FIRST PASSAGE PROBABILITIES FOR A LINEAR HARMONIC OSCILLATOR

by

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ABSTRACT

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A LINEAR HARMONIC OSCILLATOR

Submitted to the Department of Mechanical Engineering on August 24, 1964 in partial fulfillment of the requirements for the degree of Doctor of Science

This thesis is essentially devoted to numerical evaluation of first passage probabilities for a linear harmonic oscillator driven by white Gaussian noise.

First passage probability $Q(t)$ is defined as follows. Let $y(t)$ and $\dot{y}(t)$ be displacement and velocity of the oscillator at time t. With a given initial probability distribution $G(y, \dot{y}, 0)$ at $t = 0$ inside a region Ω in the phase plane (y - $\dot{y}$ plane), $Q(t)$ = probability that $(y(\tau), \dot{y}(\tau)) \in \Omega$ for $0 \leq \tau \leq t$, that is the probability that the oscillator will not leave Ω between 0 and t.

The regions Ω considered are (1) $|y| \leq a$, that is, first passage through an absolute displacement level (2) $y^2 + \frac{\dot{y}^2}{\gamma^2} \leq a^2$, that is, first passage through an envelope or energy level. γ is the natural frequency of the oscillator.

The initial distributions $G(y, \dot{y}, 0)$ considered are (1) $G(y, \dot{y}, 0) = \delta(y)\ \delta(\dot{y})$, that is, oscillator excited from rest and (2) $G(y, \dot{y}, 0)$ = stationary distribution (inside a part or whole of Ω) for the process.

The first passage probabilities for all these cases are found to approach an exponential distribution for sufficiently high values of t. The initial transience is found to be of a cyclic nature, with frequency of 2γ.

The computational procedure is based on repetitive distributions of probability masses according to theoretically established transition probabilities for the oscillator output.

Excitations other than white Gaussian noise have also been considered.

ACKNOWLEDGEMENTS

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Section 1

INTRODUCTION

1.1 Preliminary Background

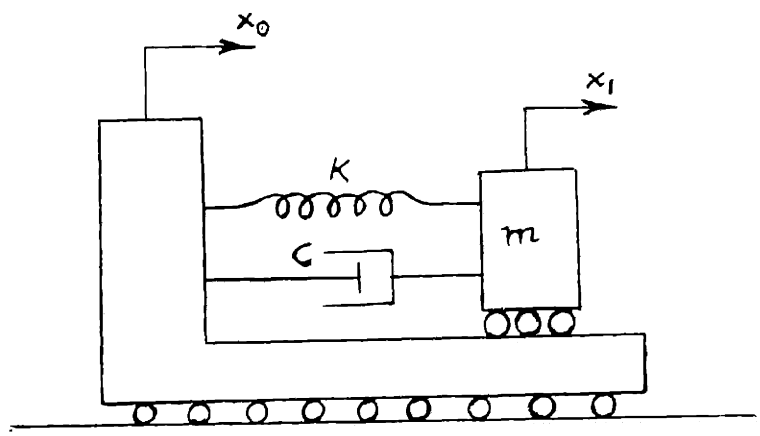


Figure 1

Consider a mass m elastically supported to a moving foundation with viscous damping, as shown in Fig. 1.

The relative displacement $y = x_1 - x_0$, satisfies the differential equation,

$$m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + Ky = -m\frac{d^2x_0}{dt^2}$$

or

$$\ddot{y} + 2\alpha\dot{y} + \gamma^2 y = -\ddot{x}_0(t) \qquad (1)$$

Where $2\alpha = \frac{c}{m}$, $\gamma^2 = \frac{K}{m}$

The solution $C(y_o, \dot{y}_o, t)$, of the corresponding homogeneous differential equation, $\ddot{y} + 2\alpha\dot{y} + \gamma^2 y = 0$, subject to initial conditions $y(t = 0) = y_o$, $\dot{y}(t = 0) = \dot{y}_o$, is

$$C(y_o, \dot{y}_o, t) = y_o\, a(t) + \dot{y}_o\, h(t) \tag{2}$$

where
$$a(t)_{t \geq 0} = e^{-\alpha t}\left(\frac{\alpha}{\beta}\sin\beta t + \cos\beta t\right) \tag{3}$$

$$h(t)_{t \geq 0} = \frac{e^{-\alpha t}}{\beta}\sin\beta t \tag{4}$$

and
$$\beta^2 = \gamma^2 - \alpha^2 \tag{5}$$

Also
$$\dot{C}(y_o, \dot{y}_o, t) = y_o\, \dot{a}(t) + \dot{y}_o\, \dot{h}(t) \tag{2'}$$

where $\dot{C} = \frac{\partial C}{\partial t}$, $\dot{a}(t) = \frac{d}{dt}a(t)$ and $\dot{h}(t) = \frac{d}{dt}h(t)$

The initial values y_o, $\dot{y}_o$ at $t = 0$, thus decay to $y(t) = C(y_o, \dot{y}_o, t)$ and $\dot{y}(t) = \dot{C}(y_o, \dot{y}_o, t)$ after time t.

One can look upon the arrangement as a linear, time invariant system with single degree of freedom, with $x(t) = -\ddot{x}_o(t)$ as input and $y(t)$ as output (Figure 2).

The unit impulse response $h(\tau)$ for this system would be,

$$h(\tau) = \frac{1}{\beta}\, e^{-\alpha\tau}\sin\beta\tau \quad \text{for } \tau \geq 0$$
$$= 0 \quad \text{for } \tau < 0$$

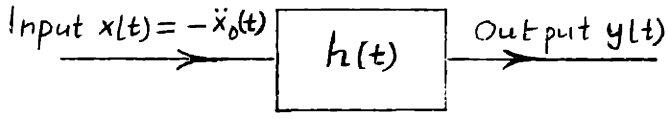


Figure 2

$h(\tau)$ is the ouptut $y(\tau)$ of the system for $\tau \geq 0$, when input $x(t)$ is simply a unit impulse at time $\tau = 0$.

Thus for any known input $x(t)$,

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau \qquad \text{(Ref. 1)}$$

$$\text{and} \quad \dot{y}(t) = \int_{-\infty}^{\infty} x(\tau)\, \dot{h}(t-\tau)\, d\tau$$

If however $x(t)$ is a random process, both $x(t)$ and $y(t)$ can at best be described by a set of probability distributions.

Suppose $x(t)$ is stationary and purely random. That is to say, probability $W(x(t))dx$, of finding $x(t)$ in the range $(x, x + dx)$ at time t does not change with t. Also for $t_1 \neq t_2 \neq t_3 \ldots$; $x(t_1)$, $x(t_2)$, $x(t_3)\ldots$; have no correlation whatsoever and are independent random variables with <u>identical</u> probability distributions.

For such an input $x(t)$, it can be shown that the output $y(t)$ is temporally homogeneous, and Markoffian in $y(t)$, $\dot{y}(t)$. That is

$$\text{Prob.}\ (y_1, \dot{y}_1, t_1; y_2, \dot{y}_2, t_2; \ldots; y_{n-1}, \dot{y}_{n-1}, t_{n-1} \mid y_n, \dot{y}_n, t_n)dy_n d\dot{y}_n$$

$$t_1 < t_2 < t_3 \ldots < t_{n-1} < t_n$$

$$= \text{Prob.}\ (y_{n-1}, \dot{y}_{n-1}, t_{n-1} \mid y_n, \dot{y}_n, t_n)dy_n d\dot{y}_n \quad \text{Markoffian property} \qquad (6)$$

$$= \text{Prob.}\ (y_{n-1}, \dot{y}_{n-1}, t_{n-1} + \tau \mid y_n, \dot{y}_n, t_n + \tau)dy_n d\dot{y}_n \qquad (6')$$

for any τ (temporal homogeneity)

The left hand side of (6) represents the conditional probability of getting $y_n \leq y(t_n) \leq y_n + dy_n$ and $\dot{y}_n \leq \dot{y}(t_n) \leq \dot{y}_n + d\dot{y}_n$, when the values of $y(t)$ and $\dot{y}(t)$ are given for $t = t_1, t_2 \ldots, t_{n-1}$.

In what follows, Prob. (A, B, C ... | X, Y, Z, y, $\dot{y}$, t) will always stand for $\longrightarrow$ given that conditions A, B, C, ... are satisfied, probability that conditions X, Y, Z be satisfied <u>and</u> $y \leqslant y(t) \leqslant y + dy$, $\dot{y} \leqslant \dot{y}(t) \leqslant \dot{y} + d\dot{y}$.

The Markoffian nature of y(t) is intuitively obvious from the physical situation, since the homogeneous solution (relations (2) and (2')) for the system can be completely described by <u>two</u> initial conditions y_0 and $\dot{y}_0$. For a more rigorous proof one must refer tc the general theorem of Doob (Ref. 3).

For such a Markoffian, temporally homogeneous output y(t), caused by a stationary, purely random excitation x(t), one can find the transition probability $P(y_0, \dot{y}_0 \mid y, \dot{y}, t)$ = Prob. $\left(y(t = 0) = y_0, \dot{y}(t = 0) = \dot{y}_0 \mid y, \dot{y}, t\right)$ = Prob. $\left(y(t = \tau) = y_0, \dot{y}(t = \tau) = \dot{y}_0 \mid y, \dot{y}, t + \tau\right)$,

by noting that,

$$y(t)_{t \geqslant 0} = C(y_0, \dot{y}_0, t) + \phi(t) \tag{7}$$

$$\dot{y}(t)_{t \geqslant 0} = \dot{C}(y_0, \dot{y}_0, t) + \psi(t) \tag{8}$$

where $$\phi(t) = \int_0^t x(\tau)\, h(t-\tau)\, d\tau \tag{9}$$

$$\psi(t) = \int_0^t x(\tau)\, \dot{h}(t-\tau)\, d\tau \tag{10}$$

$C(y_0, \dot{y}_0, t)$ and $C(y_0, \dot{y}_0, t)$ are completely determined by y_0, $\dot{y}_0$ and t. $\phi(t)$ and $\psi(t)$ are random variables whose probability distributions depend on t, $h(\tau)$ and probability distribution W(x) of input x(t).

In particular if x(t) is white Gaussian noise with zero mean and spectral density S_0 (or autocorrelation function $R_{xx}(\tau) = 2\pi S_0\, \delta(\tau)$), it can be shown that $\phi(t)$ and $\psi(t)$ are also Gaussianly distributed.

Also

$$E[\phi(t)] = E[\psi(t)] = 0$$

$$E[\phi^2(t)] = 2\pi S_0 \int_0^t h^2(\tau)\, d\tau$$

$$E[\psi^2(t)] = 2\pi S_0 \int_0^t \dot{h}^2(\tau)\, d\tau$$

$$E[\phi(t)\psi(t)] = 2\pi S_0 \int_0^t h(\tau)\, \dot{h}(\tau)\, d\tau$$

[See Ref. 2 for relations for a linear system.]

Hence from relations 7 and 8,

$$E[y(t)] = C(y_0, \dot{y}_0, t) = C \text{ for short}$$

$$E[\dot{y}(t)] = \dot{C}(y_0, \dot{y}_0, t) = \dot{C} \text{ for short}$$

σ_t^2 = variance of y(t) = $E\left[\left(y(t) - E[y(t)]\right)^2\right]$

$$= E[\phi^2(t)] = 2\pi S_0 \int_0^t h^2(\tau)\, d\tau$$

$$= \sigma^2 + \frac{\sigma^2}{\beta^2} e^{-2\alpha t}\left(-\gamma^2 + \alpha^2 \cos 2\beta t - \alpha\beta \sin 2\beta t\right) \tag{11}$$

where $$\sigma^2 = \frac{\pi S_0}{2\alpha\gamma^2} \tag{12}$$

$$\dot{\sigma}_t^2 = \text{variance of } \dot{y}(t) = E\left[\left(\dot{y}(t) - E[\dot{y}(t)]\right)^2\right]$$

$$= E\left[\psi^2(t)\right] = 2\pi S_0 \int_0^t h^2(\tau)\, d\tau$$

$$= \dot{\sigma}^2 + \frac{\dot{\sigma}^2}{\beta^2} e^{-2\alpha t}\left(-\gamma^2 + \alpha^2 \cos 2\beta t + \alpha\beta \sin 2\beta t\right) \quad (13)$$

where $$\dot{\sigma}^2 = \frac{\pi S_0}{2\alpha} \quad (14)$$

Also correlation coefficient

$$\rho_t = E\left[\left(y(t) - E[y(t)]\right)\left(\dot{y}(t) - E[\dot{y}(t)]\right)\right] \Big/ \sigma_t \dot{\sigma}_t$$

$$= \frac{E\left[\Phi(t)\psi(t)\right]}{\sigma_t \dot{\sigma}_t} = 2\pi S_0 \int_0^t h(\tau)\, \dot{h}(\tau)\, d\tau$$

$$= \frac{\pi S_0 h^2(t)}{\sigma_t \dot{\sigma}_t} \quad (15)$$

This gives enough information to write down the transition probability

$$P(y_0, \dot{y}_0 \mid y, \dot{y}, t) =$$

$$\frac{1}{2\pi \sigma_t \dot{\sigma}_t (1-\rho_t^2)^{1/2}} e^{-\frac{1}{2(1-\rho_t^2)}\left[\frac{(y-c)^2}{\sigma_t^2} - \frac{2\rho_t}{\sigma_t \dot{\sigma}_t}(y-c)(\dot{y}-\dot{c}) + \frac{(\dot{y}-\dot{c})^2}{\dot{\sigma}_t^2}\right]} \quad (16)$$

1.2 Definition of First Passage Probabilities of Interest.

The transition probability (16) is the most general information one could find regarding the random motion of a harmonic oscillator subject to white Gaussian excitation. The first passage probabilities we are about to define should therefore be uniquely determined by the transition probability (16) for the process.

Two first passage probabilities of interest are

(i) $Q_a(y_0, \dot{y}_0 \mid t) = \text{Prob.}\left(y(0) = y_0, |y_0| < a, \dot{y}(0) = \dot{y}_0 \;\middle|\; |y(\tau)| < a \text{ for } 0 \leq \tau \leq t \right)$

(ii) $Q_r(y_0, \dot{y}_0 \mid t) = \text{Prob.}\left(y(0) = y_0, \dot{y}(0) = \dot{y}_0, y_0^2 + \frac{\dot{y}_0^2}{\gamma^2} < r^2 \;\middle|\; y^2(\tau) + \frac{\dot{y}^2(\tau)}{\gamma^2} < r^2 \text{ for } 0 \leq \tau \leq t \right)$

If $f_a(y_0, \dot{y}_0 \mid t) = -\frac{\partial}{\partial t} Q_a(y_0, \dot{y}_0 \mid t)$ and $f_r(y_0, \dot{y}_0 \mid t) = -\frac{\partial}{\partial t} Q_r(y_0, \dot{y}_0 \mid t)$, then $f_a(y_0, \dot{y}_0 \mid t)\,dt$ or $f_r(y_0, \dot{y}_0 \mid t)\,dt$ represents the probability of leaving the region of interest (i.e. $|y| < a$ or $y^2 + \dot{y}^2/\gamma^2 < r^2$ respectively) for the <u>first time</u> between t and t + dt.

The applications of above first passage probabilities would be numerous. A variety of mechanical (and electrical) systems approximately behave like a damped oscillator, subject to random excitation. The probability Q_a would be useful in designing for the safe maximum displacement (or current) or other quantities which are dependent on displacement, such as stress, strain, etc. Q_r would similarly be useful when one is concerned about total energy in the system. The quantity $\left[y^2(t) + \dot{y}^2(t)/\gamma^2 \right]^{1/2}$ is called the envelope of the process y(t) and is a measure of total energy in the oscillator.

We are specially interested in obtaining Q_a and Q_r for $y_0 = 0$, $\dot{y}_0 = 0$. This would correspond to an oscillator being excited from rest.

Section 2

GENERAL DISCUSSION OF THE PROBLEM AND THE NUMERICAL APPROACH

2.1 Backward and Forward (Fokker-Planck) Equations

Here we shall describe the differential equations satisfied by the transition and first passage probabilities.

For this purpose we define a general first passage probability as follows.

Let Ω be a region in the phase plane (i.e. y, $\dot{y}$ plane) for the process (see Figure 3).

Then $Q(y_0, \dot{y}_0 | t)$ = Prob. $\Big(y(0) = y_0 \,,\ \dot{y}(0) = \dot{y}_0 \,,$

$$(y_0 \,,\ \dot{y}_0) \in \Omega \,\Big|\, (y(\tau), \dot{y}(\tau)) \in \Omega \text{ for } 0 \leqslant \tau \leqslant t \Big)$$

For Q_a, $\Omega = \Omega_a \equiv |y| < a$

For Q_r, $\Omega = \Omega_r \equiv y^2 + \frac{\dot{y}^2}{\gamma^2} < r^2$

Further, we define the probability $R(y_0, \dot{y}_0 \,|\, y, \dot{y}, t)\, dy\, d\dot{y}$

$$= \text{Prob.} \Big(y(0) = y_0 \,,\ \dot{y}(0) = \dot{y}_0 \,,\ (y_0, \dot{y}_0) \in \Omega \,\Big|\, (y(\tau), \dot{y}(\tau)) \in \Omega \text{ for } 0 \leqslant \tau \leqslant t \,;\ y, \dot{y}, t \Big)$$

Note that

$$Q(y_0, \dot{y}_0 \,|\, t) = \iint_{\Omega} R(y_0, \dot{y}_0 \,|\, y, \dot{y}, t)\, dy\, d\dot{y}$$

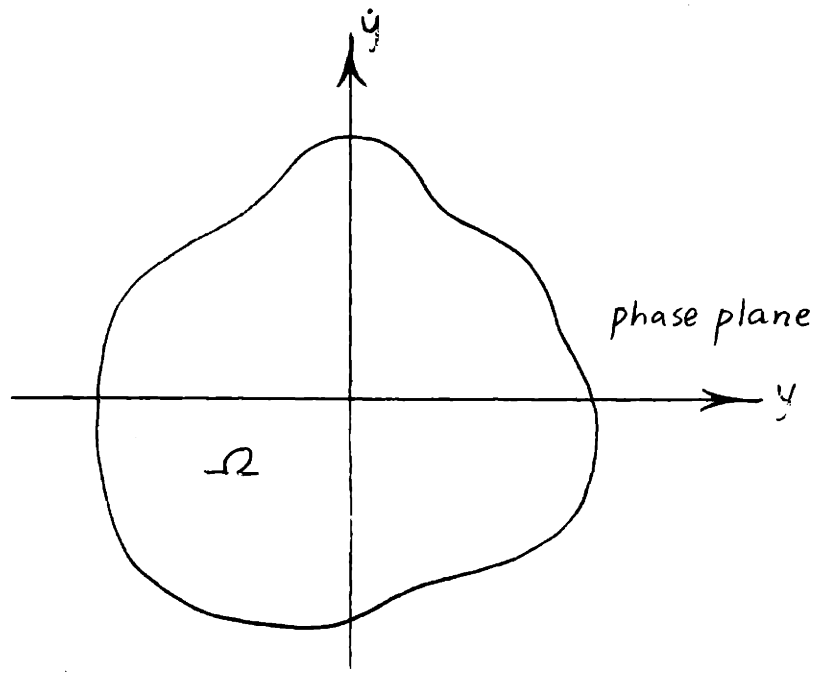


Figure 3

Since y(t) is Markoffian in y and ẏ, we have

$$R(y_0, \dot{y}_0 \mid y, \dot{y}, t) = \iint_{\Omega} R(y_0, \dot{y}_0 \mid y_1, \dot{y}_1, \tau)\ R(y_1, \dot{y}_1 \mid y, \dot{y}, t-\tau)\, dy_1\, d\dot{y}_1$$
$$\text{for } 0 < \tau < t \qquad (17)$$

This is just Chapman-Kolomogrov identity extended to a process Markoffian in two parameters y and ẏ. In particular

$$R(y_0, \dot{y}_0 \mid y, \dot{y}, t) = \iint_{\Omega} R(y_0, \dot{y}_0 \mid y_1, \dot{y}_1, \Delta t)\ R(y_1, \dot{y}_1, t-\Delta t)\, dy_1\, d\dot{y}_1 \qquad (18)$$

For small Δt it is reasonable to use the approximation,

$$R(y_0, \dot{y}_0 \mid y_1, \dot{y}_1, \Delta t) \sim P(y_0, \dot{y}_0 \mid y_1, \dot{y}_1, \Delta t),$$ the transition probability for the process.

Thus

$$R(y_0, \dot{y}_0 | y, \dot{y}, t) = \iint_{\Omega} P(y_0, \dot{y}_0 | y_1, \dot{y}_1, \Delta t)\, R(y_1, \dot{y}_1 | y, \dot{y}, t-\Delta t)\, dy_1 d\dot{y}_1 \quad (19)$$

Similarly we have

$$R(y_0, \dot{y}_0 | y, \dot{y}, t) = \iint_{\Omega} R(y_0, \dot{y}_0 | y_1, \dot{y}_1, t-\Delta t)\, P(y_1, \dot{y}_1 | y, \dot{y}, \Delta t)\, dy_1 d\dot{y}_1 \quad (20)$$

From these integral relations it is possible to derive differential equations for Q and R (Ref. 3, 5).

Relation (19) yields,

$$\frac{\partial Q}{\partial t} = \dot{y}_0 \frac{\partial Q}{\partial y_0} - (2\alpha \dot{y}_0 + \gamma^2 y_0) \frac{\partial Q}{\partial \dot{y}_0} + 2\alpha\gamma^2\sigma^2 \frac{\partial^2 Q}{\partial \dot{y}_0^2} \quad (21)$$

where Q stands for $Q(y_0, \dot{y}_0 | t)$. This is the backward equation for $Q(y_0, \dot{y}_0 | t)$ [backward, because it is in terms of the initial parameters y_0, $\dot{y}_0$].

Relation (20) yields,

$$\frac{\partial R}{\partial t} = -\dot{y} \frac{\partial R}{\partial y} + \frac{\partial}{\partial \dot{y}} \left[(2\alpha \dot{y} + \gamma^2 y) R \right] + 2\alpha\gamma^2\sigma^2 \frac{\partial^2 R}{\partial \dot{y}^2} \quad (22)$$

where R stands for $R(y_0, \dot{y}_0 \mid y, \dot{y}, t)$. This is the forward (or Fokker-Planck) equation for $R(y_0, \dot{y}_0 \mid y, \dot{y}, t)$ [forward, because it is in terms of the final parameters y and $\dot{y}$].

The transition probability $P(y_0, \dot{y}_0 \mid y, \dot{y}, t)$ also satisfies these forward and backward differential equations. Their derivation is along the same lines. In fact, one way to determine $P(y_0, \dot{y}_0 \mid y, \dot{y}, t)$ (which we found (relation 16) in section 1 through a systems appraoch) is to solve the Fokker-Planck equation (22) for P with initial condition

$$P(y_0, \dot{y}_0 \mid y, \dot{y}, 0) = \delta(y - y_0)\,\delta(\dot{y} - \dot{y}_0)$$

and boundary conditions

$$P(y_0, \dot{y}_0 \mid y, \dot{y}, t) \longrightarrow 0 \quad \text{as} \quad \begin{matrix} y \longrightarrow \pm\infty \\ \dot{y} \longrightarrow \pm\infty \end{matrix} \qquad \text{(Ref. 3)}$$

One way to obtain the first passage probabilities Q and R would be to solve the differential equations (21) and (22) with appropriate initial and boundary conditions. This method has, in fact, been successfully employed for stochastic processes Markoffian in one parameter (Ref. 6, 7). However, oscillator output is Markoffian in two parameters, y and $\dot{y}$. To illustrate the kind of difficulty one experiences with boundary conditions, let us consider first passage probabilities Q and R for $\Omega \equiv y < a, \; a > 0$.

Fig. 4 shows the boundary of Ω, $y = a$ in the phase plane. Note that in the upper half plane ($\dot{y} > 0$) all oscillator paths must necessarily travel in the direction of increasing y, similarly in the lower half plane they must travel towards decreasing y. A possible path from (0, 0) is shown in Fig. 4.

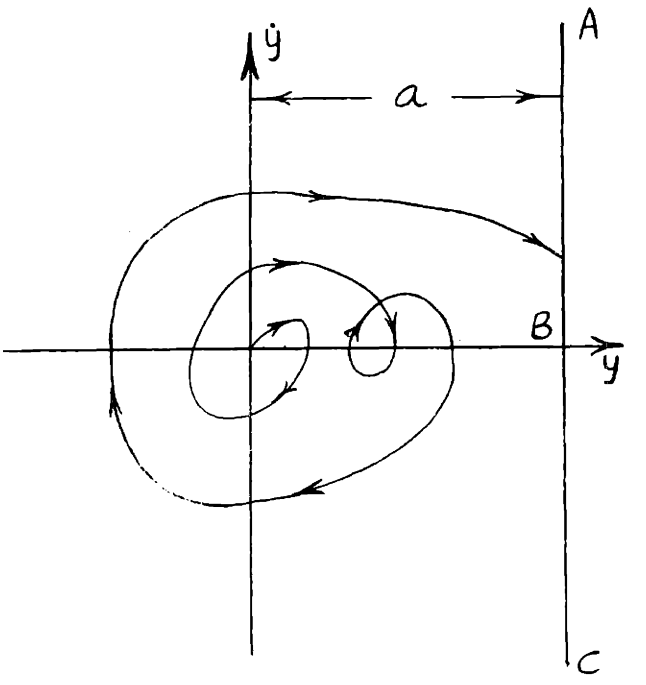


Figure 4

$Q(y_o , \dot{y}_o \mid t)$ satisfies differential equation (21). Initial condition is of course

$$Q(y_o, \dot{y}_o \mid 0) = 1 \tag{23}$$

This is because $(y_o , \dot{y}_o) \in \Omega$. Referring to Fig. 4, it is fairly clear that a path in Ω which has $\dot{y} > 0$ and is very close to the boundary $y = a$, is bound to leave Ω in a short time. Thus the boundary condition along AB is,

$$\lim_{y_o \to a} Q(y_o, \dot{y}_o \mid t) = 0 \quad \text{for } \dot{y}_o > 0 \tag{24}$$

However, nothing much can be said about the boundary condition along BC, except that

$$\lim_{y_o \to a} Q(y_o, \dot{y}_o \mid t) \neq 0 \quad \text{for } \dot{y}_o < 0$$

Very possibly the conditions (23) and (24) are sufficient to ensure a unique solution for $Q(y_o , \dot{y}_o \mid t)$. But no one has yet succeeded in solving this problem.

Similarly $R(y_o , \dot{y}_o \mid y, \dot{y}, t)$ satisfies differential equation (22). Initial condition is,

$$R(y_o, \dot{y}_o \mid y, \dot{y}, 0) = \delta(y - y_o)\, \delta(\dot{y} - \dot{y}_o) \tag{25}$$

Again referring to Fig. 4, it is intuitively obvious that no path originating from inside Ω can reach line BC without leaving Ω (through AB). Thus the boundary condition along BC is,

$$\lim_{y \to a} R(y_o, \dot{y}_o \mid y, \dot{y}, t) = 0 \quad \text{for} \quad \dot{y} < 0 \tag{26}$$

Integrating (22) over Ω, one easily gets

$$\frac{\partial}{\partial t} Q(y_o, \dot{y}_o \mid t) = -\int_0^\infty \dot{y} \left[\lim_{y \to a} R(y_o, \dot{y}_o \mid y, \dot{y}, t) \right] d\dot{y} \tag{27}$$

Since in general, $Q(y_o , \dot{y}_o \mid t)$ cannot be constant in time (it decreases with t),

$$\lim_{y \to a} R(y_o, \dot{y}_o \mid y, \dot{y}, t) \neq 0 \quad \text{for} \quad \dot{y} > 0$$

Again the same remarks apply for the solution of $R(y_o , \dot{y}_o \mid y, \dot{y}, t)$.

Even the numerical solution of didferential equations (21) and (22) is not possible since the boundary conditions cannot be fully prescribed.

2.2 Discussion of the Numerical Approach

Now we return to the integral relation (20) (which in fact is the source of the Fokker-Planck equation 22).

Let initial probability distribution inside Ω at $t = 0$ be $G(y, \dot{y}, 0)$.

$G(y, \dot{y}, 0)dy\,d\dot{y}$ = Prob. $(y \leqslant y(0) \leqslant y + dy,\ \dot{y} \leqslant \dot{y}(0) \leqslant \dot{y} + dy)$

We define the probabilities $G(y, \dot{y}, t)$ and $Q(t)$ as follows.

$G(y, \dot{y}, t)\,dy\,d\dot{y}$ = Prob. $\Big($At $t = 0$, $\big(y(0), \dot{y}(0)\big)$ is distributed in Ω according to distribution $G(y, \dot{y}, 0)$ $\Big|$ $\big(y(\tau), \dot{y}(\tau)\big) \in \Omega$ for $0 \leqslant \tau \leqslant t$; $y, \dot{y}, t$ $\Big)$

and $Q(t) = \iint_{\Omega} G(y, \dot{y}, t)\,dy\,d\dot{y}$

= Prob. $\Big($At $t = 0$, $\big(y(0), \dot{y}(0)\big)$ is distributed in Ω according to distribution $G(y, \dot{y}, 0)$ $\Big|$ $\big(y(\tau), \dot{y}(\tau)\big) \in \Omega$ for $0 \leqslant \tau \leqslant t$ $\Big)$

It is clear that,

$$G(y, \dot{y}, t) = \iint_{\Omega} G(y_o, \dot{y}_o, 0)\, R(y_o, \dot{y}_o \mid y, \dot{y}, t)\,dy_o\,d\dot{y}_o \qquad (28)$$

Using relation (20), we have,

$$G(y, \dot{y}, t) = \iint_{\Omega} G(y_1, \dot{y}_1, t - \Delta t)\, p(y_1, \dot{y}_1 \mid y, \dot{y}, \Delta t)\,dy_1\,d\dot{y}_1 \qquad (29)$$

Now we start from $t = 0$ and examine the process at $t = \Delta t$, $2\,\Delta t$, $3\,\Delta t$ and so on. For $t = N\,\Delta t$, relation (29) yields

$$G(y, \dot{y}, N\Delta t) = \iint_{\Omega} G(y_1, \dot{y}_1, (N-1)\Delta t)\, P(y_1, \dot{y}_1 \mid y, \dot{y}, \Delta t)\, dy_1\, d\dot{y}_1 \tag{30}$$

For the Kernel $P(y_1, \dot{y}_1 \mid y, \dot{y}, \Delta t)$ in Ω, let us assume eigen values $\rho_1 > \rho_2 > \rho_3$ ------, and corresponding eigen functions $H_1(y, \dot{y})$, $H_2(y, \dot{y})$, $H_3(y, \dot{y})$------, such that

$$\rho_i H_i(y, \dot{y}) = \iint_{\Omega} H_i(y_1, \dot{y}_1)\, P(y_1, \dot{y}_1 \mid y, \dot{y}, \Delta t)\, dy_1\, d\dot{y}_1 \tag{31}$$

$i = 1, 2, 3$-----

(Ref. 9)

Note that the argument Δt in $P(y_1, \dot{y}_1 \mid y, \dot{y}, \Delta t)$ is taken to be a constant rather than a variable.

Further we assume that

$$G(y, \dot{y}, 0) = \sum_{i=1}^{\infty} C_i H_i(y, \dot{y}) \tag{32}$$

where C_i are some constant.

Then from (30), (31) and (32), it is clear that

$$G(y, \dot{y}, N\Delta t) = \sum_{i=1}^{\infty} \rho_i^N C_i H_i(y, \dot{y}) \tag{33}$$

and $$Q(N\Delta t) = \sum_{i=1}^{\infty} D_i \rho_i^N \tag{34}$$

where $$D_i = C_i \iint_{\Omega} H_i(y, \dot{y})\, dy\, d\dot{y} \tag{35}$$

Defining $$\lambda_i = -\frac{\pi}{(\gamma \Delta t)} \ln \rho_i \tag{36}$$

where γ is the natural frequency of the oscillator, we have

$$Q(t) = \sum_{i=1}^{\infty} D_i e^{-\lambda_i \frac{\gamma}{\pi} t} \tag{37}$$

Unless the region Ω consists of the entire phase plane, each $\rho_i < 1$ and each $\lambda_i > 0$, thus

$$\lim_{t \to \infty} Q(t) = D_1 e^{-\lambda_1 \frac{\gamma}{\pi} t} \tag{38}$$

λ_1 being the smallest exponential factor (or ρ_1 being the largest eigen value).

The numerical method chosen for evaluation of Q(t) (for a particular Ω and $G(y, \dot{y}, o)$ is to calculate $G(y, \dot{y}, N\Delta t)$ for N = 1, 2, 3 ---, by numerical integration of relation (30). The region Ω of the phase plane is divided into rectangular elements of size Δy, $\Delta \dot{y}$ for this purpose. The computation is continued until the asymptotic exponential behavior indicated by relation (38) is approached. Thus in addition to the dominant solution (38) one gets full transient behavior (37), through individual values of D_2, D_3,--- and λ_2, λ_3, --- can not be found.

A slightly different way of looking at the process of numerical integration of (30) is to consider probability masses distributed initially at t = 0 according to the distribution $G(y, \dot{y}, 0)$ among

the rectangular elements of the region Ω of the phase plane. One keeps on distributing these masses after each Δt, according to the transition probability $P(y_1, \dot{y}_1 \mid y, \dot{y}, t)$, agreeing to "lose" any masses which leave the region Ω.

To elucidate the computational procedure and to gain some insight into the behavior, convergence and accuracy of the solutions, an excitation much simpler than white Gaussian noise is considered in the next section.

Section 3

OSCILLATOR EXCITED BY PERIODIC IMPULSES

3.1 One Dimensional Process.

Consider the harmonic oscillator being excited by a noise x(t) which consists of a series of periodic impulses, the time between two consecutive impulses being $\frac{\pi}{\beta}$ (where $\beta^2 = \gamma^2 - \alpha^2$). Let each impulse be an independent random variable, the area A under it being Gaussianly distributed.

$$\text{Thus } p(A) = \frac{1}{\sqrt{2\pi}\,\sigma_A} e^{-\frac{A^2}{2\sigma_A^2}} \tag{39}$$

It is obvious that autocorrelation $R_{xx}(t)$ of x(t) = 0, for $t \neq 0$ (since all impulses are independent with zero mean).

To find $R_{xx}(0)$ we consider a process v(t) obtained by replacing each impulse in x(t) with a rectangular pulse of height H and duration dt at the same time, such that the area Hdt under each rectangular pulse in v(t) = area A under the corresponding impulse in x(t).

It is clear that the height H of any pulse will be Gaussianly distributed with probability density g(H) say,

where g(H) dH = Prob. (H $\leqslant$ height of a pulse $\leqslant$ H + dH)

= Prob. (Hdt $\leqslant$ area under the pulse $\leqslant$ (H + dH) dt)

= Prob. (Hdt $\leqslant$ area A under the corresponding impulse in process x(t) $\leqslant$ (H + dH)dt)

= P(A = Hdt) dHdt

Thus $g(H) = \frac{1}{\sqrt{2\pi}\,\sigma_A} e^{-\frac{H^2 dt^2}{2\sigma_A^2}} \, dt$ [use relation 39 for P]

Also probability of finding a rectangular pulse in process v(t) at any time $t = dt/\frac{\pi}{\beta}$ and probability of not finding any pulse at time $t = 1 - \left(dt/\frac{\pi}{\beta}\right)$

Hence $E\left[v^2(t)\right] = \left(1 - dt/\frac{\pi}{\beta}\right)(0) + \frac{dt}{\pi/\beta} \int_{-\infty}^{\infty} H^2 g(H)\, dH$

$$= \frac{\beta dt}{\pi} \int_{-\infty}^{\infty} \frac{H^2}{\sqrt{2\pi}\,\sigma_A} e^{-\frac{H^2 dt^2}{2\sigma_A^2}} \, dH\, dt \quad [\text{put } Hdt = u]$$

$$= \frac{\beta dt}{\pi} \int_{-\infty}^{\infty} \frac{u^2}{dt^2 \sqrt{2\pi}\,\sigma_A} e^{-\frac{u^2}{2\sigma_A^2}} \, du$$

$$= \frac{\beta}{\pi} \frac{\sigma_A^2}{dt}$$

As the height H of each reactangular pulse in v(t) tends to ∞, while its duration dt $\longrightarrow$ 0, such taht Hdt = constant for each pulse, process v(t) $\longrightarrow$ process x(t)

Hence $R_{xx}(0) = \overline{x^2(t)}$

$$= \lim_{\substack{H \to \infty \\ dt \to 0 \\ Hdt = \text{constant}}} \overline{v^2(t)} = \lim_{dt \to 0} \frac{\beta \sigma_A^2}{\pi\, dt} \qquad (40a)$$

and as mentioned previously,

$$R_{xx}(t) = 0 \qquad \text{for } t \neq 0 \tag{40b}$$

Combining (40a) and (40b) we get,

$$R_{xx}(t) = \frac{\beta}{\pi}\,\sigma_A^2\,\delta(t) \tag{41}$$

Thus x(t) is white noise with its non-zero (impulse) values Gaussianly distributed.

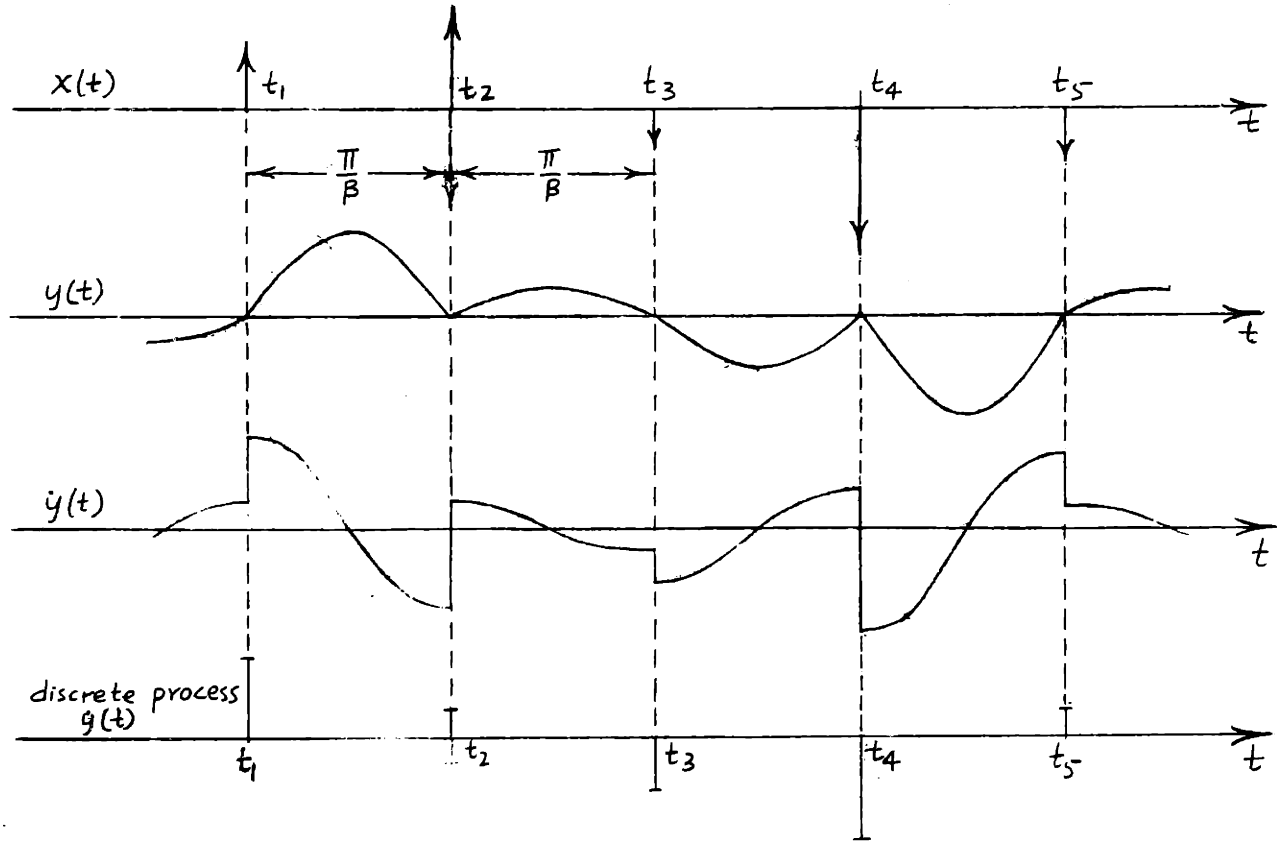


Figure 5

Now each x(t) impulse with area A under it simultaneously gives rise to velocity $\dot{y} = A$ as the output from the oscillator (see Fig. 5). Noting that $C\left(y_0, \dot{y}_0, \frac{\pi}{\beta}\right) = -y_0\, e^{-\frac{\pi\alpha}{\beta}}$, and $\dot{C}\left(y_0, \dot{y}_0, \frac{\pi}{\beta}\right) = -\dot{y}_0\, e^{-\frac{\pi\alpha}{\beta}}$

(see relations (2) and (2')), it is clear that if the oscillator was excited from rest (however far back in the past), the random changes in $\dot{y}$ will always occur at intervals of $\frac{\pi}{\beta}$, at times when $y = 0$ (Fig. 5). Hence the first passage problems related to $\Omega_a \equiv (|y| < a)$ or to $\Omega_r \equiv \left(y^2 + \frac{\dot{y}^2}{\gamma^2} < r^2 \right)$ can be described in terms of first passage problems for $\Omega_b \equiv (|\dot{y}| < b)$.

Also, the critical values of $\dot{y}$ occur at instants (t_1, t_2, t_3, t_4... in Fig. 5) when $\dot{y}$ is subject to random changes. In between these instants (which are $\frac{\pi}{\beta}$ apart) $|\dot{y}|$ just decays to smaller values according to relation (2'). Hence to consider the first passage problem for $\Omega_b \equiv (|\dot{y}| < b)$, we only need consider a process $\dot{y}(t)$ which is discrete in time and records values of $\dot{y}(t)$ only at times t_1, t2, t_3 ... (Fig. 5), the random change at each instant being <u>included</u> in the observation for the same instant, as illustrated in Fig. 5.

From now on $\dot{y}(t)$ would denote this discrete process.

Transition probability and (discrete) autocorrelation for $\dot{y}(t)$: - If $\dot{y}(0) = \dot{y}_0$, it decays to $(-1)^n \dot{y}_0 e^{-\frac{\pi \alpha n}{\beta}}$ at time $t = \frac{\pi n}{\beta}$ $(n = 0, 1, 2, \cdots)$. Also at each interveinign instant $t = \frac{\pi m}{\beta}$ $(m = 1, 2, \cdots, n)$, there is an independent random contribution $\dot{y}_r$ to the velocity and

$$P(\dot{y}_r) = \frac{1}{\sqrt{2\pi}\,\sigma_A} e^{-\frac{\dot{y}_r^2}{2\sigma_A^2}}$$

This contribution $\dot{y}_r$ at $t = \frac{\pi p}{\beta}$ decays to $(-1)^{n-p} e^{-\frac{\pi\alpha}{\beta}(n-p)}$ at time $t = \frac{\pi n}{\beta}$ $(n \gg p)$

Calling $-e^{-\frac{\pi\alpha}{\beta}} = K$ we get,

$$\dot{y}\left(t = \frac{\pi n}{\beta}\right) = \dot{y}_n = \dot{y}_0 K^n + \dot{y}_r K^{n-1} + \dot{y}_r K^{n-2} + \text{-------} + \dot{y}_r K + \dot{y}_r$$

$\dot{y}_0 K^n$: determinate component

$\dot{y}_r K^{n-1} + \dot{y}_r K^{n-2} + \text{-------} + \dot{y}_r K + \dot{y}_r$: random components

Since all the random components are independent and Gaussian with zero mean, $\dot{y}_n$ is Gaussian too.

And $E[\dot{y}_n] = \dot{y}_0 K^n$

$$E\left[\left(\dot{y}_n - E[\dot{y}_n]\right)^2\right] = \dot{\sigma}_n^2 = \sigma_A^2 \left(K^{2n-2} + K^{2n-4} + \text{------} + K^2 + 1\right)$$

$$\therefore \dot{\sigma}_n^2 = \sigma_A^2 \frac{1 - K^{2n}}{1 - K^2} \tag{42}$$

$$\text{where} \quad K = -e^{-\frac{\pi\alpha}{\beta}} \tag{43}$$

∴ the transition probability

$$P\left(\dot{y}_0 \mid \dot{y}, t = \frac{\pi n}{\beta}\right) = \frac{1}{\sqrt{2\pi}\,\dot{\sigma}_n} e^{\frac{-\left(\dot{y} - \dot{y}_0 K^n\right)^2}{2\dot{\sigma}_n^2}} \tag{44}$$

and the stationary probability distribution

$$P(\dot{y}) = \lim_{n\to\infty} P\left(y_o \mid \dot{y}, t = \frac{\pi n}{\beta}\right) = \frac{1}{\sqrt{2\pi}\,\dot{\sigma}} e^{-\frac{\dot{y}^2}{2\dot{\sigma}^2}} \tag{45}$$

where stationary variance $\dot{\sigma}^2 = \lim_{n\to\infty} \dot{\sigma}_n^2$

$$= \frac{\sigma_A^2}{1-K^2} \tag{47}$$

The autocorrelation $R_{\dot{y}\dot{y}}\left(t = \frac{\pi n}{\beta}\right)$ is clearly $= \dot{\sigma}^2 K^{|n|}$ (48)

3.2 Computation of First Passage Probabilities.

Now suppose we want to compute the first passage probability Q(t) for this process $\dot{y}(t)$. Q(t) is defined as follows,

Q(t) = Prob. (At t = 0, $\dot{y}(0)$ is distributed in Ω according to initial distribution $G(\dot{y}, 0)$ $\mid$ $\dot{y}(\tau) \in \Omega$ for $0 \leqslant \tau \leqslant t$)

We take $\Omega \equiv |\dot{y}| < a\dot{\sigma}$, $a > 0$; and $G(\dot{y}, 0)$, the initial distribution at t = 0, = $\delta(\dot{y})$.

Analogous to relation (30) for the two dimensional process, here we have $\Delta t = \frac{\pi}{\beta}$ and

$$G\left(\dot{y}, \frac{N\pi}{\beta}\right) = \int_{-a\dot{\sigma}}^{a\dot{\sigma}} G\left(\dot{y}_1, (N-1)\frac{\pi}{\beta}\right) P\left(\dot{y}_1 \mid \dot{y}, \frac{\pi}{\beta}\right) d\dot{y}_1 \tag{49}$$

From relations (42) and (44), the unit step transition probability

$$P\left(\dot{y}_1 \mid \dot{y}, \frac{\pi}{\beta}\right) = \frac{1}{\sqrt{2\pi}\,\sigma_A} e^{-\frac{(\dot{y} - \dot{y}_1 K)}{2\sigma_A^2}} \tag{50}$$

Now we divide the region $\Omega \equiv -a\dot{\sigma} \leqslant \dot{y} \leqslant a\dot{\sigma}$ into $2M+1$ equal elements each of length $\Delta\dot{y} = \frac{2a\dot{\sigma}}{2M+1}$ (see Fig. 6). The middle points of these elements have coordinates $0, \pm \Delta\dot{y}, \pm 2\Delta\dot{y}, \ldots \pm M\,\Delta\dot{y}$.

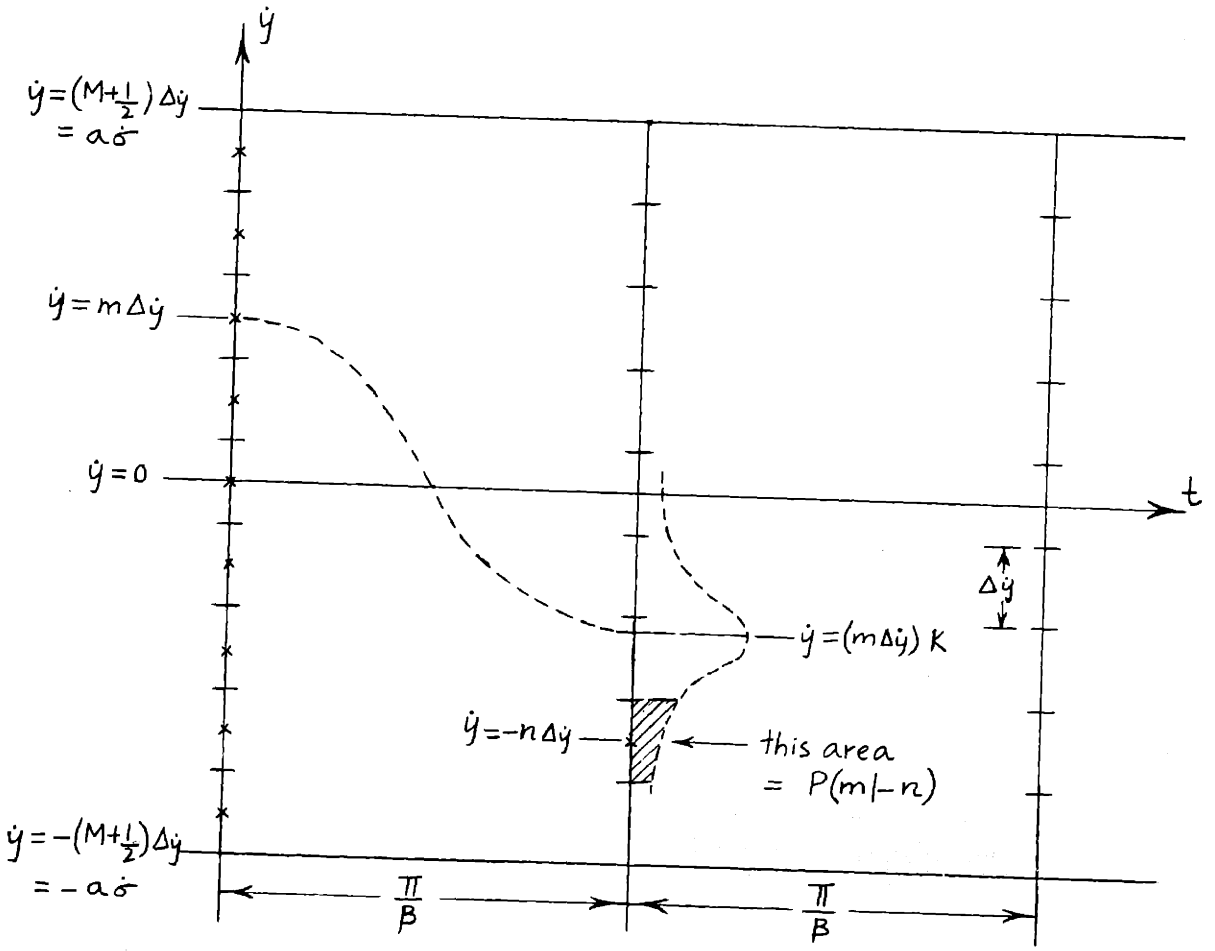


Figure 6

We shall consider the probability masses in these elements to be concentrated at their middle points. Under this assumption we compute the unit step transition matrix

$$P(m|n) = \int_{n\Delta\dot{y}-\Delta\dot{y}/2}^{n\Delta\dot{y}+\Delta\dot{y}/2} P\left(\dot{y}_1 = m\Delta\dot{y} \mid \dot{y}, \frac{\pi}{\beta}\right) d\dot{y} \quad , \qquad (51)$$

using relation (50) for the integrand and performing the numerical integration with the use of the parabolic formula.

Starting with the initial distribution $G(\dot{y}, 0)$ similarly concentrated as impulses at the middle points of the elements, we compute successives G's according to the relation

$$G\left(n\Delta\dot{y}\,,\ N\frac{\pi}{\beta}\right) = \sum_{m=-M}^{m=M} G\left(m\Delta\dot{y}\,,\ (N-1)\frac{\pi}{\beta}\right) P(m|n) \qquad (52)$$

This is just discrete counterpart of relation (49). At every step $Q(N\pi/\beta)$ is obtained by merely summing up $G(n\,\Delta\dot{y}, \frac{N\pi}{\beta})$ over Ω. Also the probability $f(\frac{N\pi}{\beta})$ of leaving Ω at the Nth step is obtained as

$$f\left(\frac{N\pi}{\beta}\right) = Q\left((N-1)\frac{\pi}{\beta}\right) - Q\left(\frac{N\pi}{\beta}\right) \qquad (53)$$

One also computes

$$\bar{P}\left(\frac{N\pi}{\beta}\right) = \frac{Q\left(N\pi/\beta\right)}{Q\left((N-1)\pi/\beta\right)} \qquad (54)$$

As reasoned out in Section 2 (relations 34 to 38), $\lim_{N\to\infty} \bar{P}\left(\frac{N\pi}{\beta}\right) = p_1$ the largest eigen value of $P(m \mid n)$. Thus the computations are continued until $\bar{P}(\frac{N\pi}{\beta})$ approaches the constant value p_1.

Analogous to relation (36), we define the exponential factor λ_1 as

$$\lambda_1 = -\ln p_1 \qquad (55)$$

Thus the asymptotic expression for Q(t) becomes,

$$Q(t) = D_1 e^{-\lambda_1 \frac{\beta}{\pi} t} \tag{56}$$

where $t = \frac{\pi}{\beta}, \frac{2\pi}{\beta}, \text{-------}.$

The factor D_1 is found as

$$D_1 = \lim_{N \to \infty} \frac{Q(N\pi/\beta)}{\left[\bar{P}(N\pi/\beta)\right]^N} \tag{57}$$

3.3 Symmetric Cases.

As long as $G(\dot{y}, 0)$ is an even function of $\dot{y}$ and Ω is symmetric about $\dot{y} = 0$, one can take advantage of the symmetry of the problem and perform the computations in only half of Ω . Thus one proceeds as follows,

(i) Consider only M + 1 elements of length $\Delta\dot{y}$ (see Fig. 6), having their middle points at $\dot{y} = 0, \Delta\dot{y}, 2\Delta\dot{y}, \ldots, M\Delta\dot{y}$.

(ii) Assign the initial distribution $G(\dot{y}, 0)$ as

$$G(m\Delta\dot{y}, 0) = \int_{m\Delta\dot{y} - \Delta\dot{y}/2}^{m\Delta\dot{y} + \Delta\dot{y}/2} G(\dot{y}, 0)\, d\dot{y} \quad \text{for } m = 1, 2, \cdots M.$$

$$= \frac{1}{2} \int_{-\Delta\dot{y}/2}^{\Delta\dot{y}/2} G(\dot{y}, 0)\, d\dot{y} \quad \text{for } m = 0 \tag{58}$$

(iii) Compute the unit step transition matrix

$$P_{sym}(m|n) = P(m|n) + P(m|-n) \quad \text{for } 0 \leqslant m \leqslant M, \quad 1 \leqslant n \leqslant M$$
$$= P(m|n) \quad \text{for } 0 \leqslant m \leqslant M, \; n = 0 \qquad (59)$$

where $P(m|n)$ is given by relation (51)

(iv) Iterate according to

$$G_1\left(n\Delta\dot{y}, \frac{N\Pi}{\beta}\right) = \sum_{m=0}^{M} G\left(m\Delta\dot{y}, (N-1)\frac{\Pi}{\beta}\right) P_{sym}(m|n) \qquad (60)$$

(v) Finally

$$Q\left(\frac{N\Pi}{\beta}\right) = 2\sum_{n=0}^{M} G\left(n\Delta\dot{y}, \frac{N\Pi}{\beta}\right) \qquad (61)$$

3.4 Tests on the Computational Procedure.

One could say with confidence that progressively better results would be obtained as the size $\Delta\dot{y}$ of the elements of Ω is decreased or as the number of elements for a fixed Ω is increased.

The nondimensional factor d, defined as the ratio of the size $\Delta\dot{y}$ to the standard deviation of the unit step transition probability, seems to be the best way of indicating this size effect.

$$d = \frac{\Delta\dot{y}}{\sigma_A} = \frac{2a\dot{\sigma}}{(2M+1)\sigma_A} = \frac{2a}{(2M+1)(1-K^2)^{1/2}} \qquad (62)$$

assuming $2M+1$ divisions in $\Omega \equiv |\dot{y}| < a\dot{\sigma}$

For $\Omega \equiv |\dot{y}| < 3\dot{\sigma}$, $G(\dot{y}, 0) = \delta(\dot{y})$ and $\frac{\alpha}{\gamma} = 0.01$,

Fig. 7 shows variation in λ_1 and D_1 with this size factor d. The variation is quite small in the neighborhood of d = 0.25.

To further check the validity of the computational procedure, the probability distributions actually obtained were compared with those from relation (44). With reference to relation (44), we chose $\dot{y}_0 = 0$, n = 60 and $\frac{\alpha}{\gamma} = 0.01$. The step by step iteration was limited to a region $\Omega \equiv |\dot{y}| < 5\dot{\sigma}$, with $G(\dot{y}, 0) = \delta(\dot{y})$ of course, and d = 0.5. The two distributions compared very favorably. For $\frac{\alpha}{\gamma} = 0.01$, $\sigma_A = 1$, $\dot{\sigma}_{60} = 4.005$ from relation (42). $\dot{\sigma}_{60}$ obtained from iteration was = 4.044.

The same test was repeated with the transition matrix $P(m \mid n)$ computed on a slightly different basis. Instead of assuming the probability masses to be concentrated at the middle points of the elemental lengths $\Delta\dot{y}$, they were assumed to be uniformly distributed over each $\Delta\dot{y}$. In that case the unit step transition matrix $P'(m \mid n)$ would be given by

$$P'(m|n) = \frac{1}{\Delta\dot{y}} \int_{m\Delta\dot{y} - \Delta\dot{y}/2}^{\dot{y}_1 = m\Delta\dot{y} + \Delta\dot{y}/2} \int_{n\Delta\dot{y} - \Delta\dot{y}/2}^{\dot{y} = n\Delta\dot{y} + \Delta\dot{y}/2} P\left(\dot{y}_1 \mid \dot{y}, \frac{\Pi}{\beta}\right) d\dot{y}_1 \, d\dot{y} \qquad (63)$$

Practically no change was observed between the two tests. $\dot{\sigma}_{60}$ obtained with this modified $P'(m|n)$ was = 4.081.

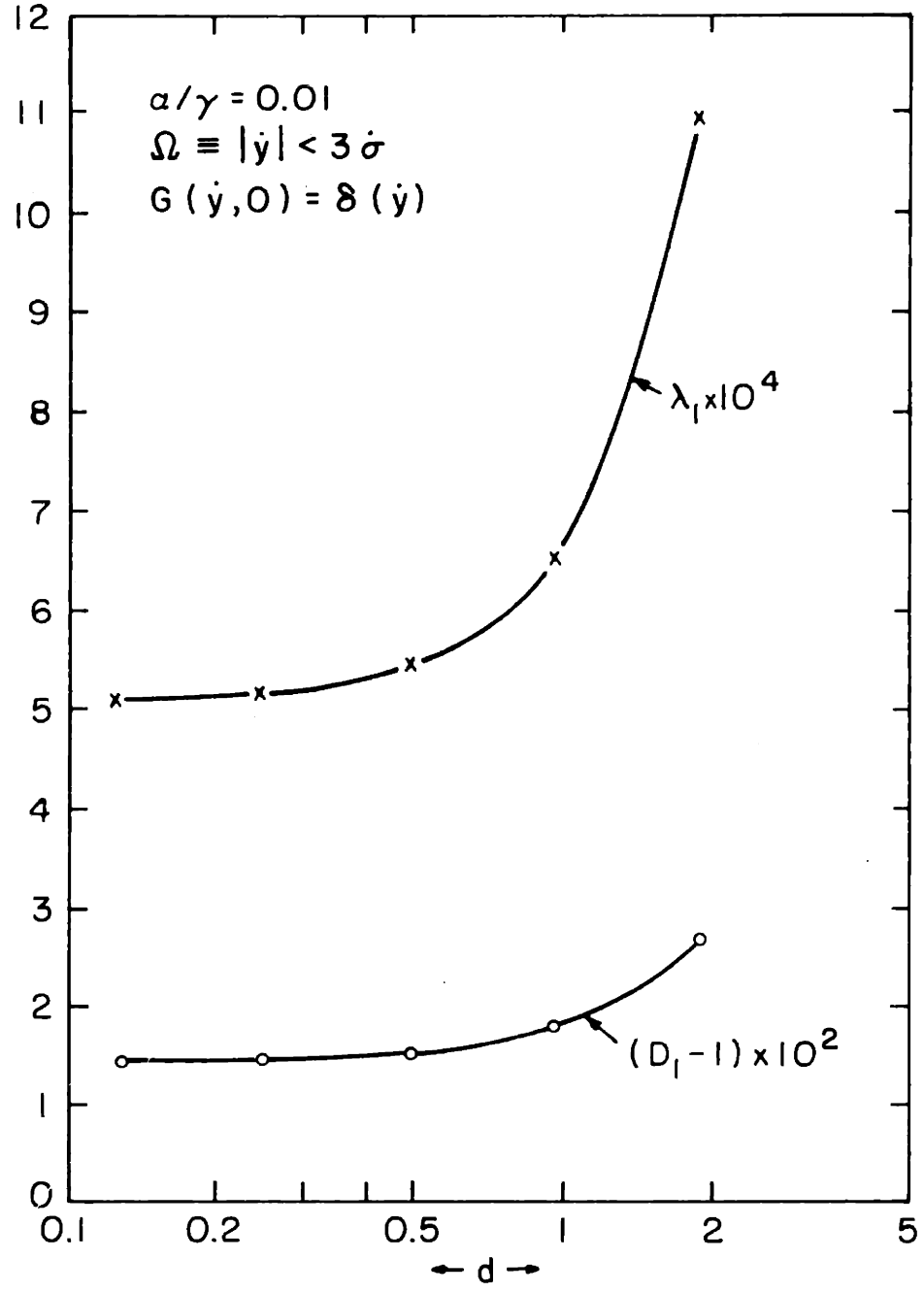


Figure 7

One dimensional process $\dot{y}$. Variation of λ_1 and D_1 with size factor d.

3.5 Results.

All the results given here were obtained with the use of the original transition matrix $P(m \mid n)$ (evaluated from relation 51).

Fig. 8 shows $Q\left(\frac{N\pi}{\beta}\right)$, $f\left(\frac{N\pi}{\beta}\right)$ and $\bar{P}\left(\frac{N\pi}{\beta}\right)$ for $\Omega \equiv |\dot{y}| < 2\dot{\sigma}$ $G(\dot{y}, 0)$ = stationary distribution (45) inside Ω, $\frac{\alpha}{\gamma} = 0.01$ and $d = 0.25$.

P_1 is found to be = 0.99433 (hence $\lambda_1 = -\ln P_1 = 0.00568$), and $D_1 = 0.9343$ as compared to $Q(0) = 0.9545$ (Note that $Q(0) = \sum_{i=1}^{\infty} D_i$).

Thus the exponential approximation

$$Q_{exp}(t) = Q(0)\, e^{-\lambda_1 \frac{\beta}{\pi} t}$$

$$= \left[\int_{\Omega} \frac{1}{\sqrt{2\pi}\,\dot{\sigma}} e^{-\frac{\dot{y}^2}{2\dot{\sigma}^2}} d\dot{y}\right] e^{-\lambda_1 \frac{\beta}{\pi} t}, \qquad (64)$$

is quite fair in this case.

$Q_{exp}\left(N\frac{\pi}{\beta}\right)$ and $f_{exp}\left(N\frac{\pi}{\beta}\right) = Q_{exp}\left((N-1)\frac{\pi}{\beta}\right) - Q_{exp}\left(N\frac{\pi}{\beta}\right)$

are also shown in Fig. 8.

Fig. 9 shows similar results with $G(\dot{y}, 0) = \delta(\dot{y})$.

Here P_1 is again = 0.99433 and $\lambda_1 = 0.00568$. This is only to be expected since the transition matrix $P(m \mid n)$ is common to both the cases. However, now $D_1 = 1.0834$ as compared to $Q(0) = 1$.

In addition to

$$Q_{exp}(t) = Q(0)\, e^{-\lambda_1 \frac{\beta}{\pi} t} = e^{-\lambda_1 \frac{\beta}{\pi} t}, \qquad (65)$$

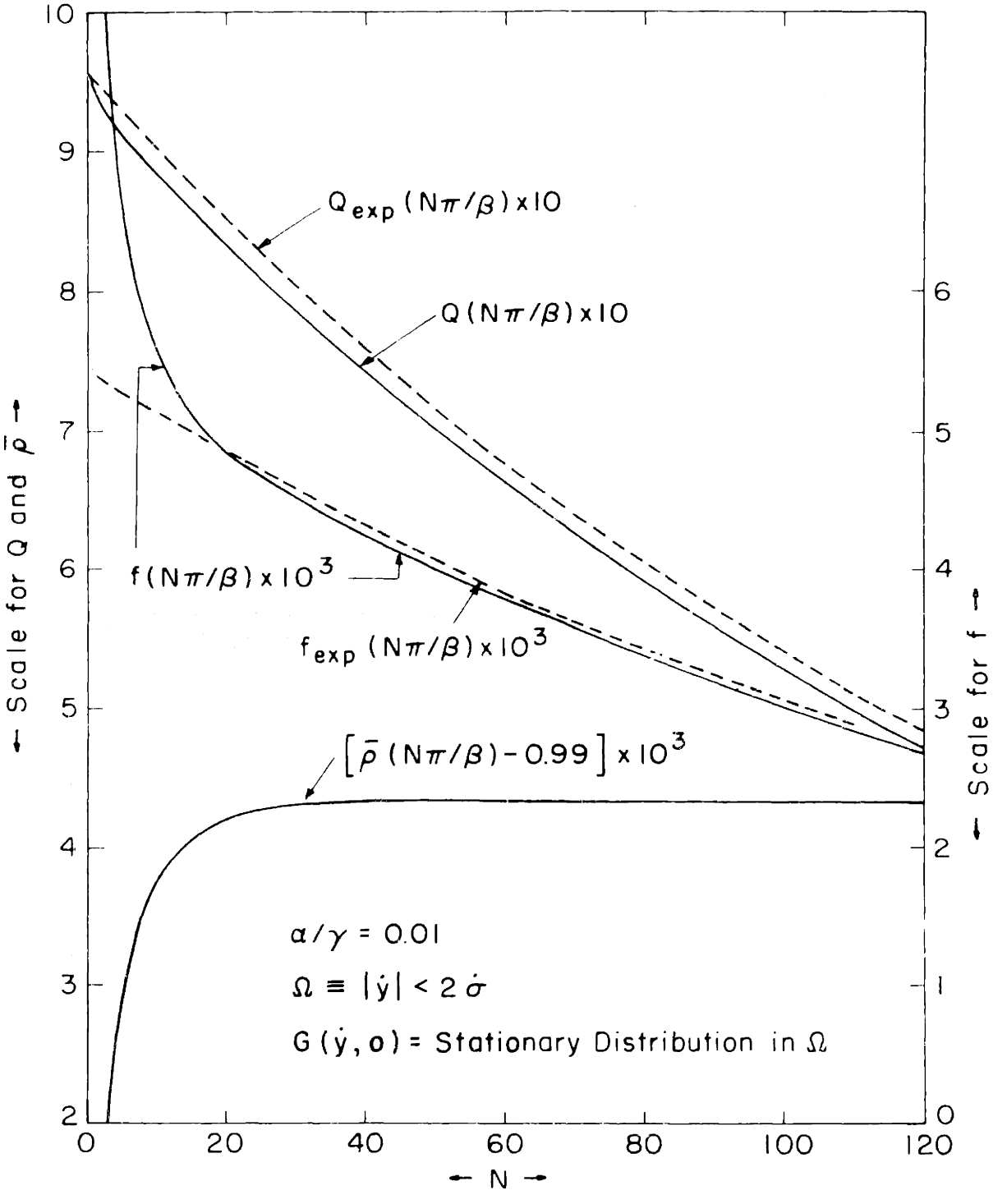


Figure 8

One dimensional process $\dot{y}$. Computed first passage probabilities and exponential approximations.

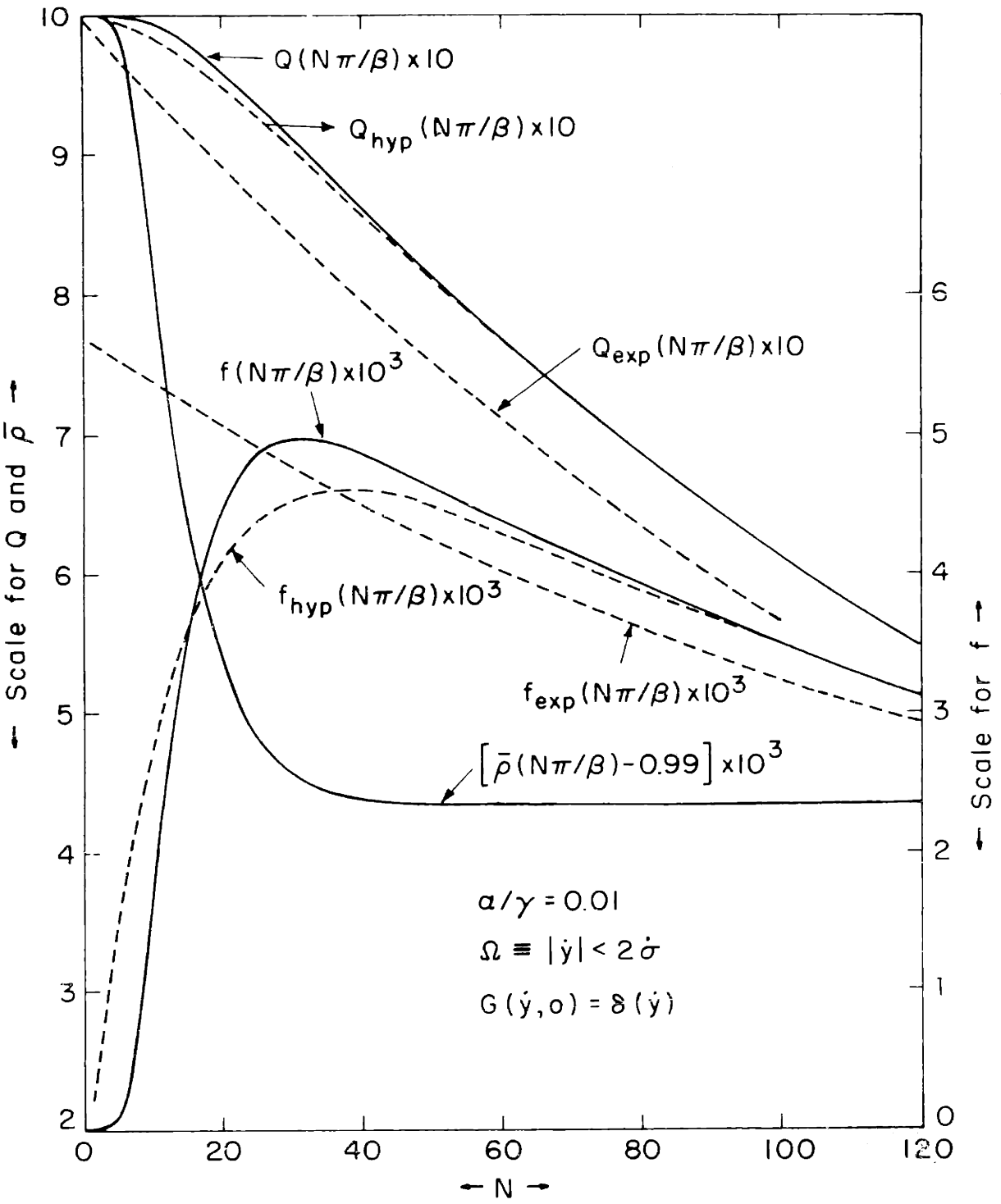


Figure 9

One dimensional process $\dot{y}$. Computed first passage probabilities, exponential and hyperexponential approximations.

Fig. 9 also shows the hyperexponential approximation

$$Q_{hyp}(t) = D_1 e^{-\lambda_1 \frac{\beta}{\pi} t} + D_2^* e^{-\lambda_2^* \frac{\beta}{\pi} t} \tag{66}$$

D_2^* and λ_2^* are determined from the conditions

$$Q_{hyp}(0) = 1$$

$$f_{hyp}(0) = -\frac{\partial}{\partial t} Q_{hyp}(t)\Big|_{t=0} = 0$$

Thus one gets

$$D_2^* = 1 - D_1 \tag{67}$$

$$\lambda_2^* = -\frac{\lambda_1 D_1}{1 - D_1} \tag{68}$$

The following values of D_1 and λ_1 were computed for $\Omega \equiv |\dot{y}| < a\dot{\sigma}$, $G(\dot{y}, 0) = \delta(\dot{y})$, $\frac{\alpha}{\gamma} = 0.01$ and d = 0.25

a	λ_1	D_1	$\lambda_1 / e^{-\frac{a^2}{2}}$
0.5	0.1697	1.19	0.1922
1.0	0.453×10^{-1}	1.1876	0.0746
2.0	0.568×10^{-2}	1.0834	0.0419
3.0	0.506×10^{-3}	1.0141	0.0455
5.0	0.440×10^{-6}	nearly 1	0.1182

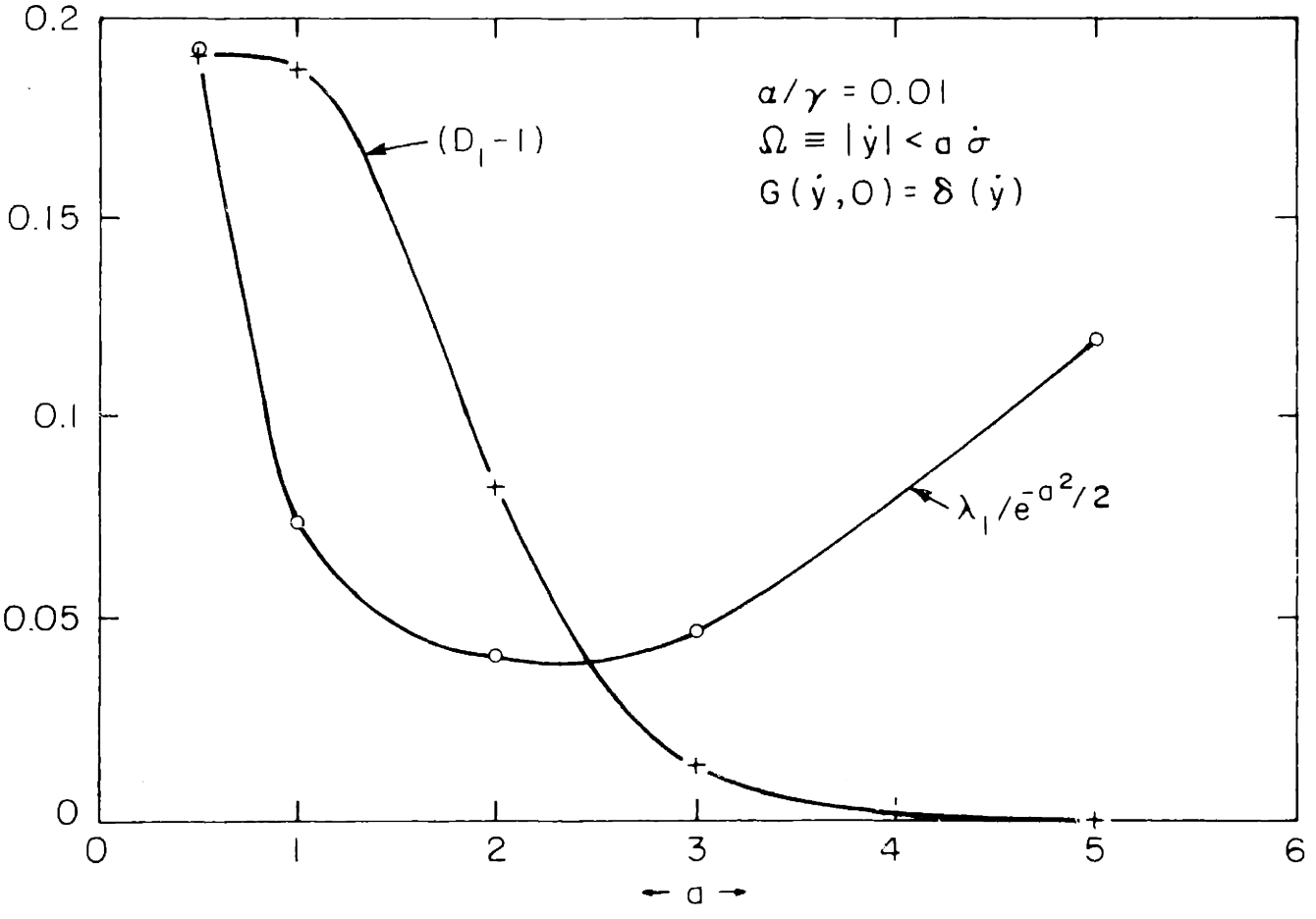


Figure 10

One dimensional process $\dot{y}$. Variation of $\lambda_1 / e^{-a^2/2}$ and D_1 with a.

Variation of λ_1 with "a" is of the order of $e^{-a^2/2}$, as seen by the values of $\lambda_1 / e^{-\frac{a^2}{2}}$. D_1 becomes quite close to 1 for $a \geqslant 3$. Fig. 10 shows the variation of $\lambda_1 / e^{-\frac{a^2}{2}}$ and D_1 with "a".

It becomes very hard to get any meaningful results for $a \geqslant 5$. This is because one computes ρ_1 rather than λ_1 directly. Since a digital computer cannot give an accuracy of more than 9 significant digits for single precision arithmetic, it is possible to compute ρ_1 only if it is less than 0.999999999. Even at $a = 5$, $\lambda_1 = 0.44 \times 10^{-6}$ or $\rho_1 = 0.999999560$ and this takes up nearly all the nine significant digits.

3.6 A Modified Computational Procedure.

With $\frac{\alpha}{\gamma} = 0.01$, $d = 0.25$, the number of elements of length $\Delta\dot{y} = d\sigma_A$ in the region $|\dot{y}| \leqslant 4\dot{\sigma}$ is given by

$$2M+1 = \frac{2a}{d(1-K^2)^{1/2}} \simeq 129 \qquad \text{(see relation 62)}$$

If advantage is taken of the symmetry of the problem, the size of the transition matrix $P_{Sym}(m/n)$ is $(M+1)^2$ or 4225. Now if one were to consider a similar problem for the two dimensional oscillator ouptut $(y(t), \dot{y}(t))$, the equivalent region Ω would be an area in the phase plane with roughly $(2M+1)^2$ rectangular elements of size Δy, $\Delta\dot{y}$. Symmetry would enable consideration of only $(M+1)^2$ elements, with the resulting transition matrix of size $(M+1)^4$ or nearly 17.8×10^6. This is well beyond the capacity of a digital

computer (IBM 7094 has a total capacity of 32,768 storage locations out of which at least 4,000 locations have to be utilized to store the programme and various routines).

With a view towards making similar computations possible for the two dimensional case, following modification of the present method was considered.

First of all one must note that the unit step transition probability $P(\dot{y}_1 \mid \dot{y}, \frac{\Pi}{\beta})$ (relation 50) involves decay of the process from initial value $\dot{y}_1$, to a mean value $\dot{y}_m = \dot{y}_1 k$, about which it is normally distributed with variance σ_A^2. Since 99.98% of the area under a normal distribution lies within ± 4 standard deviations, the same percentage of a probability mass at $\dot{y}_1$ would be distributed (after one step of the process) within $\frac{8}{d}$ consecutive elements of length $\Delta\dot{y}$. (Note that $d = \frac{\Delta\dot{y}}{\sigma_A}$).

Secondly, let each element of length $\Delta\dot{y}$ be divided into s equal divisions of length Δ_s, such that $s\Delta_s = \Delta\dot{y}$. Consider the element $\Delta\dot{y}$ with its middle point at $\dot{y} = n\Delta\dot{y}$ (Fig. 11). Let its s subdivisions of length Δ_s be denoted by n1, n2, n3, ---, ns, as shown in Fig. 11, coordinates of their middle points being denoted by $\dot{y}_{n1}$, $\dot{y}_{n2}$, ---, $\dot{y}_{ns}$ respectively. Let nu and nv be two such subdivisions symmetrically placed with respect to the middle point $\dot{y} = n\Delta\dot{y}$, that is $u + v = s + 1$.

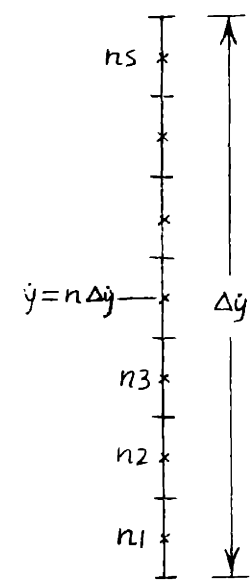


Figure 11

Because of the symmetry of the normal distribution it is clear that

$$P_n(u,r) = P_n(v,-r) \tag{69}$$

and $$P_m(u,r) = P_n(u,r) \tag{70}$$

where $P_n(u,r)$ is defined as

$$P_n(u,r) = \int_{\omega(n+r)} \frac{1}{\sqrt{2\pi}\,\sigma_A} e^{-\frac{(\dot{y}^2 - \dot{y}_{nu})^2}{2\sigma_A^2}} d\dot{y} \tag{71}$$

where u = 1, 2 ---, s; n and r are any integers and (n + r) represents an element of length $\Delta\dot{y}$ with its middle point at $\dot{y} = (n + r)\Delta\dot{y}$.

Thus if the (reflected) mean value $\dot{y}_m = |K| m \Delta\dot{y}$ $(m = 0, 1, 2, \cdots M)$ is approximated to the nearest point in the collection $\dot{y}_{nu}$ (n = 0,1,2, --- M, u = 1, 2, --- s) and further if the probability mass at y = m $\Delta\dot{y}$ is distributed after one step within $\frac{8}{d}$ consecutive elements (of length $\Delta\dot{y}$) around the approximated mean, the total number of storage locations required in the computer to carry out the computation would be considerably reduced.

Thus the modified procedure would be as follows,

(i) Decide on $\frac{\alpha}{\gamma}$, d, a (so that $\Omega \equiv |\dot{y}| < a\dot{\sigma}$) and s.

(ii) Determine the nearest integer (2M + 1) which best satisfies relation (62). Get d_{ef}, the effective d from the same relation. Then $\Delta\dot{y} = d_{ef}$ (with $\sigma_A = 1$, a convenient normalization).

(iii) Determine the nearest integer $Z \simeq \frac{4}{d_{ef}}$, the number of elements of length $\Delta\dot{y}$ within 4 standard deviations of the unit step transition probability.

(iv) Compute the probability areas

$$P_0(u,r) = \int_{\omega(r)} \frac{1}{\sqrt{2\pi}\,\sigma_A} e^{-\frac{(\dot{y}-\dot{y}_{ou})}{2\sigma_A^2}} d\dot{y} \qquad (72)$$

$$u = 1, 2, \text{-----} S$$

$$r = 0, 1, 2, \text{----} Z$$

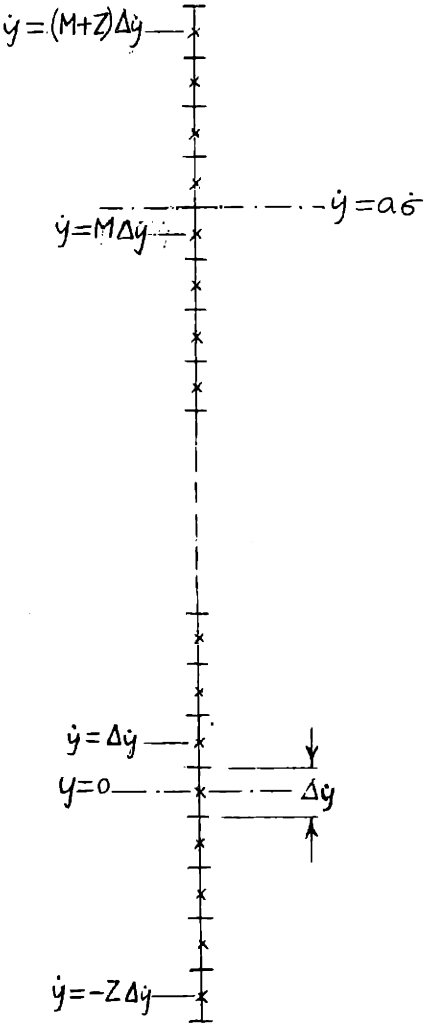
This is just relation (71) with n = 0.

(v) Set up two arrays A(m), B(m) each of size (M + 1 + 2Z) as shown in Fig. 12.

Assign the initial distribution $G(\dot{y}, 0)$ in array A according to relation (58).

(vi) Set the whole array B = 0.

Do the following for m = 0, 1, 2, ---, M. For each m approximate the mean $|K|\, m\Delta\dot{y}$ to the nearest $\dot{y}_{nu}$, then distribute the probability mass at A(m) in the array B as follows,



Array A or B

Figure 12

$$B(n+r) = A(m)\ P_0(u,r) \quad \text{for } r = 0, 1, 2, \text{----} Z$$

$$B(n-r) = A(m)\ P_0(u,r) \quad \text{for } r = 1, 2, \text{-----} Z \qquad (73)$$

(vii) Discard the probability masses in B(m) for $m > M + 1$. Add up the probability mass B(-m) to B(m) and set B(-m) = 0, for m = 1, 2, ---,Z.

(viii) Now B(m), m = 0, 1, 2, ---, M, represents $G(m\,\Delta\dot{y}, \frac{\pi}{\beta})$. To obtain $G(n\,\Delta\dot{y}, \frac{2\pi}{\beta})$ one goes back to step (vi), and repeates everything with A and B interchanged.

Of course at every step $Q(\frac{N\pi}{\beta}) = 2\sum_{m=0}^{M} G(m\Delta\dot{y}, \frac{N\pi}{\beta})$

For $\frac{\alpha}{\gamma} = 0.01$, d = 0.25, a = 3 and s = 4; 2M + 1 = 97, and Z = 16 and total number of constants $P_0(u,r) = sZ = 64$. Hence total number of storage locations required are

$$= 2(M + 1 + 2Z) + sZ = 226$$

The exponential factor λ_1 for this case was found to be $= 0.539 \times 10^{-3}$. This is in good agreement with corresponding result from the original procedure (0.541×10^{-3}), which by the way required $2(M + 1) + (M + 1)^2 = 2499$ storage locations.

The purpose of providing an extra margin of Z elements on each end of arrays A and B is to avoid cumbersome and time consuming logic required to continuously check whether in relations (73), $l+r > M$ (in which case probability masses leave Ω) and whether $l-r < 0$ (in which case probability masses have to be "reflected" back to the posi-

tive half of Ω).

Before passing on to the case of white Gaussian excitation, a brief mention must be made of computations made with excitation to the oscillator consisting of more than 2 (8 or 16 say) periodic Gaussian impulses per cycle. In that case the output $\left(y(t), \dot{y}(t)\right)$ would be two dimensional. The computational procedure was quite similar to one described in the next section. The distributions $G(y, \dot{y}, t)$ inside the two dimensional region Ω were however found to be unsatisfactory in the y - direction, presumably because the impulses in the excitation would cause direct random changes only in velocity $\dot{y}$. Thus the approximate locating of mean positions $(y_m , \dot{y}_m)$ at every step involved large errors in the y - direction. Surprisingly though, the exponential factors λ_1 computed by this procedure turned out to be in fairly good agreement with subsequent results with white Gaussian excitation.

Section 4

WHITE GAUSSIAN EXCITATION

4.1 The Discussion of the Computational Procedure.

For white Gaussian excitation the transition probability is given by relation (16). This is bivariate Gaussian distribution with positive cross-correlation between the two random variables y and $\dot{y}$.

Regions Ω of interest are (i) $|y| < a\sigma$, that is, first passage through an absolute displacement barrier and (ii) $y^2 + \dot{y}^2/\gamma^2 < a^2\sigma^2$, that is, first passage through an energy (or envelope) level. σ^2 and $\dot{\sigma}^2$ are the stationary mean square values of y(t) and $\dot{y}(t)$ respectively (relations 12 and 14).

To make the computations feasible for $\Omega \equiv |y| < a\sigma$, Ω has to be bounded in the direction of $\dot{y}$ as well. These bounds, $|\dot{y}| < \bar{a}\dot{\sigma}$ can be chosen such that $\bar{a} \simeq 2a$. In that case, probability of approaching $\dot{y}$ - boundaries is much smaller than the probability of approaching y - boundaries. Further, if the oscillator output approaches an absolute velocity of $\bar{a}\dot{\sigma}$, probability that the absolute displacement $|y|$ exceeds $a\sigma$ within a short time would be very high.

For envelope boundary of course the region Ω is already finite.

The first passage probabilities are again computed by distribution of probability masses in Ω after each time element Δt. For example $\Delta t = \pi/4\gamma$ would correspond to 8 iterations per natural cycle of th oscillator. For convenience we shall write τ instead of Δt.

The variances σ_τ^2, $\dot{\sigma}_\tau^2$ and cross-correlation coefficient ρ_τ for the unit step transition probability $P(y_0, \dot{y}_0 \mid y, \dot{y}, \tau)$ are given by relations (11), (13) and (15) respectively.

The region Ω under consideration is divided into rectangular elements of size Δy, $\Delta \dot{y}$. For example, for $\Omega \equiv (|y| < a\sigma, |\dot{y}| < \bar{a}\dot{\sigma})$ there are $2M+1$ divisions in the y - direction and $2\bar{M}+1$ elements in the $\dot{y}$ - direction, so that

$$\Delta y = \frac{2a\sigma}{2M+1} \tag{74}$$

$$\Delta \dot{y} = \frac{2\bar{a}\dot{\sigma}}{2\bar{M}+1} \tag{75}$$

The center points of these rectangular elements have the coordinates $(n\,\Delta y,\ \bar{n}\,\Delta \dot{y})$, $n = 0, \pm 1, \pm 2, \cdots, \pm M$; $\bar{n} = 0, \pm 1, \pm 2, \cdots, \pm \bar{M}$.

Analogous to relation (62), the size factors here are defined as

$$d_y = \frac{\Delta y}{\sigma_\tau} = \frac{2a\sigma}{(2M+1)\sigma_\tau} \tag{76}$$

$$d_{\dot{y}} = \frac{\Delta \dot{y}}{\dot{\sigma}_\tau} = \frac{2\bar{a}\dot{\sigma}}{(2\bar{M}+1)\dot{\sigma}_\tau} \tag{77}$$

$$\text{and } d = d_y d_{\dot{y}} \tag{78}$$

Now consider the unit step transition probability,

$$P(y_0, \dot{y}_0 \mid y, \dot{y}, \tau) =$$

$$\frac{1}{2\pi \sigma_\tau \dot{\sigma}_\tau (1-\rho_\tau^2)^{1/2}}\, e^{-\frac{1}{2(1-\rho_\tau^2)}\left[\frac{(y-C)^2}{\sigma_\tau^2} - \frac{2\rho_\tau}{\sigma_\tau \dot{\sigma}_\tau}(y-C)(\dot{y}-\dot{C}) + \frac{(\dot{y}-\dot{C})^2}{\dot{\sigma}_\tau^2}\right]} \tag{79}$$

The mean position $\left(E[y], E[\dot{y}]\right) = (C, \dot{C})$ where C and $\dot{C}$ are given by relations (2) and (2').

For an elliptic region Γ defined as

$$\Gamma \equiv \left\{ \frac{1}{(1-\rho_\tau^2)} \left[\frac{(y-C)^2}{\sigma_\tau^2} - \frac{2\rho_\tau}{\sigma_\tau \dot{\sigma}_\tau} (y-C)(\dot{y}-\dot{C}) + \frac{(\dot{y}-\dot{C})^2}{\dot{\sigma}_\tau^2} \right] \leq R^2 \right\}, \quad (81)$$

the probability volume

$$\iint_\Gamma P(y_0, \dot{y}_0 \mid y, \dot{y}, \tau)\, dy\, d\dot{y} = 1 - e^{-R^2/2} \quad (82)$$

This integration can be performed by a transformation to coordinates r, θ where

$$r\cos\theta = \frac{1}{[2(1+\rho_\tau)]^{1/2}} \left[\frac{y-C}{\sigma_\tau} + \frac{\dot{y}-\dot{C}}{\dot{\sigma}_\tau} \right]$$

$$r\sin\theta = \frac{1}{[2(1-\rho_\tau)]^{1/2}} \left[\frac{y-C}{\sigma_\tau} - \frac{\dot{y}-\dot{C}}{\dot{\sigma}_\tau} \right]$$

Further let all the elemental lengths Δy and $\Delta \dot{y}$ be subdivided into s and $\bar{s}$ equal divisions of lengths Δ_s and $\Delta_{\bar{s}}$ respectively, such that $s\Delta_s = \Delta y$, $\bar{s}\Delta_{\bar{s}} = \Delta \dot{y}$. Consider the rectangular element $\Delta y\, \Delta \dot{y}$ with its center point having the coordinates $(n\Delta y, \bar{n}\Delta \dot{y})$.

Let its $s\bar{s}$ rectangular subdivisions (each of area $\Delta_s \Delta_{\bar{s}}$) be denoted as $n\bar{n}$ 11, $n\bar{n}$ 12, ---, $n\bar{n}$ 1s; $n\bar{n}$ 21, ---, $n\bar{n}$ 2s; ---; ---; $n\bar{n}$ $\bar{s}$1, ---, $n\bar{n}$ $\bar{s}\bar{s}$, as shown in Fig. 13, the coordinates of the centerpoint of subdivision $n\bar{n}$ $u\bar{u}$ being denoted by $(y_{nu} , \dot{y}_{\bar{n}\bar{u}})$.

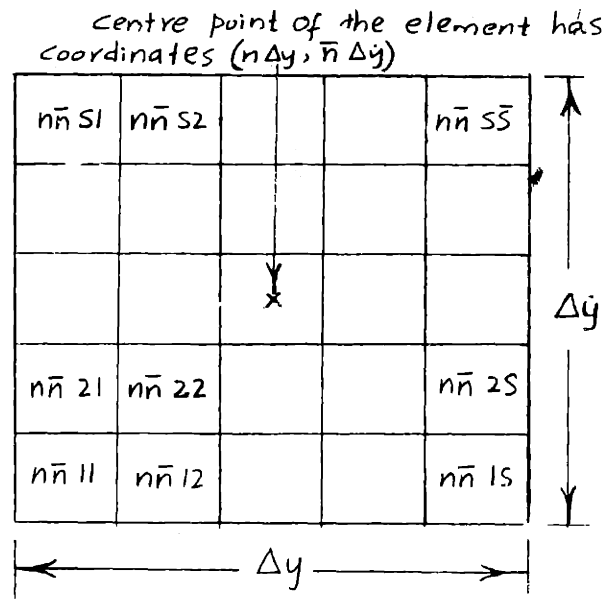


Figure 13

Let $n\bar{n}$ $u\bar{u}$ and $n\bar{n}$ $v\bar{v}$ be two such rectangular subelements, symmetrically placed with respect to the center point $(n\Delta y , \bar{n}\Delta\dot{y})$ of their parent element, that is, $u + v = s + 1$, $\bar{u} + \bar{v} = \bar{s} + 1$.

Because of the symmetry of the bivariate Gaussian distribution about its mean, it is clear that

$$P_{n,\bar{n}} (u, \bar{u} ; r, \bar{r}) = P_{n,\bar{n}} (v, \bar{v} ; -r, -\bar{r}) , \qquad (82)$$

and

$$P_{m,\bar{m}} (u, \bar{u} ; r, \bar{r}) = P_{n,\bar{n}} (u, \bar{u} ; r, \bar{r}) , \qquad (83)$$

where $P_{n,\bar{n}} (u, \bar{u} ; r, \bar{r})$ is defined as,

$$P_{n,\bar{n}}\left(u,\bar{u};r,\bar{r}\right) = \frac{1}{2\pi\,\sigma_\eta\,\dot{\sigma}_\eta\left(1-\rho_\eta^{\,2}\right)^{1/2}} \times$$

$$\iint_{\omega(n+r,\,\bar{n}+\bar{r})} e^{-\frac{1}{2(1-\rho_\eta^{\,2})}\left[\frac{(y-y_{nu})^2}{\sigma_\eta^{\,2}} - \frac{2\rho_\eta}{\sigma_\eta\dot{\sigma}_\eta}(y-y_{nu})(\dot{y}-\dot{y}_{\bar{n}\bar{u}}) + \frac{(\dot{y}-\dot{y}_{\bar{n}\bar{u}})^2}{\dot{\sigma}_\eta^{\,2}}\right]} dy\,d\dot{y} \qquad (84)$$

$$u = 1, 2, ----, 5$$

$$\bar{u} = 1, 2, ----, \bar{5}$$

$u, \bar{u}, r, \bar{r}$ are integers and $\omega(n+r, \bar{n}+\bar{r})$

represents a rectangular element of area $\Delta y\ \Delta\dot{y}$ with its center point having coordinates $\left((n+r)\,\Delta y\,,\,(\bar{n}+\bar{r})\,\Delta\dot{y}\right)$.

For all the cases considered the region Ω and the initial probability distribution in Ω, $G(y,\dot{y},0)$ are symmetric with respect to the origin (0, 0) of the phase plane. The whole problem, therefore, possesses the same symmetry. Hence it is necessary to consider only half of Ω, say the half for which $y \geqslant 0$.

For computations the probability masses in rectangular elements $\Delta y\ \Delta\dot{y}$ are assumed to be concentrated as impulses at their center points. After each time step τ, each of these probability masses is distributed in the half of the region Ω (for which $y \geqslant 0$) according to the transition probability (79), its mean position $(C, \dot{C})$ being approximated to the nearest point in the collection $(\ y_{nu}, \dot{y}_{\bar{n}\bar{u}}\ )$, n = 1, 2, ---, M; $\bar{n}$ = 1, 2, ---, $\bar{M}$;

$u = 1, 2, \cdots s$; $\bar{u} = 1, 2, \cdots \bar{s}$; and its distribution being restricted to the elliptic region Γ (see relation 80). For $R^2 = 15.2$ in relation (81), 99.95% of the probability mass will be distributed within Γ.

4.2 The Computational Procedure in Detail.

The detailed computational procedure for $\Omega \equiv (|y| < a\sigma, |\dot{y}| < \bar{a}\dot{\sigma})$ is as follows,

(i) Decide on $\frac{\alpha}{\gamma}$, τ, $d_y, d_{\dot{y}}$ (see relations 76, 77), $s, \bar{s}$, and R. For example let $\frac{\alpha}{\gamma} = 0.04$, $\tau = \frac{\pi}{4\gamma}$, $d_y = 0.65, d_{\dot{y}} = 0.52, s = \bar{s} = 4$, and $R = 3.4$.

(ii) Compute $\sigma_\tau, \dot{\sigma}_\tau$ and ρ_τ. For the values specified above,

$$\sigma_\tau = 0.1477\sigma, \quad \dot{\sigma}_\tau = 0.3114\dot{\sigma}, \quad \rho_\tau = 0.8172.$$

Graphically plot the boundary of the elliptical region Γ',

$$\Gamma' \equiv \left\{ \frac{1}{(1-\rho_\tau^2)} \left[y'^2 d_y^2 - 2\rho_\tau (y' d_y)(\dot{y}' d_{\dot{y}}) + \dot{y}'^2 d_{\dot{y}}^2 \right] \leqslant R^2 \right\} \quad (85)$$

Γ' is obtained from the region Γ (relation 80) by first shifting the center $(C, \dot{C})$ to $(0, 0)$ and then substituting $y = y' \Delta y$, $\dot{y} = \dot{y}' \Delta\dot{y}$.

For the values specified, the region Γ' (for $\dot{y}' \geqslant 0$) is shown in Fig. 14. The rectangular elements in the same figure are of the size 1,1. Element number 0 is centered around the origin (0, 0). The dotted lines enclose any extra area covered by the boundary of Γ' if the whole region Γ' is shifted around, such that its center can occupy any possible position in element number 0. The outermost boundary made up of straight segments determines the necessary arrangement of rectangular elements of size 1,1 which are required to completely enclose the elliptical region Γ' with its center anywhere in element number 0. These

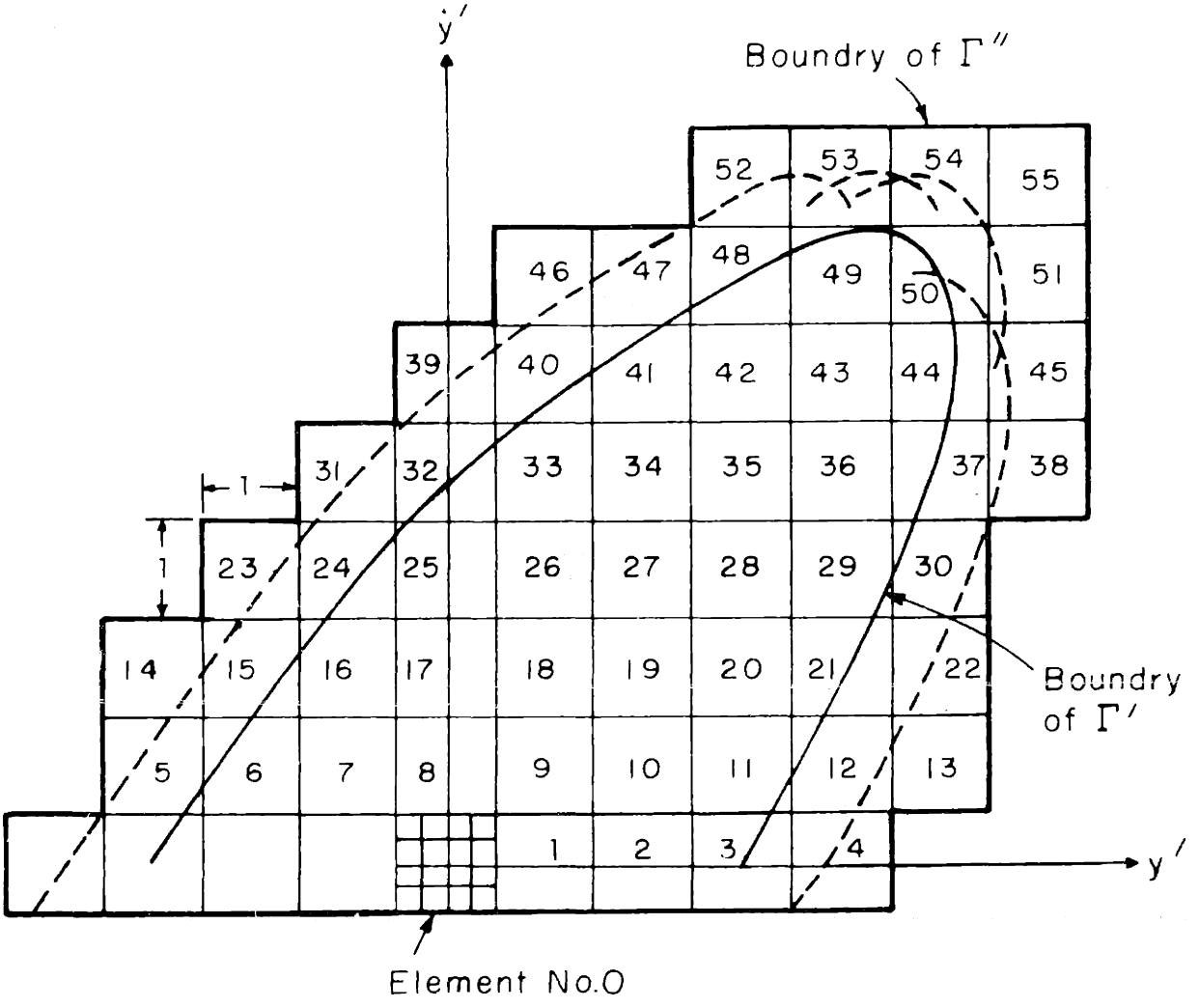


Figure 14

Regions Γ' and Γ''.

elements are suitably numbered as shown. Let this arrangement of elements be denoted by Γ''. Because of the symmetry of the situation, Γ'' for $\dot{y}' \leqslant 0$ will be symmetric (with respect to the origin) to its positive counterpart.

Let $(2Z + 1)$ and $(2\bar{Z} + 1)$ represent the maximum number of elements spanned by the whole of Γ'' (including its half for $\dot{y}' \leqslant 0$) in y' and $\dot{y}'$ directions respectively. Also let total number of elements in Γ'' be denoted by $2J + 1$. For Fig. 14, $2Z + 1 = 13$, $2\bar{Z} + 1 = 15$, and $2J + 1 = 111$.

Let the coordinates of the center point of element number j be denoted by $(k_j , \bar{k}_j)$. Also let element number 0 be subdivided into ss rectangular subdivisions in exactly the same way as shown in Fig. 13, the coordinates of the center point of subidivision oo $u\bar{u}$ being denoted by $(y'_{ou} , \dot{y}'_{o\bar{u}})$. Then it is easy to see that

$$P'(u,\bar{u},j) = P_{o,o}(u,\bar{u};\ k_j , \bar{k}_j)$$

where the right hand side is given by relation (84) and the left hand side is defined as

$$P'(u,\bar{u},j) = \frac{d_y d_{\dot{y}}}{2\pi (1-\rho_T^2)^{1/2}} \times$$

$$\iint_{\omega(j)} e^{-\frac{1}{2(1-\rho_T^2)}\left[(y'-y'_{ou})^2 d_y^2 - 2\rho_T (y'-y'_{ou})(\dot{y}'-\dot{y}'_{o\bar{u}}) d_y d_{\dot{y}} + (\dot{y}'-\dot{y}'_{ou})^2 d_{\dot{y}}^2\right]} dy' d\dot{y}' \qquad (86)$$

$u = 1,2, \text{---} s$

$\bar{u} = 1,2, \text{---} \bar{s}$

$j = 0,1,2, \text{---} J$

$\omega(j) \equiv$ rectangular element number j(of size 1,1) in Γ''.

(iii) Now the computations can be performed with a single computer programme for any general values of a and $\bar{a}$.

With given values of a and $\bar{a}$, determine the smallest integer $(2M+1)$ which best satisfies relation (76), then determine $d_{y_{ef}}$, the effective dy from the same relation. Similarly determine $2\bar{M}+1$ and $d_{\dot{y}_{ef}}$. This procedure ensures that $dy_{ef} \geqslant dy$, $d\dot{y}_{ef} \geqslant d\dot{y}$.

(iv) Compute probability volumes $P'(u,\bar{u},j)$ given by relation (86) (with effective values dy_{ef} and $d\dot{y}_{ef}$), using parabolic rule for numerical integration.

With $dy_{ef} \times d\dot{y}_{ef} \geqslant dy\, dy$, one is assured that the actual probability volume enclosed in the whole of Γ'' is $\geqslant 1 - e^{-R^2/2}$.

(v) Compute the effective probability volumes

$$P(u,\bar{u},j) = P'(u,\bar{u},j)/\mu(u,\bar{u}) \tag{87}$$

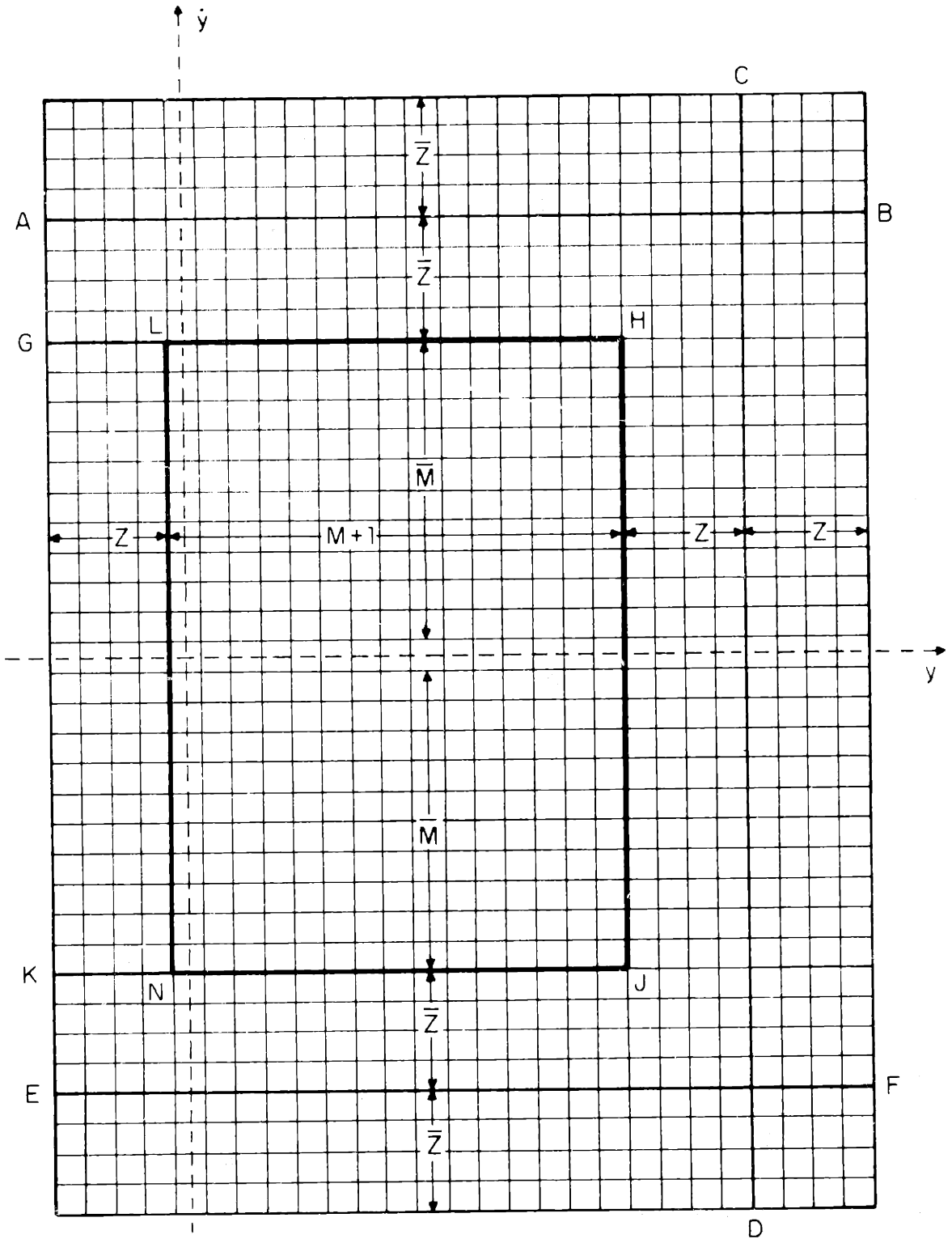
$$\text{where } \mu(u,\bar{u}) = P'(u,\bar{u},0) + 2\sum_{j=1}^{J} P'(u,\bar{u},j) \tag{88}$$

Thus the probabilities $P'(u,\bar{u},j)$ are normalized to $P(u,\bar{u},j)$ so that their sum over $\Gamma'' = 1$.

Also compute

$$\mu = \frac{1}{s\bar{s}} \sum_{u=1}^{s} \sum_{\bar{u}=1}^{\bar{s}} \mu(u,\bar{u}) \tag{89}$$

(vi) Set up 2 two-dimensional arrays $A(m,\bar{m})$, $B(m,\bar{m})$ each of size $(M + 1 + 3Z,\ 2\bar{M} + 1 + 4\bar{Z})$ as shown in Fig. 15.



Array A or B (drawn for $M = 14$, $\bar{M} = 10$, $Z = \bar{Z} = 4$)

Figure 15

Two dimensional array A or B.

In each of these arrays, an element $(m, \bar{m})$ corresponds to a rectangular element of size $\Delta y, \Delta \dot{y}$ with its center having the coordinates $(m\ \Delta y, \bar{m}\ \Delta \dot{y})$.

(vii) Assign the initial distribution $G(y, \dot{y}, 0)$ in array A as follows

$$A(m, \bar{m}) = \int_{m\Delta y - \Delta y/2}^{m\Delta y + \Delta y/2} \int_{\bar{m}\Delta \dot{y} - \Delta \dot{y}/2}^{\bar{m}\Delta \dot{y} + \Delta \dot{y}/2} G(y, \dot{y}, 0)\, dy\, d\dot{y}$$

$$\text{for } 1 \leqslant m \leqslant M$$

$$-\bar{M} \leqslant \bar{m} \leqslant \bar{M} \qquad (90)$$

$$= \int_{-\Delta y/2}^{\Delta y/2} \int_{\bar{m}\Delta \dot{y} - \Delta \dot{y}/2}^{\bar{m}\Delta \dot{y} + \Delta \dot{y}/2} G(y, \dot{y}, 0)\, dy\, d\dot{y}$$

$$\text{for } m = 0$$

$$-\bar{M} \leqslant \bar{m} \leqslant \bar{M}$$

(viii) Set the whole array B = 0.

Do the following for m = 0, 1, 2, --- M; $\bar{m}$ = $-\bar{M}$, ---1, 0, 1, ---$\bar{M}$ (total of $(M + 1)(2\bar{M}+1)$ times)

Compute the mean position of $(m\,\Delta y, m\ \Delta \dot{y})$ after time τ from

relations (2) and (2'). Let this mean position be denoted by $\nu_{\bar{m}}$ with coordinates ($C_m \Delta y$, $C_{\bar{m}} \Delta \dot{y}$).

If $\nu_{\bar{m}}$ lies in the half of Ω for which y $\leq$ 0 (i.e. $C_m \leq 0$), "reflect" $\nu_{\bar{m}}$ across the origin to position ν_{mr} with coordinates $(-C_m \Delta y , -C_{\bar{m}} \Delta \dot{y})$

With reference to Fig. 15, if $\nu_{\bar{m}}$ (or ν_{mr}) lies to the right of line CD (i.e., $C_m > M + Z + 1/2$) or if $\nu_{\bar{m}}$ lies above line AB or below line EF (i.e., $|C_{\bar{m}}| > \bar{M} + \bar{Z} + 1/2$), discard the contribution of the probability mass at position A(m, $\bar{m}$)

Otherwise, approximate point $\nu_{\bar{m}}$ (or ν_{mr}) to the nearest point in the collection (y_{nu} , $\dot{y}_{\bar{n}\bar{u}}$); $0 \leq n \leq M$, $-\bar{M} \leq \bar{n} \leq \bar{M}$, $1 \leq u \leq s$, $1 \leq \bar{u} \leq \bar{s}$. (see Fig. 13).

Distribute the probability mass at A(m, m) in the array B as follows,

$$B(n + k_j , \bar{n} + \bar{k}_j) = A(m, \bar{m})\ P(u, \bar{u}, j) \qquad \text{for } j = 0, 1, 2, \text{----} J \tag{91}$$

$$B(n - k_j , \bar{n} - \bar{k}_j) = A(m, \bar{m})\ P(s-u, \bar{s}-\bar{u}, j) \qquad \text{for } j = 1, 2, \text{----} J$$

Here P(u, $\bar{u}$, j) is given by relations (86), (87) and (k_j , $\bar{k}_j$) are the coordinates of the center point of element number j in the region Γ''.

All the (2J+ 1) equations indicated by relation (91) are written out in full in the computer programme with numerical values of k_j , $\bar{k}_j$ (determined from the region Γ'', previously established independently of the programme) inserted in them. This was found to be the most

efficient way of conducting the computations. For this highly repetitive part of distributing the probability masses in Γ'', the programme has no decision-making logic. (A series of similar equations, like those in relation (91) can be written in a FORTRAN programme as a combination of DO LOOPS. However this would implicitly involve extra decision making locig on the part of the computer.)

(ix) Discard the probability contributions in Array B which lie outside the region GHJK (Fig. 15), that is, for which $m > M$ or $|\bar{m}| > \bar{M}$.

Add up the probability $B(-m, \bar{m})$ to the probabilitiy $B(m, -\bar{m})$ and set $B(-m, \bar{m}) = 0$, for $m = 1, 2, ---, Z$; $\bar{m} = -\bar{M}, ---, -1, 0, 1, ---, \bar{M}$.

(x) Now the array $B(m, \bar{m})$, $m = 0, 1, ---, M$; $m = 0, \pm 1, ---, \pm\bar{M}$; represents the probability distribution $G(m\Delta y, \bar{m}\Delta\dot{y}, \tau)$ in Ω.

To obtain $G(m\Delta y, \bar{m}\Delta\dot{y}, 2\tau)$, one repeats from step (vii) with A and B interchanged.

At every step,

$$Q(N\tau) = 2 \sum_{m=0}^{M} \sum_{\bar{m}=-\bar{M}}^{\bar{M}} G(m\Delta y, \bar{m}\Delta\dot{y}, N\tau) \tag{92}$$

$$f(N\tau) = Q\left((N-1)\tau\right) - Q(N\tau) \tag{93}$$

$$\text{and} \quad \bar{P}(N\tau) = Q(N\tau) \Big/ Q\left((N-1)\tau\right) \tag{94}$$

Computations are continued until $\bar{P}(N\tau)$ approaches a constant value.

Defining,

$$\rho_1 = \lim_{N\to\infty} \bar{P}(N\tau) \tag{95}$$

$$\lambda_1 = -\frac{\pi}{\gamma\tau} \ln \rho_1 \tag{96}$$

$$\text{and } D_1 = \lim_{N\to\infty} \frac{Q(N\tau)}{\left[\bar{P}(N\tau)\right]^N} \tag{97}$$

the asymptotic expression for Q(t) becomes

$$Q(t) = D_1 e^{-\lambda_1 \frac{\gamma}{\pi} t} \tag{98}$$

4.3 Tests on the Computational Procedure.

The test case chosen was as follows,

$$\frac{\alpha}{\gamma} = 0.04, \quad \Omega \equiv \left(|y| < a\sigma, \ |\dot{y}| < \bar{a}\dot{\sigma}\right),$$

$$G(y, \dot{y}, 0) = \delta(y)\,\delta(\dot{y})$$

(A) For $\tau = \frac{\pi}{4\gamma}$, $a = 1.5$, $\bar{a} = 3$, $d_y = 0.65$, $d_{\dot{y}} = 0.52$ and $s = \bar{s} = 4$, Fig. 16 shows variation of the exponential factor λ_1 with $(1 - \mu)$, the volume under unit step transition probability which lies outside the region Γ'' (see relation 89). All subsequent tests are performed with $1 - \mu = 0.0001$ to 0.0002.

(B) Variation of λ_1 with $\bar{a}$ is shown below for a = 1.5, $1 - \mu$ = 0.0001 and all other input parameters as in test (A).

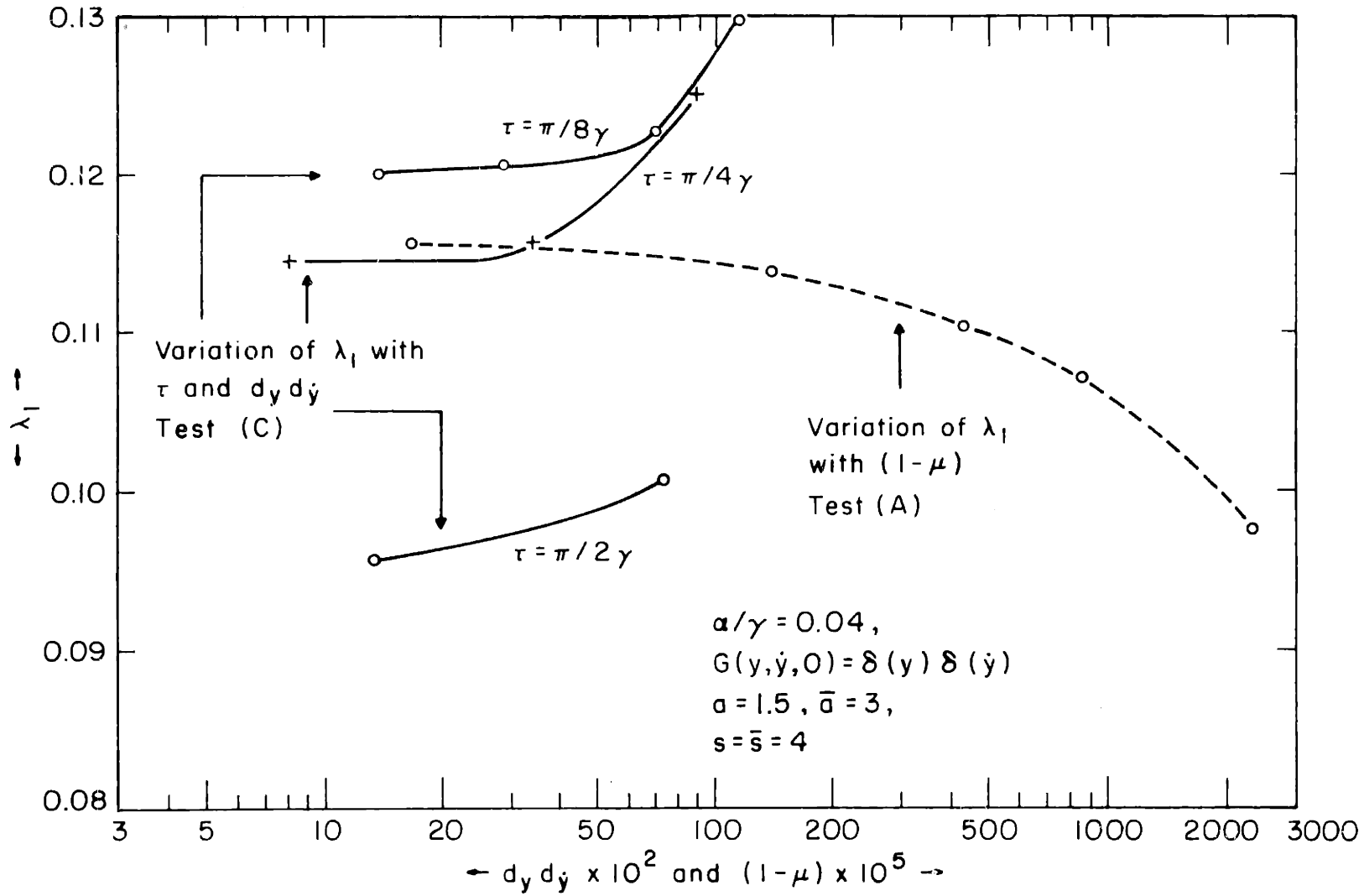


Figure 16

White Gaussian excitation. Tests (A) and (C).

a	λ_1
4	0.11716
5	0.11688
6	0.11642

(C) For a = 1.5, $\bar{a}$ = 3, s = $\bar{s}$ = 4, Fig. (16) shows variation of λ_1 with the size factor dy dẏ (effective values) and with the time element τ . Variation of λ_1 with dy dẏ is of the same nature as that in Fig. 7 for one dimensional process. There is a large variation in λ_1 between $\tau = \frac{\pi}{2\gamma}$ (4 iterations/cycle) and $\tau = \frac{\pi}{4\gamma}$ (8 iterations/cycle). The variation reduces sharply between $\tau = \frac{\pi}{4\gamma}$ and $\tau = \frac{\pi}{8\gamma}$. λ_1 (and hence the probability of leaving Ω) increases as τ decreases. This can be easily explained. The numerical iteration after every time step τ , corresponds to examining the random oscillator output after each time interval τ . By doing this one entirely misses all the cases where oscillator leaves Ω and comes back in, all within time τ between two consecutive observations. Thus the probability of "catching" exits from Ω decreases as τ increases.

Fig. 17 shows how in the phase plane the mean position (C, Ċ) (see relations 2 and 2') of an oscillator path originating at (y_o , $\dot{y}_o$) decays ultimately to (0, 0). It is quite clear that if (y_o , $\dot{y}_o$) lies in the region $\Omega_E \equiv \left(y^2 + \frac{\dot{y}^2}{\gamma^2} < a^2\right)$, (C, Ċ) will always lie in Ω_E . However if (y_o , $\dot{y}_o$) lies in the region $\Omega_D \equiv (|y| < b)$, (at least for

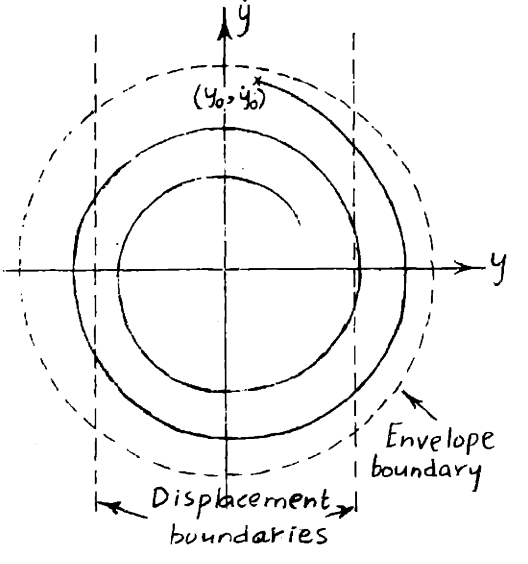


Figure 17

some positions (y_o , $\dot{y}_o$), like the one shown in Fig. 17) $(C,\dot{C})$ may very well leave Ω_D and be back in again. Thus one might say that the probability of having a first passage and re-entry within a small time τ is higher for Ω_D than for Ω_E . Hence the size of the time element τ in computations should introduce lesser error in the first passage probability for Ω_E than in the first passage probability for Ω_D .

(D) Final check on the computational procedure was to compare the probability distributions obtained from numerical iteration, with those from relation (16). For this test $\frac{\alpha}{\gamma} = 0.08$, $\tau = \frac{\pi}{4\gamma}$, dy = 0.65, dy = 0.52, $s = \bar{s} = 4$, $1-\mu = 0.0002$.

The iteration was limited to a region $\Omega \equiv \left(y^2 + \frac{\dot{y}^2}{\gamma^2} < (\bar{s}\sigma)^2 \right)$ and $G(y, \dot{y}, 0) = \delta(y)\,\delta(\dot{y})$. The envelope region was approximated by a suitable collection of rectangular elements of size Δy, $\Delta \dot{y}$ as shown in Fig. 18. The computational procedure for the envelope boundary was along exactly the same lines as the one detailed out previously for the displacement boundaries. The iterated distribution obtained after 5 natural cycles (that is after 40 iterations) was compared with the theoretical distribution (16). From relations (11), (13) and (15), for $t = 5 \times \frac{2\pi}{\gamma}$, $\sigma_t = 0.9968\,\sigma$, $\dot{\sigma}_t = 0.9967\,\dot{\sigma}$, $\rho_t = 0.1075 \times 10^{-4}$. The corresponding averages from the computed distribution were $\sigma_t = 1.008\,\sigma$, $\dot{\sigma}_t = 1.007\,\dot{\sigma}$, $\rho_t = 0.121 \times 10^{-2}$. The actual distributions obtained are compared in Fig. 18. For this comparison the theoretical distribution (16) was also integrated on ractangular elements of size $\Delta y\ \Delta \dot{y}$ and assumed to be in the form of impulses at the centers of these elements.

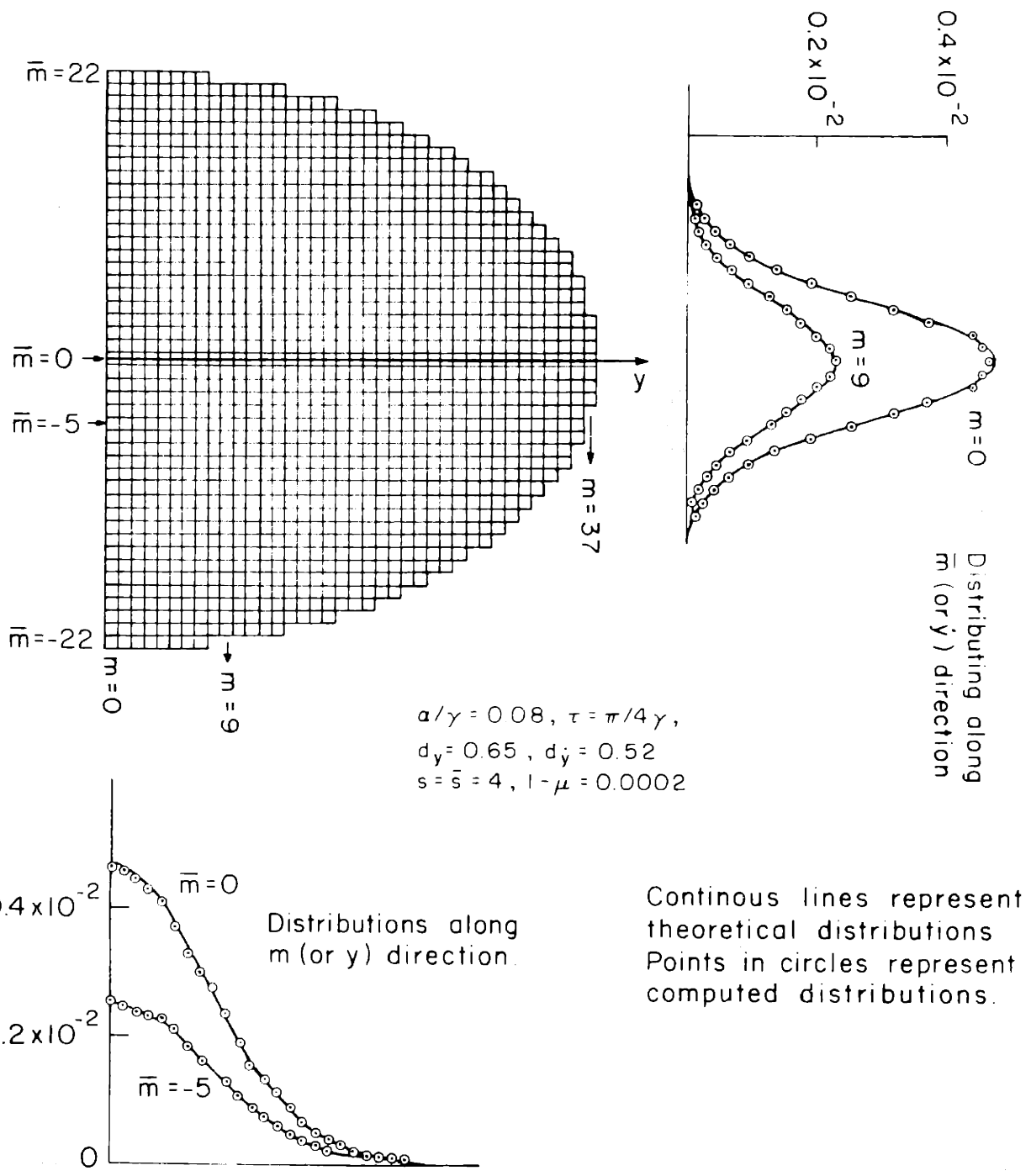


Figure 18

White Gaussian excitation. Test D.

4.4 The Final Results and Discussion.

Table 1 shows the final results obtained. For all these results $\tau = \frac{\pi}{4\gamma}$. Unfortunately it was found to be quite uneconomical to perform more than 8 iterations per cycle. The exponential factors λ_1 given below, therefore may be looked upon as lower bounds to the correct values. The difference is expected to be of the order of 5% for the envelope boundaries and 10% for the displacement boundaries.

TABLE 1

For all the results given below $\tau = \frac{\pi}{4\gamma}$, $s = \bar{s} = 4$, $1 - \mu \leqslant 0.0002$, dy (effective) = 0.65 to 0.7, dy (effective) = 0.5 to 0.7 and $\bar{a} \simeq 2a$.

D represents $\Omega \equiv (|y| < a\sigma)$, that is, first passage through symmetric displacement barriers.

E represents $\Omega \equiv (y^2 + \frac{\dot{y}^2}{\gamma^2} < a^2\sigma^2)$, that is, first passage through an evelope barrier.

G1 represents $G(y, \dot{y}, 0) = \delta(y)\ \delta(\dot{y})$, that is, the oscillator is initially at rest.

G2 represents G(y, ẏ, 0) = stationary distribution inside whole of Ω

G3 represents G(y, ẏ, 0) = stationary distribution inside a square portion of Ω ; $|y| < a\sigma$, $|\dot{y}| < a\dot{\sigma}$

$Q(0) = \iint_{\Omega} G(y, \dot{y}, 0)\, dy\, d\dot{y}$, and is given to aid comparison with D_1. The asymptotic expression for the first passage probability Q(t) is given by

$$Q(t) = D_1 e^{-\lambda_1 \frac{\gamma}{\pi} t}$$

Result No.	$\frac{\alpha}{\gamma}$	Type of Region Ω	$G(y, \dot{y}, 0)$	a	λ_1	D_1	$Q(0)$	$\lambda_1 / e^{-\frac{a^2}{2}}$
1	0.08	D	G1	1	0.4678	1.3588	1	0.767
2	"	"	"	1.5	0.1835	1.2588	1	0.565
3	"	"	"	2	0.6946×10^{-1}	1.1487	1	0.512
4	"	"	"	2.5	0.2319×10^{-1}	1.0696	1	0.528
5	"	"	"	3	0.6380×10^{-2}	1.0241	1	0.574
6	"	"	"	4	0.2181×10^{-3}	1.0011	1	0.653
7	0.08	E	G1	1	0.7010	1.3365	1	1.271
8	"	"	"	1.5	0.2720	1.2749	1	0.839
9	"	"	"	2	0.1106	1.1853	1	0.816
10	"	"	"	2.5	0.3966×10^{-1}	1.0976	1	0.903
11	"	"	"	3	0.1208×10^{-1}	1.0393	1	1.085
12	"	"	"	4	0.5981×10^{-3}	1.0028	1	1.780
13	0.04	D	G1	1	0.3035	1.4127	1	0.499
14	"	"	"	1.5	0.1148	1.2892	1	0.354
15	"	"	"	2	0.4479×10^{-1}	1.1782	1	0.331
16	"	"	"	2.5	0.1553×10^{-1}	1.0851	1	0.354
17	"	"	"	3	0.4492×10^{-2}	1.0319	1	0.404
18	"	"	"	4	0.180×10^{-3}	1.0019	1	0.537
19	0.01	D	G3	1	0.1074	0.3843	0.4534	0.177
20	"	"	"	2	0.1545×10^{-1}	0.8400	0.9081	0.114
21	"	"	"	3	0.1780×10^{-2}	0.9882	0.9946	0.160
22	0.01	D	G3	4	0.828×10^{-4}	1	1	0.247
23	0.01	E	G2	1	0.1310	0.3345	0.3951	0.216
24	"	"	"	2	0.1901×10^{-1}	0.8027	0.8656	0.140
25	"	"	"	3	0.2450×10^{-2}	0.9819	0.9890	0.220
26	"	"	"	4	0.140×10^{-3}	0.9997	0.9997	0.417

Figures (19) and (20) show results number 5 and 11. Shown in these figures are $Q(N\tau)$,

$$f(N\tau) = Q\left((N-1)\tau\right) - Q(N\tau),$$

and $$\bar{f}(N\tau) = Q(N\tau)/Q\left((N-1)\tau\right)$$

Also shown are the exponential and the hyperexponential approximations

$$Q_{exp}(t) = Q(0)\, e^{-\lambda_1 \frac{\gamma}{\pi} t} = e^{-\lambda_1 \frac{\gamma}{\pi} t} \tag{99}$$

and $$Q_{hyp}(t) = D_1 e^{-\lambda_1 \frac{\gamma}{\pi} t} + D_2^* e^{-\lambda_2^* \frac{\gamma}{\pi} t} \tag{100}$$

D_2^* and λ_2^* are determined from the conditions

$$Q_{hyp}(0) = Q(0) = 1$$

$$\frac{\partial}{\partial t} Q_{hyp}(t)\Big|_{t=0} = 0$$

These approximations are plotted as

$$Q_{exp}(N\tau) = e^{-\lambda_1 \frac{N}{4}} \tag{101}$$

$$f_{exp}(N\tau) = Q_{exp}\left((N-1)\tau\right) - Q_{exp}(N\tau) \tag{102}$$

$$Q_{hyp}(N\tau) = D_1 e^{-\lambda_1 \frac{N}{4}} + D_2^* e^{-\lambda_2^* \frac{N}{4}} \tag{103}$$

$$f_{hyp}(N\tau) = Q_{hyp}\left((N-1)\tau\right) - Q_{hyp}(N\tau) \tag{104}$$

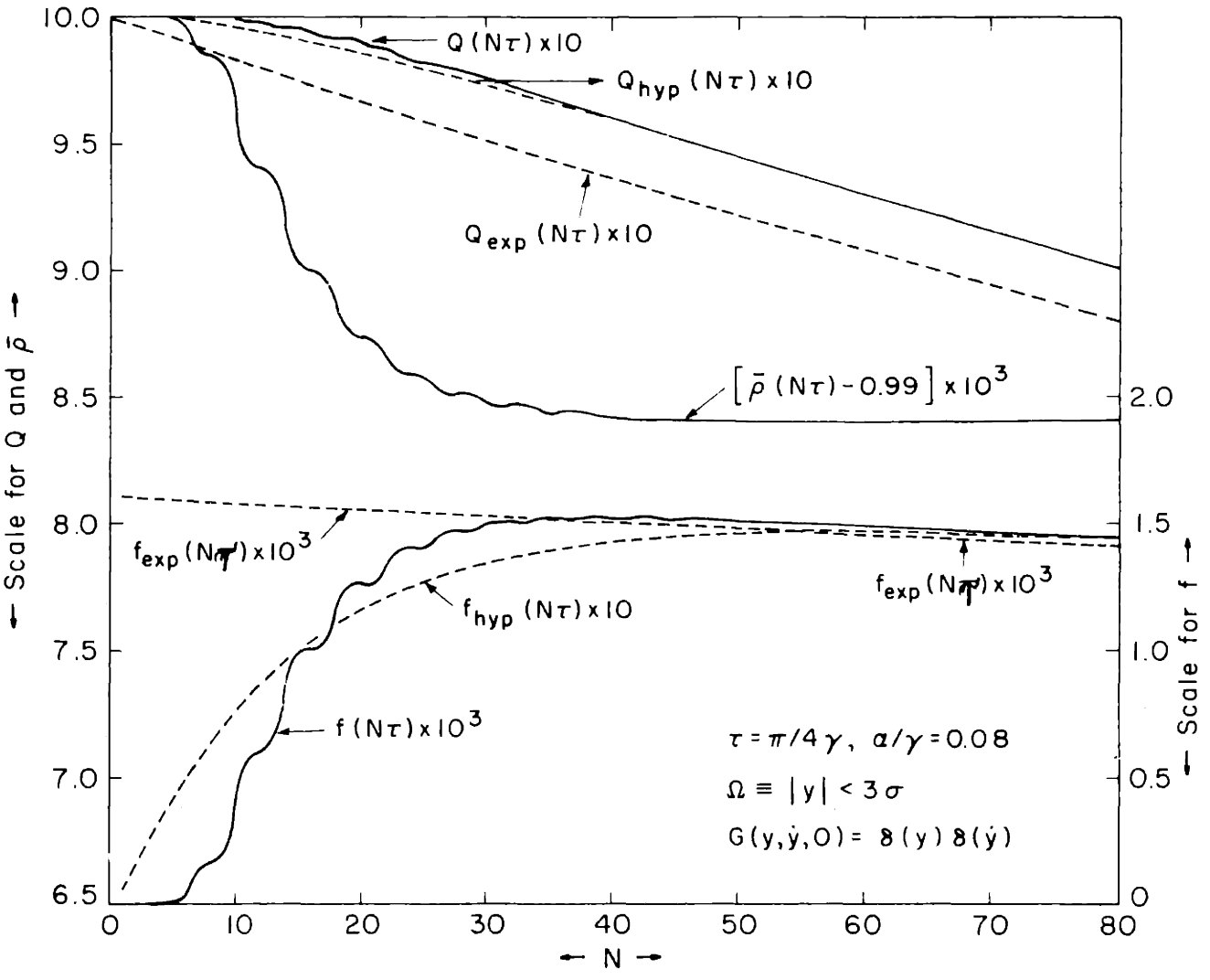


Figure 19

White Gaus ian excitation. Computed first passage probabilities, exponential and hyper-exponential approximations.

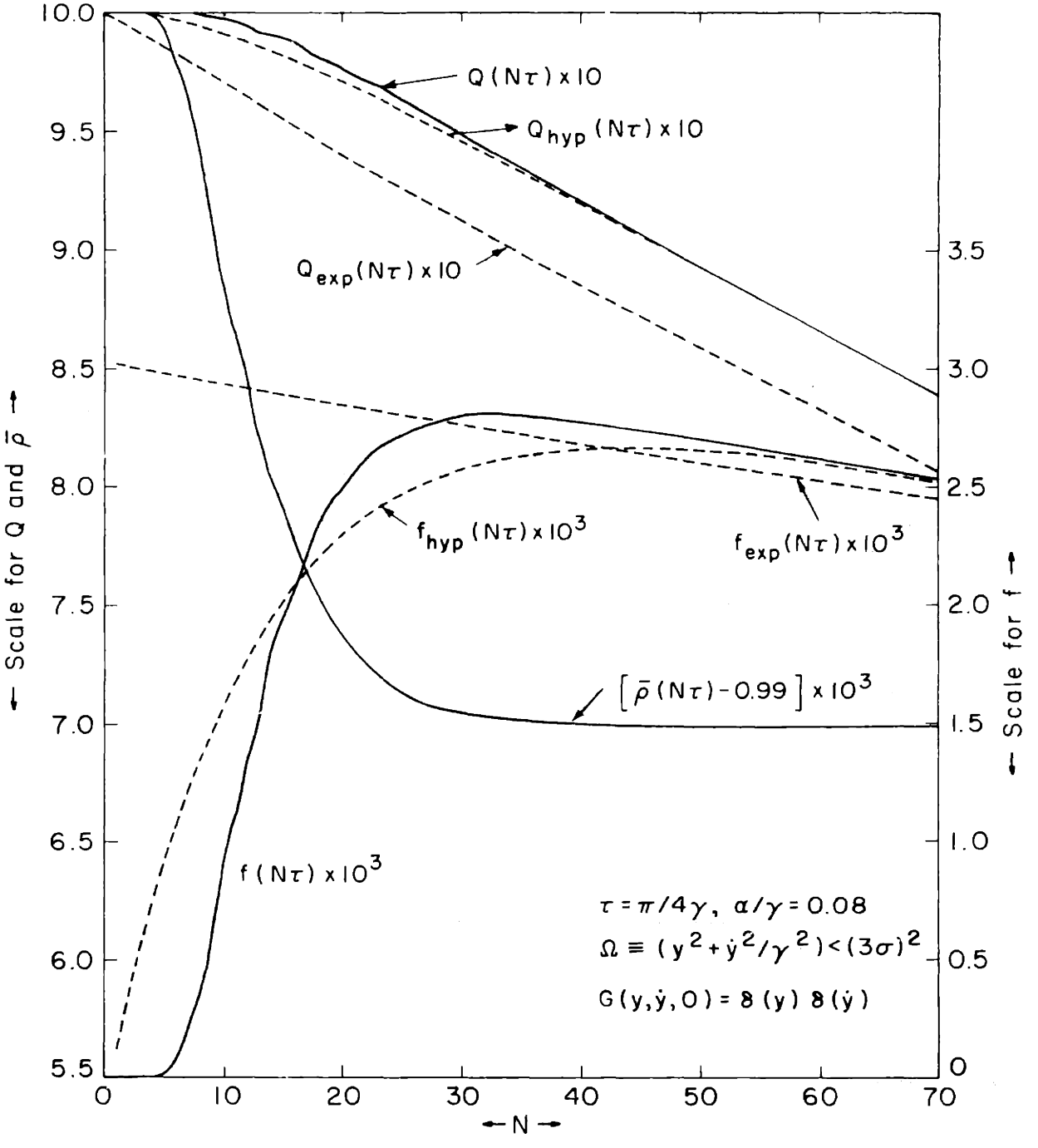


Figure 20

White Gaussian excitation. Computed first passage probabilities, exponential and hyper-exponential approximations.

There is a marked cyclic fluctuation in the first passage probability $f(N\tau)$ in Fig. 19. The frequency of fluctuation is twice the natural frequency γ of the oscillator. Note that the region Ω for this result is of the type D(displacement barriers). With reference to Fig. 17 it is easy to see that an oscillator path starting from $(y_0 , \dot{y}_0)$ would have a relatively high probability of leaving the region $\Omega \equiv (|y| < a\sigma)$, whenever its mean position $(C, \dot{C})$ approaches or crosses the boundaries $y = \pm a\sigma$ and this happens with the frequency of 2γ. Similarly the path would have a relatively low probability of leaving Ω if $(C, \dot{C})$ is well inside Ω.

Since for a path starting from inside of an envelope region Ω, the mean position $(C, \dot{C})$ can never leave Ω, the cyclic fluctuation in the first passage probability for an envelope barrier should be comparatively less. This is indeed so, as seen in Fig. 20 (for the envelope boundary). Whatever small cyclic fluctuation in $f(N\tau)$ is observed, is due to the fact that for small τ, the transition probability $P(y_0, \dot{y}_0 \mid y, \dot{y}, \tau)$ is unsymmetric. For example in Fig. 21 two oscillator paths starting from positions A and A_1 in the region $\Omega \equiv \left(y^2 + \dot{y}^2/\gamma^2 < a^2\sigma^2 \right)$, have the mean positions M and M_1 after $\tau = \frac{\pi}{4\gamma}$. The elliptical regions Γ and Γ_1 around M and M_1 are of exactly the same shape and size, and enclose areas of the phase plane in which paths

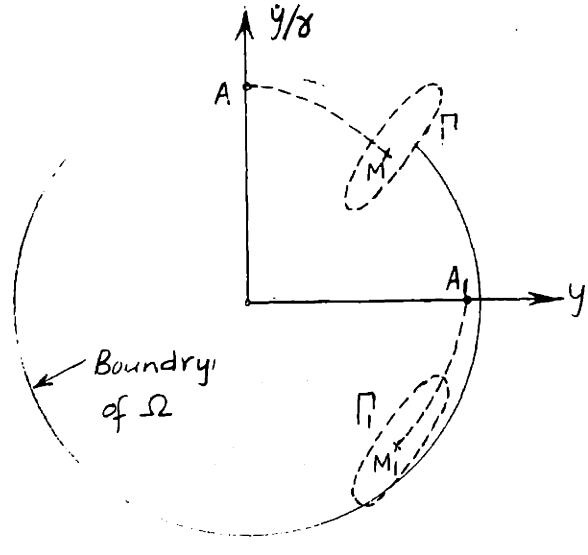


Figure 21

from A and A_1 will most likely end up after τ (say with probability 0.99). Yet Γ and Γ_1 are quite differently oriented with respect to the boundary of Ω .

For all the cases where $G(y, \dot{y}, 0) = \delta(y)\ \delta(\dot{y})$, the first passage probability f(t) starts from 0, attains a maximum value and then approaches exponential behavior shortly, as already illustrated in Figures 19 and 20. The time required by f(t) to attain a maximum value increases as the size of the region Ω (of type E or D) increases, and as the damping factor $\frac{\alpha}{\gamma}$ decreases. This is because both these factors (increase in Ω or decrease in $\frac{\alpha}{\gamma}$) increase the time required by the oscillator (which initially has zero energy, since $G(y, \dot{y}, 0) = \delta(y)\,\delta(\dot{y})$) to attain high enough displacements or energies to cross Ω . This shift in the maximum of f(t) is illustrated in Figures 22 and 23.

Fig. 22 shows results number 4 and 16 (see table 1). For both these cases $\Omega \equiv (|y| < 2.5\sigma)$ and $G(y, \dot{y}, 0) = \delta(y)\ \ \delta(\dot{y})$. The damping factors $\frac{\alpha}{\gamma}$ however are different.

Fig. 23 shows results number 1, 3, 5, and 6. For all these cases $\frac{\alpha}{\gamma} = 0.08$ and $G(y, \dot{y}, 0) = \delta(y)\ \delta(\dot{y})$. The parameter "a", indicating the size of $\Omega \equiv (|y| < a\sigma)$, is different for each case.

Fig. 24 shows result number 20. The initial stationary distribution inside the square of size $a\sigma$ was considered so that Q(t) and f(t) may converge towards their exponential forms more rapidly.

In addition to $Q(N\tau)$ and $f(N\tau)$

$$\bar{\bar{P}}(N\tau) = Q(N\tau)/Q((N-4)\tau) \quad , \tag{105}$$

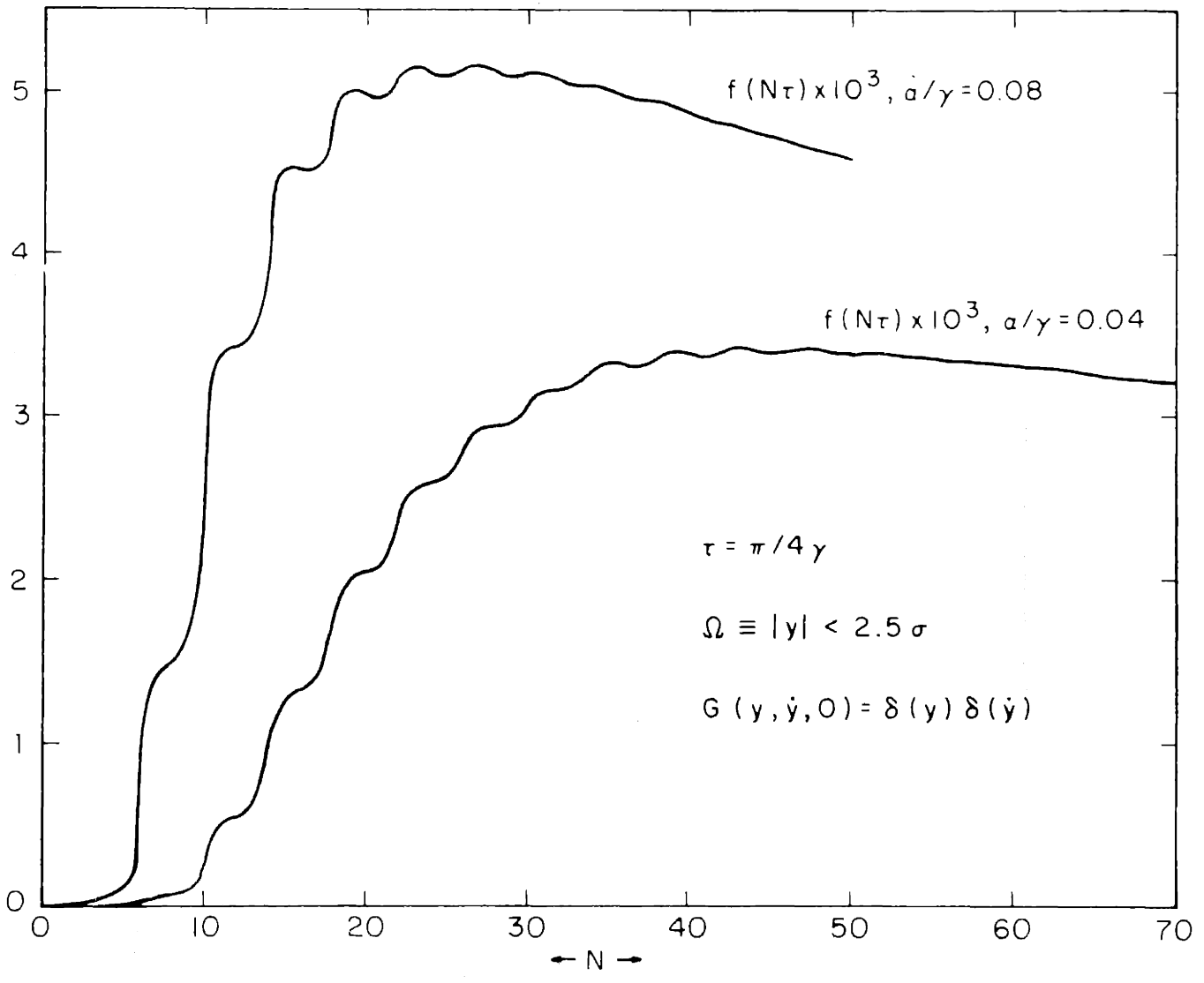


Figure 22

White Gaussian excitation. Computed first passage probabilities.

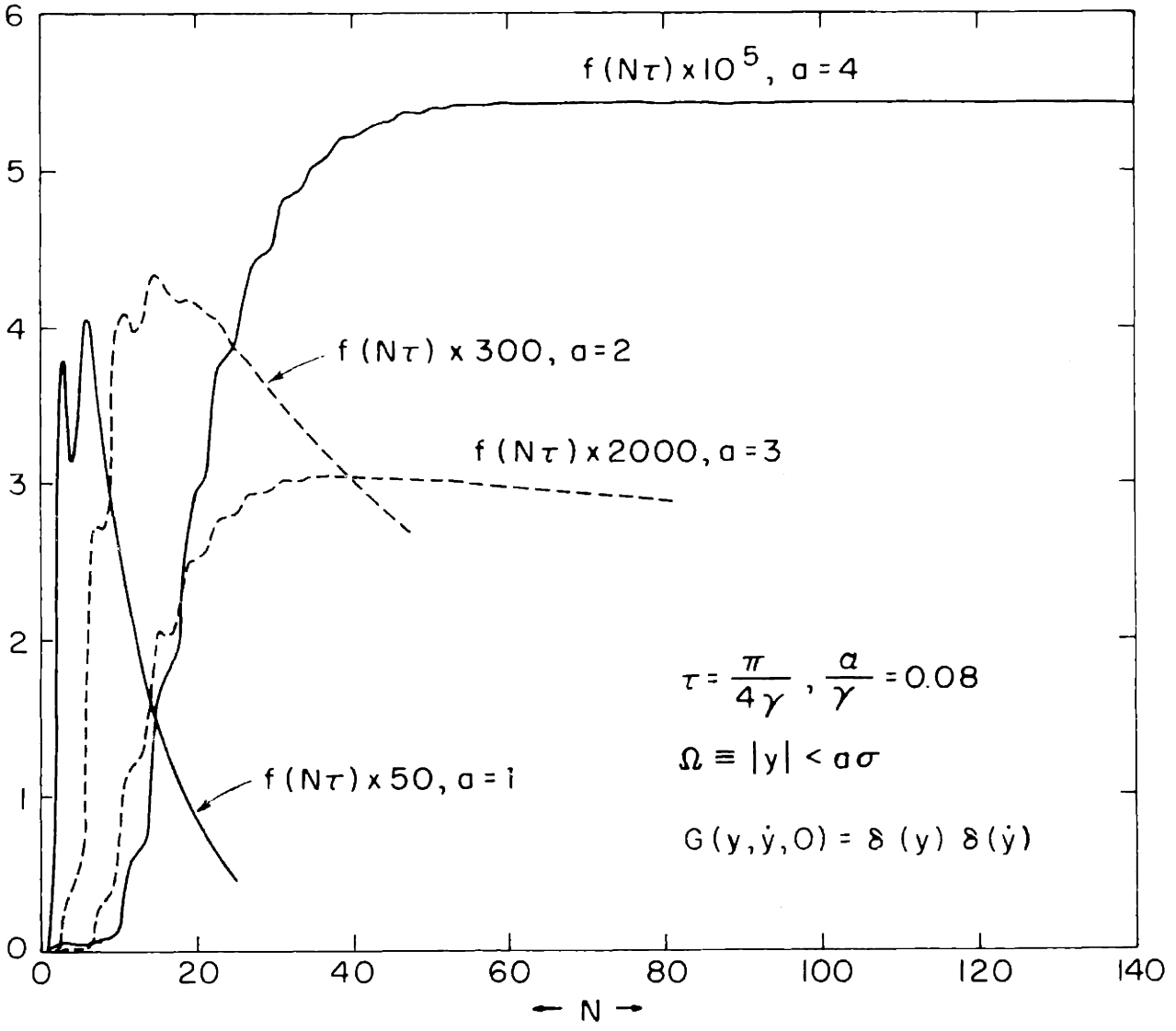


Figure 23

White Gaussian excitation. Computed first passage probabilities.

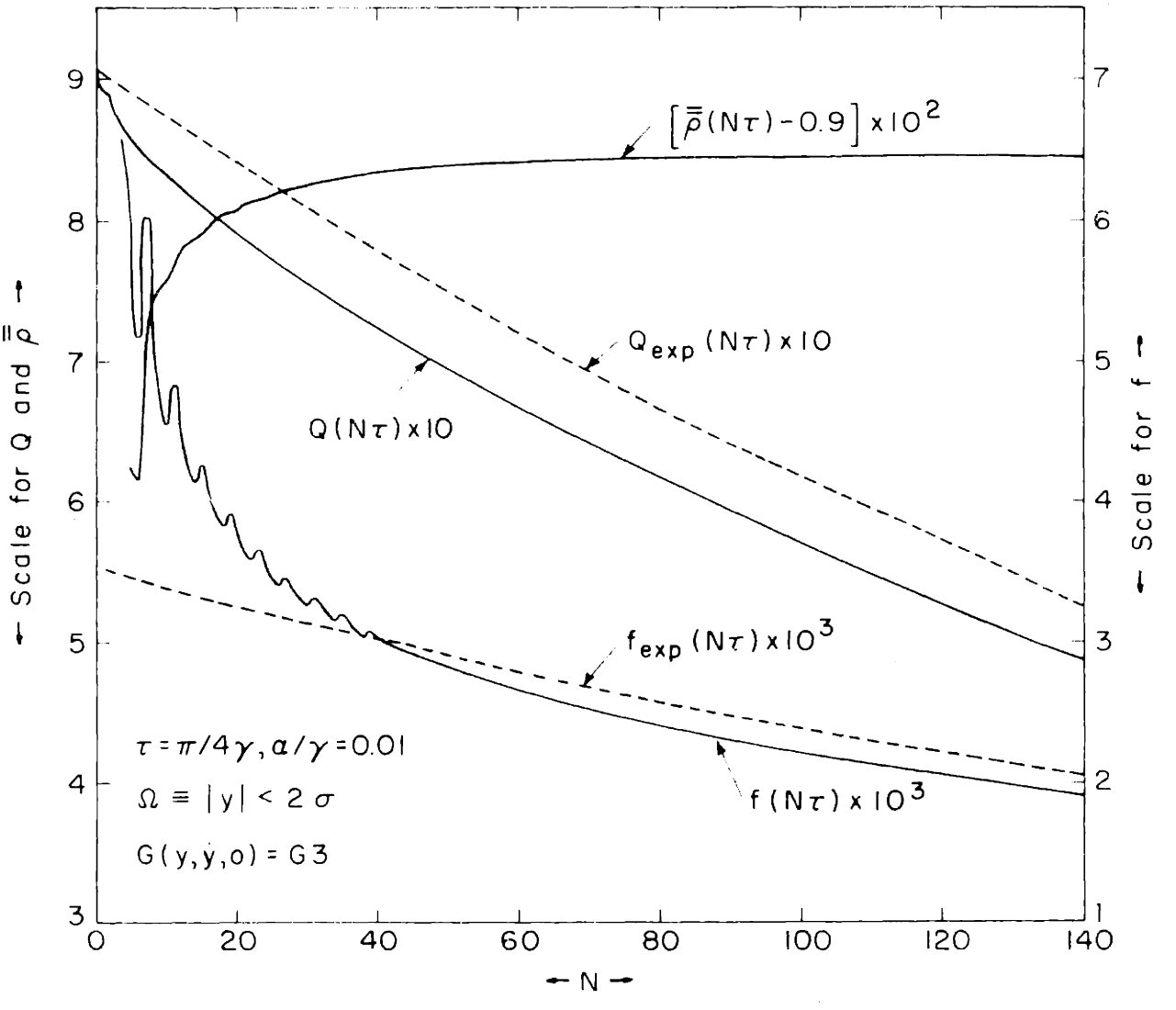


Figure 24

White Gaussian excitation. Computed first passage probabilities and exponential approximations.

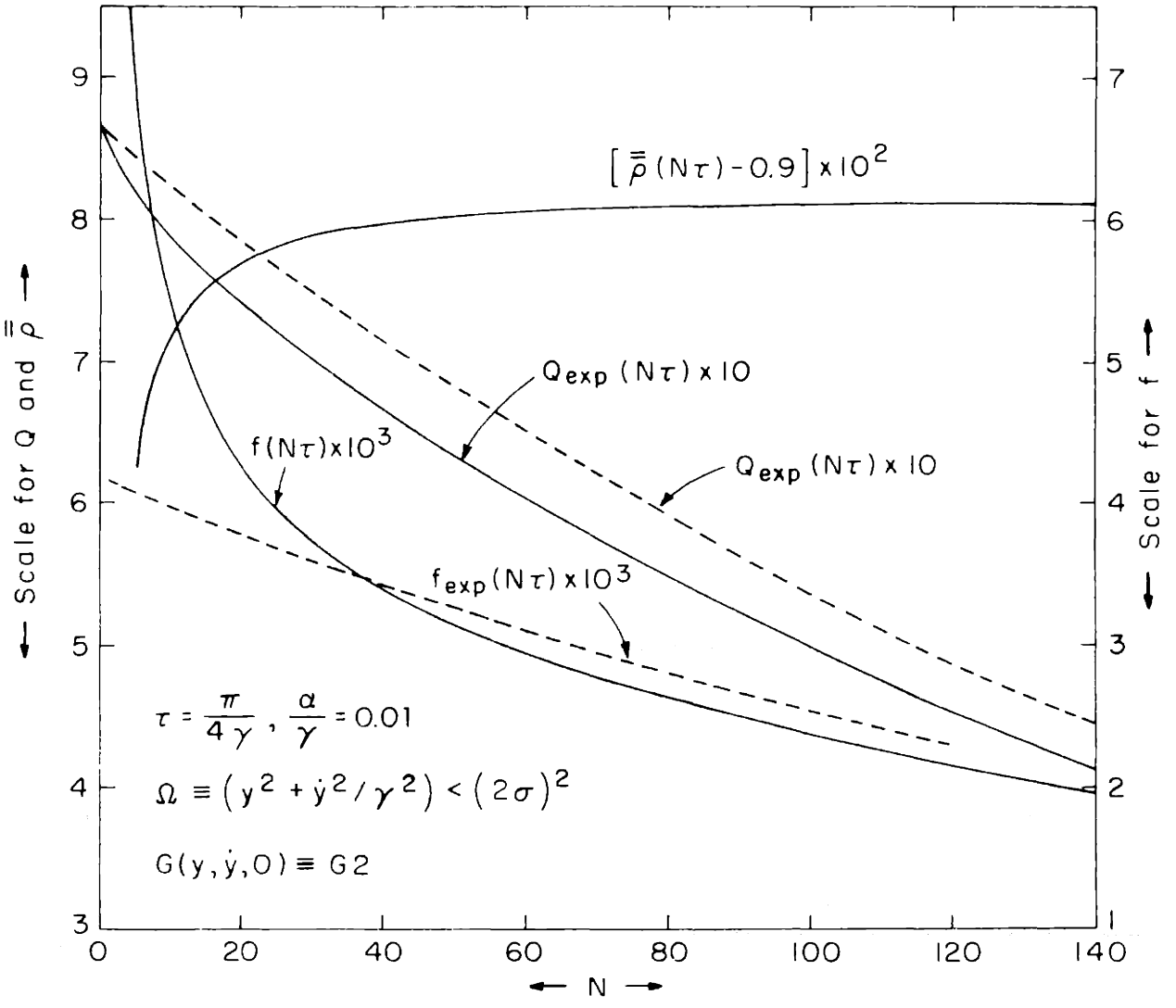


Figure 25

White Gaussian excitation. Computed first passage probabilities and exponential approximations.

is also shown.

$\bar{\bar{P}}(N\tau)$ was considered in place of $\bar{P}(N\tau) = Q(N\tau)/Q((N-1)\tau)$, because this enabled somewhat more accurate determination of the exponential factor λ_1 . Since $\bar{\bar{P}}(N\tau)$ is the ratio of two values of Q(t) separated by half a cycle, it shows practically no cyclic fluctuation (as compared to $\bar{P}(N\tau)$ in Fig. 19). Also shown in Fig. 24 are the exponential approximations

$$Q_{exp}(N\tau) = Q(0)\, e^{-\lambda_1 \frac{N}{4}}$$

$$f_{exp}(N\tau) = Q_{exp}\left((N-1)\tau\right) - Q_{exp}(N\tau)$$

The initial decrease in $Q(N\tau)$ (and corresponding high initial values of $f(N\tau)$) is due to the fact that the initial distribution $G(y, \dot{y}, 0)$ in this case is quite well spread in Ω . As a result, there are always some oscillator paths with high enough energies initially to leave Ω within a very short time.

Fig. 25 shows result number 24 for the corresponding envelope boundary. The initial distribution inside the whole envelope region is stationary. Again $Q(N\tau)$, $f(N\tau)$, $\bar{\bar{P}}(N\tau)$, $Q_{exp}(N\tau)$ and $f_{exp}(N\tau)$ are shown.

Fig. 26 shows variation of λ_1 with $\frac{\alpha}{\delta}$ and the size of the region Ω . Since variation of λ_1 with "a" is of the order of $e^{-a^2/2}$ the plot shows variation of $\lambda_1 / e^{-\frac{a^2}{2}}$ rather than that of λ_1 . D indicates values of λ_1 for the displacement barriers and E indicates those for the envelope barriers (as in table 1). The straight

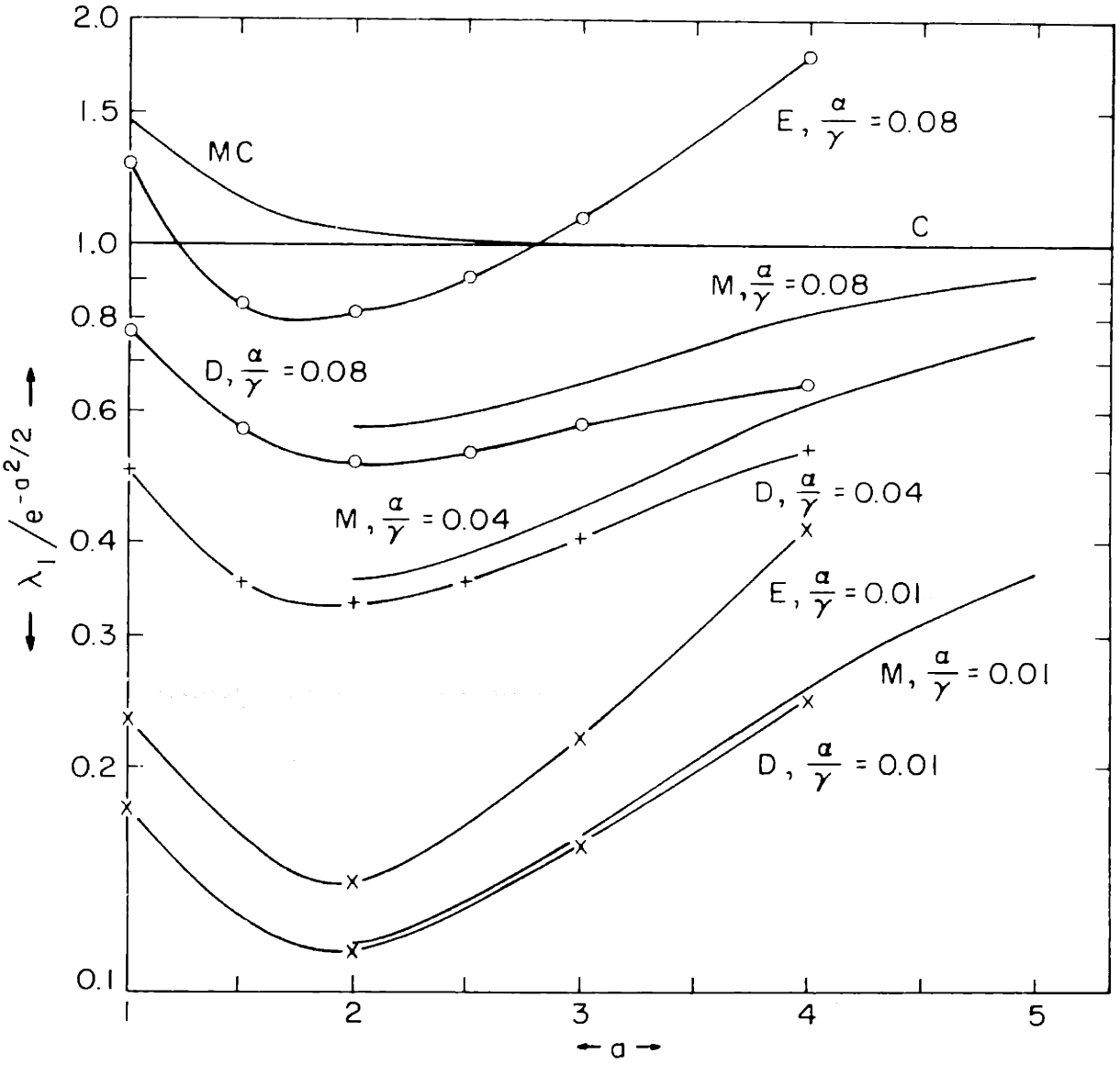


Figure 26

White Gaussian excitation. Variation of $\lambda_1 / e^{-a^2/2}$ with a and α/γ.

line C indicates values of λ_1 according to Crandall's approximate theory, curve MC indicates a modification of it (see Appendix A). This modified solution MC is suggested with the hope of explaining the similar downward trend of computed values of $\lambda_1 / e^{-\frac{a^2}{2}}$ for low values of "a". Curves M denote values of λ_1 recently computed by William D. Mark. His complete results have not been published yet, those presented in Fig. 26 were communicated personally. A rough idea of his approach may be gleaned from Appendix B.

From Fig. 26 it is clear that all computed solutions for the displacement boundaries tend towards Crandall's solution C as "a" increases. However one cannot say whether the correct values of λ_1 for displacement barriers would merge with C or settle down below it. As explained in Appendix A, these values should always be less than Crandall's solution C (or the modified solution MC for small values of "a").

Finally Fig. 27 shows variation of factor D_1 with $\frac{\alpha}{\gamma}$ and "a". These do not seem to vary too much with $\frac{\alpha}{\gamma}$ or the type of boundary for Ω. As mentioned earlier, for distribution G1 (that is, $G(y, \dot{y}, 0) = \delta(y)\ \delta(\dot{y})$) $\frac{\partial Q}{\partial t}\Big|_{t=0} = 0$. Thus factors D_1 for this distribution are ≥ 1. For distributions G2 and G3, because of the sharp initial decrease in Q(t), factors D_1 are ≤ 1. For $a \geq 3$, $D_1 \simeq 1$, thus the exponential approximation $Q_{exp}(t) = Q(0) e^{-\lambda_1 \frac{\gamma}{\pi} t}$ is indeed quite good.

It may be instructive to note that the above results for white Gaussian excitation show the same general trends as those for one dimensional model discussed in Section 3, though the specific values for the two cases are quite different.

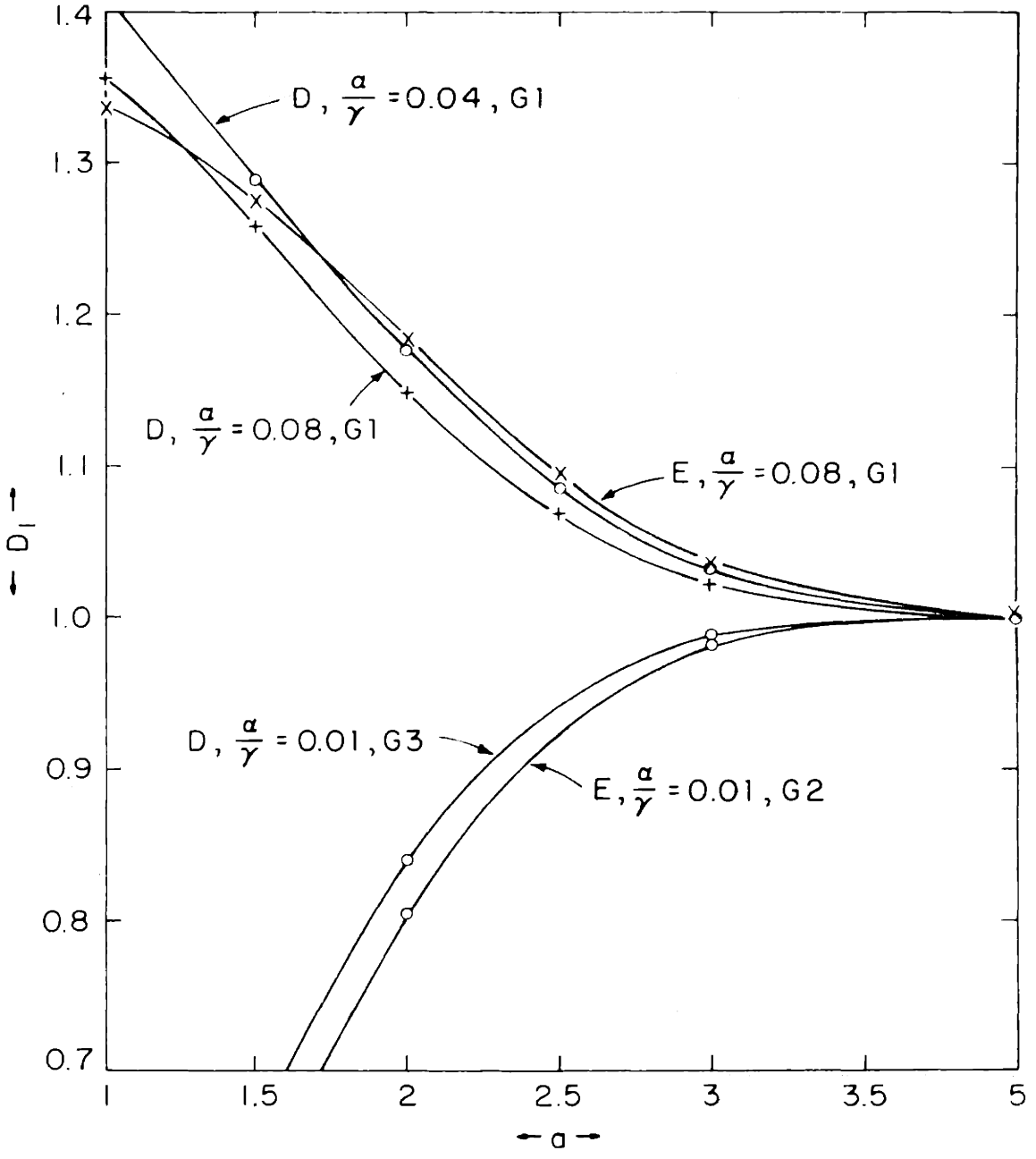


Figure 27

White Gaussian excitation. Variation of D_1 with a and α/γ.

Lastly we discuss the possibility of determining the first passage probabilities for still lower values of the damping factor $\frac{\alpha}{\gamma}$ (of the order of 0.001). Unfortunately the iterative method presented here is quite unable to cope with such extreme cases. Both the computing time and the necessary storage capacity mount fantastically as $\frac{\alpha}{\gamma}$ decreases or "a" increases. However with this appraoch we have been able to show that the multiplying factors D_1 are roughly invariant with $\frac{\alpha}{\gamma}$ and further that $D_1 \simeq 1$ for $a \geqslant 3$. Thus to get a fair estimate of the first passage probabilities for lower values of $\frac{\alpha}{\gamma}$, one only needs to know the exponential factor λ_1 corresponding to dominant eigen value β_1.

M. Kac (Ref. 5) has found a variational formula for determining the mean first passage time for leaving the region $\Omega \equiv (y > 0)$, given that one initially starts with the stationary distribution in Ω. This approach can be extended to apply for $\Omega \equiv (|y| < a\sigma)$. The knowledge of the mean first passage time would enable a fairly good estimate of λ_1 based on a suitable exponential or hyperexponential approximation involving known values of D_1.

A second possibility arises from the fact that the exponential factors λ_1 for corresponding displacement and envelope boundaries, $\Omega_D \equiv (|y| < a\sigma)$ and $\Omega_E \equiv \left(y^2 + \dot{y}^2/\gamma^2 < a^2\sigma^2\right)$ respectively, approach each other (for same size "a") as the damping factor $\frac{\alpha}{\gamma}$ decreases. This can be seen in Fig. 26 where the vertical separation between solutions E, M and D is lesser for $\frac{\alpha}{\gamma} = 0.01$ than for $\frac{\alpha}{\gamma} = 0.08$.

This behavior can be easily explained. The random oscillator

process becomes more and more correlated as damping decreases. For low enough values of $\frac{\alpha}{\gamma}$, the process would trace out almost circular paths $\left(y^2 + \dot{y}^2/\gamma^2 = \text{constant} \right)$ in the phase plane, with circular frequencies very closely distributed around the natural frequency γ of the oscillator.

Thus the first passage probability for any arbitrary region Ω would be effectively determined by the largest envelope region Ω_E that can be fitted inside Ω .

As indicated in Appendix B, the envelope process $r(t) = \left(y^2(t) + \frac{\dot{y}^2(t)}{\gamma^2} \right)^{1/2}$ can be described as a temporally discrete one-dimensional Markoffian process, for $t = \frac{n\pi}{\beta}$, $n = 0, 1, 2 \ldots$.

This simpler model has been used by W.D. Mark. As explained earlier, the effect of examining the process after relatively large time intervals $\tau = \frac{\pi}{\beta}$ would be to decrease λ_1 . However the extent of this error would decrease with damping factor $\frac{\alpha}{\gamma}$. For a highly correlated oscillator process, the probability of leaving and re-entering an envelope region within $\tau = \frac{\pi}{\beta}$, would be quite small.

As a result, for $\frac{\alpha}{\gamma} \leqslant 0.01$, Mark's estimates of λ_1 for envelope barriers would apply equally well to the corresponding displacement barriers.

Appendix A

An Approximate Solution to the First Passage Problem

An approximate solution for the first passage probability Q(t) for $\Omega \equiv (|y| < a\sigma)$, and G(y, $\dot{y}$, 0) ≡ stationary distribution inside Ω , was found by Crandall and Mark (Ref. 8).

The stationary probability distribution $W(y, \dot{y})$ for the oscillator output is obtained as $\lim_{t\to\infty} P(y_0 , \dot{y}_0 \mid y, \dot{y}, t)$ (relation 16),

Thus,

$$W(y,\dot{y}) = \frac{1}{2\pi\sigma\dot{\sigma}}\, e^{-\frac{1}{2}\left(\frac{y^2}{\sigma^2} + \frac{\dot{y}^2}{\dot{\sigma}^2}\right)} \tag{106}$$

where $\sigma^2 = E[y^2]_{\text{stationary}} = \lim_{t\to\infty} \sigma_t^2 = \frac{\pi S_0}{2\alpha\gamma^2}$ (107)

and $\dot{\sigma}^2 = E[\dot{y}^2]_{\text{stationary}} = \lim_{t\to\infty} \dot{\sigma}_t^2 = \frac{\pi S_0}{2\alpha} = \gamma^2\sigma^2$ (108)

From Rice's formula (Ref. 4), expected rate of crossings of level $y=a\sigma$ with $\dot{y} > 0$ is given by

$$V_a^+ = \int_0^\infty \dot{y}\, W(a\sigma, \dot{y})\, d\dot{y} = \frac{1}{2\pi}\,\frac{\dot{\sigma}}{\sigma}\, e^{-a^2/2} \tag{109}$$

Similarly the expected rate of crossing of level $y=-a\sigma$ with $\dot{y}<0$ is given by

$$\nu_a^- = \frac{1}{2\pi}\,\frac{\dot{\sigma}}{\sigma}\,e^{-a^2/2} \tag{110}$$

Making the approximation that the event E ≡ (crossing the level $y = a\sigma$ with $\dot{y} > 0$ or the level $y = -a\sigma$ with $\dot{y} < 0$) is purely random in time, with average rate of occurrence $\nu_a = \nu_a^+ + \nu_a^-$, one gets a Poisson distribution, $P_n(t) = \frac{(\nu_a t)^n}{n!}\,e^{-\nu_a t}$, for the probability of n events E occurring in time t. Neglecting small number of paths originating from outside Ω at time t = 0, one gets

Q(t) = probability of no event E in time t

$$= P_0(t) = e^{-\nu_a t}$$

since

$$\nu_a = \nu_a^+ + \nu_a^- = \frac{1}{\pi}\,\frac{\dot{\sigma}}{\sigma}\,e^{-a^2/2} = \frac{\gamma}{\pi}\,e^{-\frac{a^2}{2}}\,, \tag{111}$$

one gets, $$Q(t) = e^{-\lambda_1 \frac{\gamma}{\pi} t} \tag{112}$$

with $\lambda_1 = e^{-a^2/2}$

Thus the straight line C in Fig. 26 represents the solution $\lambda_1 = e^{-a^2/2}$.

If the time between events E be denoted by random variable l; $\bar{l}$, the mean value of l is clearly $= \frac{1}{\nu_a}$.

Thus a slightly different way of deriving relation (112) would be to assume that $f(t) = -\frac{\partial Q(t)}{\partial t}$, is exponential and the mean first passage time is $\bar{l}$.

thus

$$f(t) = \frac{1}{\bar{l}} e^{-\frac{t}{\bar{l}}}$$

$$\text{or } Q(t) = e^{-\frac{t}{\bar{l}}} = e^{-V_a t} \text{ as before.}$$

However if instead of assuming mean first passage time to be equal to $\bar{l}$, one assumes it to be equal to $\bar{g}$, where g is the random variable representing that portion of l which lies inside Ω (see Fig. 28),

one gets,

$$f(t) = \frac{1}{\bar{g}} e^{-\frac{t}{\bar{g}}}$$

It is quite clear that

$$\bar{g} = K\bar{l}$$

where

$$K = \iint_{\Omega} W(y, \dot{y})\, dy\, d\dot{y}$$

$$= 2\int_0^a e^{-y^2/2}\, dy \qquad (113)$$

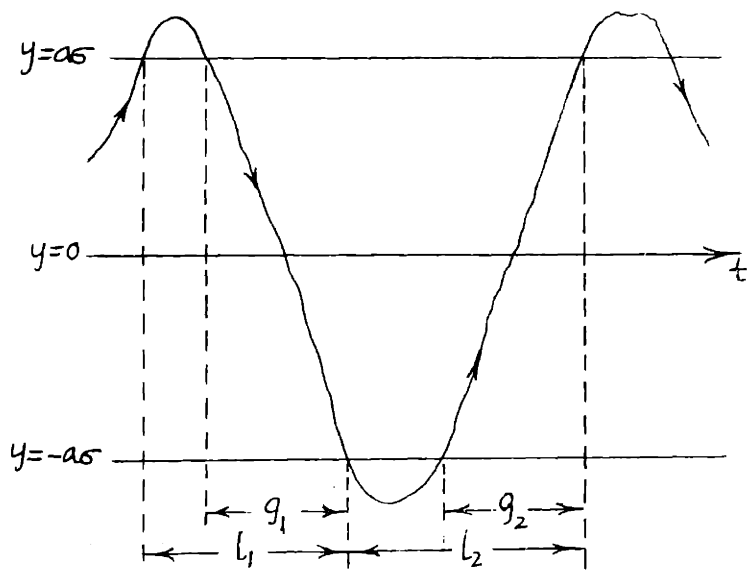


Figure 28

Thus $f(t) = \frac{1}{K\bar{l}} e^{-\frac{t}{K\bar{l}}}$

$$\text{or } Q(t) = e^{-\frac{t}{K\bar{l}}} = e^{-\frac{V_a t}{K}} = e^{-\lambda_1 \frac{\gamma}{\pi} t} \qquad (114)$$

$$\text{with } \lambda_1 = \frac{e^{-a^2/2}}{K}$$

This is the modified solution and the curve MC in Fig. 26 represents

$$\lambda_1 = \frac{e^{-a^2/2}}{2\int_0^a e^{-y^2/2}\,dy} \tag{115}$$

These solutions (C and MC) can be expected to predict higher first passage probabilities (that is higher values of λ_1) than the correct values because no account is taken of any correlation between consecutive crossings of levels $y = \pm a\sigma$. This correlation (which is quite strong for low values of $\frac{\alpha}{\gamma}$) tends to clump these crossings together in time. The average separation (in time) between such clumps is of course considerably larger than $\bar{t}$, thus resulting in a smaller λ_1.

The curves MC and C (for high enough values of "a") provide upper bounds for λ_1, only for displacement barriers. As seen in Fig. 26, λ_1 for the envelope boundary ($y^2 + \dot{y}^2/\gamma^2 < a^2\sigma^2$) can easily exceed $e^{-a^2/2}$.

Appendix B

A One Dimensional Model for the Envelope Process

Consider a radially symmetric probability distribution $G(r, 0)$ at time $t = 0$, in a circular element of radius r and width dr in the phase plane $(y, \dot{y}/\gamma)$ or (r, θ) where $y = r \cos\theta$, $\frac{\dot{y}}{\gamma} = \sin\theta$. (see Fig. 29)

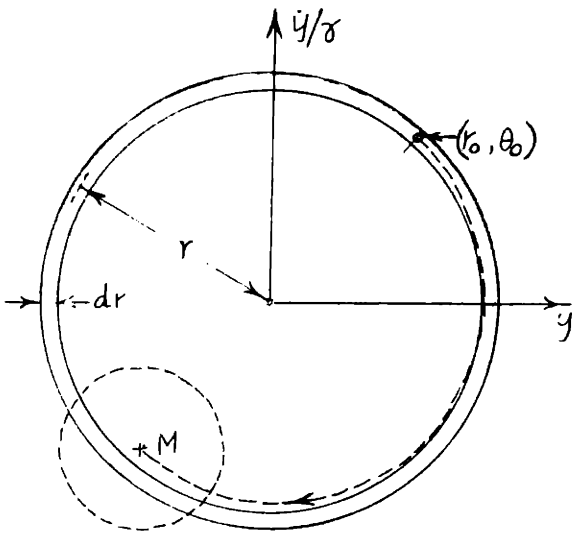


Figure 29

After time $t = \frac{\pi}{\beta}$, an oscillator path starting from position (r_0, θ_0) would have its mean position at point M, diametrically opposite to (r_0, θ_0) and at a radius of $r_0 e^{-\frac{\pi\alpha}{\beta}}$ (see relations 2, 2'). Its distribution around mean M will be totally symmetric with respect to y and $\frac{\dot{y}}{\gamma}$, with zero cross-correlation. This is indicated in Fig. 29 by circular region around M. Arguing for all positions in the circular ring, it follows that the resulting distribution at time $t = \frac{\pi}{\beta}$ would be radially symmetric too, say $G(r, \frac{\pi}{\beta})$.

Thus as long as one starts with a radially symmetric distribution at $t = 0$, and examines the process at $t = \frac{n\pi}{\beta}$, $n = 1, 2, 3, \ldots$, one obtains a one-dimensional process $r(t)$ [instead of two dimensional $(r(t), \theta(t))$]. It can be shown that the transition probability $P(r_0 | r, \frac{n\pi}{\beta})$ for this process is given by,

$$P\left(r_0 \mid r, \frac{n\pi}{\beta}\right) = \frac{r}{\sigma_n^2}\, e^{-\frac{\left(r^2 + r_0^2 K^{2n}\right)}{2\sigma_n^2}}\, I_0\left(\frac{r r_0 K^n}{\sigma_n^2}\right)$$

$$\text{where} \quad \sigma_n^2 = \sigma^2\left(1 - K^{2n}\right) \tag{116}$$

$$\sigma^2 = \frac{\pi S_0}{2\alpha\gamma^2} \quad \text{(relation 12)}$$

$$K = e^{-\frac{\pi\alpha}{\beta}}$$

and $I_0(x)$ is the modified Bessel function of the first kind and zero order,

$$I_0(x) = 1 + \sum_{n=2,4,6,\ldots} \frac{x^n}{n!}\, \frac{1\cdot 3\cdot 5 \cdots (n-1)}{2\cdot 4\cdot 6 \cdots n} \tag{117}$$

The transition probability (116) is obtained by transforming relation (16) (for $t = \frac{n\pi}{\beta}$) to polar coordinates and integrating it with respect to the angular coordinate θ from 0 to 2π. It can be shown that the process defined by (116) is Markoffian.

W.D. Mark has solved the first passage problem for envelope boundaries, essentially using the above one dimensional model. His findings will be published shortly.

On substituting $\frac{n\pi}{\beta} \doteq t$ and $K = e^{-\frac{\alpha t}{n}}$ in relation (116), one obtains the transition probability for a temporally continuous Markoffian process r(t). This process has been discussed in References (10) and (11).

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BIOGRAPHICAL NOTE

The author was born in Hyderabad Sind, Pakistan on December 11, 1937. He received a Bachelor of Mechanical Engineering in 1958 from Maharaja Sayajirao University of Baroda, India and a Master of Science in Mechanical Engineering in 1960 from Massachusetts Institute of Technology.