Dynamics of a Rigid Body

Particle: $\vec{F} = m\vec{a}$ (N-2'nd Law)

$\vec{F} \longleftrightarrow \vec{\alpha}$? (ang. accel)

Torques

Body rotates because there are torques acting.

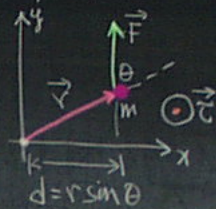
Force $\vec{F}$ on particle with position vector $\vec{r}$.

$$\boxed{\vec{\tau} = \vec{r} \times \vec{F}} \quad [N \cdot m]$$

$\tau = r F \sin\theta$

$\vec{\tau} \perp \vec{r} \times \vec{F}$ Plane

RH Rule



$d = r\sin\theta$: Moment (lever arm)

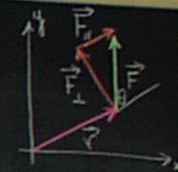
$\perp$ dis. from axis of rot. to line of action of $\vec{F}$.

$\tau = dF$

$F_t = F_\perp = F\sin\theta$

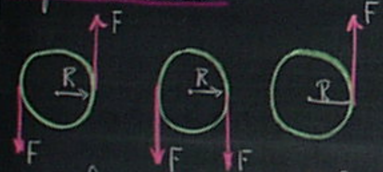
$F_\parallel = F\cos\theta$ [No Rotation!]

Torque $= r F_t$: Radial Distance × Transverse Force



①

Torque and Force



$\tau = 2Rf$	$\tau = 0$	$\tau = Rf$
$\Sigma F = 0$	$F = 2f$	$F = f$
$a = 0$	$a \neq 0$	$a \neq 0$

Angular Momentum $\vec{L}$ and Torque $\vec{\tau}$

$\vec{L} = \vec{r} \times \vec{p}$ For a particle

$\vec{L} = \sum \vec{r}_i \times \vec{p}_i$ For many part's.

$$\frac{d\vec{L}}{dt} = \sum_i \frac{d\vec{r}_i}{dt} \times \vec{p}_i + \sum \vec{r}_i \times \frac{d\vec{p}_i}{dt}$$

$$= \sum_i \cancel{\vec{v}_i \times m\vec{v}_i} + \sum \vec{r}_i \times \vec{F}_i$$

$\therefore \frac{d\vec{L}}{dt} = \sum \vec{r}_i \times \vec{F}_i = \sum \vec{\tau}_i$

Time rate of change of $\vec{L}$ is equal to the net applied torques.

If $\sum \vec{\tau}_i = \vec{\tau} \equiv 0$

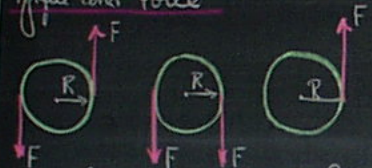
$\frac{d\vec{L}}{dt} = 0 \Rightarrow \vec{L} \equiv$ constant / Conserved!!!

$\vec{L}, \vec{\tau}$: Always Common Origin!

②

Lecture 23 Blackboard #1

Torque and Force



$\tau = 2Rf$, $\Sigma F = 0$, $a = 0$

$\tau = 0$, $F = 2f$, $a \neq 0$

$\tau = Rf$, $F = f$, $a \neq 0$

Angular Momentum $\vec{L}$ and Torque $\vec{\tau}$

$\vec{L} = \vec{r} \times \vec{p}$ For a particle

$\vec{L} = \Sigma \vec{r}_i \times \vec{p}_i$ For many part's.

$$\frac{d\vec{L}}{dt} = \sum_i \frac{d\vec{r}_i}{dt} \times \vec{p}_i + \sum \vec{r}_i \times \frac{d\vec{p}_i}{dt}$$

$$= \sum_i \cancel{\vec{v}_i \times m\vec{v}_i} + \sum \vec{r}_i \times \vec{F}_i$$

$$\therefore \frac{d\vec{L}}{dt} = \Sigma \vec{r}_i \times \vec{F}_i = \Sigma \vec{\tau}_i$$

Time rate of change of $\vec{L}$ is equal to the net applied torques.

If $\Sigma \vec{\tau}_i = \vec{\tau} \equiv 0$

$\frac{d\vec{L}}{dt} = 0 \Rightarrow \vec{L} \equiv$ constant / Conserved!!!

$\vec{L}, \vec{\tau}$: Always Common Origin!

(2)

Internal Forces - Isolated System

(3)

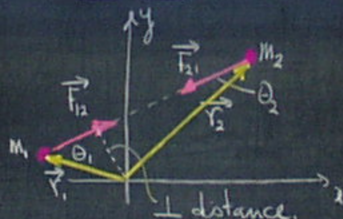
- 3'rd Law - Equal + Opposite forces.
- Assume forces along line of particles

$\vec{r}_1 \times \vec{F}_{12} = F_{12} r_1 \sin\theta_1$ (mag)

$\vec{r}_2 \times \vec{F}_{21} = F_{21} r_2 \sin\theta_2$ (mag)

$r_1 \sin\theta_1 = r_2 \sin\theta_2$

$\perp$ dist. from origin to line of action of forces.



$\vec{F}_{12} = -\vec{F}_{21}$

Torques cancel internally!!

Example: Rotating Skew Rod

Light rod, angle θ to z-axis.

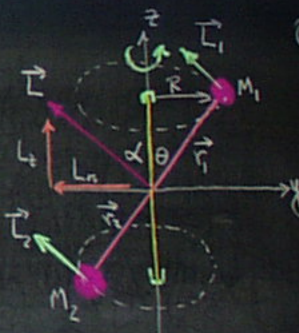
$M_1 = M_2 = M$ $r_1 = r_2 = r$

$R = r \sin\theta$

$v = \omega R = \omega r \sin\theta$

$|L_1| = |L_2| = m(\vec{r} \times \vec{v}) = mrv = m\omega r^2 \sin\theta$

Total $\vec{L} = \vec{L}_1 + \vec{L}_2 = 2m\omega r^2 \sin\theta$



Lecture 23 Blackboard #2

Total $\vec{L}$ makes angle $(90-\theta)$ to z-axis
precesses around $\hat{z}$ as particles move

$L_z = L\cos(90-\theta) = 2m\omega r^2 \sin^2\theta$

$L_z = 2m\omega R^2 = I_z\omega$!!!

$\vec{L}_z$ = constant in time.

$\vec{L}_r$ = precesses with rotation.

Torques on Skew Rod

$\vec{L} = L_z\hat{k} + L_r\hat{r}$

$\vec{\tau} = \frac{d\vec{L}}{dt} = \frac{d\vec{L}_r}{dt}\hat{\phi} \neq 0$

$|\Delta\vec{L}_r| = |\vec{L}_r|\Delta\varphi$

$\frac{dL_r}{dt} = L_r\frac{d\phi}{dt} = L_r\omega$

$\tau = \frac{dL_r}{dt} = L_r\omega$

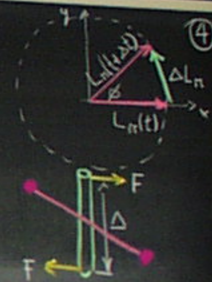
$= \omega L\sin\alpha$

Torque lies in tangential direction rotating in horizontal plane with rod.

Force on bearings produces required torque.

$\vec{\tau} = \vec{F}\times\vec{\Delta}$

④



Rotational Dynamics: Torques → α

Particle of mass m, moving in a circle of radius r. Assume a tangential force $\vec{F}_t$

$\vec{F}_t = m\vec{a}_t$

$\tau = F_t r = (ma_t)r$

$a_t = r\alpha$

$\vec{r} \perp \vec{F}_t$

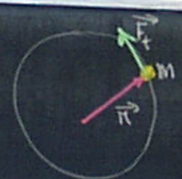
$\therefore \tau = (mr^2)\alpha$

$\boxed{\vec{\tau} = I\vec{\alpha}}$

Rotations

$\boxed{\vec{F} = m\vec{a}}$

Translations



$\vec{F}_c = \frac{mv^2}{R}$

↓

No Torque

Rigid Body Dynamics

- Mass element dm.
- Tangential F_t on mass el.
- Rotates with r about O.

$d\vec{F}_t = dm\,\vec{a}_t$

$d\vec{\tau} = r(dF_t)\hat{k} = r(dm)a_t\hat{k}$

$a_t = r\alpha$

$\therefore d\tau = r\,dm\,r\alpha$

$= r^2 dm\,\alpha$

$\vec{\tau}_{Net} = \int r^2 dm\ \alpha$

$\alpha \equiv$ same for all mass dm.

$\therefore \vec{\tau}_{Net} = \alpha\int r^2\,dm\ \hat{k}$

$\boxed{\vec{\tau}_{Net} = I\vec{\alpha}}$

Eq. of Motion of a Rigid Body!!

⑤

Lecture 23 Blackboard #3

Rotational Dynamics: Torques → α

Particle of mass m, moving in a circle of radius r. Assume a tangential force $\vec{F}_t$

$\vec{F}_t = m\vec{a}_t$

$\tau = F_t r = (m a_t) r$ $\quad \vec{r} \perp \vec{F}_t$

$a_t = r\alpha$

$\therefore \tau = (mr^2)\alpha$

$\boxed{\vec{\tau} = I\vec{\alpha}}$ Rotations

$\boxed{\vec{F} = m\vec{a}}$ Translations

$\vec{F}_c = \frac{mv^2}{R}$

↓ No Torque

Rigid Body Dynamics

⑤

· Mass element dm.

· Tangential F_t on mass el.

· Rotates with r about O.

$d\vec{F}_t = dm\,\vec{a}_t$

$d\vec{\tau} = r(dF_t)\hat{k} = r(dm)a_t\hat{k}$

$a_t = r\alpha$

$\therefore d\tau = r\,dm\,r\alpha = r^2 dm\,\alpha$

$\vec{\tau}_{Net} = \int r^2 dm\,\alpha$

$\alpha \equiv$ same for all masses dm.

$\therefore \vec{\tau}_{Net} = \alpha\int r^2 dm\,\hat{k}$

$\boxed{\vec{\tau}_{Net} = I\vec{\alpha}}$ Eq. of Motion of a Rigid Body!!

Example: Rotating Cylinder.

⑥

Cylinder: Mass = M, Radius R

Constant Tension, T.

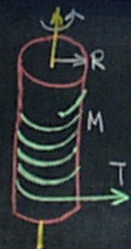
$\vec{\tau} = I\vec{\alpha}$

$RT = \frac{1}{2}MR^2\alpha$

$\alpha = \frac{2T}{M} = \frac{2\times 2}{15\times .06} = 4.4\,r/s^2$

$\omega = \alpha t = 4.44\times 2 = 8.8\,r/s$

$\nu = \omega/2\pi = 1.3\,Hz$



M = 15 kg

R = 6 cm

T = 2N

what is ω at t=2s?

Example: Pivoted Rod

Uniform Rod: length = L

Mass = M

$\tau_0 = Mg\frac{L}{2} =$ Weight × Mom. Arm

$I = \frac{1}{3}ML^2$ Rod at end.

$N_V, N_H \equiv$ No torques about 'O'.

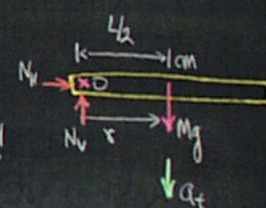
$\therefore Mg\frac{L}{2} = I\alpha$ Eq. of Motion.

$\alpha = \frac{MgL/2}{\frac{1}{3}ML^2} = \frac{3}{2}\frac{g}{L}$

$a_t = r\alpha = \frac{3}{2}\frac{g}{L}r$

If $r > \frac{2}{3}L$ $\quad a > g$!!!

If $r = L$ $\quad a_t = \frac{3}{2}g$



Lecture 23 Blackboard #4

Example: MIT Wheel / Rotational Dynamics

Cylinder: Mass = M; Radius R

Weight: Mass = m

$\alpha = ?$, $a = ?$; $T = ?$

① $mg - T = ma$ $(\vec{F} = m\vec{a})$

② $RT = \frac{1}{2}MR^2\alpha$ $(\vec{\tau} = I\vec{\alpha})$

③ $a = R\alpha$ (Geometry)

③ → ② $T = Ma/2$ ④

④ → ① $mg - \frac{Ma}{2} = ma$

$$a = \frac{g}{1+M/2m} \qquad \alpha = \frac{a}{R} = \frac{g/R}{1+M/2m}$$

$$T = mg - ma = \frac{Mg/2}{1+2m/M}$$

If $M = 0$ $T = 0$
$a = g$

$M = 5.223$ kg

$m = .050$ kg

$R = 0.50$ m

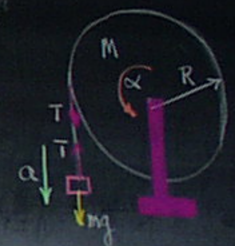
$H = 1.0$ m

$$a = \frac{9.81}{1 + 5.223/2\times.050} = 0.1843\ \text{m/s}^2$$

$$T_{Th} = \sqrt{\frac{2H}{a}} = \sqrt{\frac{2\times 1}{.1843}} = 3.29\ \text{s}$$

$T_{Exp} =$

⑦



Torque due to Gravity

⑧

· Body of mass M

· Origin at A

· Mass m_j ; $\vec{r}_j$

$$\vec{\tau}_j = \vec{r}_j \times m_j\vec{g}$$

$$\vec{\tau} = \sum \vec{\tau}_j = \sum \vec{r}_j \times m_j\vec{g}$$

$$\vec{\tau} = \sum(m_j\vec{r}_j) \times \vec{g}$$

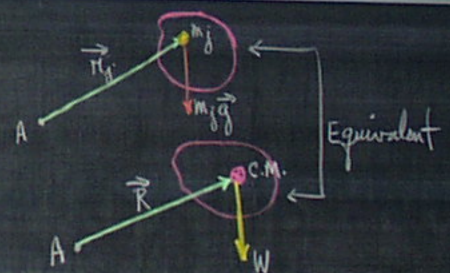
$$\sum m_j\vec{r}_j = M\vec{R}$$

$\vec{R}$: pos. vector of CM.

$$\therefore \vec{\tau} = M\vec{R} \times \vec{g}$$

$$= \vec{R} \times M\vec{g} = \vec{R} \times \vec{W}$$

To balance object need pivot at 'O'.



Torque due to Gravity

8

- Body of mass M
- Origin at A
- Mass m_j : $\vec{r}_j$

$\vec{\tau}_j = \vec{r}_j \times m_j \vec{g}$

$\vec{\tau} = \sum \vec{\tau}_j = \sum \vec{r}_j \times m_j \vec{g}$

$\vec{\tau} = \sum (m_j \vec{r}_j) \times \vec{g}$

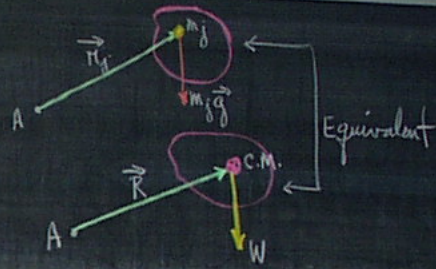
$\sum m_j \vec{r}_j = M \vec{R}$

$\vec{R}$ = pos. vector of CM.

$\therefore \vec{\tau} = M\vec{R} \times \vec{g}$

$= \vec{R} \times M\vec{g} = \vec{R} \times \vec{W}$

To balance object need pivot at 'O'.



Rigid Body L_z:

9

- mass element dm
- Radial component of ang mom: l_ρ
- z-component of ang mom: l_z

$l_\rho \Rightarrow$ forces on bearings
precesses with time.

$\frac{dl_\rho}{dt} \neq 0 \Rightarrow$ torques.

$dl_z = \rho^2 \omega \, dm$

$L_z = \int dl_z = \omega \int \rho^2 dm$

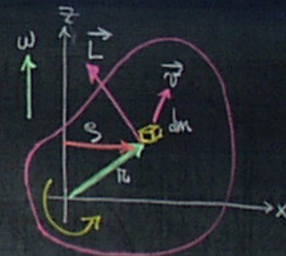
$L_z = I_z \omega \qquad \omega = \frac{L_z}{I_z}$

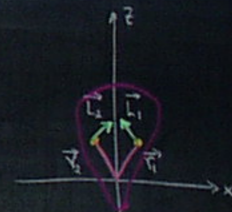
$KE = \frac{1}{2} I_z \omega^2 = \frac{1}{2} I_z \left(\frac{L_z}{I_z}\right)^2 = \frac{L_z^2}{2 I_z}$

Symmetric Body:

$\sum \vec{L}_\rho \equiv 0$

No Forces, Balanced!!!





$\sum \vec{L}_\rho \equiv 0$

Lecture 23 Blackboard #6