Experiment 09: Angular momentum

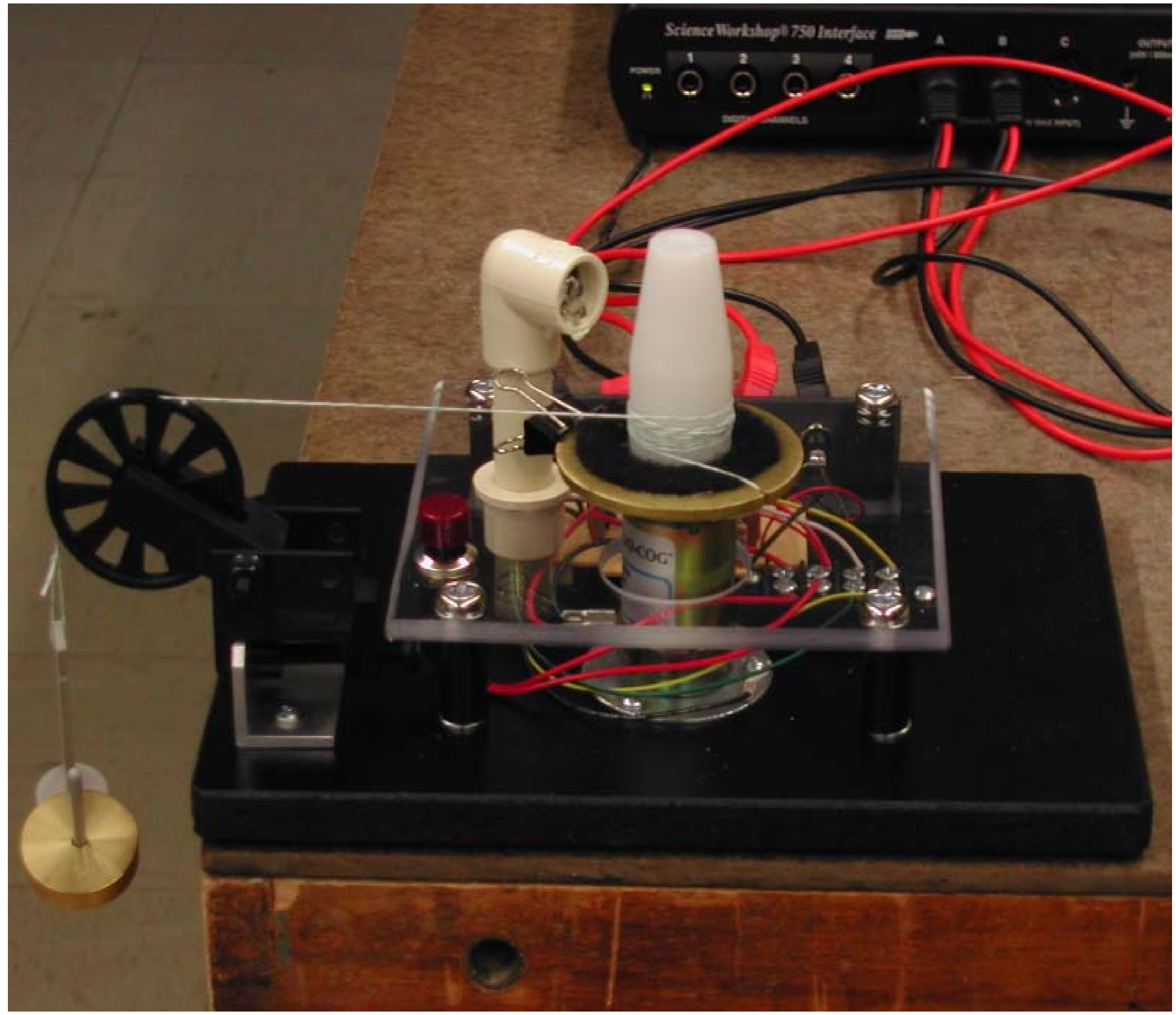
TEAL
MIT Studio Physics
8.01T

Goals

- Investigate conservation of angular momentum and kinetic energy in rotational collisions.
- Measure and calculate moments of inertia.
- Measure and calculate non-conservative work in an inelastic collision.

Apparatus

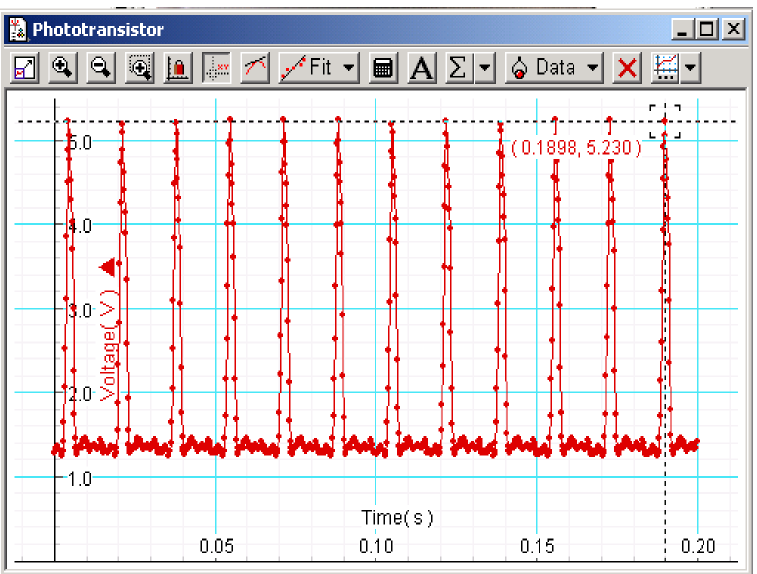
- Connect output of phototransistor to channel A of 750.
- Connect output of tachometer generator to channel B of 750.
- Connect power supply.
- Red button is pressed: Power is applied to motor.
- Red button is released: Rotor coasts: Read output voltage using DataStudio.

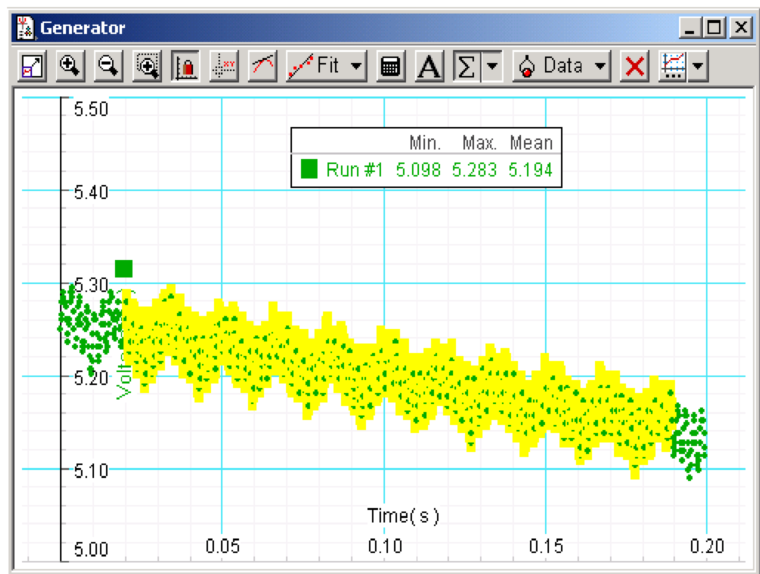


- Use black sticker or tape on white plastic rotor for generator calibration.

Calibrate tachometer-generator

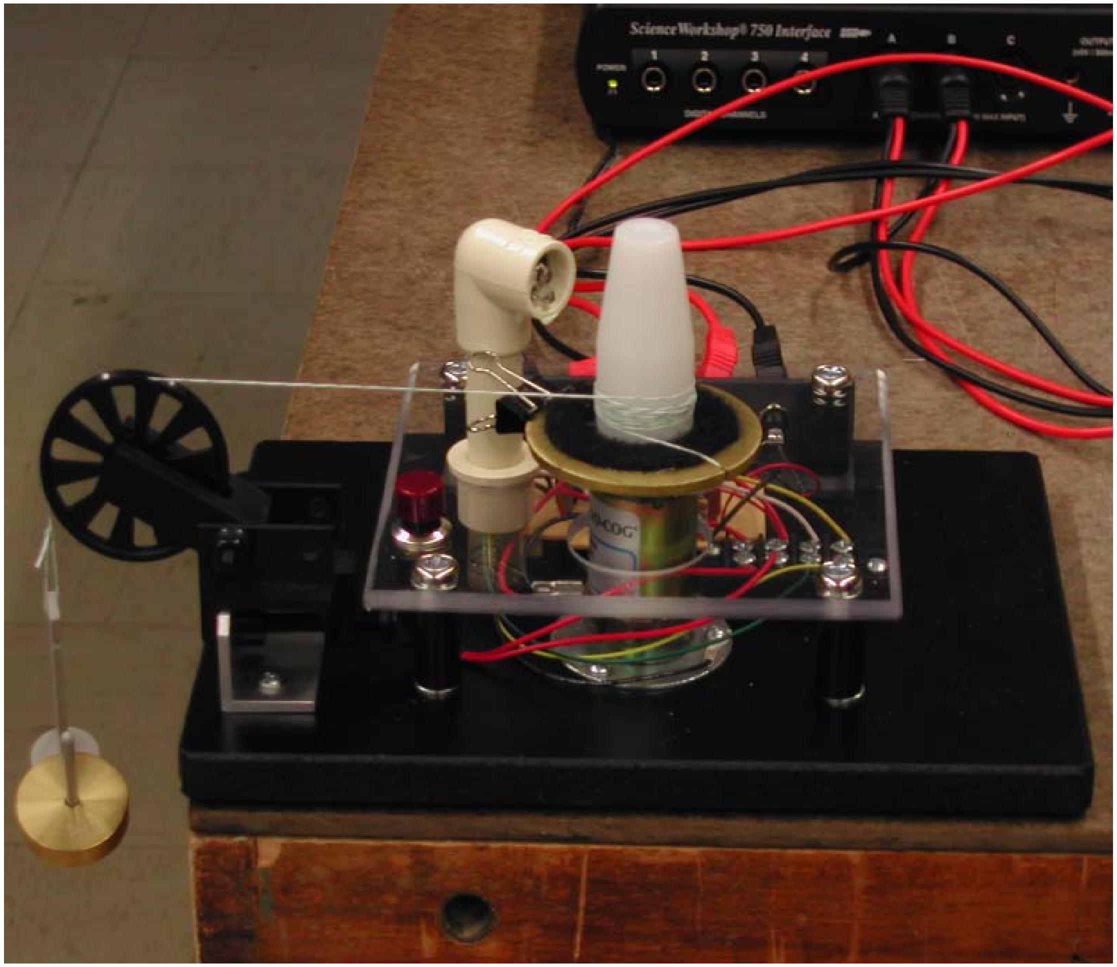
- Spin motor up to full speed, let it coast. Measure and plot voltages for 0.25 s period. Sample Rate: 5000 Hz, and Sensitivity: Low.





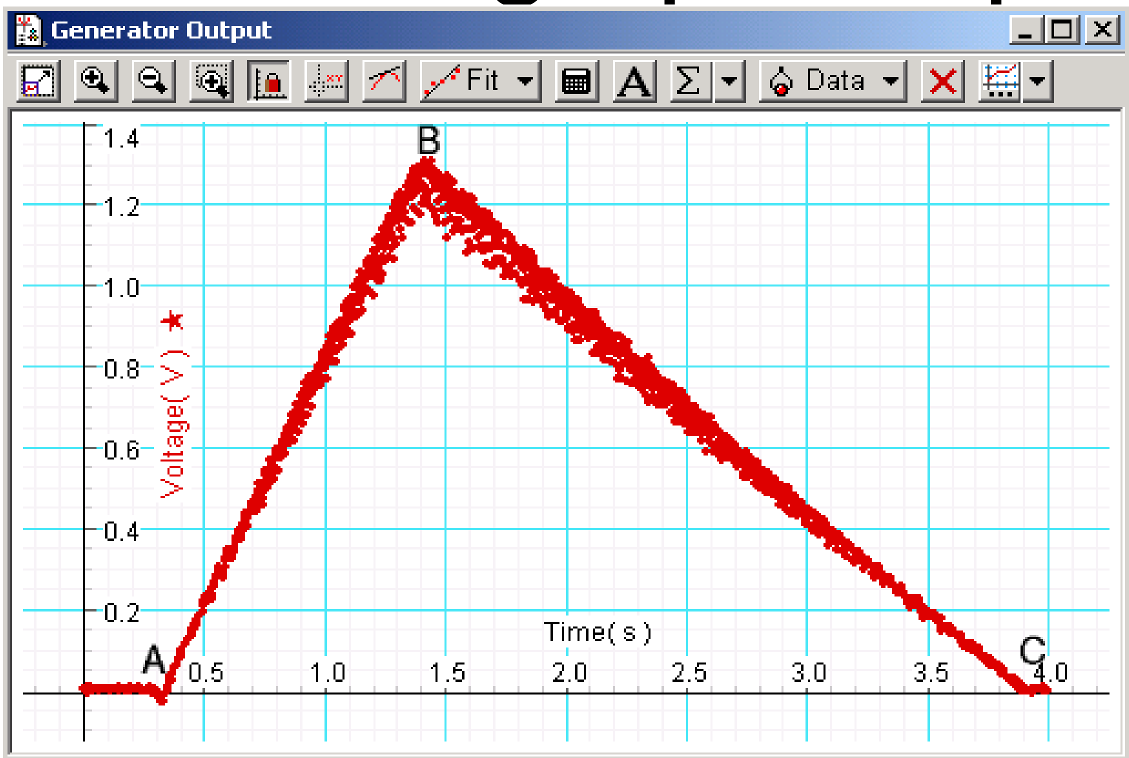
- Time 10 periods to measure ω.
- Then calculate ω for 1 V output.
- Average the output voltage over the same 10 periods.

Measure rotor I_R



- Plot only the generator voltage for rest of experiment.
- Use a 55 gm weight to accelerate the rotor.
- Settings:
  - Sensitivity: Low
  - Sample rate 500 Hz.
  - Delayed start: None
  - Auto Stop: 4 seconds
- Start DataStudio and let the weight drop.

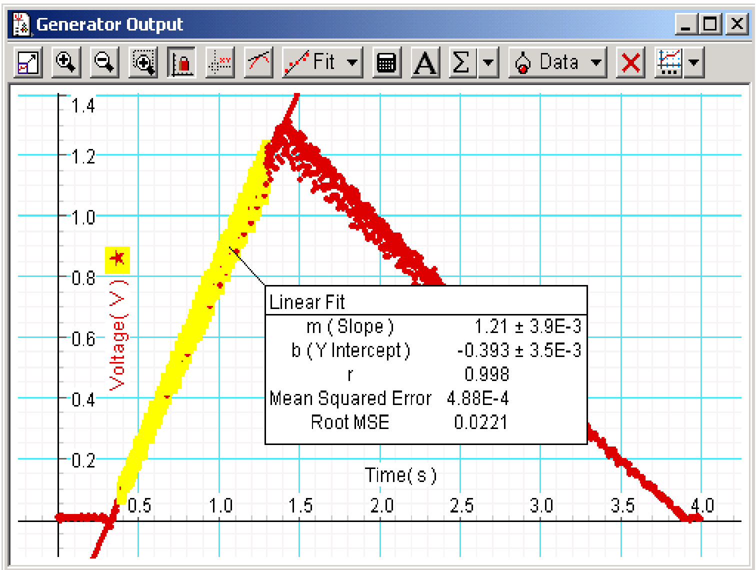
Understand graph output to

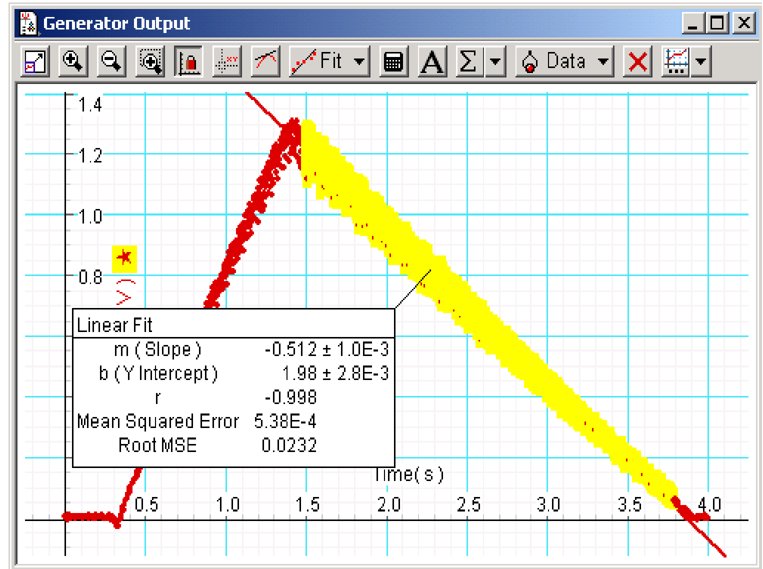


- Generator voltage while measuring I_R. What is happening:
  1. Along line A-B ?
  2. At point B ?
  3. Along line B-C ?

- How do you use this graph to find I_R ?

Measure I_R results



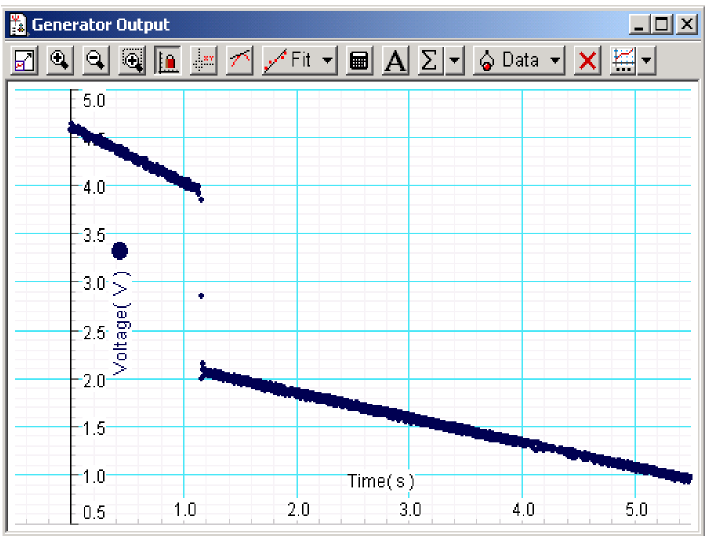


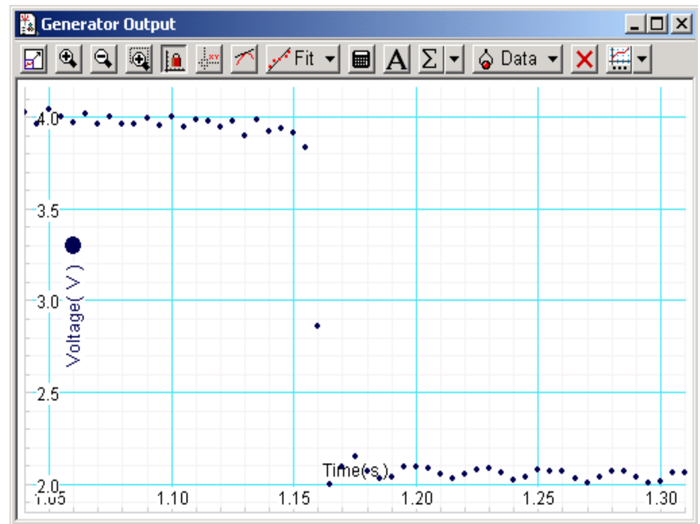
- Measure and record α_{up} and α_{down}.
- For your report, calculate I_R:

$$\tau_f = I_R \alpha_{\text{down}} \qquad I_R = \frac{mr(g - r\alpha_{\text{up}})}{\alpha_{\text{up}} - |\alpha_{\text{down}}|}$$

Fast collision

Sensitivity	Sample Rate	Delayed Start	Auto Stop
Low	200 Hz	1 sec	Falls below 0.5V



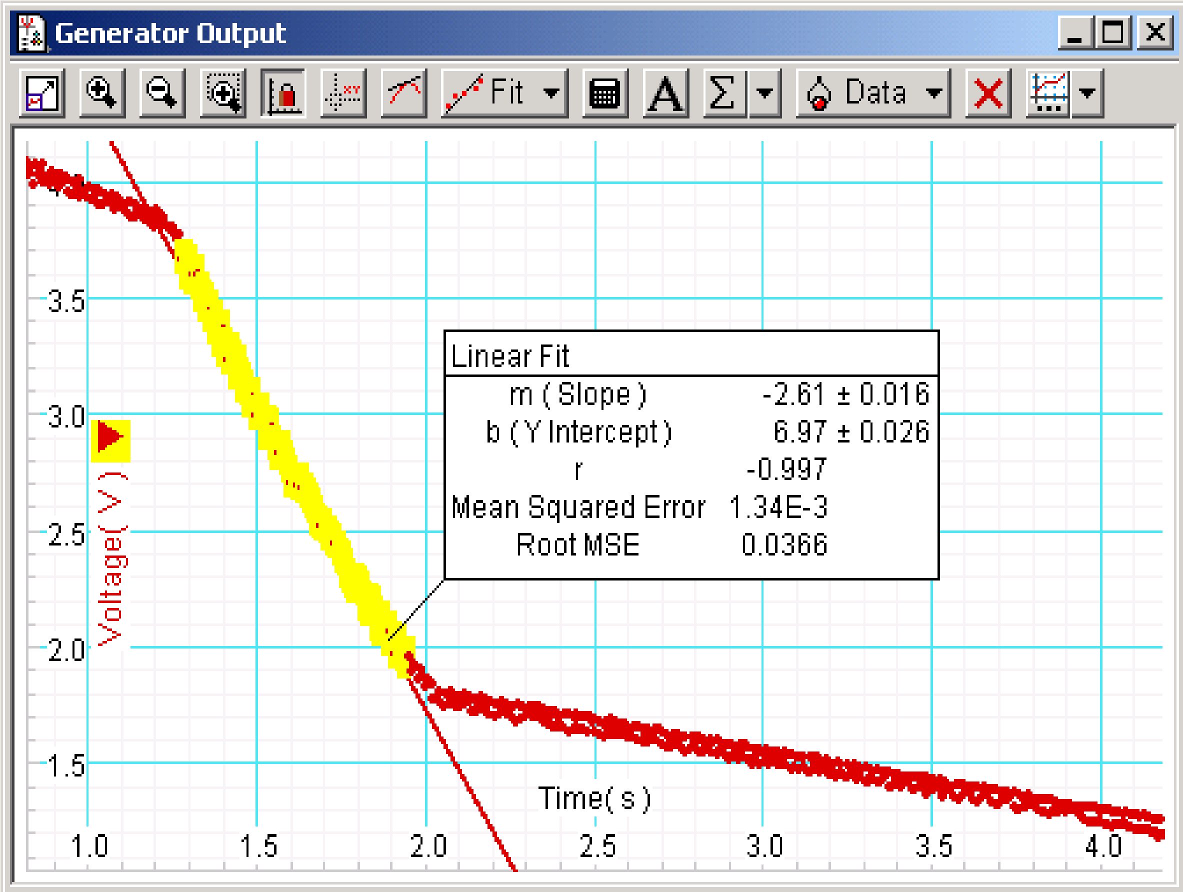


Find ω $_1$ (before) and ω $_2$ (after), estimate δt for collision.

Calculate

$$I_W = \frac{1}{2} m\omega (r_o^{\ 2} + r_i^{\ 2})$$

Slow collision



- Find ω_1 and ω_2, measure δt, fit or measure to find a_c.
- Keep a copy of your results for the homework problem.

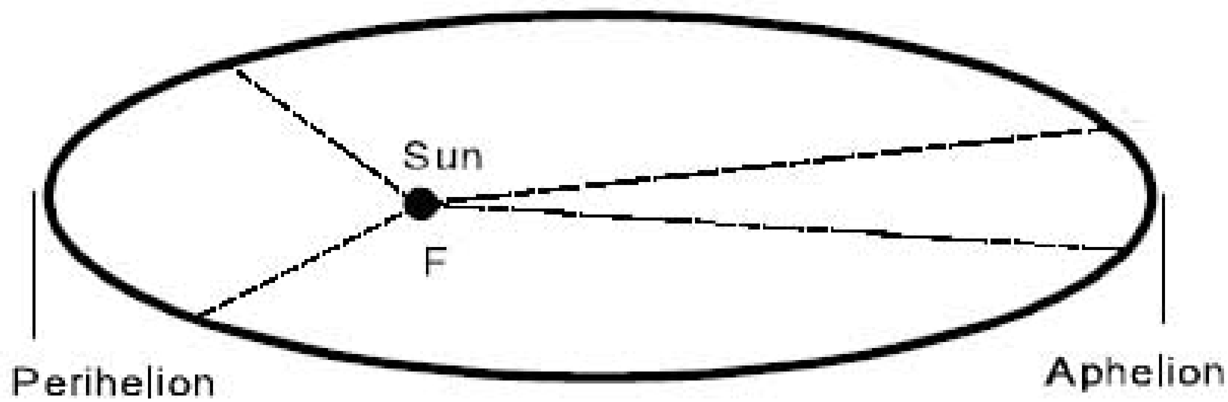
Kepler Problem and Planetary Motion

8.01t

Nov 15, 2004

Kepler’s Laws

1. The orbits of planets are ellipses; and the sun is at one focus



2. The radius vector sweeps out equal areas in equal time

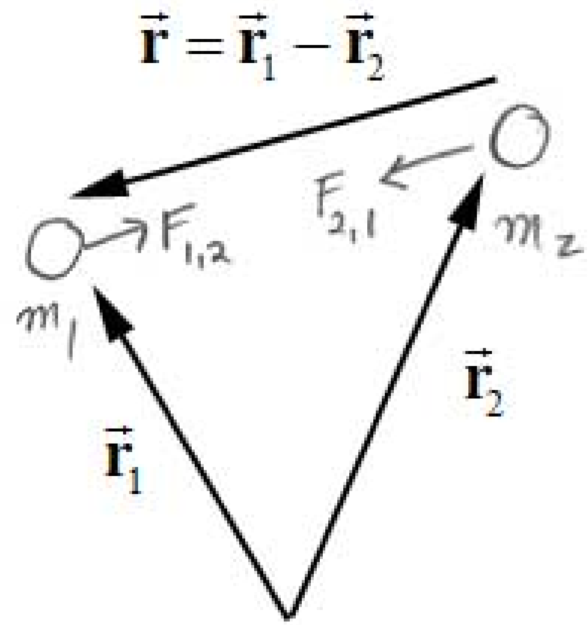
3. The period T is proportional to the radius to the 3/2 power

$$T \sim r^{3/2}$$

Kepler Problem

- Find the motion of two bodies under the influence of a gravitational force using Newtonian mechanics

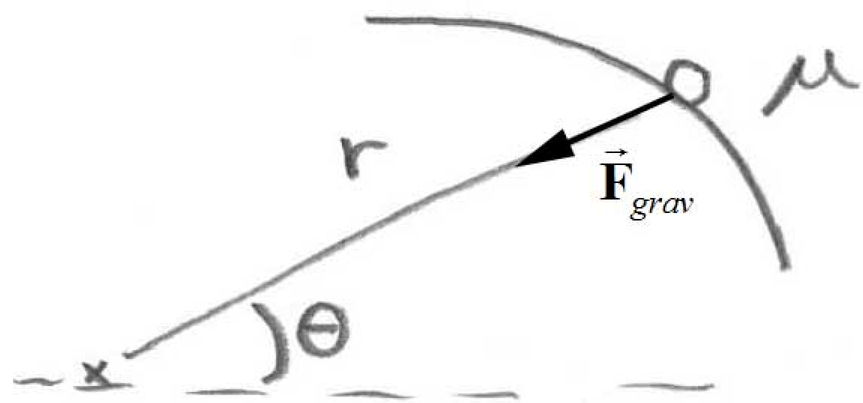
$$\vec{\mathbf{F}}_{1,2}\left(r\right)=-G\frac{m_1 m_2}{r^2}\hat{\mathbf{r}}$$



Reduction of Two Body Problem

- Reduce two body problem to one body of mass μ moving about a central point under the influence of gravity with position vector corresponding to the vector from mass m_2 to mass m_1

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$



$$\mathrm{F}_{1,2}\left(\vec{\mathbf{r}}\right)\hat{\mathbf{r}} = \mu \frac{d^2\vec{\mathbf{r}}}{dt^2}$$

Solution of One Body Problem

- Solving the problem means finding the distance from the origin $r(t)$ and angle $\theta(t)$ as functions of time
- Equivalently, finding the distance from the center as a function of angle $r(\theta)$
- Solution:

$$r = \frac{r_0}{1 - \varepsilon \cos\theta}$$

Constants of the Motion

- Velocity $\vec{\mathbf{v}} = \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\theta}{dt}\hat{\boldsymbol{\theta}}$ $\quad v^2 = \left(\frac{dr}{dt}\right)^2 + \left(r\frac{d\theta}{dt}\right)^2$
- Angular Momentum $L = \mu r v_{tangential} = \mu r^2 \frac{d\theta}{dt}$
- Energy

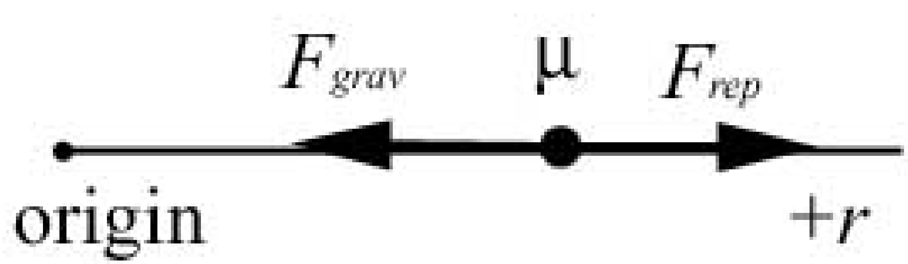
$$E = \frac{1}{2}\mu v^2 - \frac{Gm_1 m_2}{r}$$

$$E = \frac{1}{2}\mu\left[\left(\frac{dr}{dt}\right)^2 + \left(r\frac{d\theta}{dt}\right)^2\right] - \frac{Gm_1 m_2}{r}$$

$$E = \frac{1}{2}\mu\left(\frac{dr}{dt}\right)^2 + \frac{L^2}{2\mu r^2} - \frac{Gm_1 m_2}{r}$$

Reduction to One Dimensional Motion

- Reduce the one body problem in two dimensions to a one body problem moving only in the r-direction but under the action of a repulsive force and a gravitational force



PRS Question

Suppose the potential energy of two particles (reduced mass μ) is given by

$$U(r) = \frac{1}{2}\frac{L^2}{\mu r^2}$$

where r is the relative distance between the particles. The force between the particles is

1. attractive and has magnitude $F = \dfrac{L^2}{2\mu r}$

2. repulsive and has magnitude $F = \dfrac{L^2}{2\mu r}$

3. attractive and has magnitude $F = \dfrac{L^2}{\mu r^3}$

4. repulsive and has magnitude $F = \dfrac{L^2}{\mu r^3}$

One Dimensional Description

- Energy $$E = \frac{1}{2}\mu\left(\frac{dr}{dt}\right)^2 + \frac{1}{2}\frac{L^2}{\mu r^2} - \frac{Gm_1m_2}{r} = K + U_{effective}$$
- Kinetic Energy $$K = \frac{1}{2}\mu\left(\frac{dr}{dt}\right)^2$$
- Effective Potential Energy $$U_{effective} = \frac{L^2}{2\mu r^2} - \frac{Gm_1m_2}{r}$$
- Repulsive Force $$F_{centrifugal} = -\frac{d}{dr}\left(\frac{L^2}{2\mu r^2}\right) = \frac{L^2}{\mu r^3}$$
- Gravitational force $$F_{gravitational} = -\frac{dU_{gravitational}}{dr} = -\frac{Gm_1m_2}{r^2}$$

Energy Diagram

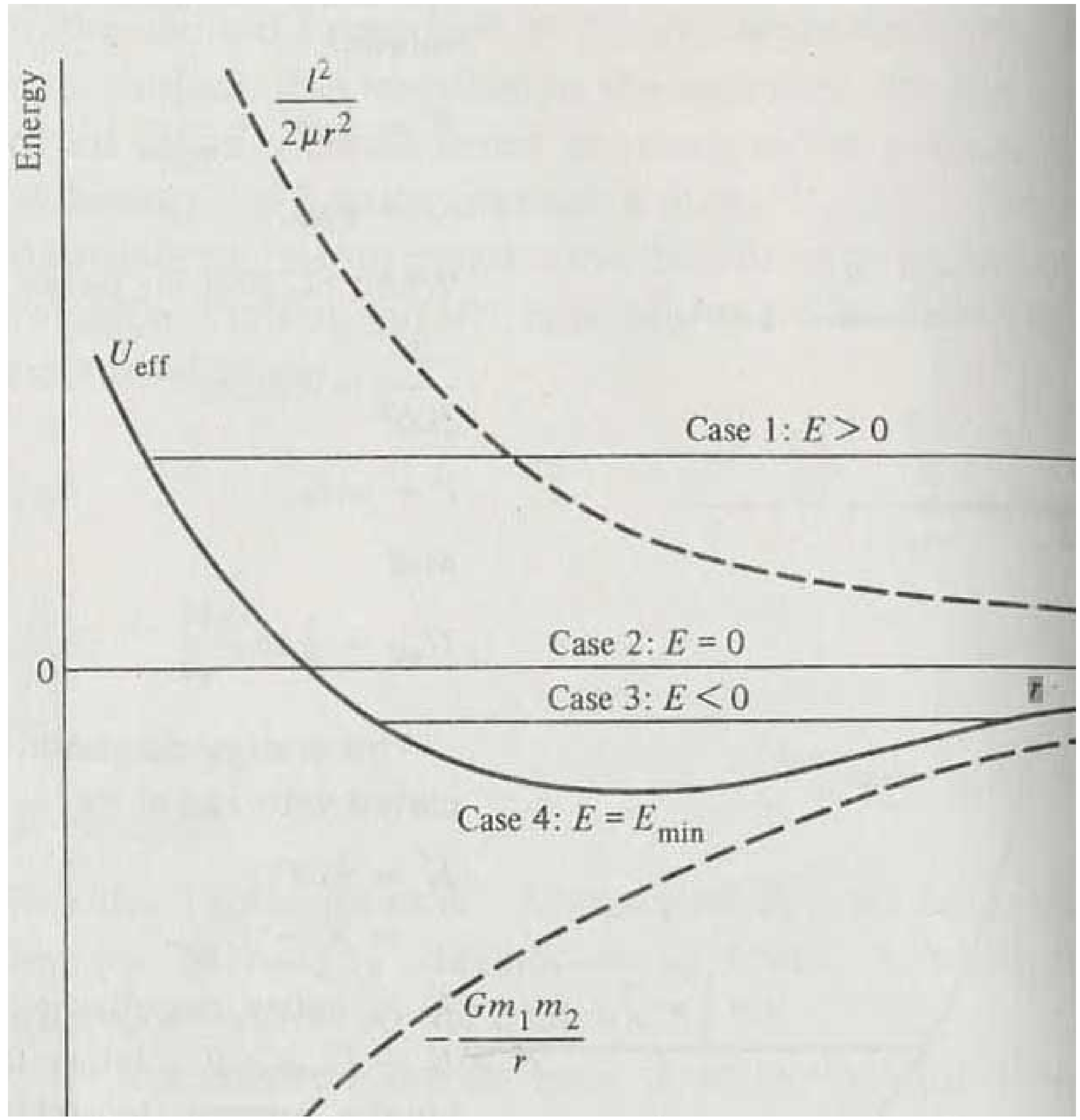
$$U_{effective} = \frac{L^2}{2\mu r^2} - \frac{Gm_1m_2}{r}$$

Case 4: Circular Orbit E = Eo

Case 3: Elliptic Orbit Eo < E < 0

Case 2: Parabolic Orbit E = 0

Case 1: Hyberbolic Orbit E > 0



PRS Question

The radius and sign of the energy for the lowest energy orbit (circular orbit) is given by

1. energy is positive, $$r_0 = \frac{L^2}{\mu G m_1 m_2}$$

2. energy is positive, $$r_0 = \frac{L^2}{2\mu G m_1 m_2}$$

3. energy is negative, $$r_0 = \frac{L^2}{\mu G m_1 m_2}$$

4. Energy is negative, $$r_0 = \frac{L^2}{2\mu G m_1 m_2}$$

PRS Answer: Circular Orbit

- The lowest energy state corresponds to a circular orbit where the radius can be found by finding the minimum of effective potential energy

$$0 = \frac{dU_{effective}}{dr} = -\frac{L^2}{\mu r^3} + \frac{Gm_1 m_2}{r^2}$$

$$r_0 = \frac{L^2}{\mu G m_1 m_2}$$

- Energy of circular orbit

$$E_0 = \left(U_{effective}\right)\Big|_{r=r_0} = -\frac{\mu\left(Gm_1 m_2\right)^2}{2L^2}$$

PRS Question

If the earth slows down due to tidal forces will the moon’s angular momentum

1. increase
2. decrease
3. cannot tell from the information given

PRS Question

If the earth slows down due to tidal forces will the radius of the moon's orbit

1. increase
2. decrease
3. cannot tell from the information given

Orbit Equation

- Solution:

$$r = \frac{r_0}{1 - \varepsilon \cos\theta}$$

where the two constants are

- radius of circular orbit

$$r_0 = \frac{L^2}{\mu G m_1 m_2}$$

- eccentricity

$$\varepsilon = \left(1 + \frac{2EL^2}{\mu \left(G m_1 m_2 \right)^2} \right)^{\frac{1}{2}}$$

Energy and Angular Momentum

- Energy: $$E = E_0 \left(1 - \varepsilon^2\right)^{\frac{1}{2}}$$

where E_0 is the energy of the 'ground state'

$$E_0 = \left(U_{effective}\right)\Big|_{r=r_0} = -\frac{\mu\left(Gm_1m_2\right)^2}{2L^2}$$

- Angular momentum

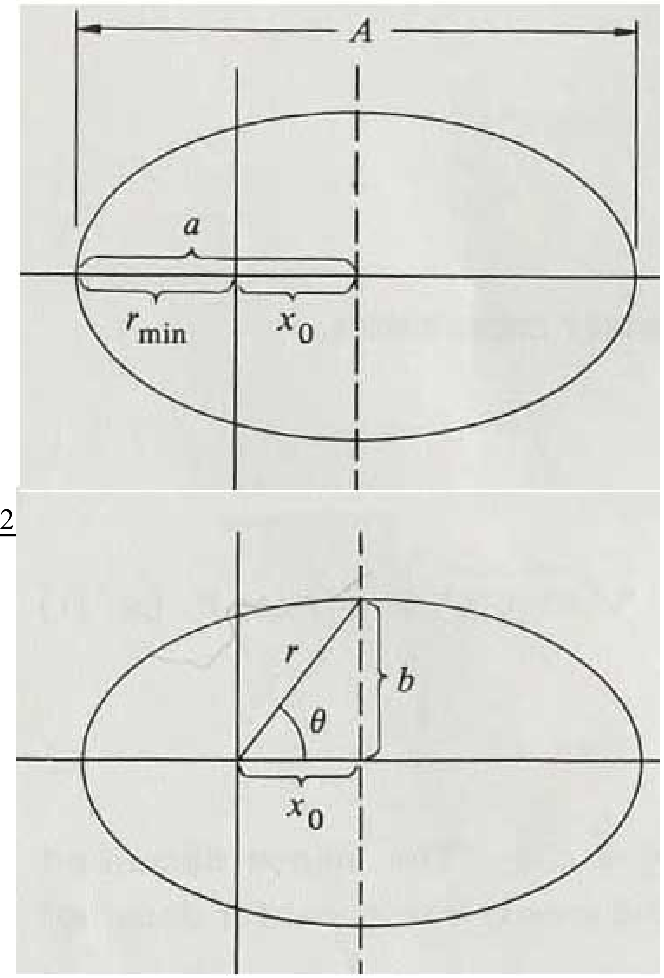
$$L = \left(r_0\mu Gm_1m_2\right)^{1/2}$$

where r_0 is the radius of the 'ground state'

Properties of Ellipse:

$$r_{minimum} = r(\theta = \pi) = \frac{r_0}{1+\varepsilon}$$

$$r_{maximum} = r(\theta = 0) = \frac{r_0}{1-\varepsilon}$$



Semi-Major axis

$$a = \frac{1}{2}\left(r_{maximum} + r_{minimum}\right) = \frac{1}{2}\left(\frac{r_0}{1-\varepsilon} + \frac{r_0}{1+\varepsilon}\right) = \frac{r_0}{1-\varepsilon^2} = -\frac{Gm_1m_2}{2E}$$

location of the center of the ellipse

$$x_0 = r_{maximum} - a = \frac{\varepsilon r_0}{1-\varepsilon^2} = \varepsilon a$$

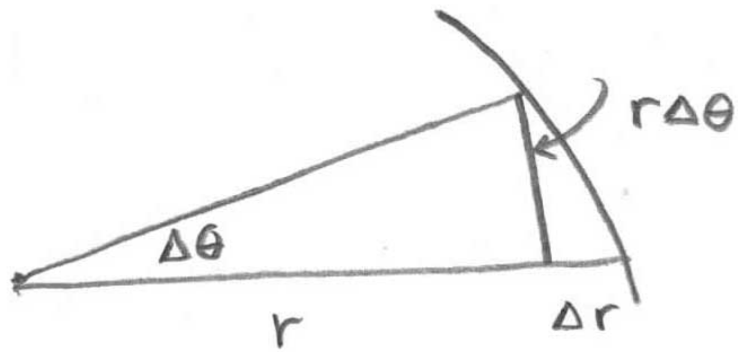
Semi-Minor axis $$b = \sqrt{\left(a^2 - x_0{}^2\right)} = a^{1/2} r_0{}^{1/2}$$

Area $$A = \pi ab = \pi a^{3/2} r_0{}^{1/2}$$

Kepler's Laws: Equal Area

- Area swept out in time Δt

$$\frac{\Delta A}{\Delta t} = \frac{1}{2}\left(r\frac{\Delta\theta}{\Delta t}\right)r + \frac{(r\Delta\theta)}{2}\frac{\Delta r}{\Delta t}$$



$$\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt} \qquad \frac{d\theta}{dt} = \frac{L}{\mu r^2}$$

- Equal Area Law:

$$\frac{dA}{dt} = \frac{L}{2\mu} = constant$$

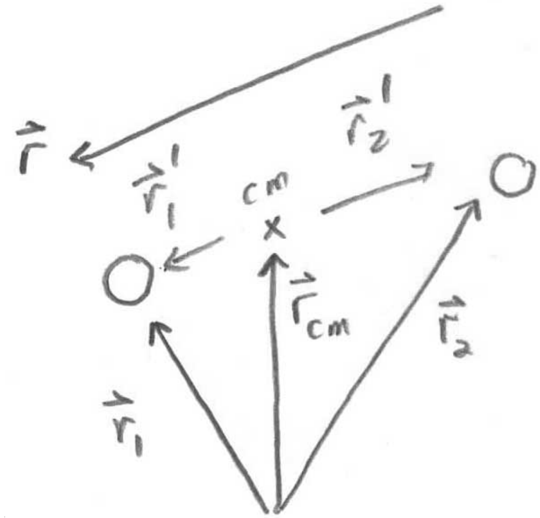
Kepler's Laws: Period

- Area $$A = \pi ab = \pi a^{3/2} r_0^{1/2}$$
- Integral of Equal Area Law $$\int_{orbit} \frac{2\mu}{L} dA = \int_0^T dt$$
- Period $$T = \frac{2\mu}{L} A = \frac{2\mu\pi a^{3/2} r_0^{1/2}}{L}$$
- Period squared proportional to cube of the major axis but depends on both masses

$$T^2 = \frac{4\mu^2}{L^2} \pi^2 a^3 r_0 = \frac{4\pi^2 \mu a^3}{G m_1 m_2} = \frac{4\pi^2 a^3}{G\left(m_1 + m_2\right)}$$

Two Body Problem Revisited

- Elliptic Case: Each mass orbits around center of mass with



$$\vec{\mathbf{r}}_1' = \vec{\mathbf{r}}_1 - \vec{\mathbf{R}}_{cm} = \vec{\mathbf{r}}_1 - \frac{m_1\vec{\mathbf{r}}_1 + m_2\vec{\mathbf{r}}_2}{m_1 + m_2} = \frac{m_2\left(\vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2\right)}{m_1 + m_2} = \frac{\mu}{m_2}\vec{\mathbf{r}}$$

$$\vec{\mathbf{r}}_2' = -\frac{\mu}{m_2}\vec{\mathbf{r}}$$