Rigid Body Kinematics

Objects in the real world are not point-like particles that we have been dealing with up to now. A real object has a <u>mass distribution</u> associated with its <u>size</u> and <u>shape</u>.

The motion of a real object involves both:

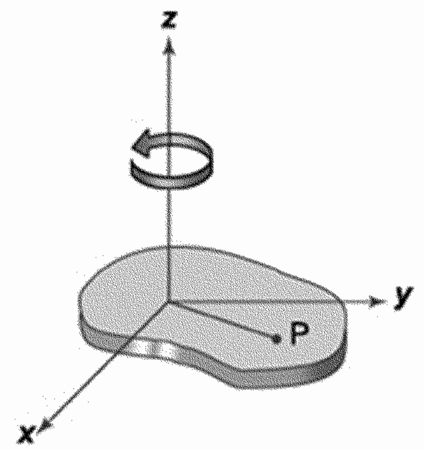
- <u>translational motion</u> of the <u>cm</u>.
- <u>rotational motion</u> about an <u>axis</u> (usually take to be an axis through the cm or some other fixed axis).

We will restrict our discussions to that of rigid bodies. A rigid body is one in which the relative coordinates connecting all the constituent particles remain constant. This is of course an idealized situation.

Rotations about a Fixed Axis

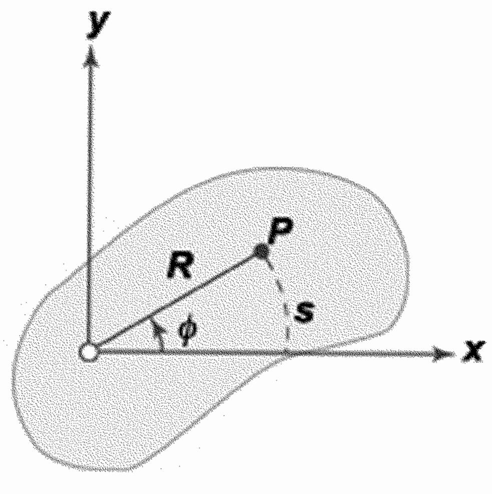
We will initially study the <u>motion</u> of a <u>rigid body</u> rotating about an <u>axis</u> that is <u>fixed in an inertial frame</u>.

Consider motion around the z-axis. Reference point P (which is not on the axis) represents the rotational motion of the body and of its angular position.



Rotation of a rigid body about a fixed axis (z axis).

Given a reference point P, its angular position is measured by the angle ϕ, between position vector $\vec{r}$ and the x-axis.



As the particle moves in a circle from the positive x-axis ($\varphi = 0$) to the point P, it moves through an arc length

$$s = R\phi$$

$$\phi(\text{rad}) = \frac{\pi}{180}\,\phi(\text{deg})$$

ϕ = positive counterclockwise

$\phi = 0 \Rightarrow x$-axis

$\phi = 2\pi \Rightarrow x$-axis again.

ϕ: is <u>not</u> a vector [rotations do not commute]

$\vec{d\phi} = d\varphi\,\hat{k}$ [infinitesimal rotation is a vector]

The rotational motion of a body is described by the rate of change of ϕ. In general the position angle is a function of time:

$$\phi = \phi(t)$$

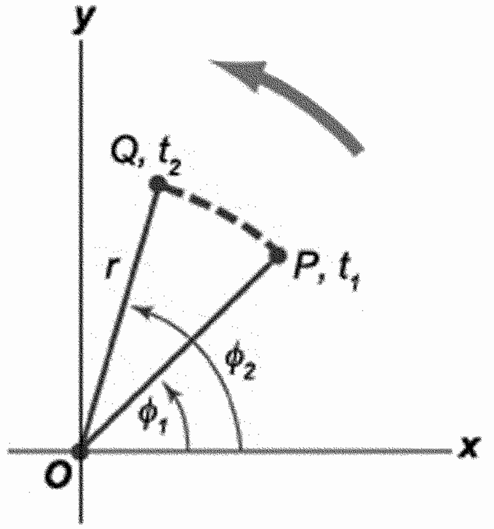
Suppose the particle moves from P to Q. The reference line OP makes an angle ϕ_1 at the time t_1, and an angle ϕ_2 at the time t_2. Define the average angular velocity of the body, $\bar{\omega}$, in the time interval $\Delta t = t_2 - t_1$

as the ratio of angular displacement $\Delta\phi = \phi_2 - \phi_1$ to Δt.

$$\bar{\omega} = \frac{\phi_2 - \phi_1}{t_2 - t_1} = \frac{\Delta\phi}{\Delta t} \quad \text{rad/s or } s^{-1}$$

$\hat{k}$ = unit vector along axis of rotation (z-axis)

$\vec{\omega}$ = points along axis of rotation.
[RHRule for sign convention]



A particle on a rotating rigid body moves from *P* to Q along the arc of a circle. In the time interval $\Delta t = t_2 - t_1$, the radius vector sweeps out an angle $\Delta\phi = \phi_2 - \phi_1$.

Analagous to linear velocity, the instantaneous angular velocity, is defined as the limit of this ratio as $\Delta t \to 0$. Becomes time rate-of-change of $\phi(t)$.

$$\vec{\omega} = \lim_{\Delta t \to 0} \frac{\Delta\phi}{\Delta t} = \frac{d\phi}{dt}\hat{k} \quad (s^{-1})$$

If the angular velocity, ω, is a constant

$$\omega = \omega_0,$$

the rate of rotation is often given in terms of the frequency, or number of revolutions per unit time.

1 revolution = $\Delta\phi = 2\pi$ radians

Time per revolution, or period $T = \dfrac{2\pi}{\omega_0}$ (s)

Frequency of revolution is $\nu = \dfrac{1}{T} = \dfrac{\omega_0}{2\pi}$ (Hz)

If the angular velocity of the body is changing with time (i.e. ω is not constant), then there is an angular acceleration.
If the angular velocities are ω_1 and ω_2 at the times t_1 and t_2, the average angular acceleration is

$$\bar{\alpha} = \frac{\omega_2 - \omega_1}{t_2 - t_1} = \frac{\Delta\omega}{\Delta t}$$

The instantaneous angular acceleration is the limit of this ratio as $\Delta t \to 0$.

$$\vec{\alpha} = \lim_{\Delta t \to 0} \frac{\Delta \omega}{\Delta t} = \frac{d\vec{\omega}}{dt} . \qquad (s^{-2})$$

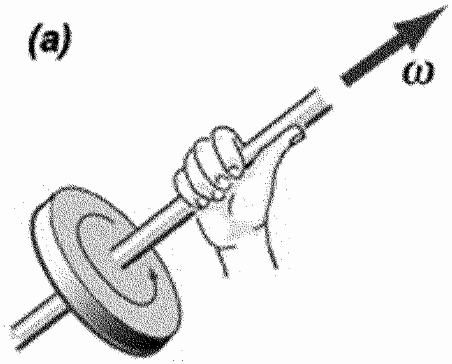
Since $\omega = \dfrac{d\phi}{dt}$, we also have

$$\vec{\alpha} = \frac{d^2\vec{\phi}}{dt^2}$$

For <u>rotation</u> about a <u>fixed axis</u>, every particle on the rigid body has the <u>same angular velocity</u> and the <u>same angular acceleration</u>.
The direction of $\vec{\alpha}$ is along the same axis as $\vec{\omega}$. If the axis of rotation is changing then $\vec{\alpha}$ is not in the same direction as $\vec{\omega}$.

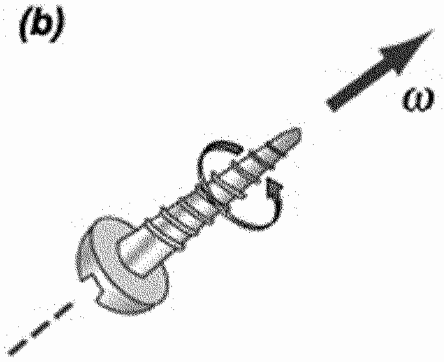
Direction - Right Hand Rule.

(a)



The right-hand rule for determining the direction of the angular velocity.

(b)



The direction of ω is in the direction of advance of a right-handed screw.

Fingers of <u>right hand</u> are wrapped along direction of rotation. Then $\vec{\omega}$ points along thumb.

Direction of $\vec{\alpha}$ is related to $\frac{d\vec{\omega}}{dt}$.

$\frac{d\vec{\omega}}{dt} > 0$ $\qquad$ $\vec{\alpha}$ same as $\vec{\omega}$

$\frac{d\vec{\omega}}{dt} < 0$ $\qquad$ $\vec{\alpha}$ opposite to $\vec{\omega}$

[Fixed-Axis Rotation]

Rotational Motion with Constant Angular Acceleration

- Assume motion along a fixed axis.
- Ignore vector notation (sign designates direction)
- Results also hold for axis in linear translation

$$\frac{d\omega}{dt} = \alpha \qquad (\alpha = \text{constant})$$

$$\int d\omega = \int \alpha \, dt$$

$$\omega = \alpha t + C$$

If $\omega = \omega_0$ at $t = 0$, $\Rightarrow C = \omega_0$.

and

$$\boxed{\omega = \omega_0 + \alpha t} \qquad (1)$$

$$\frac{d\phi}{dt} = \omega = \omega_0 + \alpha t$$

$$\int d\phi = \int \omega_0 \, dt + \alpha \int t \, dt$$

$$\phi = \omega_0 t + \frac{1}{2}\alpha t^2 + C$$

If $\phi = \phi_0$ at $t = 0$, $\Rightarrow C = \phi_0$.

and

$$\boxed{\phi = \phi_0 + \omega_0 t + \frac{\alpha t^2}{2}} \qquad (2)$$

Solve Eq. (1) for t and substitute in Eq. (2)

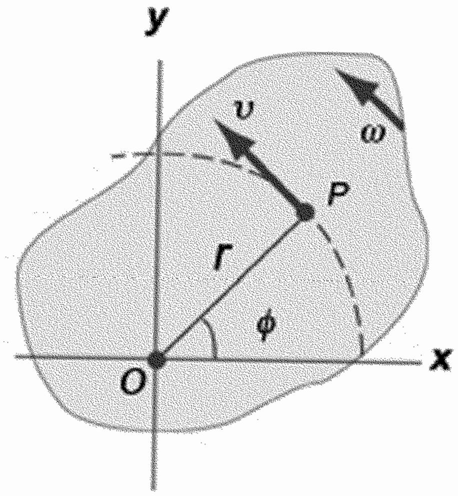
$$\boxed{\omega^2 = \omega_0^2 + 2\alpha(\phi - \phi_0)} \qquad (3)$$

Motion with constant linear acceleration	Motion with constant angular acceleration
a = constant	α = constant
$v = v_0 + at$	$\omega = \omega_0 + \alpha t$
$x = x_0 + v_0 t + \frac{1}{2}at^2$	$\phi = \phi_0 + \omega_0 t + \frac{1}{2}\alpha t^2$
$v^2 = v_0^2 + 2a(x - x_0)$	$\omega^2 = \omega_0^2 + 2\alpha(\phi - \phi_0)$

Relation between Angular and Linear Velocity and Acceleration

As a rigid body rotates about a fixed axis, every particle in the body moves in a circle the center of which is on the axis of rotation.

Consider the point P. P moves in a circle, the linear velocity vector is thus tangent to this circle. Magnitude is ds/dt, where s is distance travelled along the circular path.



As a rigid body rotates around the fixed axis through O, the point P has a linear velocity v, which is always tangent to the circular path of radius r.

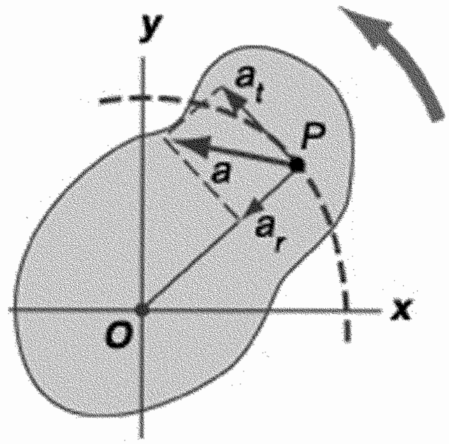
$$s = r\phi \qquad [\phi \text{ in radians}]$$

$$v = \frac{ds}{dt} = r\frac{d\phi}{dt}$$

$$v = r\omega$$

Speed of the particle is directly proportional to its distance from the axis of rotation. The further from the axis the higher its velocity.

To relate the linear acceleration of the point P to the angular acceleration of the rigid body about a fixed axis, we take the time derivative of v:



As a rigid body rotates about a fixed axis through *O*, the point *P* experiences a tangential component of acceleration, a_t, and a centripetal component of acceleration, a_r. The total acceleration of this point is $\mathbf{a} = \mathbf{a}_t + \mathbf{a}_r$.

$$a_t = a_{\parallel} = \frac{dv}{dt} = r\frac{d\omega}{dt} = r\alpha$$

This is the tangential (parallel) component of the linear acceleration of a point at a distance r from the axis of rotation. It is related to the change in speed of the particle.

Since the particle moves in a circle, we have seen that it also has a radial or centripetal acceleration due to the changing direction of its velocity.

$$a_r = a_{\perp} = \frac{v^2}{r} = r\omega^2$$

Total linear acceleration of the particle is $\vec{a}$:

$$\vec{a} = \vec{a}_t + \vec{a}_r$$

$$a = \sqrt{a_t^2 + a_r^2} = r\sqrt{\alpha^2 + \omega^4} \qquad \text{m/s}^2$$

Note:

All points in a rotating rigid body have the same value of ω and the same value of α. Points that are different distances from the axis have different values of v and different values of a_t and a_c.

Example - Rotating Turntable

Record player rotates at 33 rev/min and takes 20s to come to rest.

a) What is angular acceleration, assuming it is uniform?

$$\omega_0 = \left(33 \frac{\text{rev}}{\text{min}}\right)\left(2\pi \frac{\text{rad}}{\text{rev}}\right) \times \left(\frac{1\text{min}}{60\text{s}}\right) = 3.46 \text{ rad/s}.$$

$$\omega = \omega_0 + \alpha t$$

$\omega = 0$ at $t = 20\text{s}$

$$\therefore \alpha = \frac{\omega - \omega_0}{t} = \frac{0 - 3.46}{20} = -0.173 \text{ rad/s}^2$$

(<0, decellerating)

b) How many rotations before it comes to rest?

$$\Delta\phi = \phi - \phi_0 = \omega_0 t + \frac{1}{2}\alpha t^2$$

$$= 3.46(20) - \frac{1}{2}(0.173)\,20^2$$

$$= 34.6 \text{ rad}$$

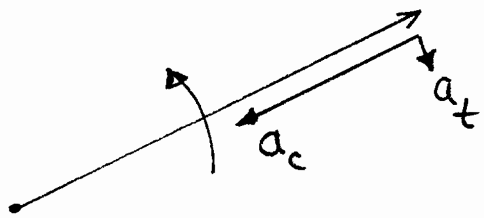
$$= 34.6/2\pi = 5.51 \text{ rev}.$$

c) If rim is at radius $r = 14\text{cm}$, what is the acceleration of a point on the rim at $t = 0$.

$$a_t = r\alpha = 14\text{ cm}\,(0.173 \text{ rad/s}^2) = 2.42 \text{ cm/s}^2$$

$$a_c = r\omega_0^2 = 14\text{cm}\,(3.46 \text{ rad/s})^2 = 168 \text{ cm/s}^2 \quad (t=0)$$

$$a = \sqrt{2.42^2 + 168^2} = 168.0 \text{ cm/s}^2$$



Velocity at rim ($t=0$):

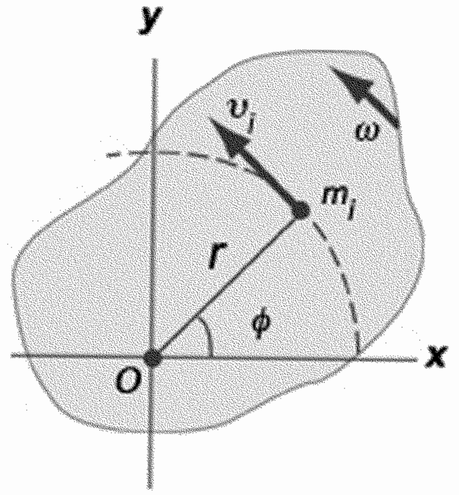
$$v = r\omega_0 = 14\text{cm} \times 3.46 \text{ rad/s}$$

$$= 48.4 \text{ cm/s}$$

Rotational Kinetic Energy

Consider a rigid body as a collection of small particles. The KE of a rotating rigid body is the sum of the individual KE's of all the particles.

$$K = \sum_{i=1}^{N} \frac{1}{2} m_i v_i^2$$



A rigid body rotating about the z axis with angular velocity ω. The kinetic energy of the particle of mass m_i is $\frac{1}{2} m_i v_i^2$. The total kinetic energy of the body is $\frac{1}{2} I\omega^2$.

Suppose the rigid body is rotating about a fixed z-axis with an angular velocity ω. All particles execute circular motion with same angular speed.

$$v_i = r_i \omega$$

r_i = particle distance from axis of rot.

$$K = \sum_{i=1}^{n} \frac{1}{2} m_i r_i^2 \omega^2$$

$$\boxed{K = \frac{1}{2} I \omega^2}$$

where $I = \sum_i m_i r_i^2$ [Moment-of-Inertia]

Particles at large r_i have higher speed and contribute more to KE

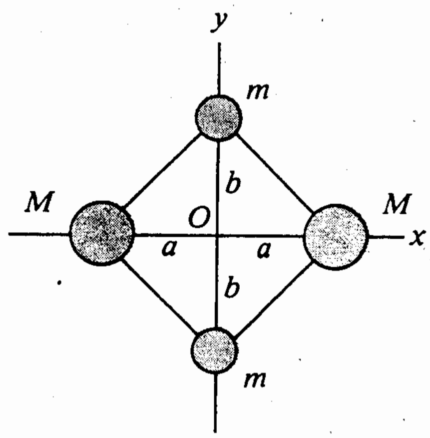
$[I] = kg \cdot m^2$ (SI)
$\quad slug \cdot ft^2$ (Br)

ω, I ← Resistance to rotational motion
v, M ← Resistance to linear motion

} Inertial quant.

Example : Four Rotating Particles

- Four point masses fastened to a very light frame. lying in xy-plane.



a) Rotation about y-axis with ang. velocity ω.

- masses m do not contribute since $r_i = 0$ for them. and they have no motion about y.!

$$I_y = \sum m_i r_i^2 = Ma^2 + Ma^2 = 2Ma^2$$

$$K = \frac{1}{2} I_y \omega^2 = \frac{1}{2}(2Ma^2)\omega^2 = Ma^2\omega^2$$

b) Rotation about z-axis, $\perp$ to xy-plane
r_i, in each case is the $\perp$ distance to axis of rot.

$$I_z = \sum m_i r_i^2 = Ma^2 + Ma^2 + mb^2 + mb^2$$

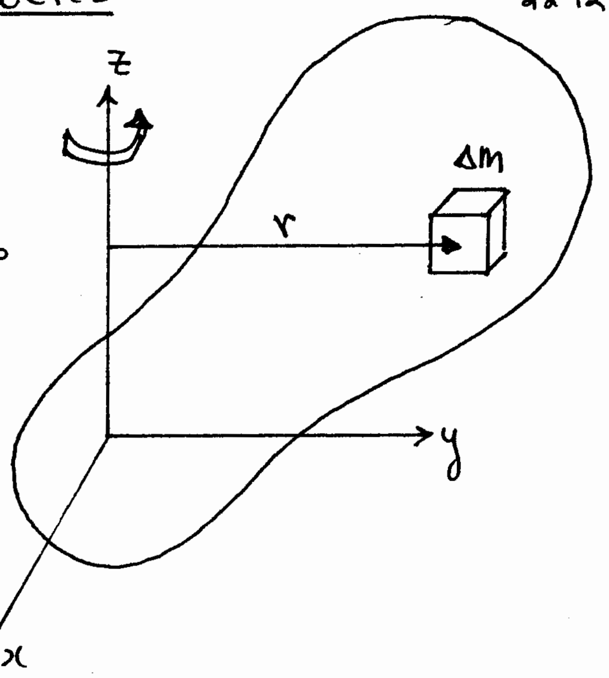
$$= 2Ma^2 + 2mb^2$$

$$K = \frac{1}{2} I_z \omega^2 = (Ma^2 + mb^2)\omega^2$$

Summary:

- Moment-of-Inertia depends on axis of rotation.
- It will take more work, for this example, to set the system into rotation about z-axis than about y-axis. Depends on the distribution of mass.

Moments of Inertia for Rigid Bodies



We evaluate I, for a rigid body rotating about a fixed axis by dividing it up into volume elements of mass Δm.
Use $I = \Sigma r^2 \Delta m$ and take the limit of this sum as $\Delta m \to 0$ we have an integral over the volume.
r: $\perp$ distance from rotation axis to Δm.

$$I = \lim_{\Delta m \to 0} \Sigma r^2 \Delta m = \int_V r^2 dm$$

dm: must be expressed in terms of its coordinates.
- For a 3-dimensional object it is convenient to do this in terms of the local volume density, i.e mass per unit volume

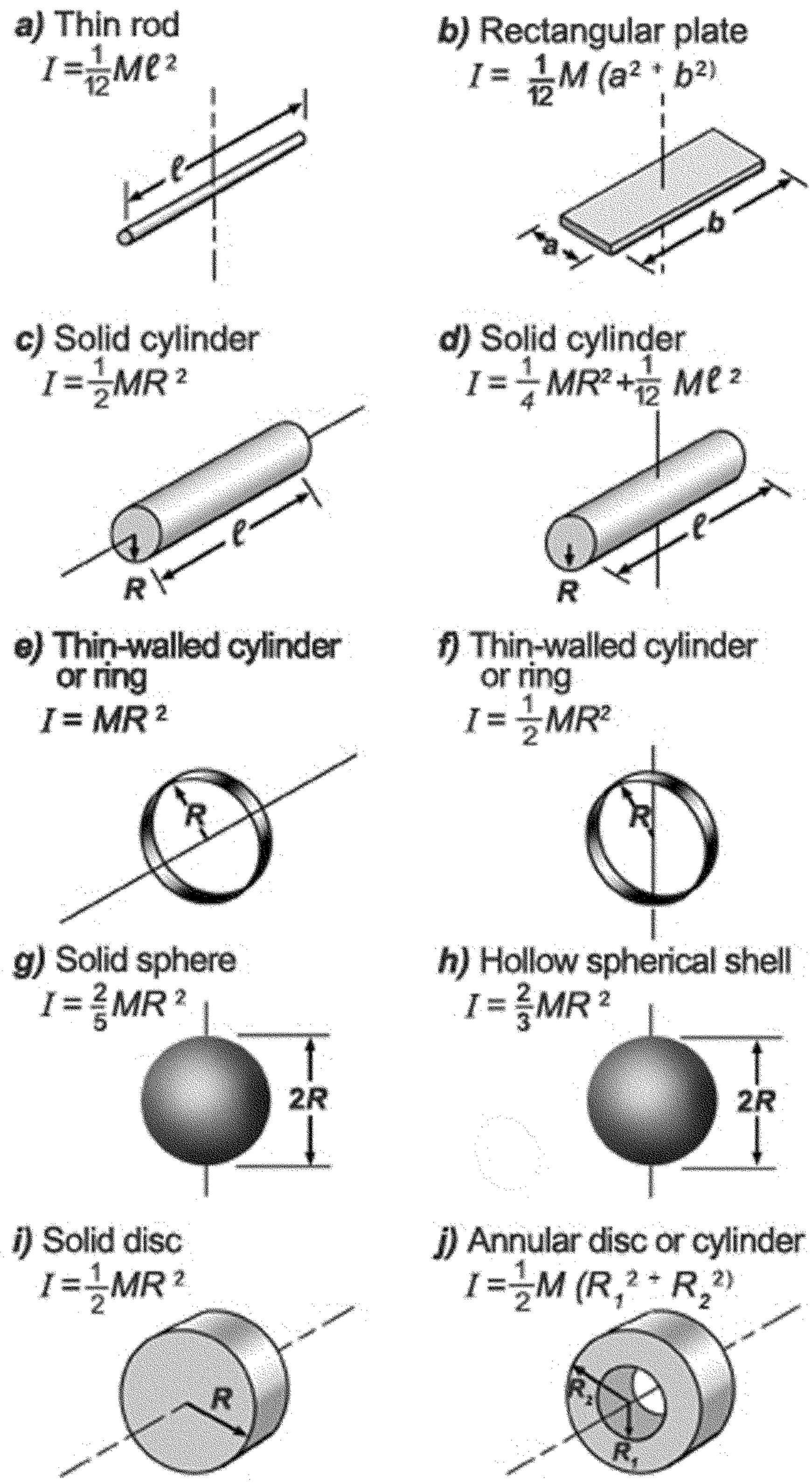
$$\rho = \lim_{\Delta V \to 0} \frac{\Delta m}{\Delta V} = \frac{dm}{dV}$$

$$\therefore dm = \rho dV$$

and $\boxed{I = \int \rho r^2 dV}$

2'ND Moment of mass distribution.

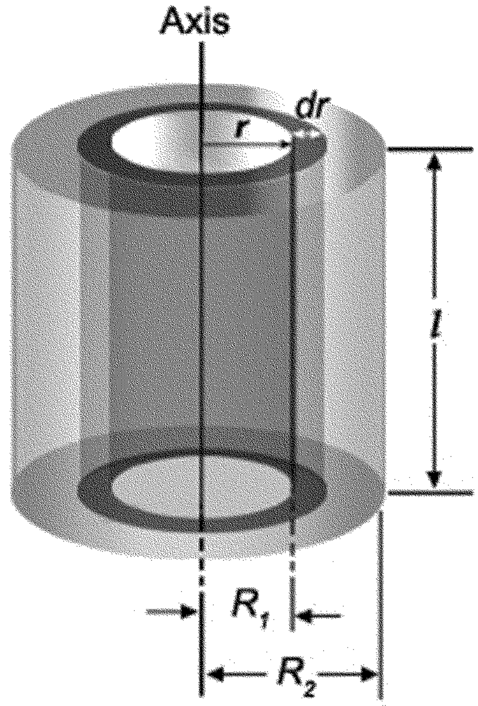
$$I = \frac{M}{V} \int r^2 dV$$, for homogeneous bodies



Rational Inertia values for various objects for the indicated axes

Example: I– Uniform Hollow Cylinder

- Always need to choose mass elements which are a fixed radius from axis of rotation.



Moment of inertia of a hollow cylinder. The mass element is a cylindrical shell of a radius *r* and thickness *dr*.

We choose a thin cylindrical shell of radius r, thickness dr, and length l.
The volume of such a shell is that of a flat sheet of length l, thickness dr and width $2\pi r$.

$$dV = 2\pi l r\,dr$$

$$dm = \rho\,dV = 2\pi \rho l r\,dr$$

$$I = \rho \int r^2 dV = 2\pi \rho l \int_{R_1}^{R_2} r^3 dr$$

$$= \frac{\pi \rho l}{2} (R_2^4 - R_1^4)$$

$$= \frac{\pi \rho l}{2} (R_2^2 - R_1^2)(R_2^2 + R_1^2)$$

Mass of Cylinder

$$M = \rho V = \pi l \rho (R_2^2 - R_1^2)$$

$$I = \frac{M}{2} (R_2^2 + R_1^2)$$

If cylinder is solid, $R_1 = 0$.

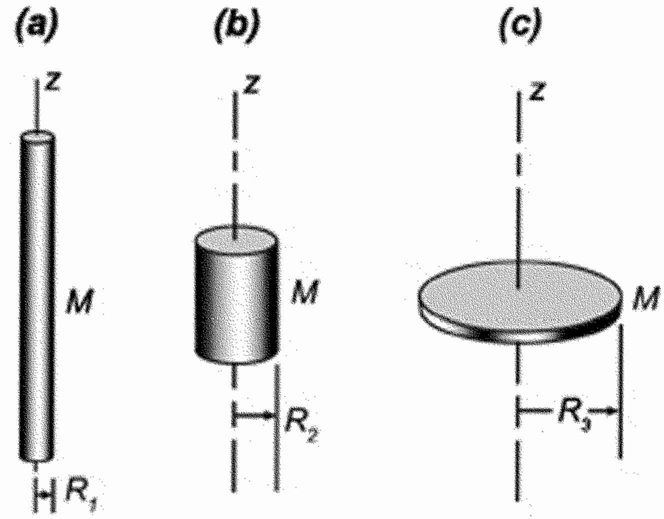
$$I = \frac{1}{2} M R^2$$

If cylinder is a very thin shell, $R_1 \sim R_2 = R$

$$I = M R^2$$

I in all cases does not depend on ℓ. The distribution along the axis does not matter. Moment-of-inertial depends on radial distribution.

Three different distributions about the z axis of the same mass M of material. The rotational inertia values are $I_z(R_1) < I_z(R_2) < I_z(R_3)$.



When using I, it is often convenient to do so in terms of a "Radius of Gyration", k

$$I = M k^2$$

It is defined such that if all of the mass of an object were located a distance k from the axis, it would have the same moment-of-inertia as the actual object.