Potential Energy: Solid Sphere + Particle

- Particle m
- Homogeneous solid sphere M, R

Case I: $r \ge R$

Force F, on shell depends only on dist. to center

$$\vec{F} = -\frac{GmM}{r^2} \quad (r \ge R)$$

$$U = -GmM/r \quad (r \ge R)$$

Case II: $r \le R$

From shell result, force due only to mass between $r=0$ and $r=R$

$$\therefore F = \frac{G m M_e(r)}{r^2}$$

$M_e(r) \equiv$ mass in $r=0 \rightarrow r$

$$M(r) = M\left(\frac{r}{R}\right)^3$$

$$\therefore \vec{F} = -\frac{GmMr}{R^3}\hat{r} \quad (r \le R)$$

$$U(r) = -\int_{\infty}^{r} \vec{F}\cdot d\vec{r}$$

$$= \int_{\infty}^{R} \frac{GmM\,dr'}{r'^2} + \int_{R}^{r} \frac{GmM\,r'\,dr'}{R^3}$$

$$= -\frac{GmM}{2R^3}\left[3R^2 - r^2\right] \quad (r \le R)$$

①

Forces between Two Spherical Objects

Force on A due to (m_i) part of B acts on CM of A.

Sum over all m_i

$$\vec{F}_{AB} = \frac{G M_A M_B}{r^2}$$

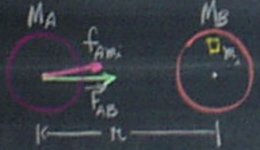
Force on A at cm A

Force on B at cm B

If B is not a sphere:

Force on B is NOT at cm B

Force on A is at cm of A



Gravitational Force on an Extended Object

Earth acting on particle m_i:

$\vec{f}_i = m_i\vec{g}$ $\quad \vec{g} \equiv$ accel. due to gravity

$$\vec{F} = \Sigma\vec{f}_i = \Sigma m_i\vec{g} = (\Sigma m_i)\vec{g} = M\vec{g}$$

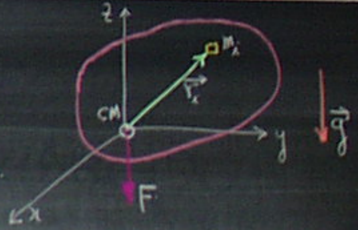
↑ total mass.

Calculate torques about CM:

$$\Sigma\vec{\tau}_{cm} = \Sigma\vec{r}_i \times \vec{f}_i = \Sigma\vec{r}_i \times m_i\vec{g} = (\Sigma m_i\vec{r}_i) \times \vec{g}$$

$$\therefore \Sigma\vec{\tau}_{cm} \equiv 0!$$

↑ $\vec{R}_{cm} \equiv 0$



$\therefore \vec{F}$ must act on CM!

$\Rightarrow$ "center-of-gravity"

②

Lecture 28 Blackboard #1

Forces between Two Spherical Objects

Force on A due to (m_i) part of B acts on cm of A.

Sum over all m_i

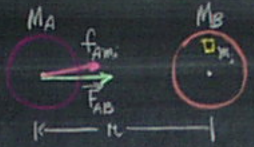
$\vec{F}_{AB} = \dfrac{G m_A M_B}{r^2}$ Force on A at cm A

Force on B at cm B

If B is not a sphere:

Force on B is NOT at cm B

Force on A is at cm of A



Gravitational Force on an Extended Object.

②

Earth acting on particle m_i:

$\vec{f}_i = m_i \vec{g}$ $\vec{g} \equiv$ accel. due to gravity

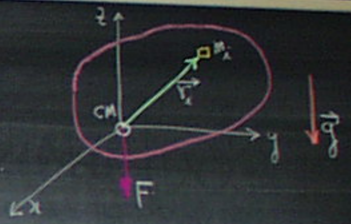
$\vec{F} = \Sigma \vec{f}_i = \Sigma m_i \vec{g} = (\Sigma m_i)\vec{g} = M\vec{g}$

total mass.

Calculate torques about CM:

$\Sigma \vec{\tau}_{cm} = \Sigma \vec{r}_i \times \vec{f}_i = \Sigma \vec{r}_i \times m_i \vec{g} = (\Sigma m_i \vec{r}_i) \times \vec{g}$

$\therefore \Sigma \vec{\tau}_{cm} \equiv 0!$ $\vec{R}_{cm} \equiv 0$



$\therefore \vec{F}$ must act on CM!

$\Rightarrow$ "center-of-gravity"

③

If $\vec{g}$ varies over an extended object, C of G cannot be defined except for spherical objects.

Tidal Forces.

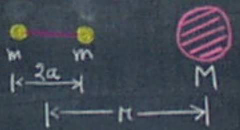
- Non-uniform gravity
- Differential Forces.

Object stretched or compressed.

Uniform

Non - Uniform

Earth Tides:



Tidal Force $\equiv$ difference between gravitation force at both ends

$$F_T = \frac{GMm}{(r-a)^2} - \frac{GMm}{(r+a)^2} = GMm\,\frac{(r+a)^2-(r-a)^2}{(r-a)^2(r+a)^2}$$

$F_T = \dfrac{4\,GMm\,a}{r^3}$ (tidal force)

For $a \ll r$

- Falls of as r^3
- Moon, though smaller, more important than sun.

Lecture 28 Blackboard #2

Weight and Gravitational Force:

$w = mg$ near earth

$mg = \frac{G M_E m}{R_E^2}$ $\qquad \vec{F} = m\vec{a}$

$g = \frac{G M_E}{R_E^2}$ definition of g!

Know G, $R_E \Rightarrow$ Weigh Earth

Suppose object not near earth

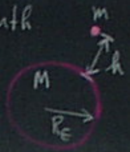
$r = h + R_E$ $\quad h$ = height

$$mg' = \frac{G M_E m}{r^2} = \frac{G M_E m}{(h+R_E)^2}$$

$$\therefore g' = \frac{G M_E}{r^2} = \frac{G M_E}{(h+R_E)^2}$$

g' decreases with altitude!

$g' \to 0$ as $h \to \infty$



PE of m at radius r:

$$U(r) = -\frac{G M_E m}{r}$$

$$U(R_E) = -\frac{G M_E m}{R_E}$$

$$\Delta U = U(R_E) - U(r)$$

$$= -G M_E m\left[\frac{1}{r} - \frac{1}{R_E}\right]$$

$$\Delta U = -G M_E m\left[\frac{r - R_E}{r R_E}\right]$$

④

Let $\left.\begin{array}{l} r - R_E = z \\ r R_E \approx R_E^2 \end{array}\right\}$ for $z \ll R_E$

$$\Delta U = \frac{G M_E m}{R_E^2} z = (mg) z$$

→ Change in PE for particle m due to change in elevation near earth's surface.

Planetary Motion

Case I: Circular Orbits

$F = \frac{G M m}{r_0^2}$ Force on planet

$m \ll M$

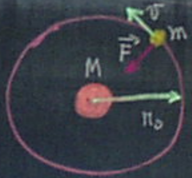
uniform speed: circles.

$$a_c = \frac{v^2}{r_0}$$

$$\frac{G M m}{r_0^2} = \frac{m v^2}{r_0}$$

$v^2 = \frac{G M}{r_0}$ Radius of orbit fixes speed.

$T = \frac{2\pi r_0}{v} = \frac{\text{Circumference}}{\text{Speed}}$ = Period - one revolution



$$T^2 = \frac{4\pi^2 r_0^2}{v^2} = \frac{4\pi^2}{GM} r_0^3$$

(Kepler's Law)

$K = \frac{1}{2} m v^2 = \frac{1}{2}\frac{G M m}{r_0}$ (KE)

$U = -\frac{G M m}{r_0}$ (PE)

⑤

$E = K + U$ Total Energy

$$= \frac{1}{2}\frac{GMm}{r_0} - \frac{GMm}{r_0}$$

$= -\frac{1}{2}\frac{GMm}{r_0}$ $[E < 0]$

$$L^2 = (m v r_0)^2 = G M m^2 r_0$$

$$r_0 = L^2 / G M m^2$$

Lecture 28 Blackboard #3

Planetary Motion

Case I: Circular Orbits

$F = \frac{GMm}{r_0^2}$ Force on planet.

$m \ll M$

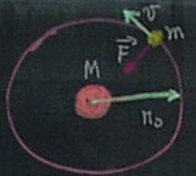
uniform speed: circles.

$$a_c = \frac{v^2}{r_0}$$

$$\frac{GMm}{r_0^2} = \frac{m v^2}{r_0}$$

$$v^2 = \frac{GM}{r_0}$$ Radius of orbit fixes speed.

$$T = \frac{2\pi r_0}{v} = \frac{\text{Circumference}}{\text{speed}} = \text{Period - one revolution}$$



$$T^2 = \frac{4\pi^2 r_0^2}{v^2} = \frac{4\pi^2}{GM} r_0^3$$

(Kepler's Law)

$$K = \tfrac{1}{2}mv^2 = \tfrac{1}{2}\frac{GMm}{r_0} \quad (KE)$$

$$U = -\frac{GMm}{r_0} \quad (PE)$$

$E = K + U$ Total Energy

$$= \tfrac{1}{2}\frac{GMm}{r_0} - \frac{GMm}{r_0}$$

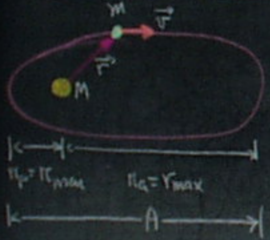
$$= -\tfrac{1}{2}\frac{GMm}{r_0} \quad [E < 0]$$

$$L^2 = (mvr_0)^2 = GMm^2 r_0$$

$$r_0 = L^2/GMm^2$$

⑤

Case II: Elliptical Orbits



$m \ll M$

Easy to understand at extremes of orbit. Then $\vec{v} \perp \vec{r}$!!

$$E = \tfrac{1}{2}mv^2 - \frac{GMm}{r} \quad (E \equiv \text{constant})$$

$$L = mvr \quad (\text{Total } L \equiv \text{constant})$$

$$K = \frac{L^2}{2I} = \frac{L^2}{2mr^2}$$

$$\therefore \frac{L^2}{2mr^2} - \frac{GMm}{r} = E$$

$$r^2 + \frac{GMm}{E} r - \frac{L^2}{2mE} = 0$$

$$r = \frac{-\frac{GMm}{E} \pm \sqrt{\left(\frac{GMm}{E}\right)^2 + \frac{4L^2}{2mE}}}{2}$$

$$r_a = r_{max} = \left[(\ \ ) + \sqrt{\quad}\right]/2$$

$$r_p = r_{min} = \left[(\ \ ) - \sqrt{\quad}\right]/2$$

$$A = r_a + r_p = -\frac{GMm}{E}$$ length of maj. axis indep. of L

same $A \Longrightarrow$ same total E.

Can show that: $\epsilon = \sqrt{1 + \frac{2EL^2}{m(GMm)^2}}$ eccentricity

Also: $T^2 = \frac{4\pi^2 a^3}{GM}$ where $\left(a = \frac{A}{2}\right)$

⑥

General Planetary Motion

Inertial Frame:

$\vec{F}_{12} = m_1 \dfrac{d^2\vec{r}_1}{dt^2}$ (1)

$\vec{F}_{21} = m_2 \dfrac{d^2\vec{r}_2}{dt^2}$ (2)

$\vec{R} \equiv$ Pos. Vector of CM.

$\vec{r}$ = relative separation.

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2}$$

$M\vec{R} = m_1\vec{r}_1 + m_2\vec{r}_2$ (3)

$M = m_1 + m_2$

$\vec{r} = \vec{r}_2 - \vec{r}_1$ (4)

Solve: $\vec{r}_1 = \vec{R} - \dfrac{m_2}{M}\vec{r}$ (5)

$\vec{r}_2 = \vec{R} + \dfrac{m_1}{M}\vec{r}$ (6)

$\vec{F}_{12} = -\vec{F}_{21}$ (3rd Law)

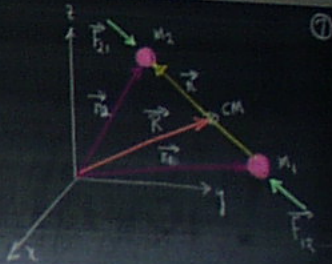
(1) + (2) $M\dfrac{d^2\vec{R}}{dt^2} = m_1\dfrac{d^2\vec{r}_1}{dt^2} + m_2\dfrac{d^2\vec{r}_2}{dt^2} \equiv 0$ $\quad \dfrac{d\vec{R}}{dt} = \vec{V}_{cm} = \text{const.}$

$m_2 \times$ Eq(1) $- m_1 \times$ Eq(2) and use (5) and (6)

$$m_1 m_2 \frac{d^2\vec{r}}{dt^2} = M\vec{F}_{21}$$

Let $\mu = \dfrac{m_1 m_2}{M} \Rightarrow \boxed{\mu \dfrac{d^2\vec{r}}{dt^2} = \vec{F}_{21}}$

(7)



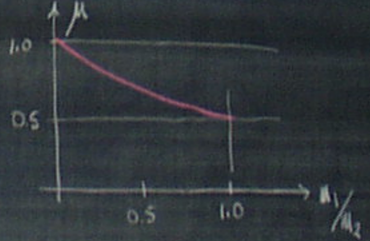
Summary:

Relative coordinate $\vec{r} = \vec{r}_2 - \vec{r}_1$ has the same behavior as the coordinate of a single particle of mass $= \mu$ moving in a force field $\vec{F}_{21}(\vec{r})$. $\vec{F}_{21}(\vec{r})$ is the actual mutual force between m_1 and m_2.

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

If $m_1 \ll m_2$
$\mu = m_1$

If $m_1 = m_2$
$\mu = \frac{1}{2} m_1$



(8)

$(v_{esc})_{Moon} = 618$ km/s.

Lecture 28 Blackboard #5

Summary:

Relative coordinate $\vec{r} = \vec{r}_2 - \vec{r}_1$ has the same behavior as the coordinate of a single particle of mass $= \mu$ moving in a force field $\vec{F}_{21}(\vec{r})$. $\vec{F}_{21}(\vec{r})$ is the actual mutual force between m_1 and m_2.

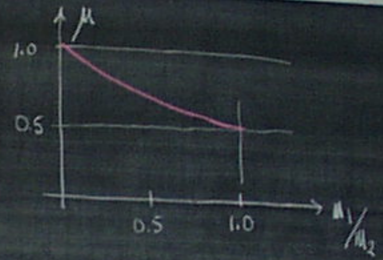
$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

If $m_1 \ll m_2$

$\mu = m_1$

If $m_1 = m_2$

$\mu = \frac{1}{2} m_1$



⑧

Example:

Initial speed $= v_i$

Max height, $h = ?$

Energy conserved

$$K_i + U_i = K_f + U_f$$

$$\frac{1}{2} m v_i^2 - \frac{G M_e m}{r_i} = \frac{1}{2} m v_f^2 - \frac{G M_e m}{r_f}$$

At $r_f = r_{max}$, $v_f = 0$

$r_i = R_e$

$$\therefore v_i^2 = 2GM_E\left(\frac{1}{R_E} - \frac{1}{r_{max}}\right)$$

$$h = r_{max} - R_E$$

Escape Velocity

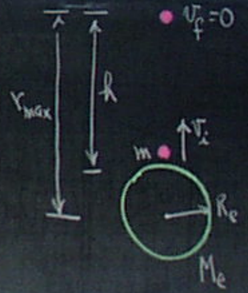
let $r_{max} = \infty$

$$v_{esc} = \sqrt{\frac{2GM_e}{R_E}}$$

$$(v_{esc})_E = \sqrt{\frac{2 \times 6.67 \times 10^{-11} \times 5.98 \times 10^{24}}{6.37 \times 10^{6}}}$$

$= 1 \times 10^4$ m/s $\quad$ 25,000 mi/h

$(v_{esc})_{Moon} = 618$ km/s.



⑨