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We investigate graphs that can be disconnected into small components by removing a vanishingly small fraction of their vertices. We show that, when a controllable quantum network is described by such a graph and the gaps in eigenfrequencies and in transition frequencies are bounded exponentially in the number of vertices, the network is efficiently controllable, in the sense that universal quantum computation can be performed using a control sequence polynomial in the size of the network while controlling a vanishingly small fraction of subsystems. We show that networks corresponding to finite-dimensional lattices are efficiently controllable and explore generalizations to percolation clusters and random graphs. We show that the classical computational complexity of estimating the ground state of Hamiltonians described by controllable graphs is polynomial in the number of subsystems or qubits.

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Controlling large quantum networks and performing universal quantum computation are two important and related problems in quantum information processing. A common goal is to perform control and computation efficiently, by accessing a minimum number of directly controlled parts. Quantum networks were introduced in Ref. [1]. In Ref. [2], it was shown that almost any quantum network with a single probe is controllable. If controls can be applied to quantum degrees of freedom in a pairwise fashion, then the control is computationally universal [3]. The connectivity of the graph of interactions plays an important role in controllability and computation [4]. Under mild assumptions about network topology and the algebra of controls, Ref. [5] gave sufficient conditions for a network to be controlled using a small number of control qubits, without regard for the efficiency of the control sequence. See [6] for a similar study of classical linear systems. In suitable systems, quantum computation is possible with only a few control qubits [7,8]. As suggested in these papers, spin chains with specific Hamiltonians can give controllability as well as the ability to enact efficient universal quantum computation on the chain. These results raise the question of when it is possible to perform universal quantum control and quantum computation efficiently on a general quantum network. This Letter shows in a general setting that it is possible to perform universal quantum control and computation in time polynomial in network size on a wide variety of controllable quantum networks with Hamiltonians whose gaps in eigenfrequencies and in transition frequencies are bounded exponentially in the number of vertices while acting on only a vanishingly small fraction of their nodes. This naturally leads one to define an interesting class of graphs, which we call *efficiently controllable graphs*, which admit efficient control by acting on a vanishingly small fraction of controlled nodes. The existence, construction, and analysis of this new class of graphs pose an intriguing problem in the graph theory. In this work, we construct several examples of such families of graphs and show that the ground states of Hamiltonians of systems whose interactions are determined by such graphs can be approximated efficiently.

Consider a quantum system consisting of subsystems interacting via local Hamiltonians. The subsystems can be represented as vertices of an interaction (hyper)graph where (hyper)edges exist only between vertices corresponding to coupled subsystems. For simplicity of exposition, we will restrict our attention to pairwise Hamiltonians and interaction graphs. However, all our results apply to general local Hamiltonians and interaction hypergraphs. Without the loss of generality, here the quantum systems are restricted to networks of qubits.

Since implementation becomes more complex as the number of controlled spins grows, a scalable implementation needs to choose the smallest number of controlled spins possible while still preserving polynomial efficiency of quantum computation. The primary purpose of this Letter is to show that there exist families of interaction graphs such that the quantum computational efficiency scales polynomially with the number of vertices and the fraction of controlled qubits approaches zero as the number of nodes in the graph goes to infinity given assumptions on the spectrum of the Hamiltonian and controllability. That is, there are scalable and efficient quantum computer architecture schemes that make use of a vanishing fraction of controlled qubits. Lattices and uniform tilings are examples of such families. Expanders and complete graphs are not likely to be such graphs. Not every family of graphs admits

such schemes; therefore, we define a new family, which we call efficiently controllable graphs. An efficiently controllable graph is a graph that can be divided into components of size poly[log(n)] by removing a vanishingly small fraction of vertices, where n is the number of vertices. Assuming controllability conditions, we prove that on a quantum network described by such a graph one can perform universal quantum computation efficiently by controlling a vanishingly small fraction of vertices in the limit that the size of the graph goes to infinity.

In a connected network architecture of n spins satisfying certain assumptions on the drift and control Hamiltonians, enacting an arbitrary unitary operation within a constant error ϵ requires $s(n) = O(2^{nx}\mathrm{poly}(1/\epsilon))$ elementary operations for some x as we show now. Consider a connected network of n spins. To make the argument simple, restrict to the Hamiltonian with a single control term $H(t) = H_0 + H_c\gamma(t)$ where $-iH_0, -iH_c \in su(d=2^n)$ are bounded. We assume that the pair (H_0, H_c) is controllable. See, for example, [4,5,9,10] for sufficient conditions for local controllability. The control problem is to find $\gamma(t)$ to drive an initial unitary, which is the identity, to the ϵ neighborhood of a final unitary. The Hamiltonian defines a flow in the set of unitaries as $\dot{U}(t) = -i[H_0 + H_c\gamma(t)]U(t)$, where $U(t) \in SU(d)$. Assume that $H_0 = \sum_k E_k P_k$ is nondegenerate to accuracy $1/d^r$, where $P_k = |k\rangle\langle k|$ is the projector onto the eigenvector $|k\rangle$ of H_0 with eigenvalue E_k. Assume also that $|\Delta_{jk}| = |E_j - E_k| > 1/d^r$ are distinct and $P_k H_c P_m \neq 0$ for all k, m (this condition can be relaxed: see [11]), and $\|H_c\| = O(1)$ where $\|U\| = \sup_{y\in\mathbb{C}^d, y^\dagger y=1}\|Uy\|$ is the operator norm. It is an interesting feature that there are few if any nontrivial many-body systems whose level statistics are proven to obey the above conditions. However, for example, extensive numerical evidence suggests that one-dimensional spin chains with randomly constructed Hamiltonians [12,13] exhibit Wigner-Dyson-like statistics and therefore energy level repulsion [14], implying that the probability of having exponentially small level spacing or transition level spacing is exponentially small in d. Drive the system with control with amplitude A, resonant frequency Δ_{jm}, and phase ϕ, so that the Hamiltonian takes the form $H(t) = H_0 + A\cos(\Delta_{jm}t + \phi)H_c$. Now go to the interaction picture by defining $U_i(t)$ via $U(t) = e^{-itH_0}U_i(t)$. Then

$$\dot{U}_i(t) = -ie^{itH_0}A\cos(\Delta_{jm}t + \phi)H_c e^{-itH_0}U_i(t).$$

The approximate solution of this equation is given by the Magnus expansion [15] as

$$\begin{aligned} U_0(T) &= \exp[-i\Omega(T)] \\ &= e^{-i\int_0^T dt \exp(itH_0)A\cos(\Delta_{jm}t+\phi)H_c\exp(-itH_0)} \end{aligned}$$

with error $\|U_0(T) - U_i(T)\| = O(|A|^2T^2\|H_c\|^2)$, where $|A|T\|H_c\| < \pi$ for the convergence of the series. We write $\Omega(T) = \Omega_1(T) + \Omega_2(T)$ as the sum of the resonant term and the off-resonant term. The resonant term is given by

$$\Omega_1(T) = \frac{A}{2}T(e^{-i\phi}P_jH_cP_m + e^{i\phi}P_mH_cP_j)$$

and $\|\Omega_2(T)\| = O(|A|d^{2+r}\|H_c\|)$. The error in neglecting $\Omega_2(T)$ is given by [16]

$$\begin{aligned} \|U_0(T) - e^{-i\Omega_1(T)}\| &\le \|e^{-i[\Omega_1(T)+\Omega_2(T)]} - e^{-i\Omega_1(T)}e^{-i\Omega_2(T)}\| \\ &\quad + \|e^{-i\Omega_1(T)}e^{-i\Omega_2(T)} - e^{-i\Omega_1(T)}\| \\ &= O(\|\Omega_1(T)\|\|\Omega_2(T)\| + \|\Omega_2(T)\|) \\ &= O(|A|^2Td^{2+r}\|H_c\|^2 + |A|d^{2+r}\|H_c\|). \end{aligned}$$

Now bound the error between $U_i(T)$ and $e^{-i\Omega_1(T)}$ using the triangle inequality as

$$\begin{aligned} \|U_i(T) - e^{-i\Omega_1(T)}\| &\le \|U_0(t) - e^{-i\Omega_1(T)}\| + \|U_i(t) - U_0(t)\| \\ &= O(|A|^2Td^{2+r}\|H_c\|^2 \\ &\quad + |A|d^{2+r}\|H_c\| + |A|^2T^2\|H_c\|^2). \end{aligned}$$

Choose A and T such that

$$|A|^2T^2\|H_c\|^2 > |A|d^{2+r}\|H_c\| > |A|^2Td^{2+r}\|H_c\|^2,$$

implying $1 > |A|T\|H_c\|$ ensuring the convergence of the Magnus series, $d^{2+r} < |A|T^2\|H_c\|$, and $T > d^{2+r}$. Such a choice is possible by making A sufficiently small, hence weak driving. Then

$$\|U_i(T) - e^{-i\Omega_1(T)}\| = O(|A|^2T^2\|H_c\|^2).$$

Note that $\Omega_1(T)$ is a single-qubit Hamiltonian acting on the subspace spanned by $|j\rangle$ and $|m\rangle$. By adjusting ϕ, one can implement

$$V_i = e^{-i(A/2)T|\langle j|H_c|k\rangle|\sigma},$$

where $\sigma = \pm\sigma_x, \pm\sigma_y$. Now any $SU(2)$ gate U_2 can be decomposed in the form $U_2 = e^{-ic_1\sigma_x}e^{-ic_2\sigma_y}e^{-ic_3\sigma_x}$ for some c_1, c_2, c_3. This is the Cartan decomposition of $SU(2)$; see, for example, [17]. Therefore, it takes $O(3/|A|T)$ gates to generate any U_2 with error $O[(3/|A|T)|A|^2T^2] = O(3|A|T)$, since the errors accumulate linearly [18]. An arbitrary unitary $U \in SU(d)$ can be implemented by at most $d(d-1)/2 = O(d^2)$ $SU(2)$ rotations [16,18]. The total error is then $\epsilon = O(3|A|Td^2)$. Therefore, it requires a total number of $O(3d^2/|A|T) = O[9d^4(1/\epsilon)]$ operations to implement any unitary with accuracy ϵ. The gate complexity can be improved to $\mathrm{poly}(d)\mathrm{poly}\log(1/\epsilon)$ by generating $SU(2)$ gates via the Solovay-Kitaev algorithm [19]. The ability to implement

any unitary in the interaction picture implies the ability to implement any unitary noting that $U(T') = e^{-iT'H_0}U_i(T')$.

Note that, since in our setting we have a drift term whose inverse cannot be reached directly, we could not invoke the discrete Solovay-Kitaev bound [18,19] or bounds relating optimal control costs to gate complexity [20,21].

Thus, the efficiency of universal quantum computation scales exponentially with the total number of spins. Are there architectures enabling control complexity to scale polynomially, or subexponentially, yet the fraction of controlled qubits vanishes with n? One such architecture is to decompose the graph into N connected blocks, b_j, each containing L qubits with boundaries of size B between them where $L \gg B$ for large n. The boundaries are promoted to be controls. And we require that for each block one can apply an additional control term, which together with the Hamiltonian of the block satisfies the assumptions of the complexity result proved above. Thus, neighboring blocks are separated by a number of controlled qubits. The ability to perform arbitrary transformations on control qubits makes it possible to completely decouple the blocks [22]. Decoupling every block but two, one can perform a quantum computation on two adjacent blocks with efficiency $s(2L)$. In an arbitrary network, two blocks are at most N blocks apart from each other. To transfer quantum information between two arbitrary blocks or equivalently to apply any quantum operation to arbitrary two blocks, one applies the following procedure. Let $b_{p(1)}b_{p(2)}\ldots b_{p(N)}$ be a path of blocks between maximally separated $b_{p(1)}$ and $b_{p(N)}$, where $p(\cdot)$ is some permutation of N blocks. Quantum information is mediated through the network by first decoupling the adjacent blocks $b_{p(1)}$ and $b_{p(2)}$ from the rest of the network and enacting a quantum transformation on it. Then $b_{p(2)}$ and $b_{p(3)}$ are decoupled from the rest, and quantum information is transferred between these blocks. Continuing this way, quantum information can be mediated between any two blocks at most using $O(N)$ pairwise decoupling operations. The total gate complexity of applying any quantum operation between any two blocks is at most $O(Ns(2L)) = O(N2^{2xL}/\epsilon)$. How can the required operations be made to depend subexponentially to n? Take a family of spin networks, $G(n)$, indexed by the total number of qubits. If each network in the family admits a decomposition into N blocks of size $L = \log N/2x$ while $n = N\log N/2x$, the complexity can be made polynomial as $O(Ns(2L)) = O(N^2/\epsilon) = O(n^2/\epsilon)$. If also the fraction of controls c/n can be made to vanish as n grows large, where c is the number of controls, one has a scalable quantum computer architecture with a small fraction of controls whose gate complexity is subexponential.

Not every family of graphs, $G(n)$, admits a decomposition into blocks such that the fraction of controls vanishes while the number of elementary operations needed scales polynomially with the number of vertices. To distinguish between efficiently controllable graphs and graphs that are not efficiently controllable, we now present a formal definition of the efficiently controllable family of graphs.

Definition: Efficiently controllable family.—A family of graphs, $G(n)$, indexed by the number of vertices, n, is called an efficiently controllable family if for every n there exists a decomposition into connected subgraphs, blocks, $G(n) = \cup_{K=1}^{N(n)} G_k$ such that $\lim_{n\to\infty}\sum_{1=j<k=N}|G_j \cap G_k|/n \to 0$, where $|G_j \cap G_k|$ is the cardinality of $G_j \cap G_k$, the controls between two blocks; in addition, we require that control complexity $D(n)s[L(n)] = O(\mathrm{poly}(n), \mathrm{poly}(1/\epsilon))$, where $L(n)$ is the maximum size of blocks and $D(n)$ is the diameter of the graph formed by the blocks. Note that the definition can be easily generalized to the control of classical networks and other complexity measures.

We give a simple example of a scalable network architecture. Quantum information can be transferred from one end to the other of a one-dimensional chain of n qubits using a fraction of them as controls. This fraction can be chosen so that it vanishes as n goes to infinity, and the number of elementary operations required scales polynomially with n. Assume N blocks of qubits of size $L-1$. Between neighboring blocks lies a single control qubit. Then the fraction of controls is $c/n = 1/L$. We choose $L = \log N/2x$ so that c/n vanishes as n goes to infinity. In order to enact arbitrary unitary operations between the blocks lying at the right and left ends, one first decouples blocks 1 and 2 from the rest of the chain and transfers quantum information coherently from 1 to 2, then decouples blocks 2 and 3, then 3 and 4, etc. Thus, it takes $O(N)$ steps to mediate quantum information between the blocks that lie at the ends. The number of elementary operations required to perform arbitrary operations with accuracy ϵ between adjacent blocks is of the order of $s(2L-1)$. Thus, the total number of elementary operations needed to couple the blocks at the ends is $O(Ns(2L-1))$. With the choice we made for L, the quantum gate complexity to enact any desired quantum logic operation between any two blocks is at most $O(n^2)$.

The previous scheme can be easily generalized to a family of d-dimensional cubic lattices. We take N^d blocks of size L^d where the total number of qubits is $n = N^dL^d$. Between two adjacent blocks lies a $d-1$ dimensional layer of control qubits. The fraction of controls is again $c/n = 1/L$. Quantum information can be transferred between blocks lying in the opposite diagonal ends by $O(dN)$ pairwise operations on blocks lying in the interior. The number of elementary operations required to enact quantum logic between adjacent blocks with accuracy ϵ is given by $s[(L-1)^{d-1}(2L-1)]$. Choosing $L = \{[(d-\frac{1}{2})/2x]\log N\}^{1/d}$, the total number of elementary quantum operations is given by $O(dNs(2L^d)) = O(dn^2/\epsilon)$ while c/n is vanishingly small in the limit of large n. Now, the generalization to lattices or uniform tilings

is evident. In the presence of symmetries [23] (existence of a subalgebra of the Lie algebra commuting with the drift and all the control terms), complete controllability is lost. However, one can generically break symmetries by perturbations in the coupling Hamiltonian or controls. Therefore, assuming controllability, efficient controllability follows.

Note that the construction of efficiently controllable families given above does not require the dimension d to be an integer. Fractals such as the Sierpinski gasket automatically generate efficiently controllable families.

An efficient way to generate efficiently controllable families is via site percolation [24]. Consider an infinite lattice of spins where spins sitting in adjacent lattice sites interact with probability p and the interaction probabilities for each edge connecting lattice sites are independent. When p is just above the percolation threshold p_c, the graph is connected with unit probability, while the structure of the cluster formed is a fractal [25]. At this point, removing a vanishingly small fraction of the spins at random separates the graph into disconnected pieces: that is, those removed spins form the interfaces between those pieces of the graph. The largest size of those disconnected pieces can be estimated as follows. Start at $p \approx p_c$ and remove a fraction δ of the spins. A group of N previously connected spins will remain connected if, by a statistical fluctuation, the fraction of connections within that set remains above p_c. Otherwise, the group will become disconnected for large N. The average fluctuation in the number of connections in the group goes as $\pm\sqrt{p_c(1-p_c)N}$. The probability that the group remains connected goes as $e^{-\delta^2 N/p_c(1-p_c)}$. Accordingly, if one removes a fraction δ of the spins, the largest connected group size goes as $O(\delta^{-2})$. This gives the same scaling for the fraction of control spins required as that for a two-dimensional lattice, where a group of size N has a boundary of size $O(\sqrt{N})$. But that family is efficiently controllable, as shown above. Consequently, a graph just above the percolation threshold realizes an efficiently controllable family: universal quantum computation can be effected by controlling a vanishingly small fraction of the spins. The same argument holds for other families of graphs with percolation thresholds, e.g., Erdos-Renyi graphs [26].

The site percolation construction above can be applied to scale-free networks characterized by the degree distribution $P(k) \sim k^{-\alpha}$, where $P(k)$ is the probability for a site to be connected to k other sites. For the random removal of sites, the percolation threshold is either 0 or finite [27]. However, for $\alpha = 2$, the removal of high degree nodes makes the percolation threshold approach 1, and removing a fraction $\sim 1/N$ of nodes is sufficient to break down the network [28] into clusters of size $\log N/\log\log N$, where N is the total number of sites [29]. Therefore, scale-free networks with $\alpha = 2$ can be made efficiently controllable. Take high degree nodes as controls for decoupling, and take one node for each decoupled cluster as the control for enacting quantum gates. The total number of controls required to perform a quantum computation efficiently is then a vanishing fraction of the total number of sites.

The purely graph-theoretic definition of an efficiently controllable family has applications outside of the quantum control theory. Consider, for example, the problem of approximating the ground state energy of a system, classical or quantum, whose interactions correspond to an efficiently controllable graph. The construction of efficiently controllable graphs shows that the problem of finding a state whose energy is within a multiplicative factor ϵ of the actual ground state energy is polynomial in the size of the system. More precisely, consider a quantum Hamiltonian described by the graph $G = (V, E)$, where each vertex corresponds to a variable and each edge to a pairwise interaction. We want to find a state whose energy is within a factor ϵ of the actual ground state. Let n be the number of variables and N the number of clusters, each of size $\log N$, so that $n = N \log N$. Disconnect and decouple the clusters of size $\log N$ by removing the control qubits, the boundaries between the clusters, to get the Hamiltonian $\tilde{H} = \sum_k H_{C_k}$, where H_{C_k} is the Hamiltonian acting on the cluster C_k. The error introduced in calculating the ground state energy is at most ϵn, where ϵ is the fraction of controls; i.e., the ground state energy of H is ϵn close to that of $\tilde{H}$. But the ground state of $\tilde{H}$ is the tensor product of the ground states of $\{H_{C_k}\}_k$. By standard matrix diagonalization techniques, the ground state energy of H_{C_k} can be found in $O(\mathrm{poly}N) = O(\mathrm{poly}n)$ steps. There are N clusters, so it still takes only polynomial steps to calculate the ground state energy of $\tilde{H}$ and therefore to approximate that of H within accuracy ϵN which vanishes as n becomes large. Note that our construction is a polynomial time approximation scheme for finding the ground state energy of a 2-local Hamiltonian [30] using clustered product states. Although our construction is in the spirit of product state approximations to ground states [31,32], we are approximating with a multiplicative error instead of an additive error.

This Letter investigated the requirements for being able to control extended systems efficiently. Quantum systems that can be controlled in time polynomial in the number of coupled variables in the system Hamiltonian, by only operating on a vanishingly small fraction of those variables, correspond to efficiently controllable families of interaction graphs. Such graphs can be divided into clusters of size $O(\mathrm{poly}(\log n))$ while removing a fraction ϵ of the n vertices, with $\epsilon \to 0$ in the limit $n \to \infty$. Canonical graph families such as regular lattices are readily shown to be also efficiently controllable. The general criterion for when families of graphs admit a polynomially efficient universal quantum computation yet using a vanishing fraction of fully controlled qubits is an open question. Other open

questions include the computational complexity of the construction of efficiently controllable families and whether existing heuristics for graph partitioning problems can be exploited to find approximate solutions.

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