- Particle velocity $\vec{u}\,(u_x, u_y)$ in x, y, z, t system
- What is $\vec{u}'$ in x', y', z', t'?
- S' velocity is v wrt S.

$$u_x = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} \qquad u_y = \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t}$$

Also

$$u_x' = \lim_{\Delta t' \to 0} \frac{\Delta x'}{\Delta t'} \qquad u_y' = \lim_{\Delta t' \to 0} \frac{\Delta y'}{\Delta t'}$$

Lorentz transformation differentials:

$$\Delta x' = \gamma(\Delta x - v\,\Delta t)$$
$$\Delta y' = \Delta y$$
$$\Delta t' = \gamma\left(\Delta t - \frac{v}{c^2}\,\Delta x\right)$$

$$\therefore \quad \frac{\Delta x'}{\Delta t'} = \frac{\gamma(\Delta x - v\,\Delta t)}{\gamma\left(\Delta t - \frac{v}{c^2}\,\Delta x\right)} = \frac{\Delta x/\Delta t - v}{1 - \frac{v}{c^2}\left(\frac{\Delta x}{\Delta t}\right)}$$

Let $\Delta t \to 0$

$$u_x' = \frac{u_x - v}{1 - \frac{v}{c^2}u_x}$$

Also $\quad u_y' = \dfrac{u_y}{\gamma\left[1 - \dfrac{v u_x}{c^2}\right]}$

$u_z' = \dfrac{u_z}{\gamma\left[1 - \dfrac{v u_x}{c^2}\right]}$

Invert to get

$u_x = \dfrac{u_x' + v}{1 + \dfrac{v u_x'}{c^2}}$

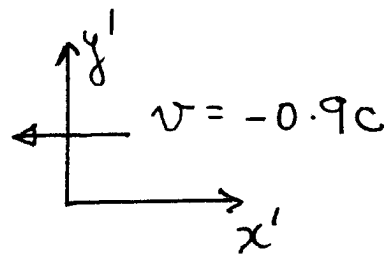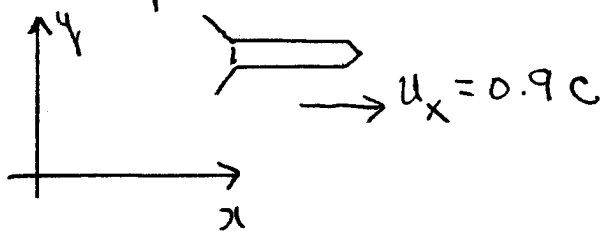$u_y = \dfrac{u_y'}{\gamma\left[1 + \dfrac{v u_x'}{c^2}\right]}$

$u_z = \dfrac{u_z'}{\gamma\left[1 + \dfrac{v u_x'}{c^2}\right]}$

If $\quad v \lll c$

$\left.\begin{array}{l} u_x' = u_x - v \\ u_y' = u_y \\ u_z' = u_z \end{array}\right\}$ Galilean Trans.

Example:

$$u_x' = \frac{u_x - v}{1 - \frac{v}{c^2}u_x} = \frac{0.9c - (-0.9c)}{1 - [(-0.9c)(0.9c)]} = \frac{1.8c}{1.81}$$

$$u_x' = 0.99c$$

Example:

Let $u_x = c$

$$u_x' = \frac{c - v}{1 - \frac{vc}{c^2}} \equiv c$$

Independent of v !!
Limiting velocity $\equiv c$

Doppler Effect
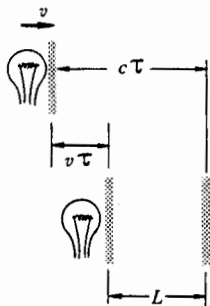
- Sound pitch increases when source approaches.
- Pitch decreases when source recedes.
- What about light?

- Source produces light flashes with period $\tilde{v}_0 = 1/\tilde{v}_0$ in its rest frame $-S'$
- Source moving with velocity v relative to S.
- Due to time dilation the period in the observers (S) rest frame is

$$\tilde{v} = \gamma \tilde{v}_0$$

- Pulse arrives at observer with speed c.

Observed frequency is

$v_D = c/L$ where L is the separation between two pulses. Since source is moving to observer

$$L = c\tilde{v} - v\tilde{v} = (c-v)\tilde{v}$$
and
$$v_D = \frac{c}{(c-v)\tilde{v}}$$

$$\nu_D = \frac{1}{1 - v/c} \frac{1}{\gamma \nu_0}$$

$$\nu_D = \nu_0 \frac{\sqrt{1 - v^2/c^2}}{1 - v/c}$$

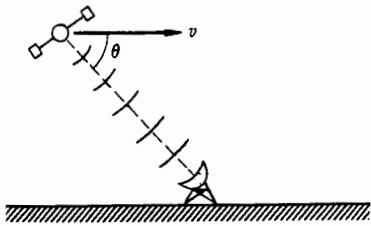$$\nu_D = \nu_0 \sqrt{\frac{1 + v/c}{1 - v/c}} \qquad \text{(source approaching)}$$

ν_D : frequency in observer's rest frame

ν_0 : frequency in source rest frame

$$\nu_D = \nu_0 \sqrt{\frac{1 - v/c}{1 + v/c}} \qquad \text{(source receding)}$$
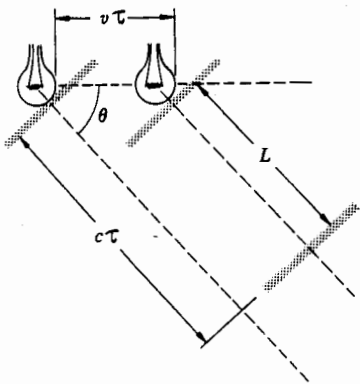
Consider source of light making an angle θ to direction of motion.

Period of flashes,
$$\tilde{\tau} = \gamma \tau_0$$

Observed frequency,
$$\vartheta_D = \frac{c}{L}$$

L: distance between flashes

$\vartheta \tilde{\tau}$: distance source moves between flashes.

$$L = c\tilde{\tau} - v\tilde{\tau} \cos\theta$$
$$= (c - v\cos\theta)\,\tilde{\tau}$$

$$\vartheta_D = \frac{c}{L} = \frac{c}{(c - v\cos\theta)\,\tau_0 \gamma}$$

$$\vartheta_D = \vartheta_0 \frac{\sqrt{1 - v^2/c^2}}{1 - \frac{v}{c}\cos\theta}$$

$$\vartheta_D (\theta = 90^\circ) = \vartheta_0 / \gamma$$
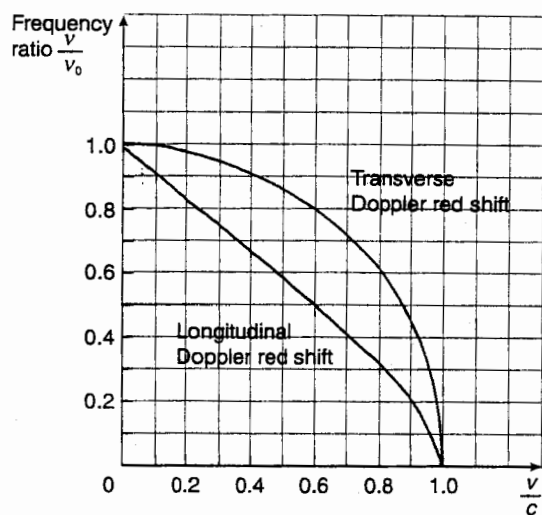
$$\vartheta_D (\text{classically}) \equiv 0.$$

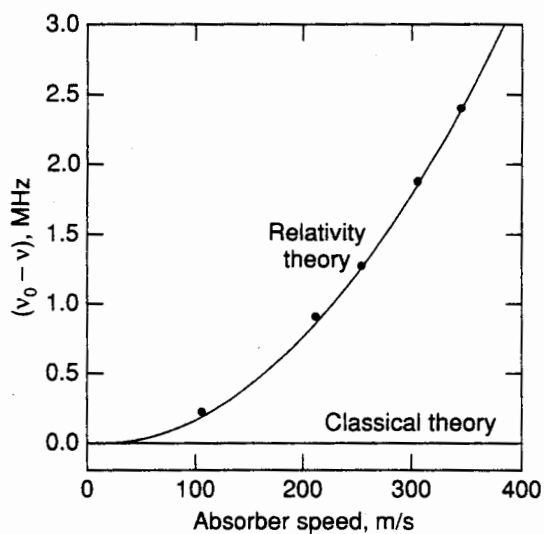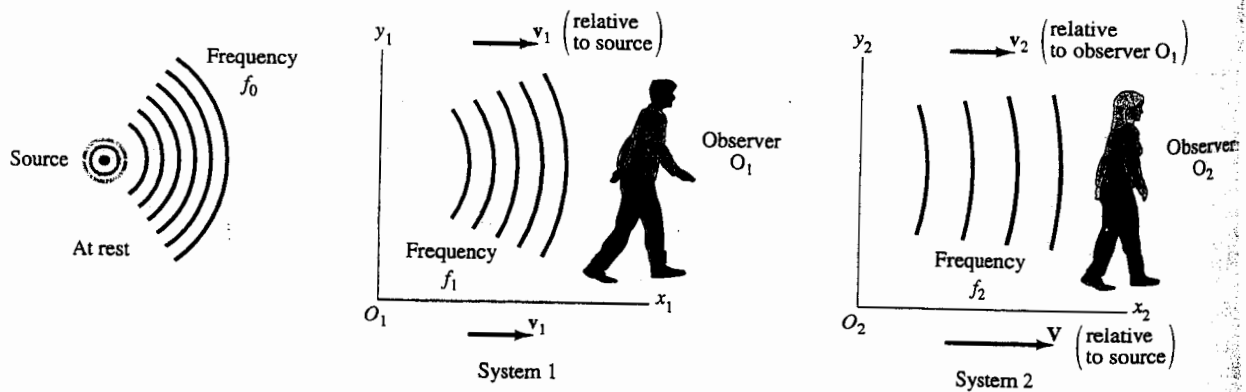FIGURE 25.43 Longitudinal and transverse Doppler effects.

FIGURE 2-12. The results of Kindig [9] on the transverse Doppler effect. The experimental points agree very well with the relativistic prediction and not at all with the classical prediction.

Two Observers / Doppler Shift

Frequency f_0 — Source — At rest

System 1: Frequency f_1, v_1 (relative to source), Observer O_1

System 2: Frequency f_2, v_2 (relative to observer O_1), V (relative to source), Observer O_2

- Light emitted with rest-frame frequency $= \partial_0$
- S' moves away with speed $\vec{v}_1$ along x-axis. measures redshifted frequency ∂_1.
- S'' moves away with speed v_2 along x-axis relative to S' and measures a frequency ∂_2.
- Assume S'' moves with velocity V relative to rest-frame. He should observe ∂_2 as frequency shift relative to ∂_0.

$$\partial_1 = \partial_0 \sqrt{\frac{1 - v_1/c}{1 + v_1/c}} \qquad [\text{Observer } S']$$

$$\partial_2 = \partial_1 \sqrt{\frac{1 - v_2/c}{1 + v_2/c}} \qquad [\text{Observer } S'']$$

$$\partial_2 = \partial_0 \sqrt{\frac{1 - v_1/c}{1 + v_1/c}} \sqrt{\frac{1 - v_2/c}{1 + v_2/c}}$$

We must also have,

$$\partial_2 = \partial_0 \sqrt{\frac{1 - V/c}{1 + V/c}}$$

$$\therefore \quad \frac{1 - V/c}{1 + V/c} = \frac{\left[1 - v_1/c\right]\left[1 - v_2/c\right]}{\left[1 + v_1/c\right]\left[1 + v_2/c\right]}$$

Solve for V

$$V = \frac{v_1 + v_2}{1 + v_1 v_2 / c^2}$$

A: Bob

B: Dave

· Twins ; one stays and one travels

· A sees B travel a distance L with velocity v

$$T = L/v$$

· B reverses direction and returns in same time T.

· Assume turn-around can be neglected.

· A observes time T_B' on B's clock is related to his time $T_A = 2T$

$$T_B' = T_A / \gamma$$

$$\frac{\text{Aging of B}}{\text{Aging of A}} = \frac{T_B'}{T_A} = \frac{1}{\gamma} \quad (\text{As viewed by A})$$

· B's Perspective :

· A travels with velocity $-v$; turns around and comes back.

$$T_B = 2T$$

$$T_A' = T_B / \gamma$$

$$\frac{\text{Aging of B}}{\text{Aging of A}} = \frac{T_B}{T_A'} = \gamma \quad (\text{As viewed by B})$$

<u>Paradox</u> : A thinks B is younger and B
thinks A is younger

• This problem elicited many papers and
discussions for many years. Is something
wrong with Special Relativity?

No !!

<u>Resolution</u> :
• Example : Bob leaves his brother i NYC.
• Travels with $v = 0.8c$ to a star, $\gamma = 5/3$.
• By his clock the travel time
$$T_B = 3 \text{ yrs}.$$

• Fires his rockets and reverses direction and speed
• At end fires rockets to stop in NYC.

• Total travel time : $T_B = 6$ yrs
• We will ignore the accelerations. Time
accelerating can be made much shorter
than travel time.
• Dave's clock reads 10 yrs.
• Both twins have identical clocks which send
out light signals at one-year intervals.
• Dave gets signals from Bob's clock and
records them against the annual signals

of his clock.

- Bob receives the signals from Dave's clock and records them against the signals from his own clock.
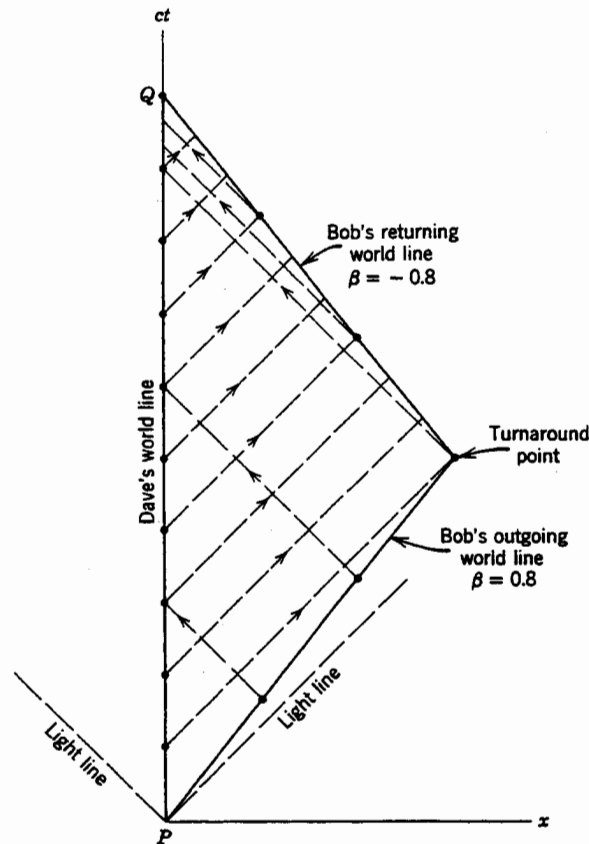

Figure B-2. Spacetime diagram of the twin paradox.

- Dave's world-line is the ct-axis. He stays at $x = 0$.
- Mark off 10 yrs on the ct-axis.
- Bob's world line is inclined, ct', for a frame moving with $v = 0.8c$. He stays at $x' = 0$.
- After 3 yrs he reverses velocity to $v = -0.8c$ and returns.

- Mark off 3 yrs going and 3 yrs returning.
- Draw light signals from Dave to Bob.
- Draw light signals from Bob to Dave.
- Signals parallel to light line.

- Bob sends 6 signals; the last on arrival.
- Dave sends 10 signals; the last on arrival.

- How to explain space-time diagram?
- Doppler effect.
- As clocks recede, their frequency is

$$\nu = \nu_0 \sqrt{\frac{1-\beta}{1+\beta}} = \frac{\nu_0}{3}$$

- Bob receives first signal from Dave after 3 yrs, just as he turns around.
- Dave receives signals at a reduced rate from Bob on the way out, once every 3 of his years. Receives three signals in nine years.

- As clocks approach, their frequencies are

$$\nu = \nu_0 \sqrt{\frac{1+\beta}{1-\beta}} = 3\nu_0$$

- Bob receives 9 signals from Dave on his return journey. Altogether he gets 10 signals.

- Dave receives 3 signals from Bob on the last year. Dave receives 6 signals.

- No disagreement on signals :
Bob sends 6 ; Dave receives 6
Dave sends 10 ; Bob receives 10

- Why are the times different?
- Dave sees Bob recede for 9 years and return for one year.
- Bob sees himself receding for 3 years and approaching for three years.

- Dave received signals at a slow rate for nine years and a fast rate for one year.
- Bob had slow signals for three years and fast signals for 3 years.

- Doppler effect shows the asymmetry in the travel
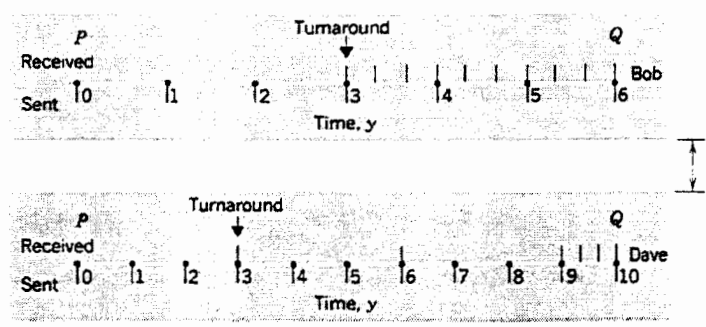- Bob is younger than Dave after trip.

Figure B-3. The signal logs for the twins.

- Confusion on appearance of rapidly moving objects. Not totally understood till ~ 1960.
- Consider a board of length L_0 and width W_0 in it's rest frame.
- Moving with speed v parallel to L_0
- Take a picture in a plane $\perp v$.
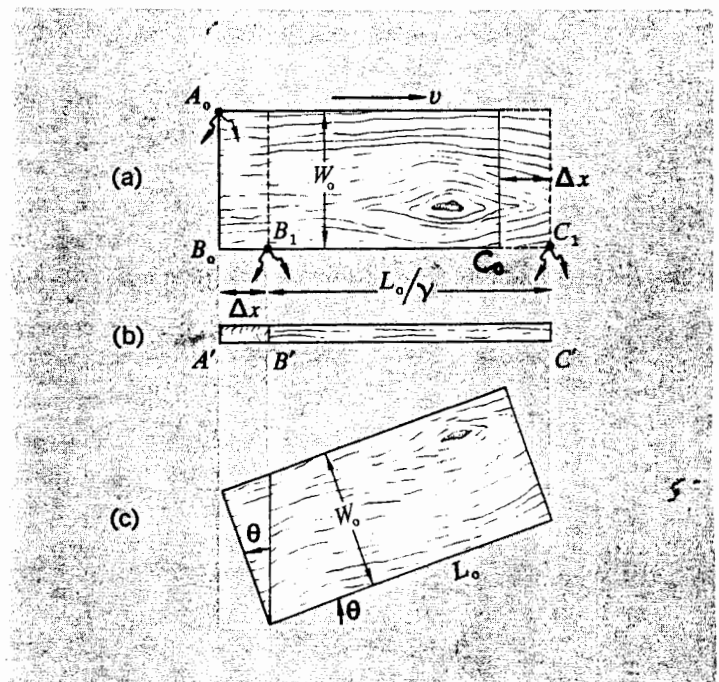- Picture: Instant simultaneous collection of light from object.

- Consider three points: A_0, B_0, C_0
- Light from B_0, C_0 arrives at film plane at same time.
- Light from A_0 must start earlier to reach film on time.

$$\Delta t = W_0/c$$

- In time Δt board moves to the right

$$\Delta x = v\Delta t = \frac{v W_0}{c}$$

Fig. 5-11   (a) A rectangular object moving at speed v parallel to x. (b) The apparent positions of the corners A, B, C as recorded at a given instant by a distant observer looking in the y direction in the plane of the object. (c) It is inferred that the object is rotated in its own plane, but not Lorentz-contracted.

Looking at moving clocks and other objects

At Δt: $\qquad B_0 \to B_1$
$\qquad\qquad\quad C_0 \to C_1$

· Board along $-x$ is Lorentz contracted

$$\therefore B_1 C_1 = L_0 / \gamma$$

Also $\qquad A'B' = v W_0 / c$

Consider board at rest but rotated through angle θ

$$A'B' = W_0 \sin \theta$$
$$B_1 C_1 = B'C' = L_0 \cos \theta$$

Appears as if board is rotated

$$\sin \theta = v/c$$

$$\cos \theta = \sqrt{1 - \frac{v^2}{c^2}}$$