HW Solutions 2 - 8.01 MIT - Prof. Kowalski

Topics: Vectors and Two dimensional motion

1) **1.38**

`Please refer to figure 1.26 p.35.`

To add the several vector displacements, we use components, referring to east and north instead of x and y. The net northward displacement is 2.6 + (3.1) (sin 45) =+ 4.8 km, and the net eastward displacement is (4.0)+ (3.1) (cos 45) = + 6.2 km. The magnitude of the net displacement is $\sqrt{(4.8)^2 + (6.2)^2} =$ 7.8 km, using Pythagoras' theroem. To find the direction, note that the ratio of northward to eastward displacement is the tangent of the angle with the east axis: $\arctan(\frac{4.8}{6.2}) =$ 38 noth of east.

* The total displacement is a vector starting from the starting point towards the end. By sketching the vectors to scale you'll see the agreement with the method of components.

2) **1.70**

`Please refer to figure 1.32 p.37.`

The displacement vectors of the three legs must sum up to give the total observed displacement. This vector equality provides TWO equations, one for each component - the net x displacement should be 5.80 km and the net y displacement should be 0:

The third leg must take the sailor east the distance: (5.80) - (3.50) cos 45 - (2.00) = 1.33 km and a distance north (3.5) sin 45 = 2.47 km.

$\Rightarrow$ magnitude of the displacement is $\sqrt{(1.33)^2 + (2.47)^2} = 2.81$ km and the direction $\arctan(\frac{2.47}{1.33}) = 62$north of east.

3) **1.50**

`Please refer to figure 1.28 p.35.`

Method : (Product of magnitudes $\times \cos\theta$)
The angles must be found from the figure.
$\mathbf{A.B} = AB\cos\theta_{\mathtt{AB}} = (12 \times 15)\cos 93 = -9.4m^2$
$\mathbf{B.C} = BC\cos\theta_{\mathtt{BC}} = (15 \times 6)\cos 80 = +15.6m^2$
$\mathbf{C.A} = CA\cos\theta_{\mathtt{CA}} = (12 \times 6)\cos 173 = -71.5m^2$

4) **1.52**

The problem as initially posed was ambiguous as to which figure was referred to." Either of these solutions will be graded as correct.

Method: The vectors in problem 1.52 are in the x-y plane, so the cross product will result in a vector in the z-direction. The general formula is just the z-component of Eq. 1.26: $(A_xB_y - A_yB_x)\hat{\mathbf{k}}$.
a) $\mathbf{A} \times \mathbf{B} = (A_xB_y - A_yB_x)\hat{\mathbf{k}} = (-2 \times -3 - 2 \times 6)\hat{\mathbf{k}} = -6\hat{\mathbf{k}}$
b) $\mathbf{A} \times \mathbf{B} = (A_xB_y - A_yB_x)\hat{\mathbf{k}}) = (3 \times 6 - 10 \times 5)\hat{\mathbf{k}} = -32\hat{\mathbf{k}}$
c) $\mathbf{A} \times \mathbf{B} = (A_xB_y - A_yB_x)\hat{\mathbf{k}} = (-4 \times 14 - 7 \times 2)\hat{\mathbf{k}} = -70\hat{\mathbf{k}}$

`Alternate Solution - Please refer to figure 1.28 p.35.`

Method : (**Magnitude:** Product of magnitudes $\times \sin\theta$
Direction: right-hand rule)

$\mathbf{A} \times \mathbf{B} = AB\sin\theta_{\mathtt{AB}}(-\hat{\mathbf{k}}) = (12 \times 15 \times \sin 93)(-\hat{\mathbf{k}}) = -179.8\hat{\mathbf{k}}\ m^2$
$\mathbf{B} \times \mathbf{C} = BC\sin\theta_{\mathtt{BC}}(-\hat{\mathbf{k}}) = (15 \times 6 \times \sin 80)(-\hat{\mathbf{k}}) = -88.6\hat{\mathbf{k}}\ m^2$
$\mathbf{C} \times \mathbf{A} = AC\sin\theta_{\mathtt{CA}}(+\hat{\mathbf{k}}) = (6 \times 12 \times \sin 173)(\hat{\mathbf{k}}) = 8.77\hat{\mathbf{k}}\ m^2$

5) **2.27**

The **natural** time and length **scales** in the speeding up and braking process is **s** and **ft** respectively.

Useful conversions:
1mi= 5280 ft
1mph= $\frac{5280}{60\times 60}$ ft/s = 1.46 ft/s $\Rightarrow$ 1 ft/s = $\frac{1}{1.46}$=0.68 mph

Eq.1: $x(t) = x(t_0) + v(t_0)(t - t_0) + \frac{1}{2}a(t - t_0)^2$
Eq.2: $v(t) = v(t_0) + a(t - t_0)$
Eq.3: $v^2(t) - v^2(t_0) = +2a[x(t) - x(t_0)]$

Take the direction of motion as *the positive* x-direction.

a)
Speeding up:
Eq.1 $\Rightarrow$ $.250 \times 5280 = 0 + 0 \times (19.9) + \frac{1}{2}\ a_+\ (19.9)^2\ \Rightarrow a_+ = +6.67$ ft/s^2

Braking:
Eq.3 $\Rightarrow$ $0^2 - (60.0 \times 1.46)^2 = 2\ a_-\ 146 \Rightarrow a_- = -26.28$ ft/s^2

b) Eq.3 $\Rightarrow v^2 - 0^2 = 2\ a_+\ \Delta x \Rightarrow v = \sqrt{2 \times 6.67 \times .250 \times 5280} = 133$ ft/s = 90 mph The discrepancy with actual final speed shows that the motion is **not** with constant acceleration. The acceleration becomes less and as the car speeds up. Therefore, the car quickly comes up to 45 or 50 mph, hence it travels further than if it increased its speed more slosly at first.

c) Eq.2 $\Rightarrow 0 - 60 \times 1.46 = 2(-26.28)\Delta t \Rightarrow \Delta t = 1.7s$

* It's good to pay attention to the fact that our results **make sense**.

6) **2.98**

Let the height be **h** and denote the 1.30 s interval as $\Delta \mathbf{t}$ and the time it takes the boulder to fall from the top of the cliff to the ground as **t**. For sign convenience I consider the downward direction as positive.

a)
...it takes Δt for the boulder to fall the last 1/3 of the way to the ground. $\Rightarrow$ ... it takes $(t - \Delta t)$ for the boulder to fall $1 - \frac{1}{3} = \frac{2}{3}$ of its way to the ground...
We should solve the simultaneous equations that are derived from two conditions of the problem:

$$h = \frac{1}{2} g t^2 \tag{1}$$

$$\frac{2}{3} h = \frac{1}{2} g \ (t - \Delta t)^2 \tag{2}$$

Dividing the two sides we get $\frac{t}{t-\Delta t} = \pm\sqrt{\frac{3}{2}}$.
Picking the positive one we get $t = \frac{\Delta t}{1-\sqrt{\frac{2}{3}}} > \Delta t$ and substituting into $h = \frac{1}{2} g t^2$ gives h=246 m.

b)
Picking the negative one ($\frac{t}{t-\Delta t} = -\sqrt{\frac{3}{2}}$) we get $t = \frac{\Delta t}{1+\sqrt{\frac{2}{3}}} < \Delta t$! and h=2.5 m, clearly not a "cliff"! The fact that $(t - \Delta t) < 0$ suggests that the boulder was already at 2/3 h! This would correspond to an object initially at 1/3 of the height of the cliff being thrown upward and taking 1.30s to rise to the top and fall to the bottom. Clearly the problem is not interested in this case.

7) **Indoor Cannon**

Let's first summarize the standard projectile problem ignoring the ceiling.

The change in v_y comes from the gravity. It takes the y-component of velocity from v_{0y} to -v_{0y} by symmetry. So the total time that the ball is in the air - before hitting the ground - is $t_{total} = \frac{(-v_{0y})-(+v_{0y})}{-g} = \frac{2v_{0y}}{g}$ During this time the ball has a motion with a constant velocity , i.e. v_{0x}, in x direction so the range is $R = v_{0x}t_{total} = \frac{2v_{0x}v_{0y}}{g}$

$v_{0x} = v_0 \cos\alpha$
$v_{0y} = v_0 \sin\alpha$
$2\sin\alpha\cos\alpha = \sin 2\alpha$

$\Rightarrow R = \frac{v_0^2 sin2\alpha}{g}$

So the maximum range is obtained for $\alpha_{max} = 45$ which is $\frac{v_0^2}{g}$.

In the presence of the ceiling, the maximum height the ball reaches - defined as ***h*** - should be less than **H**. Otherwise the ball would hit the ceiling.

The next step is to find h for $\alpha_{max} = 45$:

Pay attention again that the top point corresponds to $v_y = 0$. Consider Eq. 3 of the problem 5 in the y direction. (h_{max} : the h corresponding to α_{max})

$0 - v_{0y}^2 = 2(-g)h \Rightarrow h_{max} = \frac{v_0^2(sin45)^2}{2g} = \frac{v_0^2}{4g}$

a) Clearly for $H \geq h_{max}$ Maximum R is independent of H and obtained at $\alpha = 45$.
$H \geq h_{max} \Rightarrow v_0 \leq \sqrt{4gH} \Rightarrow v_{\mathbf{crit}} = \sqrt{4gH}$

b) For $v > v_{\mathbf{crit}}$ we should set the angle in such a way to have $v_y = 0$ at y=H. Otherwise the ball will hit the ceiling which we want to avoid. I represent this v_y as $\tilde{v}_y$. Again using Eq. 3 of the problem 5 along y direction we get:
$\tilde{v}_y = \sqrt{2gH}$
This gives the y-component of the initial velocity. We need the x-component, and it can be obtained from Pythagoras' theorem: $v_0^2 = \tilde{v}_y^2 + \tilde{v}_x^2$, which implies $\tilde{v}_x = \sqrt{v_0^2 - \tilde{v}_y^2}$ The range can now be determined from the equation above: $R(H) = \frac{2v_{0x}v_{0y}}{g} = \frac{2\sqrt{2gH(v_0^2 - 2gH)}}{g}$
It is more elegant to write this in terms of $v_{\mathbf{crit}}$.
$R(H) = \frac{v_{\mathbf{crit}}^2\sqrt{2(\frac{v_0}{v_{\mathbf{crit}}})^2 - 1}}{g} \Longrightarrow R(H) = 4H\sqrt{2(\frac{v_0}{v_{\mathbf{crit}}})^2 - 1}$
This obviously gives the correct range when $v_0 = v_{\mathbf{crit}}$ and moreover shows that the range increases only linearly with v_0 for large values of v_0 (rather than quadratically as it does without the ceiling).
c) $v_{\mathbf{crit}} = \sqrt{4gH} = 10\sqrt{2}$ m/s.
$R_0 = \frac{v_0^2}{g} = 40$ m.
$R(5) = 4 \times 5\sqrt{2(\frac{20}{10\sqrt{2}})^2 - 1} = 20\sqrt{3} \simeq 35$ m.

$R_{lost} \simeq 5$ m.