Experiment 09: Angular momentum
Goals

- Investigate conservation of angular momentum and kinetic energy in rotational collisions.

- Measure and calculate moments of inertia.

- Measure and calculate non-conservative work in an inelastic collision.
Apparatus

- Connect output of phototransistor to channel A of 750.
- Connect output of tachometer generator to channel B of 750.
- Connect power supply.
- Red button is pressed: Power is applied to motor.
- Red button is released: Rotor coasts: Read output voltage using DataStudio.
- Use black sticker or tape on white plastic rotor for generator calibration.
Calibrate tachometer-generator

- Spin motor up to full speed, let it coast. Measure and plot voltages for 0.25 s period. Sample Rate: 5000 Hz, and Sensitivity: Low.

- Time 10 periods to measure $\omega$.

- Then calculate $\omega$ for 1 V output.

- Average the output voltage over the same 10 periods.
Measure rotor $I_R$

- Plot only the generator voltage for rest of experiment.
- Use a 55 gm weight to accelerate the rotor.
- Settings:
  - Sensitivity: Low
  - Sample rate 500 Hz.
  - Delayed start: None
  - Auto Stop: 4 seconds

- Start DataStudio and let the weight drop.
Understanding graph output to measure IR generator voltage while measuring $I_R$. What is happening:

1. Along line A-B?
2. At point B?
3. Along line B-C?

How do you use this graph to find $I_R$?
Measure $I_R$ results

- Measure and record $\alpha_{up}$ and $\alpha_{down}$.

- For your report, calculate $I_R$:

\[
\tau_f = I_R \alpha_{down} \quad I_R = \frac{mr(g - r\alpha_{up})}{\alpha_{up} - |\alpha_{down}|}
\]
Fast collision

<table>
<thead>
<tr>
<th>Sensitivity</th>
<th>Sample Rate</th>
<th>Delayed Start</th>
<th>Auto Stop</th>
</tr>
</thead>
<tbody>
<tr>
<td>Low</td>
<td>200 Hz</td>
<td>1 sec</td>
<td>Falls below 0.5V</td>
</tr>
</tbody>
</table>

Find $\omega_1$ (before) and $\omega_2$ (after), estimate $\delta t$ for collision.

Calculate

$$I_W = \frac{1}{2} m \omega (r_o^2 + r_i^2)$$
Slow collision

- Find $\omega_1$ and $\omega_2$, measure $\delta t$, fit or measure to find $a_c$.
- Keep a copy of your results for the homework problem.
Kepler Problem and Planetary Motion

8.01t
Nov 15, 2004
Kepler’s Laws

1. The orbits of planets are ellipses; and the sun is at one focus

2. The radius vector sweeps out equal areas in equal time

3. The period $T$ is proportional to the radius to the $3/2$ power

$$T \sim r^{3/2}$$
Kepler Problem

- Find the motion of two bodies under the influence of a gravitational force using Newtonian mechanics.

\[ \vec{F}_{1,2}(r) = -G \frac{m_1 m_2}{r^2} \hat{r} \]
Reduction of Two Body Problem

• Reduce two body problem to one body of mass $\mu$ moving about a central point under the influence of gravity with position vector corresponding to the vector from mass $m_2$ to mass $m_1$

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

$$F_{1,2}(\vec{r})\dot{\vec{r}} = \mu \frac{d^2\vec{r}}{dt^2}$$
Solution of One Body Problem

• Solving the problem means finding the distance from the origin \( r(t) \) and angle \( \theta(t) \) as functions of time.

• Equivalently, finding the distance from the center as a function of angle \( r(\theta) \).

• Solution:

\[
r = \frac{r_0}{1 - \varepsilon \cos \theta}
\]
Constants of the Motion

- Velocity \( \vec{v} = \frac{dr}{dt} \hat{r} + r \frac{d\theta}{dt} \hat{\theta} \quad v^2 = \left( \frac{dr}{dt} \right)^2 + \left( r \frac{d\theta}{dt} \right)^2 \)

- Angular Momentum \( L = \mu rv_{tangential} = \mu r^2 \frac{d\theta}{dt} \)

- Energy
  \[
  E = \frac{1}{2} \mu v^2 - \frac{Gm_1m_2}{r}
  \]
  \[
  E = \frac{1}{2} \mu \left[ \left( \frac{dr}{dt} \right)^2 + \left( r \frac{d\theta}{dt} \right)^2 \right] - \frac{Gm_1m_2}{r}
  \]
  \[
  E = \frac{1}{2} \mu \left( \frac{dr}{dt} \right)^2 + \frac{L^2}{2\mu r^2} - \frac{Gm_1m_2}{r}
  \]
Reduction to One Dimensional Motion

- Reduce the one body problem in two dimensions to a one body problem moving only in the r-direction but under the action of a repulsive force and a gravitational force.
Suppose the potential energy of two particles (reduced mass $\mu$) is given by

$$U(r) = \frac{1}{2} \frac{L^2}{\mu r^2}$$

where $r$ is the relative distance between the particles. The force between the particles is

1. attractive and has magnitude $F = \frac{L^2}{2 \mu r}$

2. repulsive and has magnitude $F = \frac{L^2}{2 \mu r}$

3. attractive and has magnitude $F = \frac{L^2}{\mu r^3}$

4. repulsive and has magnitude $F = \frac{L^2}{\mu r^3}$
One Dimensional Description

• Energy
\[ E = \frac{1}{2} \mu \left( \frac{dr}{dt} \right)^2 + \frac{1}{2} \frac{L^2}{\mu r^2} - \frac{G m_1 m_2}{r} = K + U_{\text{effective}} \]

• Kinetic Energy
\[ K = \frac{1}{2} \mu \left( \frac{dr}{dt} \right)^2 \]

• Effective Potential Energy
\[ U_{\text{effective}} = \frac{L^2}{2\mu r^2} - \frac{G m_1 m_2}{r} \]

• Repulsive Force
\[ F_{\text{centrifugal}} = -d \left( \frac{L^2}{2\mu r^2} \right) = \frac{L^2}{\mu r^3} \]

• Gravitational force
\[ F_{\text{gravitational}} = -\frac{dU_{\text{gravitational}}}{dr} = -\frac{G m_1 m_2}{r^2} \]
Energy Diagram

\[ U_{\text{effective}} = \frac{L^2}{2\mu r^2} - \frac{Gm_1m_2}{r} \]

Case 4: Circular Orbit \ E = E_0

Case 3: Elliptic Orbit \ E_0 < E < 0

Case 2: Parabolic Orbit \ E = 0

Case 1: Hyperbolic Orbit \ E > 0
The radius and sign of the energy for the lowest energy orbit (circular orbit) is given by

1. energy is positive, \[ r_0 = \frac{L^2}{\mu G m_1 m_2} \]

2. energy is positive, \[ r_0 = \frac{L^2}{2\mu G m_1 m_2} \]

3. energy is negative, \[ r_0 = \frac{L^2}{\mu G m_1 m_2} \]

4. Energy is negative, \[ r_0 = \frac{L^2}{2\mu G m_1 m_2} \]
PRS Answer: Circular Orbit

- The lowest energy state corresponds to a circular orbit where the radius can be found by finding the minimum of effective potential energy

\[
0 = \frac{dU_{\text{effective}}}{dr} = -\frac{L^2}{\mu r^3} + \frac{Gm_1m_2}{r^2}
\]

\[
r_0 = \frac{L^2}{\mu Gm_1m_2}
\]

- Energy of circular orbit

\[
E_0 = \left(U_{\text{effective}}\right)_{r=r_0} = -\frac{\mu (Gm_1m_2)^2}{2L^2}
\]
If the earth slows down due to tidal forces will the moon’s angular momentum

1. increase
2. decrease
3. cannot tell from the information given
PRS Question

If the earth slows down due to tidal forces will the radius of the moon’s orbit

1. increase
2. decrease
3. cannot tell from the information given
Orbit Equation

• Solution:

\[ r = \frac{r_0}{1 - \varepsilon \cos \theta} \]

where the two constants are

• radius of circular orbit

\[ r_0 = \frac{L^2}{\mu G m_1 m_2} \]

• eccentricity

\[ \varepsilon = \left(1 + \frac{2EL^2}{\mu (Gm_1 m_2)^2}\right)^{\frac{1}{2}} \]
Energy and Angular Momentum

- **Energy:**
  \[ E = E_0 \left(1 - \varepsilon^2\right)^{\frac{1}{2}} \]
  where \( E_0 \) is the energy of the ‘ground state’

- **Angular momentum**
  \[ L = \left(r_0 \mu G m_1 m_2\right)^{\frac{1}{2}} \]
  where \( r_0 \) is the radius of the ‘ground state’
Properties of Ellipse:

\[ r_{\text{minimum}} = r(\theta = \pi) = \frac{r_0}{1 + \varepsilon} \]

\[ r_{\text{maximum}} = r(\theta = 0) = \frac{r_0}{1 - \varepsilon} \]

Semi-Major axis

\[ a = \frac{1}{2} \left( r_{\text{maximum}} + r_{\text{minimum}} \right) = \frac{1}{2} \left( \frac{r_0}{1 - \varepsilon} + \frac{r_0}{1 + \varepsilon} \right) = \frac{r_0}{1 - \varepsilon^2} = -\frac{G m_1 m_2}{2E} \]

location of the center of the ellipse

\[ x_0 = r_{\text{maximum}} - a = \frac{\varepsilon r_0}{1 - \varepsilon^2} = \varepsilon a \]

Semi-Minor axis

\[ b = \sqrt{a^2 - x_0^2} = a^{1/2} r_0^{1/2} \]

Area

\[ A = \pi a b = \pi a^{3/2} r_0^{1/2} \]
Kepler’s Laws: Equal Area

• Area swept out in time $\Delta t$

$$\frac{\Delta A}{\Delta t} = \frac{1}{2} \left( r \frac{\Delta \theta}{\Delta t} \right) r + \frac{(r\Delta \theta)}{2} \frac{\Delta r}{\Delta t}$$

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt}$$

$$\frac{d\theta}{dt} = \frac{L}{\mu r^2}$$

• Equal Area Law:

$$\frac{dA}{dt} = \frac{L}{2\mu} = \text{constant}$$
Kepler’s Laws: Period

• Area

\[ A = \pi ab = \pi a^{3/2} r_0^{1/2} \]

• Integral of Equal Area Law

\[ \int_{\text{orbit}} \frac{2\mu}{L} dA = \int_0^T dt \]

• Period

\[ T = \frac{2\mu}{L} A = \frac{2\mu\pi a^{3/2} r_0^{1/2}}{L} \]

• Period squared proportional to cube of the major axis but depends on both masses

\[ T^2 = \frac{4\mu^2}{L^2} \pi^2 a^3 r_0 = \frac{4\pi^2 \mu a^3}{G m_1 m_2} = \frac{4\pi^2 a^3}{G (m_1 + m_2)} \]
Two Body Problem Revisited

- Elliptic Case: Each mass orbits around center of mass with

\[
\ddot{\mathbf{r}}_1 = \mathbf{r}_1 - \mathbf{R}_{cm} = \mathbf{r}_1 - \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{m_1 + m_2} = \frac{m_2 (\mathbf{r}_1 - \mathbf{r}_2)}{m_1 + m_2} = \frac{\mu}{m_2} \mathbf{r}
\]

\[
\ddot{\mathbf{r}}_2 = -\frac{\mu}{m_2} \mathbf{r}
\]