HW Solutions # 10 - 8.01 MIT - Prof. Kowalski

Angular Energy and Angular Momentum.

1) *10.70*

Please refer to figure 10.54 p.399.

The general strategy to solve this problem is to figure out the torque around a point and the direction of the torque will tell you which way it rotates.

Choose the z direction to be perpendicular to the plane shown in the figure and pointing *up*. If the direction of the torque is *up* it means that it rotates *counterclockwise* or it moves to the *left*. If the direction of the torque is *down* it means that it rotates *clockwise* or it moves to the *right*.

The smartest choice of the point you want to write the torque about is the contact point with the ground because the torque due to friction and normal force is Zero around this point since they are acting *at* this point. Weight vector is also passing through this point so it doesn't generate any torque around it.
In the following calculation regard the contact point as origin:

$$\overrightarrow{\mathbf{r}}_{contact\ point} = \mathbf{0}$$

a) *F to right; bottom of axle(# 1)*:

$$\overrightarrow{\tau}_c = \overrightarrow{\mathbf{r}}_{Fc} \times \overrightarrow{\mathbf{F}} \propto -\hat{\mathbf{z}}$$

(Where c denotes the *contact* point; $\overrightarrow{\tau}_c$: torque around the contact point c; $\overrightarrow{\mathbf{r}}_{Fc}$: $\overrightarrow{\mathbf{r}}$ to force from contact point) So it rotates to the right. Now if you write the torque around *center of mass* the only force that has torque around it is $\overrightarrow{\mathbf{f}}$. Consider z component of $\sum \overrightarrow{\tau}_{cm}$ about center of mass. To roll to right, torque must be in $-\hat{\mathbf{z}}$, since F gives a $+\hat{\mathbf{z}}$ torque, f (friction) must give $-\hat{\mathbf{z}}$ torque hence points to *left*.

F to right; top of axle(# 2):

$$\overrightarrow{\tau}_c = \overrightarrow{\mathbf{r}}_{Fc} \times \overrightarrow{\mathbf{F}} \propto -\hat{\mathbf{z}}$$

So it rotates to the right. With the same argument friction can point to the left(if axle radius is small) or to the right (if axle radius is large).

F points up; left of axle(# 3):

$$\overrightarrow{\tau}_c = \overrightarrow{\mathbf{r}}_{Fc} \times \overrightarrow{\mathbf{F}} \propto -\hat{\mathbf{z}}$$

So it rotates to the right. Friction is the *only* force acting in horizontal direction and is responsible from Newton's second law in horizontal direction for this acceleration so it points to the *right.*

b) From the definition:

$$\overrightarrow{\tau}_c = \overrightarrow{\mathbf{r}}_{Fc} \times \overrightarrow{\mathbf{F}}$$

There are three ways to make $\overrightarrow{\tau}$ zero: *1*)r=0; *2*)F=0; *3*)ϕ=0 (where ϕ is the angle between the two vectors $\overrightarrow{\mathbf{r}}_{Fc}$ and $\overrightarrow{\mathbf{F}}$.) Here you don't have r=0 or F=0. You can make the angle between them Zero. This configuration is shown in the figure 2.

c) Please see the right triangle shown in the figure two with the indicated parameters.

$$\sin\alpha = \frac{r}{R} \Rightarrow \alpha = \arcsin(\frac{r}{R})$$

d) Let's analyze the Newton's law $\overrightarrow{\mathbf{F}} = m\overrightarrow{\mathbf{a}}$ here:

$$\sum F_y = ma_y = 0 \Rightarrow \; N - mg - F\cos\alpha = 0 \Rightarrow \;\; N = mg - F\cos\alpha$$

So by increasing f you'll decrease N. Because $f_{max} = \mu_s N$ you will also decrease f_{max}. Now if you write Newton's law in the x direction:

$$\sum F_x = ma_x \Rightarrow F \sin\alpha - f = ma_x$$

So as you pull harder, F increases and f_{max} decreases and you'll reach a point that f can not cancel completely $F \sin\alpha$ and the Yo-Yo starts to move to the left, but *without rotating.*

2) ***10.78*** *Ball rolling down track over edge.*

The assumption is there is no friction present here so:

$$K_h + U_h = E_h = E_0 = K_0 + U_0 \tag{1}$$

There two parts contributing to K: one is the *rotation* and the *center of mass motion.*

$$K = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

Since the ball rotates without slipping we have $w = \frac{v}{R}$ so

$$K = \frac{1}{2}m(1 + \frac{I}{mR^2})v^2 \tag{2}$$

For a sphere you have

$$I_{ball} = \frac{2}{5}mR^2$$

you have:

$$K = \frac{7}{10}mv^2$$

I will measure the gravitation potential energy with respect to the height of the table so $y_h = h$ and $y_0 = 0$.

a) Setting up equation (1) we have:

$$0 + mgh = \frac{7}{10}mv^2 + 0$$

$$\boxed{v = \sqrt{\frac{10gh}{7}}}$$

Setting up the familiar kinematic equations for free fall with constant gravity:

$$y(t) = y_0 + v_0 t - \frac{1}{2}gt^2$$

$$x(t) = v_0 t$$

The time T the ball is in the air is $(y(T) = \; -y)$:

$$T = \sqrt{\frac{2y}{g}}$$

$$\boxed{x \equiv x(T) = v_0 T = \sqrt{\frac{20hy}{7}}}$$

b) x does not depend on g, so the result should be the same on the moon.

c) No system is perfect: energy is lost to noise generation, crushing dirt on the track air resistance, etc.

d) For dollar coin you have:

$$I_{_{coin}} = \frac{1}{2}mR^2$$

from equation (2) you have

$$K_{_{coin}} = \frac{3}{4}mv^2$$

Repeating the same thing we did for the ball you'll get:

$$\boxed{x_{_{coin}} = \sqrt{\frac{8hy}{3}}} \qquad (slightly\ less\ far)$$

3) ***10.87*** *Bullet hits pivot rod.*
There is no external torque acting on the system of bullet plus rod if you take torques about the pivot point, so that the forces on the axle don't generate any torque:

$$\frac{d\overrightarrow{\mathbf{L}}_P}{dt} = \sum \overrightarrow{\tau}_P = 0$$

We conclude that the total angular momentum about P is *conserved*
a) Angular momentum $\overrightarrow{\mathbf{L}} = \overrightarrow{\mathbf{r}} \times \overrightarrow{\mathbf{p}}$ is in z direction. Factor out the $\hat{\mathbf{z}}$ from your equations:

$$L_1 = mv\frac{L}{2}$$

$$L_2 = \tilde{I}_P\omega = [I_P + m(\frac{L}{2})^2]\omega$$

($\tilde{I}_P$ denotes the total I_P)

For a rod I around the pivot is

$$I_P = \frac{1}{3}ML^2$$

where M is the mass of the rod.

$$\frac{mvL}{2} = \frac{1}{3}ML^2\omega + \frac{mL^2\omega}{4}$$

So we get

$$\omega = \frac{\frac{m}{2}}{\frac{M}{3} + \frac{m}{4}}\frac{v}{L}$$

Here $m = \frac{M}{4}$:

$$\boxed{\omega = \frac{6}{19}\frac{v}{L}}$$

b)

$$K_{after} = \frac{1}{2}\tilde{I}\omega^2 \qquad (3)$$

Where $\tilde{I}$ denotes the *total* I:

$$\tilde{I} = \frac{1}{3}ML^2 + m(\frac{L}{2})^2 = (\frac{4}{3} + \frac{1}{4})mL^2 = \frac{19}{12}mL^2 \tag{4}$$

$$K_{before} = \frac{1}{2}mv^2 \tag{5}$$

So the ratio ϵ is:

$$\epsilon = \frac{\frac{1}{2}\tilde{I}\omega^2}{\frac{1}{2}mv^2} \tag{6}$$

combine equations (3), (4), (5) and the boxed result of part **a** you'll get:

$$\boxed{\epsilon = \frac{3}{19}}$$

4) ***10.101*** *Rolling without slipping.*

Please refer to figure 3. z is perpendicular to the plane and it's pointing up. Without loss of generality assume that the disk is rotating *counterclockwise.* From the free body diagram shown in the figure:

$$\sum F_y = ma_y = 0 \Rightarrow N - Mg = 0 \Rightarrow N = Mg \tag{7}$$

$$\sum F_x = Ma_x \Rightarrow \; f_k = Ma_x \tag{8}$$

$$f_k = \mu_k N = \mu_k Mg \tag{9}$$

(where as shown in the figure x is pointing to the left)
Combine (7), (8) and (9):

$$a_x = \frac{f_k}{M} = \frac{\mu_k Mg}{M} \Rightarrow \boxed{a_x{=}\mu_k g} \tag{10}$$

This acceleration is *constant* and from $v_x(t) = v_0 + a_x t$ we have

$$\boxed{v_x(t) = \mu_k g t} \tag{11}$$

The equation for ω can be derived from:

$$\sum \tau_z = I\alpha_z \tag{12}$$

Where we write τ_z around the center of mass. Around this point the only force that has torque is f_k:

$$\sum \overrightarrow{\tau}_c = \overrightarrow{\mathbf{r}}_{fc} \times \overrightarrow{\mathbf{f}}_k = (-R\hat{\mathbf{y}}) \times (-f_k\hat{\mathbf{x}}) = -Rf_k\hat{\mathbf{z}}$$

Combine (12) and (13):

$$\boxed{\alpha_z = -\frac{Rf_k}{I_{cm}} = -\frac{R\mu_k Mg}{I_{cm}}} \tag{13}$$

α_z is *constant* so:

$$\boxed{\omega_z(t) = \omega_0 - \frac{Rf_k}{I_{cm}}t} \tag{14}$$

Let's denote T as the time that we reach the criteria for no sliding:

$$v_x(T) = R\omega_z(T) \tag{15}$$

Combine (15) with (11) and (14):

$$\mu_k g T = R(\omega_0 - \frac{Rf_k}{I_{cm}}T)$$

$$T(\mu_k g + \frac{R^2 f_k}{I_{cm}}) = R\omega_0$$

Replacing $f_k = \mu_k Mg$ and $I = \frac{1}{2}MR^2$ you'll get:

$$\boxed{T = \frac{R\omega_0}{3\mu_k g}} \tag{16}$$

The distance it travels during this time can be derived from *kinematics:*

$$x(T) = x_0 + v_{0_x}T + \frac{1}{2}a_x T^2 = \frac{1}{2}\mu_k g T^2$$

$$x(T) = \frac{1}{2}\mu_k g(\frac{R\omega_0}{3\mu_k g})^2$$

$$\boxed{x(T) = \frac{R^2\omega_0^2}{18\mu_k g}} \quad (17)$$

From energy conservation equation :

$$K_0 + U_0 + W_{f_k} = K_T + U_T \quad (18)$$

Where K_T denotes the Kinetic energy when there is no slipping:(The equation is similar to *10.78* part **d**)

$$K_T = \frac{1}{2}I_{cm}\omega^2 + \frac{1}{2}Mv^2 = \frac{3}{4}Mv(T)^2$$

$$= \frac{3}{4}M(a_x T)^2 = \frac{3}{4}M(\mu_k g\frac{R\omega_0}{3\mu_k g})^2 \quad (19)$$

$$K_0 = \frac{1}{2}I_{cm}\omega_0^2 = \frac{1}{4}MR^2\omega_0^2 \quad (20)$$

(Where I dropped "cm" from v_{cm} throughout)
Combine (18) with (19), (20) and $U_0 = U_T$:

$$W_{f_k} = K_T - K_0 = \frac{3}{4}(\mu_k g\frac{R\omega_0}{3\mu_k g})^2 - \frac{1}{4}MR^2\omega_0^2 = -\frac{1}{6}MR^2\omega_0^2$$

$$\boxed{W_{f_k} = -\frac{1}{6}MR^2\omega_0^2}$$

NOTE: This is more than

$$-f_k x(T) = -\mu_k Mg\frac{R^2\omega_0^2}{18\mu_k g} = -\frac{1}{18}MR^2\omega_0^2$$

in magnitude because the coin is spinning rapidly (especially at first) so that that relative distance of slipping of the coin edge and the surface is a factor of 3 greater than $x(T)$.

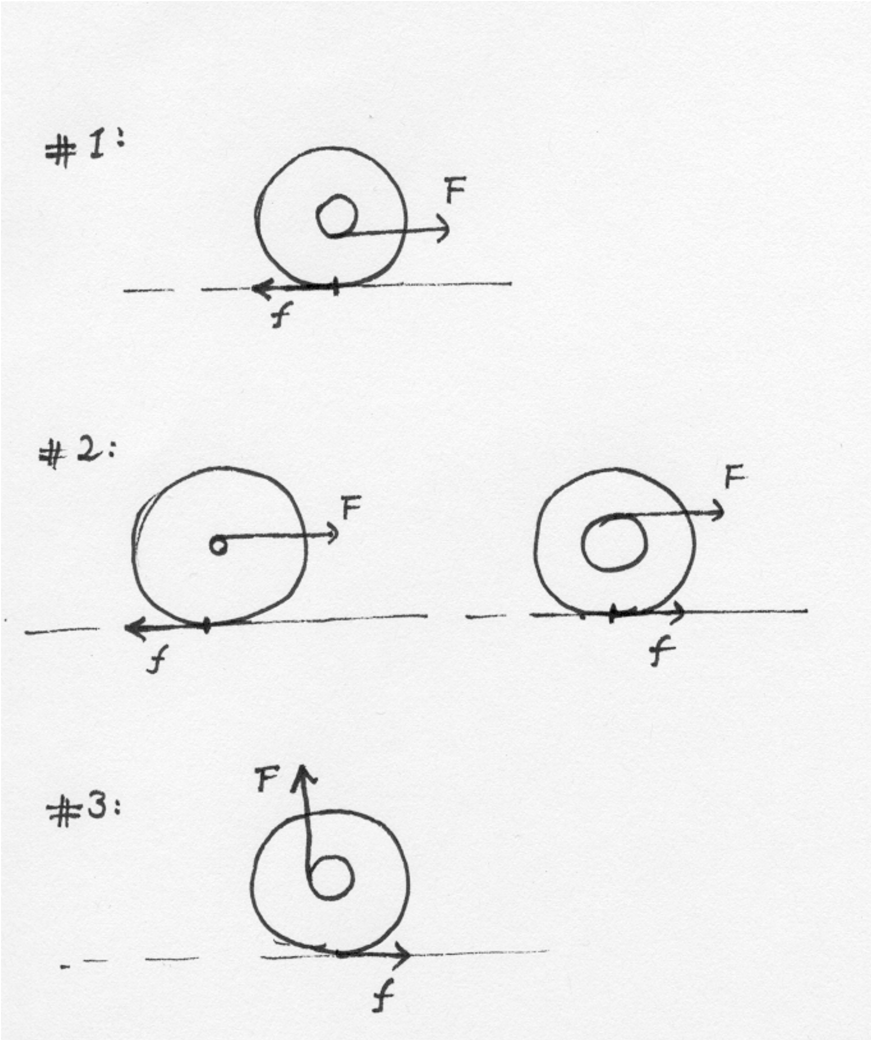

Figure 1: **10.70 part a**

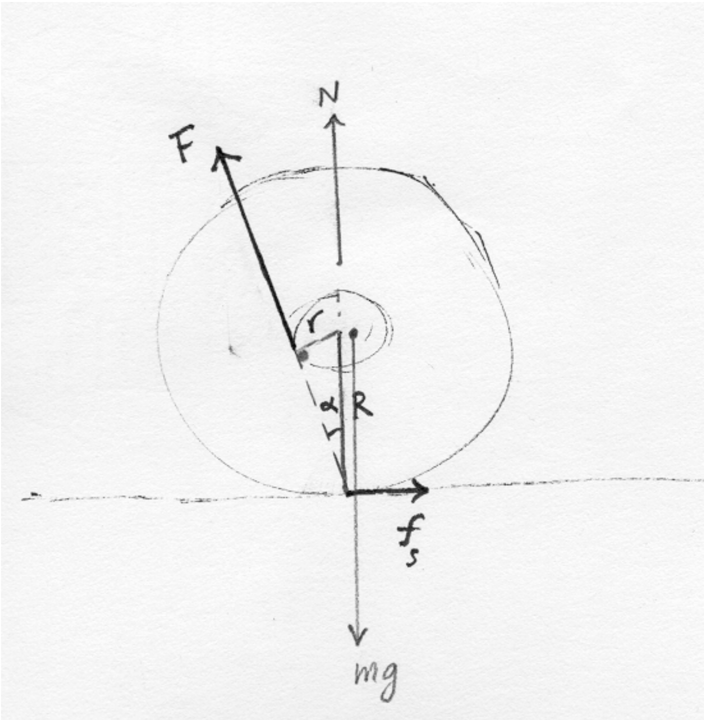

Figure 2: **10.70 part b, c, d**

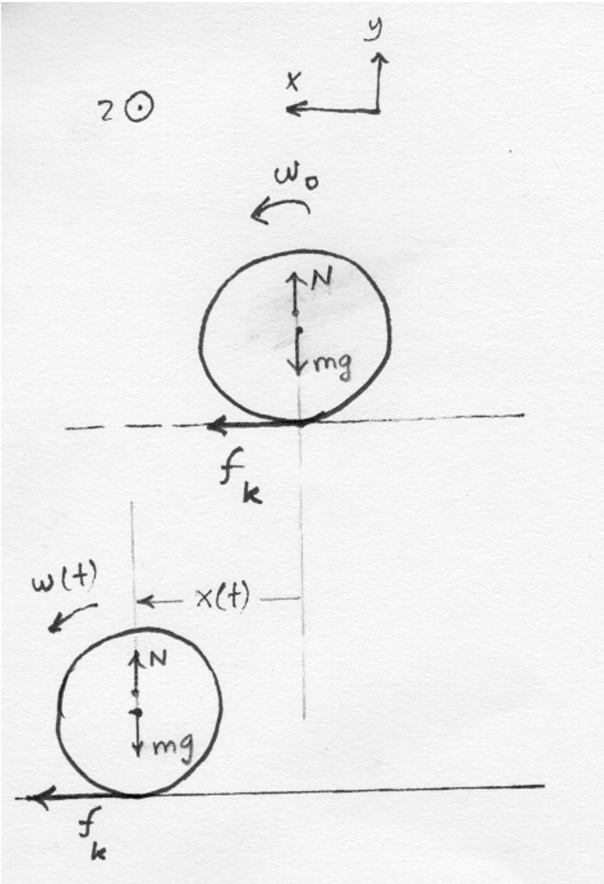

Figure 3: **10.101**