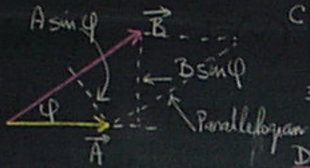
Vector Multiplication

Dot Product: [Scalar]

$\vec{A}\cdot\vec{B} = AB\cos\varphi$

Cross Product: [Vector]

$\vec{C} = \vec{A}\times\vec{B}$



$C = A\,\underbrace{B\sin\varphi}_{\text{Comp } \vec{B}\perp\vec{A}}$ (mag.)

$C = B\underbrace{(A\sin\varphi)}_{\text{comp } \vec{A}\perp\vec{B}}$

$\equiv$ Area of Parallelogram

Direction: Right Hand Rule

$\vec{C}\perp$ plane of $\vec{A}$ and $\vec{B}$

Along normal to plane.

$\vec{C}\perp\vec{A}$ and $\vec{C}\perp\vec{B}$ at the same time ①

$\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$

Parallel Vectors: $\varphi = 0^\circ$

$\vec{A}\times\vec{A} = AA\sin\varphi \equiv 0.$

Test for parallel vectors!

Unit Vectors:

$\hat{i}\times\hat{i} = 0 \qquad \hat{i}\times\hat{j} = -\hat{j}\times\hat{i} = \hat{k}$

$\hat{j}\times\hat{j} = 0 \qquad \hat{k}\times\hat{i} = -\hat{i}\times\hat{k} = \hat{j}$

$\hat{k}\times\hat{k} = 0 \qquad \hat{j}\times\hat{k} = -\hat{k}\times\hat{j} = \hat{i}$

$\vec{A}\times\vec{B} = (A_x\hat{i} + A_y\hat{j} + A_z\hat{k})\times(B_x\hat{i} + B_y\hat{j} + B_z\hat{k})$

$= (A_yB_z - A_zB_y)\hat{i} + (A_zB_x - A_xB_z)\hat{j}$

$+ (A_xB_y - A_yB_x)\hat{k}$

$$\vec{A}\times\vec{B} \equiv \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Example Cross Product

$\vec{A} = 3\hat{i} + 7\hat{j} - \hat{k}$

$\vec{B} = \hat{i} - \hat{j}$

$$\vec{C} = \vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 7 & -1 \\ 1 & -1 & 0 \end{vmatrix}$$

$\vec{C} = -\hat{i} - \hat{j} - 10\hat{k}$

$\vec{C}\cdot\vec{A} = (-\hat{i} - \hat{j} - 10\hat{k})\cdot(3\hat{i} + 7\hat{j} - \hat{k})$ ②

$= -3 - 7 + 10 = 0$

$\vec{C}\cdot\vec{B} = (-\hat{i} - \hat{j} - 10\hat{k})\cdot(\hat{i} - \hat{j})$

$= -1 + 1 = 0$

$\therefore \vec{C}\perp\vec{A}$ and $\vec{C}\perp\vec{B}$!!!

$\sin\theta = \frac{C}{AB} = \frac{\sqrt{102}}{\sqrt{59}\sqrt{2}} = \sqrt{\frac{102}{118}} \Rightarrow 70.8^\circ$

Lecture 5, Blackboard #1

$$\vec{A}\times\vec{B} = (A_x\hat{i} + A_y\hat{j} + A_z\hat{k})\times(B_x\hat{i} + B_y\hat{j} + B_z\hat{k})$$

$$= (A_yB_z - A_zB_y)\hat{i} + (A_zB_x - A_xB_z)\hat{j} + (A_xB_y - A_yB_x)\hat{k}$$

$$\vec{A}\times\vec{B} \equiv \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Example Cross Product

$$\vec{A} = 3\hat{i} + 7\hat{j} - \hat{k}$$

$$\vec{B} = \hat{i} - \hat{j}$$

$$\vec{C} = \vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 7 & -1 \\ 1 & -1 & 0 \end{vmatrix}$$

$$\vec{C} = -\hat{i} - \hat{j} - 10\hat{k}$$

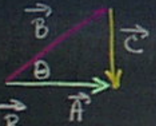
$$\vec{C}\cdot\vec{A} = (-\hat{i} - \hat{j} - 10\hat{k})\cdot(3\hat{i} + 7\hat{j} - \hat{k}) = -3 - 7 + 10 = 0$$

$$\vec{C}\cdot\vec{B} = (-\hat{i} - \hat{j} - 10\hat{k})\cdot(\hat{i} - \hat{j}) = -1 + 1 = 0$$

$$\therefore \vec{C}\perp\vec{A} \text{ and } \vec{C}\perp\vec{B} \;!!!$$

$$\sin\theta = \frac{C}{AB} = \frac{\sqrt{102}}{\sqrt{59}\sqrt{2}} = \sqrt{\frac{102}{118}} \Rightarrow 70.8^\circ$$

Law of Cosines



$$\vec{C} = \vec{A} - \vec{B}$$

$$\vec{C}\cdot\vec{C} = \vec{A}\cdot\vec{A} - 2\vec{A}\cdot\vec{B} + \vec{B}\cdot\vec{B}$$

$$C^2 = A^2 - 2AB\cos\theta + B^2$$

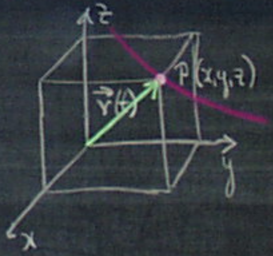
$$\cos\theta = \frac{A^2 + B^2 - C^2}{2AB}$$

3-D Kinematics

- Particle moving in 3D
- Like 3 1-D motions simult.

$\vec{r}(t) \equiv$ inst. position vector for point $P(x,y,z)$

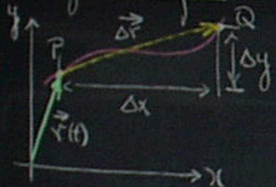
$$\vec{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k}$$



③

Lecture 5, Blackboard #2

Average Velocity: 3-D



$\vec v$: direction of $\vec{\Delta r}$
: magnitude $\Delta r/\Delta t$

$\vec{\Delta r} = \Delta x\,\hat i + \Delta y\,\hat j + \Delta z\,\hat k$

$\vec{\bar v} = \frac{\vec{\Delta r}}{\Delta t} = \frac{\Delta x}{\Delta t}\hat i + \frac{\Delta y}{\Delta t}\hat j + \frac{\Delta z}{\Delta t}\hat k$

$\bar v_x = \Delta x/\Delta t \quad \bar v_y = \frac{\Delta y}{\Delta t} \quad \bar v_z = \frac{\Delta z}{\Delta t}$

$$\boxed{\vec{\bar v} = \bar v_x\,\hat i + \bar v_y\,\hat j + \bar v_z\,\hat k}$$

Instantaneous Velocity: 3-D

- Take limit as $\Delta t \to 0$ and Q moves closer to P.

$\vec v(t) = \lim_{\Delta t\to 0} \frac{\vec{\Delta r}}{\Delta t} = \frac{d\vec r}{dt}$ { tangent to path at P

$\vec v(t) = \frac{d}{dt}(x\,\hat i + y\,\hat j + z\,\hat k)$
$= \frac{dx}{dt}\hat i + \frac{dy}{dt}\hat j + \frac{dz}{dt}\hat k$

$\vec v(t) = v_x\,\hat i + v_y\,\hat j + v_z\,\hat k$

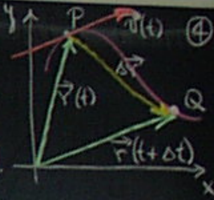
$v_x = \frac{dx}{dt}$, etc

$|v| = \sqrt{v_x^2 + v_y^2 + v_z^2}$

Average Speed = $\frac{\text{dist. travel.}}{\text{Time}}$

Inst. Speed = Magnitude of $|\vec v|$

Velocity: Magnitude + Direction



Acceleration: 3-D

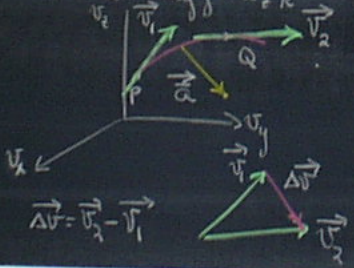
Average $\vec a$:

$\vec{\Delta v} = \vec v_2 - \vec v_1$ { mag. dir.

$\vec a = \frac{\vec{\Delta v}}{\Delta t} = \frac{\vec v_2 - \vec v_1}{\Delta t} = \frac{\vec v(t+\Delta t) - \vec v(t)}{\Delta t}$

$\vec a = \frac{\Delta \vec v_x}{\Delta t} + \frac{\Delta \vec v_y}{\Delta t} + \frac{\Delta \vec v}{\Delta t}$

$\vec{\bar a} = \bar a_x\,\hat i + \bar a_y\,\hat j + \bar a_z\,\hat k$



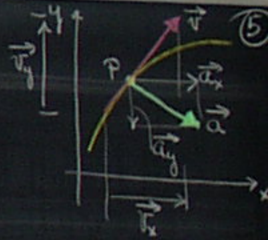
Instantaneous $\vec a(t)$: 3-D

$\vec a(t) = \lim_{\Delta t\to 0} \frac{\vec{\Delta v}}{\Delta t} = \frac{d\vec v}{dt} = \frac{d^2\vec r}{dt}$

$= a_x\,\hat i + a_y\,\hat j + a_z\,\hat k$

$a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2} \qquad a_y = \frac{dv_y}{dt} = \frac{d^2y}{dt^2}$

$a_z = \frac{dv_z}{dt} = \frac{d^2z}{dt^2}$



Lecture 5, Blackboard #3

Acceleration: 3-D

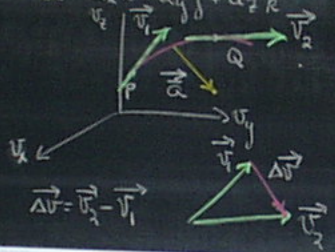
Average $\vec{a}$:

$\Delta\vec{v} = \vec{v}_2 - \vec{v}_1$ { mag. dir.

$$\vec{a} = \frac{\Delta\vec{v}}{\Delta t} = \frac{\vec{v}_2 - \vec{v}_1}{\Delta t} = \frac{\vec{v}(t+\Delta t) - \vec{v}(t)}{\Delta t}$$

$$\vec{a} = \frac{\Delta\vec{v}_x}{\Delta t} + \frac{\Delta\vec{v}_y}{\Delta t} + \frac{\Delta\vec{v}_z}{\Delta t}$$

$$\vec{a} = \vec{a}_x\hat{i} + \vec{a}_y\hat{j} + \vec{a}_z\hat{k}$$



$\Delta\vec{v} = \vec{v}_2 - \vec{v}_1$

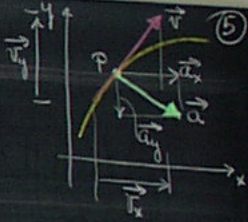
Instantaneous $\vec{a}(t)$: 3-D

$$\vec{a}(t) = \lim_{\Delta t \to 0} \frac{\Delta\vec{v}}{\Delta t} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt}$$

$$= a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$$

$$a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2} \qquad a_y = \frac{dv_y}{dt} = \frac{d^2y}{dt^2}$$

$$a_z = \frac{dv_z}{dt} = \frac{d^2z}{dt^2}$$



Example: 2-D Motion

$x = 1 + 2t^2$ (m)

$y = 2t + t^3$ (m)

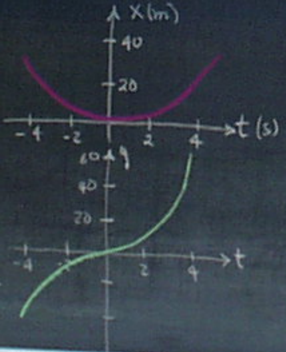
Find $\vec{r}, \vec{v}, \vec{a}$ at $t = 2s$

$x(t=2) = 1 + 2(2)^2 = 9m$

$y(t=2) = 2(2) + 2^3 = 12m$

$\vec{r}(t=2) = 9\hat{i} + 12\hat{j}$ Pos. Vector

$r = \sqrt{x^2+y^2} = \sqrt{9^2+12^2} = 15m$



$\theta = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{12}{9} = 53.1°$

Velocity:

$v_x = \frac{dx}{dt} = 4t$ m/s

$v_y = \frac{dy}{dt} = 2 + 3t^2$ m/s

$v_x(t=2) = 8$ m/s

$v_y(t=2) = 14$ m/s

$\vec{v} = 8\hat{i} + 14\hat{j}$

$|v| = \sqrt{8^2+14^2} = 16.1$ m/s

Acceleration:

$a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2} = 4\ m/s^2$

$a_y = \frac{dv_y}{dt} = \frac{d^2y}{dt^2} = 6t\ m/s^2$

$a_x(t=2) = 4$

$a_y(t=2) = 12$

$\vec{a} = 4\hat{i} + 12\hat{j}$

$a = \sqrt{4^2+12^2} = 12.6\ m/s^2$

$\theta_v = \tan^{-1}\frac{14}{8} = 60.3°$

$\theta_a = \tan^{-1}\frac{12}{4} = 71.6°$

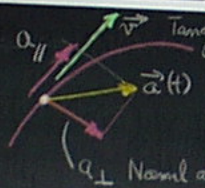
Acceleration: $a_\parallel$ and $a_\perp$

$a_\perp$: Normal to Path

$\Rightarrow$ change in dir. of $\vec{v}$

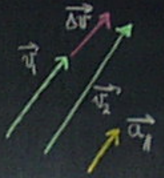
$a_\parallel$: Parallel to Path

$\Rightarrow$ change in mag. of $\vec{v}$



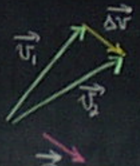
$\vec{a} = \frac{d\vec{v}}{dt}$ $\quad d\vec{v} = \vec{a}\,dt$

$\vec{a} \parallel \vec{v}$



magnitude change

$\vec{a} \perp \vec{v}$



direction change

⑦

$\vec{a}\cdot\vec{v} > 0$ vel increases

$\vec{a}\cdot\vec{v} < 0$ vel decreases

$\vec{a}\cdot\vec{v} = 0$ vel changes dir.

mag. stays const.

– circular motion

Constant $\vec{a}$ Motion: 3-D

$\vec{a} = \frac{\Delta\vec{v}}{\Delta t}$ $\quad \Delta\vec{v} = \vec{v} - \vec{v}_0$

$\Delta t = t - 0$

Let $\vec{a}(t) = \vec{a} \equiv$ constant

$\vec{v} = \vec{v}_0 + \vec{a}t$ ①

$v_x = v_{0x} + a_x t$

$v_y = v_{0y} + a_y t$ ②

$v_z = v_{0z} + a_z t$

$\vec{r} = \vec{r}_0 + \vec{v}_0 t + \frac{1}{2}\vec{a}t^2$ Pos. Vector

– same arguments as 1-D

$x(t) = x_0 + v_{0x}t + \frac{1}{2}a_x t^2$

$y(t) = y_0 + v_{0y}t + \frac{1}{2}a_y t^2$

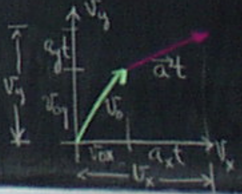
$z(t) = z_0 + v_{0z}t + \frac{1}{2}a_z t^2$

Motions are completely Independent.

x-accel → x-velocity

x-vel, x-accel } → x-position

↓ etc.



⑧