8 $Z_2 \times Z_2^f$-SPT phases, 2 $Z_8^f$-SPT phases, and so on. Besides, we show that two topological orders with identical bulk excitations and central charge always differ by the stacking of the SPT states of the same symmetry.

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## I. INTRODUCTION

Topological order [1–3] is a new kind of order beyond the symmetry breaking orders [4] in gapped quantum systems. Topological orders are patterns of *long-range entanglement* [5] in *gapped quantum liquids* (GQL) [6]. Based on the unitary modular tensor category (UMTC) theory for non-Abelian statistics [7–9], in Refs. [10,11], it is proposed that 2+1D bosonic topological orders are classified by {UMTC} × {iTO$_B$}, where {UMTC} is the set of UMTCs and {iTO$_B$} is the set of invertible topological orders (iTO) [10,12] for 2+1D boson systems. In fact, {iTO$_B$} $= \mathbb{Z}$, which is generated by the $E_8$ bosonic quantum Hall (QH) state, and a table of UMTCs was obtained in Refs. [11,13]. Thus we have a table (and a classification) of 2+1D bosonic topological orders.

In a recent work [14], we show that 2+1D fermionic topological orders are classified by {UMTC$_{/\text{sRep}(Z_2^f)}$} × {iTO$_F$}, where {UMTC$_{/\text{sRep}(Z_2^f)}$} is the set of nondegenerate unitary braided fusion categories (UBFC) over the symmetric fusion category (SFC) sRep($Z_2^f$) (see definition II B). We also require UMTC$_{/\text{sRep}(Z_2^f)}$s to have modular extensions. {iTO$_F$} is the set of invertible topological orders for 2+1D fermion systems. In fact, {iTO$_F$} $= \mathbb{Z}$, which is generated by the $p + \mathrm{i}p$ superconductor. In Ref. [14], we computed the table for UMTC$_{/\text{sRep}(Z_2^f)}$s, and obtained a table (and a classification) of 2+1D fermionic topological orders.

In Ref. [14], we also point out the importance of modular extensions. If a UMTC$_{/\text{sRep}(Z_2^f)}$ does not have a modular extension, it means that the fermion-number-parity symmetry is not on-site (i.e., anomalous [15]). On the other hand, if a UMTC$_{/\text{sRep}(Z_2^f)}$ does have modular extensions, then the UMTC$_{/\text{sRep}(Z_2^f)}$ is realizable by a lattice model of fermions. In this case, a given UMTC$_{/\text{sRep}(Z_2^f)}$ may have several modular extensions. We found that different modular extensions of UMTC$_{/\text{sRep}(Z_2^f)}$ contain information of iTO$_F$s.

Our result on fermionic topological orders can be easily generalized to describe bosonic/fermionic topological orders with symmetry. This will be the main topic of this paper. (Some of the results are announced in Ref. [14]. In this paper, we will consider symmetric GQL phases for 2+1D bosonic/fermionic systems. The notion of GQL was defined in Ref. [6]. The symmetry group of GQL is $G$ (for bosonic systems) or $G^f$ (for fermionic systems). If a symmetric GQL has long-range entanglement (as defined in Refs. [5,6]), it corresponds to a symmetry enriched topological (SET) order [5]. If a symmetric GQL has short-range entanglement, it corresponds to a symmetry protected trivial (SPT) order [which is also known as symmetry protected topological (SPT) order] [16–20].

In this paper, we are going to show that, 2+1D symmetric GQLs are classified by UMTC$_{/\mathcal{E}}$ plus their modular extensions and chiral central charge. In other words, GQLs are labeled by three UBFCs $\mathcal{E} \subset \mathcal{C} \subset \mathcal{M}$ plus the central charge $c$ (see Figs. 1 and 2). Roughly speaking, a UBFC can be viewed as a set of quasiparticle types, plus the data on quasiparticle fusion and braiding.

(1) $\mathcal{E}$ is a special kind of UBFC called SFC where all the quasiparticles have trivial mutual statistics between each other. Such a SFC $\mathcal{E}$ describes the local excitations (i.e., the excitations that can be created by local operators). The types of those local excitations are described the representations of the

 

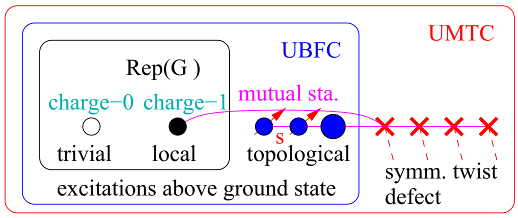


FIG. 1. Bosonic topological orders with symmetry $G$ are classified by three unitary categories: SFC $\mathcal{E} = \text{Rep}(G) \subset$ UBFC $\mathcal{C} \subset$ UMTC $\mathcal{M}$, which describe quasiparticle excitations and symmetry-twist defects. The particles connected by lines have nontrivial mutual statistics between them.

symmetry group. Thus $\mathcal{E}$ is given by $\mathcal{E} = \text{Rep}(G)$ for bosonic cases, or $\mathcal{E} = \text{sRep}(G^f)$ for fermionic cases.

(2) The UBFC $\mathcal{C}$ contains both local excitations and topological excitations (i.e., the excitations that cannot be created by local operators), and thus $\mathcal{E} \subset \mathcal{C}$. Those topological excitations can carry fractional statistics and fractional angular momentum $s$, which will be called *topological spin*. The topological excitations may also have symmetry fractionalization (such as fractional symmetry quantum numbers). We also require $\mathcal{E}$ to include all the excitations that have trivial mutual statistics with every excitation in $\mathcal{C}$ (which can be viewed as an operational definition of the so called *local excitation*), which leads to a mathematical notion of *UBFC over SFC* $\mathcal{E}$ (denoted as $\text{UMTC}_{/\mathcal{E}}$).

(3) The UBFC $\mathcal{M}$ contains both quasiparticle excitations and symmetry-twist defects [21–23], and thus $\mathcal{C} \subset \mathcal{M}$. We require that every particle in $\mathcal{M}$ (except the trivial one) has a nontrivial mutual statistics with at least one particle in $\mathcal{M}$. A UBFC satisfying such a condition is called UMTC, and we call the extension from $\mathcal{C}$ to $\mathcal{M}$ a modular extension. (To be more precise, a modular extension of $\mathcal{C}$, $\mathcal{M}$, is a UMTC with a fully faithful embedding $\mathcal{C} \to \mathcal{M}$. In particular, even if the UMTC $\mathcal{M}$ is fixed, different embeddings correspond to different modular extensions.) The existence of modular extensions for $\mathcal{C}$ is an anomaly-free condition for $\mathcal{C}$: the quasiparticles described by $\mathcal{C}$ can be realized by a well defined local lattice model with on-site-symmetry in the same dimension [15]. The chiral central charge $c$ for the edge states describes the invertible topological orders, which have trivial bulk excitations.

We like to remark that symmetry charges by carried topological excitations are in general not well defined. In other words, a topological excitation may not carry a representation of the symmetry group. This phenomenon is called symmetry fractionalization. In general, a topological excitation may not even carry a projective representation of the symmetry group (which corresponds to fractionalized symmetry quantum numbers). In other words, a topological excitations can carry something more exotic than projective representations of the symmetry group. For example, in a gauge theory with gauge group $K$ and symmetry group $G$, a topological excitation (a gauge charge) may carry a representation of group $H$ which satisfies $H/K = G$. So symmetry fractionalization can be more general than fractionalized quantum numbers and projective representations of the symmetry group.

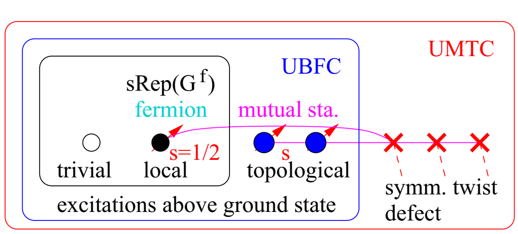


FIG. 2. Fermionic topological orders with symmetry $G^f$ are classified by three unitary categories: SFC $\mathcal{E} = \text{sRep}(G^f) \subset$ UBFC $\mathcal{C} \subset$ UMTC $\mathcal{M}$.

One example of the classified bosonic SET (see Table VI) is given by the $Z_2^{\text{gauge}}$ spin liquid [24,25] with excitations $1,e,m,f$, where 1 is the trivial excitation, $e$ the $Z_2^{\text{gauge}}$ charge, $m$ the $Z_2^{\text{gauge}}$ vortex, and $f$ the bound state of $e$ and $m$. The excitation $1,e,m$ are bosons and $f$ is a fermion. There is also a $Z_2^{\text{sym}}$ symmetry which exchanges $e$ and $m$ [26–28]. The excitations in such a SET state are labeled by $1_+$, $1_-$, $f_+$, $f_-$, $e \oplus m$, which form the UBFC $\mathcal{C}$. They have topological spins $s_i = 0,0,\frac{1}{2},\frac{1}{2},0$ and quantum dimensions $d_i = 1,1,1,1,2$. $1_+$ and $1_-$ are the local excitations with $Z_2^{\text{sym}}$ charge 0 and 1. The two excitations $1_+$ and $1_-$ form the SFC $\mathcal{E} = \text{Rep}(Z_2^{\text{sym}})$. $f_+$ and $f_-$ are topological fermionic excitations with $Z_2^{\text{sym}}$ charges 0 and 1. $e \oplus m$ is a doublet excitation that corresponds to degenerate $e$ and $m$ (just like the spin-1/2 doublet that corresponds to degenerate spin-up and spin-down). This is why $e \oplus m$ has a quantum dimension 2. The modular extension is obtained by adding the $Z_2^{\text{sym}}$-symmetry twist defect, as well as its bound states with excitations $f_+$, $f_-$, $e \oplus m$. Figure 1 happens to describe such a SET.

As a second example, Fig. 2 describes the topological order $\mathcal{F}_{(A_1,6)}$ in Table I of Ref. [14], which has a $G^f = Z_2^f$ symmetry. The state has two types of local excitations with $Z_2^f$-charge 0 (a boson) and 1 (a fermion) that form the SFC $\mathcal{E} = \text{sRep}(Z_2^f)$. They have topological spin $s_i = 0,\frac{1}{2}$. The state also has two types of topological excitations with topological spin $s_i = \frac{1}{4},-\frac{1}{4}$ and quantum dimension $d_i = 1+\sqrt{2},1+\sqrt{2}$. The local and topological excitations form the UBFC $\mathcal{C}$. The modular extension is obtained by adding the $Z_2^f$-symmetry twist defect, as well as its bound state with the excitations in $\mathcal{C}$, which gives rise to three types of symmetry twist defects.

There is another more precise and mathematical way to phrase our result: we find that the structure $\mathcal{E} \hookrightarrow \mathcal{C} \hookrightarrow \mathcal{M}$ (plus the chiral central charge $c$) classifies the 2+1D GQLs with symmetry $\mathcal{E}$, where $\hookrightarrow$ represents the embeddings and $\mathcal{E}_{\mathcal{M}}^{\text{cen}} = \mathcal{C}$ (see definition II B).

As a special case of the above result, we find that bosonic 2+1D SPT phase with symmetry $G$ are classified by the modular extensions of $\text{Rep}(G)$, while fermionic 2+1D SPT phase with symmetry $G^f$ are classified by the modular extensions of $\text{sRep}(G^f)$ that have central charge $c = 0$.

We like to mention that Ref. [29] has classified bosonic GQLs with symmetry $G$, using $G$-crossed UMTCs. This paper uses a different approach so that we can classify both bosonic and fermionic GQLs with symmetry. For bosonic systems, the two approaches produces identical classification. We also like to mention that there is a mathematical companion Ref. [30] of this paper, where one can find detailed proof and explanations for related mathematical results.

The paper is organized as the following. In Sec. II, we review the notion of topological order and introduce category theory as a theory of quasiparticle excitations in a GQL. We will introduce a categorical way to view the symmetry. In Sec. III, we discuss invertible GQLs and their classification based on modular extensions. In Secs. IV and V, we generalize the above results and propose a classification of all GQLs. Section VI investigates the stacking operation from physical and mathematical points of view. Section VII describes how to numerically calculate the modular extensions and Sec. VIII discusses some simple examples. For people with physics background, one way to read this paper is to start with Secs. II and V, and then go to Sec. VIII for the examples. Table I summarizes some important mathematical concepts and their physical correspondences.

## II. GAPPED QUANTUM LIQUIDS, TOPOLOGICAL ORDER, AND SYMMETRY

### A. The finite on-site symmetry and symmetric fusion category

In this paper, we consider physical systems with an on-site symmetry described by a finite group $G$. For fermionic systems, we further require that $G$ contains a central $Z_2$ fermion-number-parity subgroup. More precisely, fermionic symmetry group is a pair $(G,f)$, where $G$ is a finite group, $f \neq 1$ is an element of $G$ satisfying $f^2=1, fg=gf, \forall g \in G$. We denote the pair $(G,f)$ as $G^f$.

There is another way to view the on-site symmetries, which is nicer because bosonic and fermionic symmetries can be formulated in the same manner. Consider a bosonic/fermionic product state $|\psi\rangle$ that does not break the symmetry $G$: $U_g|\psi\rangle = |\psi\rangle$, $g \in G$. Then the new way to view the symmetry is to use the properties of the excitations above the product state to encode the information of the symmetry $G$.

The product state contain only local excitations that can be created by acting local operators $O$ on the ground state $O|\psi\rangle$. For any group action $U_g$, $U_g O|\psi\rangle = U_g O U_g^\dagger U_g|\psi\rangle = U_g O U_g^\dagger|\psi\rangle$ is an excited state with the same energy as $O|\psi\rangle$. Since we assume the symmetry to be on-site, $U_g O U_g^\dagger$ is also a local operator. Therefore $U_g O U_g^\dagger|\psi\rangle$ and $O|\psi\rangle$ correspond to the degenerate local excitations. We see that local excitations "locally" carry group representations. In other words, different types of local excitations are labeled by irreducible representations of the symmetry group.

By looking at how the local excitations (more precisely, their group representations) fuse and braid with each other, we arrive at the mathematical structure called symmetric fusion categories (SFC). By definition a SFC is a braided fusion category where all the objects (the excitations) have trivial mutual statistics (i.e., centralize each other, see next section). A SFC is automatically a unitary braided fusion category.

In fact, there are only two kinds of SFCs: one is representation category of $G$: Rep$(G)$, with the usual braiding (all representations are bosonic); the other is sRep$(G^f)$ where an irreducible representation is bosonic if $f$ is represented trivially $(+1)$, and fermionic if $f$ is represented nontrivially $(-1)$.

It turns out that SFC (or the fusion and braiding properties of the local excitations) fully characterize the symmetry group (which is known as Tannaka duality [31]). Therefore a finite on-site symmetry is equivalently given by a SFC $\mathcal{E}$. Also, by checking the braiding in $\mathcal{E}$ we know whether it is bosonic or fermionic. This is the new way, the categorical way, to view the symmetry. Such a categorical view of bosonic/fermionic symmetry allows us to obtain a classification of symmetric topological/SPT orders.

### B. Categorical description of topological excitations with symmetry

In symmetric GQLs with topological order (i.e., with long range entanglement), there can be particlelike excitations with local energy density, but they cannot be created by local operators. They are known as (nontrivial) topological excitations. Topological excitations do not necessarily carry group representations. Nevertheless, we can still study how they fuse and braid with each other; so we have a unitary braided fusion category (UBFC) to describe the particlelike excitations. To proceed, we need the following key definition on "centralizers".

*Definition 1.* The objects $X,Y$ in a UBFC $\mathcal{C}$ are said to *centralize* (mutually local to) each other if

$$c_{Y,X} \circ c_{X,Y} = \mathrm{id}_{X\otimes Y}, \tag{1}$$

where $c_{X,Y} : X \otimes Y \cong Y \otimes X$ is the braiding in $\mathcal{C}$.

Physically, we say that $X$ and $Y$ have trivial mutual statistics.

*Definition 2.* Given a subcategory $\mathcal{D} \subset \mathcal{C}$, its *centralizer* $\mathcal{D}_\mathcal{C}^{\text{cen}}$ in $\mathcal{C}$ is the full subcategory of objects in $\mathcal{C}$ that centralize all the objects in $\mathcal{D}$.

We may roughly view a category as a "set" of particlelike excitations. So the centralizer $\mathcal{D}_\mathcal{C}^{\text{cen}}$ is the "subset" of particles in $\mathcal{C}$ that have trivial mutual statistics with all the particles in $\mathcal{D}$.

*Definition 3.* A UBFC $\mathcal{E}$ is a *symmetric* fusion category if $\mathcal{C}_\mathcal{C}^{\text{cen}} = \mathcal{E}$. A UBFC $\mathcal{C}$ with a fully faithful embedding $\mathcal{E} \hookrightarrow \mathcal{C}_\mathcal{C}^{\text{cen}}$ is called a UBFC over $\mathcal{E}$. Moreover, $\mathcal{C}$ is called a nondegenerate UBFC over $\mathcal{E}$, or UMTC$_{/\mathcal{E}}$, if $\mathcal{C}_\mathcal{C}^{\text{cen}} = \mathcal{E}$.

*Definition 4.* Two UBFCs over $\mathcal{E}$, $\mathcal{C}$, and $\mathcal{C}'$ are equivalent if there is a unitary braided equivalence $F : \mathcal{C} \to \mathcal{C}'$ such that it preserves the embeddings, i.e., the following diagram commute:

$$\begin{array}{ccc} \mathcal{E} & \hookrightarrow & \mathcal{C} \\ \| & & \downarrow F \\ \mathcal{E} & \hookrightarrow & \mathcal{C}' \end{array} \tag{2}$$

We denote the category of unitary braided autoequivalences of $\mathcal{C}$ by $\mathcal{A}\mathrm{ut}(\mathcal{C})$ and its underlining group by Aut$(\mathcal{C})$.

We recover the usual definition of UMTC when $\mathcal{E}$ is trivial, i.e., the category of Hilbert spaces, denoted by Vec = Rep$(\{1\})$. In this case, the subscript is omitted.

Physically, a UBFC $\mathcal{C}$ is the collection of all bulk topological excitations plus their fusion and braiding data. Requiring $\mathcal{C}$ to be a UMTC$_{/\mathcal{E}}$ means (1) the set of local excitations, $\mathcal{E}$ (which is the set of all the irreducible representations of the symmetry group), is included in $\mathcal{C}$ as a subcategory; (2)

TABLE I. Some mathematical concepts and their physical correspondences, as well as the meaning of some notations.

| Mathematical term | Physical correspondence |
|---|---|
| UBFC (unitary braided fusion category) $\mathcal{C}$ | Set of excitations that can braid and fuse |
| SFC (symmetric fusion category) $\mathcal{E}$, which is a special kind of UBFC | Set of local excitations carrying representations of symmetry group |
| UMTC (unitary modular tensor category) $\mathcal{M}$, which is a special kind of UBFC | Set of excitations such that every nontrivial excitation has a nontrivial mutual statistics with at least one excitation |
| $\text{UMTC}_{/\mathcal{E}}$ (UBFC over $\mathcal{E}$) a special kind of UBFC | Set of excitations that contain a subset SFC $\mathcal{E}$, where $\mathcal{E}$ is formed by the excitations that have trivial mutual statistics with all excitations |
| Modular extension | Adding symmetry-twist defects (i.e., gauging the symmetry) |
| Chiral central charge $c$ | The number of right-moving edge modes minus the number of left-moving edge modes ($c$ can be fractional) |
| Topological spin $s_i$ | Fractional part of 2D angular momentum of the quasiparticle $i$ |
| Quantum dimension $d_i$ | The effective dimension of the Hilbert space for the internal degrees of freedom of the quasiparticle $i$ ($d_i$ can be noninteger) |
| $N$ | Number of particle types (also called rank of category) |
| $D$ | $\sqrt{\sum_i d_i^2}$ (total quantum dim.) |
| $\Theta$ | $D^{-1}\sum_i e^{2\pi i s_i} d_i^2 = \lvert\Theta\rvert e^{2\pi i c/8}$ |
| $N_c^{\lvert\Theta\rvert}$ | A short label of topological orders |
| $N_c^B$ | When $\lvert\Theta\rvert = 1$, rewrite $N_c^{\lvert\Theta\rvert}$ as $N_c^B$ |
| $\zeta_n^m$ | $\sin\frac{\pi(m+1)}{n+2}/\sin\frac{\pi}{n+2}$ |
| $(A_n,k)$ | Topological order of SU$(n+1)$ level-$k$ Chern-Simons theory |
| $(B_n,k)$ | Topological order of $SO(2n+1)$ level-$k$ Chern-Simons theory |
| $(C_n,k)$ | Topological order of $Sp(2n)$ level-$k$ Chern-Simons theory |
| $(D_n,k)$ | Topological order of $SO(2n)$ level-$k$ Chern-Simons theory |
| $\boxtimes$ | Stacking of two states |
| $\otimes$ | Fusion of two particles |

$\mathcal{C}$ is anomaly-free, i.e., all the topological excitations (the ones not in $\mathcal{E}$) can be detected by mutual braiding [10]. In other words, every topological excitation must have nontrivial mutual statistics with some excitations. Those excitations that cannot be detected by mutual braiding (i.e., $\mathcal{C}_{\mathcal{C}}^{\text{cen}}$) are exactly the local excitations in $\mathcal{E}$. Moreover, we want the symmetry to be on-site (gaugeable), which requires the existence of modular extensions (see definition 6). Such an understanding leads to the following conjecture.

*Conjecture 1.* Bulk topological excitations of topological orders with symmetry $\mathcal{E}$ are classified by $\text{UMTC}_{/\mathcal{E}}$'s that have modular extensions.

We like to remark that $\text{UMTC}_{/\mathcal{E}}$'s fail to classify topological orders. This is because two different topologically ordered phases may have bulk topological excitations with the same non-Abelian statistics (i.e., described by the same $\text{UMTC}_{/\mathcal{E}}$). However, $\text{UMTC}_{/\mathcal{E}}$'s, with modular extensions, do classify topological orders up to invertible ones. See next section for details. The relation between anomaly and modular extension will also be discussed later.

## III. INVERTIBLE GQLS AND MODULAR EXTENSION

### A. Invertible GQLs

There exist nontrivial topological ordered states that have only trivial topological excitations in the bulk (but nontrivial edge states). They are "invertible" under the stacking operation [10,12] (see Sec. VI for details). More generally, we define the following.

*Definition 5.* A GQL is invertible if its bulk topological excitations are all trivial (i.e., can all be created by local operators).

Consider some invertible GQLs with the same symmetry $\mathcal{E}$. The bulk excitations of those invertible GQLs are the same which are described by the same SFC $\mathcal{E}$. Now the question is, how to distinguish those invertible GQLs?

First, we believe that invertible bosonic topological orders with no symmetry are generated by the $E_8$ QH state (with central charge $c = 8$) via time-reversal and stacking, and form a $\mathbb{Z}$ group. Stacking with an $E_8$ QH state only changes the central charge by 8, and does not change the bulk excitations or the symmetry. So the only data we need to know to determine the invertible bosonic topological order with no symmetry is the central charge $c$. The story is parallel for invertible fermionic topological orders with no symmetry, which are believed to be generated by the $p + ip$ superconductor state with central charge $c = 1/2$.

Second, invertible bosonic GQLs with symmetry are generated by bosonic SPT states and invertible bosonic topological orders (i.e., $E_8$ states) via stacking. We know that the bosonic SPT states with symmetry $G$ are classified by the 3-cocycles in $H^3[G,\text{U}(1)]$. Therefore bosonic invertible GQLs with symmetry $G$ are classified by $H^3[G,\text{U}(1)] \times \mathbb{Z}$ (where $\mathbb{Z}$ corresponds to layers of $E_8$ states).

However, this result and this point of view is not natural to generalize to fermionic cases or noninvertible GQLs. Thus we introduce an equivalent point of view, which can cover boson, fermion, and noninvertible GQLs in the same fashion.

### B. Modular extension

First, we introduce the notion of modular extension of a $\mathrm{UMTC}_{/\mathcal{E}}$.

*Definition 6.* Given a $\mathrm{UMTC}_{/\mathcal{E}}$ $\mathcal{C}$, its *modular extension* is a UMTC $\mathcal{M}$, together with a fully faithful embedding $\iota_{\mathcal{M}}:\mathcal{C}\hookrightarrow\mathcal{M}$, such that $\mathcal{E}_{\mathcal{M}}^{\mathrm{cen}}=\mathcal{C}$, equivalently $\dim(\mathcal{M})=\dim(\mathcal{C})\dim(\mathcal{E})$.

Two modular extensions $\mathcal{M}$ and $\mathcal{M}'$ are equivalent if there is an equivalence between the UMTCs $F:\mathcal{M}\to\mathcal{M}'$ that preserves the embeddings, i.e., the following diagram commute:

$$\begin{array}{ccc} \mathcal{C} & \hookrightarrow & \mathcal{M} \\ \| & & \downarrow F \\ \mathcal{C} & \hookrightarrow & \mathcal{M}' \end{array} \tag{3}$$

We denote the set of equivalent classes of modular extensions of $\mathcal{C}$ by $\mathcal{M}_{\mathrm{ext}}(\mathcal{C})$.

*Remark 1.* Since the total quantum dimension of modular extensions of a given $\mathcal{C}$ is fixed, there are only finitely many different modular extensions, due to Ref. [32]. In principle, we can always perform a finite search to exhaust all the modular extensions.

Remember that $\mathcal{C}$ describes the particlelike excitations in our topological state. Some of those excitations are local that have trivial mutual statistics with all other excitations. Those local excitation form $\mathcal{E}\subset\mathcal{C}$. The modular extension $\mathcal{M}$ of $\mathcal{C}$ is obtained as adding particles that have nontrivial mutual statistics with the local excitations in $\mathcal{E}$, so that every particle in $\mathcal{M}$ will always have nontrivial mutual statistics with some particles in $\mathcal{M}$. Since the particles in $\mathcal{E}$ carry "charges" (i.e., the irreducible representations of $G$), the added particles correspond to "flux" (i.e., the symmetry twists of $G$). So the modular extension correspond to gauging [21] the on-site symmetry $G$. Since we can use the gauged symmetry to detect SPT orders [23], we like to propose the following conjecture.

*Conjecture 2.* Invertible bosonic GQLs with symmetry $\mathcal{E}=\mathrm{Rep}(G)$ are classified by $(\mathcal{M},c)$ where $\mathcal{M}$ is a modular extension of $\mathcal{E}$ and $c=0$ mod 8.

### C. Classify 2+1D bosonic SPT states

Invertible bosonic GQLs described by $(\mathcal{M},c)$ include both bosonic SPT states and bosonic topological orders. Among those, $(\mathcal{M},c=0)$ classify bosonic SPT states. In other words:

*Corollary 1.* 2+1D bosonic SPT states with symmetry $G$ are classified by the modular extensions of $\mathrm{Rep}(G)$ (which always have $c=0$).

In Ref. [18–20], it was shown that 2+1D bosonic SPT states are classified by $H^3[G,\mathrm{U}(1)]$. Such a result agrees with our conjecture, due to the following theorem, which follows immediately from results in Ref. [33].

*Theorem 1.* The modular extensions of $\mathrm{Rep}(G)$ 1-to-1 correspond to 3-cocycles in $H^3[G,\mathrm{U}(1)]$. The central charge of these modular extensions are $c=0$ mod 8.

*Remark 2.* In Sec. VI D, we give more detailed explanation of the 1-to-1 correspondence in theorem 2. Moreover, we will prove a stronger result in theorem 11. It turns out that the set $\mathcal{M}_{\mathrm{ext}}(\mathrm{Rep}(G))$ of modular extensions of $\mathrm{Rep}(G)$ is naturally equipped with a physical stacking operation such that $\mathcal{M}_{\mathrm{ext}}(\mathrm{Rep}(G))$ forms an Abelian group, which is isomorphic to the group $H^3[G,\mathrm{U}(1)]$.

*Remark 3.* $c/8$ determines the number of layers of the $E_8$ QH states, which is the topological order part of invertible bosonic symmetric GQLs. In other words

$$\begin{aligned}&\{\text{invertible bosonic symmetric GQLs}\}\\&\quad=\{\text{bosonic SPT states}\}\times\{\text{layers of } E_8 \text{ states}\}.\end{aligned} \tag{4}$$

### D. Classify 2+1D fermionic SPT states

The above approach also apply to fermionic case. Note that, the invertible fermionic GQLs with symmetry $G^f$ have bulk excitations described by SFC $\mathcal{E}=\mathrm{sRep}(G^f)$. So we would like to conjecture that

*Conjecture 3.* Invertible fermionic GQLs with symmetry $G^f$ are classified by $(\mathcal{M},c)$, where $\mathcal{M}$ is a modular extension of $\mathcal{E}=\mathrm{sRep}(G^f)$, and $c$ is the central charge determining the layers of $\nu=8$ IQH states.

*Remark 4.* Note that the central charge $c$ mod 8 is determined by $\mathcal{M}$, while $(c-\mathrm{mod}(c,8))/8$ determines the number of layers of the $\nu=8$ IQH states.

*Remark 5.* Invertible fermionic symmetric GQLs include both fermion SPT states and fermionic topological orders. $(\mathcal{M},c)$ with $c=0$ classify fermionic SPT states.

In other words:

*Corollary 3.* 2+1D fermionic SPT states with symmetry $G$ are classified by the $c=0$ modular extensions of $\mathrm{sRep}(G^f)$.

*Remark 6.* Unlike the bosonic case, in general

$$\begin{aligned}&\{\text{invertible fermionic symmetric GQLs}\}\\&\quad\neq\{\text{fermionic SPT states}\}\times\{\text{layers of } p+\mathrm{i}p \text{ states}\}.\end{aligned} \tag{5}$$

For example (see Table XV),

$$\begin{aligned}&\{\text{invertible } Z_4^f \text{ fermionic symmetric GQLs}\}\\&\quad=\{\text{fermionic } Z_4^f\text{-SPT states}\}\\&\qquad\times\{\text{layers of } \nu=1 \text{ integer quantum Hall states}\}.\end{aligned} \tag{6}$$

However, we have

$$\begin{aligned}&\{\text{invertible fermionic symmetric GQLs}\}\\&\quad=\{\text{invertible fermionic symmetric GQLs with } c\in[0,8)\}\\&\qquad\times\{\text{layers of } E_8 \text{ states}\}.\end{aligned} \tag{7}$$

Or when $G^f=G_b\times Z_2^f$

$$\begin{aligned}&\{\text{invertible fermionic symmetric GQLs}\}\\&\quad=\{\text{fermionic SPT states}\}\times\{\text{layers of } p+\mathrm{i}p \text{ states}\},\end{aligned} \tag{8}$$

where the fermions in the $p+\mathrm{i}p$ states are $G_b$-invariant.

When there is no symmetry, the invertible fermionic GQLs become the invertible fermionic topological order, which have bulk excitations described by $\mathcal{E}=\mathrm{sRep}(Z_2^f)$. $\mathrm{sRep}(Z_2^f)$ has 16 modular extensions, with central charges $c=n/2, n=0,1,2,\ldots,15$. There is only one modular extension with $c=0$, which correspond to a trivial product state. Thus there is no nontrivial fermionic SPT state when there is no symmetry, as expected.

The modular extensions with $c=n/2$ correspond to invertible fermionic topological order formed by $n$ layers of $p+\mathrm{i}p$

states. Since the modular extensions can only determine $c$ mod 8, in order for the above picture to be consistent, we need to show the following.

*Theorem 4.* The stacking of 16 layers $c = 1/2\ p + \mathrm{i}p$ states is equivalent to a $\nu = 8$ IQH state, which is in turn equivalent to a $E_8$ bosonic QH state stacked with a trivial fermionic product state.

*Proof.* First, two layers of $p + \mathrm{i}p$ states is equal to one layer of $\nu = 1$ IQH state. Thus 16 layers $c = 1/2\ p + \mathrm{i}p$ states is equivalent to a $\nu = 8$ IQH state. To show that $\nu = 8$, IQH state is equivalent to $E_8$ bosonic QH states stacked with a trivial fermionic product state, we note that the $\nu = 8$ IQH state is described by $K$ matrix $K_{\nu=8} = I_{8\times 8}$, which is an 8-by-8 identity matrix. While the $E_8$ bosonic QH state stacked with a trivial fermionic product state is described by $K$ matrix $K_{E_8 \boxtimes \mathcal{F}_0} = K_{E_8} \oplus \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, where $K_{E_8}$ is the matrix that describe the $E_8$ root lattice. We also know that two odd[1] $K$ matrices $K_1$ and $K_2$ describe the same fermionic topological order if after direct summing with proper number of $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$'s:

$$K_1' = K_1 \oplus \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \oplus \cdots ,$$

$$K_2' = K_2 \oplus \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \oplus \cdots , \tag{9}$$

$K_1'$ and $K_2'$ become equivalent, i.e.,

$$K_1' = U K_2' U^T, \quad U \in SL(N,\mathbb{Z}). \tag{10}$$

Notice that $K_{\nu=8} \oplus \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $K_{E_8 \boxtimes \mathcal{F}_0}$ have the same determinant $-1$ and the same signature. Using the result that odd matrices with $\pm 1$ determinants are equivalent if they have the same signature, we find that $K_{\nu=8} \oplus \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $K_{E_8 \boxtimes \mathcal{F}_0}$ are equivalent. Therefore $\nu = 8$ IQH state is equivalent to $E_8$ bosonic QH state stacked with a trivial fermionic product state. ■

## IV. A FULL CLASSIFICATION OF 2+1D GQLS WITH SYMMETRY

We have seen that all invertible GQLs with symmetry $G$ (or $G^f$) have the same kind of bulk excitations, described by Rep($G$) [or sRep($G^f$)]. To classify distinct invertible GQLs that share the same kind of bulk excitations, we need to compute the modular extensions of Rep($G$) [or sRep($G^f$)]. This result can be generalized to noninvertible topological orders.

In general, the bulk excitations of a 2+1D bosonic/fermionic SET are described by a UMTC$_{/\mathcal{E}}$ $\mathcal{C}$. However, there can be many distinct SET orders that have the same kind of bulk excitations described by the same $\mathcal{C}$. To classify distinct invertible SET orders that shared the same kind of bulk excitations $\mathcal{C}$, we need to compute the modular extensions of $\mathcal{C}$. This leads to the following.

*Conjecture 4.* 2+1D GQLs with symmetry $\mathcal{E}$ (i.e., the 2+1D SET orders) are classified by $(\mathcal{C},\mathcal{M},c)$, where $\mathcal{C}$ is a UMTC$_{/\mathcal{E}}$ describing the bulk topological excitations, $\mathcal{M}$ is a modular extension of $\mathcal{C}$ describing the edge state up to $E_8$ states, and $c$ is the central charge determining the layers of $E_8$ states.

Let $\mathcal{M}$ be a modular extension of a UMTC$_{/\mathcal{E}}$ $\mathcal{C}$. We note that all the simple objects (particles) in $\mathcal{C}$ are contained in $\mathcal{M}$ as simple objects. Assume that the particle labels of $\mathcal{M}$ are $\{i,j,\ldots,x,y,\ldots\}$, where $i,j,\ldots$ correspond to the particles in $\mathcal{C}$ and $x,y,\ldots$ the additional particles (not in $\mathcal{C}$). Physically, the additional particles $x,y,\ldots$ correspond to the symmetry twists of the on-site symmetry [22]. The modular extension $\mathcal{M}$ describes the fusion and the braiding of original particles $i,j,\ldots$ with the symmetry twists. In other words, the modular extension $\mathcal{M}$ is the resulting topological order after we gauge the on-site symmetry [21].

Now, it is clear that the existence of modular extension is closely related to the on-site symmetry (i.e., anomaly-free symmetry) which is gaugable (i.e., allows symmetry twists). For non-on-site symmetry (i.e., anomalous symmetry [15]), the modular extension does not exist since the symmetry is not gaugable (i.e., does not allow symmetry twists). We also have

*Conjecture 5.* 2+1D GQLs with anomalous symmetry [15] $\mathcal{E}$ are classified by UMTC$_{/\mathcal{E}}$'s that have no modular extensions.

It is also important to clarify the equivalence relation between the triples $(\mathcal{C},\mathcal{M},c)$. Two triples $(\mathcal{C},\mathcal{M},c)$ and $(\mathcal{C}',\mathcal{M}',c')$ are equivalent if: (1) $c = c'$; (2) there exists braided equivalences $F_\mathcal{C} : \mathcal{C} \to \mathcal{C}'$ and $F_\mathcal{M} : \mathcal{M} \to \mathcal{M}'$ such that all the embeddings are preserved, i.e., the following diagram commutes:

$$\begin{array}{ccccc} \mathcal{E} & \hookrightarrow & \mathcal{C} & \hookrightarrow & \mathcal{M} \\ \| & & \downarrow F_\mathcal{C} & & \downarrow F_\mathcal{M} \\ \mathcal{E} & \hookrightarrow & \mathcal{C}' & \hookrightarrow & \mathcal{M}' \end{array} \tag{11}$$

The equivalence classes will be in one-to-one correspondence with GQLs (i.e., SET orders and SPT orders).

Note that the group of the automorphisms of a UMTC$_{/\mathcal{E}}$ $\mathcal{C}$, denoted by Aut($\mathcal{C}$) (recall definition 4), naturally acts on the modular extensions $\mathcal{M}_{\text{ext}}(\mathcal{C})$ by changing the embeddings, i.e., $F \in$ Aut($\mathcal{C}$) acts as follows:

$$(\mathcal{C} \hookrightarrow \mathcal{M}) \mapsto (\mathcal{C} \xrightarrow{F} \mathcal{C} \hookrightarrow \mathcal{M}).$$

For a fixed $\mathcal{C}$, the above equivalence relation amounts to say that GQLs with bulk excitations described by a fixed $\mathcal{C}$ are in one-to-one correspondence with the quotient $\mathcal{M}_{\text{ext}}(\mathcal{C})/\text{Aut}(\mathcal{C})$ plus a central charge $c$. When $\mathcal{C} = \mathcal{E}$, the GQLs with bulk excitations described by $\mathcal{E}$ and central charge $c = 0$ are SPT phases. In this case, the group Aut($\mathcal{E}$), where $\mathcal{E}$ is viewed as the trivial UMTC$_{/\mathcal{E}}$, is trivial. Thus SPT phases are classified by the modular extensions of $\mathcal{E}$ with $c = 0$.

## V. ANOTHER DESCRIPTION OF 2+1D GQLS WITH SYMMETRY

Although the above result has a nice mathematical structure, it is hard to implement numerically to produce a table of GQLs. To fix this problem, we propose a different description of 2+1D GQLs. The second description is motivated by a conjecture that the fusion and the spins of the particles, $(\mathcal{N}^{IJ}_K,\mathcal{S}_I)$, completely characterize a UMTC. We conjecture that

[1] An odd matrix is a symmetric integer matrix with at least one of its diagonal elements being odd.

*Conjecture 6.* The data $(\tilde N^{ab}_c,\tilde s_a;N^{ij}_k,s_i;\mathcal{N}^{IJ}_K,\mathcal{S}_I;c)$, up to some equivalence relations, give a one-to-one classification of 2+1D GQLs with symmetry $G$ (for boson) or $G^f$ (for fermion), with a restriction that the symmetry group can be fully characterized by the fusion ring of its irreducible representations. The data $(\tilde N^{ab}_c,\tilde s_a;N^{ij}_k,s_i;\mathcal{N}^{IJ}_K,\mathcal{S}_I;c)$ satisfy the conditions described in Appendix C (see Ref. [11] for UMTCs).

Here, $(\tilde N^{ab}_c,\tilde s_a;N^{ij}_k,s_i;\mathcal{N}^{IJ}_K,\mathcal{S}_I;c)$ is closely related to $(\mathcal{E};\mathcal{C};\mathcal{M};c)$ discussed above. The data $(\tilde N^{ab}_c,\tilde s_a)$ describe the symmetry (i.e., the SFC $\mathcal{E}$): $a=1,\dots,\tilde N$ label the irreducible representations and $\tilde N^{ab}_c$ are the fusion coefficients of irreducible representations. $\tilde s_a=0$ or $1/2$ depending on if the fermion-number-parity transformation $f$ is represented trivially or nontrivially in the representation $a$. The data $(N^{ij}_k,s_i)$ describe fusion and the spins of the bulk particles $i=1,\dots,N$ in the GQL. The data $(N^{ij}_k,s_i)$ contain $(\tilde N^{ab}_c,\tilde s_a)$ as a subset, where $a$ is identified with the first $\tilde N$ particles of the GQL. The data $(\mathcal{N}^{IJ}_K,\mathcal{S}_I)$ describe fusion and the spins of a UMTC, and it includes $(N^{ij}_k,s_i)$ as a subset, where $i$ is identified with the first $N$ particles of the UMTC. Also among all the particles in UMTC, only the first $N$ ($i.e.$ $I=1,\dots,N$) have trivial mutual statistics with first $\tilde N$ particles ($i.e.$ $I=1,\dots,\tilde N$). Last, $c$ is the chiral central charge of the edge state.

If the data $(\tilde N^{ab}_c,\tilde s_a;N^{ij}_k,s_i)$ fully characterized the UMTC$_{/\mathcal{E}}$, then conjecture 6 would be equivalent to conjecture 4. However, for nonmodular tensor category, $(\tilde N^{ab}_c,\tilde s_a;N^{ij}_k,s_i)$ fails to to fully characterize a UMTC$_{/\mathcal{E}}$. In other words, there are different UMTC$_{/\mathcal{E}}$'s that have the same data $(\tilde N^{ab}_c,\tilde s_a;N^{ij}_k,s_i)$. We need to include the extra data, such as the $F$ tensor and the $R$ tensor, to fully characterize the UMTC$_{/\mathcal{E}}$.

In Appendix A, we list the data $(\tilde N^{ab}_c,\tilde s_a;N^{ij}_k,s_i)$ that satisfy the conditions in Appendix C (without the modular extension condition) in many tables. These tables include all the UMTC$_{/\mathcal{E}}$'s (up to certain total quantum dimensions), but the tables are not perfect: (1) some entries in the tables may be fake and do not correspond to any UMTC$_{/\mathcal{E}}$ (for the conditions are only necessary) and (2) some entries in the tables may correspond to more then one UMTC$_{/\mathcal{E}}$ [since $(\tilde N^{ab}_c,\tilde s_a;N^{ij}_k,s_i)$ does not fully characterize a UMTC$_{/\mathcal{E}}$].

We then continue to compute $(\mathcal{N}^{IJ}_K,\mathcal{S}_I;c)$, the modular extensions of $(\tilde N^{ab}_c,\tilde s_a;N^{ij}_k,s_i)$. We find that the modular extensions can fix the imperfectness mentioned above. First, we find that the fake entries do not have modular extensions, and are ruled out. Second, as we will show in Sec. VI, all UMTC$_{/\mathcal{E}}$'s have the same numbers of modular extensions (if they exist); therefore, the entry that corresponds to more UMTC$_{/\mathcal{E}}$'s has more modular extensions. The modular extensions can tell us which entries correspond to multiple UMTC$_{/\mathcal{E}}$'s. This leads to the conjecture that the full data $(\tilde N^{ab}_c,\tilde s_a;N^{ij}_k,s_i;\mathcal{N}^{IJ}_K,\mathcal{S}_I;c)$ give rise to an one-to-one classification of 2+1D GQLs, and allows us to calculate the tables of 2+1D GQLs, which include 2+1D SET states and 2+1D SPT states. Those are given in Sec. VIII.

As for the equivalence relation, we only need to consider $(\mathcal{N}^{IJ}_K,\mathcal{S}_I;c)$, since the data $(\tilde N^{ab}_c,\tilde s_a;N^{ij}_k,s_i)$ are included in $(\mathcal{N}^{IJ}_K,\mathcal{S}_I;c)$. Two such data $(\mathcal{N}^{IJ}_K,\mathcal{S}_I;c)$ and $(\bar{\mathcal{N}}^{IJ}_K,\bar{\mathcal{S}}_I;\bar c)$ are called equivalent if $c=\bar c$, and $(\mathcal{N}^{IJ}_K,\mathcal{S}_I)$ and $(\bar{\mathcal{N}}^{IJ}_K,\bar{\mathcal{S}}_I)$ are related by two permutations of indices in the range $N_{\mathcal{M}}\geqslant I>N$ and in the range $N\geqslant I>\tilde N$, where $N_{\mathcal{M}}$ is the range of $I$. Such an equivalence relation corresponds to the one in Eq. (11) and will be called the TO-equivalence relation. We use the TO-equivalence relation to count the number of GQL phases (i.e., the number of SET orders and SPT orders).

We can also define another equivalence relation, called ME-equivalence relation: we say $(\mathcal{N}^{IJ}_K,\mathcal{S}_I;c)$ and $(\bar{\mathcal{N}}^{IJ}_K,\bar{\mathcal{S}}_I;\bar c)$ to be ME-equivalent if $c=\bar c$ and they only differ by a permutation of indices in range $I>N$. The ME-equivalence relation is closely related to the one defined in Eqs. (3). We use the ME-equivalence relation to count the number of modular extensions of a *fixed* $\mathcal{C}$.

Last, let us explain the restriction on the symmetry group. In conjecture 6, we try to use the fusion $\tilde N^{ab}_c$ of the irreducible representations to characterize the symmetry group. However, it is known that certain different groups may have identical fusion ring for their irreducible representations. So we need to restrict the symmetry group to be the group that can be fully characterized by its fusion ring. Those groups include simple groups and Abelian groups [34]. If we do not impose such a restriction, then conjecture 6 gives rise to GQLs with a given symmetry fusion ring, instead of a given symmetry group.

## VI. THE STACKING OPERATION OF GQLS

### A. Stacking operation

Consider two GQLs $\mathcal{C}_1$ and $\mathcal{C}_2$. If we stack them together (without introducing interactions between them), we obtain another GQL, which is denoted by $\mathcal{C}_1\boxtimes\mathcal{C}_2$. The stacking operation $\boxtimes$ makes the set of GQLs into a monoid. $\boxtimes$ does not makes the set of GQLs into a group, because in general, a GQL $\mathcal{C}$ may not have an inverse under $\boxtimes$, i.e., there is no GQL $\mathcal{D}$ such that $\mathcal{C}\boxtimes\mathcal{D}$ becomes a trivial product state. This is because when a GQL have nontrivial topological excitations, stacking it with another GQL can never cancel out those topological excitations.

When we are considering GQLs with symmetry $\mathcal{E}$, the simple stacking $\boxtimes$ will "double" the symmetry, leads to a GQL with symmetry $\mathcal{E}\boxtimes\mathcal{E}$ [Rep$(G\times G)$ or sRep$(G^f\times G^f)$]. In general we allow local interactions between the two layers to break some symmetry such that the resulting system only has the original symmetry $\mathcal{E}$ [in terms of the symmetry group, keep only the subgroup $G\hookrightarrow G\times G$ with the diagonal embedding $g\mapsto(g,g)$]. This leads to the stacking between GQLs with symmetry $\mathcal{E}$, denoted by $\boxtimes_{\mathcal{E}}$. Similarly, $\boxtimes_{\mathcal{E}}$ makes GQLs with symmetry $\mathcal{E}$ a monoid, but in general not all GQLs are invertible.

However, if the bulk excitations of $\mathcal{C}$ are all local (i.e., all described by SFC $\mathcal{E}$), then $\mathcal{C}$ will have an inverse under the stacking operation $\boxtimes_{\mathcal{E}}$, and this is why we call such GQL invertible. Those invertible GQLs include invertible topological orders and SPT states.

### B. The group structure of bosonic SPT states

We have proposed that 2+1D SPT states are classified by $c=0$ modular extensions of the SFC $\mathcal{E}$ that describes the symmetry. Since SPT states are invertible, they form a group under the stacking operation $\boxtimes_{\mathcal{E}}$. This implies that the modular extensions of the SFC should also form a group under the stacking operation. So checking if the modular extensions of the SFC have a group structure is a way to find support for our conjecture.

However, in this section, we will first discuss such stacking operation and group structure from a physical point of view. We will only consider bosonic SPT states.

It has been proposed that the bosonic SPT states are described by group cohomology $\mathcal{H}^{d+1}[G,\mathrm{U}(1)]$ [18–20]. However, it has not been shown that those bosonic SPT states form a group under stacking operation. Here we will fill this gap. An ideal bosonic SPT state of symmetry $G$ in $d+1$D is described the following path integral:

$$Z = \sum_{\{g_i\}} \prod_{\{i,j,\ldots\}} \nu_{d+1}(g_i,g_j,\ldots), \tag{12}$$

where $\nu_{d+1}(g_i,g_j,\ldots)$ is a function $G^{d+1} \to \mathrm{U}(1)$, which is a cocycle $\nu_{d+1} \in \mathcal{H}^{d+1}[G,\mathrm{U}(1)]$. Here, the space-time is a complex whose vertices are labeled by $i,j,\ldots$, and $\prod_{\{i,j,\ldots\}}$ is the product over all the simplices of the space-time complex. Also $\sum_{\{g_i\}}$ is a sum over all $g_i$ on each vertex.

Now consider the stacking of two SPT states described by cocycle $\nu'_{d+1}$ and $\nu''_{d+1}$:

$$Z = \sum_{\{g'_i,g''_i\}} \prod_{\{i,j,\ldots\}} \nu'_{d+1}(g'_i,g'_j,\ldots)\nu''_{d+1}(g''_i,g''_j,\ldots). \tag{13}$$

Such a stacked state has a symmetry $G \times G$ and is a $G \times G$ SPT state.

Now let us add a term to break the $G \times G$ symmetry to $G$ symmetry and consider

$$Z = \sum_{\{g'_i,g''_i\}} \prod_{\{i,j,\ldots\}} \nu'_{d+1}(g'_i,g'_j,\ldots)\nu''_{d+1}(g''_i,g''_j,\ldots) \times \prod_i \mathrm{e}^{-U|g'_i-g''_i|^2}, \tag{14}$$

where $|g'-g''|$ is an invariant distance between group elements. As we change $U=0$ to $U=+\infty$, the stacked system changes into the system for an ideal SPT state described by the cocycle $\nu_{d+1}(g_i,g_j,\ldots) = \nu'_{d+1}(g_i,g_j,\ldots)\nu''_{d+1}(g_i,g_j,\ldots)$. If such a deformation does not cause any phase transition, then we can show that the stacking of a $\nu'_{d+1}$-SPT state with a $\nu''_{d+1}$-SPT state give rise to a $\nu_{d+1} = \nu'_{d+1}\nu''_{d+1}$-SPT state. Thus the key to show the stacking operation to give rise to the group structure for the SPT states, is to show the theory Eq. (14) has no phase transition as we change $U=0$ to $U=+\infty$.

To show there is no phase transition, we put the system on a closed space-time with no boundary, say $S^{d+1}$. In this case, $\prod_{\{i,j,\ldots\}} \nu'_{d+1}(g'_i,g'_j,\ldots)\nu''_{d+1}(g''_i,g''_j,\ldots) = 1$, since $\nu'_{d+1}$ and $\nu''_{d+1}$ are cocycles. Thus the path integral (14) is reduced to

$$Z = \sum_{\{g'_i,g''_i\}} \prod_i \mathrm{e}^{-U|g'_i-g''_i|^2} = \left(|G| \sum_g \mathrm{e}^{-U|1-g|^2}\right)^{N_v}, \tag{15}$$

where $N_v$ is the number of vertices and $|G|$ the order of the symmetry group. We see that the free energy density

$$f = -\lim_{N_v\to\infty} \ln Z/N_v \tag{16}$$

is a smooth function of $U$ for $U \in [0,\infty)$. There is indeed no phase transition.

The above result is highly non trivial from a categorical point of view. Consider two 2+1D bosonic SPT states described by two modular extensions $\mathcal{M}'$ and $\mathcal{M}''$ of Rep($G$). The natural tensor product $\mathcal{M}' \boxtimes \mathcal{M}''$ is not a modular extension of Rep($G$), but a modular extension of Rep($G$) $\boxtimes$ Rep($G$) = Rep($G \times G$). So, $\mathcal{M}' \boxtimes \mathcal{M}''$ describes a $G \times G$-SPT state. According to the above discussion, we need to break the $G \times G$ symmetry down to the $G$ symmetry to obtain the $G$-SPT state. Such a symmetry breaking process correspond to the so-called "anyon condensation" in category theory. We will discuss such anyon condensation later. The stacking operation $\boxtimes_{\mathcal{E}}$, with such a symmetry breaking process included, is the correct stacking operation that maintains the symmetry $G$. In Ref. [30], we also discussed more general symmetry breaking processes, from $G$ to any subgroup $H$.

### C. Mathematical construction of the stacking operation

We have conjectured that a 2+1D topological order with symmetry $\mathcal{E}$ is classified by $(\mathcal{C},\mathcal{M}_{\mathcal{C}},c)$, where $\mathcal{C}$ is a UMTC$_{/\mathcal{E}}$, $\mathcal{M}_{\mathcal{C}}$ is a modular extension of $\mathcal{C}$, and $c$ is the central charge. If we have another topological order of the same symmetry $\mathcal{E}$ described by $(\mathcal{C}',\mathcal{M}_{\mathcal{C}'},c')$, stacking $(\mathcal{C},\mathcal{M}_{\mathcal{C}},c)$ and $(\mathcal{C}',\mathcal{M}_{\mathcal{C}'},c')$ should give a third topological order described by similar data $(\mathcal{C}'',\mathcal{M}_{\mathcal{C}''},c'')$:

$$(\mathcal{C},\mathcal{M}_{\mathcal{C}},c) \boxtimes_{\mathcal{E}} (\mathcal{C}',\mathcal{M}_{\mathcal{C}'},c') = (\mathcal{C}'',\mathcal{M}_{\mathcal{C}''},c''). \tag{17}$$

In this section, we will show that such a stacking operation can be defined mathematically. This is an evidence supporting our conjecture 4. We like to point out that a special case of the above result for $\mathcal{C} = \mathcal{C}' = \mathcal{C}'' = \mathcal{E} = \mathrm{Rep}(G)$ was discussed in Sec. VI B.

To define $\boxtimes_{\mathcal{E}}$ mathematically, first, we like to introduce

*Definition 7.* A *condensable algebra* in a UBFC $\mathcal{C}$ is a triple $(A,m,\eta)$, $A \in \mathcal{C}$, $m : A \otimes A \to A$, $\eta : 1 \to A$ satisfying (1) associative: $m(\mathrm{id}_A \otimes m) = m(m \otimes \mathrm{id}_A)$; (2) unit: $m(\eta \otimes \mathrm{id}_A) = m(\mathrm{id}_A \otimes \eta) = \mathrm{id}_A$; (3) isometric: $mm^\dagger = \mathrm{id}_A$; (4) connected: $\mathrm{Hom}(1,A) = \mathbb{C}$; and (5) commutative: $mc_{A,A} = m$.

Note that in the unitary case, $(A,m,\eta)$ is automatically a special symmetric Frobenius algebra [35]. Physically, such a condensable algebra $A$ is a composite self-bosonic anyon satisfies additional conditions such that one can condense $A$ to obtain another topological phase.

*Definition 8.* A (left) *module* over a condensable algebra $(A,m,\eta)$ in $\mathcal{C}$ is a pair $(X,\rho)$, $X \in \mathcal{C}$, $\rho : A \otimes X \to X$ satisfying

$$\rho(\mathrm{id}_A \otimes \rho) = \rho(m \otimes \mathrm{id}_M),$$
$$\rho(\eta \otimes \mathrm{id}_M) = \mathrm{id}_M. \tag{18}$$

It is further a *local* module if

$$\rho c_{M,A} c_{A,M} = \rho.$$

We denote the category of left $A$-modules by $\mathcal{C}_A$. A left module $(X,\rho)$ is turned into a right module via the braiding, $(X,\rho c_{X,A})$ or $(X,\rho c^{-1}_{A,X})$, and thus an $A$-$A$–bimodule. The relative tensor functor $\otimes_A$ of bimodules then turns $\mathcal{C}_A$ into a fusion category. (This is known as $\alpha$-induction in subfactor context.) In general, there can be two monoidal structures on $\mathcal{C}_A$, since there are two ways to turn a left module into a bimodule (usually we pick one for definiteness when considering $\mathcal{C}_A$ as a fusion category). The two monoidal structures coincide for the fusion subcategory $\mathcal{C}_A^0$ of local $A$-modules. Moreover, $\mathcal{C}_A^0$ inherited the braiding from $\mathcal{C}$ and is also a UBFC. The local modules are nothing but the anyons in the topological phases after condensing $A$.

*Lemma 1.* (DMNO [36])

$$\dim(\mathcal{C}_A) = \frac{\dim(\mathcal{C})}{\dim(A)}.$$

If $\mathcal{C}$ is a UMTC, then so is $\mathcal{C}_A^0$, and

$$\dim(\mathcal{C}_A^0) = \frac{\dim(\mathcal{C})}{\dim(A)^2}.$$

A noncommutative algebra $A$ is also of interest. We have the left center $A_l$ of $A$, the maximal subalgebra such that $mc_{A_l,A} = m$, and the right center $A_r$, the maximal subalgebra such that $mc_{A,A_r} = m$. $A_l$ and $A_r$ are commutative subalgebras, thus condensable.

*Theorem 5.* (FFRS [37]) There is a canonical equivalence between the categories of local modules over the left and right centers, $\mathcal{C}_{A_l}^0 = \mathcal{C}_{A_r}^0$.

*Definition 9.* The Drinfeld center $Z(\mathcal{A})$ of a monoidal category $\mathcal{A}$ is a monoidal category with objects as pairs $(X \in \mathcal{A}, b_{X,-})$, where $b_{X,-} : X \otimes - \to - \otimes X$ are half-braidings that satisfy similar conditions as braidings. Morphisms and the tensor product are naturally defined.

$Z(\mathcal{A})$ is a braided monoidal category. There is a forgetful tensor functor $for_{\mathcal{A}} : Z(\mathcal{A}) \to \mathcal{A}, (X, b_{X,-}) \mapsto X$ that forgets the half-braidings.

*Theorem 6.* (Müger [38]) $Z(\mathcal{A})$ is a UMTC if $\mathcal{A}$ is a unitary fusion category and $\dim(Z(\mathcal{A})) = \dim(\mathcal{A})^2$.

*Definition 10.* Let $\mathcal{C}$ be a braided fusion category and $\mathcal{A}$ a fusion category, a tensor functor $F : \mathcal{C} \to \mathcal{A}$ is called a central functor if it factorizes through $Z(\mathcal{A})$, i.e., there exists a braided tensor functor $F' : \mathcal{C} \to Z(\mathcal{A})$ such that $F = F' for_{\mathcal{A}}$.

*Lemma 2.* (DMNO [36]) Let $F : \mathcal{C} \to \mathcal{A}$ be a central functor, and $R : \mathcal{A} \to \mathcal{C}$ the right adjoint functor of $F$. Then the object $A = R(1) \in \mathcal{C}$ has a canonical structure of condensable algebra. $\mathcal{C}_A$ is monoidally equivalent to the image of $F$, i.e., the smallest fusion subcategory of $\mathcal{A}$ containing $F(\mathcal{C})$.

*Example 1.* If $\mathcal{C}$ is a UBFC, it is naturally embedded into $Z(\mathcal{C})$, so is $\overline{\mathcal{C}}$. Therefore there is a braided monoidal functor $\mathcal{C} \boxtimes \overline{\mathcal{C}} \to Z(\mathcal{C})$. Compose this functor with the forgetful functor $for_{\mathcal{C}} : Z(\mathcal{C}) \to \mathcal{C}$ we get a central functor

$$\mathcal{C} \boxtimes \overline{\mathcal{C}} \to \mathcal{C},$$
$$X \boxtimes Y \mapsto X \otimes Y.$$

Let $R$ be its right adjoint functor, we obtain a condensable algebra $L_{\mathcal{C}} := R(1) \cong \oplus_i (i \boxtimes \bar{i}) \in \mathcal{C} \boxtimes \overline{\mathcal{C}}$ ($\bar{i}$ denotes the dual object, or antiparticle of $i$) and $\mathcal{C} = (\mathcal{C} \boxtimes \overline{\mathcal{C}})_{L_{\mathcal{C}}}$, $\dim(L_{\mathcal{C}}) = \dim(\mathcal{C})$. In particular, for a symmetric category $\mathcal{E}$, $L_{\mathcal{E}}$ is a condensable algebra in $\mathcal{E} \boxtimes \mathcal{E}$, and $\mathcal{E} = (\mathcal{E} \boxtimes \mathcal{E})_{L_{\mathcal{E}}} = (\mathcal{E} \boxtimes \mathcal{E})_{L_{\mathcal{E}}}^0$ for $\mathcal{E}$ is symmetric, all $L_{\mathcal{E}}$-modules are local. Condensing $L_{\mathcal{E}}$ is nothing but breaking the symmetry from $\mathcal{E} \boxtimes \mathcal{E}$ to $\mathcal{E}$.

Now, we are ready to define the stacking operation for UMTC$_{/\mathcal{E}}$'s as well as their modular extensions.

*Definition 11.* Let $\mathcal{C},\mathcal{D}$ be UMTC$_{/\mathcal{E}}$'s, and $\mathcal{M}_{\mathcal{C}},\mathcal{M}_{\mathcal{D}}$ their modular extensions. The stacking is defined by

$$\mathcal{C} \boxtimes_{\mathcal{E}} \mathcal{D} := (\mathcal{C} \boxtimes \mathcal{D})_{L_{\mathcal{E}}}^0,$$
$$\mathcal{M}_{\mathcal{C}} \boxtimes_{\mathcal{E}} \mathcal{M}_{\mathcal{D}} := (\mathcal{M}_{\mathcal{C}} \boxtimes \mathcal{M}_{\mathcal{D}})_{L_{\mathcal{E}}}^0.$$

Note that in Ref. [39], the tensor product $\boxtimes_{\mathcal{E}}$ for UMTC$_{/\mathcal{E}}$'s is defined as $(\mathcal{C} \boxtimes \mathcal{D})_{L_{\mathcal{E}}}$. For UMTC$_{/\mathcal{E}}$'s the two definitions coincide $(\mathcal{C} \boxtimes \mathcal{D})_{L_{\mathcal{E}}}^0 = (\mathcal{C} \boxtimes \mathcal{D})_{L_{\mathcal{E}}}$, for $L_{\mathcal{E}}$ lies in the centralizer of $\mathcal{C} \boxtimes \mathcal{D}$ which is $\mathcal{E} \boxtimes \mathcal{E}$. But for the modular extensions we have to take the unusual definition above.

*Theorem 7.* $\mathcal{C} \boxtimes_{\mathcal{E}} \mathcal{D}$ is a UMTC$_{/\mathcal{E}}$, and $\mathcal{M}_{\mathcal{C}} \boxtimes_{\mathcal{E}} \mathcal{M}_{\mathcal{D}}$ is a modular extension of $\mathcal{C} \boxtimes_{\mathcal{E}} \mathcal{D}$.

*Proof.* The embeddings $\mathcal{E} = (\mathcal{E} \boxtimes \mathcal{E})_{L_{\mathcal{E}}}^0 \hookrightarrow (\mathcal{C} \boxtimes \mathcal{D})_{L_{\mathcal{E}}}^0 = \mathcal{C} \boxtimes_{\mathcal{E}} \mathcal{D} \hookrightarrow (\mathcal{M}_{\mathcal{C}} \boxtimes \mathcal{M}_{\mathcal{D}})_{L_{\mathcal{E}}}^0 = \mathcal{M}_{\mathcal{C}} \boxtimes_{\mathcal{E}} \mathcal{M}_{\mathcal{D}}$ are obvious. So $\mathcal{C} \boxtimes_{\mathcal{E}} \mathcal{D}$ is a UBFC over $\mathcal{E}$. Also

$$\dim(\mathcal{C} \boxtimes_{\mathcal{E}} \mathcal{D}) = \frac{\dim(\mathcal{C} \boxtimes \mathcal{D})}{\dim(L_{\mathcal{E}})} = \frac{\dim(\mathcal{C})\dim(\mathcal{D})}{\dim(\mathcal{E})},$$

and $\mathcal{M}_{\mathcal{C}} \boxtimes_{\mathcal{E}} \mathcal{M}_{\mathcal{D}}$ is a UMTC,

$$\dim(\mathcal{M}_{\mathcal{C}} \boxtimes_{\mathcal{E}} \mathcal{M}_{\mathcal{D}}) = \frac{\dim(\mathcal{M}_{\mathcal{C}} \boxtimes \mathcal{M}_{\mathcal{D}})}{\dim(L_{\mathcal{E}})^2} = \dim(\mathcal{C})\dim(\mathcal{D}).$$

Thus $\mathcal{M}_{\mathcal{C}} \boxtimes_{\mathcal{E}} \mathcal{M}_{\mathcal{D}}$ is a modular extension of $\mathcal{C} \boxtimes_{\mathcal{E}} \mathcal{D}$. ∎

Take $\mathcal{D} = \mathcal{E}$. Note that $\mathcal{C} \boxtimes_{\mathcal{E}} \mathcal{E} = \mathcal{C}$. Therefore, for any modular extension $\mathcal{M}_{\mathcal{E}}$ of $\mathcal{E}$, $\mathcal{M}_{\mathcal{C}} \boxtimes_{\mathcal{E}} \mathcal{M}_{\mathcal{E}}$ is still a modular extension of $\mathcal{C}$. In the following, we want to show the inverse, that one can extract the "difference", a modular extension of $\mathcal{E}$, between two modular extensions of $\mathcal{C}$.

*Lemma 3.* We have $(\mathcal{C} \boxtimes \overline{\mathcal{C}})_{L_{\mathcal{C}}}^0 = \mathcal{C}_{\mathcal{C}}^{\text{cen}}$.

*Proof.* $(\mathcal{C} \boxtimes \overline{\mathcal{C}})_{L_{\mathcal{C}}}$ is equivalent to $\mathcal{C}$ (as a fusion category). Moreover, for $X \in \mathcal{C}$, the equivalence gives the free module $L_{\mathcal{C}} \otimes (X \boxtimes 1) \cong L_{\mathcal{C}} \otimes (1 \boxtimes X)$. $L_{\mathcal{C}} \otimes (X \boxtimes 1)$ is a local $L_{\mathcal{C}}$-module if and only if $X \boxtimes 1$ centralize $L_{\mathcal{C}}$. This is the same as $X \in \mathcal{C}_{\mathcal{C}}^{\text{cen}}$. Therefore we have $(\mathcal{C} \boxtimes \overline{\mathcal{C}})_{L_{\mathcal{C}}}^0 = \mathcal{C}_{\mathcal{C}}^{\text{cen}}$. ∎

*Theorem 8.* let $\mathcal{M}$ and $\mathcal{M}'$ be two modular extensions of the UMTC$_{/\mathcal{E}}$ $\mathcal{C}$. There exists a unique $\mathcal{K} \in \mathcal{M}_{\text{ext}}(\mathcal{E})$ such that $\mathcal{K} \boxtimes_{\mathcal{E}} \mathcal{M} = \mathcal{M}'$. Such $\mathcal{K}$ is given by

$$\mathcal{K} = (\mathcal{M}' \boxtimes \overline{\mathcal{M}})_{L_{\mathcal{C}}}^0.$$

*Proof.* $\mathcal{K}$ is a modular extension of $\mathcal{E}$. This follows Lemma 3 that $\mathcal{E} = \mathcal{C}_{\mathcal{C}}^{\text{cen}} = (\mathcal{C} \boxtimes \overline{\mathcal{C}})_{L_{\mathcal{C}}}^0$ is a full subcategory of $\mathcal{K}$. $\mathcal{K}$ is a UMTC by construction, and $\dim(\mathcal{K}) = \frac{\dim(\mathcal{M})\dim(\mathcal{M}')}{\dim(L_{\mathcal{C}})^2} = \dim(\mathcal{E})^2$.

To show that $\mathcal{K} = (\mathcal{M}' \boxtimes \overline{\mathcal{M}})_{L_{\mathcal{C}}}$ satisfies $\mathcal{M}' = \mathcal{K} \boxtimes_{\mathcal{E}} \mathcal{M}$, note that $\mathcal{M}' = \mathcal{M}' \boxtimes \text{Vec} = \mathcal{M}' \boxtimes (\overline{\mathcal{M}} \boxtimes \mathcal{M})_{L_{\overline{\mathcal{M}}}}^0$. It suffices that

$$(\mathcal{M}' \boxtimes \overline{\mathcal{M}} \boxtimes \mathcal{M})_{1 \boxtimes L_{\overline{\mathcal{M}}}}^0 = [(\mathcal{M}' \boxtimes \overline{\mathcal{M}})_{L_{\mathcal{C}}}^0 \boxtimes \mathcal{M}]_{L_{\mathcal{E}}}^0$$
$$= (\mathcal{M}' \boxtimes \overline{\mathcal{M}} \boxtimes \mathcal{M})_{(L_{\mathcal{C}} \boxtimes 1) \otimes (1 \boxtimes L_{\mathcal{E}})}^0.$$

This follows that $1 \boxtimes L_{\overline{\mathcal{M}}}$ and $(L_{\mathcal{C}} \boxtimes 1) \otimes (1 \boxtimes L_{\mathcal{E}})$ are left and right centers of the algebra $(L_{\mathcal{C}} \boxtimes 1) \otimes (1 \boxtimes L_{\overline{\mathcal{M}}})$.

If $\mathcal{M}' = \mathcal{K} \boxtimes_{\mathcal{E}} \mathcal{M} = (\mathcal{K} \boxtimes \mathcal{M})_{L_{\mathcal{E}}}^0$, then

$$\mathcal{K} = (\mathcal{K} \boxtimes \mathcal{M} \boxtimes \overline{\mathcal{M}})_{1 \boxtimes L_{\mathcal{M}}}^0 = (\mathcal{K} \boxtimes \mathcal{M} \boxtimes \overline{\mathcal{M}})_{(L_{\mathcal{E}} \boxtimes 1) \otimes (1 \boxtimes L_{\mathcal{C}})}^0$$
$$= [(\mathcal{K} \boxtimes_{\mathcal{E}} \mathcal{M}) \boxtimes \overline{\mathcal{M}}]_{L_{\mathcal{C}}}^0 = (\mathcal{M}' \boxtimes \overline{\mathcal{M}})_{L_{\mathcal{C}}}^0.$$

It is similar here that $1 \boxtimes L_{\mathcal{M}}$ and $(L_{\mathcal{E}} \boxtimes 1) \otimes (1 \boxtimes L_{\mathcal{C}})$ are the left and right centers of the algebra $(L_{\mathcal{E}} \boxtimes 1) \otimes (1 \boxtimes L_{\mathcal{M}})$. This proves the uniqueness of $\mathcal{K}$. ■

Let us list several consequences of theorem 8.

*Theorem 9.* $\mathcal{M}_{\text{ext}}(\mathcal{E})$ forms a finite Abelian group.

*Proof.* Firstly, there exists at least one modular extension of a symmetric fusion category $\mathcal{E}$, the Drinfeld center $Z(\mathcal{E})$. So the set $\mathcal{M}_{\text{ext}}(\mathcal{E})$ is not empty. The multiplication is given by the stacking $\boxtimes_{\mathcal{E}}$. It is easy to verify that the stacking $\boxtimes_{\mathcal{E}}$ for modular extensions is associative and commutative. To show that they form a group we only need to find out the identity and inverse. In this case $\mathcal{K} = (\mathcal{M}' \boxtimes \overline{\mathcal{M}})^0_{L_{\mathcal{E}}} = \mathcal{M}' \boxtimes_{\mathcal{E}} \overline{\mathcal{M}}$, theorem 8 becomes $\mathcal{M}' \boxtimes_{\mathcal{E}} \overline{\mathcal{M}} \boxtimes_{\mathcal{E}} \mathcal{M} = \mathcal{M}'$, for any modular extensions $\mathcal{M},\mathcal{M}'$ of $\mathcal{E}$. Thus $\overline{\mathcal{M}'} \boxtimes_{\mathcal{E}} \mathcal{M}' = \overline{\mathcal{M}'} \boxtimes_{\mathcal{E}} \mathcal{M}' \boxtimes_{\mathcal{E}} \overline{\mathcal{M}} \boxtimes_{\mathcal{E}} \mathcal{M} = \overline{\mathcal{M}} \boxtimes_{\mathcal{E}} \mathcal{M}$, i.e., $\overline{\mathcal{M}} \boxtimes_{\mathcal{E}} \mathcal{M}$, is the same category for any extension $\mathcal{M}$, which turns out to be $Z(\mathcal{E})$. It is exactly the identity element. It is then obvious that the inverse of $\mathcal{M}$ is $\overline{\mathcal{M}}$. The finiteness follows from Ref. [32]. ■

*Example 2.* For bosonic case, we find that $\mathcal{M}_{\text{ext}}(\text{Rep}(G)) = H^3(G,\text{U}(1))$, which is discussed in more detail in the next section. For fermionic case, a general group cohomological classification is still lacking. We know some simple ones such as $\mathcal{M}_{\text{ext}}(\text{sRep}(\mathbb{Z}_2^f)) = \mathbb{Z}_{16}$, which agrees with Kitaev's 16-fold way [9].

*Theorem 10.* For a $\text{UMTC}_{/\mathcal{E}}$ $\mathcal{C}$, if the modular extensions exist, $\mathcal{M}_{\text{ext}}(\mathcal{C})$ form a $\mathcal{M}_{\text{ext}}(\mathcal{E})$-torsor. In particular, $|\mathcal{M}_{\text{ext}}(\mathcal{C})| = |\mathcal{M}_{\text{ext}}(\mathcal{E})|$.

*Proof.* The action is given by the stacking $\boxtimes_{\mathcal{E}}$. For any two extensions $\mathcal{M},\mathcal{M}'$, there is a unique extension $\mathcal{K}$ of $\mathcal{E}$, such that $\mathcal{M} \boxtimes_{\mathcal{E}} \mathcal{K} = \mathcal{M}'$. To see $Z(\mathcal{E})$ acts trivially, note that $\mathcal{M}' \boxtimes_{\mathcal{E}} Z(\mathcal{E}) = \mathcal{M} \boxtimes_{\mathcal{E}} \mathcal{K} \boxtimes_{\mathcal{E}} Z(\mathcal{E}) = \mathcal{M} \boxtimes_{\mathcal{E}} \mathcal{K} = \mathcal{M}'$ holds for any $\mathcal{M}'$. Due to uniqueness we also know that only $\mathcal{Z}_{\mathcal{E}}$ acts trivially. Thus the action is free and transitive. ■

This means that for any modular extension of $\mathcal{C}$, stacking with a nontrivial modular extensions of $\mathcal{E}$, one always obtains a different modular extension of $\mathcal{C}$; on the other hand, starting with a particular modular extension of $\mathcal{C}$, all the other modular extensions can be generated by staking modular extensions of $\mathcal{E}$ (in other words, there is only on orbit). However, in general, there is no preferred choice of the starting modular extension, unless $\mathcal{C}$ is the form $\mathcal{C}_0 \boxtimes \mathcal{E}$ where $\mathcal{C}_0$ is a UMTC.

### D. Modular extensions of Rep($G$)

We set $\mathcal{E} = \text{Rep}(G)$ throughout this subsection. Let $(\mathcal{M},\iota_{\mathcal{M}})$ be a modular extension of $\text{Rep}(G)$. $\iota_{\mathcal{M}}$ is the embedding $\iota_{\mathcal{M}} : \mathcal{E} \hookrightarrow \mathcal{M}$ that we need to consider explicitly in this subsection. The algebra $A = \text{Fun}(G)$ is a condensable algebra in $\text{Rep}(G)$ and also a condensable algebra in $\mathcal{M}$. Moreover, $A$ is a Lagrangian algebra in $\mathcal{M}$ because $(\dim A)^2 = |G|^2 = (\dim \text{Rep}(G))^2 = \dim \mathcal{M}$. Therefore $\mathcal{M} \simeq Z(\mathcal{M}_A)$, where $\mathcal{M}_A$ is the category of right $A$-modules in $\mathcal{M}$. In other words, $\mathcal{M}$ describes the bulk excitations in a 2+1D topological phase with a gapped boundary (see Fig. 3). Moreover, the fusion category $\mathcal{M}_A$ is pointed and equipped with a canonical fully faithful $G$ grading [33], which means that

$$\mathcal{M}_A = \oplus_{g\in G}(\mathcal{M}_A)_g, \quad (\mathcal{M}_A)_g \simeq \text{Vec}, \quad \forall g \in G,$$
$$\text{and} \quad \otimes : (\mathcal{M}_A)_g \boxtimes (\mathcal{M}_A)_h \xrightarrow{\simeq} (\mathcal{M}_A)_{gh}.$$

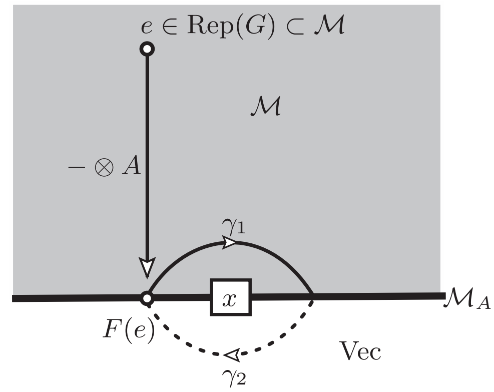


FIG. 3. Consider a physical situation in which the excitations in the $2+1$D bulk are given by a modular extension $\mathcal{M}$ of $\text{Rep}(G)$, and those on the gapped boundary by the UFC $\mathcal{M}_A$. Consider a simple particle $e \in \text{Rep}(G)$ in the bulk moving toward the boundary. The bulk-to-boundary map is given by the central functor $- \otimes A : \mathcal{M} \to \mathcal{M}_A$, which restricted to $\text{Rep}(G)$ is nothing but the forgetful functor $F : \text{Rep}(G) \to \text{Vec}$. Let $x$ be a simple excitation in $\mathcal{M}_A$ sitting next to $F(e)$. We move $F(e)$ along the semicircle $\gamma_1$ (defined by the half-braiding), then move along the semicircle $\gamma_2$ (defined by the symmetric braiding in the trivial phase Vec).

Let us recall the construction of this $G$ grading. The physical meaning of acquiring a $G$ grading on $\mathcal{M}_A$ after condensing the algebra $A = \text{Fun}(G)$ in $\mathcal{M}$ is depicted in Fig. 3. The process in Fig. 3 defines the isomorphism $F(e) \otimes_A x \xrightarrow{z_{e,x}} x \otimes_A F(e) = F(e) \otimes_A x$, which further gives a monoidal automorphism $\phi(x) \in \text{Aut}(F) = G$ of the fiber functor $F : \text{Rep}(G) \to \text{Vec}$.

Since $\phi$ is an isomorphism, the associator of the monoidal category $\mathcal{M}_A$ determines a unique $\omega_{(\mathcal{M},\iota_{\mathcal{M}})} \in H^3(G,\text{U}(1))$ such that $\mathcal{M}_A \simeq \text{Vec}_G^{\omega}$ as $G$-graded fusion categories.

*Theorem 11.* The map $(\mathcal{M},\iota_{\mathcal{M}}) \mapsto \omega_{(\mathcal{M},\iota_{\mathcal{M}})}$ defines a group isomorphism $\mathcal{M}_{\text{ext}}(\text{Rep}(G)) \simeq H^3(G,\text{U}(1))$. In particular, we have

$$\left(Z\left(\text{Vec}_G^{\omega_1}\right),\iota_{\omega_1}\right) \boxtimes_{\mathcal{E}} \left(Z\left(\text{Vec}_G^{\omega_2}\right),\iota_{\omega_2}\right) \simeq \left(Z\left(\text{Vec}_G^{\omega_1+\omega_2}\right),\iota_{\omega_1+\omega_2}\right).$$

For the proof and more related details, see also Ref. [30].

### E. Relation to numerical calculations

In Sec. V, we proposed another way to characterise GQLs, using the data $(\tilde{N}^{ab}_c,\tilde{s}_a;N^{ij}_k,s_i;\mathcal{N}^{IJ}_K,\mathcal{S}_I;c)$, which is more friendly in numerical calculations. We would like to investigate how to calculate the stacking operation in terms of these data.

Assuming that $\mathcal{C}$ and $\mathcal{C}'$ can be characterized by data $(N^{ij}_k,s_i)$ and $(N'^{ij}_k,s'_i)$. Let $(N^{\mathcal{D},ij}_k,s^{\mathcal{D}}_i)$ be the data that characterize the stacked $\text{UMTC}_{/\mathcal{E}}$ $\mathcal{D} = \mathcal{C} \boxtimes_{\mathcal{E}} \mathcal{C}'$.

To calculate $(N^{\mathcal{D},ij}_k,s^{\mathcal{D}}_i)$, let us first construct

$$N^{ii',jj'}_{kk'} = N^{ij}_k N'^{i'j'}_{k'}, \quad s_{ii'} = s_i + s'_{i'}. \tag{19}$$

Note that, the above data describe a $\text{UMTC}_{/\mathcal{E}\boxtimes\mathcal{E}}$ $\mathcal{D}' = \mathcal{C} \boxtimes \mathcal{C}'$ (i.e., with centralizer $\mathcal{E} \boxtimes \mathcal{E}$), which is not what we want. We need reduce centralizer from $\mathcal{E} \boxtimes \mathcal{E}$ to $\mathcal{E}$. This is the $G \times G$ to $G$ process and $\mathcal{C}$-$\mathcal{C}'$ coupling, or condensing the $L_{\mathcal{E}}$ algebra, as discussed above

To do the $\mathcal{E}\boxtimes\mathcal{E}$ to $\mathcal{E}$ reduction (i.e., to obtain the real stacking operation $\boxtimes_{\mathcal{E}}$), we can introduce an equivalence relation. Noting that the excitations in $\mathcal{D}'=\mathcal{C}\boxtimes\mathcal{C}'$ are labeled by $ii'=i\boxtimes i'$, the equivalence relation is

$$ii'\sim jj',\quad \text{if } ii'\otimes L_{\mathcal{E}}=jj'\otimes L_{\mathcal{E}}, \tag{20}$$

where $L_{\mathcal{E}}=\oplus_a a\bar{a}, a\in\mathcal{E}$. In the simple case of Abelian groups, where all the $a$'s are Abelian particles, the equivalence relation reduces to

$$(a\otimes i)i'\sim i(a\otimes i'),\quad \forall\ i\in\mathcal{C},\quad i'\in\mathcal{C}',\quad a\in\mathcal{E}. \tag{21}$$

Mathematically, this amounts to considering only the free local $L_{\mathcal{E}}$-modules. The equivalent classes $[ii']$ are then some composite anyons in $\mathcal{D}=\mathcal{C}\boxtimes_{\mathcal{E}}\mathcal{C}'$

$$[ii']=k\oplus l\oplus\cdots,\quad \text{for some } k,l,\dots\in\mathcal{D}. \tag{22}$$

In other words, they form a fusion sub ring of $\mathcal{D}$. Moreover, the spin of $ii'$ is the same as the direct summands

$$s_{ii'}=s_k^{\mathcal{D}}=s_l^{\mathcal{D}}=\cdots. \tag{23}$$

Since it is limited to a subset of data of UMTC$_{/\mathcal{E}}$'s, we can only give these necessary conditions. However, as we already give a large list of GQLs in terms of these data, they are usually enough to pick the resulting $\mathcal{C}\boxtimes_{\mathcal{E}}\mathcal{C}'$ from the list.

## VII. HOW TO CALCULATE THE MODULAR EXTENSION OF A UMTC$_{/\mathcal{E}}$

### A. A naive calculation

How do we calculate the modular extension $\mathcal{M}$ of UMTC$_{/\mathcal{E}}$ $\mathcal{C}$ from the data of $\mathcal{C}$? Actually, we do not know how to do that. So here, we will follow a closely related conjecture V, and calculate instead $(\mathcal{N}_K^{IJ},\mathcal{S}_I,c)$ (that fully characterize $\mathcal{M}$) from the data $(\tilde{N}_c^{ab},\tilde{s}_a;N_k^{ij},s_i)$ (that partially characterize $\mathcal{C}$). In this section, we will describe such a calculation.

We note that all the simple objects (particles) in $\mathcal{C}$ are contained in $\mathcal{M}$ as simple objects, and $\mathcal{M}$ may contain some extra simple objects. Assume that the particle labels of $\mathcal{M}$ are $\{I,J,\dots\}=\{i,j,\dots,x,y,\dots\}$, where we use $i,j,\dots$ to label the particles in $\mathcal{C}$ and $x,y,\dots$ to label the additional particles (not in $\mathcal{C}$). Also let us use $a,b,\dots$ to label the simple objects in the centralizer of $\mathcal{C}$: $\mathcal{E}=\mathcal{C}_{\mathcal{C}}^{\text{cen}}$. Let $\mathcal{N}_K^{IJ}$, $\mathcal{S}_I$ be the fusion coefficients and the spins for $\mathcal{M}$, and $N_k^{ij}$, $s_i$ be the fusion coefficients and the spins for $\mathcal{C}$. The idea is to find as many conditions on $(\mathcal{N}_K^{IJ},\mathcal{S}_I)$ as possible, and use those conditions to solve for $(\mathcal{N}_K^{IJ},\mathcal{S}_I)$. Since the data $(\mathcal{N}_K^{IJ},\mathcal{S}_I)$ describe the UMTC $\mathcal{M}$, they should satisfy all the conditions discussed in Ref. [11]. On the other hand, as a modular extension of $\mathcal{C}$, $(\mathcal{N}_K^{IJ},\mathcal{S}_I)$ also satisfy some additional conditions. Here, we will discuss those additional conditions.

First, the modular extension $\mathcal{M}$ has a fixed total quantum dimension:

$$\dim(\mathcal{M})=\dim(\mathcal{E})\dim(\mathcal{C}). \tag{24}$$

In other words,

$$\sum_{I\in\mathcal{M}}d_I^2=\sum_{a\in\mathcal{E}}d_a^2\sum_{i\in\mathcal{C}}d_i^2. \tag{25}$$

Physically, the modular extension $\mathcal{M}$ is obtained by "gauging" the symmetry $\mathcal{E}$ in $\mathcal{C}$ (i.e., adding the symmetry twists of $\mathcal{E}$). So the additional particles $x,y,\dots$ correspond to the symmetry twists. Fusing an original particle $i\in\mathcal{C}$ to a symmetry twist $x\notin\mathcal{C}$ still give us a symmetry twist. Thus

$$\mathcal{N}_j^{ix}=\mathcal{N}_j^{xi}=\mathcal{N}_x^{ij}=0. \tag{26}$$

Therefore $\mathcal{N}_i$ for $i\in\mathcal{C}$ is block diagonal:

$$\mathcal{N}_i=N_i\oplus\hat{N}_i, \tag{27}$$

where $(N_i)_{jk}=\mathcal{N}_k^{ij}=N_k^{ij}$ and $(\hat{N}_i)_{xy}=\mathcal{N}_x^{iy}$.

If we pick a charge conjugation for the additional particles $x\mapsto\bar{x}$, the conditions for fusion rules reduce to

$$\mathcal{N}_y^{ix}=\mathcal{N}_y^{xi}=\mathcal{N}_i^{\bar{x}y}=\mathcal{N}_{\bar{x}}^{i\bar{y}},$$
$$\sum_{k\in\mathcal{C}}N_k^{ij}\mathcal{N}_y^{kx}=\sum_{z\notin\mathcal{C}}\mathcal{N}_x^{iz}\mathcal{N}_z^{jy}. \tag{28}$$

With a choice of charge conjugation, it is enough to construct (or search for) the matrices $\hat{N}_i$ and $\mathcal{N}_z^{xy}$ to determine all the extended fusion rules $\mathcal{N}_K^{IJ}$.

Besides the general condition (28), there are also some simple constraints on $\hat{N}_i$ that may speed up the numerical search. Firstly, observe that (28) is the same as

$$\hat{N}_i\hat{N}_j=\sum_{k\in\mathcal{C}}N_k^{ij}\hat{N}_k, \tag{29}$$

where $i,j,k\in\mathcal{F}$. This means that $\hat{N}_i$ satisfy the same fusion algebra as $N_i$, and $N_k^{ij}=\mathcal{N}_k^{ij}$ is the structure constant; therefore the eigenvalues of $\hat{N}_i$ must be a subset of the eigenvalues of $N_i$.

Secondly, since $\sum_{y\notin\mathcal{C}}\mathcal{N}_y^{ix}d_y=d_id_x$, by Perron-Frobenius theorem, we know that $d_i$ is the largest eigenvalue of $\hat{N}_i$, with eigenvector $v,v_x=d_x$. ($d_i$ is also the largest absolute values of the eigenvalues of $\hat{N}_i$.) Note that $\hat{N}_{\bar{i}}\hat{N}_i=\hat{N}_i\hat{N}_{\bar{i}}$, $\hat{N}_{\bar{i}}=\hat{N}_i^\dagger$. Thus $d_i^2$ is the largest eigenvalue of the positive semidefinite Hermitian matrix $\hat{N}_i^\dagger\hat{N}_i$. For any unit vector $v$, we have $v^\dagger\hat{N}_i^\dagger\hat{N}_iv\leqslant d_i^2$, in particular,

$$\left(\hat{N}_i^\dagger\hat{N}_i\right)_{xx}=\sum_y\left(\mathcal{N}_y^{ix}\right)^2\leqslant d_i^2. \tag{30}$$

The above result is very helpful to reduce the scope of numerical search.

Once we find the fusion rules, $\mathcal{N}_K^{IJ}$, we can then use the rational conditions and other conditions to determine the spins $\mathcal{S}_I$ (for details, see Ref. [11]). The set of data $(\mathcal{N}_K^{IJ},\mathcal{S}_I)$ that satisfy all the conditions give us the set of modular extensions.

The above proposed calculation for modular extensions is quite expensive. If the quantum dimensions of the particles in $\mathcal{C}$ are all equal to 1: $d_i=1$, then there is another much cheaper way to calculate the fusion coefficient $\mathcal{N}_K^{IJ}$ of the modular extension $\mathcal{M}$. Such an approach is explained in Appendix B. We will also use such an approach in our calculation.

Last, we would like to mention that two sets of data $(\mathcal{N}_K^{IJ},\mathcal{S}_I)$ and $(\bar{\mathcal{N}}_K^{IJ},\bar{\mathcal{S}}_I)$ describe the same modular extension of $\mathcal{C}$, if they only differ by a permutation of indices $x\in\mathcal{M}$ but $x\notin\mathcal{C}$. So some times, two sets of data $(\mathcal{N}_K^{IJ},\mathcal{S}_I)$ and $(\bar{\mathcal{N}}_K^{IJ},\bar{\mathcal{S}}_I)$ can describe different modular extensions, even through they describe the same UMTC. [Two sets of data $(\mathcal{N}^{IJ}_K,\mathcal{S}_I)$ and $(\bar{\mathcal{N}}^{IJ}_K,\bar{\mathcal{S}}_I)$ describe the same UMTC, if they are only different by a permutation of indices $I \in \mathcal{M}$.]

Why we use such a permutation in the calculation of modular extensions? This is because when we considering modular extensions, the particle $x \in \mathcal{M}$ but $x \notin \mathcal{C}$ correspond to symmetry twists. They are extrinsic excitations that do not appear in the finite energy spectrum of the Hamiltonian. While the particle $i \in \mathcal{C}$ are intrinsic excitations that do appear in the finite energy spectrum of the Hamiltonian. So $x \notin \mathcal{C}$ and $i \in \mathcal{C}$ are physically distinct and we do not allow permutations that mix them. Also we should not permute the particles $a \in \mathcal{E}$, because they correspond to symmetries. We should not mix, for example, the $Z_2$ symmetry of exchange layers and the $Z_2$ symmetry of 180° spin rotation.

### B. The limitations of the naive calculation

Since a UMTC$_{/\mathcal{E}}$ $\mathcal{C}$ is not modular, the data $(\tilde{N}^{ab}_c,\tilde{s}_a;N^{ij}_k,s_i)$ may not fully characterize $\mathcal{C}$. To fully characterize $\mathcal{C}$, we need to use additional data, such as the $F$ tensor and the $R$ tensor [9,11].

In this paper, we will not use those additional data. As a result, the data $(\tilde{N}^{ab}_c,\tilde{s}_a;N^{ij}_k,s_i)$ may correspond to several different UMTC$_{/\mathcal{E}}$ $\mathcal{C}$'s. In other words, $(\tilde{N}^{ab}_c,\tilde{s}_a;N^{ij}_k,s_i)$ is a one-to-many labeling of UMTC$_{/\mathcal{E}}$'s.

So in our naive calculation, when we calculate the modular extensions of $(\tilde{N}^{ab}_c,\tilde{s}_a;N^{ij}_k,s_i)$, we may actually calculate the modular extension of several different $\mathcal{C}$'s that are described by the same data $(\tilde{N}^{ab}_c,\tilde{s}_a;N^{ij}_k,s_i)$. However, for UMTC$_{/\mathcal{E}}$'s that can be fully characterized by the data $(\tilde{N}^{ab}_c,\tilde{s}_a;N^{ij}_k,s_i)$, our calculation produce the modular extensions of a single $\mathcal{C}$. For example, the naive calculation can obtain the correct modular extensions of $\mathcal{C} = \text{Rep}(G)$ and $\mathcal{C} = \text{sRep}(G^f)$, when $G$ and $G^f$ are Abelian groups, or simple finite groups [34].

If the $(\tilde{N}^{ab}_c,\tilde{s}_a;N^{ij}_k,s_i)$ happen to describe two different UMTC$_{/\mathcal{E}}$'s, we find that our naive calculation will produce the modular extensions for both of UMTC$_{/\mathcal{E}}$'s (see Sec. VIII D). So by computing the modular extensions of $(\tilde{N}^{ab}_c,\tilde{s}_a;N^{ij}_k,s_i)$, we can tell if $(\tilde{N}^{ab}_c,\tilde{s}_a;N^{ij}_k,s_i)$ corresponds to none, one, two, etc UMTC$_{/\mathcal{E}}$'s. This leads to conjecture V that $(\tilde{N}^{ab}_c,\tilde{s}_a;N^{ij}_k,s_i,\mathcal{N}^{IJ}_K,\mathcal{S}_I;c)$ can fully and one-to-one classify GQLs in 2+1D.

## VIII. EXAMPLES OF 2+1D SET ORDERS AND SPT ORDERS

In this section, we will discuss simple examples of UMTC$_{/\mathcal{E}}$ $\mathcal{C}$'s, and their modular extensions $\mathcal{M}$. The triple $(\mathcal{C},\mathcal{M},c)$ describe a topologically ordered or SPT phase. A single UMTC$_{/\mathcal{E}}$ $\mathcal{C}$ only describes the set of bulk topological excitations, which correspond to topologically ordered states up to invertible ones.

However, in this section we will not discuss examples of UMTC$_{/\mathcal{E}}$ $\mathcal{C}$. What we really do is to discuss examples of the solutions $(\tilde{N}^{ab}_c,\tilde{s}_a;N^{ij}_k,s_i)$ (which are not really UMTC$_{/\mathcal{E}}$'s, but closely related). We will also discuss the modular extensions $(\mathcal{N}^{IJ}_K,\mathcal{S}_I;c)$ of $(\tilde{N}^{ab}_c,\tilde{s}_a;N^{ij}_k,s_i)$. $(\tilde{N}^{ab}_c,\tilde{s}_a;N^{ij}_k,s_i)$ will correspond to UMTC$_{/\mathcal{E}}\mathcal{C}$ if it has modular extensions $(\mathcal{N}^{IJ}_K,\mathcal{S}_I;c)$. This allows us to classify GQLs in terms of the data $(\tilde{N}^{ab}_c,\tilde{s}_a;N^{ij}_k,s_i,\mathcal{N}^{IJ}_K,\mathcal{S}_I;c)$.

TABLE II. The bottom two rows correspond to the two modular extensions of Rep$(Z_2)$ (denoted by $N_c^{|\Theta|} = 2_0^{\zeta_2^1}$). Thus we have two different trivial topological orders with $Z_2$ symmetry in 2+1D (i.e., two $Z_2$ SPT states). We use $N_c^{|\Theta|}$ to label UMTC$_{/\mathcal{E}}$'s, where $\Theta = D^{-1}\sum_i e^{2\pi i s_i} d_i^2 = |\Theta| e^{2\pi i c/8}$ and $D^2 = \sum_i d_i^2$.

| $N_c^{\vert\Theta\vert}$ | $D^2$ | $d_1,d_2,\ldots$ | $s_1,s_2,\ldots$ | comment |
|---|---|---|---|---|
| $2_0^{\zeta_2^1}$ | 2 | 1,1 | 0,0 | Rep$(Z_2)$ |
| $4_0^B$ | 4 | 1,1,1,1 | $0,0,0,\frac{1}{2}$ | $Z_2$ gauge |
| $4_0^B$ | 4 | 1,1,1,1 | $0,0,\frac{1}{4},\frac{3}{4}$ | double semion |

### A. $Z_2$ bosonic SPT states

Tables XXII, XXIII, and XXIV list the solutions $(\tilde{N}^{ab}_c,\tilde{s}_a;N^{ij}_k,s_i)$ when $(\tilde{N}^{ab}_c,\tilde{s}_a)$ describes a SFC Rep$(Z_2)$. The table contains all UMTC$_{/\text{Rep}(Z_2)}$'s but may contain extra fake entries. Physically, they describe possible sets of bulk excitations for $Z_2$-SET orders of bosonic systems. The sets of bulk excitations are listed by their quantum dimensions $d_i$ and spins $s_i$.

For example, let us consider the entry $N_c^{|\Theta|} = 2_0^{\zeta_2^1}$ in Table XXII. Such an entry has a central charge $c = 0$. Also $N = 2$, hence the $Z_2$-SET state has two types of bulk excitations both with $d_i = 1$ and $s_i = 0$. Both types of excitations are local excitations; one is the trivial type and the other carries a $Z_2$ charge.

The first question that we like to ask is that "is such an entry a fake entry, or it corresponds to some $Z_2$-symmetric GQL's?" If it corresponds to some $Z_2$-symmetric GQL's, how many distinct $Z_2$-symmetric GQL phases that it corresponds to? In other word, how many distinct $Z_2$-symmetric GQL phases are there, that share the same set of bulk topological excitations described by the entry $2_0^{\zeta_2^1}$?

Both questions can be answered by computing the modular extensions of $2_0^{\zeta_2^1}$ [which is also denoted as Rep$(Z_2)$]. We find that the modular extensions exist, and thus Rep$(Z_2)$ does correspond to some $Z_2$-symmetric GQL's. In fact, one of the $Z_2$-symmetric GQL's is the trivial product state with $Z_2$ symmetry. Other $Z_2$-symmetric GQL's are $Z_2$ SPT states.

After a numerical calculation, we find that there are only two different modular extensions of Rep$(Z_2)$ (see Table II). Thus there are two distinct $Z_2$-symmetric GQL phases whose bulk excitations are described by the Rep$(Z_2)$. The first one corresponds to the trivial product states whose modular extension is the $Z_2$ gauge theory which has four types of particles with $(d_i,s_i) = (1,0),(1,0),(1,0),(1,\frac{1}{2})$. (Gauging the $Z_2$ symmetry of the trivial product state gives rise to a $Z_2$ gauge theory.) The second one corresponds to the only nontrivial $Z_2$ bosonic SPT state in 2+1D, whose modular extension is the double-semion theory which has four types of particles with $(d_i,s_i) = (1,0),(1,0),(1,\frac{1}{4}),(1,-\frac{1}{4})$. (Gauging the $Z_2$ symmetry of the $Z_2$-SPT state gives rise to a double-semion theory [21].) So the $Z_2$-SPT phases are classified by $\mathbb{Z}_2$, reproducing the group cohomology result [18–20]. In

TABLE III. The two modular extensions of $N_c^{|\Theta|} = 3_2^{\zeta_2^1}$. $3_2^{\zeta_2^1}$ has a centralizer Rep($Z_2$). Thus we have two topological orders with $Z_2$ symmetry in 2+1D, which has only one type of spin-1/3 topological excitations.

| $N_c^{\|\Theta\|}$ | $D^2$ | $d_1, d_2, \dots$ | $s_1, s_2, \dots$ | comment |
|---|---|---|---|---|
| $3_2^{\zeta_2^1}$ | 6 | 1,1,2 | $0,0,\frac{1}{3}$ | $K = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ |
| $5_2^B$ | 12 | $1,1,2,\zeta_4^1,\zeta_4^1 = \sqrt{3}$ | $0,0,\frac{1}{3},\frac{1}{8},\frac{5}{8}$ | $(A_1,4)$ |
| $5_2^B$ | 12 | $1,1,2,\zeta_4^1,\zeta_4^1$ | $0,0,\frac{1}{3},\frac{3}{8},\frac{7}{8}$ | |

general, the modular extensions of Rep($G$) correspond to the bosonic SPT states in 2+1D with symmetry $G$.

### B. $Z_2$-SET orders for bosonic systems

The entry $N_c^{|\Theta|} = 3_2^{\zeta_2^1}$ in Table XXII corresponds to more nontrivial UMTC$_{/\mathrm{Rep}(Z_2)}$. It describes the bulk excitations of $Z_2$-SET orders, which has only one type of nontrivial topological excitation (with quantum dimension $d = 2$ and spin $s = 1/3$, see Table IV). The other two types of excitations are local excitations with $Z_2$-charge 0 and 1. We find that $3_2^{\zeta_2^1}$ has modular extensions and hence is not a fake entry.

To see how many SET orders that have such set of bulk excitations, we need to compute how many modular extensions are there for $3_2^{\zeta_2^1}$. We find that $3_2^{\zeta_2^1}$ has two modular extensions (see Table III). Thus there are two $Z_2$-SET orders with the above mentioned bulk excitations. It is not an accident that the number of $Z_2$-SET orders with the same set of bulk excitations is the same as the number of $Z_2$ SPT states. This is because the different $Z_2$-SET orders with a fixed set of bulk excitations are generated by stacking with $Z_2$ SPT states.

We would like to point out that for any $G$-SET state, if we break the symmetry, the $G$-SET state will reduce to a topologically ordered state described by a UMTC. In fact, the different $G$-SET states described by the same UMTC$_{/\mathcal{E}}$ (i.e., with the same set of bulk excitations) will reduce to the same topologically ordered state (i.e., the same UMTC). In Appendix D, we discussed such a symmetry breaking process and how to compute UMTC from UMTC$_{/\mathcal{E}}$. We found that the two $Z_2$-SET orders from $3_2^{\zeta_2^1}$ reduce to an Abelian topological order described by a $K$-matrix $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$. This is indicated by SB:$K = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ in the comment column of Table XXII. In other place, we use SB:$N_c^B$ or SB:$N_c^F(^a_b)$ to indicate the reduced topological order after the symmetry breaking (for bosonic or fermionic cases). (The topological orders described by $N_c^B$ or $N_c^F(^a_b)$ are given by the tables in Ref. [11] or Ref. [14].)

As we have mentioned, there are two $Z_2$-SET orders with the same bulk excitations. But how to realize those $Z_2$-SET orders? We find that one of the $Z_2$-SET orders is the double layer FQH state with $K$ matrix $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ (same as the reduced topological order after symmetry breaking), where the $Z_2$ symmetry is the layer-exchange symmetry. The quasiparticles are labeled by the $l$ vectors $l = \begin{pmatrix} l_1 \\ l_2 \end{pmatrix}$. The two nontrivial quasiparticles are given by

$$l = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{31}$$

whose spins are all equal to $\frac{1}{3}$.

Since the layer-exchange $Z_2$ symmetry exchanges $l_1$ and $l_2$, we see that the two excitations $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ always have the same energy. Despite the $Z_2$ symmetry has no two-dimentional irreducible representations, the above spin-1/3 topological excitations has an exact twofold degeneracy due to the $Z_2$ layer-exchange symmetry. This effect is an interplay between the long-range entanglement and the symmetry: degeneracy in excitations may not always arise from high dimensional irreducible representations of the symmetry.

Such two degenerate excitations are viewed as one type of topological excitations with quantum dimension $d = 2$ (for the twofold degeneracy) and spin $s = \frac{1}{3}$ (see Table XXII). The $Z_2$ symmetry twist in such a double-layer state carry a non-Abelian statistics with quantum dimension $d = \sqrt{3}$. In fact, there are two such $Z_2$ symmetry twists whose spin differ by 1/2. The other $Z_2$-SET order can be viewed as the above

TABLE IV. The fusion rule of the $N_c^{|\Theta|} = 3_2^{\zeta_2^1}$ $Z_2$-SET order. The particle **1** carries the $Z_2$-charge 0, and the particle $s$ carries the $Z_2$-charge 1. From the table, we see that $\sigma \otimes \sigma = \mathbf{1} \oplus s \oplus \sigma$.

| | | | |
|---|---|---|---|
| $s_i$ | 0 | 0 | $\frac{1}{3}$ |
| $d_i$ | 1 | 1 | 2 |
| $3_2^{\zeta_2^1}$ | **1** | $s$ | $\sigma$ |
| **1** | **1** | $s$ | $\sigma$ |
| $s$ | $s$ | **1** | $\sigma$ |
| $\sigma$ | $\sigma$ | $\sigma$ | $\mathbf{1} \oplus s \oplus \sigma$ |

TABLE V. The fusion rules of the two $N_c^{|\Theta|} = 4_1^{\zeta_2^1}$ $Z_2$ symmetry enriched topological orders with identical $d_i$ and $s_i$. We see that one has a $Z_2 \times Z_2$ fusion rule and the other has a $Z_4$ fusion rule.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $s_i$ | 0 | 0 | $\frac{1}{4}$ | $\frac{1}{4}$ | $s_i$ | 0 | 0 | $\frac{1}{4}$ | $\frac{1}{4}$ |
| $d_i$ | 1 | 1 | 1 | 1 | $d_i$ | 1 | 1 | 1 | 1 |
| $4_1^{\zeta_2^1}$ | **00** | **01** | **10** | **11** | $4_1^{\zeta_2^1}$ | **0** | **2** | **1** | **3** |
| **00** | **00** | **01** | **10** | **11** | **0** | **0** | **2** | **1** | **3** |
| **01** | **01** | **00** | **11** | **10** | **2** | **2** | **0** | **3** | **1** |
| **10** | **10** | **11** | **00** | **01** | **1** | **1** | **3** | **2** | **0** |
| **11** | **11** | **10** | **01** | **00** | **3** | **3** | **1** | **0** | **2** |

TABLE VI. The four modular extensions of $N_c^{|\Theta|} = 5_0^{\zeta_2^1}$ with $Z_2 \times Z_2$ fusion. $5_0^{\zeta_2^1}$ has a centralizer Rep($Z_2$). The first and the second pairs turn out to be equivalent. The fusion rules are listed in Table VIII.

| $N_c^{\|\Theta\|}$ | $D^2$ | $d_1, d_2, \dots$ | $s_1, s_2, \dots$ | comment |
|---|---|---|---|---|
| $5_0^{\zeta_2^1}$ | 8 | $1 \times 4, 2$ | $0,0,\frac{1}{2},\frac{1}{2},0$ | |
| $9_0^B$ | 16 | $1 \times 4, 2, \zeta_2^1 \times 4$ | $0,0,\frac{1}{2},\frac{1}{2},0,\frac{15}{16},\frac{1}{16},\frac{7}{16},\frac{9}{16}$ | $3^B_{-1/2} \boxtimes 3^B_{1/2}$ |
| $9_0^B$ | 16 | $1 \times 4, 2, \zeta_2^1 \times 4$ | $0,0,\frac{1}{2},\frac{1}{2},0,\frac{3}{16},\frac{13}{16},\frac{11}{16},\frac{5}{16}$ | $3^B_{3/2} \boxtimes 3^B_{-3/2}$ |
| $9_0^B$ | 16 | $1 \times 4, 2, \zeta_2^1 \times 4$ | $0,0,\frac{1}{2},\frac{1}{2},0,\frac{1}{16},\frac{15}{16},\frac{9}{16},\frac{7}{16}$ | $3^B_{1/2} \boxtimes 3^B_{-1/2}$ |
| $9_0^B$ | 16 | $1 \times 4, 2, \zeta_2^1 \times 4$ | $0,0,\frac{1}{2},\frac{1}{2},0,\frac{13}{16},\frac{3}{16},\frac{5}{16},\frac{11}{16}$ | $3^B_{-3/2} \boxtimes 3^B_{3/2}$ |

TABLE VII. The four modular extensions of $N_c^{|\Theta|} = 5_1^{\zeta_2^1}$ with $Z_2 \times Z_2$ fusion. $5_1^{\zeta_2^1}$ has a centralizer Rep($Z_2$). The fusion rules are listed in Tables IX and X.

| $N_c^{\|\Theta\|}$ | $D^2$ | $d_1, d_2, \dots$ | $s_1, s_2, \dots$ | comment |
|---|---|---|---|---|
| $5_1^{\zeta_2^1}$ | 8 | $1 \times 4, 2$ | $0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{8}$ | |
| $9_1^B$ | 16 | $1 \times 4, 2, \zeta_2^1 \times 4$ | $0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{8}, \frac{1}{16}, \frac{1}{16}, \frac{9}{16}, \frac{9}{16}$ | $3_{1/2}^B \boxtimes 3_{1/2}^B$ |
| $9_1^B$ | 16 | $1 \times 4, 2, \zeta_2^1 \times 4$ | $0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{8}, \frac{13}{16}, \frac{13}{16}, \frac{5}{16}, \frac{5}{16}$ | $3_{-3/2}^B \boxtimes 3_{5/2}^B$ |
| $9_1^B$ | 16 | $1 \times 4, 2, \zeta_2^1 \times 4$ | $0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{8}, \frac{15}{16}, \frac{3}{16}, \frac{7}{16}, \frac{11}{16}$ | $3_{-1/2}^B \boxtimes 3_{3/2}^B$ |
| $9_1^B$ | 16 | $1 \times 4, 2, \zeta_2^1 \times 4$ | $0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{8}, \frac{3}{16}, \frac{15}{16}, \frac{11}{16}, \frac{7}{16}$ | $3_{3/2}^B \boxtimes 3_{-1/2}^B$ |

double layer FQH state $K = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ stacked with a $Z_2$ SPT state.

### C. Two other $Z_2$-SET orders for bosonic systems

The fourth and fifth entries in Table XXII describe the bulk excitations of two other $Z_2$-SET orders. Those bulk excitations have identical $s_i$ and $d_i$, but they have different fusion rules $N_k^{ij}$ (see Table V).

Both entries have two modular extensions, and correspond to two SET orders. Among the two SET orders for the $Z_2 \times Z_2$ fusion rule, one of them is obtained by stacking a $Z_2$ *neutral* $\nu = 1/2$ Laughlin state with a trivial $Z_2$ product state. The other is obtained by stacking a $Z_2$ neutral $\nu = 1/2$ Laughlin state with a nontrivial $Z_2$ SPT state.

The entry with $Z_4$ fusion rule also correspond to two SET orders. They are obtained by stacking a $Z_2$ *charged* $\nu = 1/2$ Laughlin state with a trivial or a nontrivial $Z_2$ SPT state. Here, *charged* means that the particles forming the $\nu = 1/2$ Laughlin state carry $Z_2$-charge 1. In this case, the anyon in the $\nu = 1/2$ Laughlin state carries a fractional $Z_2$-charge $1/2$. So the fusion of two such anyons give us a $Z_2$-charge 1 excitation instead of a trivial neutral excitation. This leads to the $Z_4$ fusion rule.

### D. The rank $N = 5$ $Z_2$-SET orders for bosonic systems

The first and the second entries in Table XXIII describe two $N = 5\text{UMTC}_{/\text{Rep}(Z_2)}$'s. They describe two different sets of bulk excitations for $Z_2$-SET orders. Those bulk excitations have identical $s_i$ and $d_i$, but they have different fusion rules $N_k^{ij}$: the 4 $d = 1$ particles have a $Z_2 \times Z_2$ fusion rule for the first entry, and they have a $Z_4$ fusion rule for the second entry (as indicated by F:$Z_2 \times Z_2$ or F:$Z_4$ in the comment column of Table XXIII).

#### 1. *The first entry in Table XXIII*

Let us compute the modular extensions of the first entry (i.e., $5_0^{\zeta_2^1}$ with $Z_2 \times Z_2$ fusion). Since the total quantum dimension of the modular extensions is $D^2 = 16$, the modular extensions must have rank $N = 13$ or less (since quantum dimension $d \geqslant 1$).

Now we would like to show $N = 13$ is not possible. If a modular extension has $N = 13$, then it must have 12 particles (labeled by $a = 1, \dots, 12$) with quantum dimension $d_a = 1$, and one particle (labeled by $x$) with quantum dimension $d_x = 2$, so that $12 \times 1^2 + 2^2 = D^2 = 16$. In this case, we must have the fusion rule

$$a \otimes x = x, \quad x \otimes x = 1 \oplus 2 \oplus 3 \oplus 4, \tag{32}$$

where $x \otimes x$ is determined by the fusion rule of the $\text{UMTC}_{/\text{Rep}(Z_2)}$. The above determines the fusion matrix $N_x$ defined as $(N_x)_{ij} \equiv N_j^{xi}$. The largest eigenvalue of $N_x$ should be 2, the quantum dimension of $x$. Indeed, we find that the

TABLE VIII. The first and the third entries in Table VI have different fusion rules, despite they have the same $(d_i, s_i)$.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $s_i$ | 0 | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 | $\frac{1}{16}$ | $\frac{7}{16}$ | $\frac{9}{16}$ | $\frac{15}{16}$ |
| $d_i$ | 1 | 1 | 1 | 1 | 2 | $\zeta_2^1$ | $\zeta_2^1$ | $\zeta_2^1$ | $\zeta_2^1$ |
| $9_0^B$ | **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** | **9** |
| **1** | **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** | **9** |
| **2** | **2** | **1** | **4** | **3** | **5** | **8** | **9** | **6** | **7** |
| **3** | **3** | **4** | **1** | **2** | **5** | **8** | **7** | **6** | **9** |
| **4** | **4** | **3** | **2** | **1** | **5** | **6** | **9** | **8** | **7** |
| **5** | **5** | **5** | **5** | **5** | **1 ⊕ 2 ⊕ 3 ⊕ 4** | **7 ⊕ 9** | **6 ⊕ 8** | **7 ⊕ 9** | **6 ⊕ 8** |
| **6** | **6** | **8** | **8** | **6** | **7 ⊕ 9** | **1 ⊕ 4** | **5** | **2 ⊕ 3** | **5** |
| **7** | **7** | **9** | **7** | **9** | **6 ⊕ 8** | **5** | **1 ⊕ 3** | **5** | **2 ⊕ 4** |
| **8** | **8** | **6** | **6** | **8** | **7 ⊕ 9** | **2 ⊕ 3** | **5** | **1 ⊕ 4** | **5** |
| **9** | **9** | **7** | **9** | **7** | **6 ⊕ 8** | **5** | **2 ⊕ 4** | **5** | **1 ⊕ 3** |
| $s_i$ | 0 | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 | $\frac{1}{16}$ | $\frac{7}{16}$ | $\frac{9}{16}$ | $\frac{15}{16}$ |
| $d_i$ | 1 | 1 | 1 | 1 | 2 | $\zeta_2^1$ | $\zeta_2^1$ | $\zeta_2^1$ | $\zeta_2^1$ |
| $9_0^B$ | **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** | **9** |
| **1** | **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** | **9** |
| **2** | **2** | **1** | **4** | **3** | **5** | **8** | **9** | **6** | **7** |
| **3** | **3** | **4** | **1** | **2** | **5** | **6** | **9** | **8** | **7** |
| **4** | **4** | **3** | **2** | **1** | **5** | **8** | **7** | **6** | **9** |
| **5** | **5** | **5** | **5** | **5** | **1 ⊕ 2 ⊕ 3 ⊕ 4** | **7 ⊕ 9** | **6 ⊕ 8** | **7 ⊕ 9** | **6 ⊕ 8** |
| **6** | **6** | **8** | **6** | **8** | **7 ⊕ 9** | **1 ⊕ 3** | **5** | **2 ⊕ 4** | **5** |
| **7** | **7** | **9** | **9** | **7** | **6 ⊕ 8** | **5** | **1 ⊕ 4** | **5** | **2 ⊕ 3** |
| **8** | **8** | **6** | **8** | **6** | **7 ⊕ 9** | **2 ⊕ 4** | **5** | **1 ⊕ 3** | **5** |
| **9** | **9** | **7** | **7** | **9** | **6 ⊕ 8** | **5** | **2 ⊕ 3** | **5** | **1 ⊕ 4** |

TABLE IX. The third and the fourth entries in Table VII have different fusion rules, despite they have the same $(d_i, s_i)$.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $s_i$ | 0 | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{8}$ | $\frac{3}{16}$ | $\frac{7}{16}$ | $\frac{11}{16}$ | $\frac{15}{16}$ |
| $d_i$ | 1 | 1 | 1 | 1 | 2 | $\zeta_2^1$ | $\zeta_2^1$ | $\zeta_2^1$ | $\zeta_2^1$ |
| $9_1^B$ | **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** | **9** |
| **1** | **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** | **9** |
| **2** | **2** | **1** | **4** | **3** | **5** | **8** | **9** | **6** | **7** |
| **3** | **3** | **4** | **1** | **2** | **5** | **8** | **7** | **6** | **9** |
| **4** | **4** | **3** | **2** | **1** | **5** | **6** | **9** | **8** | **7** |
| **5** | **5** | **5** | **5** | **5** | **1** ⊕ **2** ⊕ **3** ⊕ **4** | **7** ⊕ **9** | **6** ⊕ **8** | **7** ⊕ **9** | **6** ⊕ **8** |
| **6** | **6** | **8** | **8** | **6** | **7** ⊕ **9** | **1** ⊕ **4** | **5** | **2** ⊕ **3** | **5** |
| **7** | **7** | **9** | **7** | **9** | **6** ⊕ **8** | **5** | **1** ⊕ **3** | **5** | **2** ⊕ **4** |
| **8** | **8** | **6** | **6** | **8** | **7** ⊕ **9** | **2** ⊕ **3** | **5** | **1** ⊕ **4** | **5** |
| **9** | **9** | **7** | **9** | **7** | **6** ⊕ **8** | **5** | **2** ⊕ **4** | **5** | **1** ⊕ **3** |
| $s_i$ | 0 | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{8}$ | $\frac{3}{16}$ | $\frac{7}{16}$ | $\frac{11}{16}$ | $\frac{15}{16}$ |
| $d_i$ | 1 | 1 | 1 | 1 | 2 | $\zeta_2^1$ | $\zeta_2^1$ | $\zeta_2^1$ | $\zeta_2^1$ |
| $9_1^B$ | **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** | **9** |
| **1** | **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** | **9** |
| **2** | **2** | **1** | **4** | **3** | **5** | **8** | **9** | **6** | **7** |
| **3** | **3** | **4** | **1** | **2** | **5** | **6** | **9** | **8** | **7** |
| **4** | **4** | **3** | **2** | **1** | **5** | **8** | **7** | **6** | **9** |
| **5** | **5** | **5** | **5** | **5** | **1** ⊕ **2** ⊕ **3** ⊕ **4** | **7** ⊕ **9** | **6** ⊕ **8** | **7** ⊕ **9** | **6** ⊕ **8** |
| **6** | **6** | **8** | **6** | **8** | **7** ⊕ **9** | **1** ⊕ **3** | **5** | **2** ⊕ **4** | **5** |
| **7** | **7** | **9** | **9** | **7** | **6** ⊕ **8** | **5** | **1** ⊕ **4** | **5** | **2** ⊕ **3** |
| **8** | **8** | **6** | **8** | **6** | **7** ⊕ **9** | **2** ⊕ **4** | **5** | **1** ⊕ **3** | **5** |
| **9** | **9** | **7** | **7** | **9** | **6** ⊕ **8** | **5** | **2** ⊕ **3** | **5** | **1** ⊕ **4** |

largest eigenvalue of $N_x$ is 2. But we also require that $N_x$ can be diagonalized by a unitary matrix (which happens to be the $S$-matrix). $N_x$ fails such a test. So $N$ cannot be 13.

$N$ also cannot be 12. If $N = 12$, then the modular extension will have ten particles (labeled by $a = 1, \ldots, 10$) with quantum dimension $d_a = 1$, one particle (labeled by $x$) with quantum dimension $d_x = 2$, and one particle (labeled by $y$) with quantum dimension $d_y = \sqrt{2}$. The fusion of ten $d_a = 1$ particles is described by an Abelian group $Z_{10}$ or $Z_2 \times Z_5$. None of them contain $Z_2 \times Z_2$ as subgroup. Thus $N = 12$ is incompatible with the $Z_2 \times Z_2$ fusion of the first four $d_a = 1$ particles.

TABLE X. The fusion rules of the first and the second entries in Table VII.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $s_i$ | 0 | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{8}$ | $\frac{1}{16}$ | $\frac{1}{16}$ | $\frac{9}{16}$ | $\frac{9}{16}$ |
| $d_i$ | 1 | 1 | 1 | 1 | 2 | $\zeta_2^1$ | $\zeta_2^1$ | $\zeta_2^1$ | $\zeta_2^1$ |
| $9_1^B$ | **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** | **9** |
| **1** | **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** | **9** |
| **2** | **2** | **1** | **4** | **3** | **5** | **8** | **9** | **6** | **7** |
| **3** | **3** | **4** | **1** | **2** | **5** | **8** | **7** | **6** | **9** |
| **4** | **4** | **3** | **2** | **1** | **5** | **6** | **9** | **8** | **7** |
| **5** | **5** | **5** | **5** | **5** | **1** ⊕ **2** ⊕ **3** ⊕ **4** | **7** ⊕ **9** | **6** ⊕ **8** | **7** ⊕ **9** | **6** ⊕ **8** |
| **6** | **6** | **8** | **8** | **6** | **7** ⊕ **9** | **1** ⊕ **4** | **5** | **2** ⊕ **3** | **5** |
| **7** | **7** | **9** | **7** | **9** | **6** ⊕ **8** | **5** | **1** ⊕ **3** | **5** | **2** ⊕ **4** |
| **8** | **8** | **6** | **6** | **8** | **7** ⊕ **9** | **2** ⊕ **3** | **5** | **1** ⊕ **4** | **5** |
| **9** | **9** | **7** | **9** | **7** | **6** ⊕ **8** | **5** | **2** ⊕ **4** | **5** | **1** ⊕ **3** |
| $s_i$ | 0 | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{8}$ | $\frac{5}{16}$ | $\frac{5}{16}$ | $\frac{13}{16}$ | $\frac{13}{16}$ |
| $d_i$ | 1 | 1 | 1 | 1 | 2 | $\zeta_2^1$ | $\zeta_2^1$ | $\zeta_2^1$ | $\zeta_2^1$ |
| $9_1^B$ | **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** | **9** |
| **1** | **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** | **9** |
| **2** | **2** | **1** | **4** | **3** | **5** | **8** | **9** | **6** | **7** |
| **3** | **3** | **4** | **1** | **2** | **5** | **8** | **7** | **6** | **9** |
| **4** | **4** | **3** | **2** | **1** | **5** | **6** | **9** | **8** | **7** |
| **5** | **5** | **5** | **5** | **5** | **1** ⊕ **2** ⊕ **3** ⊕ **4** | **7** ⊕ **9** | **6** ⊕ **8** | **7** ⊕ **9** | **6** ⊕ **8** |
| **6** | **6** | **8** | **8** | **6** | **7** ⊕ **9** | **1** ⊕ **4** | **5** | **2** ⊕ **3** | **5** |
| **7** | **7** | **9** | **7** | **9** | **6** ⊕ **8** | **5** | **1** ⊕ **3** | **5** | **2** ⊕ **4** |
| **8** | **8** | **6** | **6** | **8** | **7** ⊕ **9** | **2** ⊕ **3** | **5** | **1** ⊕ **4** | **5** |
| **9** | **9** | **7** | **9** | **7** | **6** ⊕ **8** | **5** | **2** ⊕ **4** | **5** | **1** ⊕ **3** |

We searched the modular extensions with $N$ up to 11. We find four $N=9$ modular extensions (see Table VI), and thus the first entry corresponds to valid $Z_2$-SET states.

In fact, one of the $Z_2$-SET states is the $Z_2$ gauge theory with a $Z_2$ global symmetry, where the $Z_2$ symmetry action exchange the $Z_2$-charge $e$ and the $Z_2$-vortex $m$. The degenerate $e$ and $m$ give rise to the $(d,s)=(2,0)$ particle (the fifth particle in the table). The bound state of $e$ and $m$ is a fermion $f$. It may carry the $Z_2$-charge 0 or 1, which correspond to the third and the fourth particle with $(d,s)=(1,1/2)$ in the table.

However, from the discussion in the last few sections, we know that a $\text{UMTC}_{/\text{Rep}(Z_2)}$ always has two modular extensions, corresponding to the two bosonic $Z_2$-SPT states in 2+1D. This seems contradictory with the above result that the $Z_2$-SET state, $5_0^{\zeta_2^1}$ with $Z_2\times Z_2$ fusion, has four different modular extensions.

In fact, there is no contradiction. Here, we only use $(N_k^{ij},s_i)$ to label different entries. However, a $\text{UMTC}_{/\mathcal{E}}$ is fully characterized by $(N_k^{ij},s_i)$ plus the $F$ and the $R$ tensors. To see this point, we note that the Ising-like UMTC $N_c^B=3_{m/2}^B$, $m=1,3,\dots,15$ (with central charge $c=m/2$) has three particles: 1, $f$ with $(d_f,s_f)=(1,1/2)$, and $\sigma$ with $(d_\sigma,s_\sigma)=(\sqrt{2},m/16)$. Its $R$ tensor is given by [9]

$$R_1^{ff}=-1,\quad R_\sigma^{\sigma f}=R_\sigma^{f\sigma}=-\mathrm{i}^m, \tag{33}$$
$$R_1^{\sigma\sigma}=(-1)^{\frac{m^2-1}{8}}\mathrm{e}^{-\mathrm{i}\frac{\pi}{8}m},\quad R_f^{\sigma\sigma}=(-1)^{\frac{m^2-1}{8}}\mathrm{e}^{\mathrm{i}\frac{3\pi}{8}m},$$

and some components of the $F$ tensor are given by

$$F_{f;1}^{f\sigma\sigma;\sigma}=F_{f;1}^{\sigma\sigma f;\sigma}=1. \tag{34}$$

The values of $R_\sigma^{\sigma f}$ and $R_\sigma^{f\sigma}$ are not gauge invariant. However, if we fix the values of the $F$ tensor to be the ones given above, this will fix the gauge, and we can treat $R_\sigma^{\sigma f}$ and $R_\sigma^{f\sigma}$ as if they are gauge invariant quantities.

If we stack $N_c^B=3_{m/2}^B$ and $N_c^B=3_{m'/2}^B$ together, the induced UMTC $3_{m/2}^B\boxtimes 3_{m'/2}^B$ contains particles $\mathbf{1}=(1,1)$, $\mathbf{2}=(f,f')$, $\mathbf{3}=(f,1)$, $\mathbf{4}=(1,f')$, $\mathbf{5}=(\sigma,\sigma')$. Those five particles are closed under the fusion, and correspond to the five particles in $\text{UMTC}_{/\text{Rep}(Z_2)}5_{m+m'}^{\zeta_2^1}$. We note that some components of the $R$ tensor of $3_{m/2}^B\boxtimes 3_{m'/2}^B$ are given by

$$R_{(\sigma,\sigma')}^{(f,1),(\sigma,\sigma')}=R_{(\sigma,\sigma')}^{(\sigma,\sigma'),(f,1)}=-\mathrm{i}^m,$$
$$R_{(\sigma,\sigma')}^{(1,f'),(\sigma,\sigma')}=R_{(\sigma,\sigma')}^{(\sigma,\sigma'),(1,f')}=-\mathrm{i}^{m'}. \tag{35}$$

Taking $(m,m')=(-1,1)$ and $(1,-1)$, it is clear the $3_{-1/2}^B\boxtimes 3_{1/2}^B$ and $3_{1/2}^B\boxtimes 3_{-1/2}^B$ give rise to two different $R$ tensors that have identical $(N_k^{ij},s_i)$. So the first entry in Table XXIII (*i.e.* $5_0^{\zeta_2^1}$ with $Z_2\times Z_2$ fusion) split into two different entries if we include the $R$ tensors. Each give rise to two modular extensions, and this is why we got four modular extensions. In Table VI, the first two modular extensions have the same $(N_k^{ij},s_i)$, $F$ and $R$ tensors when restricted to the first five particles. The second pair of modular extensions also have the same $(N_k^{ij},s_i)$, $F$ and $R$ tensors when restricted to the first five particles, but their $R$ tensor is different from that of the first pair. However, note that under the exchange of the two fermions, the $R$ tensor of the first pair becomes that of the second pair.

We like to stress that Table VI is obtained using the ME-equivalence relation, i.e., the different entries are different under the ME-equivalence relation (see Sec. V). We see that for each fixed $\text{UMTC}_{/\text{Rep}(Z_2)}$ (i.e., for each fixed set of $(N_k^{ij},s_i)$, $F$ and $R$ tensors), there are two modular extensions, which agrees with our general result for modular extensions. However, if we ignore $F$ and $R$ tensors, then for each fixed set of $(N_k^{ij},s_i)$, we get four modular extensions. This is because $(N_k^{ij},s_i)$ is only a partial description of a $\text{UMTC}_{/\text{Rep}(Z_2)}$, and as discussed above, in this case there are two ways to assign $F$ and $R$

TABLE XI. The three modular extensions of $\text{Rep}(Z_3)$.

| $N_c^{\lvert\Theta\rvert}$ | $D^2$ | $d_1,d_2,\dots$ | $s_1,s_2,\dots$ | comment |
|---|---|---|---|---|
| $3_0^{\zeta_4^1}$ | 3 | 1,1,1 | 0,0,0 | $\text{Rep}(Z_3)$ |
| $9_0^B$ | 9 | $1\times 9$ | $0,0,0,0,0,\frac{1}{3},\frac{1}{3},\frac{2}{3},\frac{2}{3}$ | $Z_3$ gauge |
| $9_0^B$ | 9 | $1\times 9$ | $0,0,0,\frac{1}{9},\frac{1}{9},\frac{4}{9},\frac{4}{9},\frac{7}{9},\frac{7}{9}$ | |
| $9_0^B$ | 9 | $1\times 9$ | $0,0,0,\frac{2}{9},\frac{2}{9},\frac{5}{9},\frac{5}{9},\frac{8}{9},\frac{8}{9}$ | |

TABLE XII. The six modular extensions of $\text{Rep}(S_3)$.

| $N_c^{\lvert\Theta\rvert}$ | $D^2$ | $d_1,d_2,\dots$ | $s_1,s_2,\dots$ | comment |
|---|---|---|---|---|
| $3_0^{\sqrt{6}}$ | 6 | 1,1,2 | 0,0,0 | $\text{Rep}(S_3)$ |
| $8_0^B$ | 36 | 1,1,2,2,2,2,3,3 | $0,0,0,0,\frac{1}{3},\frac{2}{3},0,\frac{1}{2}$ | $S_3$ gauge |
| $8_0^B$ | 36 | 1,1,2,2,2,2,3,3 | $0,0,0,0,\frac{1}{3},\frac{2}{3},\frac{1}{4},\frac{3}{4}$ | |
| $8_0^B$ | 36 | 1,1,2,2,2,2,3,3 | $0,0,0,\frac{1}{9},\frac{4}{9},\frac{7}{9},0,\frac{1}{2}$ | $(B_4,2)$ |
| $8_0^B$ | 36 | 1,1,2,2,2,2,3,3 | $0,0,0,\frac{1}{9},\frac{4}{9},\frac{7}{9},\frac{1}{4},\frac{3}{4}$ | |
| $8_0^B$ | 36 | 1,1,2,2,2,2,3,3 | $0,0,0,\frac{2}{9},\frac{5}{9},\frac{8}{9},0,\frac{1}{2}$ | $(B_4,-2)$ |
| $8_0^B$ | 36 | 1,1,2,2,2,2,3,3 | $0,0,0,\frac{2}{9},\frac{5}{9},\frac{8}{9},\frac{1}{4},\frac{3}{4}$ | |

TABLE XIII. The 16 modular extensions of $\text{sRep}(Z_2^f)$.

| $N_c^{\lvert\Theta\rvert}$ | $D^2$ | $d_1,d_2,\dots$ | $s_1,s_2,\dots$ | comment |
|---|---|---|---|---|
| $2_0^0$ | 2 | 1,1 | $0,\frac{1}{2}$ | $\text{sRep}(Z_2^f)$ |
| $4_0^B$ | 4 | 1,1,1,1 | $0,\frac{1}{2},0,0$ | $Z_2$ gauge |
| $4_1^B$ | 4 | 1,1,1,1 | $0,\frac{1}{2},\frac{1}{8},\frac{1}{8}$ | F:$Z_4$ |
| $4_2^B$ | 4 | 1,1,1,1 | $0,\frac{1}{2},\frac{1}{4},\frac{1}{4}$ | F:$Z_2\times Z_2$ |
| $4_3^B$ | 4 | 1,1,1,1 | $0,\frac{1}{2},\frac{3}{8},\frac{3}{8}$ | F:$Z_4$ |
| $4_4^B$ | 4 | 1,1,1,1 | $0,\frac{1}{2},\frac{1}{2},\frac{1}{2}$ | F:$Z_2\times Z_2$ |
| $4_{-3}^B$ | 4 | 1,1,1,1 | $0,\frac{1}{2},\frac{5}{8},\frac{5}{8}$ | F:$Z_4$ |
| $4_{-2}^B$ | 4 | 1,1,1,1 | $0,\frac{1}{2},\frac{3}{4},\frac{3}{4}$ | F:$Z_2\times Z_2$ |
| $4_{-1}^B$ | 4 | 1,1,1,1 | $0,\frac{1}{2},\frac{7}{8},\frac{7}{8}$ | F:$Z_4$ |
| $3_{1/2}^B$ | 4 | $1,1,\zeta_2^1$ | $0,\frac{1}{2},\frac{1}{16}$ | $p+\mathrm{i}p$ SC |
| $3_{3/2}^B$ | 4 | $1,1,\zeta_2^1$ | $0,\frac{1}{2},\frac{3}{16}$ | |
| $3_{5/2}^B$ | 4 | $1,1,\zeta_2^1$ | $0,\frac{1}{2},\frac{5}{16}$ | |
| $3_{7/2}^B$ | 4 | $1,1,\zeta_2^1$ | $0,\frac{1}{2},\frac{7}{16}$ | |
| $3_{-7/2}^B$ | 4 | $1,1,\zeta_2^1$ | $0,\frac{1}{2},\frac{9}{16}$ | |
| $3_{-5/2}^B$ | 4 | $1,1,\zeta_2^1$ | $0,\frac{1}{2},\frac{11}{16}$ | |
| $3_{-3/2}^B$ | 4 | $1,1,\zeta_2^1$ | $0,\frac{1}{2},\frac{13}{16}$ | |
| $3_{-1/2}^B$ | 4 | $1,1,\zeta_2^1$ | $0,\frac{1}{2},\frac{15}{16}$ | |

TABLE XIV. The five modular extensions of Rep($Z_5$).

| $N_c^{\|\Theta\|}$ | $D^2$ | $d_1,d_2,\dots$ | $s_1,s_2,\dots$ | comment |
|---|---|---|---|---|
| $5_0^{\sqrt{5}}$ | 5 | $1\times 5$ | $0,0,0,0,0$ | |
| $25_0^B$ | 25 | $1\times 25$ | $0,0,0,0,0,0,0,0,0,\frac{1}{5},\frac{1}{5},\frac{1}{5},\frac{1}{5},\frac{2}{5},\frac{2}{5},\frac{2}{5},\frac{2}{5},\frac{3}{5},\frac{3}{5},\frac{3}{5},\frac{3}{5},\frac{4}{5},\frac{4}{5},\frac{4}{5},\frac{4}{5}$ | $5_0^B\boxtimes 5_0^B$ |
| $25_0^B$ | 25 | $1\times 25$ | $0,0,0,0,0,\frac{1}{25},\frac{1}{25},\frac{4}{25},\frac{4}{25},\frac{6}{25},\frac{6}{25},\frac{9}{25},\frac{9}{25},\frac{11}{25},\frac{11}{25},\frac{14}{25},\frac{14}{25},\frac{16}{25},\frac{16}{25},\frac{19}{25},\frac{19}{25},\frac{21}{25},\frac{21}{25},\frac{24}{25},\frac{24}{25}$ | |
| $25_0^B$ | 25 | $1\times 25$ | $0,0,0,0,0,\frac{1}{25},\frac{1}{25},\frac{4}{25},\frac{4}{25},\frac{6}{25},\frac{6}{25},\frac{9}{25},\frac{9}{25},\frac{11}{25},\frac{11}{25},\frac{14}{25},\frac{14}{25},\frac{16}{25},\frac{16}{25},\frac{19}{25},\frac{19}{25},\frac{21}{25},\frac{21}{25},\frac{24}{25},\frac{24}{25}$ | |
| $25_0^B$ | 25 | $1\times 25$ | $0,0,0,0,0,\frac{2}{25},\frac{2}{25},\frac{3}{25},\frac{3}{25},\frac{7}{25},\frac{7}{25},\frac{8}{25},\frac{8}{25},\frac{12}{25},\frac{12}{25},\frac{13}{25},\frac{13}{25},\frac{17}{25},\frac{17}{25},\frac{18}{25},\frac{18}{25},\frac{22}{25},\frac{22}{25},\frac{23}{25},\frac{23}{25}$ | |
| $25_0^B$ | 25 | $1\times 25$ | $0,0,0,0,0,\frac{2}{25},\frac{2}{25},\frac{3}{25},\frac{3}{25},\frac{7}{25},\frac{7}{25},\frac{8}{25},\frac{8}{25},\frac{12}{25},\frac{12}{25},\frac{13}{25},\frac{13}{25},\frac{17}{25},\frac{17}{25},\frac{18}{25},\frac{18}{25},\frac{22}{25},\frac{22}{25},\frac{23}{25},\frac{23}{25}$ | |

tensors to them. This is why each fixed $(N_k^{ij},s_i)$ has four modular extensions, while each fixed $(N_k^{ij},s_i,F,R)$ has only two modular extensions.

On the other hand, under the TO-equivalence relation (see Sec. V), the two ways to assign and $R$ tensors are actually equivalent (related by exchanging the two fermions), and the first entry in Table XXIII corresponds to only one $\text{UMTC}_{/\text{Rep}(Z_2)}$. Thus the first entry is equivalent to the third entry, and the second entry is equivalent to the fourth entry in Table VI. So the four entries of Table VI in fact represent only two distinct $Z_2$-SET orders.

One of the two $Z_2$-SET orders have been studied extensively. It corresponds to $Z_2$ gauge theory with a $\mathbb{Z}_2$ global symmetry that exchanges the $Z_2$-gauge-charge $e$ and the $Z_2$-gauge-vortex $m$ [26,27].

#### 2. The second entry in Table XXIII

Next, we compute the modular extensions of the second entry in Table XXIII (i.e., $5_0^{\zeta_2^1}$ with $Z_4$ fusion). Again, we can use the same argument to show that modular extensions of rank 12 and above do not exist. We searched the modular extensions with $N$ up to 11, and find that there is no modular extensions. So the second entry is not realizable and does not correspond to any valid bosonic $Z_2$-SET in 2+1D. This is indicated by NR in the comment column of Table XXIII.

Naively, the (none existing) state from the second entry is very similar to that from the first entry. It is also a $Z_2$ gauge theory with a $Z_2$ global symmetry that exchange $e$ and $m$. However, for the second entry, the $f$ particles (the third and the fourth particles) are assigned fraction $Z_2$-charge of $\pm 1/2$. This leads to the $Z_4$ fusion rule. Our result implies that such an assignment is not realizable (or is illegal). It turns out that all the $5_c^{\zeta_2^1}$'s with $Z_4$ fusion do not have modular extensions. They are not realizable, and do not correspond to any 2+1D bosonic $Z_2$-SET orders.

#### 3. The third entry in Table XXIII

Third, let us compute the modular extensions of the third entry in Table XXIII (i.e., $5_1^{\zeta_2^1}$ with $Z_2\times Z_2$ fusion). We find that the entry has four modular extensions. In fact, the entry corresponds to two different $\text{UMTC}_{/\text{Rep}(Z_2)}$s, each with two modular extensions, as implied by the two $Z_2$-SPT states. The two $\text{UMTC}_{/\text{Rep}(Z_2)}$s have identical $(N_k^{ij},s_i,c)$, but different $F$ and $R$ tensors. Sometimes two different $\text{UMTC}_{/\mathcal{E}}$'s (with different $F$ and the $R$ tensors) can have the same $(N_k^{ij},s_i)$'s. The third, seventh, etc., entries of Table XXIII provide such examples. We like to stress that this is different from the first entry in Table XXIII which corresponds to one $\text{UMTC}_{/\text{Rep}(Z_2)}$.

To see those different $F$ and $R$ tensors, we note that one of the two $5_1^{\zeta_2^1}$ with $Z_2\times Z_2$ fusion has modular extensions given by $3_{1/2}^B\boxtimes 3_{1/2}^B$ and $3_{-3/2}^B\boxtimes 3_{5/2}^B$. We find the $R$ tensor for this first $5_1^{\zeta_2^1}$ with $Z_2\times Z_2$ fusion is given by

$$\begin{aligned} R^{(f,1),(\sigma,\sigma')}_{(\sigma,\sigma')} &= R^{(\sigma,\sigma'),(f,1)}_{(\sigma,\sigma')} = -\mathrm{i},\\ R^{(1,f'),(\sigma,\sigma')}_{(\sigma,\sigma')} &= R^{(\sigma,\sigma'),(1,f')}_{(\sigma,\sigma')} = -\mathrm{i}. \end{aligned} \tag{36}$$

The second $5_1^{\zeta_2^1}$ with $Z_2\times Z_2$ fusion has modular extensions given by $3_{-1/2}^B\boxtimes 3_{3/2}^B$ and $3_{3/2}^B\boxtimes 3_{-1/2}^B$. We find the $R$ tensor for the second $5_1^{\zeta_2^1}$ with $Z_2\times Z_2$ fusion is given by

$$\begin{aligned} R^{(f,1),(\sigma,\sigma')}_{(\sigma,\sigma')} &= R^{(\sigma,\sigma'),(f,1)}_{(\sigma,\sigma')} = \mathrm{i},\\ R^{(1,f'),(\sigma,\sigma')}_{(\sigma,\sigma')} &= R^{(\sigma,\sigma'),(1,f')}_{(\sigma,\sigma')} = \mathrm{i}. \end{aligned} \tag{37}$$

TABLE XV. All the eight modular extensions of sRep($Z_4^f$).

| $N_c^{\|\Theta\|}$ | $D^2$ | $d_1,d_2,\dots$ | $s_1,s_2,\dots$ | comment |
|---|---|---|---|---|
| $4_0^0$ | 4 | 1,1,1,1 | $0,0,\frac{1}{2},\frac{1}{2}$ | sRep($Z_4^f$) |
| $16_0^B$ | 16 | 1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1 | $0,0,\frac{1}{2},\frac{1}{2},0,0,0,0,0,0,\frac{1}{4},\frac{1}{4},\frac{1}{2},\frac{1}{2},\frac{3}{4},\frac{3}{4}$ | |
| $16_1^B$ | 16 | 1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1 | $0,0,\frac{1}{2},\frac{1}{2},\frac{1}{32},\frac{1}{32},\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{9}{32},\frac{9}{32},\frac{17}{32},\frac{17}{32},\frac{25}{32},\frac{25}{32}$ | |
| $16_2^B$ | 16 | 1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1 | $0,0,\frac{1}{2},\frac{1}{2},\frac{1}{16},\frac{1}{16},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{5}{16},\frac{5}{16},\frac{9}{16},\frac{9}{16},\frac{13}{16},\frac{13}{16}$ | $8_1^B\boxtimes 2_1^B$ |
| $16_3^B$ | 16 | 1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1 | $0,0,\frac{1}{2},\frac{1}{2},\frac{3}{32},\frac{3}{32},\frac{11}{32},\frac{11}{32},\frac{3}{8},\frac{3}{8},\frac{3}{8},\frac{3}{8},\frac{19}{32},\frac{19}{32},\frac{27}{32},\frac{27}{32}$ | |
| $16_4^B$ | 16 | 1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1 | $0,0,\frac{1}{2},\frac{1}{2},\frac{1}{8},\frac{1}{8},\frac{3}{8},\frac{3}{8},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{5}{8},\frac{5}{8},\frac{7}{8},\frac{7}{8}$ | $4_3^B\boxtimes 4_1^B$ |
| $16_{-3}^B$ | 16 | 1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1 | $0,0,\frac{1}{2},\frac{1}{2},\frac{5}{32},\frac{5}{32},\frac{13}{32},\frac{13}{32},\frac{5}{8},\frac{5}{8},\frac{5}{8},\frac{5}{8},\frac{21}{32},\frac{21}{32},\frac{29}{32},\frac{29}{32}$ | |
| $16_{-2}^B$ | 16 | 1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1 | $0,0,\frac{1}{2},\frac{1}{2},\frac{3}{16},\frac{3}{16},\frac{7}{16},\frac{7}{16},\frac{11}{16},\frac{11}{16},\frac{3}{4},\frac{3}{4},\frac{3}{4},\frac{3}{4},\frac{15}{16},\frac{15}{16}$ | $8_{-1}^B\boxtimes 2_{-1}^B$ |
| $16_{-1}^B$ | 16 | 1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1 | $0,0,\frac{1}{2},\frac{1}{2},\frac{7}{32},\frac{7}{32},\frac{15}{32},\frac{15}{32},\frac{23}{32},\frac{23}{32},\frac{7}{8},\frac{7}{8},\frac{7}{8},\frac{7}{8},\frac{31}{32},\frac{31}{32}$ | |

TABLE XVI. The two $c=0$ modular extensions of sRep($Z_8^f$) imply that the $Z_8^f$ fermionic SPT phases are described by $\mathbb{Z}_2$. All other modular extensions only appear for integer $c$ and are all Abelian (two modular extensions for each integer $c$).

| $N_c^{\|\Theta\|}$ | $D^2$ | $d_1,d_2,\dots$ | $s_1,s_2,\dots$ | comment |
|---|---|---|---|---|
| $8_0^0$ | 8 | $1\times 8$ | $0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2}$ | |
| $64_0^B$ | 64 | $1\times 64$ | $0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{3}{8},\frac{3}{8},\frac{3}{8},\frac{3}{8},$ $\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{5}{8},\frac{5}{8},\frac{5}{8},\frac{5}{8},\frac{3}{4},\frac{3}{4},\frac{3}{4},\frac{3}{4},\frac{3}{4},\frac{3}{4},\frac{3}{4},\frac{3}{4},\frac{7}{8},\frac{7}{8},\frac{7}{8},\frac{7}{8}$ | |
| $64_0^B$ | 64 | $1\times 64$ | $0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},0,0,0,0,0,0,0,0,0,0,0,0,\frac{1}{16},\frac{1}{16},\frac{1}{16},\frac{1}{16},\frac{3}{16},\frac{3}{16},\frac{3}{16},\frac{3}{16},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{5}{16},\frac{5}{16},\frac{5}{16},\frac{5}{16},$ $\frac{7}{16},\frac{7}{16},\frac{7}{16},\frac{7}{16},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{9}{16},\frac{9}{16},\frac{9}{16},\frac{9}{16},\frac{11}{16},\frac{11}{16},\frac{11}{16},\frac{11}{16},\frac{3}{4},\frac{3}{4},\frac{3}{4},\frac{3}{4},\frac{13}{16},\frac{13}{16},\frac{13}{16},\frac{13}{16},\frac{15}{16},\frac{15}{16},\frac{15}{16},\frac{15}{16}$ | |

We see that the two $5_1^{\zeta_2^1}$'s with $Z_2\times Z_2$ fusion are really different UMTC$_{/\mathrm{Rep}(Z_2)}$. Each $5_1^{\zeta_2^1}$ has two modular extensions, and that is why we have four entries in Table VII.

Again, Table VII is obtained using the ME-equivalence relation, and is not a table of GQLs. Under the TO-equivalence relation, the third entry is equivalent to the fourth entry of Table VII. So the four entries in Table VII actually describe

TABLE XVII. All the 32 modular extensions of sRep($Z_2\times Z_2^f$) with $N=9$.

| $N_c^{\|\Theta\|}$ | $D^2$ | $d_1,d_2,\dots$ | $s_1,s_2,\dots$ | comment |
|---|---|---|---|---|
| $4_0^0$ | 4 | 1,1,1,1 | $0,0,\frac{1}{2},\frac{1}{2}$ | sRep($Z_2\times Z_2^f$) |
| $9_0^B$ | 16 | $1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1,2$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{1}{16},\frac{7}{16},\frac{9}{16},\frac{15}{16},0$ | $3_{-1/2}^B\boxtimes 3_{1/2}^B$ |
| $9_0^B$ | 16 | $1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1,2$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{1}{16},\frac{7}{16},\frac{9}{16},\frac{15}{16},0$ | $3_{-1/2}^B\boxtimes 3_{1/2}^B$ |
| $9_0^B$ | 16 | $1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1,2$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{3}{16},\frac{5}{16},\frac{11}{16},\frac{13}{16},0$ | $3_{-3/2}^B\boxtimes 3_{3/2}^B$ |
| $9_0^B$ | 16 | $1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1,2$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{3}{16},\frac{5}{16},\frac{11}{16},\frac{13}{16},0$ | $3_{-3/2}^B\boxtimes 3_{3/2}^B$ |
| $9_1^B$ | 16 | $1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1,2$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{1}{16},\frac{1}{16},\frac{9}{16},\frac{9}{16},\frac{1}{8}$ | $3_{1/2}^B\boxtimes 3_{1/2}^B$ |
| $9_1^B$ | 16 | $1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1,2$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{3}{16},\frac{7}{16},\frac{11}{16},\frac{15}{16},\frac{1}{8}$ | $3_{-1/2}^B\boxtimes 3_{3/2}^B$ |
| $9_1^B$ | 16 | $1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1,2$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{3}{16},\frac{7}{16},\frac{11}{16},\frac{15}{16},\frac{1}{8}$ | $3_{-1/2}^B\boxtimes 3_{3/2}^B$ |
| $9_1^B$ | 16 | $1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1,2$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{5}{16},\frac{5}{16},\frac{13}{16},\frac{13}{16},\frac{1}{8}$ | $3_{-3/2}^B\boxtimes 3_{5/2}^B$ |
| $9_2^B$ | 16 | $1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1,2$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{1}{16},\frac{3}{16},\frac{9}{16},\frac{11}{16},\frac{1}{4}$ | $3_{3/2}^B\boxtimes 3_{1/2}^B$ |
| $9_2^B$ | 16 | $1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1,2$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{1}{16},\frac{3}{16},\frac{9}{16},\frac{11}{16},\frac{1}{4}$ | $3_{3/2}^B\boxtimes 3_{1/2}^B$ |
| $9_2^B$ | 16 | $1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1,2$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{5}{16},\frac{7}{16},\frac{13}{16},\frac{15}{16},\frac{1}{4}$ | $3_{-1/2}^B\boxtimes 3_{5/2}^B$ |
| $9_2^B$ | 16 | $1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1,2$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{5}{16},\frac{7}{16},\frac{13}{16},\frac{15}{16},\frac{1}{4}$ | $3_{-1/2}^B\boxtimes 3_{5/2}^B$ |
| $9_3^B$ | 16 | $1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1,2$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{1}{16},\frac{5}{16},\frac{9}{16},\frac{13}{16},\frac{3}{8}$ | $3_{5/2}^B\boxtimes 3_{1/2}^B$ |
| $9_3^B$ | 16 | $1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1,2$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{1}{16},\frac{5}{16},\frac{9}{16},\frac{13}{16},\frac{3}{8}$ | $3_{5/2}^B\boxtimes 3_{1/2}^B$ |
| $9_3^B$ | 16 | $1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1,2$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{3}{16},\frac{3}{16},\frac{11}{16},\frac{11}{16},\frac{3}{8}$ | $3_{3/2}^B\boxtimes 3_{3/2}^B$ |
| $9_3^B$ | 16 | $1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1,2$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{7}{16},\frac{7}{16},\frac{15}{16},\frac{15}{16},\frac{3}{8}$ | $3_{-1/2}^B\boxtimes 3_{7/2}^B$ |
| $9_4^B$ | 16 | $1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1,2$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{1}{16},\frac{7}{16},\frac{9}{16},\frac{15}{16},\frac{1}{2}$ | $3_{7/2}^B\boxtimes 3_{1/2}^B$ |
| $9_4^B$ | 16 | $1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1,2$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{1}{16},\frac{7}{16},\frac{9}{16},\frac{15}{16},\frac{1}{2}$ | $3_{7/2}^B\boxtimes 3_{1/2}^B$ |
| $9_4^B$ | 16 | $1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1,2$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{3}{16},\frac{5}{16},\frac{11}{16},\frac{13}{16},\frac{1}{2}$ | $3_{5/2}^B\boxtimes 3_{3/2}^B$ |
| $9_4^B$ | 16 | $1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1,2$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{3}{16},\frac{5}{16},\frac{11}{16},\frac{13}{16},\frac{1}{2}$ | $3_{5/2}^B\boxtimes 3_{3/2}^B$ |
| $9_{-3}^B$ | 16 | $1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1,2$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{1}{16},\frac{1}{16},\frac{9}{16},\frac{9}{16},\frac{5}{8}$ | $3_{-7/2}^B\boxtimes 3_{1/2}^B$ |
| $9_{-3}^B$ | 16 | $1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1,2$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{3}{16},\frac{7}{16},\frac{11}{16},\frac{15}{16},\frac{5}{8}$ | $3_{7/2}^B\boxtimes 3_{3/2}^B$ |
| $9_{-3}^B$ | 16 | $1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1,2$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{3}{16},\frac{7}{16},\frac{11}{16},\frac{15}{16},\frac{5}{8}$ | $3_{7/2}^B\boxtimes 3_{3/2}^B$ |
| $9_{-3}^B$ | 16 | $1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1,2$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{5}{16},\frac{5}{16},\frac{13}{16},\frac{13}{16},\frac{5}{8}$ | $3_{5/2}^B\boxtimes 3_{5/2}^B$ |
| $9_{-2}^B$ | 16 | $1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1,2$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{1}{16},\frac{3}{16},\frac{9}{16},\frac{11}{16},\frac{3}{4}$ | $3_{-5/2}^B\boxtimes 3_{1/2}^B$ |
| $9_{-2}^B$ | 16 | $1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1,2$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{1}{16},\frac{3}{16},\frac{9}{16},\frac{11}{16},\frac{3}{4}$ | $3_{-5/2}^B\boxtimes 3_{1/2}^B$ |
| $9_{-2}^B$ | 16 | $1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1,2$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{5}{16},\frac{7}{16},\frac{13}{16},\frac{15}{16},\frac{3}{4}$ | $3_{7/2}^B\boxtimes 3_{5/2}^B$ |
| $9_{-2}^B$ | 16 | $1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1,2$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{5}{16},\frac{7}{16},\frac{13}{16},\frac{15}{16},\frac{3}{4}$ | $3_{7/2}^B\boxtimes 3_{5/2}^B$ |
| $9_{-1}^B$ | 16 | $1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1,2$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{1}{16},\frac{5}{16},\frac{9}{16},\frac{13}{16},\frac{7}{8}$ | $3_{-3/2}^B\boxtimes 3_{1/2}^B$ |
| $9_{-1}^B$ | 16 | $1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1,2$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{1}{16},\frac{5}{16},\frac{9}{16},\frac{13}{16},\frac{7}{8}$ | $3_{-3/2}^B\boxtimes 3_{1/2}^B$ |
| $9_{-1}^B$ | 16 | $1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1,2$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{3}{16},\frac{3}{16},\frac{11}{16},\frac{11}{16},\frac{7}{8}$ | $3_{-5/2}^B\boxtimes 3_{3/2}^B$ |
| $9_{-1}^B$ | 16 | $1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1,2$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{7}{16},\frac{7}{16},\frac{15}{16},\frac{15}{16},\frac{7}{8}$ | $3_{7/2}^B\boxtimes 3_{7/2}^B$ |

TABLE XVIII. The first 32 modular extensions of sRep($Z_2 \times Z_2^f$) with $N = 12$.

| $N_c^{\|\Theta\|}$ | $D^2$ | $d_1,d_2,\ldots$ | $s_1,s_2,\ldots$ | comment |
|---|---|---|---|---|
| $4_0^0$ | 4 | 1,1,1,1 | $0,0,\frac{1}{2},\frac{1}{2}$ | sRep($Z_2 \times Z_2^f$) |
| $12_{1/2}^B$ | 16 | $1,1,1,1,1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1$ | $0,0,\frac{1}{2},\frac{1}{2},0,0,\frac{1}{2},\frac{1}{2},\frac{1}{16},\frac{1}{16},\frac{1}{16},\frac{9}{16}$ | $4_0^B \boxtimes 3_{1/2}^B$ |
| $12_{1/2}^B$ | 16 | $1,1,1,1,1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1$ | $0,0,\frac{1}{2},\frac{1}{2},0,0,\frac{1}{2},\frac{1}{2},\frac{1}{16},\frac{1}{16},\frac{1}{16},\frac{9}{16}$ | $4_0^B \boxtimes 3_{1/2}^B$ |
| $12_{1/2}^B$ | 16 | $1,1,1,1,1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{1}{8},\frac{1}{8},\frac{5}{8},\frac{5}{8},\frac{1}{16},\frac{1}{16},\frac{7}{16},\frac{15}{16}$ | $4_{-3}^B \boxtimes 3_{7/2}^B$ |
| $12_{1/2}^B$ | 16 | $1,1,1,1,1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{1}{8},\frac{1}{8},\frac{5}{8},\frac{5}{8},\frac{1}{16},\frac{1}{16},\frac{7}{16},\frac{15}{16}$ | $4_{-3}^B \boxtimes 3_{7/2}^B$ |
| $12_{1/2}^B$ | 16 | $1,1,1,1,1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{1}{4},\frac{1}{4},\frac{3}{4},\frac{3}{4},\frac{1}{16},\frac{1}{16},\frac{5}{16},\frac{13}{16}$ | $6_{-1/2}^B \boxtimes 2_1^B$ |
| $12_{1/2}^B$ | 16 | $1,1,1,1,1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{1}{4},\frac{1}{4},\frac{3}{4},\frac{3}{4},\frac{1}{16},\frac{1}{16},\frac{5}{16},\frac{13}{16}$ | $6_{-1/2}^B \boxtimes 2_1^B$ |
| $12_{1/2}^B$ | 16 | $1,1,1,1,1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{3}{8},\frac{3}{8},\frac{7}{8},\frac{7}{8},\frac{1}{16},\frac{1}{16},\frac{3}{16},\frac{11}{16}$ | $4_{-1}^B \boxtimes 3_{3/2}^B$ |
| $12_{1/2}^B$ | 16 | $1,1,1,1,1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{3}{8},\frac{3}{8},\frac{7}{8},\frac{7}{8},\frac{1}{16},\frac{1}{16},\frac{3}{16},\frac{11}{16}$ | $4_{-1}^B \boxtimes 3_{3/2}^B$ |
| $12_{3/2}^B$ | 16 | $1,1,1,1,1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1$ | $0,0,\frac{1}{2},\frac{1}{2},0,0,\frac{1}{2},\frac{1}{2},\frac{3}{16},\frac{3}{16},\frac{3}{16},\frac{11}{16}$ | $4_0^B \boxtimes 3_{3/2}^B$ |
| $12_{3/2}^B$ | 16 | $1,1,1,1,1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1$ | $0,0,\frac{1}{2},\frac{1}{2},0,0,\frac{1}{2},\frac{1}{2},\frac{3}{16},\frac{3}{16},\frac{3}{16},\frac{11}{16}$ | $4_0^B \boxtimes 3_{3/2}^B$ |
| $12_{3/2}^B$ | 16 | $1,1,1,1,1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{1}{8},\frac{1}{8},\frac{5}{8},\frac{5}{8},\frac{1}{16},\frac{3}{16},\frac{3}{16},\frac{9}{16}$ | $4_1^B \boxtimes 3_{1/2}^B$ |
| $12_{3/2}^B$ | 16 | $1,1,1,1,1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{1}{8},\frac{1}{8},\frac{5}{8},\frac{5}{8},\frac{1}{16},\frac{3}{16},\frac{3}{16},\frac{9}{16}$ | $4_1^B \boxtimes 3_{1/2}^B$ |
| $12_{3/2}^B$ | 16 | $1,1,1,1,1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{1}{4},\frac{1}{4},\frac{3}{4},\frac{3}{4},\frac{3}{16},\frac{3}{16},\frac{7}{16},\frac{15}{16}$ | $6_{1/2}^B \boxtimes 2_1^B$ |
| $12_{3/2}^B$ | 16 | $1,1,1,1,1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{1}{4},\frac{1}{4},\frac{3}{4},\frac{3}{4},\frac{3}{16},\frac{3}{16},\frac{7}{16},\frac{15}{16}$ | $6_{1/2}^B \boxtimes 2_1^B$ |
| $12_{3/2}^B$ | 16 | $1,1,1,1,1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{3}{8},\frac{3}{8},\frac{7}{8},\frac{7}{8},\frac{3}{16},\frac{3}{16},\frac{5}{16},\frac{13}{16}$ | $4_{-1}^B \boxtimes 3_{5/2}^B$ |
| $12_{3/2}^B$ | 16 | $1,1,1,1,1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{3}{8},\frac{3}{8},\frac{7}{8},\frac{7}{8},\frac{3}{16},\frac{3}{16},\frac{5}{16},\frac{13}{16}$ | $4_{-1}^B \boxtimes 3_{5/2}^B$ |
| $12_{5/2}^B$ | 16 | $1,1,1,1,1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1$ | $0,0,\frac{1}{2},\frac{1}{2},0,0,\frac{1}{2},\frac{1}{2},\frac{5}{16},\frac{5}{16},\frac{5}{16},\frac{13}{16}$ | $4_0^B \boxtimes 3_{5/2}^B$ |
| $12_{5/2}^B$ | 16 | $1,1,1,1,1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1$ | $0,0,\frac{1}{2},\frac{1}{2},0,0,\frac{1}{2},\frac{1}{2},\frac{5}{16},\frac{5}{16},\frac{5}{16},\frac{13}{16}$ | $4_0^B \boxtimes 3_{5/2}^B$ |
| $12_{5/2}^B$ | 16 | $1,1,1,1,1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{1}{8},\frac{1}{8},\frac{5}{8},\frac{5}{8},\frac{3}{16},\frac{5}{16},\frac{5}{16},\frac{11}{16}$ | $4_1^B \boxtimes 3_{3/2}^B$ |
| $12_{5/2}^B$ | 16 | $1,1,1,1,1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{1}{8},\frac{1}{8},\frac{5}{8},\frac{5}{8},\frac{3}{16},\frac{5}{16},\frac{5}{16},\frac{11}{16}$ | $4_1^B \boxtimes 3_{3/2}^B$ |
| $12_{5/2}^B$ | 16 | $1,1,1,1,1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{1}{4},\frac{1}{4},\frac{3}{4},\frac{3}{4},\frac{1}{16},\frac{5}{16},\frac{5}{16},\frac{9}{16}$ | $6_{3/2}^B \boxtimes 2_1^B$ |
| $12_{5/2}^B$ | 16 | $1,1,1,1,1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{1}{4},\frac{1}{4},\frac{3}{4},\frac{3}{4},\frac{1}{16},\frac{5}{16},\frac{5}{16},\frac{9}{16}$ | $6_{3/2}^B \boxtimes 2_1^B$ |
| $12_{5/2}^B$ | 16 | $1,1,1,1,1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{3}{8},\frac{3}{8},\frac{7}{8},\frac{7}{8},\frac{5}{16},\frac{5}{16},\frac{7}{16},\frac{15}{16}$ | $4_{-1}^B \boxtimes 3_{7/2}^B$ |
| $12_{5/2}^B$ | 16 | $1,1,1,1,1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{3}{8},\frac{3}{8},\frac{7}{8},\frac{7}{8},\frac{5}{16},\frac{5}{16},\frac{7}{16},\frac{15}{16}$ | $4_{-1}^B \boxtimes 3_{7/2}^B$ |
| $12_{7/2}^B$ | 16 | $1,1,1,1,1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1$ | $0,0,\frac{1}{2},\frac{1}{2},0,0,\frac{1}{2},\frac{1}{2},\frac{7}{16},\frac{7}{16},\frac{7}{16},\frac{15}{16}$ | $4_0^B \boxtimes 3_{7/2}^B$ |
| $12_{7/2}^B$ | 16 | $1,1,1,1,1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1$ | $0,0,\frac{1}{2},\frac{1}{2},0,0,\frac{1}{2},\frac{1}{2},\frac{7}{16},\frac{7}{16},\frac{7}{16},\frac{15}{16}$ | $4_0^B \boxtimes 3_{7/2}^B$ |
| $12_{7/2}^B$ | 16 | $1,1,1,1,1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{1}{8},\frac{1}{8},\frac{5}{8},\frac{5}{8},\frac{5}{16},\frac{7}{16},\frac{7}{16},\frac{13}{16}$ | $4_1^B \boxtimes 3_{5/2}^B$ |
| $12_{7/2}^B$ | 16 | $1,1,1,1,1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{1}{8},\frac{1}{8},\frac{5}{8},\frac{5}{8},\frac{5}{16},\frac{7}{16},\frac{7}{16},\frac{13}{16}$ | $4_1^B \boxtimes 3_{5/2}^B$ |
| $12_{7/2}^B$ | 16 | $1,1,1,1,1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{1}{4},\frac{1}{4},\frac{3}{4},\frac{3}{4},\frac{3}{16},\frac{7}{16},\frac{7}{16},\frac{11}{16}$ | $6_{5/2}^B \boxtimes 2_1^B$ |
| $12_{7/2}^B$ | 16 | $1,1,1,1,1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{1}{4},\frac{1}{4},\frac{3}{4},\frac{3}{4},\frac{3}{16},\frac{7}{16},\frac{7}{16},\frac{11}{16}$ | $6_{5/2}^B \boxtimes 2_1^B$ |
| $12_{7/2}^B$ | 16 | $1,1,1,1,1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{3}{8},\frac{3}{8},\frac{7}{8},\frac{7}{8},\frac{1}{16},\frac{7}{16},\frac{7}{16},\frac{9}{16}$ | $4_3^B \boxtimes 3_{1/2}^B$ |
| $12_{7/2}^B$ | 16 | $1,1,1,1,1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{3}{8},\frac{3}{8},\frac{7}{8},\frac{7}{8},\frac{1}{16},\frac{7}{16},\frac{7}{16},\frac{9}{16}$ | $4_3^B \boxtimes 3_{1/2}^B$ |

*three* different $Z_2$-SET orders. This has a very interesting consequence. The $Z_2$-SET state described by the third (or fourth) entry in Table VII, after stacked with a $Z_2$-SPT state, still remains in the same phase. This is an example of the following general statement made previously. The GQLs with bulk excitations described by $\mathcal{C}$ are in one-to-one correspondence with the quotient $\mathcal{M}_{\text{ext}}(\mathcal{C})/\text{Aut}(\mathcal{C})$ plus a central charge $c$. In such an example Aut($\mathcal{C}$) is nontrivial.

It is worth noting here that for the second $5_1^{\zeta_2^1}$, two modular extensions $3_{-1/2}^B \boxtimes 3_{3/2}^B$ and $3_{3/2}^B \boxtimes 3_{-1/2}^B$ are actually equivalent UMTCs. This is an example that different embedings leads to different modular extensions. For $3_{-1/2}^B \boxtimes 3_{3/2}^B$, the first fermion in $5_1^{\zeta_2^1}$ is embedded into $3_{-1/2}^B$ and the second fermion is embedded into $3_{3/2}^B$, while for $3_{3/2}^B \boxtimes 3_{-1/2}^B$, the first fermion is embedded into $3_{3/2}^B$ and the second fermion is embedded into $3_{-1/2}^B$. The equivalence between $3_{-1/2}^B \boxtimes 3_{3/2}^B$ and $3_{3/2}^B \boxtimes 3_{-1/2}^B$ that exchanges both fermions and symmetry twists fails to relate the two embeddings, as they differ by a nontrivial automorphism of $5_1^{\zeta_2^1}$ that exchanges only the two fermions. This is an example that the Aut($\mathcal{C}$) action permutes the modular extensions, as discussed in Sec. IV.

### E. $Z_3$, $Z_5$, and $S_3$ SPT orders for bosonic systems

We also find that Rep($Z_3$) has three modular extensions (see Table XI), Rep($Z_5$) has five modular extensions (see Table XIV), and Rep($S_3$) has six modular extensions (see Table XII). They correspond to the three $Z_3$-SPT, the five $Z_5$-SPT, and the six $S_3$-SPT states respectively. These results agree with those from group cohomology theory [19].

We note that for Rep($Z_2$), Rep($Z_3$), and Rep($S_3$), their modular extensions all correspond to distinct UMTCs. However, for Rep($Z_5$), its five modular extensions only correspond to

TABLE XIX. The second 32 modular extensions of sRep($Z_2 \times Z_2^f$) with $N = 12$.

| $N_c^{\|\Theta\|}$ | $D^2$ | $d_1,d_2,\dots$ | $s_1,s_2,\dots$ | comment |
|---|---|---|---|---|
| $4_0^0$ | 4 | $1,1,1,1$ | $0,0,\frac{1}{2},\frac{1}{2}$ | sRep($Z_2 \times Z_2^f$) |
| $12_{-7/2}^B$ | 16 | $1,1,1,1,1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1$ | $0,0,\frac{1}{2},\frac{1}{2},0,0,\frac{1}{2},\frac{1}{2},\frac{1}{16},\frac{9}{16},\frac{9}{16},\frac{9}{16}$ | $4_4^B \boxtimes 3_{1/2}^B$ |
| $12_{-7/2}^B$ | 16 | $1,1,1,1,1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1$ | $0,0,\frac{1}{2},\frac{1}{2},0,0,\frac{1}{2},\frac{1}{2},\frac{1}{16},\frac{9}{16},\frac{9}{16},\frac{9}{16}$ | $4_4^B \boxtimes 3_{1/2}^B$ |
| $12_{-7/2}^B$ | 16 | $1,1,1,1,1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{1}{8},\frac{1}{8},\frac{5}{8},\frac{5}{8},\frac{7}{16},\frac{9}{16},\frac{9}{16},\frac{15}{16}$ | $4_1^B \boxtimes 3_{7/2}^B$ |
| $12_{-7/2}^B$ | 16 | $1,1,1,1,1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{1}{8},\frac{1}{8},\frac{5}{8},\frac{5}{8},\frac{7}{16},\frac{9}{16},\frac{9}{16},\frac{15}{16}$ | $4_1^B \boxtimes 3_{7/2}^B$ |
| $12_{-7/2}^B$ | 16 | $1,1,1,1,1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{1}{4},\frac{1}{4},\frac{3}{4},\frac{3}{4},\frac{5}{16},\frac{9}{16},\frac{9}{16},\frac{13}{16}$ | $6_{7/2}^B \boxtimes 2_1^B$ |
| $12_{-7/2}^B$ | 16 | $1,1,1,1,1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{1}{4},\frac{1}{4},\frac{3}{4},\frac{3}{4},\frac{5}{16},\frac{9}{16},\frac{9}{16},\frac{13}{16}$ | $6_{7/2}^B \boxtimes 2_1^B$ |
| $12_{-7/2}^B$ | 16 | $1,1,1,1,1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{3}{8},\frac{3}{8},\frac{7}{8},\frac{7}{8},\frac{3}{16},\frac{9}{16},\frac{9}{16},\frac{11}{16}$ | $4_3^B \boxtimes 3_{3/2}^B$ |
| $12_{-7/2}^B$ | 16 | $1,1,1,1,1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{3}{8},\frac{3}{8},\frac{7}{8},\frac{7}{8},\frac{3}{16},\frac{9}{16},\frac{9}{16},\frac{11}{16}$ | $4_3^B \boxtimes 3_{3/2}^B$ |
| $12_{-5/2}^B$ | 16 | $1,1,1,1,1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1$ | $0,0,\frac{1}{2},\frac{1}{2},0,0,\frac{1}{2},\frac{1}{2},\frac{3}{16},\frac{11}{16},\frac{11}{16},\frac{11}{16}$ | $4_4^B \boxtimes 3_{3/2}^B$ |
| $12_{-5/2}^B$ | 16 | $1,1,1,1,1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1$ | $0,0,\frac{1}{2},\frac{1}{2},0,0,\frac{1}{2},\frac{1}{2},\frac{3}{16},\frac{11}{16},\frac{11}{16},\frac{11}{16}$ | $4_4^B \boxtimes 3_{3/2}^B$ |
| $12_{-5/2}^B$ | 16 | $1,1,1,1,1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{1}{8},\frac{1}{8},\frac{5}{8},\frac{5}{8},\frac{1}{16},\frac{9}{16},\frac{11}{16},\frac{11}{16}$ | $4_{-3}^B \boxtimes 3_{1/2}^B$ |
| $12_{-5/2}^B$ | 16 | $1,1,1,1,1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{1}{8},\frac{1}{8},\frac{5}{8},\frac{5}{8},\frac{1}{16},\frac{9}{16},\frac{11}{16},\frac{11}{16}$ | $4_{-3}^B \boxtimes 3_{1/2}^B$ |
| $12_{-5/2}^B$ | 16 | $1,1,1,1,1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{1}{4},\frac{1}{4},\frac{3}{4},\frac{3}{4},\frac{7}{16},\frac{11}{16},\frac{11}{16},\frac{15}{16}$ | $6_{-7/2}^B \boxtimes 2_1^B$ |
| $12_{-5/2}^B$ | 16 | $1,1,1,1,1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{1}{4},\frac{1}{4},\frac{3}{4},\frac{3}{4},\frac{7}{16},\frac{11}{16},\frac{11}{16},\frac{15}{16}$ | $6_{-7/2}^B \boxtimes 2_1^B$ |
| $12_{-5/2}^B$ | 16 | $1,1,1,1,1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{3}{8},\frac{3}{8},\frac{7}{8},\frac{7}{8},\frac{5}{16},\frac{11}{16},\frac{11}{16},\frac{13}{16}$ | $4_3^B \boxtimes 3_{5/2}^B$ |
| $12_{-5/2}^B$ | 16 | $1,1,1,1,1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{3}{8},\frac{3}{8},\frac{7}{8},\frac{7}{8},\frac{5}{16},\frac{11}{16},\frac{11}{16},\frac{13}{16}$ | $4_3^B \boxtimes 3_{5/2}^B$ |
| $12_{-3/2}^B$ | 16 | $1,1,1,1,1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1$ | $0,0,\frac{1}{2},\frac{1}{2},0,0,\frac{1}{2},\frac{1}{2},\frac{5}{16},\frac{13}{16},\frac{13}{16},\frac{13}{16}$ | $4_4^B \boxtimes 3_{5/2}^B$ |
| $12_{-3/2}^B$ | 16 | $1,1,1,1,1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1$ | $0,0,\frac{1}{2},\frac{1}{2},0,0,\frac{1}{2},\frac{1}{2},\frac{5}{16},\frac{13}{16},\frac{13}{16},\frac{13}{16}$ | $4_4^B \boxtimes 3_{5/2}^B$ |
| $12_{-3/2}^B$ | 16 | $1,1,1,1,1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{1}{8},\frac{1}{8},\frac{5}{8},\frac{5}{8},\frac{3}{16},\frac{11}{16},\frac{13}{16},\frac{13}{16}$ | $4_{-3}^B \boxtimes 3_{3/2}^B$ |
| $12_{-3/2}^B$ | 16 | $1,1,1,1,1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{1}{8},\frac{1}{8},\frac{5}{8},\frac{5}{8},\frac{3}{16},\frac{11}{16},\frac{13}{16},\frac{13}{16}$ | $4_{-3}^B \boxtimes 3_{3/2}^B$ |
| $12_{-3/2}^B$ | 16 | $1,1,1,1,1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{1}{4},\frac{1}{4},\frac{3}{4},\frac{3}{4},\frac{1}{16},\frac{9}{16},\frac{13}{16},\frac{13}{16}$ | $6_{-5/2}^B \boxtimes 2_1^B$ |
| $12_{-3/2}^B$ | 16 | $1,1,1,1,1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{1}{4},\frac{1}{4},\frac{3}{4},\frac{3}{4},\frac{1}{16},\frac{9}{16},\frac{13}{16},\frac{13}{16}$ | $6_{-5/2}^B \boxtimes 2_1^B$ |
| $12_{-3/2}^B$ | 16 | $1,1,1,1,1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{3}{8},\frac{3}{8},\frac{7}{8},\frac{7}{8},\frac{7}{16},\frac{13}{16},\frac{13}{16},\frac{15}{16}$ | $4_3^B \boxtimes 3_{7/2}^B$ |
| $12_{-3/2}^B$ | 16 | $1,1,1,1,1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{3}{8},\frac{3}{8},\frac{7}{8},\frac{7}{8},\frac{7}{16},\frac{13}{16},\frac{13}{16},\frac{15}{16}$ | $4_3^B \boxtimes 3_{7/2}^B$ |
| $12_{-1/2}^B$ | 16 | $1,1,1,1,1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1$ | $0,0,\frac{1}{2},\frac{1}{2},0,0,\frac{1}{2},\frac{1}{2},\frac{7}{16},\frac{15}{16},\frac{15}{16},\frac{15}{16}$ | $4_4^B \boxtimes 3_{7/2}^B$ |
| $12_{-1/2}^B$ | 16 | $1,1,1,1,1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1$ | $0,0,\frac{1}{2},\frac{1}{2},0,0,\frac{1}{2},\frac{1}{2},\frac{7}{16},\frac{15}{16},\frac{15}{16},\frac{15}{16}$ | $4_4^B \boxtimes 3_{7/2}^B$ |
| $12_{-1/2}^B$ | 16 | $1,1,1,1,1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{1}{8},\frac{1}{8},\frac{5}{8},\frac{5}{8},\frac{5}{16},\frac{13}{16},\frac{15}{16},\frac{15}{16}$ | $4_{-3}^B \boxtimes 3_{5/2}^B$ |
| $12_{-1/2}^B$ | 16 | $1,1,1,1,1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{1}{8},\frac{1}{8},\frac{5}{8},\frac{5}{8},\frac{5}{16},\frac{13}{16},\frac{15}{16},\frac{15}{16}$ | $4_{-3}^B \boxtimes 3_{5/2}^B$ |
| $12_{-1/2}^B$ | 16 | $1,1,1,1,1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{1}{4},\frac{1}{4},\frac{3}{4},\frac{3}{4},\frac{3}{16},\frac{11}{16},\frac{15}{16},\frac{15}{16}$ | $6_{-3/2}^B \boxtimes 2_1^B$ |
| $12_{-1/2}^B$ | 16 | $1,1,1,1,1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{1}{4},\frac{1}{4},\frac{3}{4},\frac{3}{4},\frac{3}{16},\frac{11}{16},\frac{15}{16},\frac{15}{16}$ | $6_{-3/2}^B \boxtimes 2_1^B$ |
| $12_{-1/2}^B$ | 16 | $1,1,1,1,1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{3}{8},\frac{3}{8},\frac{7}{8},\frac{7}{8},\frac{1}{16},\frac{9}{16},\frac{15}{16},\frac{15}{16}$ | $4_{-1}^B \boxtimes 3_{1/2}^B$ |
| $12_{-1/2}^B$ | 16 | $1,1,1,1,1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1,\zeta_2^1$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{3}{8},\frac{3}{8},\frac{7}{8},\frac{7}{8},\frac{1}{16},\frac{9}{16},\frac{15}{16},\frac{15}{16}$ | $4_{-1}^B \boxtimes 3_{1/2}^B$ |

three distinct UMTCs. Rep($Z_5$) has five modular extensions because Rep($Z_5$) can be embedded into the same UMTC in different ways. The different embeddings correspond to different modular extensions.

### F. Invertible fermionic topological orders

We find that sRep($Z_2^f$) has 16 modular extensions (see Table XIII) which correspond to invertible fermionic topological orders in 2+1D. One might thought that the invertible fermionic topological orders are classified by $\mathbb{Z}_{16}$. However, in fact, the invertible fermionic topological orders are classified by $\mathbb{Z}$, obtained by stacking the $c = 1/2$ $p + \mathrm{i}p$ states. The discrepancy is due to the fact that the modular extensions cannot see the $c = 8E_8$ states. The 16 modular extensions exactly correspond to the invertible fermionic topological orders modulo the $E_8$ states.

We also find that the modular extensions with $c =$ even have a $Z_2 \times Z_2$ fusion rule, while the modular extensions with $c =$ odd have a $Z_4$ fusion rule (indicated by F:$Z_2 \times Z_2$ or F:$Z_4$ in the comment column of Table).

The $Z_2^f$-SPT states for fermions is given by the modular extensions with zero central charge. We see that there is only one modular extension with central charge $c = 0$. Thus there is no nontrivial 2+1D fermionic SPT states with $Z_2^f$ symmetry. In general, the modular extensions of sRep($G^f$) with zero central charge correspond to the fermionic SPT states in 2+1D with symmetry $G^f$.

To calculate the $Z_2 \times Z_2^f$ SPT orders for fermionic systems, we first compute the modular extensions for sRep($Z_2 \times Z_2^f$). We note that sRep($Z_2 \times Z_2^f$) = sRep($Z_2^f \times \tilde{Z}_2^f$). Thus the modular extensions for sRep($Z_2 \times Z_2^f$) is the modular extensions of sRep($Z_2^f \times \tilde{Z}_2^f$). Some of the modular extensions

TABLE XX. All the 32 modular extensions of $\mathrm{sRep}(Z_2 \times Z_2^f)$ with $N = 16$.

| $N_c^{\lvert\Theta\rvert}$ | $D^2$ | $d_1,d_2,\ldots$ | $s_1,s_2,\ldots$ | comment |
|---|---|---|---|---|
| $4_0^0$ | 4 | 1,1,1,1 | $0,0,\frac{1}{2},\frac{1}{2}$ | $\mathrm{sRep}(Z_2 \times Z_2^f)$ |
| $16_0^B$ | 16 | 1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1 | $0,0,\frac{1}{2},\frac{1}{2},0,0,0,0,0,0,0,0,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}$ | $4_0^B \boxtimes 4_0^B$ |
| $16_0^B$ | 16 | 1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1 | $0,0,\frac{1}{2},\frac{1}{2},0,0,0,0,\frac{1}{8},\frac{1}{8},\frac{3}{8},\frac{3}{8},\frac{5}{8},\frac{5}{8},\frac{7}{8},\frac{7}{8}$ | $4_{-1}^B \boxtimes 4_1^B$ |
| $16_0^B$ | 16 | 1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1 | $0,0,\frac{1}{2},\frac{1}{2},0,0,0,0,\frac{1}{8},\frac{1}{8},\frac{3}{8},\frac{3}{8},\frac{5}{8},\frac{5}{8},\frac{7}{8},\frac{7}{8}$ | $4_{-1}^B \boxtimes 4_1^B$ |
| $16_0^B$ | 16 | 1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1 | $0,0,\frac{1}{2},\frac{1}{2},0,0,0,0,\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{3}{4},\frac{3}{4},\frac{3}{4},\frac{3}{4}$ | $8_{-1}^B \boxtimes 2_1^B$ |
| $16_1^B$ | 16 | 1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1 | $0,0,\frac{1}{2},\frac{1}{2},0,0,\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{1}{2},\frac{1}{2},\frac{5}{8},\frac{5}{8}$ | $4_1^B \boxtimes 4_0^B$ |
| $16_1^B$ | 16 | 1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1 | $0,0,\frac{1}{2},\frac{1}{2},0,0,\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{1}{2},\frac{1}{2},\frac{5}{8},\frac{5}{8}$ | $4_1^B \boxtimes 4_0^B$ |
| $16_1^B$ | 16 | 1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1 | $0,0,\frac{1}{2},\frac{1}{2},\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{1}{4},\frac{1}{4},\frac{3}{8},\frac{3}{8},\frac{3}{4},\frac{3}{4},\frac{7}{8},\frac{7}{8}$ | $8_0^B \boxtimes 2_1^B$ |
| $16_1^B$ | 16 | 1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1 | $0,0,\frac{1}{2},\frac{1}{2},\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{1}{4},\frac{1}{4},\frac{3}{8},\frac{3}{8},\frac{3}{4},\frac{3}{4},\frac{7}{8},\frac{7}{8}$ | $8_0^B \boxtimes 2_1^B$ |
| $16_2^B$ | 16 | 1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1 | $0,0,\frac{1}{2},\frac{1}{2},0,0,\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{2},\frac{1}{2},\frac{3}{4},\frac{3}{4}$ | $8_1^B \boxtimes 2_1^B$ |
| $16_2^B$ | 16 | 1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1 | $0,0,\frac{1}{2},\frac{1}{2},0,0,\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{2},\frac{1}{2},\frac{3}{4},\frac{3}{4}$ | $8_1^B \boxtimes 2_1^B$ |
| $16_2^B$ | 16 | 1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1 | $0,0,\frac{1}{2},\frac{1}{2},\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{5}{8},\frac{5}{8},\frac{5}{8},\frac{5}{8}$ | $4_1^B \boxtimes 4_1^B$ |
| $16_2^B$ | 16 | 1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1 | $0,0,\frac{1}{2},\frac{1}{2},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{3}{8},\frac{3}{8},\frac{3}{8},\frac{3}{8},\frac{7}{8},\frac{7}{8},\frac{7}{8},\frac{7}{8}$ | $4_{-1}^B \boxtimes 4_3^B$ |
| $16_3^B$ | 16 | 1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1 | $0,0,\frac{1}{2},\frac{1}{2},0,0,\frac{3}{8},\frac{3}{8},\frac{3}{8},\frac{3}{8},\frac{3}{8},\frac{3}{8},\frac{1}{2},\frac{1}{2},\frac{7}{8},\frac{7}{8}$ | $4_3^B \boxtimes 4_0^B$ |
| $16_3^B$ | 16 | 1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1 | $0,0,\frac{1}{2},\frac{1}{2},0,0,\frac{3}{8},\frac{3}{8},\frac{3}{8},\frac{3}{8},\frac{3}{8},\frac{3}{8},\frac{1}{2},\frac{1}{2},\frac{7}{8},\frac{7}{8}$ | $4_3^B \boxtimes 4_0^B$ |
| $16_3^B$ | 16 | 1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1 | $0,0,\frac{1}{2},\frac{1}{2},\frac{1}{8},\frac{1}{8},\frac{1}{4},\frac{1}{4},\frac{3}{8},\frac{3}{8},\frac{3}{8},\frac{3}{8},\frac{5}{8},\frac{5}{8},\frac{3}{4},\frac{3}{4}$ | $8_2^B \boxtimes 2_1^B$ |
| $16_3^B$ | 16 | 1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1 | $0,0,\frac{1}{2},\frac{1}{2},\frac{1}{8},\frac{1}{8},\frac{1}{4},\frac{1}{4},\frac{3}{8},\frac{3}{8},\frac{3}{8},\frac{3}{8},\frac{5}{8},\frac{5}{8},\frac{3}{4},\frac{3}{4}$ | $8_2^B \boxtimes 2_1^B$ |
| $16_4^B$ | 16 | 1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1 | $0,0,\frac{1}{2},\frac{1}{2},0,0,0,0,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}$ | $4_4^B \boxtimes 4_0^B$ |
| $16_4^B$ | 16 | 1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1 | $0,0,\frac{1}{2},\frac{1}{2},\frac{1}{8},\frac{1}{8},\frac{3}{8},\frac{3}{8},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{5}{8},\frac{5}{8},\frac{7}{8},\frac{7}{8}$ | $4_3^B \boxtimes 4_1^B$ |
| $16_4^B$ | 16 | 1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1 | $0,0,\frac{1}{2},\frac{1}{2},\frac{1}{8},\frac{1}{8},\frac{3}{8},\frac{3}{8},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{5}{8},\frac{5}{8},\frac{7}{8},\frac{7}{8}$ | $4_3^B \boxtimes 4_1^B$ |
| $16_4^B$ | 16 | 1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1 | $0,0,\frac{1}{2},\frac{1}{2},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{3}{4},\frac{3}{4},\frac{3}{4},\frac{3}{4}$ | $8_3^B \boxtimes 2_1^B$ |
| $16_{-3}^B$ | 16 | 1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1 | $0,0,\frac{1}{2},\frac{1}{2},0,0,\frac{1}{8},\frac{1}{8},\frac{1}{2},\frac{1}{2},\frac{5}{8},\frac{5}{8},\frac{5}{8},\frac{5}{8},\frac{5}{8},\frac{5}{8}$ | $4_{-3}^B \boxtimes 4_0^B$ |
| $16_{-3}^B$ | 16 | 1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1 | $0,0,\frac{1}{2},\frac{1}{2},0,0,\frac{1}{8},\frac{1}{8},\frac{1}{2},\frac{1}{2},\frac{5}{8},\frac{5}{8},\frac{5}{8},\frac{5}{8},\frac{5}{8},\frac{5}{8}$ | $4_{-3}^B \boxtimes 4_0^B$ |
| $16_{-3}^B$ | 16 | 1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1 | $0,0,\frac{1}{2},\frac{1}{2},\frac{1}{4},\frac{1}{4},\frac{3}{8},\frac{3}{8},\frac{5}{8},\frac{5}{8},\frac{5}{8},\frac{5}{8},\frac{3}{4},\frac{3}{4},\frac{7}{8},\frac{7}{8}$ | $8_4^B \boxtimes 2_1^B$ |
| $16_{-3}^B$ | 16 | 1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1 | $0,0,\frac{1}{2},\frac{1}{2},\frac{1}{4},\frac{1}{4},\frac{3}{8},\frac{3}{8},\frac{5}{8},\frac{5}{8},\frac{5}{8},\frac{5}{8},\frac{3}{4},\frac{3}{4},\frac{7}{8},\frac{7}{8}$ | $8_4^B \boxtimes 2_1^B$ |
| $16_{-2}^B$ | 16 | 1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1 | $0,0,\frac{1}{2},\frac{1}{2},0,0,\frac{1}{4},\frac{1}{4},\frac{1}{2},\frac{1}{2},\frac{3}{4},\frac{3}{4},\frac{3}{4},\frac{3}{4},\frac{3}{4},\frac{3}{4}$ | $8_{-3}^B \boxtimes 2_1^B$ |
| $16_{-2}^B$ | 16 | 1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1 | $0,0,\frac{1}{2},\frac{1}{2},0,0,\frac{1}{4},\frac{1}{4},\frac{1}{2},\frac{1}{2},\frac{3}{4},\frac{3}{4},\frac{3}{4},\frac{3}{4},\frac{3}{4},\frac{3}{4}$ | $8_{-3}^B \boxtimes 2_1^B$ |
| $16_{-2}^B$ | 16 | 1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1 | $0,0,\frac{1}{2},\frac{1}{2},\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{5}{8},\frac{5}{8},\frac{5}{8},\frac{5}{8},\frac{3}{4},\frac{3}{4},\frac{3}{4},\frac{3}{4}$ | $4_{-3}^B \boxtimes 4_1^B$ |
| $16_{-2}^B$ | 16 | 1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1 | $0,0,\frac{1}{2},\frac{1}{2},\frac{3}{8},\frac{3}{8},\frac{3}{8},\frac{3}{8},\frac{3}{4},\frac{3}{4},\frac{3}{4},\frac{3}{4},\frac{7}{8},\frac{7}{8},\frac{7}{8},\frac{7}{8}$ | $4_3^B \boxtimes 4_3^B$ |
| $16_{-1}^B$ | 16 | 1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1 | $0,0,\frac{1}{2},\frac{1}{2},0,0,\frac{3}{8},\frac{3}{8},\frac{1}{2},\frac{1}{2},\frac{7}{8},\frac{7}{8},\frac{7}{8},\frac{7}{8},\frac{7}{8},\frac{7}{8}$ | $4_{-1}^B \boxtimes 4_0^B$ |
| $16_{-1}^B$ | 16 | 1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1 | $0,0,\frac{1}{2},\frac{1}{2},0,0,\frac{3}{8},\frac{3}{8},\frac{1}{2},\frac{1}{2},\frac{7}{8},\frac{7}{8},\frac{7}{8},\frac{7}{8},\frac{7}{8},\frac{7}{8}$ | $4_{-1}^B \boxtimes 4_0^B$ |
| $16_{-1}^B$ | 16 | 1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1 | $0,0,\frac{1}{2},\frac{1}{2},\frac{1}{8},\frac{1}{8},\frac{1}{4},\frac{1}{4},\frac{5}{8},\frac{5}{8},\frac{3}{4},\frac{3}{4},\frac{7}{8},\frac{7}{8},\frac{7}{8},\frac{7}{8}$ | $8_{-2}^B \boxtimes 2_1^B$ |
| $16_{-1}^B$ | 16 | 1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1 | $0,0,\frac{1}{2},\frac{1}{2},\frac{1}{8},\frac{1}{8},\frac{1}{4},\frac{1}{4},\frac{5}{8},\frac{5}{8},\frac{3}{4},\frac{3}{4},\frac{7}{8},\frac{7}{8},\frac{7}{8},\frac{7}{8}$ | $8_{-2}^B \boxtimes 2_1^B$ |

of $\mathrm{sRep}(Z_2^f \times \tilde{Z}_2^f)$ are given by the modular extensions of $\mathrm{sRep}(Z_2^f)$ stacked (under $\boxtimes$) with the modular extensions of $\mathrm{sRep}(\tilde{Z}_2^f)$. Some of the modular extensions of $\mathrm{sRep}(Z_2 \times Z_2^f)$ are given by the modular extensions for $\mathrm{Rep}(Z_2)$ stacked (under $\boxtimes$) with the modular extensions of $\mathrm{sRep}(Z_2^f)$.

The above mathematical statements correspond to the following physical picture. Some fermionic GQLs with $Z_2 \times Z_2^f$ symmetry can be viewed as bosonic GQLs with $Z_2$ symmetry stacked with fermionic GQLs with $Z_2^f$ symmetry. Also some fermionic GQLs with $Z_2^f \times \tilde{Z}_2^f$ symmetry can be viewed as fermionic GQLs with $Z_2^f$ symmetry stacked with fermionic GQLs with $\tilde{Z}_2^f$ symmetry.

Using Eq. (12), we find that the modular extensions for $Z_2 \times Z_2^f$ symmetry must have ranks 7, 9, 10, 12, and 16. By direct search for those ranks, we find that the modular extensions of $\mathrm{sRep}(Z_2 \times Z_2^f)$ are given by Tables XVII, XVIII, XIX, and XX. The $N = 9$ modular extensions of $\mathrm{sRep}(Z_2 \times Z_2^f)$ in Table XVII are given by the stacking of the $N = 3$ modular extensions of $\mathrm{sRep}(Z_2^f)$ and the $N = 3$ modular extensions of $\mathrm{sRep}(\tilde{Z}_2^f)$. The $N = 16$ modular extensions of $\mathrm{sRep}(Z_2 \times Z_2^f)$ in Table XX are given by the stacking of the $N = 4$ modular extensions of $\mathrm{sRep}(Z_2^f)$ and the $N = 4$ modular extensions of $\mathrm{sRep}(\tilde{Z}_2^f)$. There are also 64 $N = 12$ modular extensions of $\mathrm{sRep}(Z_2 \times Z_2^f)$ given by the stacking of the $N = 4$ ($N = 3$) modular extensions of $\mathrm{sRep}(Z_2^f)$ and the $N = 3$ ($N = 4$) modular extensions of $\mathrm{sRep}(\tilde{Z}_2^f)$.

Many of the modular extensions have nontrivial topological orders since the central charge $c$ is nonzero. There are eight modular extensions for each central charge $c = 0,1/2,1,3/2,\ldots,15/2$, and in total $8 \times 16 = 128$ modular extensions. Those eight with $c = 0$ correspond to the $Z_2 \times Z_2^f$

TABLE XXI. All the modular extensions of sRep($Z_6^f$) = sRep($Z_3 \times Z_2^f$).

| $N_c^{\|\Theta\|}$ | $D^2$ | $d_1,d_2,\dots$ | $s_1,s_2,\dots$ |
|---|---|---|---|
| $6^0_0$ | 6 | 1,1,1,1,1,1 | $0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2}$ sRep($Z_6^f$) |
| $36^B_0$ | 36 | $1\times 36$ | $0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},0,0,0,0,0,0,0,0,0,0,0,0,\frac{1}{6},\frac{1}{6},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{2},\frac{1}{2},\frac{2}{3},\frac{2}{3},\frac{2}{3},\frac{2}{3},\frac{2}{3},\frac{2}{3},\frac{5}{6},\frac{5}{6}$ |
| $36^B_0$ | 36 | $1\times 36$ | $0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},0,0,0,0,0,0,\frac{1}{18},\frac{1}{18},\frac{2}{9},\frac{2}{9},\frac{2}{9},\frac{2}{9},\frac{2}{9},\frac{2}{9},\frac{7}{18},\frac{7}{18},\frac{5}{9},\frac{5}{9},\frac{5}{9},\frac{5}{9},\frac{5}{9},\frac{5}{9},\frac{13}{18},\frac{13}{18},\frac{8}{9},\frac{8}{9},\frac{8}{9},\frac{8}{9},\frac{8}{9},\frac{8}{9}$ |
| $36^B_0$ | 36 | $1\times 36$ | $0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},0,0,0,0,0,0,\frac{1}{9},\frac{1}{9},\frac{1}{9},\frac{1}{9},\frac{1}{9},\frac{1}{9},\frac{5}{18},\frac{5}{18},\frac{4}{9},\frac{4}{9},\frac{4}{9},\frac{4}{9},\frac{4}{9},\frac{4}{9},\frac{11}{18},\frac{11}{18},\frac{7}{9},\frac{7}{9},\frac{7}{9},\frac{7}{9},\frac{7}{9},\frac{7}{9},\frac{17}{18},\frac{17}{18}$ |
| $36^B_1$ | 36 | $1\times 36$ | $0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},0,0,\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{1}{6},\frac{1}{6},\frac{1}{3},\frac{1}{3},\frac{11}{24},\frac{11}{24},\frac{11}{24},\frac{11}{24},\frac{1}{2},\frac{1}{2},\frac{2}{3},\frac{2}{3},\frac{19}{24},\frac{19}{24},\frac{19}{24},\frac{19}{24},\frac{5}{6},\frac{5}{6}$ |
| $36^B_1$ | 36 | $1\times 36$ | $0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},\frac{1}{72},\frac{1}{72},\frac{1}{72},\frac{1}{72},\frac{1}{18},\frac{1}{18},\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{2}{9},\frac{2}{9},\frac{25}{72},\frac{25}{72},\frac{25}{72},\frac{25}{72},\frac{7}{18},\frac{7}{18},\frac{5}{9},\frac{5}{9},\frac{49}{72},\frac{49}{72},\frac{49}{72},\frac{49}{72},\frac{13}{18},\frac{13}{18},\frac{8}{9},\frac{8}{9}$ |
| $36^B_1$ | 36 | $1\times 36$ | $0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},\frac{1}{9},\frac{1}{9},\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{17}{72},\frac{17}{72},\frac{17}{72},\frac{17}{72},\frac{5}{18},\frac{5}{18},\frac{4}{9},\frac{4}{9},\frac{41}{72},\frac{41}{72},\frac{41}{72},\frac{41}{72},\frac{11}{18},\frac{11}{18},\frac{7}{9},\frac{7}{9},\frac{65}{72},\frac{65}{72},\frac{65}{72},\frac{65}{72},\frac{17}{18},\frac{17}{18}$ |
| $36^B_2$ | 36 | $1\times 36$ | $0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},0,0,\frac{1}{6},\frac{1}{6},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{3},\frac{1}{3},\frac{1}{2},\frac{1}{2},\frac{7}{12},\frac{7}{12},\frac{7}{12},\frac{7}{12},\frac{2}{3},\frac{2}{3},\frac{5}{6},\frac{5}{6},\frac{11}{12},\frac{11}{12},\frac{11}{12},\frac{11}{12}$ |
| $36^B_2$ | 36 | $1\times 36$ | $0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},\frac{1}{36},\frac{1}{36},\frac{1}{36},\frac{1}{36},\frac{1}{9},\frac{1}{9},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{5}{18},\frac{5}{18},\frac{13}{36},\frac{13}{36},\frac{13}{36},\frac{13}{36},\frac{4}{9},\frac{4}{9},\frac{11}{18},\frac{11}{18},\frac{25}{36},\frac{25}{36},\frac{25}{36},\frac{25}{36},\frac{7}{9},\frac{7}{9},\frac{17}{18},\frac{17}{18}$ |
| $36^B_2$ | 36 | $1\times 36$ | $0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},\frac{1}{18},\frac{1}{18},\frac{5}{36},\frac{5}{36},\frac{5}{36},\frac{5}{36},\frac{2}{9},\frac{2}{9},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{7}{18},\frac{7}{18},\frac{17}{36},\frac{17}{36},\frac{17}{36},\frac{17}{36},\frac{5}{9},\frac{5}{9},\frac{13}{18},\frac{13}{18},\frac{29}{36},\frac{29}{36},\frac{29}{36},\frac{29}{36},\frac{8}{9},\frac{8}{9}$ |
| $36^B_3$ | 36 | $1\times 36$ | $0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},0,0,\frac{1}{24},\frac{1}{24},\frac{1}{24},\frac{1}{24},\frac{1}{6},\frac{1}{6},\frac{1}{3},\frac{1}{3},\frac{3}{8},\frac{3}{8},\frac{3}{8},\frac{3}{8},\frac{3}{8},\frac{3}{8},\frac{3}{8},\frac{3}{8},\frac{3}{8},\frac{3}{8},\frac{1}{2},\frac{1}{2},\frac{2}{3},\frac{2}{3},\frac{17}{24},\frac{17}{24},\frac{17}{24},\frac{17}{24},\frac{5}{6},\frac{5}{6}$ |
| $36^B_3$ | 36 | $1\times 36$ | $0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},\frac{1}{18},\frac{1}{18},\frac{2}{9},\frac{2}{9},\frac{19}{72},\frac{19}{72},\frac{19}{72},\frac{19}{72},\frac{3}{8},\frac{3}{8},\frac{3}{8},\frac{3}{8},\frac{3}{8},\frac{3}{8},\frac{7}{18},\frac{7}{18},\frac{5}{9},\frac{5}{9},\frac{43}{72},\frac{43}{72},\frac{43}{72},\frac{43}{72},\frac{13}{18},\frac{13}{18},\frac{8}{9},\frac{8}{9},\frac{67}{72},\frac{67}{72},\frac{67}{72},\frac{67}{72}$ |
| $36^B_3$ | 36 | $1\times 36$ | $0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},\frac{1}{9},\frac{1}{9},\frac{11}{72},\frac{11}{72},\frac{11}{72},\frac{11}{72},\frac{5}{18},\frac{5}{18},\frac{3}{8},\frac{3}{8},\frac{3}{8},\frac{3}{8},\frac{3}{8},\frac{3}{8},\frac{4}{9},\frac{4}{9},\frac{35}{72},\frac{35}{72},\frac{35}{72},\frac{35}{72},\frac{11}{18},\frac{11}{18},\frac{7}{9},\frac{7}{9},\frac{59}{72},\frac{59}{72},\frac{59}{72},\frac{59}{72},\frac{17}{18},\frac{17}{18}$ |
| $36^B_4$ | 36 | $1\times 36$ | $0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},0,0,\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{3},\frac{1}{3},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{2}{3},\frac{2}{3},\frac{5}{6},\frac{5}{6},\frac{5}{6},\frac{5}{6},\frac{5}{6},\frac{5}{6}$ |
| $36^B_4$ | 36 | $1\times 36$ | $0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},\frac{1}{18},\frac{1}{18},\frac{1}{18},\frac{1}{18},\frac{1}{18},\frac{1}{18},\frac{2}{9},\frac{2}{9},\frac{7}{18},\frac{7}{18},\frac{7}{18},\frac{7}{18},\frac{7}{18},\frac{7}{18},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{5}{9},\frac{5}{9},\frac{13}{18},\frac{13}{18},\frac{13}{18},\frac{13}{18},\frac{13}{18},\frac{13}{18},\frac{8}{9},\frac{8}{9}$ |
| $36^B_4$ | 36 | $1\times 36$ | $0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},\frac{1}{9},\frac{1}{9},\frac{5}{18},\frac{5}{18},\frac{5}{18},\frac{5}{18},\frac{5}{18},\frac{5}{18},\frac{4}{9},\frac{4}{9},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{11}{18},\frac{11}{18},\frac{11}{18},\frac{11}{18},\frac{11}{18},\frac{11}{18},\frac{7}{9},\frac{7}{9},\frac{17}{18},\frac{17}{18},\frac{17}{18},\frac{17}{18},\frac{17}{18},\frac{17}{18}$ |
| $36^B_{-3}$ | 36 | $1\times 36$ | $0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},0,0,\frac{1}{6},\frac{1}{6},\frac{7}{24},\frac{7}{24},\frac{7}{24},\frac{7}{24},\frac{1}{3},\frac{1}{3},\frac{1}{2},\frac{1}{2},\frac{5}{8},\frac{5}{8},\frac{5}{8},\frac{5}{8},\frac{5}{8},\frac{5}{8},\frac{5}{8},\frac{5}{8},\frac{5}{8},\frac{5}{8},\frac{2}{3},\frac{2}{3},\frac{5}{6},\frac{5}{6},\frac{23}{24},\frac{23}{24},\frac{23}{24},\frac{23}{24}$ |
| $36^B_{-3}$ | 36 | $1\times 36$ | $0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},\frac{1}{18},\frac{1}{18},\frac{13}{72},\frac{13}{72},\frac{13}{72},\frac{13}{72},\frac{2}{9},\frac{2}{9},\frac{7}{18},\frac{7}{18},\frac{37}{72},\frac{37}{72},\frac{37}{72},\frac{37}{72},\frac{5}{9},\frac{5}{9},\frac{5}{8},\frac{5}{8},\frac{5}{8},\frac{5}{8},\frac{5}{8},\frac{5}{8},\frac{13}{18},\frac{13}{18},\frac{61}{72},\frac{61}{72},\frac{61}{72},\frac{61}{72},\frac{8}{9},\frac{8}{9}$ |
| $36^B_{-3}$ | 36 | $1\times 36$ | $0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},\frac{5}{72},\frac{5}{72},\frac{5}{72},\frac{5}{72},\frac{1}{9},\frac{1}{9},\frac{5}{18},\frac{5}{18},\frac{29}{72},\frac{29}{72},\frac{29}{72},\frac{29}{72},\frac{4}{9},\frac{4}{9},\frac{11}{18},\frac{11}{18},\frac{5}{8},\frac{5}{8},\frac{5}{8},\frac{5}{8},\frac{5}{8},\frac{5}{8},\frac{53}{72},\frac{53}{72},\frac{53}{72},\frac{53}{72},\frac{7}{9},\frac{7}{9},\frac{17}{18},\frac{17}{18}$ |
| $36^B_{-2}$ | 36 | $1\times 36$ | $0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},0,0,\frac{1}{12},\frac{1}{12},\frac{1}{12},\frac{1}{12},\frac{1}{6},\frac{1}{6},\frac{1}{3},\frac{1}{3},\frac{5}{12},\frac{5}{12},\frac{5}{12},\frac{5}{12},\frac{1}{2},\frac{1}{2},\frac{2}{3},\frac{2}{3},\frac{3}{4},\frac{3}{4},\frac{3}{4},\frac{3}{4},\frac{3}{4},\frac{3}{4},\frac{3}{4},\frac{3}{4},\frac{3}{4},\frac{3}{4},\frac{5}{6},\frac{5}{6}$ |
| $36^B_{-2}$ | 36 | $1\times 36$ | $0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},\frac{1}{18},\frac{1}{18},\frac{2}{9},\frac{2}{9},\frac{11}{36},\frac{11}{36},\frac{11}{36},\frac{11}{36},\frac{7}{18},\frac{7}{18},\frac{5}{9},\frac{5}{9},\frac{23}{36},\frac{23}{36},\frac{23}{36},\frac{23}{36},\frac{13}{18},\frac{13}{18},\frac{3}{4},\frac{3}{4},\frac{3}{4},\frac{3}{4},\frac{3}{4},\frac{3}{4},\frac{8}{9},\frac{8}{9},\frac{35}{36},\frac{35}{36},\frac{35}{36},\frac{35}{36}$ |
| $36^B_{-2}$ | 36 | $1\times 36$ | $0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},\frac{1}{9},\frac{1}{9},\frac{7}{36},\frac{7}{36},\frac{7}{36},\frac{7}{36},\frac{5}{18},\frac{5}{18},\frac{4}{9},\frac{4}{9},\frac{19}{36},\frac{19}{36},\frac{19}{36},\frac{19}{36},\frac{11}{18},\frac{11}{18},\frac{3}{4},\frac{3}{4},\frac{3}{4},\frac{3}{4},\frac{3}{4},\frac{3}{4},\frac{7}{9},\frac{7}{9},\frac{31}{36},\frac{31}{36},\frac{31}{36},\frac{31}{36},\frac{17}{18},\frac{17}{18}$ |
| $36^B_{-1}$ | 36 | $1\times 36$ | $0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},0,0,\frac{1}{6},\frac{1}{6},\frac{5}{24},\frac{5}{24},\frac{5}{24},\frac{5}{24},\frac{1}{3},\frac{1}{3},\frac{1}{2},\frac{1}{2},\frac{13}{24},\frac{13}{24},\frac{13}{24},\frac{13}{24},\frac{2}{3},\frac{2}{3},\frac{5}{6},\frac{5}{6},\frac{7}{8},\frac{7}{8},\frac{7}{8},\frac{7}{8},\frac{7}{8},\frac{7}{8},\frac{7}{8},\frac{7}{8},\frac{7}{8},\frac{7}{8}$ |
| $36^B_{-1}$ | 36 | $1\times 36$ | $0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},\frac{1}{18},\frac{1}{18},\frac{7}{72},\frac{7}{72},\frac{7}{72},\frac{7}{72},\frac{2}{9},\frac{2}{9},\frac{7}{18},\frac{7}{18},\frac{31}{72},\frac{31}{72},\frac{31}{72},\frac{31}{72},\frac{5}{9},\frac{5}{9},\frac{13}{18},\frac{13}{18},\frac{55}{72},\frac{55}{72},\frac{55}{72},\frac{55}{72},\frac{7}{8},\frac{7}{8},\frac{7}{8},\frac{7}{8},\frac{7}{8},\frac{7}{8},\frac{8}{9},\frac{8}{9}$ |
| $36^B_{-1}$ | 36 | $1\times 36$ | $0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},\frac{1}{9},\frac{1}{9},\frac{5}{18},\frac{5}{18},\frac{23}{72},\frac{23}{72},\frac{23}{72},\frac{23}{72},\frac{4}{9},\frac{4}{9},\frac{11}{18},\frac{11}{18},\frac{47}{72},\frac{47}{72},\frac{47}{72},\frac{47}{72},\frac{7}{9},\frac{7}{9},\frac{7}{8},\frac{7}{8},\frac{7}{8},\frac{7}{8},\frac{7}{8},\frac{7}{8},\frac{17}{18},\frac{17}{18},\frac{71}{72},\frac{71}{72},\frac{71}{72},\frac{71}{72}$ |
| $27^B_{1/2}$ | 36 | $1\times 18,\zeta_2^1\times 9$ | $0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},0,0,\frac{1}{6},\frac{1}{6},\frac{1}{3},\frac{1}{3},\frac{1}{2},\frac{1}{2},\frac{2}{3},\frac{2}{3},\frac{5}{6},\frac{5}{6},\frac{1}{16},\frac{1}{16},\frac{1}{16},\frac{1}{16},\frac{1}{16},\frac{19}{48},\frac{19}{48},\frac{35}{48},\frac{35}{48}$ |
| $27^B_{1/2}$ | 36 | $1\times 18,\zeta_2^1\times 9$ | $0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},\frac{1}{18},\frac{1}{18},\frac{2}{9},\frac{2}{9},\frac{7}{18},\frac{7}{18},\frac{5}{9},\frac{5}{9},\frac{13}{18},\frac{13}{18},\frac{8}{9},\frac{8}{9},\frac{1}{16},\frac{1}{16},\frac{1}{16},\frac{41}{144},\frac{41}{144},\frac{89}{144},\frac{89}{144},\frac{137}{144},\frac{137}{144}$ |
| $27^B_{1/2}$ | 36 | $1\times 18,\zeta_2^1\times 9$ | $0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},\frac{1}{9},\frac{1}{9},\frac{5}{18},\frac{5}{18},\frac{4}{9},\frac{4}{9},\frac{11}{18},\frac{11}{18},\frac{7}{9},\frac{7}{9},\frac{17}{18},\frac{17}{18},\frac{1}{16},\frac{1}{16},\frac{1}{16},\frac{25}{144},\frac{25}{144},\frac{73}{144},\frac{73}{144},\frac{121}{144},\frac{121}{144}$ |
| $27^B_{3/2}$ | 36 | $1\times 18,\zeta_2^1\times 9$ | $0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},0,0,\frac{1}{6},\frac{1}{6},\frac{1}{3},\frac{1}{3},\frac{1}{2},\frac{1}{2},\frac{2}{3},\frac{2}{3},\frac{5}{6},\frac{5}{6},\frac{3}{16},\frac{3}{16},\frac{3}{16},\frac{3}{16},\frac{3}{16},\frac{25}{48},\frac{25}{48},\frac{41}{48},\frac{41}{48}$ |
| $27^B_{3/2}$ | 36 | $1\times 18,\zeta_2^1\times 9$ | $0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},\frac{1}{18},\frac{1}{18},\frac{2}{9},\frac{2}{9},\frac{7}{18},\frac{7}{18},\frac{5}{9},\frac{5}{9},\frac{13}{18},\frac{13}{18},\frac{8}{9},\frac{8}{9},\frac{11}{144},\frac{11}{144},\frac{3}{16},\frac{3}{16},\frac{3}{16},\frac{59}{144},\frac{59}{144},\frac{107}{144},\frac{107}{144}$ |
| $27^B_{3/2}$ | 36 | $1\times 18,\zeta_2^1\times 9$ | $0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},\frac{1}{9},\frac{1}{9},\frac{5}{18},\frac{5}{18},\frac{4}{9},\frac{4}{9},\frac{11}{18},\frac{11}{18},\frac{7}{9},\frac{7}{9},\frac{17}{18},\frac{17}{18},\frac{3}{16},\frac{3}{16},\frac{3}{16},\frac{43}{144},\frac{43}{144},\frac{91}{144},\frac{91}{144},\frac{139}{144},\frac{139}{144}$ |
| $27^B_{5/2}$ | 36 | $1\times 18,\zeta_2^1\times 9$ | $0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},0,0,\frac{1}{6},\frac{1}{6},\frac{1}{3},\frac{1}{3},\frac{1}{2},\frac{1}{2},\frac{2}{3},\frac{2}{3},\frac{5}{6},\frac{5}{6},\frac{5}{16},\frac{5}{16},\frac{5}{16},\frac{5}{16},\frac{5}{16},\frac{31}{48},\frac{31}{48},\frac{47}{48},\frac{47}{48}$ |
| $27^B_{5/2}$ | 36 | $1\times 18,\zeta_2^1\times 9$ | $0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},\frac{1}{18},\frac{1}{18},\frac{2}{9},\frac{2}{9},\frac{7}{18},\frac{7}{18},\frac{5}{9},\frac{5}{9},\frac{13}{18},\frac{13}{18},\frac{8}{9},\frac{8}{9},\frac{29}{144},\frac{29}{144},\frac{5}{16},\frac{5}{16},\frac{5}{16},\frac{77}{144},\frac{77}{144},\frac{125}{144},\frac{125}{144}$ |
| $27^B_{5/2}$ | 36 | $1\times 18,\zeta_2^1\times 9$ | $0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},\frac{1}{9},\frac{1}{9},\frac{5}{18},\frac{5}{18},\frac{4}{9},\frac{4}{9},\frac{11}{18},\frac{11}{18},\frac{7}{9},\frac{7}{9},\frac{17}{18},\frac{17}{18},\frac{13}{144},\frac{13}{144},\frac{5}{16},\frac{5}{16},\frac{5}{16},\frac{61}{144},\frac{61}{144},\frac{109}{144},\frac{109}{144}$ |
| $27^B_{7/2}$ | 36 | $1\times 18,\zeta_2^1\times 9$ | $0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},0,0,\frac{1}{6},\frac{1}{6},\frac{1}{3},\frac{1}{3},\frac{1}{2},\frac{1}{2},\frac{2}{3},\frac{2}{3},\frac{5}{6},\frac{5}{6},\frac{5}{48},\frac{5}{48},\frac{7}{16},\frac{7}{16},\frac{7}{16},\frac{7}{16},\frac{7}{16},\frac{37}{48},\frac{37}{48}$ |
| $27^B_{7/2}$ | 36 | $1\times 18,\zeta_2^1\times 9$ | $0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},\frac{1}{18},\frac{1}{18},\frac{2}{9},\frac{2}{9},\frac{7}{18},\frac{7}{18},\frac{5}{9},\frac{5}{9},\frac{13}{18},\frac{13}{18},\frac{8}{9},\frac{8}{9},\frac{47}{144},\frac{47}{144},\frac{7}{16},\frac{7}{16},\frac{7}{16},\frac{95}{144},\frac{95}{144},\frac{143}{144},\frac{143}{144}$ |
| $27^B_{7/2}$ | 36 | $1\times 18,\zeta_2^1\times 9$ | $0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},\frac{1}{9},\frac{1}{9},\frac{5}{18},\frac{5}{18},\frac{4}{9},\frac{4}{9},\frac{11}{18},\frac{11}{18},\frac{7}{9},\frac{7}{9},\frac{17}{18},\frac{17}{18},\frac{31}{144},\frac{31}{144},\frac{7}{16},\frac{7}{16},\frac{7}{16},\frac{79}{144},\frac{79}{144},\frac{127}{144},\frac{127}{144}$ |
| $27^B_{-7/2}$ | 36 | $1\times 18,\zeta_2^1\times 9$ | $0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},0,0,\frac{1}{6},\frac{1}{6},\frac{1}{3},\frac{1}{3},\frac{1}{2},\frac{1}{2},\frac{2}{3},\frac{2}{3},\frac{5}{6},\frac{5}{6},\frac{11}{48},\frac{11}{48},\frac{9}{16},\frac{9}{16},\frac{9}{16},\frac{9}{16},\frac{9}{16},\frac{43}{48},\frac{43}{48}$ |
| $27^B_{-7/2}$ | 36 | $1\times 18,\zeta_2^1\times 9$ | $0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},\frac{1}{18},\frac{1}{18},\frac{2}{9},\frac{2}{9},\frac{7}{18},\frac{7}{18},\frac{5}{9},\frac{5}{9},\frac{13}{18},\frac{13}{18},\frac{8}{9},\frac{8}{9},\frac{17}{144},\frac{17}{144},\frac{65}{144},\frac{65}{144},\frac{9}{16},\frac{9}{16},\frac{9}{16},\frac{113}{144},\frac{113}{144}$ |
| $27^B_{-7/2}$ | 36 | $1\times 18,\zeta_2^1\times 9$ | $0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},\frac{1}{9},\frac{1}{9},\frac{5}{18},\frac{5}{18},\frac{4}{9},\frac{4}{9},\frac{11}{18},\frac{11}{18},\frac{7}{9},\frac{7}{9},\frac{17}{18},\frac{17}{18},\frac{1}{144},\frac{1}{144},\frac{49}{144},\frac{49}{144},\frac{9}{16},\frac{9}{16},\frac{9}{16},\frac{97}{144},\frac{97}{144}$ |
| $27^B_{-5/2}$ | 36 | $1\times 18,\zeta_2^1\times 9$ | $0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},0,0,\frac{1}{6},\frac{1}{6},\frac{1}{3},\frac{1}{3},\frac{1}{2},\frac{1}{2},\frac{2}{3},\frac{2}{3},\frac{5}{6},\frac{5}{6},\frac{1}{48},\frac{1}{48},\frac{17}{48},\frac{17}{48},\frac{11}{16},\frac{11}{16},\frac{11}{16},\frac{11}{16},\frac{11}{16}$ |
| $27^B_{-5/2}$ | 36 | $1\times 18,\zeta_2^1\times 9$ | $0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},\frac{1}{18},\frac{1}{18},\frac{2}{9},\frac{2}{9},\frac{7}{18},\frac{7}{18},\frac{5}{9},\frac{5}{9},\frac{13}{18},\frac{13}{18},\frac{8}{9},\frac{8}{9},\frac{35}{144},\frac{35}{144},\frac{83}{144},\frac{83}{144},\frac{11}{16},\frac{11}{16},\frac{11}{16},\frac{131}{144},\frac{131}{144}$ |
| $27^B_{-5/2}$ | 36 | $1\times 18,\zeta_2^1\times 9$ | $0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},\frac{1}{9},\frac{1}{9},\frac{5}{18},\frac{5}{18},\frac{4}{9},\frac{4}{9},\frac{11}{18},\frac{11}{18},\frac{7}{9},\frac{7}{9},\frac{17}{18},\frac{17}{18},\frac{19}{144},\frac{19}{144},\frac{67}{144},\frac{67}{144},\frac{11}{16},\frac{11}{16},\frac{11}{16},\frac{115}{144},\frac{115}{144}$ |
| $27^B_{-3/2}$ | 36 | $1\times 18,\zeta_2^1\times 9$ | $0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},0,0,\frac{1}{6},\frac{1}{6},\frac{1}{3},\frac{1}{3},\frac{1}{2},\frac{1}{2},\frac{2}{3},\frac{2}{3},\frac{5}{6},\frac{5}{6},\frac{7}{48},\frac{7}{48},\frac{23}{48},\frac{23}{48},\frac{13}{16},\frac{13}{16},\frac{13}{16},\frac{13}{16},\frac{13}{16}$ |
| $27^B_{-3/2}$ | 36 | $1\times 18,\zeta_2^1\times 9$ | $0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},\frac{1}{18},\frac{1}{18},\frac{2}{9},\frac{2}{9},\frac{7}{18},\frac{7}{18},\frac{5}{9},\frac{5}{9},\frac{13}{18},\frac{13}{18},\frac{8}{9},\frac{8}{9},\frac{5}{144},\frac{5}{144},\frac{53}{144},\frac{53}{144},\frac{101}{144},\frac{101}{144},\frac{13}{16},\frac{13}{16},\frac{13}{16}$ |
| $27^B_{-3/2}$ | 36 | $1\times 18,\zeta_2^1\times 9$ | $0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},\frac{1}{9},\frac{1}{9},\frac{5}{18},\frac{5}{18},\frac{4}{9},\frac{4}{9},\frac{11}{18},\frac{11}{18},\frac{7}{9},\frac{7}{9},\frac{17}{18},\frac{17}{18},\frac{37}{144},\frac{37}{144},\frac{85}{144},\frac{85}{144},\frac{13}{16},\frac{13}{16},\frac{13}{16},\frac{133}{144},\frac{133}{144}$ |
| $27^B_{-1/2}$ | 36 | $1\times 18,\zeta_2^1\times 9$ | $0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},0,0,\frac{1}{6},\frac{1}{6},\frac{1}{3},\frac{1}{3},\frac{1}{2},\frac{1}{2},\frac{2}{3},\frac{2}{3},\frac{5}{6},\frac{5}{6},\frac{13}{48},\frac{13}{48},\frac{29}{48},\frac{29}{48},\frac{15}{16},\frac{15}{16},\frac{15}{16},\frac{15}{16},\frac{15}{16}$ |
| $27^B_{-1/2}$ | 36 | $1\times 18,\zeta_2^1\times 9$ | $0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},\frac{1}{18},\frac{1}{18},\frac{2}{9},\frac{2}{9},\frac{7}{18},\frac{7}{18},\frac{5}{9},\frac{5}{9},\frac{13}{18},\frac{13}{18},\frac{8}{9},\frac{8}{9},\frac{23}{144},\frac{23}{144},\frac{71}{144},\frac{71}{144},\frac{119}{144},\frac{119}{144},\frac{15}{16},\frac{15}{16},\frac{15}{16}$ |
| $27^B_{-1/2}$ | 36 | $1\times 18,\zeta_2^1\times 9$ | $0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},\frac{1}{9},\frac{1}{9},\frac{5}{18},\frac{5}{18},\frac{4}{9},\frac{4}{9},\frac{11}{18},\frac{11}{18},\frac{7}{9},\frac{7}{9},\frac{17}{18},\frac{17}{18},\frac{7}{144},\frac{7}{144},\frac{55}{144},\frac{55}{144},\frac{103}{144},\frac{103}{144},\frac{15}{16},\frac{15}{16},\frac{15}{16}$ |

fermionic SPT states. Those are all the $Z_2 \times Z_2^f$ fermionic SPT states [40,41].

### G. $Z_{2n}^f$ SPT orders for fermionic systems

We also find the modular extensions for sRep($Z_4^f$), sRep($Z_6^f$), and sRep($Z_8^f$) (see Tables XV, XXI, and XVI). Again, many of them has nontrivial topological orders since the central charge $c$ is nonzero.

For $Z_4^f$ group, only one of them have $c = 0$. So there is no nontrivial $Z_4^f$ fermionic SPT states. For $Z_6^f$ group, only three of them have $c = 0$. So, the $Z_6^f$ fermionic SPT states are described by $\mathbb{Z}_3$. For $Z_8^f$ group, only two of them have $c = 0$. So, the $Z_8^f$ fermionic SPT states are described by $\mathbb{Z}_2$. Those results are consistent with the results in Refs. [42]. However, the calculation present here is more complete.

## IX. SUMMARY

GQLs contain both topologically ordered states and SPT states. In this paper, we present a theory that classify GQLs in 2+1D for bosonic/fermionic systems with symmetry.

We propose that the possible non-Abelian statistics (or sets of bulk quasiparticles excitations) in 2+1D GQLs are classified by UMTC$_{/\mathcal{E}}$, where $\mathcal{E} = \text{Rep}(G)$ or sRep($G^f$) describing the symmetry in bosonic or fermionic systems. However, UMTC$_{/\mathcal{E}}$'s fail to classify GQLs, since different GQL phases can have identical non-Abelian statistics, which correspond to identical UMTC$_{/\mathcal{E}}$.

To fix this problem, we introduce the notion of modular extensions for a UMTC$_{/\mathcal{E}}$. We propose to use the triple $(\mathcal{C},\mathcal{M},c)$ to classify 2+1D GQLs with symmetry $G$ (for boson) or $G^f$ (for fermion). Here, $\mathcal{C}$ is a UMTC$_{/\mathcal{E}}$ with $\mathcal{E} = \text{Rep}(G)$ or sRep($G^f$), $\mathcal{M}$ is a modular extension of $\mathcal{C}$ and $c$ is the chiral central charge of the edge state. We show that the modular extensions of a UMTC$_{/\mathcal{E}}$ has a one-to-one correspondence with the modular extensions of $\mathcal{E}$. So the number of the modular extensions is solely determined by the symmetry $\mathcal{E}$. Also, the $c = 0$ modular extensions of a $\mathcal{E}$ ($\mathcal{E} = \text{Rep}(G)$ or sRep($G^f$)) classify the 2+1D SPT states for bosons or fermions with symmetry $G$ or $G^f$.

Although the above result has a nice mathematical structure, it is hard to implement numerically to produce a table of GQLs. To fix this problem, we propose a different description of 2+1D GQLs. We propose to use the data $(\tilde{N}_c^{ab},\tilde{s}_a;N_k^{ij},s_i;\mathcal{N}_K^{IJ},\mathcal{S}_I;c)$, up to some permutations of the indices, to describe 2+1D GQLs with symmetry $G$ (for boson) or $G^f$ (for fermion), with a restriction that the symmetry group $G$ can be fully characterized by the fusion ring of its irreducible representations (for example, for simple groups or Abelian groups). Here, the data $(\tilde{N}_c^{ab},\tilde{s}_a)$ describe the symmetry and the data $(N_k^{ij},s_i)$ describe fusion and the spins of the bulk particles in the GQL. The modular extensions are obtained by "gauging" the symmetry $G$ or $G^f$. The data $(\mathcal{N}_K^{IJ},\mathcal{S}_I)$ describe fusion and the spins of the bulk particles in the "gauged" theory. Last, $c$ is the chiral central charge of the edge state.

In this paper (see Appendix C) and in Ref. [11], we list the necessary and the sufficient conditions on the data $(\tilde{N}_c^{ab},\tilde{s}_a;N_k^{ij},s_i;\mathcal{N}_K^{IJ},\mathcal{S}_I;c)$, which allow us to obtain a list of GQLs. However, in this paper, we did not give the list of GQLs directly. We first give a list of $(\tilde{N}_c^{ab},\tilde{s}_a;N_k^{ij},s_i)$, which is an imperfect list of UMTC$_{/\mathcal{E}}$'s. We then compute the modular extensions $(\mathcal{N}_K^{IJ},\mathcal{S}_I;c)$ for each entry $(\tilde{N}_c^{ab},\tilde{s}_a;N_k^{ij},s_i)$, which allows us to obtain a perfect list of GQLs (for certain symmetry groups). As a special case, we calculated the bosonic/fermionic SPT states for some groups in 2+1D.

In Ref. [30], we will give a more mathematical description of our theory. Certainly we hope to generalize the above framework to higher dimensions. We also hope to develop more efficient numerical codes to obtain bigger tables of GQLs.

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## APPENDIX A: TABLES FOR THE SOLUTIONS OF $(\tilde{N}_c^{ab},\tilde{s}_a;N_k^{ij},s_i)$: IMPERFECT TABLES FOR UMTC$_{/\mathcal{E}}$

In this Appendix, we list UMTC$_{/\mathcal{E}}$'s for various symmetry $\mathcal{E}$, which can also be viewed as the list of 2+1D SET orders (up to invertible ones) with symmetry $\mathcal{E}$. Those lists are created using a naive calculation, by checking the necessary conditions on the data $(\tilde{N}_c^{ab},\tilde{s}_a;N_k^{ij},s_i)$ (for details, see Appendix C). So those lists should contain all UMTC$_{/\mathcal{E}}$'s (i.e., all SET orders). However, since the conditions are only known to be necessary, the lists may contain fake entries that do not correspond to any UMTC$_{/\mathcal{E}}$ (or any SET order). In other words, some entries in the lists have no modular extensions and those entries do not correspond any real 2+1D SET order.

The entries with known decomposition $N_c^B \boxtimes \text{Rep}(G)$ or $N_c^B \boxtimes \text{sRep}(G^f)$, or with given $K$ matrix in the comment column all correspond to existing 2+1D SET orders. (The topological orders described by $N_c^B$ are given by the tables in Ref. [11].) Other entries may or may not correspond to existing 2+1D SET orders, which need to be determined by checking the existence of modular extensions.

Even for the entries that have modular extensions, some times they may correspond to more than one UMTC$_{/\mathcal{E}}$'s. This is because $(\tilde{N}_c^{ab},\tilde{s}_a;N_k^{ij},s_i)$ cannot distinguish all different UMTC$_{/\mathcal{E}}$'s.

### 1. $Z_2$-SET orders

Tables XXII, XXIII, and XXIV list the $Z_2$-SET orders (up to invertible ones) for 2+1D bosonic systems. For bosonic systems the central charge is determined up to eight by the bulk excitations. The $3_2^{\zeta_2^1}$ states and the two $4_1^{\zeta_2^1}$ states in Table XXII are discussed in the main text.

TABLE XXII. $Z_2$-SET orders (or UMTC$_{/\text{Rep}(Z_2)}$) for bosonic systems labeled in terms of sets of topological excitations. The list contains all topological orders with $N = 3,4$ and $D^2 \leqslant 100$. All the topologically orders in this list are anomaly free (i.e., have modular extensions), and are realizable by 2+1D bosonic systems. We use $N_c^{|\Theta|}$ to label UMTC$_{/\mathcal{E}}$'s, where $\Theta = D^{-1}\sum_i e^{2\pi i s_i} d_i^2 = |\Theta| e^{2\pi i c/8}$ and $D^2 = \sum_i d_i^2$.

| $N_c^{\|\Theta\|}$ | $D^2$ | $d_1,d_2,\ldots$ | $s_1,s_2,\ldots$ | comment |
|---|---|---|---|---|
| $2_0^{\zeta_2^1}$ | 2 | 1,1 | 0,0 | $\mathcal{E} = \text{Rep}(Z_2)$ |
| $3_2^{\zeta_2^1}$ | 6 | 1,1,2 | $0,0,\frac{1}{3}$ | SB:$K = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ |
| $3_{-2}^{\zeta_2^1}$ | 6 | 1,1,2 | $0,0,\frac{2}{3}$ | SB:$K = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$ |
| $4_1^{\zeta_2^1}$ | 4 | 1,1,1,1 | $0,0,\frac{1}{4},\frac{1}{4}$ | $2_1^B \boxtimes \text{Rep}(Z_2)$ |
| $4_1^{\zeta_2^1}$ | 4 | 1,1,1,1 | $0,0,\frac{1}{4},\frac{1}{4}$ | $2_1^B \boxtimes' \text{Rep}(Z_2)$ |
| $4_{-1}^{\zeta_2^1}$ | 4 | 1,1,1,1 | $0,0,\frac{3}{4},\frac{3}{4}$ | $2_{-1}^B \boxtimes \text{Rep}(Z_2)$ |
| $4_{-1}^{\zeta_2^1}$ | 4 | 1,1,1,1 | $0,0,\frac{3}{4},\frac{3}{4}$ | $2_{-1}^B \boxtimes' \text{Rep}(Z_2)$ |
| $4_{14/5}^{\zeta_2^1}$ | 7.2360 | $1,1,\zeta_3^1,\zeta_3^1$ | $0,0,\frac{2}{5},\frac{2}{5}$ | $2_{14/5}^B \boxtimes \text{Rep}(Z_2)$ |
| $4_{-14/5}^{\zeta_2^1}$ | 7.2360 | $1,1,\zeta_3^1,\zeta_3^1$ | $0,0,\frac{3}{5},\frac{3}{5}$ | $2_{-14/5}^B \boxtimes \text{Rep}(Z_2)$ |
| $4_0^{\zeta_2^1}$ | 10 | 1,1,2,2 | $0,0,\frac{1}{5},\frac{4}{5}$ | SB:$K = \begin{pmatrix} 2 & -3 \\ -3 & 2 \end{pmatrix}$ |
| $4_4^{\zeta_2^1}$ | 10 | 1,1,2,2 | $0,0,\frac{2}{5},\frac{3}{5}$ | SB:$K = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \end{pmatrix}$ |

TABLE XXIII. $Z_2$-SET orders for bosonic systems labeled in terms of sets of topological excitations. The list contains all topological orders with $N = 5$ and $D^2 \leqslant 100$.

| $N_c^{\|\Theta\|}$ | $D^2$ | $d_1,d_2,\ldots$ | $s_1,s_2,\ldots$ | comment |
|---|---|---|---|---|
| $2_0^{\zeta_2^1}$ | 2 | 1,1 | 0,0 | $\mathcal{E} = \text{Rep}(Z_2)$ |
| $5_0^{\zeta_2^1}$ | 8 | 1,1,1,1,2 | $0,0,\frac{1}{2},\frac{1}{2},0$ | SB:$4_0^B$ F:$Z_2 \times Z_2$ |
| $5_0^{\zeta_2^1}$ | 8 | 1,1,1,1,2 | $0,0,\frac{1}{2},\frac{1}{2},0$ | SB:$4_0^B$ F:$Z_4$ NR |
| $5_1^{\zeta_2^1}$ | 8 | 1,1,1,1,2 | $0,0,\frac{1}{2},\frac{1}{2},\frac{1}{8}$ | SB:$4_1^B$ F:$Z_2 \times Z_2$ |
| $5_1^{\zeta_2^1}$ | 8 | 1,1,1,1,2 | $0,0,\frac{1}{2},\frac{1}{2},\frac{1}{8}$ | SB:$4_1^B$ F:$Z_4$ NR |
| $5_2^{\zeta_2^1}$ | 8 | 1,1,1,1,2 | $0,0,\frac{1}{2},\frac{1}{2},\frac{1}{4}$ | SB:$4_2^B$ F:$Z_2 \times Z_2$ |
| $5_2^{\zeta_2^1}$ | 8 | 1,1,1,1,2 | $0,0,\frac{1}{2},\frac{1}{2},\frac{1}{4}$ | SB:$4_2^B$ F:$Z_4$ NR |
| $5_3^{\zeta_2^1}$ | 8 | 1,1,1,1,2 | $0,0,\frac{1}{2},\frac{1}{2},\frac{3}{8}$ | SB:$4_3^B$ F:$Z_2 \times Z_2$ |
| $5_3^{\zeta_2^1}$ | 8 | 1,1,1,1,2 | $0,0,\frac{1}{2},\frac{1}{2},\frac{3}{8}$ | SB:$4_3^B$ F:$Z_4$ NR |
| $5_4^{\zeta_2^1}$ | 8 | 1,1,1,1,2 | $0,0,\frac{1}{2},\frac{1}{2},\frac{1}{2}$ | SB:$4_4^B \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{pmatrix}$ |
| $5_4^{\zeta_2^1}$ | 8 | 1,1,1,1,2 | $0,0,\frac{1}{2},\frac{1}{2},\frac{1}{2}$ | SB:$4_4^B$ F:$Z_4$ NR |
| $5_{-3}^{\zeta_2^1}$ | 8 | 1,1,1,1,2 | $0,0,\frac{1}{2},\frac{1}{2},\frac{5}{8}$ | SB:$4_{-3}^B$ F:$Z_2 \times Z_2$ |
| $5_{-3}^{\zeta_2^1}$ | 8 | 1,1,1,1,2 | $0,0,\frac{1}{2},\frac{1}{2},\frac{5}{8}$ | SB:$4_{-3}^B$ F:$Z_4$ NR |
| $5_{-2}^{\zeta_2^1}$ | 8 | 1,1,1,1,2 | $0,0,\frac{1}{2},\frac{1}{2},\frac{3}{4}$ | SB:$4_{-2}^B$ F:$Z_2 \times Z_2$ |
| $5_{-2}^{\zeta_2^1}$ | 8 | 1,1,1,1,2 | $0,0,\frac{1}{2},\frac{1}{2},\frac{3}{4}$ | SB:$4_{-2}^B$ F:$Z_4$ NR |
| $5_{-1}^{\zeta_2^1}$ | 8 | 1,1,1,1,2 | $0,0,\frac{1}{2},\frac{1}{2},\frac{7}{8}$ | SB:$4_{-1}^B$ F:$Z_2 \times Z_2$ |
| $5_{-1}^{\zeta_2^1}$ | 8 | 1,1,1,1,2 | $0,0,\frac{1}{2},\frac{1}{2},\frac{7}{8}$ | SB:$4_{-1}^B$ F:$Z_4$ NR |
| $5_2^{\zeta_2^1}$ | 14 | 1,1,2,2,2 | $0,0,\frac{1}{7},\frac{2}{7},\frac{4}{7}$ | SB:$7_2^B$ |
| $5_{-2}^{\zeta_2^1}$ | 14 | 1,1,2,2,2 | $0,0,\frac{3}{7},\frac{5}{7},\frac{6}{7}$ | SB:$7_{-2}^B$ |
| $5_{12/5}^{\zeta_2^1}$ | 26.180 | $1,1,\zeta_8^2,\zeta_8^2,\zeta_8^4$ | $0,0,\frac{1}{5},\frac{1}{5},\frac{3}{5}$ | SB:$4_{12/5}^B$ |
| $5_{-12/5}^{\zeta_2^1}$ | 26.180 | $1,1,\zeta_8^2,\zeta_8^2,\zeta_8^4$ | $0,0,\frac{4}{5},\frac{4}{5},\frac{2}{5}$ | SB:$4_{-12/5}^B$ |

TABLE XXIV. $Z_2$-SET orders for bosonic systems labeled in terms of sets of topological excitations. The list contains all topological orders with $N = 6$ $D^2 \leqslant 50$.

| $N_c^{\lvert\Theta\rvert}$ | $D^2$ | $d_1,d_2,\dots$ | $s_1,s_2,\dots$ | comment |
|---|---|---|---|---|
| $2_0^{\zeta_2^1}$ | 2 | 1,1 | 0,0 | $\mathcal{E} = \text{Rep}(Z_2)$ |
| $6_2^{\zeta_2^1}$ | 6 | 1,1,1,1,1,1 | $0,0,\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3}$ | $3_2^B \boxtimes \text{Rep}(Z_2)$ |
| $6_{-2}^{\zeta_2^1}$ | 6 | 1,1,1,1,1,1 | $0,0,\frac{2}{3},\frac{2}{3},\frac{2}{3},\frac{2}{3}$ | $3_{-2}^B \boxtimes \text{Rep}(Z_2)$ |
| $6_{1/2}^{\zeta_2^1}$ | 8 | $1,1,1,1,\zeta_2^1,\zeta_2^1$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{1}{16},\frac{1}{16}$ | $3_{1/2}^B \boxtimes \text{Rep}(Z_2)$ |
| $6_{1/2}^{\zeta_2^1}$ | 8 | $1,1,1,1,\zeta_2^1,\zeta_2^1$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{1}{16},\frac{1}{16}$ | SB:$3_{1/2}^B$ |
| $6_{3/2}^{\zeta_2^1}$ | 8 | $1,1,1,1,\zeta_2^1,\zeta_2^1$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{3}{16},\frac{3}{16}$ | $3_{3/2}^B \boxtimes \text{Rep}(Z_2)$ |
| $6_{3/2}^{\zeta_2^1}$ | 8 | $1,1,1,1,\zeta_2^1,\zeta_2^1$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{3}{16},\frac{3}{16}$ | SB:$3_{3/2}^B$ |
| $6_{5/2}^{\zeta_2^1}$ | 8 | $1,1,1,1,\zeta_2^1,\zeta_2^1$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{5}{16},\frac{5}{16}$ | $3_{5/2}^B \boxtimes \text{Rep}(Z_2)$ |
| $6_{5/2}^{\zeta_2^1}$ | 8 | $1,1,1,1,\zeta_2^1,\zeta_2^1$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{5}{16},\frac{5}{16}$ | SB:$3_{5/2}^B$ |
| $6_{7/2}^{\zeta_2^1}$ | 8 | $1,1,1,1,\zeta_2^1,\zeta_2^1$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{7}{16},\frac{7}{16}$ | $3_{7/2}^B \boxtimes \text{Rep}(Z_2)$ |
| $6_{7/2}^{\zeta_2^1}$ | 8 | $1,1,1,1,\zeta_2^1,\zeta_2^1$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{7}{16},\frac{7}{16}$ | SB:$3_{7/2}^B$ |
| $6_{-7/2}^{\zeta_2^1}$ | 8 | $1,1,1,1,\zeta_2^1,\zeta_2^1$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{9}{16},\frac{9}{16}$ | $3_{-7/2}^B \boxtimes \text{Rep}(Z_2)$ |
| $6_{-7/2}^{\zeta_2^1}$ | 8 | $1,1,1,1,\zeta_2^1,\zeta_2^1$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{9}{16},\frac{9}{16}$ | SB:$3_{-7/2}^B$ |
| $6_{-5/2}^{\zeta_2^1}$ | 8 | $1,1,1,1,\zeta_2^1,\zeta_2^1$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{11}{16},\frac{11}{16}$ | $3_{-5/2}^B \boxtimes \text{Rep}(Z_2)$ |
| $6_{-5/2}^{\zeta_2^1}$ | 8 | $1,1,1,1,\zeta_2^1,\zeta_2^1$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{11}{16},\frac{11}{16}$ | SB:$3_{-5/2}^B$ |
| $6_{-3/2}^{\zeta_2^1}$ | 8 | $1,1,1,1,\zeta_2^1,\zeta_2^1$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{13}{16},\frac{13}{16}$ | $3_{-3/2}^B \boxtimes \text{Rep}(Z_2)$ |
| $6_{-3/2}^{\zeta_2^1}$ | 8 | $1,1,1,1,\zeta_2^1,\zeta_2^1$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{13}{16},\frac{13}{16}$ | SB:$3_{-3/2}^B$ |
| $6_{-1/2}^{\zeta_2^1}$ | 8 | $1,1,1,1,\zeta_2^1,\zeta_2^1$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{15}{16},\frac{15}{16}$ | $3_{-1/2}^B \boxtimes \text{Rep}(Z_2)$ |
| $6_{-1/2}^{\zeta_2^1}$ | 8 | $1,1,1,1,\zeta_2^1,\zeta_2^1$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{15}{16},\frac{15}{16}$ | SB:$3_{-1/2}^B$ |
| $6_1^{\zeta_2^1}$ | 12 | 1,1,1,1,2,2 | $0,0,\frac{3}{4},\frac{3}{4},\frac{1}{12},\frac{1}{3}$ | SB:$6_1^B$ |
| $6_3^{\zeta_2^1}$ | 12 | 1,1,1,1,2,2 | $0,0,\frac{1}{4},\frac{1}{4},\frac{1}{3},\frac{7}{12}$ | SB:$6_3^B$ |
| $6_{-3}^{\zeta_2^1}$ | 12 | 1,1,1,1,2,2 | $0,0,\frac{3}{4},\frac{3}{4},\frac{5}{12},\frac{2}{3}$ | SB:$6_{-3}^B$ |
| $6_{-1}^{\zeta_2^1}$ | 12 | 1,1,1,1,2,2 | $0,0,\frac{1}{4},\frac{1}{4},\frac{2}{3},\frac{11}{12}$ | SB:$6_{-1}^B$ |
| $6_0^{\zeta_2^1}$ | 18 | 1,1,2,2,2,2 | $0,0,0,0,\frac{1}{3},\frac{2}{3}$ | SB:$9_0^B$ |
| $6_0^{\zeta_2^1}$ | 18 | 1,1,2,2,2,2 | $0,0,0,\frac{1}{9},\frac{4}{9},\frac{7}{9}$ | SB:$9_0^B$ |
| $6_0^{\zeta_2^1}$ | 18 | 1,1,2,2,2,2 | $0,0,0,\frac{2}{9},\frac{5}{9},\frac{8}{9}$ | SB:$9_0^B$ |
| $6_4^{\zeta_2^1}$ | 18 | 1,1,2,2,2,2 | $0,0,\frac{1}{3},\frac{1}{3},\frac{2}{3},\frac{2}{3}$ | SB:$9_4^B$ |
| $6_{8/7}^{\zeta_2^1}$ | 18.591 | $1,1,\zeta_5^1,\zeta_5^1,\zeta_5^2,\zeta_5^2$ | $0,0,\frac{6}{7},\frac{6}{7},\frac{2}{7},\frac{2}{7}$ | $3_{8/7}^B \boxtimes \text{Rep}(Z_2)$ |
| $6_{-8/7}^{\zeta_2^1}$ | 18.591 | $1,1,\zeta_5^1,\zeta_5^1,\zeta_5^2,\zeta_5^2$ | $0,0,\frac{1}{7},\frac{1}{7},\frac{5}{7},\frac{5}{7}$ | $3_{-8/7}^B \boxtimes \text{Rep}(Z_2)$ |
| $6_{4/5}^{\zeta_2^1}$ | 21.708 | $1,1,\zeta_3^1,\zeta_3^1,2,\zeta_8^4$ | $0,0,\frac{2}{5},\frac{2}{5},\frac{2}{3},\frac{1}{15}$ | $2_{14/5}^B \boxtimes 3_{-2}^{\zeta_2^1}$ |
| $6_{16/5}^{\zeta_2^1}$ | 21.708 | $1,1,\zeta_3^1,\zeta_3^1,2,\zeta_8^4$ | $0,0,\frac{3}{5},\frac{3}{5},\frac{2}{3},\frac{4}{15}$ | $2_{-14/5}^B \boxtimes 3_{-2}^{\zeta_2^1}$ |
| $6_{-16/5}^{\zeta_2^1}$ | 21.708 | $1,1,\zeta_3^1,\zeta_3^1,2,\zeta_8^4$ | $0,0,\frac{2}{5},\frac{2}{5},\frac{1}{3},\frac{11}{15}$ | $2_{14/5}^B \boxtimes 3_2^{\zeta_2^1}$ |
| $6_{-4/5}^{\zeta_2^1}$ | 21.708 | $1,1,\zeta_3^1,\zeta_3^1,2,\zeta_8^4$ | $0,0,\frac{3}{5},\frac{3}{5},\frac{1}{3},\frac{14}{15}$ | $2_{-14/5}^B \boxtimes 3_2^{\zeta_2^1}$ |

All the $Z_2$-SET orders in Table XXII are realizable. Some of the them are realized as $N_c^B \boxtimes \text{Rep}(Z_2)$, as indicated in the comment column. Here, $N_c^B$ describes a neutral bosonic topological order (which was denoted as $N_c^B$ in Ref. [11]) with rank $N$ and central charge $c$, which does not transform under the $Z_2$ symmetry. For example, $2_1^B$ is the $\nu = 1/2$ bosonic Laughlin state, and $2_{14/5}^B$ is the bosonic Fibonacci state [11]. $\text{Rep}(Z_2)$ describes a product state with $Z_2$ symmetry of $Z_2$ charged bosons. $N_c^B \boxtimes \text{Rep}(Z_2)$ is simply the stacking of the neutral bosonic topological order $N_c^B$ with the $Z_2$ symmetric product state.

We also introduced $N_c^B \boxtimes^t \text{Rep}(Z_2)$ which describe a state similar to $N_c^B \boxtimes \text{Rep}(Z_2)$, except here the bosons that form the topological order $N_c^B$ also carries a $Z_2$ charge. The $3_2^{\zeta_2^1}$

TABLE XXV. $Z_3$-SET orders for bosonic systems labeled in terms of sets of topological excitations. The list contains all topological orders with $N = 4,5,6$ $D^2 \leqslant 100$, $N = 7$ $D^2 \leqslant 60$, $N = 8$ $D^2 \leqslant 40$, and $N = 9$ $D^2 \leqslant 28$.

| $N_c^{\lvert\Theta\rvert}$ | $D^2$ | $d_1,d_2,\dots$ | $s_1,s_2,\dots$ | comment |
|---|---|---|---|---|
| $3_0^{\zeta_4^1}$ | 3 | 1,1,1 | 0,0,0 | $\mathcal{E} = \text{Rep}(Z_3)$ |
| $4_4^{\zeta_4^1}$ | 12 | 1,1,1,3 | $0,0,0,\frac{1}{2}$ | SB:$K = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{pmatrix}$ |
| $6_1^{\zeta_4^1}$ | 6 | 1,1,1,1,1,1 | $0,0,0,\frac{1}{4},\frac{1}{4},\frac{1}{4}$ | $2_1^B \boxtimes \text{Rep}(Z_3)$ |
| $6_{-1}^{\zeta_4^1}$ | 6 | 1,1,1,1,1,1 | $0,0,0,\frac{3}{4},\frac{3}{4},\frac{3}{4}$ | $2_{-1}^B \boxtimes \text{Rep}(Z_3)$ |
| $6_{14/5}^{\zeta_4^1}$ | 10.854 | $1,1,1,\zeta_3^1,\zeta_3^1,\zeta_3^1$ | $0,0,0,\frac{2}{5},\frac{2}{5},\frac{2}{5}$ | $2_{14/5}^B \boxtimes \text{Rep}(Z_3)$ |
| $6_{-14/5}^{\zeta_4^1}$ | 10.854 | $1,1,1,\zeta_3^1,\zeta_3^1,\zeta_3^1$ | $0,0,0,\frac{3}{5},\frac{3}{5},\frac{3}{5}$ | $2_{-14/5}^B \boxtimes \text{Rep}(Z_3)$ |
| $8_3^{\zeta_4^1}$ | 24 | 1,1,1,1,1,1,3,3 | $0,0,0,\frac{3}{4},\frac{3}{4},\frac{3}{4},\frac{1}{4},\frac{1}{2}$ | $2_{-1}^B \boxtimes 4_4^{\zeta_4^1}$ |
| $8_{-3}^{\zeta_4^1}$ | 24 | 1,1,1,1,1,1,3,3 | $0,0,0,\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{2},\frac{3}{4}$ | $2_1^B \boxtimes 4_4^{\zeta_4^1}$ |
| $8_{6/5}^{\zeta_4^1}$ | 43.416 | $1,1,1,\zeta_3^1,\zeta_3^1,\zeta_3^1,3,\frac{3+\sqrt{45}}{2}$ | $0,0,0,\frac{3}{5},\frac{3}{5},\frac{3}{5},\frac{1}{2},\frac{1}{10}$ | $2_{-14/5}^B \boxtimes 4_4^{\zeta_4^1}$ |
| $8_{-6/5}^{\zeta_4^1}$ | 43.416 | $1,1,1,\zeta_3^1,\zeta_3^1,\zeta_3^1,3,\frac{3+\sqrt{45}}{2}$ | $0,0,0,\frac{2}{5},\frac{2}{5},\frac{2}{5},\frac{1}{2},\frac{9}{10}$ | $2_{14/5}^B \boxtimes 4_4^{\zeta_4^1}$ |
| $9_2^{\zeta_4^1}$ | 9 | 1,1,1,1,1,1,1,1,1 | $0,0,0,\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3}$ | SB:$3_2^B$ F:$Z_9$ |
| $9_2^{\zeta_4^1}$ | 9 | 1,1,1,1,1,1,1,1,1 | $0,0,0,\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3}$ | $3_2^B \boxtimes \text{Rep}(Z_3)$ F:$Z_3 \times Z_3$ |
| $9_{-2}^{\zeta_4^1}$ | 9 | 1,1,1,1,1,1,1,1,1 | $0,0,0,\frac{2}{3},\frac{2}{3},\frac{2}{3},\frac{2}{3},\frac{2}{3},\frac{2}{3}$ | SB:$3_{-2}^B$ F:$Z_9$ |
| $9_{-2}^{\zeta_4^1}$ | 9 | 1,1,1,1,1,1,1,1,1 | $0,0,0,\frac{2}{3},\frac{2}{3},\frac{2}{3},\frac{2}{3},\frac{2}{3},\frac{2}{3}$ | $3_{-2}^B \boxtimes \text{Rep}(Z_3)$ F:$Z_3 \times Z_3$ |
| $9_{1/2}^{\zeta_4^1}$ | 12 | $1,1,1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1$ | $0,0,0,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{16},\frac{1}{16},\frac{1}{16}$ | $3_{1/2}^B \boxtimes \text{Rep}(Z_3)$ |
| $9_{3/2}^{\zeta_4^1}$ | 12 | $1,1,1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1$ | $0,0,0,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{3}{16},\frac{3}{16},\frac{3}{16}$ | $3_{3/2}^B \boxtimes \text{Rep}(Z_3)$ |
| $9_{5/2}^{\zeta_4^1}$ | 12 | $1,1,1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1$ | $0,0,0,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{5}{16},\frac{5}{16},\frac{5}{16}$ | $3_{5/2}^B \boxtimes \text{Rep}(Z_3)$ |
| $9_{7/2}^{\zeta_4^1}$ | 12 | $1,1,1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1$ | $0,0,0,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{7}{16},\frac{7}{16},\frac{7}{16}$ | $3_{7/2}^B \boxtimes \text{Rep}(Z_3)$ |
| $9_{-7/2}^{\zeta_4^1}$ | 12 | $1,1,1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1$ | $0,0,0,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{9}{16},\frac{9}{16},\frac{9}{16}$ | $3_{-7/2}^B \boxtimes \text{Rep}(Z_3)$ |
| $9_{-5/2}^{\zeta_4^1}$ | 12 | $1,1,1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1$ | $0,0,0,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{11}{16},\frac{11}{16},\frac{11}{16}$ | $3_{-5/2}^B \boxtimes \text{Rep}(Z_3)$ |
| $9_{-3/2}^{\zeta_4^1}$ | 12 | $1,1,1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1$ | $0,0,0,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{13}{16},\frac{13}{16},\frac{13}{16}$ | $3_{-3/2}^B \boxtimes \text{Rep}(Z_3)$ |
| $9_{-1/2}^{\zeta_4^1}$ | 12 | $1,1,1,1,1,1,\zeta_2^1,\zeta_2^1,\zeta_2^1$ | $0,0,0,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{15}{16},\frac{15}{16},\frac{15}{16}$ | $3_{-1/2}^B \boxtimes \text{Rep}(Z_3)$ |
| $9_{8/7}^{\zeta_4^1}$ | 27.887 | $1,1,1,\zeta_5^1,\zeta_5^1,\zeta_5^1,\zeta_5^2,\zeta_5^2,\zeta_5^2$ | $0,0,0,\frac{6}{7},\frac{6}{7},\frac{6}{7},\frac{2}{7},\frac{2}{7},\frac{2}{7}$ | $3_{8/7}^B \boxtimes \text{Rep}(Z_3)$ |
| $9_{-8/7}^{\zeta_4^1}$ | 27.887 | $1,1,1,\zeta_5^1,\zeta_5^1,\zeta_5^1,\zeta_5^2,\zeta_5^2,\zeta_5^2$ | $0,0,0,\frac{1}{7},\frac{1}{7},\frac{1}{7},\frac{5}{7},\frac{5}{7},\frac{5}{7}$ | $3_{-8/7}^B \boxtimes \text{Rep}(Z_3)$ |

state can be realized by double-layer FQH state with $K$-matrix $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$, which is discussed in the main text.

Since we did not use the condition of the existence of modular extensions when we calculate the tables, some the entries in the tables may not by realizable by any 2+1D bosonic systems. We use NR in the comment column to indicate such entries (see Table XXIII).

### 2. $Z_3$-SET orders

Table XXV lists the $Z_3$-SET orders (up to invertible ones) for 2+1D bosonic systems.

The $Z_3$-SET state $4_4^{\zeta_4^1}$ in the table becomes the $K = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{pmatrix}$ 4-layer FQH state after we break the $Z_3$-symmetry. We can add the $Z_3$-symmetry back to obtain the $Z_3$-SET state. The $Z_3$-symmetry is the cyclic permutation of the second, the third, and the fourth layers.

Without the symmetry, the $K = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{pmatrix}$ state has four types of particles, a trivial boson and three nontrivial fermions. With the symmetry, the three fermions become degenerate and is combined into the $d = 3$ particle (the fourth particle) for the $4_4^{\zeta_4^1}$ state. The first three particles for the $4_4^{\zeta_4^1}$ state all come from the trivial boson. They carry different $Z_3$ charges: 0,1,2, in the presence of the symmetry.

### 3. $S_3$-SET orders

Tables XXVII and XXVIII list the $S_3$-SET orders (up to invertible ones) for 2+1D bosonic systems. Table XXVII has three $5_4^{\sqrt{6}}$ entries that have identical $(d_i,s_i)$. However, the three entries have different fusion rules (see Table XXVI). If we break the symmetry, the three entries all reduce to the

TABLE XXVI. The fusion rules for the three $5_4^{\sqrt{6}}$ entries in Table XXVII. The three entries have identical $(d_i,s_i)$ but different fusions rules. $\mathbf{1}$, $a$, $b$ are the three irreducible representations of $S_3$ with dimension 1, 1, 2.

| | | | | | |
|---|---|---|---|---|---|
| $s_i$ | 0 | 0 | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $d_i$ | 1 | 1 | 2 | 3 | 3 |
| $5_4^{\sqrt{6}}$ | $\mathbf{1}$ | $a$ | $b$ | $\sigma$ | $\tau$ |
| $\mathbf{1}$ | $\mathbf{1}$ | $a$ | $b$ | $\sigma$ | $\tau$ |
| $a$ | $a$ | $\mathbf{1}$ | $b$ | $\tau$ | $\sigma$ |
| $b$ | $b$ | $b$ | $\mathbf{1}\oplus a\oplus b$ | $\sigma\oplus\tau$ | $\sigma\oplus\tau$ |
| $\sigma$ | $\sigma$ | $\tau$ | $\sigma\oplus\tau$ | $\mathbf{1}\oplus b\oplus 2\sigma$ | $a\oplus b\oplus 2\tau$ |
| $\tau$ | $\tau$ | $\sigma$ | $\sigma\oplus\tau$ | $a\oplus b\oplus 2\tau$ | $\mathbf{1}\oplus b\oplus 2\sigma$ |
| $s_i$ | 0 | 0 | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $d_i$ | 1 | 1 | 2 | 3 | 3 |
| $5_4^{\sqrt{6}}$ | $\mathbf{1}$ | $a$ | $b$ | $\sigma$ | $\tau$ |
| $\mathbf{1}$ | $\mathbf{1}$ | $a$ | $b$ | $\sigma$ | $\tau$ |
| $a$ | $a$ | $\mathbf{1}$ | $b$ | $\tau$ | $\sigma$ |
| $b$ | $b$ | $b$ | $\mathbf{1}\oplus a\oplus b$ | $\sigma\oplus\tau$ | $\sigma\oplus\tau$ |
| $\sigma$ | $\sigma$ | $\tau$ | $\sigma\oplus\tau$ | $\mathbf{1}\oplus b\oplus\sigma\oplus\tau$ | $a\oplus b\oplus\sigma\oplus\tau$ |
| $\tau$ | $\tau$ | $\sigma$ | $\sigma\oplus\tau$ | $a\oplus b\oplus\sigma\oplus\tau$ | $\mathbf{1}\oplus b\oplus\sigma\oplus\tau$ |
| $s_i$ | 0 | 0 | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $d_i$ | 1 | 1 | 2 | 3 | 3 |
| $5_4^{\sqrt{6}}$ | $\mathbf{1}$ | $a$ | $b$ | $\sigma$ | $\tau$ |
| $\mathbf{1}$ | $\mathbf{1}$ | $a$ | $b$ | $\sigma$ | $\tau$ |
| $a$ | $a$ | $\mathbf{1}$ | $b$ | $\tau$ | $\sigma$ |
| $b$ | $b$ | $b$ | $\mathbf{1}\oplus a\oplus b$ | $\sigma\oplus\tau$ | $\sigma\oplus\tau$ |
| $\sigma$ | $\sigma$ | $\tau$ | $\sigma\oplus\tau$ | $a\oplus b\oplus\sigma\oplus\tau$ | $\mathbf{1}\oplus b\oplus\sigma\oplus\tau$ |
| $\tau$ | $\tau$ | $\sigma$ | $\sigma\oplus\tau$ | $\mathbf{1}\oplus b\oplus\sigma\oplus\tau$ | $a\oplus b\oplus\sigma\oplus\tau$ |

$K=\begin{pmatrix}2&1&1&1\\1&2&0&0\\1&0&2&0\\1&0&0&2\end{pmatrix}$ four-layer state. So we expect the $S_3$ symmetry is the permutation symmetry of the second, the third, and the fourth layers.

The second $5_4^{\sqrt{6}}$ entry can be realized by the $K=\begin{pmatrix}2&1&1&1\\1&2&0&0\\1&0&2&0\\1&0&0&2\end{pmatrix}$ four-layer state. The two $d=3$ fermions are the direct sum of the three degenerate fermions in

TABLE XXVII. $S_3$-SET orders for bosonic systems labeled in terms of sets of topological excitations. The list contains all topological orders with $N=4,5,6$ $D^2\leqslant 100$, $N=7$ $D^2\leqslant 60$, and $N=8$ $D^2\leqslant 40$. (In fact, we fail to find any bosonic $S_3$-SET orders with $N=4,7,8$.)

| $N_c^{\lvert\Theta\rvert}$ | $D^2$ | $d_1,d_2,\dots$ | $s_1,s_2,\dots$ | comment |
|---|---|---|---|---|
| $3_0^{\sqrt{6}}$ | 6 | 1,1,2 | 0,0,0 | $\mathcal{E}=\text{Rep}(S_3)$ |
| $5_4^{\sqrt{6}}$ | 24 | 1,1,2,3,3 | $0,0,0,\frac{1}{2},\frac{1}{2}$ | SB:$4_4^B$ |
| $5_4^{\sqrt{6}}$ | 24 | 1,1,2,3,3 | $0,0,0,\frac{1}{2},\frac{1}{2}$ | SB:$4_4^B\begin{pmatrix}2&1&1&1\\1&2&0&0\\1&0&2&0\\1&0&0&2\end{pmatrix}$ |
| $5_4^{\sqrt{6}}$ | 24 | 1,1,2,3,3 | $0,0,0,\frac{1}{2},\frac{1}{2}$ | SB:$4_4^B$ |
| $6_1^{\sqrt{6}}$ | 12 | 1,1,2,1,1,2 | $0,0,0,\frac{1}{4},\frac{1}{4},\frac{1}{4}$ | $2_1^B\boxtimes\text{Rep}(S_3)$ |
| $6_1^{\sqrt{6}}$ | 12 | 1,1,2,1,1,2 | $0,0,0,\frac{1}{4},\frac{1}{4},\frac{1}{4}$ | SB:$2_1^B$ |
| $6_{-1}^{\sqrt{6}}$ | 12 | 1,1,2,1,1,2 | $0,0,0,\frac{3}{4},\frac{3}{4},\frac{3}{4}$ | $2_{-1}^B\boxtimes\text{Rep}(S_3)$ |
| $6_{-1}^{\sqrt{6}}$ | 12 | 1,1,2,1,1,2 | $0,0,0,\frac{3}{4},\frac{3}{4},\frac{3}{4}$ | SB:$2_{-1}^B$ |
| $6_2^{\sqrt{6}}$ | 18 | 1,1,2,2,2,2 | $0,0,0,\frac{1}{3},\frac{1}{3},\frac{1}{3}$ | SB:$3_2^B$ |
| $6_2^{\sqrt{6}}$ | 18 | 1,1,2,2,2,2 | $0,0,0,\frac{1}{3},\frac{1}{3},\frac{1}{3}$ | SB:$3_2^B$ |
| $6_{-2}^{\sqrt{6}}$ | 18 | 1,1,2,2,2,2 | $0,0,0,\frac{2}{3},\frac{2}{3},\frac{2}{3}$ | SB:$3_{-2}^B$ |
| $6_{-2}^{\sqrt{6}}$ | 18 | 1,1,2,2,2,2 | $0,0,0,\frac{2}{3},\frac{2}{3},\frac{2}{3}$ | SB:$3_{-2}^B$ |
| $6_{14/5}^{\sqrt{6}}$ | 21.708 | $1,1,2,\zeta_3^1,\zeta_3^1,\zeta_8^4$ | $0,0,0,\frac{2}{5},\frac{2}{5},\frac{2}{5}$ | $2_{14/5}^B\boxtimes\text{Rep}(S_3)$ |
| $6_{-14/5}^{\sqrt{6}}$ | 21.708 | $1,1,2,\zeta_3^1,\zeta_3^1,\zeta_8^4$ | $0,0,0,\frac{3}{5},\frac{3}{5},\frac{3}{5}$ | $2_{-14/5}^B\boxtimes\text{Rep}(S_3)$ |

TABLE XXVIII. $S_3$-SET orders for bosonic systems labeled in terms of sets of topological excitations. The list contains all topological orders with $N = 9$ $D^2 \leqslant 30$.

| $N_c^{\lvert\Theta\rvert}$ | $D^2$ | $d_1, d_2, \ldots$ | $s_1, s_2, \ldots$ | comment |
|---|---|---|---|---|
| $3_0^{\sqrt{6}}$ | 6 | $1,1,2$ | $0,0,0$ | $\mathcal{E} = \text{Rep}(S_3)$ |
| $9_2^{\sqrt{6}}$ | 18 | $1,1,2,1,1,1,1,2,2$ | $0,0,0,\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3}$ | $3_2^B \boxtimes \text{Rep}(S_3)$ |
| $9_{-2}^{\sqrt{6}}$ | 18 | $1,1,2,1,1,1,1,2,2$ | $0,0,0,\frac{2}{3},\frac{2}{3},\frac{2}{3},\frac{2}{3},\frac{2}{3},\frac{2}{3}$ | $3_{-2}^B \boxtimes \text{Rep}(S_3)$ |
| $9_0^{\sqrt{6}}$ | 24 | $1,1,2,1,1,2,2,2,2$ | $0,0,0,\frac{1}{2},\frac{1}{2},0,0,0,\frac{1}{2}$ | SB:$4_0^B$ |
| $9_0^{\sqrt{6}}$ | 24 | $1,1,2,1,1,2,2,2,2$ | $0,0,0,\frac{1}{2},\frac{1}{2},0,0,0,\frac{1}{2}$ | SB:$4_0^B$ |
| $9_1^{\sqrt{6}}$ | 24 | $1,1,2,1,1,2,2,2,2$ | $0,0,0,\frac{1}{2},\frac{1}{2},\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{1}{2}$ | SB:$4_1^B$ |
| $9_1^{\sqrt{6}}$ | 24 | $1,1,2,1,1,2,2,2,2$ | $0,0,0,\frac{1}{2},\frac{1}{2},\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{1}{2}$ | SB:$4_1^B$ |
| $9_2^{\sqrt{6}}$ | 24 | $1,1,2,1,1,2,2,2,2$ | $0,0,0,\frac{1}{2},\frac{1}{2},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{2}$ | SB:$4_2^B$ |
| $9_2^{\sqrt{6}}$ | 24 | $1,1,2,1,1,2,2,2,2$ | $0,0,0,\frac{1}{2},\frac{1}{2},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{2}$ | SB:$4_2^B$ |
| $9_3^{\sqrt{6}}$ | 24 | $1,1,2,1,1,2,2,2,2$ | $0,0,0,\frac{1}{2},\frac{1}{2},\frac{3}{8},\frac{3}{8},\frac{3}{8},\frac{1}{2}$ | SB:$4_3^B$ |
| $9_3^{\sqrt{6}}$ | 24 | $1,1,2,1,1,2,2,2,2$ | $0,0,0,\frac{1}{2},\frac{1}{2},\frac{3}{8},\frac{3}{8},\frac{3}{8},\frac{1}{2}$ | SB:$4_3^B$ |
| $9_4^{\sqrt{6}}$ | 24 | $1,1,2,1,1,2,2,2,2$ | $0,0,0,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}$ | SB:$4_4^B$ |
| $9_4^{\sqrt{6}}$ | 24 | $1,1,2,1,1,2,2,2,2$ | $0,0,0,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}$ | SB:$4_4^B$ |
| $9_{-3}^{\sqrt{6}}$ | 24 | $1,1,2,1,1,2,2,2,2$ | $0,0,0,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{5}{8},\frac{5}{8},\frac{5}{8}$ | SB:$4_{-3}^B$ |
| $9_{-3}^{\sqrt{6}}$ | 24 | $1,1,2,1,1,2,2,2,2$ | $0,0,0,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{5}{8},\frac{5}{8},\frac{5}{8}$ | SB:$4_{-3}^B$ |
| $9_{-2}^{\sqrt{6}}$ | 24 | $1,1,2,1,1,2,2,2,2$ | $0,0,0,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{3}{4},\frac{3}{4},\frac{3}{4}$ | SB:$4_{-2}^B$ |
| $9_{-2}^{\sqrt{6}}$ | 24 | $1,1,2,1,1,2,2,2,2$ | $0,0,0,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{3}{4},\frac{3}{4},\frac{3}{4}$ | SB:$4_{-2}^B$ |
| $9_{-1}^{\sqrt{6}}$ | 24 | $1,1,2,1,1,2,2,2,2$ | $0,0,0,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{7}{8},\frac{7}{8},\frac{7}{8}$ | SB:$4_{-1}^B$ |
| $9_{-1}^{\sqrt{6}}$ | 24 | $1,1,2,1,1,2,2,2,2$ | $0,0,0,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{7}{8},\frac{7}{8},\frac{7}{8}$ | SB:$4_{-1}^B$ |
| $9_{5/2}^{\sqrt{6}}$ | 24 | $1,1,2,1,1,\zeta_2^1,\zeta_2^1,2,\sqrt{8}$ | $0,0,0,\frac{1}{2},\frac{1}{2},\frac{5}{16},\frac{5}{16},\frac{1}{2},\frac{5}{16}$ | $3_{5/2}^B \boxtimes \text{Rep}(S_3)$ |
| $9_{5/2}^{\sqrt{6}}$ | 24 | $1,1,2,1,1,\zeta_2^1,\zeta_2^1,2,\sqrt{8}$ | $0,0,0,\frac{1}{2},\frac{1}{2},\frac{5}{16},\frac{5}{16},\frac{1}{2},\frac{5}{16}$ | SB:$3_{5/2}^B$ |
| $9_{1/2}^{\sqrt{6}}$ | 24 | $1,1,2,1,1,\zeta_2^1,\zeta_2^1,2,\sqrt{8}$ | $0,0,0,\frac{1}{2},\frac{1}{2},\frac{1}{16},\frac{1}{16},\frac{1}{2},\frac{1}{16}$ | $3_{1/2}^B \boxtimes \text{Rep}(S_3)$ |
| $9_{1/2}^{\sqrt{6}}$ | 24 | $1,1,2,1,1,\zeta_2^1,\zeta_2^1,2,\sqrt{8}$ | $0,0,0,\frac{1}{2},\frac{1}{2},\frac{1}{16},\frac{1}{16},\frac{1}{2},\frac{1}{16}$ | SB:$3_{1/2}^B$ |
| $9_{3/2}^{\sqrt{6}}$ | 24 | $1,1,2,1,1,\zeta_2^1,\zeta_2^1,2,\sqrt{8}$ | $0,0,0,\frac{1}{2},\frac{1}{2},\frac{3}{16},\frac{3}{16},\frac{1}{2},\frac{3}{16}$ | $3_{3/2}^B \boxtimes \text{Rep}(S_3)$ |
| $9_{3/2}^{\sqrt{6}}$ | 24 | $1,1,2,1,1,\zeta_2^1,\zeta_2^1,2,\sqrt{8}$ | $0,0,0,\frac{1}{2},\frac{1}{2},\frac{3}{16},\frac{3}{16},\frac{1}{2},\frac{3}{16}$ | SB:$3_{3/2}^B$ |
| $9_{7/2}^{\sqrt{6}}$ | 24 | $1,1,2,1,1,\zeta_2^1,\zeta_2^1,2,\sqrt{8}$ | $0,0,0,\frac{1}{2},\frac{1}{2},\frac{7}{16},\frac{7}{16},\frac{1}{2},\frac{7}{16}$ | $3_{7/2}^B \boxtimes \text{Rep}(S_3)$ |
| $9_{7/2}^{\sqrt{6}}$ | 24 | $1,1,2,1,1,\zeta_2^1,\zeta_2^1,2,\sqrt{8}$ | $0,0,0,\frac{1}{2},\frac{1}{2},\frac{7}{16},\frac{7}{16},\frac{1}{2},\frac{7}{16}$ | SB:$3_{7/2}^B$ |
| $9_{-7/2}^{\sqrt{6}}$ | 24 | $1,1,2,1,1,\zeta_2^1,\zeta_2^1,2,\sqrt{8}$ | $0,0,0,\frac{1}{2},\frac{1}{2},\frac{9}{16},\frac{9}{16},\frac{1}{2},\frac{9}{16}$ | $3_{-7/2}^B \boxtimes \text{Rep}(S_3)$ |
| $9_{-7/2}^{\sqrt{6}}$ | 24 | $1,1,2,1,1,\zeta_2^1,\zeta_2^1,2,\sqrt{8}$ | $0,0,0,\frac{1}{2},\frac{1}{2},\frac{9}{16},\frac{9}{16},\frac{1}{2},\frac{9}{16}$ | SB:$3_{-7/2}^B$ |
| $9_{-5/2}^{\sqrt{6}}$ | 24 | $1,1,2,1,1,\zeta_2^1,\zeta_2^1,2,\sqrt{8}$ | $0,0,0,\frac{1}{2},\frac{1}{2},\frac{11}{16},\frac{11}{16},\frac{1}{2},\frac{11}{16}$ | $3_{-5/2}^B \boxtimes \text{Rep}(S_3)$ |
| $9_{-5/2}^{\sqrt{6}}$ | 24 | $1,1,2,1,1,\zeta_2^1,\zeta_2^1,2,\sqrt{8}$ | $0,0,0,\frac{1}{2},\frac{1}{2},\frac{11}{16},\frac{11}{16},\frac{1}{2},\frac{11}{16}$ | SB:$3_{-5/2}^B$ |
| $9_{-3/2}^{\sqrt{6}}$ | 24 | $1,1,2,1,1,\zeta_2^1,\zeta_2^1,2,\sqrt{8}$ | $0,0,0,\frac{1}{2},\frac{1}{2},\frac{13}{16},\frac{13}{16},\frac{1}{2},\frac{13}{16}$ | $3_{-3/2}^B \boxtimes \text{Rep}(S_3)$ |
| $9_{-3/2}^{\sqrt{6}}$ | 24 | $1,1,2,1,1,\zeta_2^1,\zeta_2^1,2,\sqrt{8}$ | $0,0,0,\frac{1}{2},\frac{1}{2},\frac{13}{16},\frac{13}{16},\frac{1}{2},\frac{13}{16}$ | SB:$3_{-3/2}^B$ |
| $9_{-1/2}^{\sqrt{6}}$ | 24 | $1,1,2,1,1,\zeta_2^1,\zeta_2^1,2,\sqrt{8}$ | $0,0,0,\frac{1}{2},\frac{1}{2},\frac{15}{16},\frac{15}{16},\frac{1}{2},\frac{15}{16}$ | $3_{-1/2}^B \boxtimes \text{Rep}(S_3)$ |
| $9_{-1/2}^{\sqrt{6}}$ | 24 | $1,1,2,1,1,\zeta_2^1,\zeta_2^1,2,\sqrt{8}$ | $0,0,0,\frac{1}{2},\frac{1}{2},\frac{15}{16},\frac{15}{16},\frac{1}{2},\frac{15}{16}$ | SB:$3_{-1/2}^B$ |
| $9_0^{\sqrt{6}}$ | 30 | $1,1,2,2,2,2,2,2,2$ | $0,0,0,\frac{1}{5},\frac{1}{5},\frac{1}{5},\frac{4}{5},\frac{4}{5},\frac{4}{5}$ | SB:$5_0^B$ |
| $9_4^{\sqrt{6}}$ | 30 | $1,1,2,2,2,2,2,2,2$ | $0,0,0,\frac{2}{5},\frac{2}{5},\frac{2}{5},\frac{3}{5},\frac{3}{5},\frac{3}{5}$ | SB:$5_4^B$ |
| $9_{8/7}^{\sqrt{6}}$ | 55.775 | $1,1,2,\zeta_5^1,\zeta_5^1,\zeta_5^2,\zeta_5^2,2\zeta_5^1,\zeta_{12}^6$ | $0,0,0,\frac{6}{7},\frac{6}{7},\frac{2}{7},\frac{2}{7},\frac{6}{7},\frac{2}{7}$ | $3_{8/7}^B \boxtimes \text{Rep}(S_3)$ |
| $9_{-8/7}^{\sqrt{6}}$ | 55.775 | $1,1,2,\zeta_5^1,\zeta_5^1,\zeta_5^2,\zeta_5^2,2\zeta_5^1,\zeta_{12}^6$ | $0,0,0,\frac{1}{7},\frac{1}{7},\frac{5}{7},\frac{5}{7},\frac{1}{7},\frac{5}{7}$ | $3_{-8/7}^B \boxtimes \text{Rep}(S_3)$ |

TABLE XXIX. $Z_2 \times Z_2$-SET orders for bosonic systems labeled in terms of sets of topological excitations. The list contains all topological orders with $N = 5$ $D^2 \leqslant 100$ and $N = 6$ $D^2 \leqslant 200$.

| $N_c^{\|\Theta\|}$ | $D^2$ | $d_1,d_2,\dots$ | $s_1,s_2,\dots$ | comment |
|---|---|---|---|---|
| $4_0^2$ | 4 | 1,1,1,1 | 0,0,0,0 | $\mathcal{E} = \text{Rep}(Z_2 \times Z_2)$ |
| $5_1^2$ | 8 | 1,1,1,1,2 | $0,0,0,0,\frac{1}{4}$ | SB:$2_1^B$ |
| $5_{-1}^2$ | 8 | 1,1,1,1,2 | $0,0,0,0,\frac{3}{4}$ | SB:$2_{-1}^B$ |
| $5_{14/5}^2$ | 14.472 | $1,1,1,1,\zeta_8^4$ | $0,0,0,0,\frac{2}{5}$ | SB:$2_{14/5}^B$ |
| $5_{-14/5}^2$ | 14.472 | $1,1,1,1,\zeta_8^4$ | $0,0,0,0,\frac{3}{5}$ | SB:$2_{-14/5}^B$ |
| $6_2^2$ | 12 | 1,1,1,1,2,2 | $0,0,0,0,\frac{1}{3},\frac{1}{3}$ | SB:$3_2^B$ |
| $6_2^2$ | 12 | 1,1,1,1,2,2 | $0,0,0,0,\frac{1}{3},\frac{1}{3}$ | SB:$3_2^B$ |
| $6_2^2$ | 12 | 1,1,1,1,2,2 | $0,0,0,0,\frac{1}{3},\frac{1}{3}$ | SB:$3_2^B$ |
| $6_2^2$ | 12 | 1,1,1,1,2,2 | $0,0,0,0,\frac{1}{3},\frac{1}{3}$ | SB:$3_2^B$ |
| $6_{-2}^2$ | 12 | 1,1,1,1,2,2 | $0,0,0,0,\frac{2}{3},\frac{2}{3}$ | SB:$3_{-2}^B$ |
| $6_{-2}^2$ | 12 | 1,1,1,1,2,2 | $0,0,0,0,\frac{2}{3},\frac{2}{3}$ | SB:$3_{-2}^B$ |
| $6_{-2}^2$ | 12 | 1,1,1,1,2,2 | $0,0,0,0,\frac{2}{3},\frac{2}{3}$ | SB:$3_{-2}^B$ |
| $6_{-2}^2$ | 12 | 1,1,1,1,2,2 | $0,0,0,0,\frac{2}{3},\frac{2}{3}$ | SB:$3_{-2}^B$ |
| $6_{1/2}^2$ | 16 | $1,1,1,1,2,\sqrt{8}$ | $0,0,0,0,\frac{1}{2},\frac{1}{16}$ | SB:$3_{1/2}^B$ |
| $6_{3/2}^2$ | 16 | $1,1,1,1,2,\sqrt{8}$ | $0,0,0,0,\frac{1}{2},\frac{3}{16}$ | SB:$3_{3/2}^B$ |
| $6_{5/2}^2$ | 16 | $1,1,1,1,2,\sqrt{8}$ | $0,0,0,0,\frac{1}{2},\frac{5}{16}$ | SB:$3_{5/2}^B$ |
| $6_{7/2}^2$ | 16 | $1,1,1,1,2,\sqrt{8}$ | $0,0,0,0,\frac{1}{2},\frac{7}{16}$ | SB:$3_{7/2}^B$ |
| $6_{-7/2}^2$ | 16 | $1,1,1,1,2,\sqrt{8}$ | $0,0,0,0,\frac{1}{2},\frac{9}{16}$ | SB:$3_{-7/2}^B$ |
| $6_{-5/2}^2$ | 16 | $1,1,1,1,2,\sqrt{8}$ | $0,0,0,0,\frac{1}{2},\frac{11}{16}$ | SB:$3_{-5/2}^B$ |
| $6_{-3/2}^2$ | 16 | $1,1,1,1,2,\sqrt{8}$ | $0,0,0,0,\frac{1}{2},\frac{13}{16}$ | SB:$3_{-3/2}^B$ |
| $6_{-1/2}^2$ | 16 | $1,1,1,1,2,\sqrt{8}$ | $0,0,0,0,\frac{1}{2},\frac{15}{16}$ | SB:$3_{-1/2}^B$ |
| $6_4^2$ | 36 | 1,1,1,1,4,4 | $0,0,0,0,\frac{1}{3},\frac{2}{3}$ | SB:$9_4^B$ |
| $6_{8/7}^2$ | 37.183 | $1,1,1,1,2\zeta_5^1,\zeta_{12}^6$ | $0,0,0,0,\frac{6}{7},\frac{2}{7}$ | SB:$3_{8/7}^B$ |
| $6_{-8/7}^2$ | 37.183 | $1,1,1,1,2\zeta_5^1,\zeta_{12}^6$ | $0,0,0,0,\frac{1}{7},\frac{5}{7}$ | SB:$3_{-8/7}^B$ |

the $K = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{pmatrix}$ state. They carry the following $S_3$ representations:

$$\sigma \to \mathbf{1} \oplus b, \quad \tau \to a \oplus b. \tag{A1}$$

It is strange that two different irreducible representations are degenerate in energy. However, this can happen for topological excitations in the presence of symmetry.

Such an assignment of the $S_3$ representations (or $S_3$ "charges") is consistent with the fusion rule (see the second table in Table XXVI). For example,

$$\begin{aligned} \sigma \otimes \sigma &\to \mathbf{1} \oplus 2b \oplus b \otimes b = \mathbf{1} \oplus 2b \oplus (\mathbf{1} \oplus a \oplus b) \\ &\to \mathbf{1} \oplus b \oplus \sigma \oplus \tau. \end{aligned} \tag{A2}$$

This is why we say that the second $5_4^{\sqrt{6}}$ entry can be realized by the $K = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{pmatrix}$ state.

However, the $S_3$-charge assignment Eq. (A1) does not work for the first and the third $5_4^{\sqrt{6}}$ entries (i.e., inconsistent with fusion rules in the first and the third tables in Table XXVI). In fact, none of the $S_3$-charge assignment works. This mean that the $d = 3$ fermions in the first and the third $5_4^{\sqrt{6}}$ entries must carry fractionalized $S_3$ charges or fractionalized $S_3$ representations. It is not clear if such fractionalized $S_3$ charges are realizable or not, since we cannot calculate the modular extensions for those entries (due to the limitation of computer power).

#### 4. $Z_2 \times Z_2$-SET orders

Tables XXIX, XXX, and XXXI list the $Z_2 \times Z_2$-SET orders (up to invertible ones) for 2+1D bosonic systems.

Table XXXII lists the fusion rules for some $Z_2 \times Z_2$-SET orders. We see that the $5_1^2$ state is a $\nu = 1/2$ bosonic Laughlin state with $Z_2 \times Z_2$ symmetry, where the only topological excitation carries the projective representation of $Z_2 \times Z_2$. We also see that the $5_{14/2}^2$ state is a bosonic Fibonacci state with $Z_2 \times Z_2$ symmetry, where the only non-Abelian topological excitation carries the projective representation of $Z_2 \times Z_2$.

#### 5. $Z_2 \times Z_2^f$-SET and $Z_4^f$-SET orders

Table XXXIII lists the $Z_2 \times Z_2^f$-SET orders (up to invertible ones) for 2+1D fermionic systems. Table XXXIV lists the $Z_4^f$-SET orders (up to invertible ones) for 2+1D fermionic systems. For fermionic systems, the central charge is determined up to $c_{\min}$ by the bulk excitations, where $c_{\min}$ is the smallest positive central charge of the modular extensions of sRep($G^f$), for example, $c_{\min} = 1/2$ for $Z_2^f, Z_2 \times Z_2^f, Z_6^f$, $c_{\min} = 1$ for $Z_4^f, Z_8^f$.

TABLE XXX. $Z_2 \times Z_2$-SET orders for bosonic systems labeled in terms of sets of topological excitations. The list contains all topological orders with $N = 7$ $D^2 \leqslant 120$.

| $N_c^{\|\Theta\|}$ | $D^2$ | $d_1, d_2, \ldots$ | $s_1, s_2, \ldots$ | comment |
|---|---|---|---|---|
| $4_0^2$ | 4 | 1,1,1,1 | 0,0,0,0 | $\mathcal{E} = \text{Rep}(Z_2 \times Z_2)$ |
| $7_0^2$ | 16 | 1,1,1,1,2,2,2 | $0,0,0,0,0,0,\frac{1}{2}$ | SB:$4_0^B$ |
| $7_0^2$ | 16 | 1,1,1,1,2,2,2 | $0,0,0,0,0,\frac{1}{4},\frac{3}{4}$ | SB:$4_0^B$ |
| $7_1^2$ | 16 | 1,1,1,1,2,2,2 | $0,0,0,0,\frac{1}{8},\frac{1}{8},\frac{1}{2}$ | SB:$4_1^B$ |
| $7_1^2$ | 16 | 1,1,1,1,2,2,2 | $0,0,0,0,\frac{1}{8},\frac{1}{8},\frac{1}{2}$ | SB:$4_1^B$ |
| $7_1^2$ | 16 | 1,1,1,1,2,2,2 | $0,0,0,0,\frac{1}{8},\frac{1}{8},\frac{1}{2}$ | SB:$4_1^B$ |
| $7_1^2$ | 16 | 1,1,1,1,2,2,2 | $0,0,0,0,\frac{1}{8},\frac{1}{8},\frac{1}{2}$ | SB:$4_1^B$ |
| $7_1^2$ | 16 | 1,1,1,1,2,2,2 | $0,0,0,0,\frac{1}{8},\frac{1}{8},\frac{1}{2}$ | SB:$4_1^B$ |
| $7_1^2$ | 16 | 1,1,1,1,2,2,2 | $0,0,0,0,\frac{1}{8},\frac{1}{8},\frac{1}{2}$ | SB:$4_1^B$ |
| $7_1^2$ | 16 | 1,1,1,1,2,2,2 | $0,0,0,0,\frac{1}{8},\frac{1}{8},\frac{1}{2}$ | SB:$4_1^B$ |
| $7_2^2$ | 16 | 1,1,1,1,2,2,2 | $0,0,0,0,\frac{1}{4},\frac{1}{4},\frac{1}{2}$ | SB:$4_2^B$ |
| $7_3^2$ | 16 | 1,1,1,1,2,2,2 | $0,0,0,0,\frac{3}{8},\frac{3}{8},\frac{1}{2}$ | SB:$4_3^B$ |
| $7_3^2$ | 16 | 1,1,1,1,2,2,2 | $0,0,0,0,\frac{3}{8},\frac{3}{8},\frac{1}{2}$ | SB:$4_3^B$ |
| $7_3^2$ | 16 | 1,1,1,1,2,2,2 | $0,0,0,0,\frac{3}{8},\frac{3}{8},\frac{1}{2}$ | SB:$4_3^B$ |
| $7_3^2$ | 16 | 1,1,1,1,2,2,2 | $0,0,0,0,\frac{3}{8},\frac{3}{8},\frac{1}{2}$ | SB:$4_3^B$ |
| $7_3^2$ | 16 | 1,1,1,1,2,2,2 | $0,0,0,0,\frac{3}{8},\frac{3}{8},\frac{1}{2}$ | SB:$4_3^B$ |
| $7_3^2$ | 16 | 1,1,1,1,2,2,2 | $0,0,0,0,\frac{3}{8},\frac{3}{8},\frac{1}{2}$ | SB:$4_3^B$ |
| $7_3^2$ | 16 | 1,1,1,1,2,2,2 | $0,0,0,0,\frac{3}{8},\frac{3}{8},\frac{1}{2}$ | SB:$4_3^B$ |
| $7_4^2$ | 16 | 1,1,1,1,2,2,2 | $0,0,0,0,\frac{1}{2},\frac{1}{2},\frac{1}{2}$ | SB:$4_4^B$ |
| $7_{-3}^2$ | 16 | 1,1,1,1,2,2,2 | $0,0,0,0,\frac{1}{2},\frac{5}{8},\frac{5}{8}$ | SB:$4_{-3}^B$ |
| $7_{-3}^2$ | 16 | 1,1,1,1,2,2,2 | $0,0,0,0,\frac{1}{2},\frac{5}{8},\frac{5}{8}$ | SB:$4_{-3}^B$ |
| $7_{-3}^2$ | 16 | 1,1,1,1,2,2,2 | $0,0,0,0,\frac{1}{2},\frac{5}{8},\frac{5}{8}$ | SB:$4_{-3}^B$ |
| $7_{-3}^2$ | 16 | 1,1,1,1,2,2,2 | $0,0,0,0,\frac{1}{2},\frac{5}{8},\frac{5}{8}$ | SB:$4_{-3}^B$ |
| $7_{-3}^2$ | 16 | 1,1,1,1,2,2,2 | $0,0,0,0,\frac{1}{2},\frac{5}{8},\frac{5}{8}$ | SB:$4_{-3}^B$ |
| $7_{-3}^2$ | 16 | 1,1,1,1,2,2,2 | $0,0,0,0,\frac{1}{2},\frac{5}{8},\frac{5}{8}$ | SB:$4_{-3}^B$ |
| $7_{-3}^2$ | 16 | 1,1,1,1,2,2,2 | $0,0,0,0,\frac{1}{2},\frac{5}{8},\frac{5}{8}$ | SB:$4_{-3}^B$ |
| $7_{-2}^2$ | 16 | 1,1,1,1,2,2,2 | $0,0,0,0,\frac{1}{2},\frac{3}{4},\frac{3}{4}$ | SB:$4_{-2}^B$ |
| $7_{-1}^2$ | 16 | 1,1,1,1,2,2,2 | $0,0,0,0,\frac{1}{2},\frac{7}{8},\frac{7}{8}$ | SB:$4_{-1}^B$ |
| $7_{-1}^2$ | 16 | 1,1,1,1,2,2,2 | $0,0,0,0,\frac{1}{2},\frac{7}{8},\frac{7}{8}$ | SB:$4_{-1}^B$ |
| $7_{-1}^2$ | 16 | 1,1,1,1,2,2,2 | $0,0,0,0,\frac{1}{2},\frac{7}{8},\frac{7}{8}$ | SB:$4_{-1}^B$ |
| $7_{-1}^2$ | 16 | 1,1,1,1,2,2,2 | $0,0,0,0,\frac{1}{2},\frac{7}{8},\frac{7}{8}$ | SB:$4_{-1}^B$ |
| $7_{-1}^2$ | 16 | 1,1,1,1,2,2,2 | $0,0,0,0,\frac{1}{2},\frac{7}{8},\frac{7}{8}$ | SB:$4_{-1}^B$ |
| $7_{-1}^2$ | 16 | 1,1,1,1,2,2,2 | $0,0,0,0,\frac{1}{2},\frac{7}{8},\frac{7}{8}$ | SB:$4_{-1}^B$ |
| $7_{-1}^2$ | 16 | 1,1,1,1,2,2,2 | $0,0,0,0,\frac{1}{2},\frac{7}{8},\frac{7}{8}$ | SB:$4_{-1}^B$ |
| $7_{9/5}^2$ | 28.944 | $1,1,1,1,2,\zeta_8^4,\zeta_8^4$ | $0,0,0,0,\frac{3}{4},\frac{3}{20},\frac{2}{5}$ | SB:$4_{9/5}^B$ |
| $7_{19/5}^2$ | 28.944 | $1,1,1,1,2,\zeta_8^4,\zeta_8^4$ | $0,0,0,0,\frac{1}{4},\frac{2}{5},\frac{13}{20}$ | SB:$4_{19/5}^B$ |
| $7_{-19/5}^2$ | 28.944 | $1,1,1,1,2,\zeta_8^4,\zeta_8^4$ | $0,0,0,0,\frac{3}{4},\frac{7}{20},\frac{3}{5}$ | SB:$4_{-19/5}^B$ |
| $7_{-9/5}^2$ | 28.944 | $1,1,1,1,2,\zeta_8^4,\zeta_8^4$ | $0,0,0,0,\frac{1}{4},\frac{3}{5},\frac{17}{20}$ | SB:$4_{-9/5}^B$ |
| $7_0^2$ | 52.360 | $1,1,1,1,\zeta_8^4,\zeta_8^4,3+\sqrt{5}$ | $0,0,0,0,\frac{2}{5},\frac{3}{5},0$ | SB:$4_0^B$ |
| $7_{12/5}^2$ | 52.360 | $1,1,1,1,\zeta_8^4,\zeta_8^4,3+\sqrt{5}$ | $0,0,0,0,\frac{3}{5},\frac{3}{5},\frac{1}{5}$ | SB:$4_{12/5}^B$ |
| $7_{-12/5}^2$ | 52.360 | $1,1,1,1,\zeta_8^4,\zeta_8^4,3+\sqrt{5}$ | $0,0,0,0,\frac{2}{5},\frac{2}{5},\frac{4}{5}$ | SB:$4_{-12/5}^B$ |
| $7_{10/3}^2$ | 76.937 | $1,1,1,1,2\zeta_7^1,2\zeta_7^2,\zeta_{16}^8$ | $0,0,0,0,\frac{1}{3},\frac{2}{9},\frac{2}{3}$ | SB:$4_{10/3}^B$ |
| $7_{-10/3}^2$ | 76.937 | $1,1,1,1,2\zeta_7^1,2\zeta_7^2,\zeta_{16}^8$ | $0,0,0,0,\frac{2}{3},\frac{7}{9},\frac{1}{3}$ | SB:$4_{-10/3}^B$ |

TABLE XXXI. $Z_2 \times Z_2$-SET orders for bosonic systems labeled in terms of sets of topological excitations. The list contains all topological orders with $N = 8$ $D^2 \leqslant 60$.

| $N_c^{\|\Theta\|}$ | $D^2$ | $d_1,d_2,\dots$ | $s_1,s_2,\dots$ | comment |
|---|---|---|---|---|
| $4_0^2$ | 4 | 1,1,1,1 | 0,0,0,0 | $\mathcal{E} = \text{Rep}(Z_2 \times Z_2)$ |
| $8_1^2$ | 8 | 1,1,1,1,1,1,1,1 | $0,0,0,0,\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}$ | $2_1^B \boxtimes \text{Rep}(Z_2 \times Z_2)$ |
| $8_1^2$ | 8 | 1,1,1,1,1,1,1,1 | $0,0,0,0,\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}$ | SB:$2_1^B$ |
| $8_1^2$ | 8 | 1,1,1,1,1,1,1,1 | $0,0,0,0,\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}$ | SB:$2_1^B$ |
| $8_1^2$ | 8 | 1,1,1,1,1,1,1,1 | $0,0,0,0,\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}$ | SB:$2_1^B$ |
| $8_{-1}^2$ | 8 | 1,1,1,1,1,1,1,1 | $0,0,0,0,\frac{3}{4},\frac{3}{4},\frac{3}{4},\frac{3}{4}$ | $2_{-1}^B \boxtimes \text{Rep}(Z_2 \times Z_2)$ |
| $8_{-1}^2$ | 8 | 1,1,1,1,1,1,1,1 | $0,0,0,0,\frac{3}{4},\frac{3}{4},\frac{3}{4},\frac{3}{4}$ | SB:$2_{-1}^B$ |
| $8_{-1}^2$ | 8 | 1,1,1,1,1,1,1,1 | $0,0,0,0,\frac{3}{4},\frac{3}{4},\frac{3}{4},\frac{3}{4}$ | SB:$2_{-1}^B$ |
| $8_{-1}^2$ | 8 | 1,1,1,1,1,1,1,1 | $0,0,0,0,\frac{3}{4},\frac{3}{4},\frac{3}{4},\frac{3}{4}$ | SB:$2_{-1}^B$ |
| $8_{14/5}^2$ | 14.472 | $1,1,1,1,\zeta_3^1,\zeta_3^1,\zeta_3^1,\zeta_3^1$ | $0,0,0,0,\frac{2}{5},\frac{2}{5},\frac{2}{5},\frac{2}{5}$ | $2_{14/5}^B \boxtimes \text{Rep}(Z_2 \times Z_2)$ |
| $8_{-14/5}^2$ | 14.472 | $1,1,1,1,\zeta_3^1,\zeta_3^1,\zeta_3^1,\zeta_3^1$ | $0,0,0,0,\frac{3}{5},\frac{3}{5},\frac{3}{5},\frac{3}{5}$ | $2_{-14/5}^B \boxtimes \text{Rep}(Z_2 \times Z_2)$ |
| $8_0^2$ | 20 | 1,1,1,1,2,2,2,2 | $0,0,0,0,\frac{1}{5},\frac{1}{5},\frac{4}{5},\frac{4}{5}$ | SB:$5_0^B$ |
| $8_0^2$ | 20 | 1,1,1,1,2,2,2,2 | $0,0,0,0,\frac{1}{5},\frac{1}{5},\frac{4}{5},\frac{4}{5}$ | SB:$5_0^B$ |
| $8_0^2$ | 20 | 1,1,1,1,2,2,2,2 | $0,0,0,0,\frac{1}{5},\frac{1}{5},\frac{4}{5},\frac{4}{5}$ | SB:$5_0^B$ |
| $8_0^2$ | 20 | 1,1,1,1,2,2,2,2 | $0,0,0,0,\frac{1}{5},\frac{1}{5},\frac{4}{5},\frac{4}{5}$ | SB:$5_0^B$ |
| $8_4^2$ | 20 | 1,1,1,1,2,2,2,2 | $0,0,0,0,\frac{2}{5},\frac{2}{5},\frac{3}{5},\frac{3}{5}$ | SB:$5_4^B$ |
| $8_4^2$ | 20 | 1,1,1,1,2,2,2,2 | $0,0,0,0,\frac{2}{5},\frac{2}{5},\frac{3}{5},\frac{3}{5}$ | SB:$5_4^B$ |
| $8_4^2$ | 20 | 1,1,1,1,2,2,2,2 | $0,0,0,0,\frac{2}{5},\frac{2}{5},\frac{3}{5},\frac{3}{5}$ | SB:$5_4^B$ |
| $8_4^2$ | 20 | 1,1,1,1,2,2,2,2 | $0,0,0,0,\frac{2}{5},\frac{2}{5},\frac{3}{5},\frac{3}{5}$ | SB:$5_4^B$ |
| $8_2^2$ | 48 | $1,1,1,1,2,\sqrt{12},\sqrt{12},4$ | $0,0,0,0,0,\frac{1}{8},\frac{5}{8},\frac{1}{3}$ | SB:$5_2^B$ |
| $8_2^2$ | 48 | $1,1,1,1,2,\sqrt{12},\sqrt{12},4$ | $0,0,0,0,0,\frac{3}{8},\frac{7}{8},\frac{1}{3}$ | SB:$5_2^B$ |
| $8_{-2}^2$ | 48 | $1,1,1,1,2,\sqrt{12},\sqrt{12},4$ | $0,0,0,0,0,\frac{1}{8},\frac{5}{8},\frac{2}{3}$ | SB:$5_{-2}^B$ |
| $8_{-2}^2$ | 48 | $1,1,1,1,2,\sqrt{12},\sqrt{12},4$ | $0,0,0,0,0,\frac{3}{8},\frac{7}{8},\frac{2}{3}$ | SB:$5_{-2}^B$ |
| $8_{16/11}^2$ | 138.58 | $1,1,1,1,2\zeta_9^1,2\zeta_9^2,2\zeta_9^3,\zeta_{20}^{10}$ | $0,0,0,0,\frac{9}{11},\frac{2}{11},\frac{1}{11},\frac{6}{11}$ | SB:$5_{16/11}^B$ |
| $8_{-16/11}^2$ | 138.58 | $1,1,1,1,2\zeta_9^1,2\zeta_9^2,2\zeta_9^3,\zeta_{20}^{10}$ | $0,0,0,0,\frac{2}{11},\frac{9}{11},\frac{10}{11},\frac{5}{11}$ | SB:$5_{-16/11}^B$ |
| $8_{18/7}^2$ | 141.36 | $1,1,1,1,\zeta_{12}^6,\zeta_{12}^6,2\zeta_{12}^2,2\zeta_{12}^4$ | $0,0,0,0,\frac{6}{7},\frac{6}{7},\frac{1}{7},\frac{3}{7}$ | SB:$5_{18/7}^B$ |
| $8_{-18/7}^2$ | 141.36 | $1,1,1,1,\zeta_{12}^6,\zeta_{12}^6,2\zeta_{12}^2,2\zeta_{12}^4$ | $0,0,0,0,\frac{1}{7},\frac{1}{7},\frac{6}{7},\frac{4}{7}$ | SB:$5_{-18/7}^B$ |

## APPENDIX B: FUSION RING FOR THE MODULAR EXTENSIONS OF Rep($G$) OR sRep($G^f$) WHEN $G$ OR $G^f$ IS ABELIAN GROUP

When the symmetry group $G$ is Abelian, the different irreducible representations, under the fusion, form the same group $G$. Thus different irreducible representations can be labeled by the group elements: $(q)$, $q \in G$. The different symmetry twists are also labeled by the group elements: $[g]$, $g \in G$. More general symmetry twists may carry some charge. We denote such charge carrying symmetry twists by $[g,q]$ where $q \in G$. In fact we can identify $(q)$ as $[1,q]$. Those irreducible representations and charged symmetry twists are particles in the modular extensions of Rep($G$) or sRep($G^f$).

Since the group is Abelian, the symmetry twists do not break the symmetry. Thus we have the following fusion rule:

$$[1,q] \otimes [g,q'] = [g,qq']. \tag{B1}$$

This means that $[g,q']$ and $[g,qq']$ differ by charge $q$. We also have

$$[g,q] \otimes [g',q'] = [gg',qq']. \tag{B2}$$

TABLE XXXII. The fusion rules for some $Z_2 \times Z_2$-SET orders.

| $s_i$ | 0 | 0 | 0 | 0 | $\frac{1}{4}$ | $s_i$ | 0 | 0 | 0 | 0 | $\frac{2}{5}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $d_i$ | 1 | 1 | 1 | 1 | 2 | $d_i$ | 1 | 1 | 1 | 1 | $2\zeta_3^1$ |
| $5_1^2$ | **1** | $a$ | $b$ | $c$ | $\phi$ | $5_{14/5}^2$ | **1** | $a$ | $b$ | $c$ | $\eta$ |
| **1** | **1** | $a$ | $b$ | $c$ | $\phi$ | **1** | **1** | $a$ | $b$ | $c$ | $\eta$ |
| $a$ | $a$ | **1** | $c$ | $b$ | $\phi$ | $a$ | $a$ | **1** | $c$ | $b$ | $\eta$ |
| $b$ | $b$ | $c$ | **1** | $a$ | $\phi$ | $b$ | $b$ | $c$ | **1** | $a$ | $\eta$ |
| $c$ | $c$ | $b$ | $a$ | **1** | $\phi$ | $c$ | $c$ | $b$ | $a$ | **1** | $\eta$ |
| $\phi$ | $\phi$ | $\phi$ | $\phi$ | $\phi$ | $\mathbf{1} \oplus a \oplus b \oplus c$ | $\eta$ | $\eta$ | $\eta$ | $\eta$ | $\eta$ | $\mathbf{1} \oplus a \oplus b \oplus c \oplus 2\eta$ |

TABLE XXXIII. $Z_2 \times Z_2^f$-SET orders (up to invertible ones) for fermionic systems. The list contains all topological orders with $N = 6$ $D^2 \leqslant 300$, $N = 8$ $D^2 \leqslant 60$, and $N = 10$ $D^2 \leqslant 20$.

| $N_c^{\vert\Theta\vert}$ | $D^2$ | $d_1,d_2,\ldots$ | $s_1,s_2,\ldots$ | comment |
|---|---|---|---|---|
| $4^0_0\binom{2}{0}$ | 4 | 1,1,1,1 | $0,0,\frac{1}{2},\frac{1}{2}$ | $\mathcal{E} = \mathrm{sRep}(Z_2 \times Z_2^f)$ |
| $6^0_0$ | 12 | 1,1,1,1,2,2 | $0,0,\frac{1}{2},\frac{1}{2},\frac{1}{6},\frac{2}{3}$ | SB:$K = \begin{pmatrix} -1 & -2 \\ -2 & -1 \end{pmatrix}$ |
| $6^0_0$ | 12 | 1,1,1,1,2,2 | $0,0,\frac{1}{2},\frac{1}{2},\frac{1}{3},\frac{5}{6}$ | SB:$K = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ |
| $8^0_0\binom{0}{0}$ | 8 | 1,1,1,1,1,1,1,1 | $0,0,\frac{1}{2},\frac{1}{2},\frac{1}{4},\frac{1}{4},\frac{3}{4},\frac{3}{4}$ | $2^B_1 \boxtimes \mathrm{sRep}(Z_2 \times Z_2^f)$ |
| $8^0_0\binom{0}{0}$ | 8 | 1,1,1,1,1,1,1,1 | $0,0,\frac{1}{2},\frac{1}{2},\frac{1}{4},\frac{1}{4},\frac{3}{4},\frac{3}{4}$ | SB:$4^F_0\binom{0}{0}$ |
| $8^0_{-14/5}\binom{\zeta_8^4}{3/20}$ | 14.472 | $1,1,1,1,\zeta_3^1,\zeta_3^1,\zeta_3^1,\zeta_3^1$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{1}{10},\frac{1}{10},\frac{3}{5},\frac{3}{5}$ | $2^B_{-14/5} \boxtimes \mathrm{sRep}(Z_2 \times Z_2^f)$ |
| $8^0_{14/5}\binom{\zeta_8^4}{-3/20}$ | 14.472 | $1,1,1,1,\zeta_3^1,\zeta_3^1,\zeta_3^1,\zeta_3^1$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{2}{5},\frac{2}{5},\frac{9}{10},\frac{9}{10}$ | $2^B_{14/5} \boxtimes \mathrm{sRep}(Z_2 \times Z_2^f)$ |
| $8^0_0\binom{2}{0}$ | 20 | 1,1,1,1,2,2,2,2 | $0,0,\frac{1}{2},\frac{1}{2},\frac{1}{10},\frac{2}{5},\frac{3}{5},\frac{9}{10}$ | SB:$10^F_0\binom{\zeta_2^1}{0}$ |
| $8^0_0\binom{2}{1/2}$ | 20 | 1,1,1,1,2,2,2,2 | $0,0,\frac{1}{2},\frac{1}{2},\frac{1}{5},\frac{3}{10},\frac{7}{10},\frac{4}{5}$ | SB:$10^F_0\binom{\zeta_2^1}{1/2}$ |
| $8^0_{1/4}\binom{\zeta_2^1\zeta_6^3}{1/2}$ | 27.313 | $1,1,1,1,\zeta_6^2,\zeta_6^2,\zeta_6^2,\zeta_6^2$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{1}{4},\frac{1}{4},\frac{3}{4},\frac{3}{4}$ | SB:$4^F_{1/4}\binom{\zeta_6^3}{1/2}$ |
| $8^0_{1/4}\binom{\zeta_2^1\zeta_6^3}{1/2}$ | 27.313 | $1,1,1,1,\zeta_6^2,\zeta_6^2,\zeta_6^2,\zeta_6^2$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{1}{4},\frac{1}{4},\frac{3}{4},\frac{3}{4}$ | SB:$4^F_{1/4}\binom{\zeta_6^3}{1/2}$ |
| $10^0_0\binom{4}{0}$ | 16 | 1,1,1,1,1,1,1,1,2,2 | $0,0,\frac{1}{2},\frac{1}{2},0,0,\frac{1}{2},\frac{1}{2},0,\frac{1}{2}$ | SB:$8^F_0\binom{\sqrt{8}}{0}$ |
| $10^0_0\binom{4}{0}$ | 16 | 1,1,1,1,1,1,1,1,2,2 | $0,0,\frac{1}{2},\frac{1}{2},0,0,\frac{1}{2},\frac{1}{2},0,\frac{1}{2}$ | SB:$8^F_0\binom{\sqrt{8}}{0}$ |
| $10^0_0\binom{\sqrt{8}}{1/8}$ | 16 | 1,1,1,1,1,1,1,1,2,2 | $0,0,\frac{1}{2},\frac{1}{2},0,0,\frac{1}{2},\frac{1}{2},\frac{1}{8},\frac{5}{8}$ | SB:$8^F_0\binom{2}{1/8}$ |
| $10^0_0\binom{\sqrt{8}}{1/8}$ | 16 | 1,1,1,1,1,1,1,1,2,2 | $0,0,\frac{1}{2},\frac{1}{2},0,0,\frac{1}{2},\frac{1}{2},\frac{1}{8},\frac{5}{8}$ | SB:$8^F_0\binom{2}{1/8}$ |
| $10^0_0\binom{0}{0}$ | 16 | 1,1,1,1,1,1,1,1,2,2 | $0,0,\frac{1}{2},\frac{1}{2},0,0,\frac{1}{2},\frac{1}{2},\frac{1}{4},\frac{3}{4}$ | SB:$8^F_0\binom{0}{0}$ |
| $10^0_0\binom{0}{0}$ | 16 | 1,1,1,1,1,1,1,1,2,2 | $0,0,\frac{1}{2},\frac{1}{2},0,0,\frac{1}{2},\frac{1}{2},\frac{1}{4},\frac{3}{4}$ | SB:$8^F_0\binom{0}{0}$ |
| $10^0_0\binom{\sqrt{8}}{-1/8}$ | 16 | 1,1,1,1,1,1,1,1,2,2 | $0,0,\frac{1}{2},\frac{1}{2},0,0,\frac{1}{2},\frac{1}{2},\frac{3}{8},\frac{7}{8}$ | SB:$8^F_0\binom{2}{-1/8}$ |
| $10^0_0\binom{\sqrt{8}}{-1/8}$ | 16 | 1,1,1,1,1,1,1,1,2,2 | $0,0,\frac{1}{2},\frac{1}{2},0,0,\frac{1}{2},\frac{1}{2},\frac{3}{8},\frac{7}{8}$ | SB:$8^F_0\binom{2}{-1/8}$ |

TABLE XXXIV. $Z_4^f$-SET orders for fermionic systems. The list contains all topological orders with $N = 6$ $D^2 \leqslant 100$, $N = 8$ $D^2 \leqslant 60$, and $N = 10$ $D^2 \leqslant 20$.

| $N_c^{\vert\Theta\vert}$ | $D^2$ | $d_1,d_2,\ldots$ | $s_1,s_2,\ldots$ | comment |
|---|---|---|---|---|
| $4^0_0$ | 4 | 1,1,1,1 | $0,0,\frac{1}{2},\frac{1}{2}$ | $\mathcal{E} = \mathrm{sRep}(Z_4^f)$ |
| $6^0_0$ | 12 | 1,1,1,1,2,2 | $0,0,\frac{1}{2},\frac{1}{2},\frac{1}{6},\frac{2}{3}$ | $K = -\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ |
| $6^0_0$ | 12 | 1,1,1,1,2,2 | $0,0,\frac{1}{2},\frac{1}{2},\frac{1}{3},\frac{5}{6}$ | $K = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ |
| $8^0_0$ | 8 | 1,1,1,1,1,1,1,1 | $0,0,\frac{1}{2},\frac{1}{2},\frac{1}{4},\frac{1}{4},\frac{3}{4},\frac{3}{4}$ | $2^B_{-1} \boxtimes \mathrm{sRep}(Z_4^f)$ |
| $8^0_0$ | 8 | 1,1,1,1,1,1,1,1 | $0,0,\frac{1}{2},\frac{1}{2},\frac{1}{4},\frac{1}{4},\frac{3}{4},\frac{3}{4}$ | $2^B_1 \boxtimes \mathrm{sRep}(Z_4^f)$ |
| $8^0_{-14/5}$ | 14.472 | $1,1,1,1,\zeta_3^1,\zeta_3^1,\zeta_3^1,\zeta_3^1$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{1}{10},\frac{1}{10},\frac{3}{5},\frac{3}{5}$ | $2^B_{-14/5} \boxtimes \mathrm{sRep}(Z_4^f)$ |
| $8^0_{14/5}$ | 14.472 | $1,1,1,1,\zeta_3^1,\zeta_3^1,\zeta_3^1,\zeta_3^1$ | $0,0,\frac{1}{2},\frac{1}{2},\frac{2}{5},\frac{2}{5},\frac{9}{10},\frac{9}{10}$ | $2^B_{14/5} \boxtimes \mathrm{sRep}(Z_4^f)$ |
| $8^0_0$ | 20 | 1,1,1,1,2,2,2,2 | $0,0,\frac{1}{2},\frac{1}{2},\frac{1}{10},\frac{2}{5},\frac{3}{5},\frac{9}{10}$ | SB:$10^F_0\binom{\zeta_2^1}{0}$ |
| $8^0_0$ | 20 | 1,1,1,1,2,2,2,2 | $0,0,\frac{1}{2},\frac{1}{2},\frac{1}{5},\frac{3}{10},\frac{7}{10},\frac{4}{5}$ | SB:$10^F_0\binom{\zeta_2^1}{1/2}$ |
| $10^0_0\binom{4}{0}$ | 16 | 1,1,1,1,1,1,1,1,2,2 | $0,0,\frac{1}{2},\frac{1}{2},0,0,\frac{1}{2},\frac{1}{2},0,\frac{1}{2}$ | SB:$8^F_0\binom{\sqrt{8}}{0}$ |
| $10^0_0\binom{4}{0}$ | 16 | 1,1,1,1,1,1,1,1,2,2 | $0,0,\frac{1}{2},\frac{1}{2},0,0,\frac{1}{2},\frac{1}{2},0,\frac{1}{2}$ | SB:$8^F_0\binom{\sqrt{8}}{0}$ |
| $10^0_0\binom{\sqrt{8}}{1/8}$ | 16 | 1,1,1,1,1,1,1,1,2,2 | $0,0,\frac{1}{2},\frac{1}{2},0,0,\frac{1}{2},\frac{1}{2},\frac{1}{8},\frac{5}{8}$ | SB:$8^F_0\binom{2}{1/8}$ |
| $10^0_0\binom{\sqrt{8}}{1/8}$ | 16 | 1,1,1,1,1,1,1,1,2,2 | $0,0,\frac{1}{2},\frac{1}{2},0,0,\frac{1}{2},\frac{1}{2},\frac{1}{8},\frac{5}{8}$ | SB:$8^F_0\binom{2}{1/8}$ |
| $10^0_0\binom{0}{0}$ | 16 | 1,1,1,1,1,1,1,1,2,2 | $0,0,\frac{1}{2},\frac{1}{2},0,0,\frac{1}{2},\frac{1}{2},\frac{1}{4},\frac{3}{4}$ | SB:$8^F_0\binom{0}{0}$ |
| $10^0_0\binom{0}{0}$ | 16 | 1,1,1,1,1,1,1,1,2,2 | $0,0,\frac{1}{2},\frac{1}{2},0,0,\frac{1}{2},\frac{1}{2},\frac{1}{4},\frac{3}{4}$ | SB:$8^F_0\binom{0}{0}$ |
| $10^0_0\binom{\sqrt{8}}{-1/8}$ | 16 | 1,1,1,1,1,1,1,1,2,2 | $0,0,\frac{1}{2},\frac{1}{2},0,0,\frac{1}{2},\frac{1}{2},\frac{3}{8},\frac{7}{8}$ | SB:$8^F_0\binom{2}{-1/8}$ |
| $10^0_0\binom{\sqrt{8}}{-1/8}$ | 16 | 1,1,1,1,1,1,1,1,2,2 | $0,0,\frac{1}{2},\frac{1}{2},0,0,\frac{1}{2},\frac{1}{2},\frac{3}{8},\frac{7}{8}$ | SB:$8^F_0\binom{2}{-1/8}$ |

However, the above fusion rule is too restrictive. Although $[g,q']$ and $[g,qq']$ differ by charge $q$, we do not know the net charge of $[g,q']$ when $g \neq 1$. Thus the more general fusion rule that still preserves charge conservation is

$$[g,q] \otimes [g',q'] = [gg',\omega_2(g,g')qq'], \quad \omega_2(g,g') \in G. \tag{B3}$$

From

$$\begin{aligned}([g_1,q_1] \otimes [g_2,q_2]) \otimes [g_3,q_3] &= [g_1g_2g_3,\omega(g_1,g_2)\omega(g_1g_2,g_3)q_1q_2q_3]\\ &= [g_1,q_1] \otimes ([g_2,q_2] \otimes [g_3,q_3])\\ &= [g_1g_2g_3,\omega(g_1,g_2g_3)\omega(g_2,g_3)q_1q_2q_3],\end{aligned} \tag{B4}$$

we see that

$$\omega(g_1,g_2)\omega(g_1g_2,g_3) = \omega(g_1,g_2g_3)\omega(g_2,g_3). \tag{B5}$$

i.e., $\omega(g_1,g_2)$ is a group 2-cocycle in $\mathcal{H}^2(G,G)$.

In the above, we have assumed that the modular extension is Abelian (i.e., all the particles in the modular extension have a quantum dimension 1). We see that the fusion rules of Abelian modular extensions are labeled by 2-cocycles in $\mathcal{H}^2(G,G)$.

However, sometimes the modular extension can be non-Abelian, such as the modular extension of sRep$(Z_2^f)$ and Rep$(Z_2 \times Z_2 \times Z_2)$. To allow such a possibility, we allow $[g,q]$ to be a many-to-one label of the particle, and define a subgroup $H_g \subset G$:

$$H_g = \{h|[g,q] = [g,hq],h \in G\}. \tag{B6}$$

The mapping $g \to H_g$ is an important data to describe the fusion. $H_g$ represents the charge ambiguity of the symmetry twist $[g,q]$. To get a one-to-one label, we can use

$$[g,qH_g]. \tag{B7}$$

Note that, when $g$ is an identity: $g = 1$, $H_g$ is trivial: $H_1 = 1$.

The fusion of $[1,q']$ and $[g,qH_g]$ is still given by

$$[1,q'] \otimes [g,qH_g] = [g,q'qH_g]. \tag{B8}$$

We also have $H_g = H_{g^{-1}}$ and

$$[g,qH_g] \otimes [g^{-1},q'H_g] = \oplus_{h \in qq'H_g}[1,h]. \tag{B9}$$

We see that the quantum dimension of $[g,qH_g]$ is $d = \sqrt{|H_g|}$.

The fusion rule should satisfy

$$\begin{aligned}&[1,q] \otimes ([g_1,q_1H_{g_1}] \otimes [g_2,q_2H_{g_2}])\\ &= ([1,q] \otimes [g_1,q_1H_{g_1}]) \otimes [g_2,q_2H_{g_2}]\\ &= [g_1,q_1H_{g_1}] \otimes ([1,q] \otimes [g_2,q_2H_{g_2}]).\end{aligned} \tag{B10}$$

We find that the following ansatz satisfy the above condition:

$$[g_1,q_1H_{g_1}] \otimes [g_2,q_2H_{g_2}] = \frac{m^{g_1g_2}}{|(H_{g_1}\vee H_{g_2})\cap H_{g_1g_2}|} \oplus_{q\in\omega(g_1,g_2)q_1q_2H_{g_1}\vee H_{g_2}} [g_1g_2,qH_{g_1g_2}], \tag{B11}$$

where $m^{g_1g_2} \in \mathbb{Z}$ and $H_{g_1} \vee H_{g_2}$ is the subgroup generated by $H_{g_1}$ and $H_{g_2}$. The above implies that

$$\sqrt{|H_{g_1}|}\sqrt{|H_{g_2}|} = m^{g_1g_2}\frac{|H_{g_1} \vee H_{g_2}|}{|(H_{g_1} \vee H_{g_2}) \cap H_{g_1g_2}|}\sqrt{|H_{g_1g_2}|}. \tag{B12}$$

We see that different fusion rules are labeled by $\omega(g_1,g_2)$ and $H_g$.

It is much easier to find all the $H_g$'s that satisfy Eq. (B12) and all the $\omega(g_1,g_2)$ that satisfy Eq. (B5). From those solutions, we can directly construct the fusion rule from Eq. (B11).

## APPENDIX C: CONDITIONS TO OBTAIN UMTC$_{/\mathcal{E}}$'s

In our simplified theory, a UMTC$_{/\mathcal{E}}$ is described by an integer tensor $N^{ij}_k$ and a mod-1 rational vector $s_i$, where $i,j,k$ run from 1 to $N$ and $N$ is called the rank of the UMTC$_{/\mathcal{E}}$. We may simply denote a UMTC$_{/\mathcal{E}}$ [the collection of data $(N^{ij}_k,s_i)$] by $\mathcal{C}$, a particle $i$ in $\mathcal{C}$ by $i \in \mathcal{C}$. Sometimes it is more convenient to use abstract labels rather than 1 to $N$; we may also abuse $\mathcal{C}$ as the set of labels (particles).

Not all $(N^{ij}_k,s_i)$ describe a valid UMTC$_{/\mathcal{E}}$ $\mathcal{C}$ with modular extensions. In order to describe a valid $\mathcal{C}$, $(N^{ij}_k,s_i)$ must satisfy the following conditions [8,13,43–45].

(1) Fusion ring. $N^{ij}_k$ for the UMTC$_{/\mathcal{E}}$ $\mathcal{C}$ are non-negative integers that satisfy

$$N^{ij}_k = N^{ji}_k, \quad N^{1i}_j = \delta_{ij}, \quad \sum_{k=1}^N N^{ik}_1 N^{kj}_1 = \delta_{ij},$$
$$\sum_m N^{ij}_m N^{mk}_l = \sum_n N^{in}_l N^{jk}_n \quad \text{or} \quad \sum_m N^{ij}_m N_m = N_i N_j, \tag{C1}$$

where the matrix $N_i$ is given by $(N_i)_{kj} = N^{ij}_k$, and the indices $i,j,k$ run from 1 to $N$. In fact $N^{ij}_1$ defines a charge conjugation $i \to \bar{i}$:

$$N^{ij}_1 = \delta_{\bar{i}j}. \tag{C2}$$

$N^{ij}_k$ satisfying the above conditions define a fusion ring which is viewed as the set (of simple objects)

$$\{1,2, \ldots ,N\}. \tag{C3}$$

(2) Charge conjugation condition:

$$\begin{aligned}N^{ij}_k &= N^{j\bar{k}}_{\bar{i}} = N^{\bar{k}i}_{\bar{j}}\\ &= N^{\bar{i}k}_{j} = N^{k\bar{j}}_{i} = N^{\bar{j}\bar{i}}_{\bar{k}}.\end{aligned} \tag{C4}$$

(3) Rational condition. $N^{ij}_k$ and $s_i$ for $\mathcal{C}$ satisfy [8,46–48]

$$\sum_r V^r_{ijkl}s_r = 0 \bmod 1, \tag{C5}$$

where

$$\begin{aligned}V^r_{ijkl} &= N^{ij}_r N^{kl}_{\bar{r}} + N^{il}_r N^{jk}_{\bar{r}} + N^{ik}_r N^{jl}_{\bar{r}}\\ &\quad - (\delta_{ir} + \delta_{jr} + \delta_{kr} + \delta_{lr})\sum_m N^{ij}_m N^{kl}_{\bar{m}}.\end{aligned} \tag{C6}$$

(4) Verlinde fusion characters. Let the topological $S$ matrix be (see Eq. (223) in Ref. [9])

$$S_{ij} = \frac{1}{D}\sum_k N^{ij}_k e^{2\pi i(s_i+s_j-s_k)}d_k, \tag{C7}$$

where $d_i$ (called quantum dimension) is the largest eigenvalue of the matrix $N_i$ and $D = \sqrt{\sum_i d_i^2}$ (called the total quantum dimension). Then [49]

$$\frac{S_{il}S_{jl}}{S_{1l}} = \sum_k N_k^{ij} S_{kl}. \tag{C8}$$

(5) Weak modularity. Let the topological $T$ matrix be

$$T_{ij} = \delta_{ij} e^{2\pi i s_i}. \tag{C9}$$

Then (see Eq. (232) in Ref. [9])

$$S^\dagger T S = \Theta T^\dagger S^\dagger T^\dagger,$$
$$\Theta = D^{-1} \sum_i e^{2\pi i s_i} d_i^2 = |\Theta| e^{2\pi i c/8}. \tag{C10}$$

The parameter $c$ mod 8 is defined via $\Theta$, if $|\Theta| \neq 0$.

(6) Charge conjugation symmetry:

$$S_{ij} = S^*_{i\bar{j}}, \quad s_i = s_{\bar{i}}, \ \text{or}\ S = S^\dagger C, \quad T = TC, \tag{C11}$$

where the charge conjugation matrix $C$ is given by $C_{ij} = N_1^{ij} = \delta_{i\bar{j}}$.

(7) The centralizer describes the symmetry. Let the centralizer of $\mathcal{C}$, $\mathcal{C}_\mathcal{C}^{\text{cen}}$, be the subset of the particle labels:

$$\mathcal{C}_\mathcal{C}^{\text{cen}} = \left\{ i \,|\, S_{ij} = \frac{d_i d_j}{D}, \forall j \in \mathcal{C} \right\}. \tag{C12}$$

Then, $\mathcal{C}_\mathcal{C}^{\text{cen}} = \mathcal{E}$.

(8) The second Frobenius-Schur indicator. Let

$$\nu_k = D^{-2} \sum_{ij} N_k^{ij} d_i d_j \cos(4\pi(s_i - s_j)), \tag{C13}$$

then $\nu_k \in \mathbb{Z}$ if $k = \bar{k}$ [50].

(9) Symmetry breaking. There is a symmetry breaking induced map $\mathcal{C} \to \mathcal{C}_0$, where $\mathcal{C}_0$ is a UMTC if $\mathcal{E} = \text{Rep}(G)$ or a UMTC$_{/\text{sRep}(Z_2)}$ if $\mathcal{E} = \text{sRep}(G^f)$. See Appendix D for details.

(10) Modular extension. The UMTC$_{/\mathcal{E}}$ $\mathcal{C}$ has modular extensions.

The above conditions are necessary and sufficient (due to condition 10) for $(N_k^{ij}, s_i)$ to describe a UMTC$_{/\mathcal{E}}$ $\mathcal{C}$ with modular extensions.

However, when we calculate the tables in Appeandix A, we do not use condition 10. So the used conditions are only necessary. As a result, the tables may contain fake entries that have no modular extensions.

To numerically solve the above conditions to obtain the classification tables, we first search for $N_k^{ij}$'s that satisfy condition 1 and 2. Then for each $N_k^{ij}$, we calculate $s_i$'s that satisfy condition 3 via the Smith normal form of integer matrix $V_{ijkl}^r$, where $ijkl$ is viewed as a single index. Last, from the obtained $N_k^{ij}, s_i$'s, we select those that satisfy all the conditions.

## APPENDIX D: SYMMETRY BREAKING

A UMTC$_{/\mathcal{E}}$ $\mathcal{C}$ describes a SET with symmetry $\mathcal{E}$ (up to invertible GQLs). If we break the symmetry $\mathcal{E}$, then the UMTC$_{/\mathcal{E}}$ will become a UMTC $\mathcal{C}_0$ if $\mathcal{E} = \text{Rep}G$ or become a UMTC$_{/Z_2^f}$ $\mathcal{C}_0$ if $\mathcal{E} = \text{sRep}G^f$. So there is a natural mapping from UMTC$_{/\mathcal{E}}$'s to UMTCs or UMTC$_{/Z_2^f}$: $\mathcal{C} \to \mathcal{C}_0$. Requiring the exitance of such map can give us some additional conditions on $(N_k^{ij}, s_i)$ of $\mathcal{C}$.

To understand such a map, we note that $\mathcal{C}$ can be viewed as a subcategory of $\mathcal{C}_0$, in the sense that the simple objects in $\mathcal{C}$ can be viewed as the simple or composite objects in $\mathcal{C}_0$:

$$i \to \oplus_I M^{iI} I, \quad i \in \mathcal{C}, \quad I \in \mathcal{C}_0. \tag{D1}$$

Physically, if we just pretend the symmetry is not there, then every particle in $\mathcal{C}$ can also be viewed as a particle in $\mathcal{C}_0$. However, a particle in $\mathcal{C}$ may be the direct sum of several degenerate particles in $\mathcal{C}_0$, where the degeneracy is due to the symmetry, as described by Eq. (D1).

In the following, we will obtain some conditions on $M^{iI}$, which will help us to calculate it. Let us label the particles in $\mathcal{C}$ as $\{i\} = \{1, a, b, \ldots, x, y, \ldots\}$. Here, $a, b, \ldots$ label the *bosonic* part of $\mathcal{E}$, and $x, y, \ldots$ label the fermionic part of $\mathcal{E}$ (if any) and the rest of nontrival topological excitations. We have also used $I$ to label the particles in $\mathcal{C}_0$. Clearly, the bosonic part of $\mathcal{E}$ are local excitations and are direct sums of $\mathbf{1} \in \mathcal{C}_0$:

$$a \to d_a \mathbf{1}, \quad \text{or} \quad M^{aI} = d_a \delta_{1I}. \tag{D2}$$

(Here, $\mathbf{1}$ is the trivial particle in $\mathcal{C}_0$.) By computing $i \otimes j$ in two different ways, we find that $M^{iI}$ must also satisfy

$$\sum_{IJ} M^{iI} M^{jJ} N_K^{IJ} = \sum_k N_k^{ij} M^{kK}. \tag{D3}$$

Taking $K = \mathbf{1}$, we obtain

$$\sum_I M^{iI} M^{j\bar{I}} = \sum_a N_a^{ij} d_a. \tag{D4}$$

Assuming the charge conjugation symmetry: $M^{iI} = M^{\bar{i}\bar{I}}$, we can rewrite the above as

$$\sum_I M^{iI} M^{jI} = \sum_a N_a^{i\bar{j}} d_a, \tag{D5}$$

which implies that

$$\sum_I (M^{iI})^2 = \sum_a N_a^{i\bar{i}} d_a. \tag{D6}$$

To obtain more properties of $M^{iI}$ and to solve the above conditions on $M^{iI}$, let us consider the fusion with $a$ particles:

$$a \otimes x = \oplus_y N_y^{ax} y. \tag{D7}$$

We define $x$ to be equivalent to $y$ if there exists $a$ such that $N_y^{ax} \neq 0$. Let $[x]$ be the equivalent class of $x$. Clearly $[1] = [a]$.

First, we like to pointed out that if $i$ and $j$ are equivalent, then $i$ and $j$ are formed by the same combination of $I$'s, up to an overall factor, such as

$$i \to I_1 \oplus 2I_2, \quad j \to 3I_1 \oplus 6I_2. \tag{D8}$$

This is because $a$ particles in $\mathcal{C}$ is mapped to the direct sum of identity in $\mathcal{C}_0$. Since $i$ and $j$ is related by fusing $a$ or identity in $\mathcal{C}_0$, then $i$ and $j$ must be formed by the same combination of $I$'s.

Second, if $i$ and $j$ are not equivalent, then the $I$'s that enter $i$ do not overlap with the $I$'s that enter $j$. This is a consequence of Eq. (D5). The right-hand side of Eq. (D5) will vanish if $i$ and $j$ are not equivalent.

Third, the $I$'s that appear in $i$ must have the same quantum dimensions and spins. This is because those $I$'s must be degenerate. This can only happen if they have the same quantum dimensions and spins.

Fourth, the $I$'s that appears in $i$ must each enter with an equal weight, such as

$$i \to 2I_1 \oplus 2I_2. \tag{D9}$$

Again, this is because those $I$'s must be degenerate. This can only happen if they can be mapped into each other by symmetry transformations. Since the symmetry transformations only permute $I$'s, each $I$ enters with an equal weight.

Combine the above results, we see that $M^{iI}$ has the following block structure. We can divide the index $I$ into groups $[I]$, such that there is one-to-one correspondence between $[i]$ and $[I]$: $[i] \leftrightarrow [I]_{[i]}$, and

$$\begin{aligned} M^{iI} &= 0 \quad \text{if} \quad i \in [i], \quad I \notin [I]_{[i]}, \\ M^{iI} &= m_i > 0 \quad \text{if} \quad i \in [i], \quad I \in [I]_{[i]}. \end{aligned} \tag{D10}$$

Therefore we have

$$m_i^2 n_{[i]} = \sum_a N_a^{i\bar{i}} d_a, \tag{D11}$$

where $n_{[i]}$ is the size of the set $[I]_{[i]}$. Since

$$i = \oplus_{I \in [I]_{[i]}} m_i I, \tag{D12}$$

we have

$$m_i m_j n_{[i]} = \sum_a N_a^{i\bar{j}} d_a, \quad i,j \in [i]. \tag{D13}$$

In other words, the matrix $\tilde{N}$ with elements $\tilde{N}_{ij} = \sum_a N_a^{i\bar{j}} d_a$ is block diagonal. Each block is formed by particles in an equivalent class $[i]$, and is given by the above expression. We see that, for $i,j \in [i]$, $\sum_a N_a^{i\bar{j}} d_a$ must be a symmetric matrix with a single nonzero eigenvalue $n_{[i]} \sum_{j \in [i]} m_j^2$ and eigenvector $(m_j)$.

We also find that

$$d_i = m_i n_{[i]} d_I, \tag{D14}$$

or

$$d_I = \frac{m_i d_i}{\sum_a N_a^{i\bar{i}} d_a} \quad \forall \quad I \in [I]_{[i]}. \tag{D15}$$

Using the fact $s_i = s_j = s_I$, $\forall i,j \in [i]$, $I \in [I]_{[i]}$, we can obtain $(d_I, s_I)$ of $\mathcal{C}_0$ from $(N_k^{ij}, s_i)$ of $\mathcal{C}$. The resulting $(d_I, s_I)$ must be the quantum dimensions and the spins of a UMTC. This gives us some extra conditions on $(N_k^{ij}, s_i)$.

## APPENDIX E: PHYSICAL AND MATHEMATICAL MEANING OF UMTC$_{/\mathcal{E}}$ AND ITS MODULAR EXTENSIONS

In the main text of the paper, we have explained why UMTC$_{/\mathcal{E}}$ describes the bulk particlelike excitations. We also explained the motivation of modular extension via "gauging" the symmetry. In this section, we will discuss a deeper meaning of UMTC$_{/\mathcal{E}}$ and its modular extensions.

We know that UMTC$_{/\mathcal{E}}$ is a very abstract way to describe the non-Abelian statistics of the excitations. It is not clear at all that why the excitations described by UMTC$_{/\mathcal{E}}$ can be realized by a local lattice model with on-site symmetry. In physics, we mainly concern about local lattice models and their properties. It appears that there is a big gap between the UMTC$_{/\mathcal{E}}$ studied in this paper and local lattice models that physicists want to study. In fact, the two are closely related. Here, we will try to explain such a connection between lattice models and UMTC$_{/\mathcal{E}}$ (with their modular extensions).

We know that the fusion-braiding properties of particles within a two-dimensional open disk can be described by a unitary braided fusion category. From this point of view, a unitary braided fusion category is a *local* theory that only encode the local properties of the fusion and braiding (i.e., on an open disk). We want to promote fusion-braiding properties to be integrable to any two-dimensional manifolds because we want those fusion-braiding properties to be realizable by some local lattice models, which can always be defined on any two-dimensional manifolds. Therefore the integrability of fusion-braiding properties to any two-dimensional manifolds is necessary for the fusion-braiding properties to be realized by some local lattice models.

Now we assume that "all two-dimensional manifolds" are the most powerful probes. This means that the integrability of the local fusion-braiding properties to global invariants (on all two-dimensional manifolds), satisfying natural physically required properties, is also sufficient for those properties to be realizable by some local lattice models.

The process of integrating the local fusion-braiding properties of particles (described by a UBFC $\mathcal{C}$) to give global invariants is defined by the so-called factorization homology [51,52]. In order to be free of framing anomaly, we need a spherical structure, which is guaranteed by the unitarity of a UBFC [9]. For general UBFCs, although the global invariants are well-defined by factorization homology [52], they do not have nice properties that allow us to give them a natural physical meaning. A stronger *integrability condition* needs to be imposed in order for the global invariants to have natural physical meanings.

For example, if $\mathcal{C}$ is assumed to be nondegenerate (i.e., UMTC), it was shown in Ref. [53] that factorization homology of a UMTC $\mathcal{C}$ over a closed two-dimensional manifold is given by the category of finite dimensional Hilbert spaces. If one inserts a finite number of particlelike excitations $x_1, \cdots, x_r$ on the closed surface, one simply obtain the Hilbert space $\text{hom}_{\mathcal{C}}(1, x_1 \otimes \cdots \otimes x_r)$, which is also the space of degenerate ground states. This result remains to be true for all closed two-dimensional manifolds with topological gapped defects and with two cells decorated by different phases [53]. This includes the cases that the topological order is defined on any surfaces with boundaries. Therefore the nondegeneracy is certainly a sufficient integrability condition, which is too strong for the purpose of this work.

In this paper, we consider something more complicated—the fusion-braiding properties of particles with symmetry. By "with symmetry", we mean to include local excitations that carry representations of the symmetry group. Mathematically, this means that the unitary braided fusion category $\mathcal{C}$ contain a SFC $\mathcal{E}$ as its Müger center, i.e., a UMTC$_{/\mathcal{E}}$. We know that either $\mathcal{E} = \text{Rep}(G)$ or $\mathcal{E} = \text{sRep}(G^f)$, where $G$ or $G^f$ is the symmetry group. In this case, we must find a proper

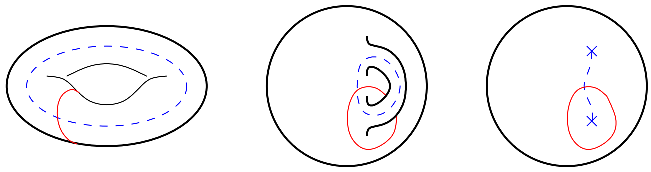


FIG. 4. (a) A torus with a flat $G$ connection (described by a symmetry twist along the dashed loop). The thin solid loop is a braiding path. (b) A handle is deformed into a very thin one. (c) A very thin handle can be viewed as two defects, and each defect corresponds to the added particle in the modular extension.

integrability condition that is weaker than the nondegeneracy of UBFC.

In order for the factorization homology of $\mathcal{C}$ on a surface, a unitary category denoted by $\mathcal{C}_\Sigma$, to have a physical meaning, we suspect that we should be able to interpret its object as finite dimensional Hilbert spaces in a natural way. This suggests that the category $\mathcal{C}_\Sigma$ should equipped with a natural functor to the category of finite dimensional Hilbert spaces, which is a factorization homology $\mathcal{M}_\Sigma$ of a UMTC $\mathcal{M}$ [53]. So we expect that we should be able to embed $\mathcal{C}$ into a UMTC $\mathcal{M}$ such that the embedding naturally descends to a functor $\mathcal{C}_\Sigma \to \mathcal{M}_\Sigma$ on factorization homologies. An arbitrary UMTC such as the Drinfeld center $Z(\mathcal{C})$ of $\mathcal{C}$ can not do the job because there is no canonical way to identify $\mathcal{C}$ in $\mathcal{M}$ (with a fixed symmetry $\mathcal{E}$) so that it is unlikely that it can be compatible with the integration process. So we expect that the condition $\mathcal{E}_{\mathcal{M}}^{\text{cen}} = \mathcal{C}$ is a natural integrability condition that replace the nondegeneracy condition in this case. This flow of thinking leads us to the concept of the modular extension of $\mathcal{C}$. It also suggests that the nonexistence of the modular extension of a given $\mathcal{C}$ means that $\mathcal{C}$ is somewhat inconsistent globally or not integrable to all two-dimensional manifolds with natural physical meanings.

This can also be viewed from a different point of view. If we require each particle to be nontrivial in some sense, then we must only consider the nondegenerate unitary braided fusion category over SFC $\mathcal{E}$. In this case, for particles not in $\mathcal{E}$, we know they are nontrivial because their nontrivial double braiding (or nontrivial mutual statistics) with some particles. But we still have trouble to know why the particles in $\mathcal{E}$ are nontrivial? From their fusion and braiding properties, they just behave like the identity or a composite of identities.

To fix this problem, we put our particles on any two-dimensional manifolds. In this case, we can find a way to understand the nontrivialness of the particle in $\mathcal{E}$. This require us to twist the symmetry $G$ or $G^f$ on the two-dimensional manifold. In other words, we equip the two-dimensional manifold with a flat $G$ connection. Since the particles in $\mathcal{E}$ all carry irreducible representations of $G$, as we move the particles along a noncontractile loop, the flat $G$ connection will induce a $G$ transformation on the particle (or more precisely, on the hom space of the particles). This allows us to probe the particles in $\mathcal{E}$ and detect their nontrivialness.

Therefore, as we put particles on a two-dimensional manifold, it is important to allow any flat $G$ connection on the manifold. Now we ask, in this case, can a nondegenerate unitary braided fusion category $\mathcal{C}$ over a SFC $\mathcal{E}$ describes the fusion-braiding properties of particles that are consistent on any two-dimensional manifolds with any flat $G$ connections?

In this paper, we propose that the answer is no. We also propose that the answer is yes if the $\mathcal{C}$ over $\mathcal{E}$ has modular extensions, which are the categorical ways of gauging the symmetry $\mathcal{E}$. So, nondegenerate unitary braided fusion categories over SFC can describe the consistent local fusion and braiding on an open disk. Only the ones with modular extensions can describe the consistent fusion and braiding on any manifolds (with any flat $G$ connections).

The intuition for the above conjecture is explained in the Fig. 4. Figure 4(a) describes a braiding of particles on a torus with flat $G$ connection. As we deform a handle into a very thin one, we may view the above braiding on torus as a braiding around the added particles in the modular extension. So the consistent fusion and braiding on any manifolds with any flat $G$ connection must be closely related to the consistent fusion and braiding on a sphere with the added particles in the modular extension. So, the mathematical meaning of the modular extension is to make the fusion and braiding to be consistent on any manifolds with any flat $G$ connection.

For a given $\mathcal{C}$ over $\mathcal{E}$, there can be several modular extensions $\mathcal{M}$. We believe that those different modular extensions describe the different structures at the boundary. This picture leads to the physical conjecture that the triple $(\mathcal{C},\mathcal{M},c)$ classify the 2+1D topological/SPT orders with symmetry $\mathcal{E}$.


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