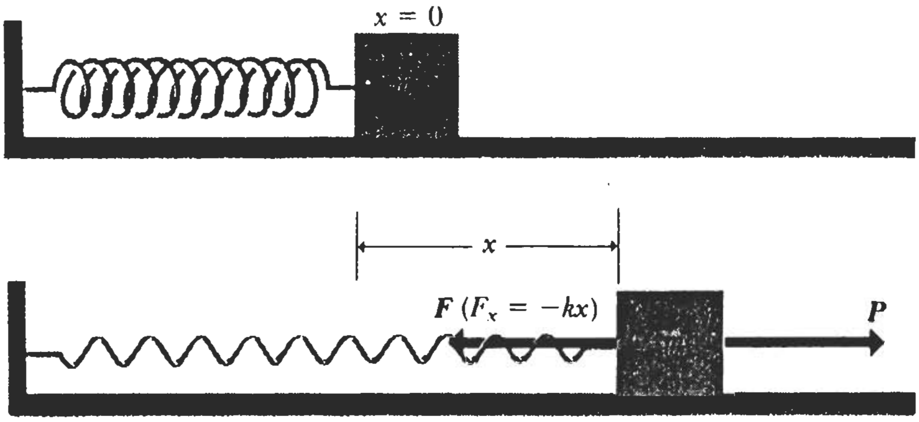
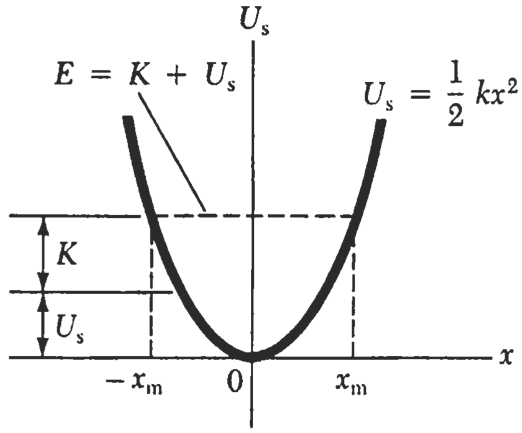
PE: Spring



For reference point we choose point P_0 at $x=0$, spring is in equilibrium.



$$\mathcal{U}(P_0) \equiv 0.$$

$$\mathcal{U}(x) = -\int_0^x F_x(x')\,dx'$$

$$= -\int_0^x (-kx')\,dx'$$

$$\mathcal{U}(x) = \frac{1}{2}kx^2$$

The total mechanical energy for the mass-spring system is:

$$E = K + \mathcal{U} = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

PE : Gravitational (General)

$$F_g = -\frac{G m M_E}{r^2}\,\hat{r}$$

$$U(P) = -\int_{P_0}^{P} \vec{F}\cdot d\vec{r} + U(P_0)$$

$$U(r) = \int_{\infty}^{r} \frac{G m M_E}{r'^2}\,dr' + U(P_0)$$

$$U(r) = -\frac{G m M_E}{r'}\Big|_{\infty}^{r} + U(P_0)$$

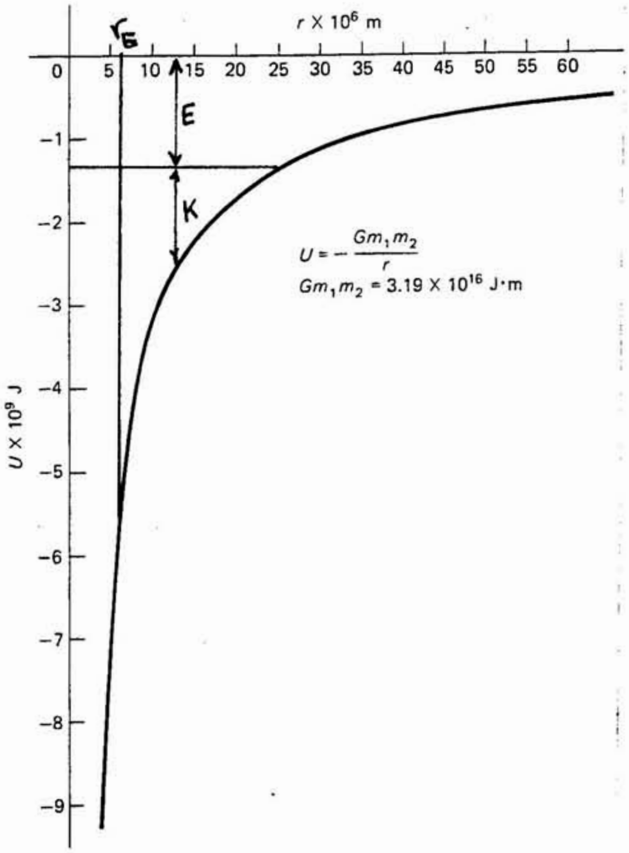
$$U(P_0) = U(\infty) = 0$$

$$\boxed{U(r) = -\frac{G m M_E}{r}}$$

Total Mechanical Energy

$$E = K + U = \frac{1}{2}mv^2 - \frac{G m M_E}{r}$$

The mutual gravitational potential energy U = - Gm1m2/r between an 80-kg object and the earth is shown versus their separation r.



Forces $\longleftrightarrow$ Potential Energy

Force	$\vec{F}(x)$	$U(x)$	x_0	$F(x_0)$
gravity (near)	$-mg\hat{j}$	mgy	$y=0$	$-mg\hat{j}$
spring	$-kx\hat{i}$	$\frac{1}{2}kx^2$	$x=0$	0
gravity (far)	$-\frac{GmM_E}{r^2}\hat{r}$	$-\frac{GmM_E}{r}$	$r=\infty$	0

often convenient to choose reference point where $\vec{F}(x_0)=0$ and $U(x_0)\equiv 0$.

$$W_{if}(\text{Cons.}) + W_{if}(\text{N-Cons}) = K_f - K_i \qquad [\text{WE Theorem}]$$

$$W_{if}(\text{NCons}) = K_f - K_i + (-W_{if}(\text{N-Cons}))$$

$$= K_f - K_i + U_f - U_i$$

$$W_{if}(\text{N.Cons}) = E_f - E_i \qquad [\text{Modified W-E}]$$

$$E = K + U \qquad \text{Total M.E.}$$

$$E_f = K_f + U_f = E_i = K_i + U_i \quad \underline{\underline{\text{if}}} \quad [W_{if}(\text{N.Cons}) \equiv 0]$$

Superposition

Several conservative forces acting on an object :

$\vec{F}_A$, $\vec{F}_B$, $\vec{F}_C$

$$W_{Total} = \int \vec{F}_A \cdot d\vec{r} + \int \vec{F}_B \cdot d\vec{r} + \int \vec{F}_C \cdot d\vec{r}$$

$$W_{Total} = \int \vec{F}_R \cdot d\vec{r}$$

$\vec{F}_R \equiv$ Resoltant Force.

$$U = U_A + U_B + U_C$$

Net Potential Energy → sum of all the individual potential energies for each force.

Total mechanical energy is conserved.

$$\boxed{K_i + \sum U_i = K_f + \sum U_f = E}$$

Example

(See notes 14-12)

$m = 0.50\,kg$

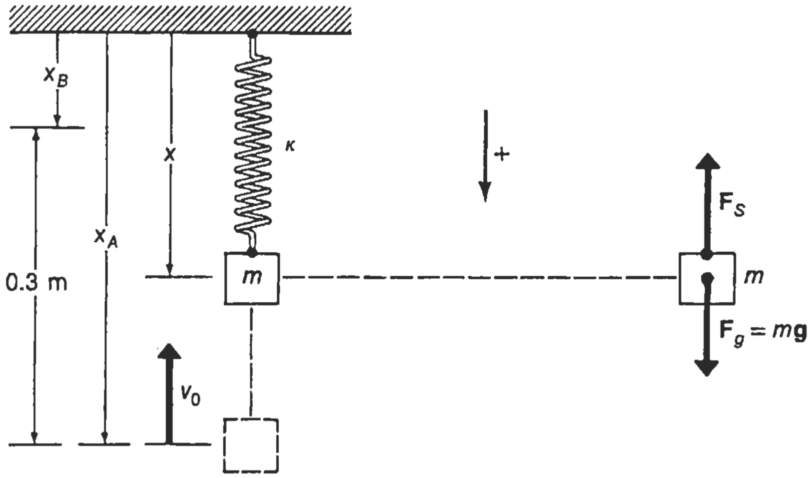
$K = 50\ N/m$

$x_A = 0.50\ m$

$x_B = 0.20\ m$

$v_0 = 2.0\ m/s \uparrow$

$v_B = ?$



Energy considerations:

2 PE's [gravity + spring]

$$\frac{1}{2} m v_A^2 + U(x_A) = \frac{1}{2} m v_B^2 + U(x_B)$$

$$\underbrace{\frac{1}{2} m v_A^2}_{KE} \underbrace{- mg x_A + \frac{1}{2} k x_A^2}_{PE} = \frac{1}{2} m v_B^2 - mg x_B + \frac{1}{2} k x_B^2$$

Solve for v_B:

$$v_B^2 = v_A^2 - 2g(x_A - x_B) + \left(\frac{k}{m}\right)(x_A^2 - x_B^2)$$

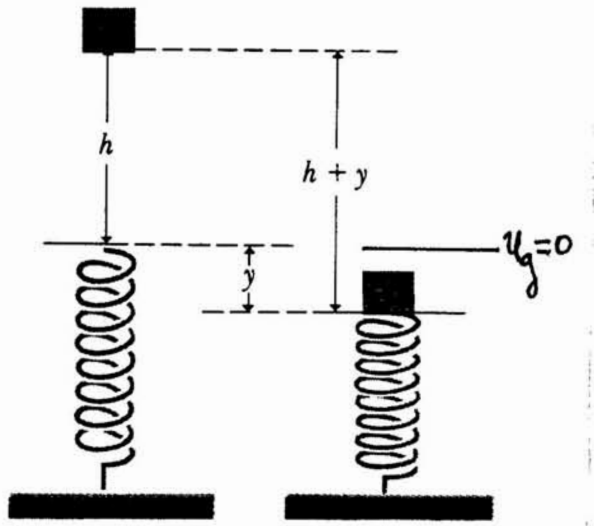
$$= 2.0^2 - 2 \times 9.8\,(0.50 - 0.20) + \frac{50}{0.5}\,(.50^2 - .20^2)$$

$$= 19.12 \qquad v_B = 4.37\,m/s$$

Example

Dropping a brick onto a spring-mounted platform: using elastic and gravitational potential energies together.

$m = 2.6\,kg$
$k = 72\,N/m$
$h = 0.55\,m$



Initial position   Maximum compression

The total fall of the block is h + y.

Mass m dropped from rest on spring. What is maximum compression of spring?

All conservative forces, PE represents all work done.

$E = K + U =$ conserved

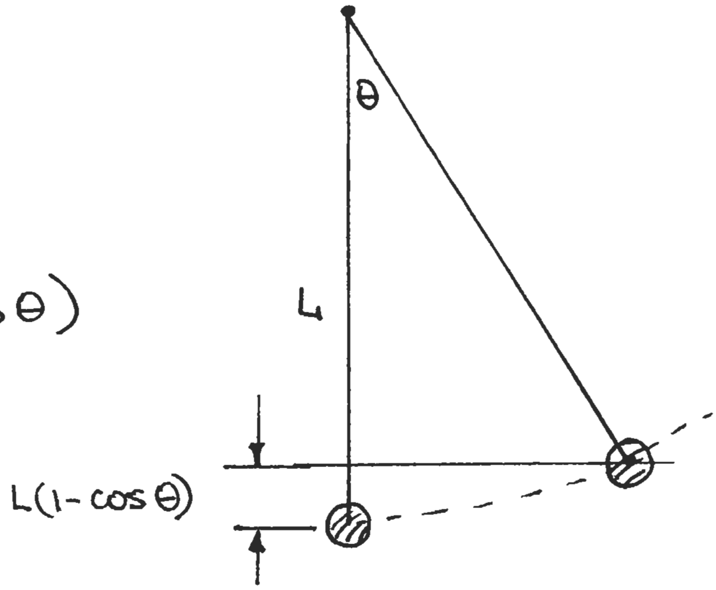
At release, brick is at rest, $v_1 = 0$ and $K_1 = 0$.
At maximum compression, brick is also momentarily at rest, $K_2 = 0$

$$K_1 + U_1 = K_2 + U_2$$
$$0 + mgh = 0 - mg(y) + \frac{1}{2}ky^2$$
$$y^2 - \frac{2mg}{k}y - \frac{2mg}{k}h = 0$$

$$y = \frac{1}{2}\left[\frac{2mg}{k} \pm \sqrt{\left(\frac{2mg}{k}\right)^2 + \frac{8mgh}{k}}\right]$$

$+ \rightarrow$ Answer.
$- \rightarrow$ [Mass + spring together str spring other side]

Example - Simple Pendulum



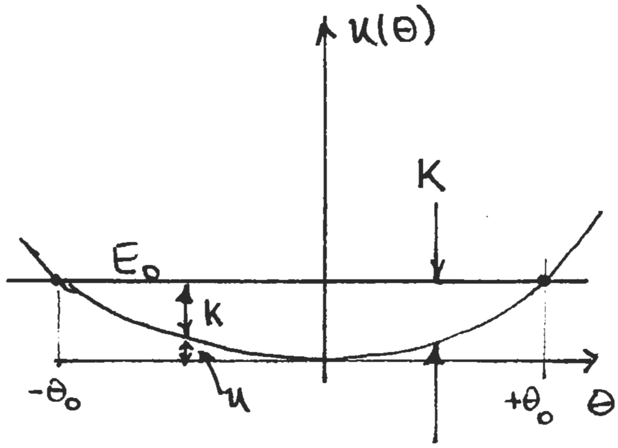
$\mathcal{U}(\theta) = mgL(1-\cos\theta)$

$E = K + \mathcal{U}$

$= \frac{1}{2}mv^2 + mgL(1-\cos\theta)$

For max. angle
$\theta = \theta_0 \quad v = 0.$

$\therefore E_0 = mgL(1-\cos\theta_0)$



$\therefore \cancel{m}gL(1-\cos\theta_0) = \frac{1}{2}\cancel{m}v^2 + \cancel{m}gL(1-\cos\theta)$

$v^2 = 2gL(\cos\theta - \cos\theta_0)$

At the bottom, $\theta = 0$

$v_B = \sqrt{2gL(1-\cos\theta_0)}$

Non-Conservative Forces

If non-conservative forces act on an object, then the change in the KE + PE of the conservative force will be equal to the work done by the friction force.

$$\Delta K + \Delta U = W_{\text{Friction}}.$$

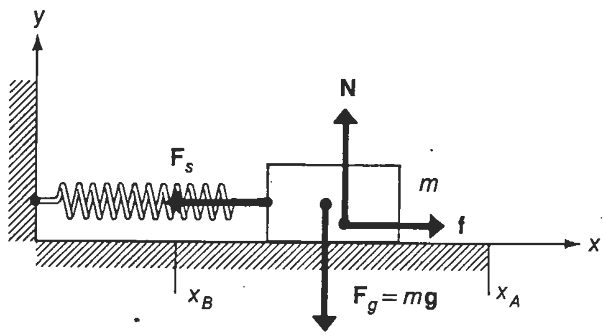
$\Delta K \equiv$ change in KE
$\Delta U \equiv$ change in PE
$W_f \equiv$ work done by frictional force.

$$E_1 = K_1 + U_1$$
$$E_2 = K_2 + U_2$$

$$(E_2 - E_1) = W_{\text{Friction}}$$

$$\frac{1}{2}mv_2^2 + U(x_2) - \left[\frac{1}{2}mv_1^2 + U(x_1)\right] = \int_{x_1}^{x_2} f\,dx$$

Example



$m = 0.50\,kg$

$k = 50\,N/m$.

$\mu_k = 0.20$

$x_A = 0.30\,m$

$x_B = 0.05\,m$

Mass released at $x = x_A$ with $v_A = 0$ [Spring is stretched]

what is velocity, v_B when $x = x_B$?

Force of friction $f = \mu_k N = \mu_k mg$

Work done by friction

$$W_f = \int_{x_A}^{x_B} \vec{f} \cdot \vec{dx} = \overbrace{+\mu_k mg\,(x_B - x_A)}^{-0.245\,J}$$

Energy Conservation:

$$\left[\tfrac{1}{2} m v_B^2 + \tfrac{1}{2} k x_B^2\right] - \left[\cancel{\tfrac{1}{2} m v_A^2} + \tfrac{1}{2} k x_A^2\right] = \mu_k mg\,(x_B - x_A)$$

$$\tfrac{1}{2} m v_B^2 = \tfrac{1}{2} k (x_A^2 - x_B^2) + \mu_k mg\,(x_B - x_A)$$

$$= \tfrac{1}{2} \times 50 (.3^2 - .05^2) - 0.245$$

$v_B = 2.788\,m/s$

Force $\longleftrightarrow$ Potential Energy

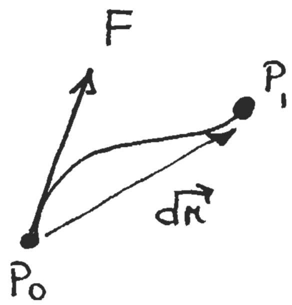
We have seen how to calculate the PE given a conservative force:

$$\mathcal{U}(P_1) - \mathcal{U}(P_0) = -\int_{P_0}^{P_1} \vec{F} \cdot d\vec{r}$$

Can we calculate the force given the PE?

Yes!!!

Assume P_0 and P_1 are separated by the infinetesimal displacement $d\vec{r}$, then differentiating expression for $\mathcal{U}(P_1)$:



$$d\mathcal{U} = \mathcal{U}(P_1) - \mathcal{U}(P_0) = -\vec{F} \cdot d\vec{r} \quad \text{[Inverse of P.E.]}$$

$$= -F_x dx - F_y dy - F_z dz$$

Assume displacement is only along x, $dy = 0$, $dz = 0$.

$$d\mathcal{U} = -F_x dx$$

or $F_x = -\dfrac{d\mathcal{U}}{dx}$ } differentiate keeping y, z constant.

Define a special derivative called a <u>partial derivative</u> for one variable at a time,

$$F_x = -\frac{\partial U}{\partial x}$$
$$F_y = -\frac{\partial U}{\partial y}$$
$$F_z = -\frac{\partial U}{\partial z}$$

Combining

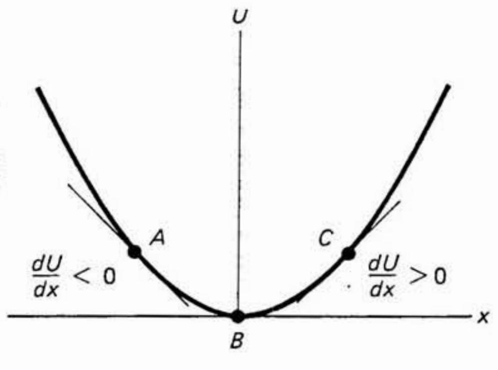
$$\vec{F} = -\left(\frac{\partial U}{\partial x}\hat{x} + \frac{\partial U}{\partial y}\hat{y} + \frac{\partial U}{\partial z}\hat{z}\right)$$
$$= -\vec{\nabla} U$$

<u>Example: Spring Force</u>

$$U(x) = \frac{1}{2}kx^2$$ PE of an elastic spring.

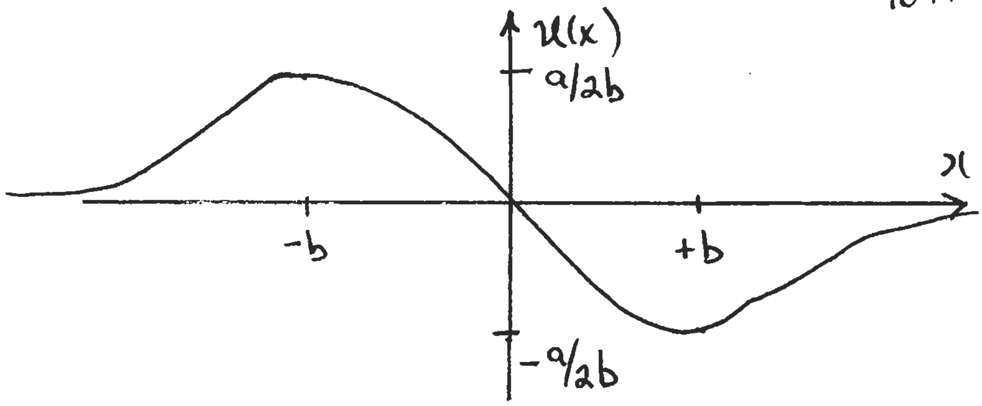
$$F_x = -\frac{\partial U}{\partial x} = -kx$$
$$F_y = -\frac{\partial U}{\partial y} = 0$$
$$F_z = -\frac{\partial U}{\partial z} = 0$$

The potential energy $U = {}^1/_2\, kx^2$ is shown for a spring. When the spring is compressed, $x<0$, the slope is negative, and the force is positive. When the spring is stretched, $x>0$, the slope is positive, and the force is negative.

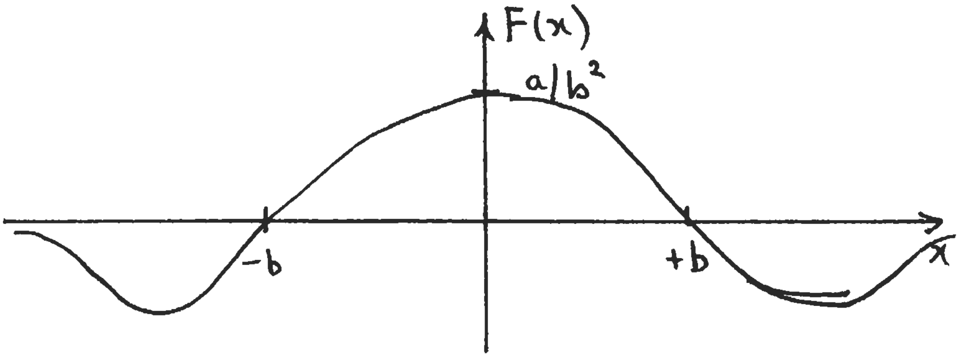


$$\vec{\nabla} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$ [Vector Operator]

Example



$$\mathcal{U}(x) = \frac{-ax}{b^2+x^2}$$



$$F(x) = -\frac{\partial \mathcal{U}(x)}{\partial x}$$

$$= \frac{a(b^2-x^2)}{(b^2+x^2)^2}$$

Example

$$\mathcal{U}(x,y) = A x^2 y^2$$

$$F_x = \frac{-\partial \mathcal{U}}{\partial x} = -2Axy^2$$

$$F_y = \frac{-\partial \mathcal{U}}{\partial y} = -2Ax^2y$$