

Energy Curves

17-1

Assume we know the PE curve for a particle moving in one-dimension. What is the description of the particle motion as a function of time?

Given: $U(x)$

Want: $x(t)$

Consider conservative forces only. Then the total mechanical energy is a constant of the motion.

$$E = K + U = \text{Constant}$$

$$= \frac{1}{2} m v^2 + U(x)$$

$$= \frac{1}{2} m \left(\frac{dx}{dt} \right)^2 + U(x)$$

Solve for v_x :

$$v_x = \frac{dx}{dt} = \sqrt{\frac{2}{m} [E - U(x)]}$$

$$\int_{x'=x_0}^x \frac{dx'}{\sqrt{\frac{2}{m} [E - U(x')]} } = \int_{t'=0}^t dt'$$

where $x' = x_0$ at $t' = 0$.

In general such an integral is not straightforward to evaluate analytically.

If we look at a PE diagram can we make some general statements about the particle motion?

Example

One dimensional potential for an alpha particle near a gold nucleus.
 $r_0 \sim 10^{-13}$ cm

$$F = -\frac{\partial U(x)}{\partial x}$$

i) At $x = x_B$ and $x = x_D$,
 $\frac{\partial U}{\partial x} = 0$ and $F(x) = 0$.

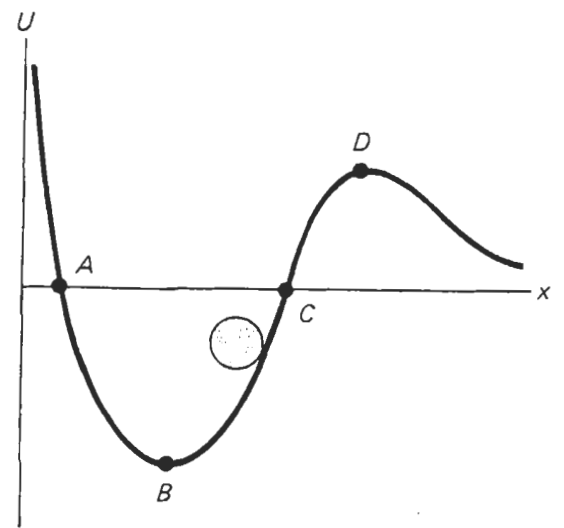
ii) If $\partial U / \partial x \rightarrow 0$ as $x \rightarrow \infty$, then $F(x) \rightarrow 0$ as $x \rightarrow \infty$.

iii) $F(x) > 0$ for $0 < x < x_B$ since $\frac{\partial U}{\partial x} < 0$. Also true for

$x > x_D$. The force is therefore repulsive (away from origin)

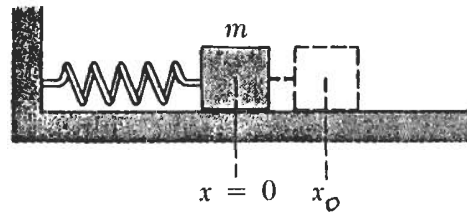
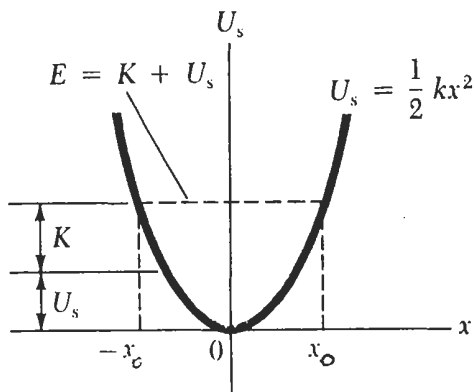
iv) $F(x) < 0$ for $x_B < x < x_D$ since slope of $U(x)$ is positive. Force is attractive (towards the origin).

Strategy: To remember force-potential energy relationship consider a ball rolling on a hilly surface under influence of gravity. The U -axis is "up". The ball is pushed in the direction of F .



Example: Mass + Spring

17-3



Mass is pulled to the right $x = x_0$ and released with zero velocity.

$E = \frac{1}{2} k x_0^2$: Initial total energy, remains constant.

$$\therefore \frac{1}{2} m v^2 + \frac{1}{2} k x^2 = \frac{1}{2} k x_0^2$$

i) $x > 0$

$$\frac{\partial U(x)}{\partial x} = kx > 0$$

$F < 0$, attractive; accelerates mass towards equilibrium position.

ii) $x < 0$

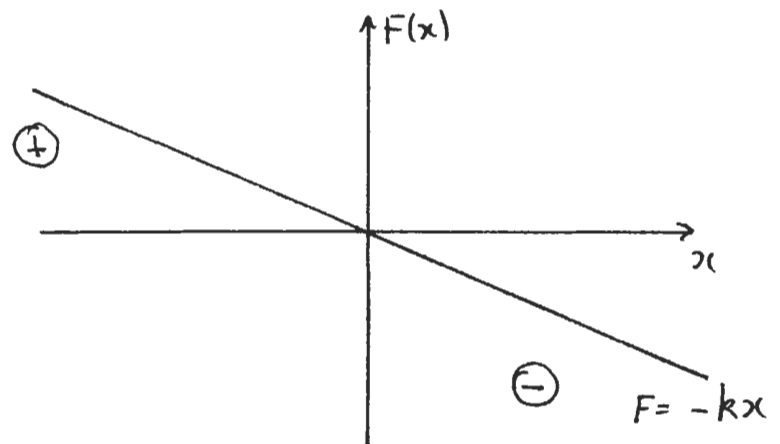
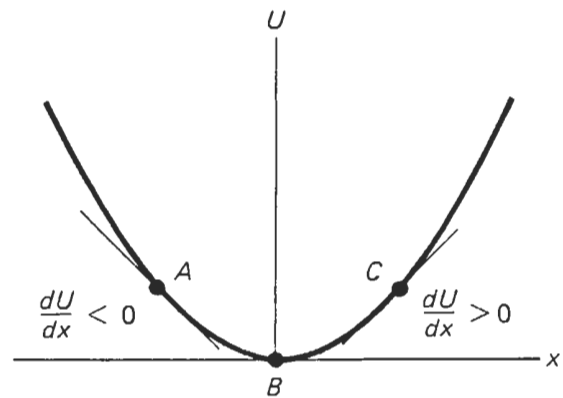
$$\frac{\partial U(x)}{\partial x} = kx < 0$$

$F > 0$, attractive; accelerates mass towards equilibrium position.

$$\text{iii) } x = 0$$

$$\frac{\partial U(x)}{\partial x} = 0$$

$$\therefore F = 0$$



Harmonic Oscillator

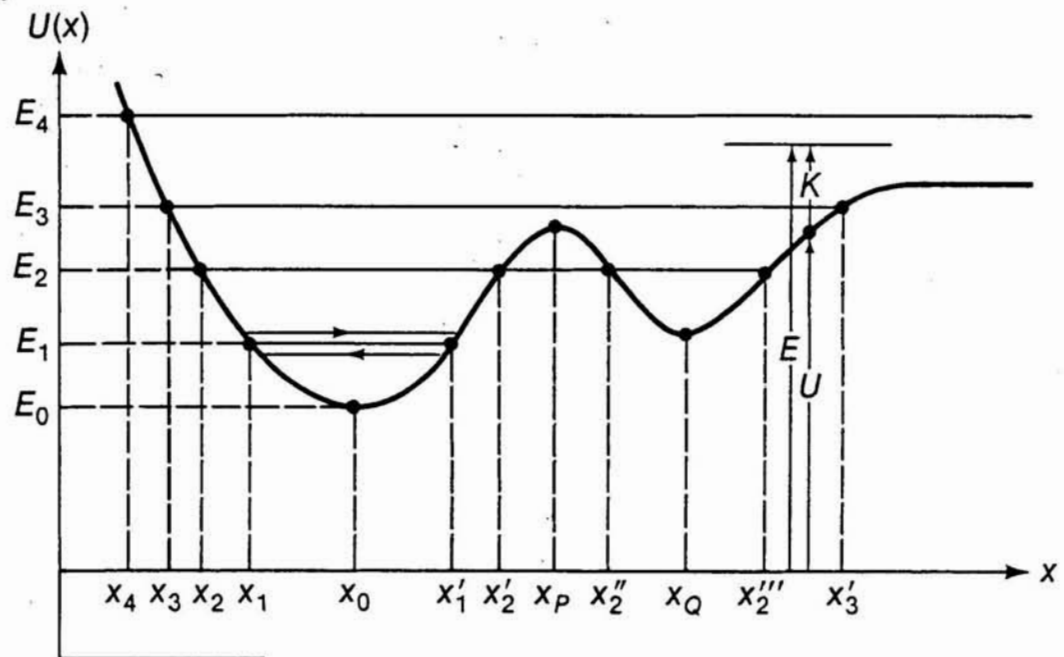
Particle moves between turning points, whose limits are defined by the total energy E . As E increases, turning points move further apart. As E increases amplitude of oscillation increases.

If $E=0$, $x=0$ always.

Motion is always completely bounded.

Example

17-5



The graphical representation of the one-dimensional constrained potential energy function $U(x)$ for an object. Also shown are various values of the total mechanical energy.

Consider the motion for particles with different total energies $E = E_0, E_1, E_2$ and E_4 . The value of E depends upon the initial conditions.

$$v_x = \sqrt{\frac{2}{m} (E - U(x))}$$

In all cases $U(x) < E$, otherwise v_x is imaginary.

i) $E = E_0$

Particle stays fixed at $x = x_0$. It has PE but can have no KE. Point of stable equilibrium. It is at bottom of lowest valley in the system.

ii) $E = E_1$

Particle oscillates back and forth between the points x_1 and x_1' . Its KE at any point is given by the difference between E_1 and $U(x)$. Starting at x_1 ,

it moves to the right [$F(x) > 0$] and increases speed up to x_0 . Continues to move to right with decreasing speed to x'_1 , where it stops, turns around and starts back.

x_1 } turning points / bounded motion [Particle is trapped
 x'_1 } in a potential well]

iii) $E = E_2$

Four turning points. Particle can move in one of two valleys, depending on where it is initially. Quantum Mechanics: Particle can jump from one valley to the other \rightarrow tunnelling. Classically not allowed!!

iv) $E = E_4$

One turning point only.

$u(x) < E_4$ for all $x > x_4$.

Particle moving to left varies in speed as it passes over the valleys. It reverses direction at $x = x_4$ and moves with $x > x_4$ indefinitely \rightarrow it never returns. Unbounded motion.

Equilibrium

$x = x_0$: Force on either side of x_0 acts to return particle to x_0 .

"Stable Equilibrium"

$x = x_p$: Force acts to move particle away from $x = x_p$

"Unstable Equilibrium."

Equilibrium/Stability.

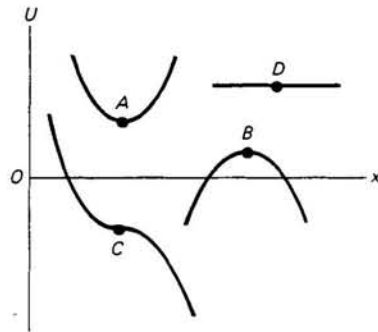
17-7

How do we tell if an equilibrium point is stable or unstable?

Mathematically: Examine sign of $\frac{d^2U}{dx^2}$!!

$$\frac{d^2U}{dx^2} > 0$$

Potential Minimum
Stable.



Points (A, B, C, D) on the potential-energy versus position graph with zero slope are equilibrium positions. The equilibrium is characterized as stable (A), unstable (B), or neutral (D), according to the graph in the neighborhood of the equilibrium point. For a curve like that near point C the equilibrium cannot be characterized simply as stable, unstable, or neutral.

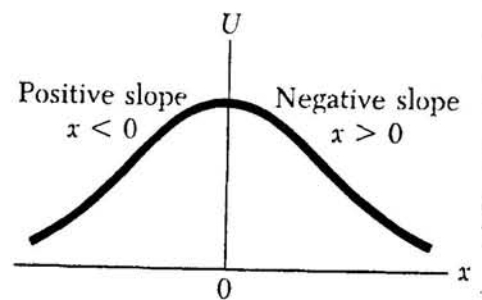
$$\frac{d^2U}{dx^2} < 0$$

Potential Maximum
Unstable.

$$\frac{d^2U}{dx^2} = 0$$

U is constant over a region.
 $F = 0$

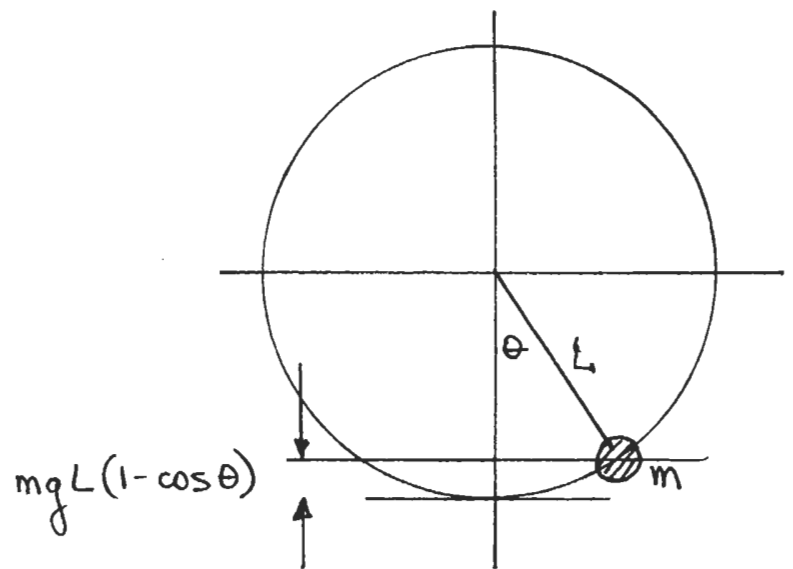
Neutral equilibrium.



A plot of U versus x for a system that has a position of unstable equilibrium, located at $x=0$. In this case, the force of the system for finite displacements is directed away from $x=0$.

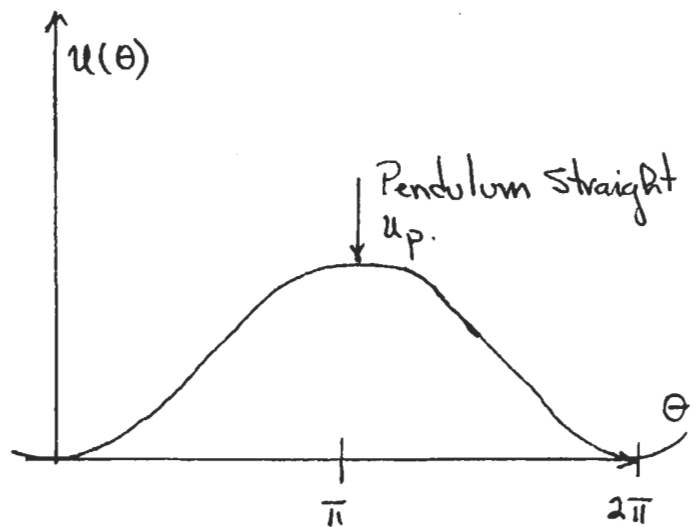
Pendulum

17-8



$$\frac{d^2u}{d\theta^2} > 0 \quad \text{at } \theta = 0 \quad (\text{stable})$$

$$\frac{d^2u}{d\theta^2} < 0 \quad \text{at } \theta = \pi \quad (\text{unstable})$$

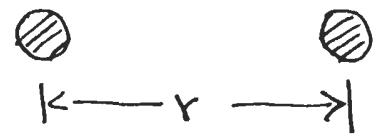


Example: Diatomic Molecule

17-9

Force between atoms in a diatomic molecule has its origin in the interactions between the electrons and nuclei in each atom. For simple atoms the potential energy is, to a good approx., represented by the Lennard-Jones potential (6, 12).

$$u(r) = u_0 \left[\left(\frac{a}{r} \right)^{12} - \left(\frac{a}{r} \right)^6 \right]$$



$r \equiv$ distance between atoms

$u_0, a \equiv$ constants.

$$\left. \begin{aligned} u_0 &= 5.6 \times 10^{-21} \text{ J} \\ a &= 3.5 \times 10^{-10} \text{ m} \end{aligned} \right\} \text{O}_2$$

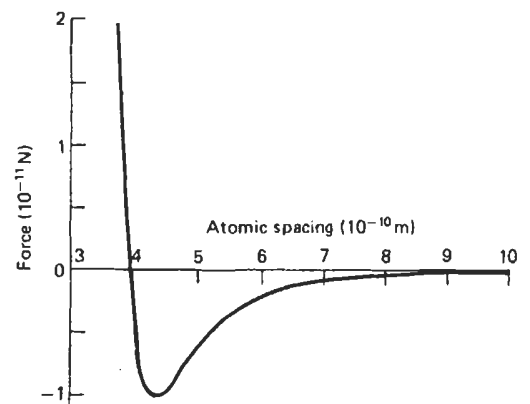
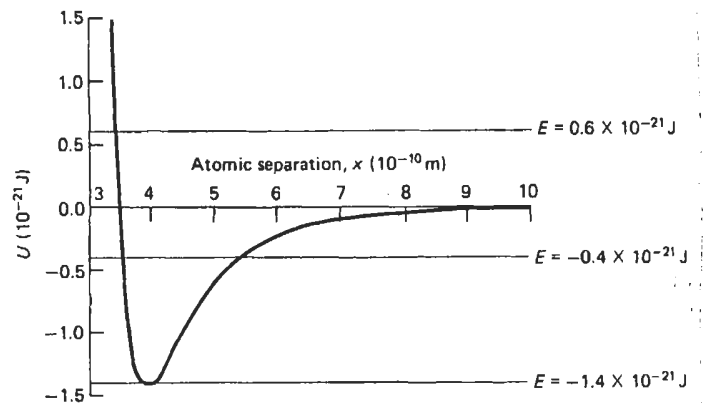
$$\begin{aligned} F &= -\frac{du}{dr} \\ &= -u_0 \left[-12 \frac{a^{12}}{r^{13}} + 6 \frac{a^6}{r^7} \right] \\ &= \frac{6u_0}{a} \left[2 \left(\frac{a}{r} \right)^{13} - \left(\frac{a}{r} \right)^7 \right] \end{aligned}$$

At equilibrium $F = 0$

$$\therefore 2 \left(\frac{a}{r_0} \right)^{13} - \left(\frac{a}{r_0} \right)^7 = 0$$

$$\left(\frac{a}{r_0} \right)^6 = \frac{1}{2}$$

$$\begin{aligned} r_0 &= 2^{1/6} a = 2^{1/6} (3.5 \times 10^{-10} \text{ m}) \\ &= 3.9 \times 10^{-10} \text{ m} \quad (\text{O}_2 \text{ molecule}) \end{aligned}$$



$$\boxed{2^{1/6} = 1.122}$$

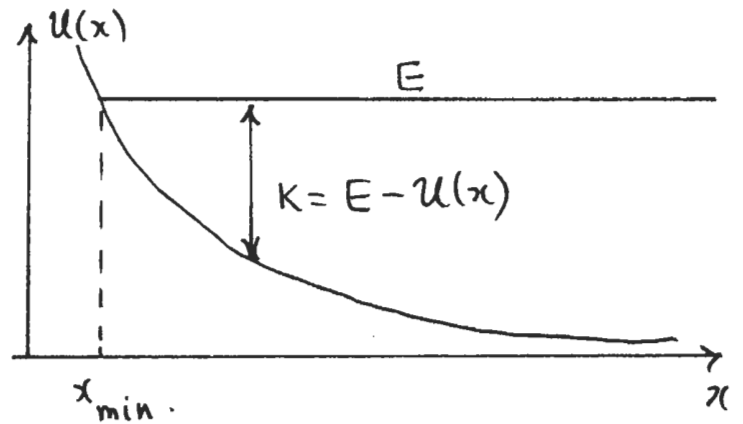
Example

Repulsive square law force. [eq. like point charges] 17-10

$$F(x) = \frac{A}{x^2}$$

$$A > 0$$

$$U = \frac{A}{x}$$



Assume particles have total energy E .

Particle approaching the origin comes as close as x_{min} , reverses direction and then moves out forever.

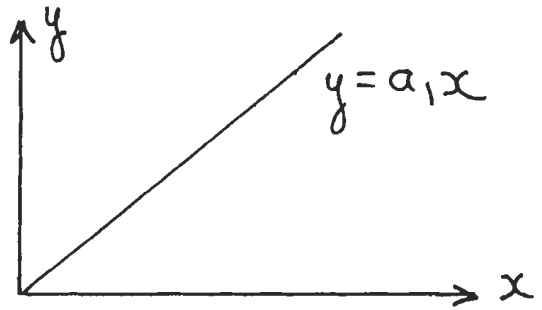
$$x_{min} = \frac{A}{E}$$

$$[i.e. U(x_{min}) = E]$$

Least Time Trajectories

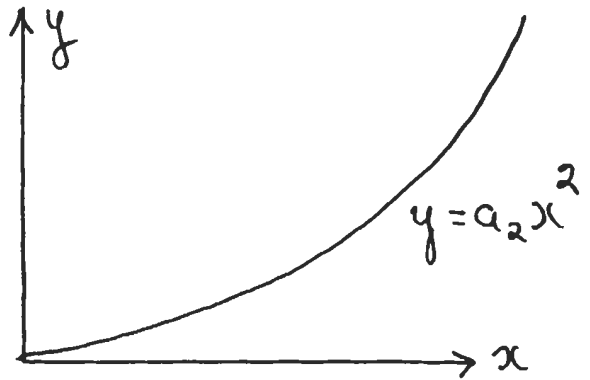
1. Straight line:

$$y = a_1 x$$



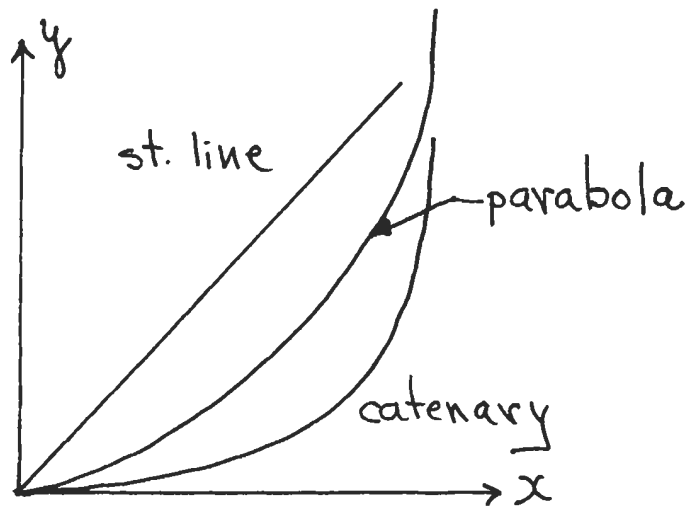
2. Parabola:

$$y = a_2 x^2$$



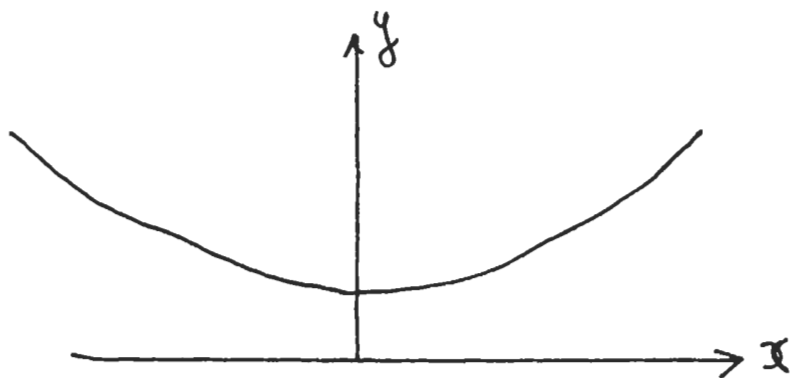
3. Catenary:

$$y = \frac{1}{a} \cosh ax$$



Catenary

- Suspend a cable between two posts and let it hang under own weight.
- The shape the cable takes is called a catenary



$$y = \frac{1}{2a} (e^{ax} + e^{-ax})$$

$$= \frac{1}{a} \cosh ax$$

$$y(x=0) = 1/a$$

- catenary has lowest average center-of-gravity
- catenary has smallest potential energy
- nature shapes curve to minimize PE

Power

17-12

- Power is defined as the time rate of doing work.

- If an external force applied to an object does an amount of work ΔW in the time interval Δt , the average power is

$$\bar{P} = \frac{\Delta W}{\Delta t}$$

- The instantaneous power, P , is the limiting value of the average power as Δt approaches zero.

$$P \equiv \lim_{\Delta t \rightarrow 0} \frac{\Delta W}{\Delta t} = \frac{dW}{dt}$$

$$[P] = \text{J/s}$$

$$= \text{Watts}$$

$$\text{British: } 1 \text{ hp} = 550 \text{ ft} \cdot \text{lbs/s}$$

$$= 745.7 \text{ W}$$

Power \longleftrightarrow Force

17-13

We can express the power in terms of the force acting and the velocity of the object.
For a small displacement $d\vec{r}$ the work done is

$$dW = \vec{F} \cdot d\vec{r}$$

$$P = \frac{dW}{dt} = \vec{F} \cdot \frac{d\vec{r}}{dt} = \vec{F} \cdot \vec{v}$$

Note: Velocity depends on the frame of reference.
Power also depends upon reference frame.

Energy \longleftrightarrow Power

$$E = \int_{t_1}^{t_2} P dt$$

$$= Pt \quad [\text{Constant } P]$$

$$1 \text{ kWh} = 3.6 \times 10^6 \text{ J} \quad [1000 \text{ W rate for 1 h}]$$

$$\left. \begin{array}{l} 1 \text{ kilocalorie} = 4.187 \times 10^3 \text{ J} \\ 1 \text{ Btu} = 1.055 \times 10^3 \text{ J} \end{array} \right\} \text{ Fuels}$$

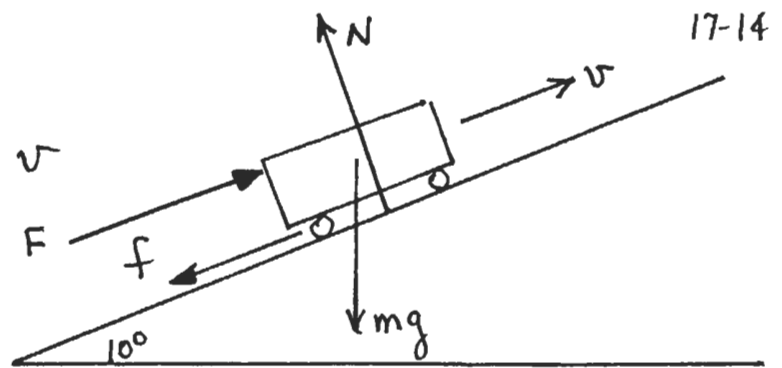
Example

Car moves at constant v
up a plane

$$m = 1400 \text{ kg}$$

$$v = 80 \text{ km/h} = 22 \text{ m/s}$$

$$f = 700 \text{ N}$$



Net force exerted by engine to move car up
at constant v :

$$F = f + mg \sin \theta$$

$$= 700 + 1400 \times 9.81 \times \sin 10^\circ$$

$$= 3100 \text{ N}$$

$$P = \vec{F} \cdot \vec{v} \quad [\vec{F} \text{ is parallel to } \vec{v}]$$

$$= 3100 \times 22 = 6.8 \times 10^4 \text{ W}$$

$$= 91 \text{ hp.}$$

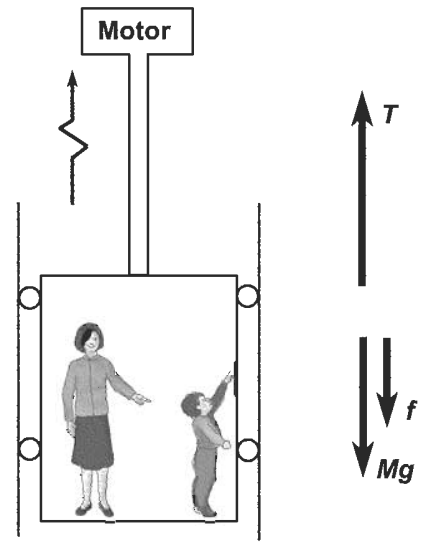
$$1 \text{ hp} = 746 \text{ W}$$

Example

$$m(\text{elevator} + \text{load}) = 1800 \text{ kg.}$$

$$f = 4000 \text{ N}$$

- a) What is horsepower required to move elevator up at constant speed of 3 m/s ?



Motor supplies required tension.

$$T - f - Mg = 0$$

$$\begin{aligned} T &= f + Mg \\ &= 4 \times 10^3 + 1.8 \times 10^3 \times 9.81 \\ &= 2.16 \times 10^4 \text{ N} \end{aligned}$$

$$\begin{aligned} P &= \vec{T} \cdot \vec{v} = T v \\ &= 2.16 \times 10^4 \times 3 \\ &= 6.48 \times 10^4 \text{ W} \\ &= 64.8 \text{ kW} \\ &= 86.9 \text{ hp.} \end{aligned}$$

- b) If elevator accelerates upward at 1.0 m/s^2 ?

$$T - f - Mg = Ma$$

$$\begin{aligned} T &= m(a + g) + f \\ &= 2.34 \times 10^4 \text{ N} \end{aligned}$$

$v \equiv$ instantaneous speed.

$$P = T v = (2.34 \times 10^4) v \text{ Watts} \quad \text{Power increases with inc. speed}$$

Forms of Energy

17-11

- electrical
- chemical
- heat
- nuclear

Atomic/Nuclear:

$$\begin{aligned} 1\text{eV} &= \text{electron-volt} \\ &= 1.602 \times 10^{-19} \text{ J} \end{aligned}$$

Power Plants:

$$1\text{-kilowatt hour} = 3.6 \times 10^6 \text{ J}$$

Fuels:

$$1 \text{ kilocalorie} = 4.187 \times 10^3 \text{ J}$$

$$1 \text{ Btu} = 1.055 \times 10^3 \text{ J}$$

Relativity/Einstein

Mass \longleftrightarrow Energy

$$E = mc^2$$

$$\uparrow c = 3 \times 10^8 \text{ m/s}$$

mass has energy } nuclear power
energy has mass } nuclear weapons
particle reactions