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We describe the nonlocal effective properties of a two-dimensional dissipative phononic crystal made by periodic arrays of rigid and motionless cylinders embedded in a viscothermal fluid such as air. The description is based on a nonlocal theory of sound propagation in stationary random fluid/rigid media that was proposed by Lafarge and Nemati [Wave Motion **50**, 1016 (2013)]. This scheme arises from a deep analogy with electromagnetism and a set of physics-based postulates including, particularly, the action-response procedures, whereby the effective density and bulk modulus are determined. Here, we revisit this approach, and clarify further its founding physical principles through presenting it in a unified formulation together with the two-scale asymptotic homogenization theory that is interpreted as the local limit. Strong evidence is provided to show that the validity of the principles and postulates within the nonlocal theory extends to high-frequency bands, well beyond the long-wavelength regime. In particular, we demonstrate that up to the third Brillouin zone including the Bragg scattering, the complex and dispersive phase velocity of the least-attenuated wave in the phononic crystal which is generated by our nonlocal scheme agrees exactly with that reproduced by a direct approach based on the Bloch theorem and multiple scattering method. In high frequencies, the effective wave and its associated parameters are analyzed by treating the phononic crystal as a random medium.

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## I. INTRODUCTION

The field of phononic crystals (PCs) for acoustic/elastic waves [1,2], which has been developed in analogy to the concept of photonic crystals related to electromagnetic (EM) waves [3], has been a subject of intensive theoretical and experimental investigations over the past two decades [4–7]. PCs are the periodic arrangement of solid/fluid inclusions embedded in a host solid/fluid material, configured to control and manipulate acoustic/elastic wave propagation that can exhibit band gaps forbidding propagation. The existence of the forbidden bands or band gaps in phononic materials, usually displayed in $\omega$-$\boldsymbol{k}$ (angular frequency-wave vector) space, is due to Bragg scattering which at specific frequencies, namely band-gap frequencies, leads to destructive interferences. Indeed, the solid/fluid periodic inclusions act as scatterers in the medium.

PCs are studied and applied across broad and various research areas, such as sound isolation [7–10], wave guiding [4,11], nanoscale thermal control [6], and quantum information processing [12]. Depending on the research area and the size of materials and devices made by the PCs, the wave frequency in these materials is ranging from sonic/audible frequencies (kHz) for sound proofing to ultrasonic imaging (MHz), hypersound (GHz) in optomechanics, and thermal applications (THz) [6]. Respectively, the material size characterized by its periodicity $L$ covers the scale from the centimeter to millimeter, micrometer, and nanometer.

We distinguish ordinary PCs, with periodic acoustic metamaterials (AMM), in that the particular wave control features of PCs are produced by the spectral bands through Bragg scattering, that occurs when the effective wavelength $\lambda$ propagating in the material is close to the period $L$. Bragg's spectral bands depend essentially on the periodic arrangements of the structural units. In contrast, the unusual macroscopic properties of the AMMs originate in local resonances leading ultimately to one or more negative effective-medium parameters such as density, bulk modulus, and elastic modulus, when $\lambda$ is several times larger than $L$. The AMM's properties rely on the internal structure of a single periodic unit including its topology and constituent materials, that characterize localized resonances. In order to design materials that exhibit the required macroscopic properties, we need to employ an appropriate effective-medium theory.

Effective-medium theories aim at establishing macroscale equations that govern the effective dynamics of a medium at the scale of measurement with heterogeneities at the microlevel. These equations typically include the field equations and constitutive relations, where the latter characterize and are specific to the medium. Within effective-medium theories, homogenization techniques are employed for upscaling or coarse graining the microscale properties, either rigorously, phenomenologically, or approximately. As a result, microscale information such as microtopology, densities, bulk moduli, volume fraction of the medium constituents are encoded in a few effective parameters at macrolevel, where the medium is assumed to be homogeneous. However, describing the effective properties of PCs and AMMs is generally restrictive within classical homogenization theories. The long-wavelength resonant behaviors of AMMs are often not captured, and the Bragg scattering in PCs is commonly thought to escape a macroscopic description as this phenomenon occurs for small wavelengths on the order of the period. Efforts have been made to capture local resonances in AMMs with elastic materials by coherent potential approximation (CPA) [13] that is based on minimizing scatterings in the long-wavelength limit. An enhanced scheme has been developed to obtain the effective properties of the same type of materials in a broader frequency band by matching the lowest-order scattering amplitudes that

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arise from the unit cell, with that of the homogenized material (metasolid) [14]. Although it was known that nonlocal effects were relevant in EM materials as a small correction when the wavelengths reduce, these effects have been also found to be impactful in EM metamaterials [15] and AMMs [16] at long wavelengths. Nonlocal homogenization approaches were formulated to derive broadband effective EM parameters by averaging the response of an appropriate distribution of sources [17] or by using the CPA method in the long-wavelength regime [18].

Here, we study the special case of heterogeneous two-phase media with a motionless nondeformable solid and a viscothermal fluid. The medium forms a connected fluid phase which is the seat of the wave propagation. At long wavelengths $\lambda \gg L$, the standard approach to define an effective medium and compute its effective properties is based on the two-scale asymptotic method of homogenization for periodic media (see, e.g., [19, Appendix A]). We show that, at the leading order, the results corresponding to this approach can be directly found from the simplifying assumption to take partially temporal dispersion effects into account, but unreservedly ignore spatial dispersion effects [20]. It means that the field dynamics at one location retains (partially) a memory of the field values at the same location, but is not affected by the neighboring values. In other words, the medium behaves locally in space by the virtue of which this approach is called here "local theory." In the absence of the spatial dispersion, we clarify that this theory presents a truncated scheme that can only be applied to the materials without widely different characteristic pore sizes. Therefore, it cannot describe the behavior of AMMs since exhibiting local resonances at long wavelengths in rigid solid/fluid media requires widely different characteristic pore sizes to be involved in the building block, e.g., materials made of Helmholtz resonators [16,21]. This approach has been extended to describe fluid/solid media with Helmholtz structures, but at the cost of separating the fluid region into different portions, in which different asymptotic expansions and rescaling are performed [22].

Furthermore and in particular, since the long-wavelength condition must be satisfied within the local approach, this theory fails to describe Bragg scattering in PCs, which is related to the core subject of this paper. To devise a macroscopic theory allowing for both temporal and spatial dispersion, and applicable regardless of the geometry and wavelength sizes, we follow a deep EM analogy within a Maxwellian nonlocal theory. This nonlocal theory is formulated to describe the wave propagation in stationary random media, such that the macroscopic fields are defined through application of ensemble-averaging operators over microscopic fields, either directly or indirectly. The theory can also be applied to the special case of periodic media by considering that the ensemble of realizations is obtained by the random translation of one reference periodic sample. However, this application of the nonlocal approach implies an ambiguity that originates in the particular choice of the periodic structural unit. Indeed, the reference period can be an arbitrary number of the irreducible period. We will refer to these types of media as the oxymoron "periodic random" to emphasize that although these media are random and therefore they are subject to the application of the nonlocal theory, they keep their periodic nature.

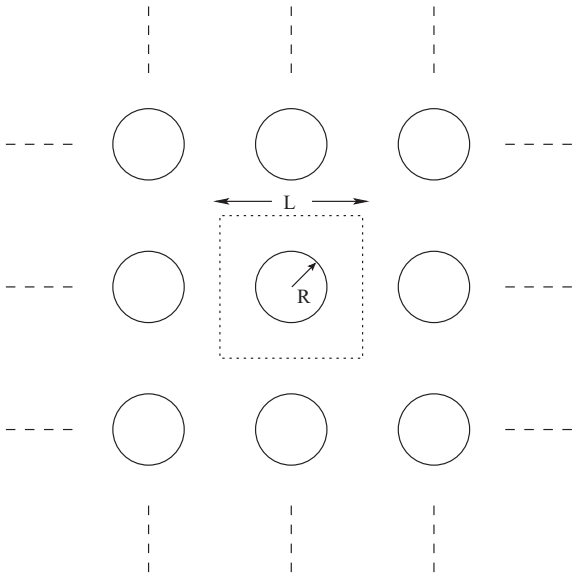


FIG. 1. Two-dimensional array of rigid cylinders with identical radius $R$, embedded in a viscothermal fluid (air). The nearest neighbors in this lattice are distanced with the length $L$. The reference periodic cell considered is shown by the square of length $L$.

For solid/solid stationary-random composites or periodic random media, a general nonlocal form of the macroscopic equations has been proposed by Willis [23]. Within the same effective-medium formulation, effective parameters have been calculated by different techniques, in one dimension (1D) [24], two dimensions (2D) [25], and three dimensions (3D) [26,27]. Interestingly, it turns out that the nonlocal Maxwellian description of fluid/rigid media that we discuss here involves equations of the same form as those of Willis, but without the coupling terms. The absence of these coupling terms is due to the different ways of defining the macroscopic fields from microscopic fields, although both schemes have nonasymptotic character and employ the ensemble-averaging concept.

In this paper, using the macroscopic equations of the Maxwellian nonlocal theory as well as a corresponding action-response homogenization method to compute the constitutive nonlocal operators, we describe accurately 1D nonlocal dynamics of PCs composed of two-dimensional periodic array of rigid cylinders permeated by air as a viscothermal fluid (Fig. 1). We demonstrate that the nonlocal approach enables us to characterize the material, that is regarded as the ensemble of random translations of one periodic sample, through a complex density and bulk modulus, or effective phase velocity and impedance, of an effective fluid in a broadband regime including the high-frequency range where the wavelength is as small as the periodicity of the PC ($\lambda \simeq L$). We refer to this effective fluid as a *phononic fluid*, in contrast to metafluids whose properties are based on local resonances. The validity and precision of the calculations are verified when the results, based on the effective-medium theory, are compared with those produced by a completely different direct Bloch wave approach (DBA) treated by multiple scattering (MS) method that incorporates the viscothermal losses. Comparison of the

phase velocity of the least-attenuated mode computed by nonlocal theory via the action-response problems, with that obtained by the source-free DBA through calculation of Bloch wave number, shows remarkable agreement between the two schemes.

The results related to the local approach or two-scale asymptotic homogenization method are also provided to present the validity domain of the latter. It is important to note that the nonlocal theory is able to predict, for a given frequency, several modes propagating and attenuating in the material, with each of which different frequency-dependent effective parameters are associated. Here, as mentioned above, we are investigating only the material properties related to the least-attenuated mode; the higher-order modes will be studied in a forthcoming paper. To the best of our knowledge, this is the first time that a dissipative phononic (sonic) crystal is precisely characterized by its effective properties in a large frequency range extending over Bragg's regime covering the entire first and second Brillouin zones, thereby the concept of phononic fluids is introduced. This is a breakthrough step towards bridging the physics of waves in materials at microscale and at homogeneous macrolevel, noting that it has been commonly presumed impossible to achieve an effective medium when the microstructure is not below the scale of measurement.

In the following, Sec. II recalls the microscopic governing equations for the linear acoustics. In Sec. III A, the local and nonlocal approaches of the macroscopic theory are presented in a unified formulation. The action-response problems involving partial differential equations (PDEs) to achieve the effective parameters are reviewed for local theory in Sec. III B and for nonlocal theory in Sec. III C. Section IV is devoted to the calculation of the phase velocity in the PC based on DBA via MS method that accounts for viscous and thermal effects. The results generated by local and nonlocal approaches and DBA are reported in Sec. V, followed by concluding remarks and discussions on future research prospects in Sec. VI.

## II. MICROSCOPIC EQUATIONS

In a heterogeneous rigid solid/fluid system, as the PC represented in Fig. 1, the governing equations consist of bulk-fluid equations and solid/fluid boundary conditions. At the microscopic scale, the linear equations governing the dynamics of small-amplitude disturbances in a homogeneous viscothermal fluid come from linearized balance equations of mass, momentum, and energy, the constitutive relations, and a general state equation of the fluid. These governing equations describe the small deviations of thermodynamic pressure $p$, density $\rho$, temperature $T$, velocity $\boldsymbol{v}$, and entropy $s$ from their rest state $p_0$, $\rho_0$, $T_0$, $\boldsymbol{v}_0 = 0$, and $s_0$, up to the terms of first order. In the framework of classical irreversible thermodynamics [28,29], the two constitutive relations are those of Stokes and Fourier. They are written as

$$\sigma'_{ij} = 2\eta\left(e_{ij} - \tfrac{1}{3}(\boldsymbol{\nabla}\cdot\boldsymbol{v})\delta_{ij}\right) + \zeta(\boldsymbol{\nabla}\cdot\boldsymbol{v})\delta_{ij}, \tag{1a}$$

$$\boldsymbol{q} = -\kappa\boldsymbol{\nabla}T. \tag{1b}$$

Stokes's law is a linear isotropic relation between the components of the viscous shear stress $\sigma'_{ij}$ and strain rate $e_{ij} = \frac{1}{2}(\partial_i v_j + \partial_j v_i)$, where $\delta_{ij}$ is the Kronecker symbol, $\eta$ and $\zeta$ are the first and second viscosity of the fluid. The heat conduction Fourier's law is a corresponding relation between $\boldsymbol{q}$ the heat flow and the temperature gradient, with $\kappa$ the coefficient of thermal conductivity.

Using Stokes's law (1a), the conservation equations of mass, and momentum in the bulk fluid $\mathcal{V}^f$ for a fluid particle yield [30,31]

$$\frac{\partial b}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{v} = 0, \tag{2a}$$

$$\rho_0\frac{\partial \boldsymbol{v}}{\partial t} = -\boldsymbol{\nabla}p + \eta\boldsymbol{\nabla}^2\boldsymbol{v} + \left(\zeta + \frac{1}{3}\eta\right)\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{v}), \tag{2b}$$

where $b \equiv \rho'/\rho_0$, $\rho'$ the density deviation. For convenience, we denote the pressure deviation as well as the absolute pressure by $p$. When we expand the thermodynamic equations of state $\rho = \rho(p,s)$ and $T = T(p,s)$ near the rest state up to the first term, and making use of the thermodynamic identities $(\partial\rho/\partial s)_p = -\rho_0\beta_0/c_p$, $(\partial T/\partial p)^s = \beta_0 T_0/\rho_0 c_p$, $c_0^2 \equiv (\partial p/\partial\rho)^s$ where $c_0$ represents the adiabatic sound speed, $\beta_0 \equiv \rho_0[\partial(1/\rho)/\partial T]_p$ and $c_p \equiv T_0(\partial s/\partial T)_p$ are the coefficients of thermal expansion and the specific heat at constant pressure, that are evaluated at the fluid rest state, then we can write the state equations as $\rho' = (1/c_0^2)p - (\rho_0\beta_0 T_0/c_p)s'$ and $T' = (T_0\beta_0/\rho_0 c_p)p' + (T_0/c_p)s'$, respectively. Omitting $s'$ in the latter equations leads to the expression of the state equation

$$\gamma\chi_0 p = b + \beta_0\tau, \tag{3}$$

where $\chi_0 \equiv \rho_0^{-1}(\partial\rho/\partial p)^s$ is the coefficient of adiabatic compressibility at rest state, $\gamma \equiv c_p/c_v$ the relative specific heats at constant pressure and constant volume, involved in the thermodynamic identity $\gamma - 1 = \beta_0^2 T_0/\rho_0 c_p$, and $\tau$ is a simpler notation for the excess temperature $T'$.

The linearized energy balance equation is reduced to the linear heat transfer equation [30,31] $\rho_0 T_0\partial s'/\partial t = \kappa\boldsymbol{\nabla}^2\tau$, where the left side is obviously the quantity of heat gained per unit volume, and for writing the right side, Fourier's law (1b) is used. This equation combined with the aforestated expression of the state equation $T = T(p,s)$ leads to the following form of energy balance equation:

$$\rho_0 c_p\frac{\partial\tau}{\partial t} = \beta_0 T_0\frac{\partial p}{\partial t} + \kappa\boldsymbol{\nabla}^2\tau, \tag{4}$$

which complements governing equations (2) and (3) in the bulk fluid.

In the (rigid) solid phase region $\mathcal{V}^s$, the energy balance equation is reduced to $\rho^s c_p^s(\partial\tau^s/\partial t) = \kappa^s\boldsymbol{\nabla}^2\tau^s$, where $\rho^s$ is the constant solid density, $\tau^s$ the solid excess temperature, and $\kappa^s$ the solid coefficient of thermal conductivity. On the fluid/solid interface $\partial\mathcal{V}$, we have the conditions of continuity of the excess temperature $\tau = \tau^s$ and the heat flux $\kappa\boldsymbol{\nabla}\tau = \kappa^s\boldsymbol{\nabla}\tau^s$. We admit that the coefficient of thermal conductivity of the solid is much larger than that of the fluid $\kappa^s \gg \kappa$, and the heat capacity at constant pressure of the solid part is much larger than that of the fluid part, i.e., $(1-\phi)\rho^s c_p^s \gg \phi\rho_0 c_p$, where $\phi$ is the fluid filling fraction (porosity). The latter assumptions combined with the Fourier heat diffusion in the solid, and the temperature and heat flux continuity relations generally result in the vanishing of the fluid excess temperature at the fluid/solid boundaries. In addition, we assume no-slip

condition on the fluid/(rigid) solid interface. The boundary conditions for the velocity and excess temperature on $\partial\mathcal{V}$ are finally written as

$$\boldsymbol{v} = \boldsymbol{0}, \quad \tau = 0. \tag{5}$$

Equations (2), (3), and (4) with boundary conditions (5) establish the governing microscale equations on the field variables $\boldsymbol{v}$, $b$, $p$, and $\tau$.

## III. LOCAL AND NONLOCAL THEORIES: PHONONIC FLUIDS

We summarize here the Maxwellian local and nonlocal macroscopic acoustics associated with a given macroscopically homogeneous fluid/rigid random medium. Then, the local and nonlocal action-response problems for determining the effective-medium parameters within the respective schemes are stated.

We consider that the medium occupies the whole space and is assumed to be macroscopically homogeneous in an ensemble-averaged sense. We imagine that we are given infinite number of samples $\boldsymbol{\omega}$ of the medium from a probability space $\boldsymbol{\Omega}$, the ensemble of which defines the homogeneous macroscopic medium. In each realization $\boldsymbol{\omega}$, the medium is composed of two regions: the void (pore) region $\mathcal{V}^f(\boldsymbol{\omega})$ which is a connected region permeated by the fluid, and the complementary solid-phase region $\mathcal{V}^s(\boldsymbol{\omega})$; the pore-wall region or fluid/(rigid) solid interface is denoted by $\partial\mathcal{V}(\boldsymbol{\omega})$. The characteristic function of the pore region is defined by

$$I(\boldsymbol{r};\boldsymbol{\omega}) = \begin{cases} 1, & \boldsymbol{r} \in \mathcal{V}^f(\boldsymbol{\omega}) \\ 0, & \boldsymbol{r} \in \mathcal{V}^s(\boldsymbol{\omega}). \end{cases} \tag{6}$$

Microscopically, the fields are of the form $a(t,\boldsymbol{r};\boldsymbol{\omega})$ and the dynamics of the system is governed by Eqs. (2)–(5) with $\mathcal{V}^f$, $\mathcal{V}^s$, and $\partial\mathcal{V}$ replaced by $\mathcal{V}^f(\boldsymbol{\omega})$, $\mathcal{V}^s(\boldsymbol{\omega})$, and $\partial\mathcal{V}(\boldsymbol{\omega})$. The ensemble-average operation at position $\boldsymbol{r}$ is denoted equivalently by $\langle\ldots\rangle(\boldsymbol{r})$ or $\langle\ldots(\boldsymbol{r})\rangle$, that gives the expectation value of the microfield at the same position. For instance, $\langle I\rangle(\boldsymbol{r}) \equiv \langle I(\boldsymbol{r})\rangle$ is the porosity $\phi$, giving the probability that the position $\boldsymbol{r}$ lies in the fluid, over an infinite number of realizations. Since we assume the solid motionless and thermally inert [Eqs. (5)], all microscopic fields $a(t,\boldsymbol{r};\boldsymbol{\omega})$ that specify the fluid motion in $\mathcal{V}^f(\boldsymbol{\omega})$ can be by convention extended to be zero in the solid $\mathcal{V}^s(\boldsymbol{\omega})$. The macroscopic mean $A(t,\boldsymbol{r})$ of the field $a(t,\boldsymbol{r};\boldsymbol{\omega})$ is defined through $A(t,\boldsymbol{r}) = \langle a(t,\boldsymbol{r};\boldsymbol{\omega})\rangle$.

We study here the case of periodic random media with the 2D PC depicted in Fig. 1 as the reference configuration. The random PC is the ensemble of realizations obtained by random translation of this reference configuration in the $x$ and $y$ directions. Thus, $\boldsymbol{\omega}$ can be regarded here as $(\omega_x,\omega_y)$ with $\omega_x$ and $\omega_y$ random variables uniformly and independently distributed in $[-L/2,L/2]$. The characteristic function of the pore region is then interpreted as $I(\boldsymbol{r};\boldsymbol{\omega}) = I(\boldsymbol{r}-\boldsymbol{\omega}) = 1$ if $\boldsymbol{r}-\boldsymbol{\omega}$ is in the fluid region, and equals 0 if $\boldsymbol{r}-\boldsymbol{\omega}$ is in the solid region of the reference configuration.

We analyze acoustic waves propagating perpendicular to the cylinders in the direction of principal $x$ axis, connecting the nearest neighbors in the lattice, whose unit vector is $\boldsymbol{e}_x$. In what follows, as the macroscopic nonlocal theory is presented in the form of Maxwellian acoustics, we present the macroscopic equations of local theory in a Maxwellian form as well, in order to compare the two frameworks and clarify further their respective properties. Also, ensemble-average conception is employed for both nonlocal and local schemes [32].

### A. Macroscopic equations: Local and nonlocal

The macroscopic or effective-medium equations in local and nonlocal approaches include field equations that are general and valid for all media and constitutive relations involving effective parameters, here effective density and bulk modulus operators, that are specific to each medium depending on the type of the fluid and the geometry of the solid structure. These equations are written in analogy to Maxwell equations in electrodynamics. The macroscopic condensation and velocity are defined as the direct ensemble averages $B = \langle b\rangle$ and $\boldsymbol{V} = \langle\boldsymbol{v}\rangle$, where $\boldsymbol{V} \equiv V\boldsymbol{e}_x$ in the present 1D macroscopic longitudinal wave propagation along the principal $x$ axis.

Since the velocity vanishes on the cylinder walls, the following direct commutation relation between averaging and divergence operators holds: $\langle\boldsymbol{\nabla}\cdot\boldsymbol{v}\rangle = \boldsymbol{\nabla}\cdot\langle\boldsymbol{v}\rangle = \boldsymbol{\nabla}\cdot\boldsymbol{V} = \partial V/\partial x$ [33]. Thus, Eq. (2a) is directly averaged to yield

$$\frac{\partial B}{\partial t} + \frac{\partial V}{\partial x} = 0. \tag{7}$$

This equation is the counterpart of the EM equation $\partial\boldsymbol{B}/\partial t + \boldsymbol{\nabla}\times\boldsymbol{E} = 0$, that is obtained in the same direct way from its respective microscopic equation. We name "Lorentz" fields, the two quantities $\boldsymbol{V}$ and $B$, that are, like their EM counterparts $\boldsymbol{E}$ and $\boldsymbol{B}$, the direct averages of the microscale fields. We note that the Lorentz fields are all true tensors, i.e., tensors of weight zero [34], when we write the above EM equation in a tensorial form and make appear the antisymmetric second-order tensor $B_{ij}$, such that, e.g., $B_{12} = B_3$. Here, the acoustic analog of $B_{ij}$ is a scalar because we study the propagation along a single principal axis.

The averaged form of Eq. (2b) will be written in a formal indirect manner. We rewrite the starting microscale equation (2b) in the form

$$\rho_0\frac{\partial\boldsymbol{v}}{\partial t} = -\chi_0^{-1}\boldsymbol{\nabla}b + \boldsymbol{j}, \tag{8}$$

where $\boldsymbol{j}$ is introduced as

$$\boldsymbol{j} = -\boldsymbol{\nabla}p + \chi_0^{-1}\boldsymbol{\nabla}b + \eta\boldsymbol{\nabla}^2\boldsymbol{v} + \left(\zeta + \tfrac{1}{3}\eta\right)\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{v}) \tag{9}$$

and we average it to get

$$\rho_0\frac{\partial\boldsymbol{V}}{\partial t} = -\chi_0^{-1}\boldsymbol{\nabla}B + \boldsymbol{J}, \tag{10}$$

where the expression

$$\boldsymbol{J} = \langle\boldsymbol{j}\rangle - \chi_0^{-1}\langle b\boldsymbol{\nabla}I\rangle \tag{11}$$

is obtained by using the commutation relation $\langle\boldsymbol{\nabla}b\rangle - \boldsymbol{\nabla}\langle b\rangle = -\langle b\boldsymbol{\nabla}I\rangle$ [see Eqs. (51) and (53) in Ref. [35]]. Now, a way to effect formally the mean operation defining $J$ in (11) suggests itself by noting that Eq. (10) is analogous to the following macroscopic EM equation in a material medium in absence of

external charges or electric currents

$$\epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} = \mu_0^{-1} \nabla \times \boldsymbol{B} - \boldsymbol{J}, \tag{12}$$

where $\epsilon_0$ and $\mu_0$ are the electric and magnetic permittivities in vacuum, and $\boldsymbol{J}$ is the EM current associated with the macroscopic motion of the particles, induced by the macroscopic EM field perturbation in the medium. Following Lorentz [36], in macroscopic EM framework, this induced bulk current is formally decomposed into a temporal derivative and a spatial derivative term: $\boldsymbol{J} = \partial \boldsymbol{P}/\partial t + \nabla \times \boldsymbol{M}$, where $\boldsymbol{P}$ and $\boldsymbol{M}$ are EM polarizations. Substituting this equation in (12) yields $\partial \boldsymbol{D}/\partial t = \nabla \times \boldsymbol{H}$, with $\boldsymbol{D} = \epsilon_0 \boldsymbol{E} + \boldsymbol{P}$ and $\boldsymbol{H} = \chi_0^{-1} \boldsymbol{B} - \boldsymbol{M}$. Similarly, in our acoustic context, we assume that the induced bulk force $\boldsymbol{J} = J \boldsymbol{e}_x$ in (11) can undergo the same type of formal decomposition

$$J = \frac{\partial P}{\partial t} + \frac{\partial M}{\partial x}. \tag{13}$$

Substituting the above equation in (10) yields

$$\frac{\partial D}{\partial t} = -\frac{\partial H}{\partial x} \tag{14}$$

with

$$D = \rho_0 \, V - P, \quad H = \chi_0^{-1} \, B - M. \tag{15}$$

The fields $D$ and $H$ thus introduced will be the effective acoustic momentum and effective acoustic pressure at the macroscopic scale. As in EM, we assume the existence of constitutive laws $P = \hat{\chi}_p V$ and $M = \hat{\chi}_m B$, that relate the "polarizations" $P$ and $M$ to the Lorentz fields $V$ and $B$, where $\hat{\chi}_p$ and $\hat{\chi}_m$ are convolution operators, such that in the most general manner, we have $P(t,x) = \int_{-\infty}^{t} dt' \int dx' \chi_p(t - t', x - x') V(t',x')$ and $M(t,x) = \int_{-\infty}^{t} dt' \int dx' \chi_m(t - t', x - x') B(t',x')$. The time invariance and macroscopic spatial homogeneity of the system result in the dependence of the kernels on the differences $t - t'$ and $x - x'$. Then, we are led to complete the field equations (7) and (14) by the following constitutive relations:

$$D = \hat{\rho} \, V, \quad H = \hat{\chi}^{-1} \, B, \tag{16}$$

where $\hat{\rho} = \rho_0 \hat{I} - \hat{\chi}_p$ and $\hat{\chi}^{-1} = \chi_0^{-1} \hat{I} - \hat{\chi}_m$, with $\hat{I}$ the identity operator. Explicitly, these are the *nonlocal constitutive equations*

$$D(t,x) = \int_{-\infty}^{t} dt' \int dx' \rho(t - t', x - x') V(t',x'), \tag{17a}$$

$$H(t,x) = \int_{-\infty}^{t} dt' \int dx' \chi^{-1}(t - t', x - x') B(t',x') \tag{17b}$$

stating that the fields $D$ and $H$ at a given time $t$ and position $x$ depend on the fields $V$ and $B$ at all previous time and all points of the space.

We assert that the EM fields $\boldsymbol{H}$, $\boldsymbol{D}$, $\boldsymbol{P}$, and $\boldsymbol{M}$, and acoustic fields $H$, $D$, $P$, and $M$, that are tensor densities of weight $-1$ [37] (substituting for $\boldsymbol{H}$ and $\boldsymbol{M}$ the corresponding antisymmetric tensors), further differ from the Lorentz fields in that they are *not* the direct average of corresponding microscale fields. This will be justified below in the acoustical case. In what follows, these are called "Maxwell" fields to be distinguished with Lorentz fields.

The constitutive laws express the Maxwell fields in terms of the Lorentz fields. The relation (17a) is written in the most general form. Indeed, it is useless to add an extra convolution term to the right-hand side of this equation, for relating $D(t,x)$ to values of the other Lorentz field, i.e., $B(t',x')$. Because the fields $V$ and $B$ are related by the field equation (7), the effect of such an additional term is already incorporated in (17a) that includes the temporal and spatial dispersion in a general manner. The second constitutive relation (17b) is also written in the most general form. Because of Eq. (7), there is no need to add an extra convolution term to the right-hand side of Eq. (17b) for connecting $H(t,\boldsymbol{r})$ with the values of $V(t',\boldsymbol{r}')$. Similar types of arguments can be found in [38], Sec. 103, in the context of electrodynamics. We note that the additional terms, which are not required in our framework, are of the same nature as the Willis coupling terms [23], which relate acoustic mean momentum (here, $D$) to mean strain (here, $B$) and, also, acoustic mean stress (here, $H$) to mean velocity. That is, if we wanted to consider the Willis coupling terms in the structure of our equations, it turns out that they would be set to zero.

The general equations (17) in the Fourier space are written as

$$D(\omega,k) = \rho(\omega,k) \, V(\omega,k), \tag{18a}$$

$$H(\omega,k) = \chi^{-1}(\omega,k) \, B(\omega,k) \tag{18b}$$

provided that

$$\rho(t,x) = \int \frac{d\omega}{2\pi} \frac{dk}{2\pi} \, \rho(\omega,k) \, e^{-i\omega t + ikx},$$

$$\chi^{-1}(t,x) = \int \frac{d\omega}{2\pi} \frac{dk}{2\pi} \, \chi^{-1}(\omega,k) \, e^{-i\omega t + ikx}.$$

Because of the medium homogeneity with respect to time and space, $D(\omega,k)$ is related to $V(\omega,k)$ and $H(\omega,k)$ is related to $B(\omega,k)$ for the same values of $\omega$ and $k$.

Now, while Eqs. (9), (11), and (13) uniquely fix the induced density field $J$, they are not determining the related polarization fields $P$ and $M$ independently. Thus, the fields $H$ and $D$ are also defined ambiguously, as yet. To fix all Maxwell fields, and thereby also the operators, we need an additional condition. Based on physical considerations inspired by the EM analogy and the thermodynamic concept of generalized susceptibilities [38,39], we postulated [35] that the field $H$ should be identified by the *acoustic part of energy current density* (or acoustic energy flux) $\boldsymbol{S} = S \boldsymbol{e}_x = \langle p \boldsymbol{v} \rangle$, by setting the "Poynting-Schoch" energetic relation as follows:

$$\langle p \boldsymbol{v} \rangle = H \langle \boldsymbol{v} \rangle. \tag{20}$$

The vector $\boldsymbol{S} = H \boldsymbol{V}$ plays the role of an acoustic Poynting vector analogous to its counterpart in EM. This relation (20) gives the relevant "macroscopic part $H$" in the microscale pressure field $p$. As $p$ is the thermodynamic excess pressure, and $p\boldsymbol{v}$ is interpreted as the acoustic energy flux, it may be viewed as a thermodynamic relation. This relation supports our previous assertion that, unlike the Lorentz fields, the Maxwell field $H$ is generally not the direct average of a corresponding

microscale field. In particular, $H$ is not exactly the mean pressure in the fluid $\langle p\rangle/\phi$.

By (indirectly) fixing the field $H$, the condition (20) also (indirectly) fixes all other Maxwell fields. Indeed, $M$ derives from $H$ and $B$, using the second equation of (15); $P$ derives from $M$ and $J$, using Eq. (13); and, finally, $D$ derives from $P$ and $V$ by the first equation of (15). The nonlocal relations (17) completed by the definition (20) provide a coherent framework to take fully into account the temporal and spatial dispersion.

As in EM, the spatial dispersion effects can be very small in the long-wavelength limit. In that case, the nonlocal constitutive relations can be practically indistinguishable from *local constitutive relations*, that are expressed as

$$D(t,x) = \int_{-\infty}^{t} dt' \rho(t-t')\, V(t',x), \tag{21a}$$

$$H(t,x) = \int_{-\infty}^{t} dt' \chi^{-1}(t-t')\, B(t',x). \tag{21b}$$

We see in the above relations that temporal dispersion effects are taken into account in the sense that the fields $D$ and $H$ at a given position $x$ and time $t$ depend on the history of the fields $V$ and $B$ at the same position. However, as we will explain below, the temporal dispersion is admitted in a limited way and is purely linked with the viscous and thermal losses. The time invariance of the system results in the dependence of the density and bulk modulus kernels on the time difference $t-t'$. Therefore, we can write Eqs. (21) in Fourier space as

$$D(\omega,x) = \rho(\omega)\, V(\omega,x), \tag{22a}$$

$$H(\omega,x) = \chi^{-1}(\omega)\, B(\omega,x) \tag{22b}$$

provided that

$$\rho(t) = \int \frac{d\omega}{2\pi}\, \rho(\omega)\, e^{-i\omega t},$$

$$\chi^{-1}(t) = \int \frac{d\omega}{2\pi}\, \chi^{-1}(\omega)\, e^{-i\omega t}.$$

The above relations correspond to an approximate modeling in the long-wavelength regime, which is meaningful to consider only when the geometries are sufficiently simple, without the involvement of very different pore sizes. This simplification makes the formalism fail to describe local-resonance behavior in the medium because it assumes that, in the limit $\lambda \gg L$, the fluid motion at the pore scale can be viewed divergence free:

$$\boldsymbol{\nabla}\cdot\boldsymbol{v} = \mathbf{0}. \tag{24}$$

In other terms, the microscopic divergence is assumed to be on the order of the macroscopic divergence, which is taken to be zero in the limit $\varepsilon \equiv L/\lambda \to 0$. In fact, as spatial nonlocality is simply ignored, the time nonlocality is not completely described, in the sense that the latter originates only in dissipative processes that occur with delays. Indeed, if we remove the viscothermal losses and assume local behavior so that the fluid is incompressible at the pore scale [Eq. (24)], then the response of the fluid to an excitation should be instantaneous. Thus, the density and compressibility kernels become proportional to the Dirac delta: $\rho(t-t') = \rho_0\alpha_\infty\delta(t-t')$ and $\chi^{-1}(t-t') = \chi_0^{-1}\delta(t-t')$, where the geometric constant $\alpha_\infty \geqslant 1$ is known as tortuosity [40], which describes an apparent increase in the inertia of the incompressible ideal fluid that is forced into the tortuous pore network. Therefore, in this case, no temporal dispersion manifests. This demonstrates that the dispersion within the limit of Eqs. (21) is wholly linked to the losses.

In presence of the losses, the simplifying assumption that in the long-wavelength limit the fluid appears as incompressible at microscale enables the separation of viscous/inertial and thermal/elastic effects. Hence, according to local theory, the viscous and inertial effects are encoded in the frequency-dependent effective density $\rho(\omega)$ [40], and the thermal and elastic effects are described by the effective bulk modulus $\chi^{-1}(\omega)$ [19]. It can be shown that, when the frequency is considered as a complex quantity, because of the fluid incompressibility, the poles and zeros of these functions are on the negative imaginary axis of the frequency [19,40,41]. On the real frequency axis, this leads to monotonic variations of these functions, excluding in particular resonant behaviors [42,43] and expressed by simple and robust models of $\rho(\omega)$ and $\chi^{-1}(\omega)$, in terms of certain geometrical parameters [44].

To elucidate further why the fluid incompressibility at the pore scale requires a simple material geometry, suppose that we want to estimate the order of magnitude of the fluid divergence $\boldsymbol{\nabla}\cdot\boldsymbol{v}$ at microlevel. Let $v$ be a characteristic amplitude of the velocity. Since the geometry is assumed to be simple, the period $L$ is also a valid estimate of the characteristic pore length. While, for a general compressible fluid motion, the magnitude of the microlevel divergence can be estimated as $v/L$, we know that in our system the correct order of the magnitude of this quantity should be $v/\lambda$. As the order of magnitude of $v/\lambda$ relative to $v/L$ is $\varepsilon(=L/\lambda)$, and the local asymptotic approach is in the limit $\varepsilon \to 0$, it is clear that the fluid moves in an incompressible manner at the pore scale [Eq. (24)]. Likewise, to estimate the order of magnitude of the fluid pressure gradient $\boldsymbol{\nabla}p$ at microlevel, let $p$ be a characteristic amplitude of the pressure represented in the form of $p = P + \delta p$, where $P \equiv \langle p\rangle/\phi$ (not to be confused with polarization field with the same notation $P$) is the mean fluid pressure that varies at the macroscopic scale, and $\delta p$ is a pressure deviation with zero mean value. Therefore, we have $\boldsymbol{\nabla}p \sim P/\lambda + \delta p/L$. If the fluid is compressible, $\delta p \sim P$ and $\boldsymbol{\nabla}p \sim P/L$, while in our system $\boldsymbol{\nabla}p \sim P/\lambda$. This means that $\delta p/L \sim P/\lambda$, i.e., the deviation amplitude $\delta p$ compared to the mean value $P$, is very small, of the order $\varepsilon$. Consequently, in the long-wavelength limit $\varepsilon \to 0$, there is no gradient for the pressure (and its time derivative); the pressure profile can be regarded as uniform at the pore scale

$$\boldsymbol{\nabla}(\partial p/\partial t) = 0. \tag{25}$$

The above equation will be used later, in particular, for the evaluation of $\chi^{-1}(\omega)$ in local theory. Finally, we note that, in local theory, because the pressure deviation is negligibly small $\delta p \sim \varepsilon P \ll P$, in Eq. (20), $p$ can be replaced by its mean value in the fluid $P = \langle p\rangle/\phi$ and extracted from the average operator. This immediately leads, in this special case, to the identification $H = \langle p\rangle/\phi = P$.

In the above discussion, we have interpreted the local constitutive relations as if there is not any difference between

the cell period $L$ and the characteristic pore lengths. Only with this feature, the application of the two-scale asymptotic homogenization method is justified. In general, when widely different characteristic pore lengths are present, the scale separation parameter $\varepsilon$ becomes ill defined owing to the arbitrariness in the choice of microlevel characteristic length. In that case, Helmholtz structures exhibiting local resonances may appear in the medium, and the nonlocal description will be required in general [16]. Another case requiring the nonlocal description is when the long-wavelength condition $\lambda \gg L$ is no longer satisfied, meaning that the fluid motion is no longer divergence free at the pore scale and, in particular, Bragg scattering may appear.

Contrary to the approximate local framework, the general relations allowing for spatial dispersion provide, at the same time, the correct and untruncated description of temporal dispersion. Particularly, in a lossless medium, the temporal dispersion effects do not completely disappear, that is, the fluid does not respond instantly to an excitation, due to its compressible motion at the pore scale [45, Sec. SI].

In summary, the definitions of the macroscopic Lorentz fields based on microscopic fields, combined with the Lorentz and Maxwell field equations (7) and (14), along with either the constitutive local relations (21) or nonlocal relations (17), and finally the Poynting thermodynamic relation of acoustic energy flux (20), establish a closed form, uniquely defined system within local approach or nonlocal approach, respectively. However, only the nonlocal system takes the full account of the microscopic equations and, therefore, applies without restrictions on geometries and frequencies. In the following, we will present the recipes to obtain the local effective functions $\rho(\omega)$ and $\chi^{-1}(\omega)$ involved in Eqs. (22), and the nonlocal effective functions $\rho(\omega,k)$ and $\chi^{-1}(\omega,k)$ introduced in Eqs. (18), based on the knowledge of microscale properties.

### B. Determination of local effective parameters

The procedure to obtain effective properties of the medium in local theory derives based on the two assumptions that there is a scale separation ($\lambda \gg L$), and also the fluid motion may be viewed as divergence free at the microlevel. Based on our previous considerations, we can directly write the two independent action-response problems, the solution of which determines the frequency-dependent density and bulk modulus. Hereafter, for convenience, we omit systematically the symbol Re[...], in writing the fields in the form Re[$fe^{-i\omega t}$].

To compute the local effective density $\rho(\omega)$ for a given real-value frequency, we consider the following action-response problem. Coherent with the assumption of the fluid incompressibility at microscale, when a harmonic bulk force $\boldsymbol{F}(t) = \boldsymbol{F}_0 e^{-i\omega t}$, where $\boldsymbol{F}_0 = F_0 \boldsymbol{e}_x$, with constant $F_0$, is applied to the fluid or, equivalently, when a uniform harmonic macroscopic pressure drop $-\boldsymbol{\nabla} P(t) = \boldsymbol{F}_0 e^{-i\omega t}$ is applied, we need to solve the following system in each realization $\boldsymbol{\omega}$ (i.e., each random positioning in space without rotation of the PC sketched in Fig. 1):

$$\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0, \tag{26a}$$

$$-i\omega\rho_0 \boldsymbol{v} = -\boldsymbol{\nabla} p + \eta \boldsymbol{\nabla}^2 \boldsymbol{v} + \boldsymbol{F}_0 \tag{26b}$$

in $\mathcal{V}^f(\boldsymbol{\omega})$,

$$\boldsymbol{v} = 0 \tag{27}$$

at $\partial\mathcal{V}(\boldsymbol{\omega})$, where the fields are the amplitudes of the response solutions in the form of $\boldsymbol{v}(t,\boldsymbol{r};\boldsymbol{\omega}) = \boldsymbol{v}(\omega,\boldsymbol{r};\boldsymbol{\omega})e^{-i\omega t}$ and $p(t,\boldsymbol{r};\boldsymbol{\omega}) = p(\omega,\boldsymbol{r};\boldsymbol{\omega})e^{-i\omega t}$. The local theory's characteristic assumption (26a) leads to the Laplacian form of the viscous terms in Eq. (26b) and is consistent with $\boldsymbol{F}_0$ treated as a spatial constant. Indeed, in the above action-response problem, the sum $-\boldsymbol{\nabla} p + \boldsymbol{F}_0$ embodies $-\boldsymbol{\nabla} p$ in the source-free wave propagation problem. In line with our previous discussion on the fluid incompressibility in local theory (Sec. III A), $\boldsymbol{F}_0$ and $p$ correspond to $-\boldsymbol{\nabla} P$ and $\delta p$, respectively. As we neglect spatial dispersion, i.e., the dependence of the medium properties on the spatial inhomogeneity of the macroscopic acoustic fields, it is consistent to treat $\boldsymbol{F}_0$ as a spatial (pore) constant.

We can find unique amplitude fields $\boldsymbol{v}(\omega,\boldsymbol{r};\boldsymbol{\omega})$ and $p(\omega,\boldsymbol{r};\boldsymbol{\omega})$, that are response solutions to Eqs. (26) and (27). In our periodic PC, these solutions are periodic with period $L$ in the $x$ direction [46]. Equations (26) and (27) in the reference configuration can be obtained by the aforementioned two-scale asymptotic homogenization method at the leading order of the asymptotic expansions [47,48]. Averaging the response field $\boldsymbol{v}(\omega,\boldsymbol{r};\boldsymbol{\omega})$ over the realizations, the local density for the effective fluid is given by

$$\rho(\omega) = -\frac{F_0}{i\omega\, V(\omega)}. \tag{28}$$

We note that, owing to the construction of the ensemble, exactly the same mean value $V(\omega)$ is obtained by solving (26) and (27) in one single realization, and then, volume averaging the response velocity in one unit cell; this is the procedure within the aforementioned standard homogenization.

To compute the local effective bulk modulus $\chi^{-1}(\omega)$ at a real $\omega$, we apply an excitation in the form of a heating rate at constant pressure $\dot{Q}(t) = \dot{Q}_0 e^{-i\omega t}$, where $\dot{Q}_0$ is a constant or, equivalently, the material is subject to a uniform time harmonic pressure, such that $\beta_0 T_0 \partial p/\partial t = \dot{Q}_0 e^{-i\omega t}$. This results in the following action-response problem for the amplitude of the excess temperature field $\tau(t,\boldsymbol{r};\boldsymbol{\omega}) = \tau(\omega,\boldsymbol{r};\boldsymbol{\omega})e^{-i\omega t}$:

$$-i\omega\rho_0 c_p \tau = \kappa \boldsymbol{\nabla}^2 \tau + \dot{Q}_0 \tag{29}$$

in $\mathcal{V}^f(\boldsymbol{\omega})$,

$$\tau = 0 \tag{30}$$

at $\partial\mathcal{V}(\boldsymbol{\omega})$. There is a unique amplitude field $\tau(\omega,\boldsymbol{r};\boldsymbol{\omega})$ response solution to Eqs. (29) and (30), which is $L$ periodic. This action-response problem in the reference configuration can also be obtained at the leading order within the classical asymptotic homogenization [19]. It is based on the physical assumption that the pressure field is a slowly variable quantity that may be viewed in first approximation as equal to the mean pressure. This assumption is incorporated in (29) in the very fact that $\dot{Q}_0$ is taken as a spatial constant. In fact, $\dot{Q}_0$ embodies the term $\beta_0 T_0 \partial p/\partial t$ in the wave propagation problem, and it is consistent to treat $\dot{Q}_0$ as a spatial constant: as we saw earlier, the divergence-free nature of the motion leads to $\beta_0 T_0 \boldsymbol{\nabla}(\partial p/\partial t) = 0$ [Eq. (25)]. Once the solution field $\tau$ is found, the factor $\rho'$ analogous to the previous $\rho$ is given as $\rho'(\omega) = -\dot{Q}_0/i\omega T(\omega)$, where $T = \langle\tau\rangle$. In the framework of

the local theory, the following direct relation exists between the two functions $\rho'$ and $\chi$: $\chi(\omega) = \phi\chi_0[\gamma - (\gamma-1)\rho_0 c_p/\rho'(\omega)]$ [45, Sec. SII]. Thus, the local bulk modulus for the effective-fluid medium is obtained as

$$\chi^{-1}(\omega) = \phi^{-1}\chi_0^{-1}\left[\gamma + (\gamma-1)\frac{i\omega\rho_0 c_p T(\omega)}{\dot{Q}_0}\right]^{-1}. \tag{31}$$

Again, the ensemble average $\langle \dots \rangle$ can be performed here by solving in one single realization and, then, volume averaging the response temperature in one unit cell; this is the procedure given by the classical homogenization.

In the local effective fluid, for a given frequency $\omega$, there is only one single normal mode that can propagate in the positive-$x$ direction. With this single mode is associated a local wave number $k_l(\omega)$ that verifies the following *local dispersion relation*:

$$\rho(\omega)\chi(\omega)\omega^2 = k_l^2 \tag{32}$$

such that $\mathrm{Im}(k_l) > 0$. The frequency-dependent complex phase velocity $c(\omega)$ and complex impedance $Z(\omega)$ are immediately written as

$$c(\omega) = \frac{\omega}{k_l(\omega)}, \quad Z(\omega) = [\rho(\omega)\ \chi^{-1}(\omega)]^{1/2}. \tag{33}$$

### C. Determination of nonlocal effective parameters

The procedure to obtain effective properties of the medium in nonlocal theory can be viewed as a generalization of the preceding local action-response problems, which accounts for the physical fact that, once the fields vary in time, they also necessarily vary in space. Thus, once external fields varying with respect to time as $e^{-i\omega t}$ are introduced, we should consider that they include spatial variations as well. We can extract one given Fourier component $e^{ikx}$ of these spatial variations.

The Fourier coefficients $\rho(\omega,k)$ and $\chi^{-1}(\omega,k)$ in (19) are directly related to the macroscopic (averaged) response of the permeating fluid subjected to a single-component $(\omega,k)$ Fourier pressure term $\mathcal{P}(t,x) = \mathcal{P}_0 e^{-i\omega t+ikx}$ that is added to the pressure, either in the Navier-Stokes equation (2b) to obtain the nonlocal effective density or to the Fourier equation (4) to obtain the nonlocal effective bulk modulus [35]. In the former action-response problem, the excitation performs inhomogeneous (variable in time and space) work per unit volume and time and in the latter it pumps an inhomogeneous amount of heat per unit volume and time. The two systems of equations to be solved in each realization are written as follows. In the fluid region $\mathcal{V}^f(\boldsymbol{\omega})$,

$$\frac{\partial b}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{v} = 0, \tag{34a}$$

$$\rho_0\frac{\partial \boldsymbol{v}}{\partial t} = -\boldsymbol{\nabla}p + \eta\boldsymbol{\nabla}^2\boldsymbol{v} + \cdots + \left(\zeta + \frac{1}{3}\eta\right)\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{v}) + \underbrace{\boldsymbol{F}_0 e^{-i\omega t+ikx}}_{\text{added for determination of density}}, \tag{34b}$$

$$\rho_0 c_p\frac{\partial \tau}{\partial t} = \beta_0 T_0\frac{\partial p}{\partial t} + \kappa\boldsymbol{\nabla}^2\tau + \underbrace{\dot{Q}_0 e^{-i\omega t+ikx}}_{\text{added for determination of bulk modulus}}, \tag{34c}$$

$$\gamma\chi_0 p = b + \beta_0\tau. \tag{34d}$$

On the fluid/solid interface $\partial\mathcal{V}(\boldsymbol{\omega})$,

$$\boldsymbol{v} = 0, \qquad \tau = 0. \tag{35}$$

The excitation amplitudes are written as $\boldsymbol{F}_0 e^{-i\omega t+ikx} = -\boldsymbol{\nabla}\mathcal{P}$ and $\dot{Q}_0 e^{-i\omega t+ikx} = \beta_0 T_0(\partial\mathcal{P}/\partial t)$, with $\mathcal{P} = \mathcal{P}_0 e^{-i\omega t+ikx}$, that are independent of the realization. It is important to emphasize that the excitation variables $\omega$ and $k$ are set as independent variables.

The solutions to the above systems for the response fields $p$, $b$, $\tau$, and components of $\boldsymbol{v}$ take the form $p(t,\boldsymbol{r};\boldsymbol{\omega}) = p(\omega,k,\boldsymbol{r};\boldsymbol{\omega})e^{-i\omega t+ikx}$, and so on, where the amplitudes are periodic functions of $x$ and proportional to the excitation amplitude $\mathcal{P}_0$. However, these solutions are not unique in the sense that the period can be chosen as any integer multiple of the irreducible period $L$. In what follows, we exclude this ambiguity by taking the period as equal to the irreducible period, i.e., requiring the amplitudes of the solutions to be periodic with period $L$. Once the above systems are solved independently in each realization, we use the fundamental relation (20) to write $\mathtt{P}\langle\boldsymbol{v}\rangle = \langle p\boldsymbol{v}\rangle$, where $\mathtt{P}(t,x) = \mathtt{P}(\omega,k)e^{-i\omega t+ikx}$ is the macroscopic part of the pressure response $p(t,\boldsymbol{r};\boldsymbol{\omega})$, whose amplitude is determined by $\mathtt{P}(\omega,k) = \{\langle p(\omega,k,\boldsymbol{r};\boldsymbol{\omega})\boldsymbol{v}(\omega,k,\boldsymbol{r};\boldsymbol{\omega})\rangle\cdot\boldsymbol{e}_x\}/V(\omega,k)$.

For determination of $\rho(\omega,k)$, we use the Fourier transform of Eq. (14). Applying Eq. (18a), and postulating that the addition of the two parts $\mathtt{P}$ and $\mathcal{P}_0$ establishes the field $H$, gives rise to the *nonlocal density of the phononic fluid,* which is valid for any $\omega$, and $k$:

$$\rho(\omega,k) = \frac{k}{\omega V(\omega,k)}[\mathtt{P}(\omega,k) + \mathcal{P}_0]. \tag{36}$$

We note that, owing to the construction of the ensemble, exactly the same mean values $V(\omega,k)$ and $\mathtt{P}(\omega,k)$ are obtained by solving (34) and (35) in one single realization, and then making the averages $\langle\dots\rangle$ by volume integration in the chosen periodic unit cell.

In order to get the response fields $p(\omega,k,\boldsymbol{r})$ and $\boldsymbol{v}(\omega,k,\boldsymbol{r})$ in the reference configuration and perform the cell averages, we explicitly solve in the reference unit cell the PDEs relating only the amplitude fields, that are periodic functions over the unit-cell boundaries. These PDEs are given in [45, Sec. SIII].

For determination of $\chi^{-1}(\omega,k)$ we use the Fourier transform of Eq. (17b). As before, admitting that the field $H$ is built upon adding the two contributions $\mathtt{P}$ and $\mathcal{P}_0$ yields $\mathtt{P}(\omega,k) + \mathcal{P}_0 = \chi^{-1}(\omega,k)B(\omega,k)$. Here, the field $B$ has yet to be identified based on microscale dynamics. We postulate that it is composed of two parts $B = \mathtt{B} + \mathcal{B}_0$: one nonisothermal response part $\mathtt{B}(\omega,k) \equiv \langle b(\omega,k,\boldsymbol{r};\boldsymbol{\omega})\rangle$ that originates in the field $b(t,\boldsymbol{r};\boldsymbol{\omega}) = b(\omega,k,\boldsymbol{r};\boldsymbol{\omega})e^{-i\omega t+ikx}$ of the action-response problem (34) and (35), and the other is an isothermal constant contribution, that can be directly written by averaging the isothermal term $\gamma\chi_0\mathcal{P}_0$, i.e., $\mathcal{B}_0 \equiv \langle\gamma\chi_0\mathcal{P}_0\rangle = \phi\gamma\chi_0\mathcal{P}_0$, where $\phi$ is the porosity of the PC. The construction of $B$ in such a manner by these two independent parts has been suggested through following the operation of the theory in a homogeneous viscothermal fluid without any solid inclusions [35]. Therefore, the *nonlocal bulk modulus of the phononic*

*fluid* is expressed as

$$\chi^{-1}(\omega,k) = \frac{1}{\mathrm{B}(\omega,k)+\mathcal{B}_0}[\mathrm{P}(\omega,k)+\mathcal{P}_0]. \quad (37)$$

Averaging the amplitudes $v(\omega,k,\boldsymbol{r};\boldsymbol{\omega})$ and $b(\omega,k,\boldsymbol{r};\boldsymbol{\omega})$ or product of amplitudes $p(\omega,k,\boldsymbol{r};\boldsymbol{\omega})v(\omega,k,\boldsymbol{r};\boldsymbol{\omega})$ can be performed equivalently over the unit-cell average in the reference configuration. Here also, in order to get the response fields, we solve the PDEs relating only the amplitude fields [45, Sec. SIII], that are periodic functions over the unit-cell boundaries. Contrary to the case of local theory, here, since we take into account spatial dispersion, several normal-mode solutions might exist, with fields varying as $e^{-i\omega t+ikx}$. Solutions must satisfy the following *nonlocal dispersion equation*:

$$\rho(\omega,k)\chi(\omega,k)\omega^2 = k^2 \quad (38)$$

which is easily derived from the Maxwellian macroscopic equations. That is, with each frequency $\omega$, several normal-mode complex wave numbers $k_n(\omega)$, $\mathrm{Im}(k_n) > 0, n = 1,2,\ldots$, solution to the nonlocal dispersion equation (38), may be associated. Furthermore, with each wave number $k_n$ are associated a frequency-dependent density and bulk modulus, such that

$$\rho_n(\omega) = \rho_n[\omega,k_n(\omega)], \quad \chi_n^{-1}(\omega) = \chi_n^{-1}[\omega,k_n(\omega)]. \quad (39)$$

Therefore, the phase velocity and impedance of the normal mode $n$ are written as

$$c_n(\omega) = \frac{\omega}{k_n(\omega)}, \quad Z_n(\omega) = \left[\rho_n(\omega)\,\chi_n^{-1}(\omega)\right]^{1/2}. \quad (40)$$

The fact that at each frequency $\omega$, we obtain several normal modes propagating and attenuating in the medium, with wave numbers $k_n(\omega)$ (equivalently phase velocities) constants of the medium, and other effective parameters, is a direct consequence of the nonlocal description. The interpretation of these wave numbers is immediate: because our ensemble of random realizations is made of the random translation of one periodic sample, the above wave numbers must coincide with the so-called Bloch wave numbers associated with the periodic geometry. In this paper, we focus only on the least-attenuated mode $n = 1$ and its associated effective parameters. The results produced by the local and nonlocal theories and respective upscaling procedures will be illustrated in Sec. V. They will be evaluated by performing an independent direct computation of the complex wave number (or phase velocity) of the least-attenuated Bloch wave propagating in the reference PC.

## IV. DIRECT BLOCH WAVE APPROACH (DBA)

Here, we aim to obtain the phase velocity of the least attenuated Bloch wave propagating in the 2D fluid/solid reference PC illustrated in Fig. 1, solving directly the source-free microscopic equations (2)–(5). Achieving the Bloch wave number $k_B(\omega)$ as eigenvalue of the medium, through DBA, is fundamentally different from the way we obtain this quantity based on local and nonlocal theories, via Eqs. (32) and (38). These theories *define* in an appropriate manner the effective susceptibilities of the media (effective density, bulk modulus) that concern macroscopic response of a medium to an applied field. Within these theories, procedures are established to determine the way in which the effective density and bulk modulus can be obtained based on microlevel "action-response" problems. Once the effective parameters are obtained, the effective wave numbers can be achieved, thus, in an *indirect* fashion. In contrast, as DBA is not based on a macroscopic theory, it cannot by itself *define*, independently, the effective susceptibilities of the material.

For the simple fluid/solid geometry illustrated in Fig. 1, a precise and relatively simple calculation of the possible Bloch wave numbers $k_n(\omega)$ is feasible by the MS approach [49]. We sketch here the generalization of the MS approach developed in [50] for a lossless host fluid and the same geometry to the present case of a viscothermal fluid. The details of calculations are given in [45, Sec. SIV]. The fluid motion corresponding to the source-free equations (2)–(5) can be described in terms of three velocity potentials: the acoustic potential $\phi^a$, entropic potential $\phi^e$, and vorticity potential $\boldsymbol{\psi}$ such that

$$\boldsymbol{v} = \boldsymbol{\nabla}(\phi^a+\phi^e) + \boldsymbol{\nabla}\times\boldsymbol{\psi}. \quad (41)$$

The vorticity potential $\boldsymbol{\psi}$ has just one component, which is directed along the $z$ axis and is denoted by $\phi^v$. In the harmonic regime, three independent Helmholtz equations

$$[\boldsymbol{\nabla}^2+(k^\alpha)^2]\phi^\alpha = 0, \quad \alpha = a,e,v \quad (42)$$

must be satisfied in $\mathcal{V}^f$, where $(k^\alpha)^2$, $\alpha = a,e,v$, are the squared wave numbers associated with acoustic, thermal, and viscous waves, respectively. The former two $(k^a)^2$ and $(k^e)^2$ are the opposite signs of the small and large solutions $\lambda_1 \equiv \lambda^a$ and $\lambda_2 \equiv \lambda^e$ of Kirchhoff-Langevin's dispersion equation [see Eq. (14) in [51], and the latter is $(k^v)^2 \equiv i\omega/\nu$, where $\nu = \eta/\rho_0$ is the kinematic viscosity. It is easy to express the excess temperature in terms of potentials, for instance, by using (41) and (42) and Eq. (12c) in [51]:

$$\frac{\beta_0\tau}{\gamma-1} = \left(\frac{\kappa}{\rho_0 c_v}+\frac{i\omega}{\lambda^a}\right)^{-1}\phi^a + \left(\frac{\kappa}{\rho_0 c_v}+\frac{i\omega}{\lambda^e}\right)^{-1}\phi^e. \quad (43)$$

The boundary conditions at the solid-fluid interface for the potentials arise from the fact that the displacement $\boldsymbol{u}$ and excess temperature fields vanish on $\partial\mathcal{V}$:

$$\boldsymbol{u} = \boldsymbol{0}, \quad \tau = 0. \quad (44)$$

These boundary conditions establish the relationship between the potentials, such that a wave carried by one potential is scattered in the three types of waves through interacting with the solid cylinders.

In this paper, we are not concerned with the terms of minor importance related to the intrinsic bulk-fluid attenuation. Thus, as a simplification we set $-\lambda^a \equiv (k^a)^2 = (\omega/c_0)^2$, that is, we neglect the damping of the acoustic mode. As it is explained in [45, Sec. SIV], this simplification leads to slowly convergent Hankel series that can, however, be replaced by rapidly convergent Schlömilch series. Moreover, we also neglect the higher-order terms governing the attenuation of entropic waves, and set as another simplification $-\lambda^e \equiv (k^e)^2 = i\omega\rho_0 c_p/\kappa$. Consistent with the first simplification, we have to remove the thermal conductivity term in the first set of parentheses in (43). After straightforward calculation using the thermodynamic identity $\gamma-1 = T_0\beta_0^2 c_0^2/c_p$, the following relation is obtained: $\tau = (T_0\beta_0/c_p)i\omega\phi^a + (\rho_0 c_p/\beta_0\kappa)\phi^e$.

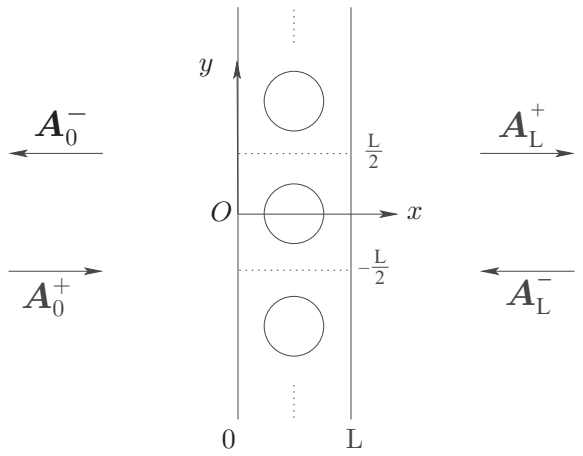


FIG. 2. One row of infinite number of rigid cylinders.

Considering one row containing an infinite number of cylinders, as is shown in Fig. 2, we expand the potentials in terms of right- and left-going plane waves

$$\phi_0^\alpha(\boldsymbol{r}) = \sum_{n=-\infty}^{\infty} \left(A_{0n}^{+\alpha} e^{i\boldsymbol{k}_n^\alpha \cdot \boldsymbol{r}} + A_{0n}^{-\alpha} e^{-i\boldsymbol{k}_n^\alpha \cdot \boldsymbol{r}}\right),$$

$$\phi_L^\alpha(\boldsymbol{r}) = \sum_{n=-\infty}^{\infty} \left(A_{Ln}^{+\alpha} e^{i\boldsymbol{k}_n^\alpha \cdot (\boldsymbol{r}-L\boldsymbol{e}_x)} + A_{Ln}^{-\alpha} e^{-i\boldsymbol{k}_n^\alpha \cdot (\boldsymbol{r}-L\boldsymbol{e}_x)}\right).$$

The ingoing or outgoing of the four types of amplitudes $A$ are illustrated in Fig. 2. The index $\alpha$ refers to the type $a$, $e$, or $v$ of potential field. It is clear that the periodicity of the potential fields with respect to $y$ coordinates implies that for each $n$, the $y$ component of the wave vectors $\boldsymbol{k}_n^\alpha$ must be $k_{ny}^\alpha = 2\pi n/L$, thus, $(k^\alpha)^2 = (k_{nx}^\alpha)^2 + (2\pi n/L)^2$. Another symmetry consideration of the problem is based on the fact that we are interested only with the solutions leading to a fluid motion symmetric around each cylinder. This restriction implies that the fields $\phi^a$ and $\phi^e$ are even functions, and $\phi^v$ an odd function, of $y$ coordinates. Thus, regarding the terms in the above equations, after combining the up and down components $n$ and $-n$, there will appear a $y$ dependence in the form of $\cos(2\pi ny/L)$ for acoustic and entropic potentials, and in the form of $\sin(2\pi ny/L)$ for vorticity potential. To account explicitly for this symmetry in the notation, we replace the above equations by the following condensed form of the potentials:

$$\phi_0^\alpha(\boldsymbol{r}) = \sum_{n=0}^{\infty} \mathcal{C}_n^\alpha(y)\left(A_{0n}^{+\alpha} e^{ik_{nx}^\alpha x} + A_{0n}^{-\alpha} e^{-ik_{nx}^\alpha x}\right),$$

$$\phi_L^\alpha(\boldsymbol{r}) = \sum_{n=0}^{\infty} \mathcal{C}_n^\alpha(y)\left(A_{Ln}^{+\alpha} e^{ik_{nx}^\alpha (x-L)} + A_{Ln}^{-\alpha} e^{-ik_{nx}^\alpha (x-L)}\right),$$

$$\text{where} \quad \mathcal{C}_n^\alpha(y) = \begin{cases} \cos(2\pi ny/L), & \alpha = a,e \\ \sin(2\pi ny/L), & \alpha = v. \end{cases}$$

Also, we note that with each $n$, $\alpha$, and $\omega$, we may associate a characteristic incidence angle $\theta_n^\alpha$, such that $k^\alpha \sin(\theta_n^\alpha) = 2\pi n/L$ and $k^\alpha \cos(\theta_n^\alpha) = k_{nx}^\alpha$. For the acoustic type $\alpha = a$, this angle is real when the frequency is such that $2\pi n/(k^\alpha L) < 1$. It is complex and equal to $\pi/2 - i\xi$ at higher frequencies, with $\xi > 0$ ensuring that $\mathrm{Im}(k_{nx}^\alpha) > 0$. For the entropic and vorticity types, this angle is complex, which is chosen to satisfy $\mathrm{Im}(k_{nx}^\alpha) > 0$. The first step in the calculation is to obtain the reflection and transmission properties of the row of cylinders, or the following scattering matrix, which relates the outgoing waves to the ingoing ones:

$$\begin{pmatrix} \boldsymbol{A}_0^- \\ \boldsymbol{A}_L^+ \end{pmatrix} = \begin{pmatrix} \boldsymbol{T} & \boldsymbol{R} \\ \boldsymbol{R} & \boldsymbol{T} \end{pmatrix} \begin{pmatrix} \boldsymbol{A}_L^- \\ \boldsymbol{A}_0^+ \end{pmatrix}, \tag{47}$$

$$\text{where} \quad \boldsymbol{A}_0^- = \begin{pmatrix} \boldsymbol{A}_0^{-a} \\ \boldsymbol{A}_0^{-e} \\ \boldsymbol{A}_0^{-v} \end{pmatrix}, \qquad \boldsymbol{A}_L^+ = \begin{pmatrix} \boldsymbol{A}_L^{+a} \\ \boldsymbol{A}_L^{+e} \\ \boldsymbol{A}_L^{+v} \end{pmatrix},$$

and so on for the vectors $\boldsymbol{A}_0^+$ and $\boldsymbol{A}_L^-$. Each of the vectors $\boldsymbol{A}_L^{+\alpha}$, $\boldsymbol{A}_0^{+\alpha}$, $\boldsymbol{A}_L^{-\alpha}$, and $\boldsymbol{A}_0^{-\alpha}$ contains the whole ensemble of plane-wave amplitudes with $\alpha = a,e,v$, each of which is indexed by $n$. The reflection and transmission matrices $\boldsymbol{R}$ and $\boldsymbol{T}$, respectively, thus have elements of the type $R_{pn}^{\alpha\beta}$ and $T_{pn}^{\alpha\beta}$, where the indexes on the right refer to ingoing waves and those on the left to outgoing waves. The presence of different elements results from the interactions and transformations of different kinds of potentials into one another, via boundary conditions (44).

To compute $\boldsymbol{R}$ and $\boldsymbol{T}$, and thereby construct the scattering matrix, the analysis of the scattering problem is divided in different elementary parts, that are combined in the end. The detail of the calculation, leading to the expressions of the scattering matrix elements $R_{pn}^{\alpha\beta}$ and $T_{pn}^{\alpha\beta}$, is given in [45, Sec. SIV].

At this point, reflection and transmission properties of one row are entirely determined. Now, we consider an infinite number of rows separated by the distance $L$ (Fig. 1). We make use of the concept of scattering matrix introduced for an arbitrary row, and apply the Bloch condition for this case of periodic medium. We have

$$\begin{pmatrix} \boldsymbol{A}_L^+ \\ \boldsymbol{A}_L^- \end{pmatrix} = e^{ik_B L} \begin{pmatrix} \boldsymbol{A}_0^+ \\ \boldsymbol{A}_0^- \end{pmatrix}, \tag{48}$$

where $k_B$ denotes the Bloch wave number to be determined. The use of scattering-matrix relation (47) and the Bloch condition (48) leads to the following eigenvalue problem:

$$\begin{pmatrix} \boldsymbol{T} & \boldsymbol{R} \\ \boldsymbol{0} & \boldsymbol{I} \end{pmatrix} \begin{pmatrix} \boldsymbol{A}_0^+ \\ \boldsymbol{A}_L^- \end{pmatrix} = e^{ik_B L} \begin{pmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{R} & \boldsymbol{T} \end{pmatrix} \begin{pmatrix} \boldsymbol{A}_0^+ \\ \boldsymbol{A}_L^- \end{pmatrix}, \tag{49}$$

where $\boldsymbol{0}$ and $\boldsymbol{I}$ are the zero and identity matrices, respectively. Since at this stage the reflection and transmission matrices $\boldsymbol{R}$ and $\boldsymbol{T}$ are known, we are able to solve the above eigenvalue problem numerically. In this manner, we get the complex eigenvalues $\mu = e^{ik_B L}$ that determine the possible Bloch wave numbers $k_B$. For each eigenvalue $\mu$ there must be an eigenvalue $\mu^{-1}$ corresponding to the opposite sign of $k_B$, i.e., the reversed direction of propagation. We restrict the solutions to forward propagation by imposing $|\mu| < 1$ and $\mathrm{Im}(k_B) > 0$. Note that the real part of the wave number is defined only modulo $2\pi/L$. Customarily, this indeterminacy issue is resolved by requiring that $-\pi/L < \mathrm{Re}(k_B) < \pi/L$, i.e., the wave number is chosen

TABLE I. Fluid (air) properties used in all computations.

| $\rho_0$ (kg/m$^3$) | $T_0$ (K) | $c_0$ (m/s) | $\eta$ (kg/ms) | $\zeta$ (kg/ms) | $\kappa$ (W/mK) | $\chi_0$ (1/Pa) | $c_p$ (J/kgK) | $\gamma$ |
|---|---|---|---|---|---|---|---|---|
| 1.2 | 293 | 343 | $1.8 \times 10^{-5}$ | $0.6\eta$[a] | $2.6 \times 10^{-2}$ | $7.1 \times 10^{-6}$ | 1005 | 1.4 |

[a]This quantity is ignored in the MS method.

to lie in the "first Brillouin zone". With each frequency $\omega$ there might be associated, in the first Brillouin zone, different mode solutions $k_{B,n}$, $n = 1,2,3,\ldots$, labeled by ascending order of the values of $\mathrm{Im}(k_{B,n})$, and characterized by complex phase velocities

$$c_n(\omega) = \frac{\omega}{k_{B,n}(\omega)}. \tag{50}$$

Here, however, we study the least-attenuated mode $n = 1$, propagating in the positive-$x$ direction, and find it convenient to express its wave number $k_B \equiv k_{B,1}(\omega)$ as a continuous function of frequency, that becomes zero when the frequency tends to zero. The wave number $k_B(\omega)$ that is defined in this manner will not always remain in the first Brillouin zone. As it will be shown in the next section, when the frequency increases sufficiently, the real part of the wave number may be found in the interval $[\pi/L,2\pi/L]$ (or upper), which means that it passes into the second (or higher) Brillouin zone. The same convention will be applied regarding the selection and presentation of the wave numbers in nonlocal theory, where, obviously, the same issues arise.

## V. ILLUSTRATION OF THE RESULTS AND DISCUSSIONS

In this section, we present the results produced by local theory, nonlocal theory, and DBA, concerning the propagation of the least-attenuated mode in 2D PCs made of a square lattice of rigid cylinders embedded in air, that acts as a viscothermal fluid.

For the least-attenuated wave, we have computed the effective macroscopic parameters based on local and nonlocal theories, that include frequency-dependent phase velocity, density, bulk modulus, and impedance. By DBA, based on the quasiexact MS method, the phase velocity of the least-attenuated Bloch wave is calculated. This serves as a measure for the domain of validity of the local and nonlocal effective-medium theories. The results are shown in a large frequency range for the porosity $\phi = 0.9$. With a fixed periodicity $L = 10$ $\mu$m, the radius of the cylinders $R = L[(1-\phi)/\pi]^{\frac{1}{2}}$ is equal to 1.78 $\mu$m. The present topology that we study here does not exhibit local resonances, thus, we expect the local theory to cover long-wavelength ($\lambda \gg L$) frequency band. The nonlocal approach is expected to be valid over the entire frequency band, without any constraint. Beyond the long-wavelength regime, the macroscopic wave and respective properties that are outcomes of the nonlocal theory should be viewed as the ensemble average of the propagation in the ensemble of media that is generated by the random translations of the reference PC illustrated in Fig. 1. Within the long wavelength regime, based on both local and nonlocal approaches, these waves can be equivalently regarded as a result of ensemble averaging, or averaging microfields over the reference unit cell. Finally, the the results provided by DBA concern the propagation of the Bloch wave in the reference PC. This direct approach provides, in principle, the dispersive Bloch wave number (or phase velocity) without defining procedures to obtain effective constitutive parameters.

The fluid properties for all computations are indicated in Table I. One of the objectives of the present theoretical analysis is to describe both viscous and thermal losses precisely from the basic microscale equations, in effective-medium approaches, as well as in DBA. Air, that is taken to be the the host fluid, produces meaningful viscous, as well as thermal, losses. The general thermodynamic identity $\gamma - 1 = \beta_0^2 T_0/\rho_0 c_p$ shows that the deviation of $\gamma$ from unity is a second-order effect on the thermal expansion coefficient $\beta_0$. For air which is a gas, $\beta_0 \simeq 1/T_0$ is not especially small and $\gamma \simeq 1.4$, while for a liquid, like water, $\beta_0$ is close to zero, which implies that $\gamma$ is very close to 1. In this case, the values of the adiabatic bulk modulus $\chi^{-1}_{0(\mathrm{adiab})}$ and isothermal bulk modulus $\chi^{-1}_{0(\mathrm{isoth})}$ are very close to each other since, in general, $\chi^{-1}_{0(\mathrm{adiab})} = \gamma\chi^{-1}_{0(\mathrm{isoth})}$. Therefore, thermal exchanges have practically no effects. Here, because $\gamma - 1$ is of order one, thermal losses can be comparable to viscous losses. Indeed, the thickness of the thermal boundary layer $\delta_t = (2\kappa/\rho_0 c_p \omega)^{\frac{1}{2}}$ is on the same order as the thickness of viscous boundary layer $\delta_v = (2\eta/\rho_0\omega)^{\frac{1}{2}}$.

As regards the small dimension of our unit cell, it has not been chosen for a special use in practice or a particular experimental investigation, but as a proof of concept. In fact, the impact of the dissipation on the results can be shown in a more pronounced way, when the size of the cell is decreased, and thereby the viscous and thermal losses are enhanced in the medium. We note that for a given value of the normalized frequency $\Omega \equiv k_0 L/\pi = \omega L/c_0\pi$, where $k_0$ is the wave number in air, decreasing the scale by a factor $a$, results in decreasing the thickness of boundary layers by only $\sqrt{a}$. Therefore, at a given $\Omega$, a decrease in the size of the structural unit leads to the increase of the unit-cell space occupied by the boundary layers; that is, $\delta_{v,t}/L$ augments like $\sqrt{a}$. Consequently, decreasing the structural size will enforce the viscous and thermal effects, at a given value of normalized frequency.

As it was mentioned before, the local theory that does not allow for the spatial dispersion predicts a single wave propagating in the medium. To obtain the equivalent-fluid parameters according to local theory, we solved by FEM the two sets of equations (26) and (27) and (29) and (30), independently, in a unit cell including a single cylinder (Fig. 1). Then, the frequency-dependent local density $\rho(\omega)$ and local bulk modulus $\chi^{-1}(\omega)$ are given through Eqs. (28)

and (31), respectively. The effective local phase velocity, that is calculated via local dispersion equation (32), is given by the left equation in (33), and the effective local impedance is achieved by the right equation in (33).

To obtain the nonlocal frequency-dependent parameters, the functions $\rho(\omega,k)$ and $\chi^{-1}(\omega,k)$ were first computed through solving independently, by FEM, the equations (S6) and (S7) and (S8) and (S9) in [45, Sec. SIII]. Based on these functions of imposed independent values of $\omega$ and $k$, the frequency-dependent wave number $k_1(\omega)$ associated with the least-attenuated normal mode was obtained by solving the nonlocal dispersion equation (38). The latter was solved by a Newton-Raphson scheme: we varied frequency step by step, taking as initial value for $k_1(\omega)$ at a given frequency the solution value obtained at the preceding frequency. Only for the starting frequency $\omega_0$ in the range of interest, we have chosen the value $k_{B,1}(\omega_0)$ with a 10% discrepancy, where $k_{B,1}$ that refers to Eq. (48) is the least-attenuated Bloch wave number produced by DBA based on the MS method. This immediately results in the nonlocal phase velocity in Eq. (40). The frequency-dependent effective density $\rho[\omega,k_1(\omega)] = \rho_1(\omega)$, and effective bulk modulus $\chi^{-1}[\omega,k_1(\omega)] = \chi_1^{-1}(\omega)$, of the corresponding principal normal mode are then calculated by replacing $k = k_1(\omega)$ in the aforementioned excitation terms in Eqs. (34b) and (34c). Subsequently, the nonlocal impedance in (40) was computed. We have performed all FEM computations, using FREEFEM++ [52], an open-source PDE solver. The weak form of the equations to be solved is first needed in order to implement the FEM simulations through this solver. Adaptive meshing was employed to deal with strong field variations in the medium.

We see in Fig. 3 that the real and imaginary parts of the phase velocity computed by nonlocal theory (40) via Newton's method converge exactly to the real and imaginary parts of Bloch phase velocity (50) which have been computed by the completely different DBA, in a very wide frequency range. The frequency range starts at $k_0L/\pi = 0.05$ corresponding to $\lambda_0 = 40L$, and ends at $k_0L/\pi = 2$, where the wavelength in air is equal to the periodicity, i.e., $\lambda_0 = L$. The effective wavelength, according to either nonlocal theory or DBA, is $\lambda \simeq 33L$ at the starting frequency and is $\lambda \simeq L$ at the ending point of the frequency band. The frequency band covers short waves up to those with wavelengths as small as the periodicity $\lambda \simeq L$. This includes the region where band gaps would appear in absence of viscothermal losses. In fact, based on the real part of the effective wave number (see Fig. S2 in [45]), we specify in Fig. 3 that the nonlocal theory predicts accurately the dispersive phase velocity, at least, for the whole first and also the entire second Brillouin zones. Here, the first Brillouin zone is, by definition, bounded by $0 < |\mathrm{Re}(k)| < \pi/L$. Thus (see Fig. S2 in [45]), the normalized frequency $k_0L/\pi$ associated with this zone lies in $0 < k_0L/\pi \lesssim 1$. Regarding the second Brillouin zone $\pi/L < |\mathrm{Re}(k)| < 2\pi/L$, we indicate in Fig. 3 that its corresponding frequency band is limited by $1 \lesssim k_0L/\pi \lesssim 1.8$. The rapid variations around $k_0L/\pi = 1$ correspond to the location of the first band gap. This may be viewed as a Bragg cell resonance, which occurs when the length of the cell is around $\lambda/2$. We see that for all frequencies, in the presence of dissipation, the propagation is possible, however, it slows down around "Bragg frequencies" through destructive

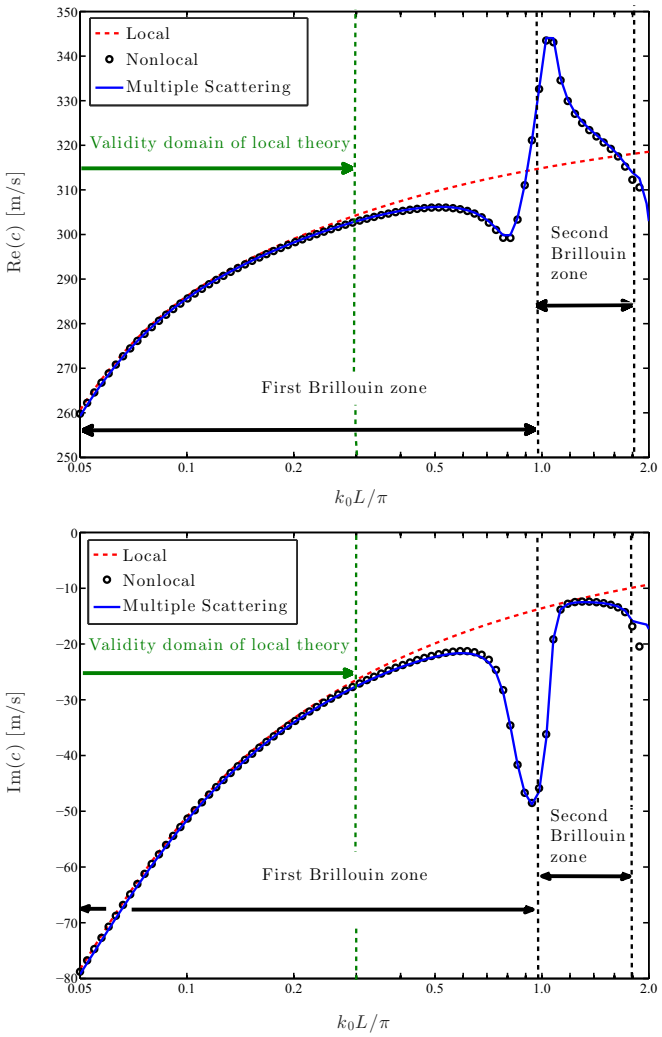


FIG. 3. Real and imaginary parts of the phase velocities of the least-attenuated wave according to local theory, nonlocal theory, and DBA for $\phi = 0.9$. The frequency domain of the validity of the local theory is shown. The Brillouin zones are determined following the values of the real part of the wave number.

interferences. In fact, losses do not allow for perfect destructive interferences.

In contrast, phase velocities predicted by the local theory cease to be valid above a certain frequency. As it was mentioned in Sec. III, this is due to the fact that the validity of the effective-medium parameters generated by the local theory is bounded up to the frequencies satisfying the condition that the fluid motions remain incompressible [Eq. (24), for the purpose of determining the microscopic velocity pattern] and the pressure field remains uniform [Eq. (25), for the purpose of determining the excess temperature pattern] at microscale. For this reason, in the validity domain of the local theory, the macroscopic pressure can be simplified as the direct average of the microscopic pressure in the fluid $H = \langle p \rangle/\phi = P \approx p$ instead of the general relation that is set in nonlocal theory: $H = \langle p\boldsymbol{v} \rangle/\langle \boldsymbol{v} \rangle$ [Eq. (20)]. These characteristics are illustrated in Fig. 4 presenting the divergence of the microscopic velocity normalized by its macroscopic value, and the microscopic pressure field normalized by its macroscopic part $H$, at two

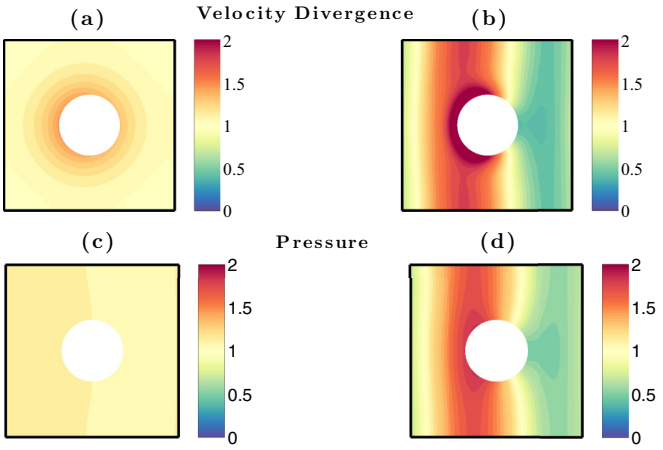


FIG. 4. Field distribution for velocity divergence (a), (b) and pressure (c), (d). Normalized velocity divergence (a), and pressure (c), at normalized frequency $k_0L/\pi = 0.1$ in the domain of validity of the local theory, where $\lambda \gg L$. Normalized velocity divergence (b), and pressure (d), at normalized frequency $k_0L/\pi = 1$ in Bragg's regime.

representative frequency points: $k_0L/\pi = 0.1$ on the left, which is seated in the validity domain of local approach, and $k_0L/\pi = 1$ on the right, which recognizably belongs to a region of Bragg scattering and rapid variations of the microscopic field patterns, where the local description is in error. Real parts of $(\nabla \cdot \boldsymbol{v})/(\nabla \cdot \boldsymbol{V})$ and $p/H$ are plotted, in order to depict noncomplex values. It is clearly illustrated that, as the local theory remains valid, the microscopic velocity divergence and pressure are distributed closely around their macroscopic values, which is a small quantity and a pore constant, respectively, owing to the long-wavelength condition. On the contrary, when the frequency is in Bragg's regime, the microscopic velocity divergence and the pressure are widely distributed around their macroscopic values. As such, the local theory fails to describe correctly the dispersive phase velocity for about one third of the first Brillouin zone, and the entire second Brillouin zone (Fig. 3). It is valid up to the frequency $k_0L/\pi \simeq 0.3$, where the real part of the wave number $\mathrm{Re}(k) \simeq 10^5$ (1/m) (see Fig. S2 in [45]).

Figures 5(a)–5(c) show the nonlocal density, bulk modulus, and impedance, as well as those based on the local theory, in functions of frequency. The local theory describes correctly the effective parameters only in the frequency range up to $k_0L/\pi \simeq 0.3$, where $\lambda_0 = 10L$ and $\lambda = 7L$, which covers only the lower one third of the first Brillouin zone.

We have performed the same computations for the case of $\phi = 0.7$ ($R = 3.1$ $\mu$m), and observed also excellent agreement between the nonlocal theory and MS method. For the more concentrated medium ($\phi = 0.7$), the discrepancies between local theory and DBA predictions are larger and commence at lower frequency. Indeed, it is known that in PCs, when the medium becomes more concentrated, the band gaps occur through larger frequency intervals (see, e.g., [4]) and the scattering phenomena become more influential. Although our analysis concerned a 2D PC, the present nonlocal approach

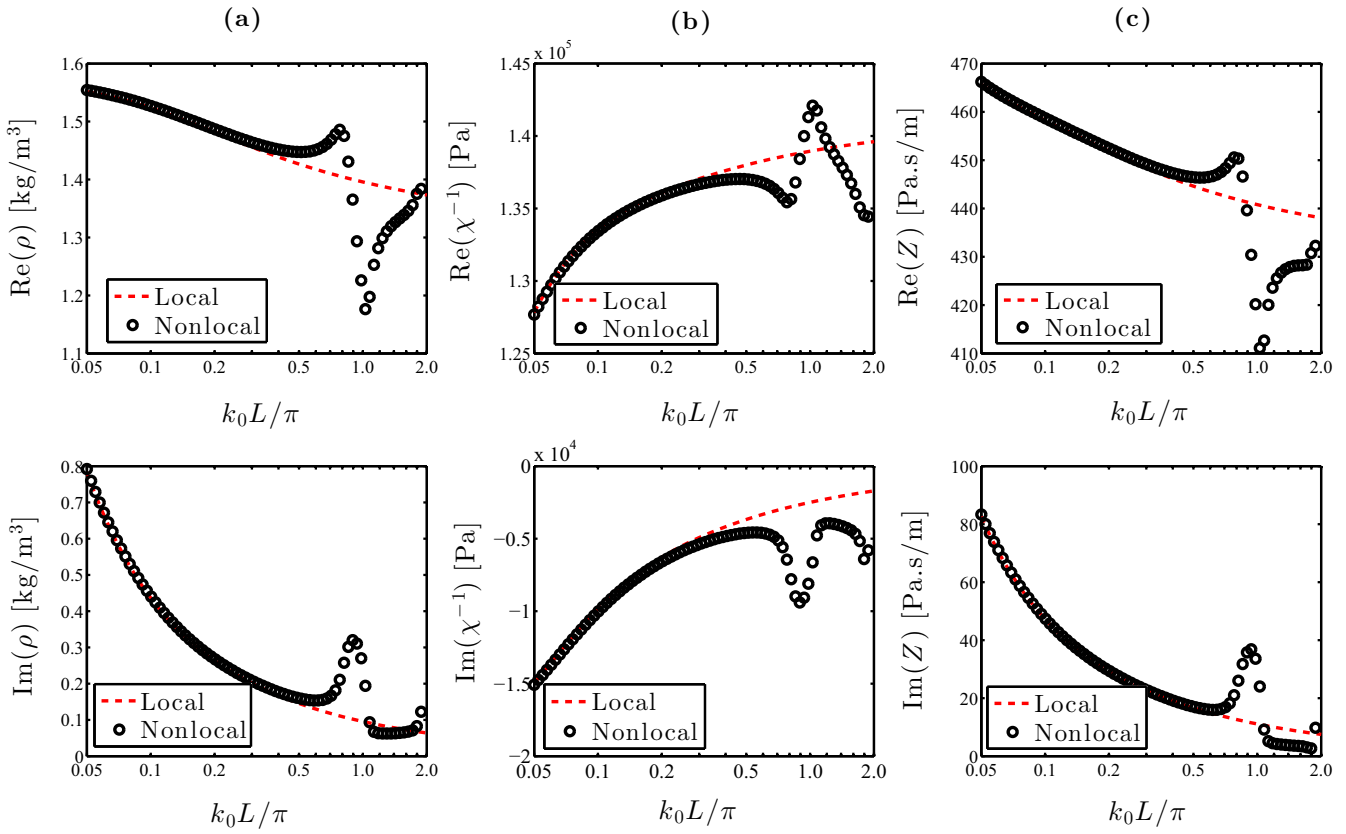


FIG. 5. Real and imaginary parts of (a) effective density, (b) effective bulk modulus, and effective (c) impedance of the least-attenuated wave according to local theory, and nonlocal theory, for porosity $\phi = 0.9$.

and its equations presented therein permit easily to consider the three-dimensional medium, e.g., with spherical inclusions, as long as the propagation is along an existing principal symmetry axis. This is only a computational issue. The reason that in this paper we restricted our study to the propagation along a principal symmetry axis is that, in this condition, the Poynting-Schoch quantity $\langle p\boldsymbol{v}\rangle$ [Eq. (20)] lies in the direction of $\langle \boldsymbol{v}\rangle$ and, thus, $H$ is a scalar field. For an arbitrary direction of propagation, a generalization could be possible, which would extend the definition of the macroscopic fields, action-response problems, and constitutive operators. Contrary to local, where indeed this generalization can be easily performed by tensor analysis, in nonlocal theory, because the Fourier components of the constitutive kernels are $\boldsymbol{k}$ dependent, a distinctive direction is generated. This feature, inherent to nonlocal approach, can engender new dynamics and mechanisms in the effective medium. This can be interesting, but requires further studies.

## VI. CONCLUDING REMARKS

We have shown that the exploited Maxwellian nonlocal approach fully describes the effective dynamics of a (rigid) solid/fluid phononic crystal, in terms of a an equivalent fluid with nonlocal properties at macroscale. Within this framework, the medium is treated as the ensemble of random translations of the crystal. In particular, this effective fluid allowing for short waves accounts also for the Bragg scattering phenomena, by virtue of which, introducing a new class of effective media, can be referred as *phononic fluids*. In addition, for the phononic fluids, we have precisely and from the microscale equations taken into account both viscous and thermal losses.

In order to demonstrate the above, we studied at the macroscopic level, 1D sound propagation in a 2D PC, made of periodic arrays of parallel rigid cylinders that are embedded in a viscothermal fluid. To investigate properly the macroscopic dissipative dynamics of the medium, the nonlocal and local approaches are presented in a unified formulation analogous to EM Maxwell equations. The temporal and spatial dispersion effects are incorporated in the constitutive relations in the same fashion as in EM. In-depth analysis is provided to situate here the classical homogenization given by the two-scale asymptotic expansion method of homogenization for periodic media, that is interpreted as a truncated local homogenization. To compute the local and nonlocal effective-fluid parameters, procedures within local and nonlocal approaches, involving action-response problems, were summarized, respectively. The dispersion equations related to local and nonlocal schemes are used to yield the frequency-dependent effective wave numbers as eigenvalues of the medium accordingly.

Furthermore, a different approach based on Floquet-Bloch theorem (DBA) was implemented to calculate the Bloch waves by solving directly and analytically the source-free microscopic equations via multiple scattering method. This quasiexact method, that takes into account the viscothermal effects, has been used to specify the validity domain of the effective-fluid schemes. Based on the local and nonlocal approaches, we have computed the frequency-dependent effective parameters associated with the least-attenuated wave. Comparisons were made between the quasiexact DBA results related to phase velocity of the normal mode and those produced by the effective theories in an air-filled medium. This showed remarkable agreement between the nonlocal theory and multiple scattering method for all frequencies, and demonstrated the range of validity of the local approach. We demonstrated that the nonlocal theory predicts accurately the effective-medium parameters for the entire first and second Brillouin zones, while the local theory is valid only for the lower one third of the first Brillouin zone. As such, the nonlocal approach has been validated to describe correctly the effective parameters of the phononic fluid in frequency bands, where the effective wavelength is equal to the periodicity, and thereby strong Bragg scattering occurs. We also discussed and illustrated the important role of the microscopic distribution of the velocity divergence of the fluid, that is indeed linked with the microscopic pressure profile, as the microlevel origin of the macroscopic spatial dispersion, and also its essential role to allow for full macroscopic temporal dispersion.

Although we studied the properties associated to the least-attenuated mode, the extension of this work to explore higher modes' properties can bring novel applications to control waves in metafluids. In addition, the nonlocal theory is expected to directly extend to the case where the separated inclusions in the fluid host are elastic solid, and the perturbation is considered to originate in the fluid. For the propagation along the principal lattice axis, the macroscopic equations remain in the same form with, obviously, modified kernel functions. To obtain the modified kernel functions, one can easily complete the microlevel action-response equations (34) by including the governing equations of the solid inclusions, as well as replacing the boundary conditions (35) by the appropriate new ones at the (elastic) solid/fluid interfaces. The forcing terms would not change in the fluid host and would be set to zero in the solid. In this case, comparison of the results of the nonlocal theory with those given by the layer-multiple-scattering method [53] would be feasible. Similarly, nonlocal properties of media made up of inclusions in the form of gas bubbles in a liquid host could be studied.

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[1] M. Sigalas and E. Economou, Solid State Commun. **86**, 141 (1993).

[2] M. S. Kushwaha, P. Halevi, L. Dobrzynski, and B. Djafari-Rouhani, Phys. Rev. Lett. **71**, 2022 (1993).

[3] E. Yablonovitch, Phys. Rev. Lett. **58**, 2059 (1987).

[4] Y. Pennec, J. Vasseur, B. Djafari-Rouhani, L. Dobrzynski, and P. A. Deymier, Surf. Sci. Rep. **65**, 229 (2010).

[5] *Acoustics Metamaterials and Phononic Crystals*, edited by P. A. Deymier (Springer, New York, 2012).

[6] M. Maldovan, Nature (London) **503**, 209 (2013).

[7] M. I. Hussein, M. J. Leamy, and M. Ruzzene, Appl. Mech. Rev. **66**, 040802 (2014).

[8] R. Martínez-Sala, J. Sancho, J. V. Sánchez, V. Gómez, J. Llinares, and F. Meseguer, Nature **378**, 241 (1995).

[9] J. V. Sánchez-Pérez, D. Caballero, R. Mártinez-Sala, C. Rubio, J. Sánchez-Dehesa, F. Meseguer, J. Llinares, and F. Gálvez, Phys. Rev. Lett. **80**, 5325 (1998).

[10] Z. Liu, X. Zhang, Y. Mao, Y. Y. Zhu, Z. Yang, C. T. Chan, and P. Sheng, Science **289**, 1734 (2000).

[11] Y. Pennec, B. Djafari Rouhani, H. Larabi, A. Akjouj, J. N. Gillet, J. O. Vasseur, and G. Thabet, Phys. Rev. B **80**, 144302 (2009).

[12] W. Aspelmeyer, T. J. Kippenberg, and F. Marquardt, Rev. Mod. Phys. **86**, 1391 (2014).

[13] Y. Wu, Y. Lai, and Z. Q. Zhang, Phys. Rev. B **76**, 205313 (2007).

[14] M. Yang, G. Ma, Y. Wu, Z. Yang, and P. Sheng, Phys. Rev. B **89**, 064309 (2014).

[15] P. A. Belov, R. Marques, S. I. Maslovski, I. S. Nefedov, M. Silveirinha, C. R. Simovski, and S. A. Tretyakov, Phys. Rev. B **67**, 113103 (2003).

[16] N. Nemati, A. Kumar, D. Lafarge, and N. X. Fang, C. R. Mecanique **343**, 656 (2015).

[17] M. G. Silveirinha, Phys. Rev. B **75**, 115104 (2007).

[18] T. Geng, S. Zhuang, J. Gao, and X. Yang, Phys. Rev. B **91**, 245128 (2015).

[19] D. Lafarge, P. Lemarinier, J. F. Allard, and V. Tarnow, J. Acoust. Soc. Am. **102**, 1995 (1997).

[20] For the general notions of temporal and spatial dispersions, see, e.g., [38], and V. M. Agranovitch and V. I. Ginzburg, *Spatial Dispersion in Crystal Optics and the Theory of Excitons* (Interscience, London, 1966).

[21] N. Fang, D. Xi, J. Xu, M. Ambati, W. Srituravanich, C. Sun, and X. Zhang, Nat. Mater. **5**, 452 (2006).

[22] C. Boutin and F. X. Becot, Wave Motion **54**, 76 (2015).

[23] G. M. Milton and J. R. Willis, Proc. R. Soc. A **463**, 855 (2007).

[24] S. Nemat-Nasser and A. Srivastava, J. Mech. Phys. Solids **59**, 1953 (2011).

[25] D. Torrent, Y. Pennec, and B. Djafari-Rouhani, Phys. Rev. B **92**, 174110 (2015).

[26] A. Srivastava and S. Nemat-Nasser, Proc. R. Soc. A **468**, 269 (2011).

[27] A. Norris, A. Shuvalov, and A. Kutsenko, Proc. R. Soc. A **468**, 1629 (2012).

[28] L. Prigogine, *Introduction to Thermodynamics of Irreversible Processes* (Interscience, New York, 1961).

[29] S. R. de Groot and P. Mazur, *Non-equilibrium Thermodynamics* (North-Holland, Amsterdam, 1962).

[30] L. D. Landau and E. Lifshitz, *Fluid Mechanics* (Pergamon, Oxford, 1987).

[31] A. D. Pierce, *Acoustics, An Introduction to Its Physical Principles and Applications* (McGraw-Hill, New York, 1981).

[32] For the local theory, it is equivalent and straightforward to consider only one periodic sample, with averaging over a unit cell. This applies also to the nonlocal theory, as long as the wavelengths remain sufficiently large.

[33] To clarify this, we replace $a, \nabla a, a\nabla$ and $\boldsymbol{n}(\boldsymbol{r}';\boldsymbol{\omega})a(t,\boldsymbol{r}';\boldsymbol{\omega})$ by $\boldsymbol{v}, \nabla\cdot\boldsymbol{v}, \boldsymbol{v}\cdot\nabla$ and $\boldsymbol{n}(\boldsymbol{r}';\boldsymbol{\omega})\cdot\boldsymbol{v}(t,\boldsymbol{r}';\boldsymbol{\omega})$ in Eqs. (51)–(53) of Ref. [35], and use the fact that the velocity vanishes on $\partial\mathcal{V}(\boldsymbol{\omega})$. Note that the commutation relation requires only that the normal component of the velocity vanishes on $\partial\mathcal{V}(\boldsymbol{\omega})$, thus it holds also for an ideal fluid.

[34] For the general notions of tensor and tensor densities, see, e.g., S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity*, (Wiley, New York, 1972), Chap. 4.

[35] D. Lafarge and N. Nemati, Wave Motion **50**, 1016 (2013).

[36] H. A. Lorentz, Proc. R. Acad. Amsterdam **5**, 254 (1902).

[37] For the tensorial nature of the fields, see, e.g., G. Fournet, *Electromagnétisme à partir des Equations Locales* (Masson, Paris, 1985); *Electromagnétisme. Traité génie électrique* (Techniques de l'ingénieur, D1020, 1993).

[38] L. D. Landau and E. Lifshitz, *Electrodynamics of Continuous Media* (Pergamon, Oxford, 1960).

[39] L. D. Landau and E. Lifshitz, *Statistical Physics* (Pergamon, Oxford, 1959).

[40] D. L. Johnson, J. Koplik, and D. Dashen, J. Fluid Mech. **176**, 379 (1987).

[41] M. Avellaneda and S. Torquato, Phys. Fluids A **3**, 2529 (1991).

[42] D. Lafarge, Propagation du son dans les matériaux poreux à structure rigide saturés par un fluide viscothermique, Ph.D. thesis, Université du Maine, Le Mans (1993).

[43] C. Boutin and C. Geindreau, J. Acoust. Soc. Am. **124**, 3576 (2008).

[44] For a recent presentation of these models, see J. Kergomard, D. Lafarge, and J. Gilbert, Acta Acust. united Ac. **99**, 557 (2013).

[45] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevB.95.224304 for a correction regarding Ref. [35] (Sec. SI), obtaining the relation between $\chi(\omega)$ and $\rho'(\omega)$ in local theory (Sec. SII), explicit action-response equations to be solved in nonlocal theory (Sec. SIII), details of calculations in multiple scattering method (Sec. SIV), and Fig. S2 that shows real and imaginary parts of the effective wave number versus normalized frequency (Sec. SV).

[46] More precisely, there is an arbitrary additive constant in the response field $p$, but it has no effect on the response field $\boldsymbol{v}$. It can be fixed by requiring that $p$, like the pressure deviation $\delta p$, is a field with zero mean value.

[47] R. Burridge and J. B. Keller, J. Acoust. Soc. Am. **70**, 1140 (1981).

[48] M. Y. Zhou and P. Sheng, Phys. Rev. B **39**, 12027 (1989).

[49] A. Duclos, D. Lafarge, and V. Pagneux, Eur. Phys. J. Appl. Phys. **45**, 11302 (2009).

[50] Y. Y. Chen and Z. Ye, Phys. Rev. Lett. **87**, 184301 (2001).

[51] N. Nemati and D. Lafarge, Wave Motion **51**, 716 (2014).

[52] F. Hecht, J. Numer. Math. **20**, 251 (2012).

[53] R. Sainidou, N. Stefanou, I. E. Psarobas, and A. Modinos, Comput. Phys. Commun. **166**, 197 (2005).