General Planetary Motion: Cont.

$$\boxed{\mu \frac{d^2 \vec r}{dt^2} = F_{12}}$$

Kinetic Energy and Momentum

$K = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$

$v_1^2 = \left(\frac{d\vec r_1}{dt}\right)^2 \qquad v_2^2 = \left(\frac{d\vec r_2}{dt}\right)^2$

Use Eq 5 + 6 for $\vec r_1$ and $\vec r_2$

$K = \frac{1}{2} M \left(\frac{d\vec R}{dt}\right)\left(\frac{d\vec R}{dt}\right) + \frac{1}{2}\mu\left(\frac{d\vec r}{dt}\right)\left(\frac{d\vec r}{dt}\right)$

$= \frac{1}{2} M V_{cm}^2 + \frac{1}{2}\mu v^2$

CM Vel. Rel Vel.

= KE of CM with vel. V

+ KE of hypothetical particle of mass μ and vel. v.

Total $\vec P$ in frame 'O':

$\vec P = \vec P_1 + \vec P_2 = M \vec V_{cm}$

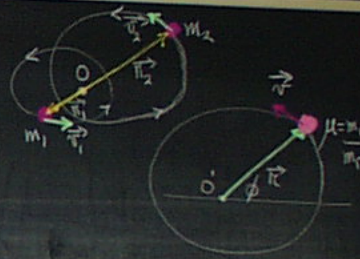
Relative linear momentum

$\mu \vec v = \frac{m_1 m_2}{M}(\vec v_2 - \vec v_1)$

$= \frac{1}{M}(m_1 \vec P_2 - m_2 \vec P_1)$

Isolated System $\vec P$ = const.

Total $\vec P$ in CM system = 0.



1

Angular Momentum

· Central Forces

· No torques

$\vec L_{cm} = \vec l_1 + \vec l_2$ = constant.

$= m_1 \vec r_1' \times \vec v_1' + m_2 \vec r_2' \times \vec v_2'$

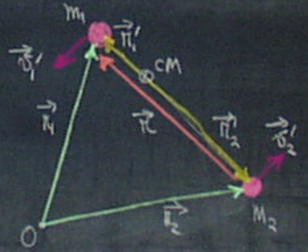
$\vec r = \vec r_1' - \vec r_2'$

$\vec r_1' = \frac{m_2}{m_1 + m_2}\vec r \qquad \vec r_2' = \frac{-m_1}{m_1 + m_2}\vec r$

$\vec L_{cm} = \frac{m_1 m_2}{m_1 + m_2}\vec r \times \vec v_1' - \frac{m_1 m_2}{m_1 + m_2}\vec r \times \vec v_2'$

$= \mu \vec r \times (\vec v_1' - \vec v_2')$

$= \mu \vec r \times \vec v = \mu r^2 \omega$



Total Energy: E

Assume F: conservative

E: K + U

$= \frac{1}{2} M V^2 + \frac{1}{2}\mu v^2 + U(r)$ = constant

For $m_1 \gg m_2$, $\mu \sim m_2$.

Motion reduces to motion of m_2 about a fixed m_1 !!!

2

Lecture 29 Blackboard #1

②

Angular Momentum

Central Forces

· No torques

$\vec{L}_{cm} = \vec{l}_1 + \vec{l}_2 = \text{constant.}$

$= m_1 \vec{r}_1{}' \times \vec{v}_1{}' + m_2 \vec{r}_2{}' \times \vec{v}_2{}'$

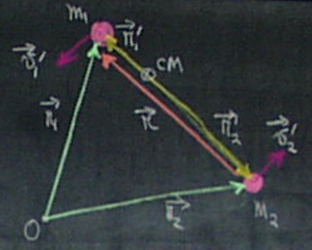
$\vec{r} = \vec{r}_1{}' - \vec{r}_2{}'$

$\vec{r}_1{}' = \frac{m_2}{m_1+m_2}\vec{r}$ $\quad \vec{r}_2{}' = \frac{-m_1}{m_1+m_2}\vec{r}$

$L_{cm} = \frac{m_1 m_2}{m_1+m_2}\vec{r}\times\vec{v}_1{}' - \frac{m_1 m_2}{m_1+m_2}\vec{r}\times\vec{v}_2{}'$

$= \mu \vec{r}\times(\vec{v}_1{}' - \vec{v}_2{}')$

$= \mu\vec{r}\times\vec{v} = \mu r^2\omega$



Total Energy: E

Assume F: conservative

$E = K + U$

$= \frac{1}{2}MV^2 + \frac{1}{2}\mu v^2 + U(r) = \text{constant}$

For $m_1 \gg m_2$, $\mu \sim m_2$.

Motion reduces to motion of m_2 about a fixed m_1 !!!

③

Planetary Motion – 2 Particle Systems

Sun + Planet

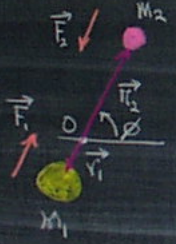
Planet + Moon

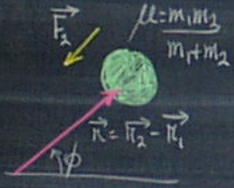
Earth + Satellites

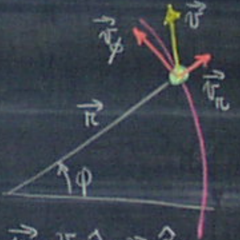
Binary Stars.

For simplicity CM coord system.

$\vec{R} \equiv 0$; $\vec{V}_{cm} = 0$







$\vec{v} = v_r\hat{r} + v_\phi\hat{\phi}$

Total Energy: E

$E = K + U = \frac{1}{2}\mu v^2 + U(r)$ [$V \equiv 0$ CM System]

$\vec{v} = v_r\hat{r} + v_\phi\hat{\phi}$

$v_\phi = v_t = r\omega = r\left(\frac{d\phi}{dt}\right)$

$L = \mu r^2\omega = \mu r^2\frac{d\phi}{dt}$

$\frac{L}{\mu r} = r\omega = v_\phi$

Lecture 29 Blackboard #2

$$E = \frac{1}{2}\mu(v_r^2 + v_\varphi^2) + U(r)$$

$$= \frac{1}{2}\mu v_r^2 + \frac{L^2}{2\mu r^2} - \frac{Gm_1m_2}{r}$$

KE due to radial motion rel. to CM.

KE due to ang. mot. about CM

Grav. PE

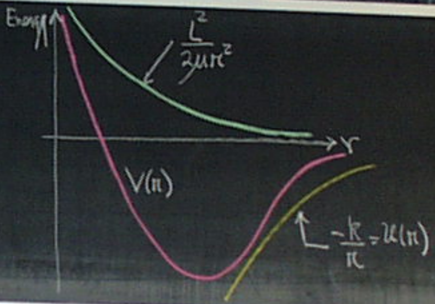
Let: $V(r) = \frac{L^2}{2\mu r^2} - \frac{Gm_1m_2}{r}$

Effective PE Function of r only.

Let $k = Gm_1m_2 = G\mu M$

$$V(r) = \frac{L^2}{2\mu r^2} - \frac{k}{r}$$

$$\boxed{E = \frac{1}{2}\mu v_r^2 + V(r)}$$



4

Solve Equations of Motion

Want: $r = f(t)$

$\phi = g(t)$

$r = h(\phi) \Rightarrow$ Trajectory

$$v_r = \frac{dr}{dt} = \sqrt{\frac{2}{\mu}(E - V(r))}$$

$$\frac{d\varphi}{dt} = \frac{L}{\mu r^2}$$

$$\therefore \frac{d\phi}{dr} = \frac{L}{\mu r^2}\frac{1}{\sqrt{\frac{2}{\mu}(E - V(r))}}$$

Solve

$$r = \frac{L^2/\mu k}{1 - \sqrt{1 + \frac{2EL^2}{\mu k^2}}\,\sin(\varphi - \varphi_0)}$$

$\varphi_0 = -\pi/2$ Convention!

$$r = \frac{r_0}{1 - \epsilon\cos\varphi}$$

$$\frac{r_0}{r} = 1 - \epsilon\cos\varphi$$

Eq. of a Conic Section

$r_0 = \frac{L^2}{\mu k}$ = radius of circular orbit with $\epsilon = 0$

$\epsilon = \sqrt{1 + \frac{2EL^2}{\mu k^2}}$ = eccentricity of ellipse.

5

Lecture 29 Blackboard #3

Solve Equations of Motion

Want: $r = f(t)$

$\phi = g(t)$

$r = h(\phi) \Rightarrow$ Trajectory

$$v_r = \frac{dr}{dt} = \sqrt{\frac{2}{\mu}\left(E - V(r)\right)}$$

$$\frac{d\phi}{dt} = \frac{L}{\mu r^2}$$

$$\therefore \frac{d\phi}{dr} = \frac{L}{\mu r^2}\,\frac{1}{\sqrt{\frac{2}{\mu}(E - V(r))}}$$

Solve

$$r = \frac{L^2/\mu k}{1 - \sqrt{1 + \frac{2EL^2}{\mu k^2}}\,\sin(\varphi - \varphi_0)}$$

$\varphi_0 = -\pi/2$ Convention!

$$r = \frac{r_0}{1 - \epsilon\cos\varphi}$$

$$\frac{r_0}{r} = 1 - \epsilon\cos\varphi$$

Eq. of a Conic Section

$r_0 = \frac{L^2}{\mu k}$ = radius of circular orbit with $\epsilon = 0$

$$\epsilon = \sqrt{1 + \frac{2EL^2}{\mu k^2}}$$

= eccentricity of ellipse

(5)

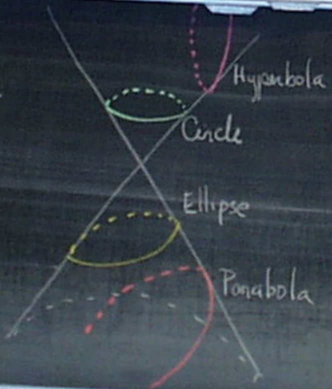
Orbit Characteristics

$\epsilon = 0$ Circular Orbits $E = E_{min}$

$\epsilon > 1$ Hyperbolic $E > 0$

$\epsilon = 1$ Parabolic $E = 0$

$0 < \epsilon < 1$ Elliptical $E < 0$



$$E = \frac{1}{2}\mu v_r^2 + V(r)$$

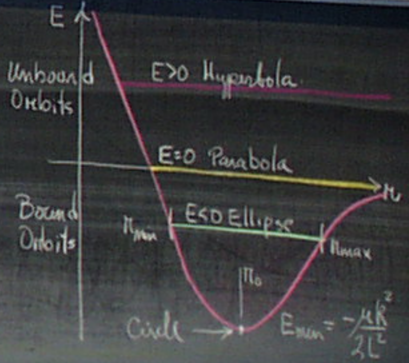
$$V(r) = \frac{L^2}{2\mu r^2} - \frac{G m_1 m_2}{r}$$

$V \to 0$ as $r \to \infty$

$$K = \frac{1}{2}\mu v_r^2 = E - V(r)$$

KE for radial motion only!!

$$E = K + V(r)$$



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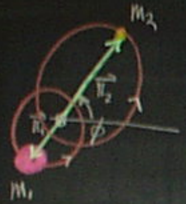
Circular Orbits



$E = E_{min}$ when $v_r = 0$

and $\frac{dV}{dr} = 0$

$$V(r) = \frac{L^2}{2\mu r^2} - \frac{k}{r}$$

$$\frac{dV(r)}{dr} = -\frac{L^2}{\mu r^3} + \frac{k}{r^2} = 0$$

$$r_0 = \frac{L^2}{\mu k} \quad \text{(radius)}$$

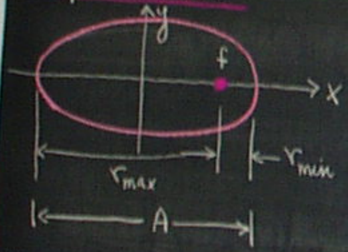
$$E_{min} = V(r_0) = -\frac{\mu k^2}{2L^2}$$

$$U(r_0) = -\frac{\mu k^2}{L^2}$$

$$E_{min} = \frac{1}{2} U(r_0)$$

⑦

Elliptical Orbits



$E < 0$

$0 < \epsilon < 1$

$$r = \frac{r_0}{1 - \epsilon \cos\varphi}$$

$$r_{max} = \frac{r_0}{1-\epsilon} \quad \varphi = 0$$

$$r_{min} = \frac{r_0}{1+\epsilon} \quad \varphi = \pi$$

Turning Points

$$A = r_{min} + r_{max}$$

$$A = r_0\left[\frac{1}{1+\epsilon} + \frac{1}{1-\epsilon}\right] = \frac{2r_0}{1-\epsilon^2}$$

$$= \frac{2L^2/\mu k}{1 - \left(1 + \frac{2EL^2}{\mu k^2}\right)} = \frac{k}{(-E)} \quad \text{Ind. of } L$$

$$\frac{r_{max}}{r_{min}} = \frac{1+\epsilon}{1-\epsilon}$$

Same $A \Rightarrow$ Same total E

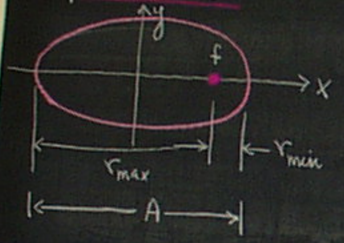
$$\epsilon = \frac{r_{max} - r_{min}}{r_{max} + r_{min}} = \sqrt{1 + \frac{2EL^2}{\mu k^2}}$$

	Planets	Earth
$r_{min} = r_p$	perihelion	perigee
$r_{max} = r_a$	aphelion	apogee

⑧

Lecture 29 Blackboard #5

Elliptical Orbits



$E<0$

$0<\epsilon<1$

$$r = \frac{r_0}{1-\epsilon\cos\varphi}$$

$$r_{max} = \frac{r_0}{1-\epsilon} \quad \varphi = 0$$

$$r_{min} = \frac{r_0}{1+\epsilon} \quad \varphi = \pi$$

} Turning Points

$$A = r_{min} + r_{max}$$

$$A = r_0\left[\frac{1}{1+\epsilon} + \frac{1}{1-\epsilon}\right] = \frac{2r_0}{1-\epsilon^2}$$

$$= \frac{2L^2/\mu k}{1-\left(1+\frac{2EL^2}{\mu k^2}\right)} = \frac{k}{(-E)}$$

Ind. of L

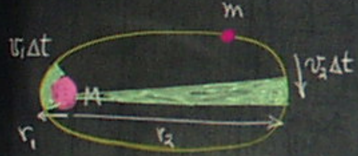
$$\frac{r_{max}}{r_{min}} = \frac{1+\epsilon}{1-\epsilon}$$

Same A ⇒ Same total E

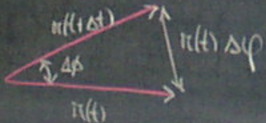
$$\epsilon = \frac{r_{max}-r_{min}}{r_{max}+r_{min}} = \sqrt{1+\frac{2EL^2}{\mu k^2}}$$ (8)

		Planets	Earth
$r_{min} = r_p$		perihelion	perigee
$r_{max} = r_a$		aphelion	apogee

Kepler's 2nd Law



Position vector for a planet sweeps out equal areas in equal time intervals. i.e. $\frac{dA}{dt}$ = constant.



$$\Delta A = \frac{1}{2}r^2\Delta\varphi$$

$$\lim_{\Delta t\to 0}\frac{\Delta A}{\Delta t} = \frac{1}{2}r^2\lim_{\Delta t\to 0}\frac{\Delta\varphi}{\Delta t}$$

$$\frac{dA}{dt} = \frac{1}{2}r^2\omega = \frac{L}{2\mu}$$

= Constant!!

Kepler's 3rd Law

$$dt = \frac{2\mu}{L}\,dA$$

$$\int dt = \frac{2\mu}{L}\int dA$$

$$T = \frac{2\mu}{L}A = \frac{2\mu}{L}\pi ab$$

$$b = a\sqrt{1-\epsilon^2}$$

$$a = A/2$$

$$a(1-\epsilon^2) = \frac{L^2}{\mu k}$$ (9)

↓ Algebra

$$T^2 = \frac{4\pi^2 a^3}{G(M+m)}$$