

## THE LQG/LTR PROCEDURE FOR MULTIVARIABLE FEEDBACK CONTROL DESIGN\*

by

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ABSTRACT

This paper provides a new perspective on the LQG/LTR design procedure for linear multivariable feedback systems. The basic design problem is interpreted as a weighted  $H^2$ -tradeoff between transfer functions in the frequency domain. This problem can be solved with a modified version of the standard LQG methodology. Properties of the solution are examined for both minimum phase and non-minimum phase systems. This leads to a formal weight augmentation procedure for the minimum phase case which permits essentially arbitrary specification of system sensitivity functions in terms of the weights. While such arbitrary specifications are not possible for non-minimum phase problems, a direct relationship between weights and sensitivities is developed for non-minimum phase SISO and certain non-minimum phase MIMO cases which guides the weight selection process.

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## 1. INTRODUCTION

One of the goals of control theory has been to capture major elements of the engineering process of feedback design under the umbrella of a formal mathematical synthesis problem. The motivation for this goal is self evident. Once formalized under such an umbrella, the elements of engineering art become rigorous tools which can be applied more or less automatically to ever more complex design situations.

Perhaps the best known example of formal mathematical synthesis is the linear-quadratic-gaussian optimal control problem (LQG) [1]. This problem formalizes a specific design situation, namely the construction of feedback compensators for finite-dimensional linear plant models, with stability and least-squares performance under additive disturbances as design objectives. Needless to say, this covers only a small subset of typical overall engineering design problems. However, linear models are applicable often enough (particularly in early stages of a project), and least-squares objectives can be manipulated cleverly enough (via free parameters in the quadratic criterion) that LQG has proven itself useful in many diverse design applications.

Research developments over the last several years have shown that the flexibility provided by the quadratic criterion of LQG is remarkably broad. Indeed, it is possible to choose free parameters in such a way that the entire formal design process can be re-interpreted not as a least-squares error minimization problem but as a "loop shaping" problem -- that is, a problem of designing feedback compensators to achieve desirable sensitivity and complementary sensitivity transfer functions at critical loop-breaking points of the feedback system. We believe that such a re-interpretation is useful enough for practical design to warrant a separate label and a separate exposition. The label which is beginning to stick is LQG/LTR, which stands for linear-quadratic-gaussian synthesis with loop transfer recovery. Its exposition was first given by Doyle and Stein in [2]. This paper provides an alternate perspective on the methodology and presents several useful new twists. These include a formal weight selection procedure which permits essentially arbitrary specification of system sensitivity functions for minimum phase problems, a classification of all recoverable functions in non-minimum phase problems, and certain direct relationships between weights and sensitivities for the latter which apply to all scalar and certain

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**multivariable cases.**

Since LQG/LTR looks most interesting under the classical frequency domain design paradigm of Nyquist and Bode [3,4], we begin in Section 2 with a very brief review of this paradigm as applied to multivariable problems. We then pose a formal  $H^2$ -optimal synthesis problem which attempts to trade-off the chief design functions of the paradigm. Comparing this  $H^2$ -problem with the standard LQG formulation then suggests specific choices of free parameters in the latter to solve the former. Under minimum phase assumptions, the resulting solutions exhibit very nice frequency domain properties. These are described in Section 4. Properties which hold for non-minimum phase cases are also discussed, along with current research questions. Section 5 discusses the non-minimum phase case in more detail and provides a brief example. Section 6 makes concluding comments.

## 2. FEEDBACK DESIGN IN THE FREQUENCY DOMAIN

Ever since the basic work of Nyquist, Bode and others, the classical approach to feedback design has followed the frequency domain perspective illustrated in Figure 1. We are given a multivariable plant described by a rational transfer function,  $G(s)$ , and wish to design a compensator,  $K(s)$ , such that the closed loop feedback system satisfies the following basic requirements:

- (1) **Stability:** bounded outputs,  $y(s)$ , for all bounded disturbances,  $d(s)$ , and bounded reference inputs,  $r(s)$
- (2) **Performance:** small errors,  $e(s)$ , in the presence of  $d(s)$  and  $r(s)$
- (3) **Robustness:** stability and performance maintained in the presence of model uncertainties,  $\delta G(s)$ , expressed in whatever form is appropriate for the real plant at hand

As is well known, the first of these requirements imposes structural constraints on certain transfer functions of the closed loop system, e.g. a Nyquist encirclement count for the function  $\det(I+GK)$  [5]. Likewise, the second requirement imposes magnitude constraints on certain transfer functions. In particular, for Figure 1 where disturbances and commands are reflected to the output loop-breaking point, the (output) sensitivity function

$$S(s) = [I + G(s)K(s)]^{-1} \quad (1)$$

must be small for all frequencies,  $s=j\omega$ , where the disturbances and/or reference commands are large. This latter statement is a fundamental frequency domain prescription for feedback design. We will refer to it as

P1: "Make  $S(j\omega)$  small whenever  $d(j\omega)$  or  $r(j\omega)$  are large."

For classical single-input single-output (SISO) systems, the meanings of "small" and "large" in (P1) are, of course, to be understood in terms of the absolute values of the respective complex numbers at each frequency. For multi-input multi-output (MIMO) systems, these meanings must be treated with greater care. Complex vectors,  $d(j\omega)$  and  $r(j\omega)$ , will be taken as small or large according to the size of their usual Euclidean norm. Complex matrices,  $S(j\omega)$ , will be taken as small if their largest singular value,  $\bar{\sigma}[S(j\omega)]$ , is small, and they will be taken as large if their smallest singular value,  $\underline{\sigma}[S(j\omega)]$ , is large. With this generalization of definitions, the design prescription (P1) applies to SISO and MIMO systems alike [2,6].

The third feedback design requirement -- tolerance for uncertainty -- can also be expressed in terms of magnitude constraints. In this case, however, the transfer functions to which the constraints apply depend upon how model uncertainties are characterized. In classical SISO problems, tradition dictates that gain margins,  $gm$ , and phase margins,  $pm$ , be used to characterize tolerable uncertainty. These margins are suitable for uncertainties in the following specific form:

$$g(s) + \delta g(s) = \exp(L)g(s) \approx (1+L)g(s) \quad (2)$$

where  $L$  is an arbitrary real scalar with  $abs(L) \leq gm \ln(10)/20$  for pure gain uncertainty, or  $L$  is an arbitrary imaginary scalar with  $abs(L) \leq pm/57.3$  for pure phase uncertainty. This characterization can be generalized to MIMO problems as follows:

$$G(s) + \delta G(s) = [I + L(s)]G(s) \quad (3a)$$

where  $L(s)$  is an arbitrary stable\* transfer function matrix with

$$\bar{\sigma}[L(j\omega)] \leq m(\omega) \quad (3b)$$

Equation (3) is sometimes called "unstructured multiplicative uncertainty". It covers simultaneous gain, phase and directions errors which are unknown but bounded in size. The bound,  $m(\omega)$ , indicates the maximum normalized magnitude which the model error can attain. It is typically small ( $m \ll 1$ ) at low frequencies but invariably rises toward unity and well above (modelling error  $\gg 100\%$ ) as frequency increases.

It can be readily verified by direct manipulation of  $\det[I+(G+\delta G)K]$  [see 6], that stability is maintained in the presence of all possible uncertainties characterized by (3) if and only if the "complementary (output) sensitivity function [7]"

$$T(s) = G(s)K(s)[I+G(s)K(s)]^{-1} \quad (4)$$

satisfies

$$\bar{\sigma}[T(j\omega)] \leq \frac{1}{m(\omega)} \quad \text{for all } \omega \quad (5)$$

This condition leads to the second fundamental frequency domain prescription for feedback design:

P2: "Keep  $T(j\omega)$  small wherever  $m(\omega)$  is large"

Note that since  $T(s)$  is nothing more than the closed loop command response transfer function, (P2) can also be interpreted as a prescription to restrict closed loop bandwidth to the frequency range over which the plant model is valid.

Taken together, the two design prescriptions (P1) and (P2) capture the basic features of feedback design as viewed from the frequency domain. We face two functions,  $S(s)$  and  $T(s)$ , which both need to be small on the  $j\omega$ -axis. However, since

$$S(s) + T(s) = I \quad (6)$$

they cannot be made small simultaneously. Rather we must trade-off the size of one function against the size of the other in accordance with the relative importance of disturbance/command power and model uncertainty at each frequency.

Feedback design is thus seen as a game of essential trade-offs between transfer functions. Of course, these trade-offs are not always as pure as the one identified above. For example, we have ignored sensor noise in Figure 1. This imposes additional magnitude constraints on  $T(s)$ . We have also restricted discussion to the output loop-breaking point. Other points give rise to trade-offs between their respective sensitivity and complementary sensitivity functions and even to trade-offs between functions taken from different points. Our objective here is not to cover all cases but simply to illustrate that the frequency domain viewpoint gives rise to certain basic trades which can be solved via LQG/LTR.

\* This assumption can be relaxed [2,6].

### 3. FORMAL $H^2$ -OPTIMIZATION VIA LQG/LTR

Trade-offs between transfer functions can be formalized by posing them as function-space optimization problems. The first thing needed is a convenient criterion of smallness. Consider

$$\bar{\sigma}[M]^2 = \lambda_{\max}[MM^H] \leq \text{Trace}[MM^H] \quad (7)$$

This shows that a matrix  $M$  will be small if  $\text{Tr}[MM^H]$  is small. Using this latter measure for the two matrices,  $S(j\omega)$  and  $T(j\omega)$ , adding weights  $W(j\omega)$  to trade one against the other, and integrating over frequency gives the following plausible optimization problem:

Given the plant  $G(s)$ , weights  $W(s)$ , and sensitivity and complementary sensitivity defined by (1) and (4), respectively, find a stabilizing compensator  $K(s)$  to minimize

$$\begin{aligned} J &= \int_0^{\infty} (\text{Tr}[S W W^H S^H] + \text{Tr}[T T^H]) d\omega \\ &= \int_0^{\infty} \text{Tr}[M M^H] d\omega \end{aligned} \quad (8a)$$

with

$$M(j\omega) = [ S(j\omega) W(j\omega) \quad T(j\omega) ] \quad (8b)$$

In mathematical terms, this represents an optimization over the Hardy space of stable transfer functions with 2-norm, i.e. an  $H^2$ -optimization problem [8].

Depending upon the properties of  $G(s)$  and the choice of  $W(s)$ , optimizing compensators for (8) are not in general finite dimensional, strictly proper, or even proper. However, under the mild restriction that  $G$  and  $W$  are themselves finite dimensional and strictly proper, it is possible to show that the standard LQG problem can be used to generate a sequence of strictly proper compensators which minimize (8) in the limit and maintain closed loop stability. To show this, consider the usual LQG setup:

Given

$$\frac{dx}{dt} = Ax(t) + Bu(t) + L\xi(t) \quad (9a)$$

$$y(t) = Cx(t) + \mu I\eta(t) \quad (9b)$$

$$z(t) = Hx(t) \quad (9c)$$

where matrices  $(A, B, C)$  form the standard  $n$ -dimensional state-space representation of  $G(s)$ , i.e.

$$G(s) = C\Phi(s)B \quad \text{with} \quad \Phi(s) = (sI - A)^{-1}, \quad (9d)$$

where  $z(t)$  is an auxiliary response variable, and where  $\xi(t)$  and  $\eta(t)$  are Gaussian white noise processes with unit intensities.

Find a controller depending only on  $y(\tau)$  and  $u(\tau)$ ,  $\tau \leq t$ , to minimize

$$J_{LQG} = E \left\{ \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T (z^T z + \rho^2 u^T u) dt \right\} \quad (10)$$

The free parameters of this problem are the noise input matrix  $L$  in (9a), the scalar  $\mu$  in (9b), the auxiliary response matrix  $H$  in (9c), and the scalar  $\rho$  in (10). In contrast to the standard LQG formulation, these parameters will not be given

*a priori* physical significance (e.g. process noise, sensor noise, controlled variables, control weights). Rather, they are free to be manipulated for other design purposes. It is well known, of course, that under mild assumptions on these parameters, the LQG problem yields a unique n-dimensional fixed-parameter stabilizing compensator as its solution [1].

Given that LQG produces a stabilizing compensator  $K(s)$ , it is possible to transform (10) into an equivalent  $H^2$ -criterion, and hence, to verify that LQG solves an  $H^2$ -problem. Note first that equations (9) can be written in the frequency domain as

$$\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} G & C\Phi L & I \\ H\Phi B & H\Phi L & 0 \end{bmatrix} \begin{bmatrix} u \\ \xi \\ \eta \end{bmatrix} \quad (11)$$

Substituting  $u = -K(s)y$  yields

$$\begin{bmatrix} z \\ u \end{bmatrix} = P(s) \begin{bmatrix} \xi \\ \eta \end{bmatrix} \quad (12a)$$

with

$$P(s) = \begin{bmatrix} H\Phi L - H\Phi BK(I+GK)^{-1}C\Phi L & -\mu H\Phi BK(I+GK)^{-1} \\ -\rho K(I+GK)^{-1}C\Phi L & \mu\rho K(I+GK)^{-1} \end{bmatrix} \quad (12b)$$

Finally, substituting (12) into (10) via Parseval's Theorem yields

$$J_{LQG} = \frac{1}{\pi} \int_0^{\infty} \text{Tr}[PP^H] d\omega \quad (13)$$

Comparing (8) and (13), it is now apparent that the only remaining step needed to solve (8) is to show that free parameters exist for (12b) such that  $P(s)$  reduces to  $M(s)$ . It is easy to verify that the following choices do the trick:

Choose  $L$  and  $\mu$  such that

$$\frac{C\Phi L}{\mu} = W(s) \quad (14a)$$

and let

$$H = C \quad \text{and} \quad \rho \rightarrow 0 \quad (14b)$$

Then

$$P(s) \rightarrow \mu \begin{bmatrix} (I+GK)^{-1}W & -GK(I+GK)^{-1} \\ 0 & 0 \end{bmatrix} \quad (14c)$$

Note that for each non-zero value of  $\rho$ , the LQG solution for these choices produces a stabilizing strictly proper controller which is  $H^2$ -optimal for criterion (13). This criterion, however, approaches (8) more and more closely as  $\rho \rightarrow 0$ . Hence, for a sequence of decreasing  $\rho$ -values, we get a sequence of controllers which optimizes (8) in the limit.

It is also easy to verify that the following alternative to (14) produces another useful transfer function trade-off:

Choose  $H$  and  $\rho$  such that

$$\frac{H\Phi B}{\rho} = W(s) \quad (15a)$$

and let

$$L = B \quad \text{and} \quad \mu \rightarrow 0 \quad (15b)$$

Then

$$P(s) \rightarrow \rho \begin{bmatrix} W(I+KG)^{-1} & 0 \\ -(I+KG)^{-1}KG & 0 \end{bmatrix} \quad (15c)$$

These choices accomplish an  $H^2$ -trade-off between the sensitivity and complementary sensitivity functions at the input loop-breaking point of Figure 1 instead of at the output. Both choices, (14) and (15), will be referred to as LQG/LTR.



#### 4. PROPERTIES OF LQG/LTR SOLUTIONS

The practical value of a formal synthesis problem rests in the qualitative properties of its solutions. We will see in this section that  $H^2$ -solutions via LQG/LTR have very nice properties for the class of systems whose models are minimum phase, i.e. for  $G(s)$ 's with no transmission zeros in the right half plane [5]. For non-minimum phase models, certain integral constraints apply to the  $H^2$ -solution (and to solutions from all other design methodologies as well) whose impact is currently only partially understood.

##### 4.1. Properties for Minimum Phase Models

The compensator produced by (14) has the following well known form:

$$K_{LQG}(s) = K_c(sI - A - BK_c - K_f C)^{-1} K_f \quad (16)$$

where  $K_f$  is the Kalman filter gain corresponding to the parameter choices (14a), and  $K_c$  is the LQ-regulator gain corresponding to (14b). Note that  $K_c$  is functionally dependent on the parameter  $\rho$ . It is shown in [2] that this functional dependence produces the following limit for square minimum phase  $G(s)$ :

$$G(s)K_{LQG}(s) \rightarrow C\Phi(s)K_f \quad \text{pointwise in } s \\ \text{as } \rho \rightarrow 0 \quad (17)$$

Equation (17) shows that the optimal (output) loop transfer function matrix of a minimum phase  $H^2$ -problem (8) corresponds to the loop transfer function of a Kalman filter with its loop broken at the residual point, as illustrated in Figure 2. Moreover, (17) shows that the sequence of LQG solutions generated by (14) converges to this function ("recovers" this function) as the design parameter  $\rho$  becomes small.

##### 4.1.1. Two-Step LQG/LTR Design

The above properties suggest a two-step approach to  $H^2$ -optimal design:

**Step(1):** Design a Kalman filter, via (14a), with desirable sensitivity, complementary sensitivity, and loop transfer functions

**Step(2):** Design a sequence of LQ-regulators, via (14b), to approximate the functions in Step(1) to whatever accuracy is needed

Both of these steps are easy design tasks. The first is easy because Kalman filter sensitivity, complementary sensitivity and loop transfer functions are explicitly related to the chosen weights  $W(j\omega)$ , as described below, and the second is easy because it involves only repeated solutions of algebraic Riccati equations followed by inspection of  $S$  and  $T$ .

##### 4.1.2. Relations between Kalman Filter Functions and Weights

As a consequence of Kalman's dual equality for filters [9] and of equation (17), the transfer functions produced by standard Kalman filter designs can be shown to exhibit the following nice properties:

**Property(1):** Designer-specified shapes

For all frequencies where the weights,  $W(j\omega) = C\Phi(j\omega)L/\mu$ , are much larger than unity, Kalman filter sensitivity, complementary sensitivity, and loop transfer functions have the shapes

$$\sigma_i[(I + C\Phi(j\omega)K_f)^{-1}] \approx \frac{1}{\sigma_i[W(j\omega)]}$$

$$\begin{aligned}\sigma_i[C\Phi(j\omega)K_f(I+C\Phi(j\omega)K_f)^{-1}] &\approx 1 \\ \sigma_i[C\Phi(j\omega)K_f] &\approx \sigma_i[W(j\omega)]\end{aligned}\quad (18)$$

for all singular values  $\sigma_i$ .

**Property(2):** Known high frequency attenuation

As  $\omega \rightarrow \infty$ , Kalman filter shapes satisfy

$$\begin{aligned}\sigma_i[(I+C\Phi(j\omega)K_f)^{-1}] &\approx 1 \\ \sigma_i[C\Phi(j\omega)K_f(I+C\Phi(j\omega)K_f)^{-1}] &\approx \frac{\sigma_i[CK_f]}{\omega} \\ \sigma_i[C\Phi(j\omega)K_f] &\approx \frac{\sigma_i[CK_f]}{\omega}\end{aligned}\quad (19)$$

for all singular value  $\sigma_i$

**Property(3):** Well-behaved crossovers

As the loop crosses over from high-gain regions (18) to low-gain regions (19), Kalman filter sensitivities and complementary sensitivities never become too large. They satisfy

$$\begin{aligned}\sigma_i[(I+C\Phi(j\omega)K_f)^{-1}] &\leq 1 \\ \sigma_i[C\Phi(j\omega)K_f(I+C\Phi(j\omega)K_f)^{-1}] &\leq 2\end{aligned}\quad (20)$$

for all singular values  $\sigma_i$  and all frequencies  $\omega$

As described shortly, the first of these properties offers essentially arbitrary freedom to shape the sensitivity function in all high-gain regions of the Kalman filter loop. The second property shows that the loop must eventually transition from high gain to low gain regions and will do so at an attenuation rate proportional to  $1/\omega$ . The third property shows that these transitions will automatically be nice, in that the loop does not amplify disturbance or command errors during the transitions and retains stability in the face of modelling uncertainties as large as 50% (compare (5) and (20)). The latter stability robustness margin is not adequate, of course, for the entire frequency range. At high frequencies where uncertainties can greatly exceed 100%, we must rely on the attenuation rate provided by Property (2) for the needed margin. As a consequence of (20), this attenuation rate actually applies not only at very high frequencies but throughout the crossover region.

#### 4.1.3. A Formal Procedure to Shape Sensitivities

As evident from (18), in order to shape the high gain regions of a Kalman filter loop we merely need to find  $L$ , the noise input matrix in our design model, such that the  $\sigma_i[C\Phi L]$ 's have the desired shapes. This can be done directly by experimenting with various  $L$ 's, and/or more formally by augmenting additional dynamics to the design model such that the choice becomes trivial. For example, whenever the desired shapes correspond to a stable function,  $W_d(s)$ , it suffices to add this function to the output of  $G(s)$  in the manner shown in Figure 3. In this figure, the plant's unstable dynamics have been collected into an all-pass factor,  $B_p(s)$ , such that

$$G(s) = B_p(s)^{-1} G_{ms}(s) \quad (21)$$

where  $G_{ms}(s)$  is minimum phase and stable, and

$$B_p(j\omega)B_p(j\omega)^H = I \quad \text{for all } \omega \quad (22)$$

Signals from  $W_d(s)$  are passed through the all-pass as shown in the figure. The overall state-space representation then remains stabilizable from  $\xi(t)$  and  $u(t)$  and detectable from  $y(t)$  and  $z(t)$  (LQG solutions exist) and has a noise input matrix  $L$  which satisfies

$$C\Phi(s)L = B_p(s)^{-1}W_d(s) \quad (23)$$

Hence, from (22)

$$\sigma_i[C\Phi(j\omega)L] = \sigma_i[W_d(j\omega)] \quad (24)$$

for all  $\omega$  and all singular values, as desired.

Note that this formal augmentation procedure\* is particularly simple when  $G(s)$  is stable, with  $B_p(s) = I$ . In addition, if the desired function  $W_d(s)$  is itself a legitimate Kalman filter loop transfer function (i.e. satisfies a dual Kalman inequality [10]), then the filter design step may be skipped entirely by letting  $K_f = L$ . Despite this simplicity, however, we caution that the procedure should not be applied indiscriminately. It produces compensators which cancel all stable dynamics of  $G(s)$  and replaces them with dynamics of  $W(s)$ . Such cancellations should be minimized by constructing the desired weights as much as possible out of dynamics of  $G(s)$ , (i.e. by choosing noise inputs which directly drive the original state-space of  $G$ ).

All of the above properties have duals based on the free parameter choices in (15). For this case, again with square minimum phase  $G(s)$ , it has been shown that [2]

$$K_{LQG}(s)G(s) \rightarrow K_c\Phi(s)B \quad \text{pointwise in } s \\ \text{as } \mu \rightarrow 0 \quad (25)$$

where  $K_c$  is the gain matrix of an LQ-regulator designed with parameters from (15a). Hence, the optimal (input) loop transfer function of an  $H^2$ -problem corresponds to the loop transfer of an LQ-regulator broken at the control input point. This function can be recovered by a sequence of Kalman filter designs with parameters from (15b). A two-step design process now starts with a single LQ-regulator which achieves desirable sensitivity, complementary sensitivity, and loop transfer functions, and is followed by a sequence of filter designs to approximate these functions to whatever needed accuracy. Nice properties dual to (18)-(24) apply which make this process easy.

## 4.2. Properties for Non-minimum Phase Models

When the model  $G(s)$  has right half plane transmission zeros, the asymptotic behavior of equations (17) or (25) does not hold, and the two-step LQG/LTR design process fails to produce the desired functions. Nevertheless, the process still offers useful options which can be exploited in design.

### 4.2.1. Option 1: Avoiding the issue

The first thing to note is that the minimum phase requirement applies to the model of a plant, not to the plant itself. A common design trick, therefore, is to approximate non-minimum phase plants with minimum phase design models. This is perfectly safe provided that the resulting (deliberate) modelling

\* Formal augmentation of desired loop shapes was first proposed for stable plants in [11]. Extensions to the unstable case similar to Figure 3 are developed in [12]. The needed all-pass pole factors can be constructed by solving an additional algebraic Riccati equation, as illustrated for completeness in Figure 4.

errors are incorporated into the uncertainty characterization of equation (3). One way to do the approximation, for example, is to collect all unstable zeros into an all-pass factor analogous to (21), i.e.

$$G(s) = B_z(s) G_m(s) \quad (26)$$

where  $G_m(s)$  is minimum phase and

$$B_z(j\omega) B_z(j\omega)^H = I \quad \text{for all } \omega \quad (27)$$

The plant  $G(s)$  is then approximated by the model  $G_m(s)$ . For a single zero at  $s = +z$ , this produces a normalized multiplicative error which satisfies

$$\bar{\sigma}[L(j\omega)] = \text{abs} \left[ \frac{j2\omega}{j\omega+z} \right] \quad (28)$$

Note that this error is small for all  $\omega \ll z$ . Indeed, whenever (28) is small compared to the existing model uncertainty,  $m(\omega)$  in (3), it can be added to  $m(\omega)$  with minor effects, and the LQG/LTR design process can proceed in normal minimum phase fashion.

#### 4.2.2. Option 2: Living with the issue

It is evident from (28), of course, that sufficiently small values of  $z$  can produce approximation errors which dominate other model uncertainties. In that case, our design trick, while still safe, becomes excessively conservative. It invokes design prescriptions, (P2) and equation (5), which ignore too much of the known phase and direction structure of the modelling error.

For plants in the latter category, the  $H^2$ -problem offers fewer known general properties. The sequences of compensators produced by (14) or (15) do not converge to nice functions such as (17) or (25). They do, however, converge to  $H^2$ -optimal functions, that is, to functions which achieve the best  $H^2$ -trade-off between  $S$  and  $T$ , subject to the inherent constraints imposed by non-minimum phase zeros. Non-minimum phase constraints have been interpreted only recently, and only for SISO problems, as integral relations over frequency applied to the function  $\log[\text{abs}S(j\omega)]$  [15]. The relations show that sensitivity improvements ( $S < 1$ ) achieved in one frequency range must be paid for with deteriorations ( $S > 1$ ) over another range, with the severity of deteriorations dependent upon right half plane zero locations. This conclusion also follows from recent results in  $H^\infty$ -optimization theory [16].

In our  $H^2$ -solutions, the frequency ranges in which improvements and deteriorations of sensitivity occur can be manipulated by the weights,  $W(s)$ . Like the non-minimum phase constraints themselves, however, the exact relationship between  $W(s)$  and  $S(s)$  is currently available only for SISO and certain limited MIMO cases. These cases are developed and illustrated by means of example in Section 5. It turns out that the shape of  $S(j\omega)$  can be directly assigned via  $W(j\omega)$ , but its magnitude level cannot (i.e.  $S(j\omega)$  is determined to within an unknown scale factor)\*\*. Analogous results for general MIMO problems remain a goal of current research.

\* All-pass zero factors can be found by solving another Riccati equation, as illustrated in Figure 5, or numerically preferred, by solving generalized eigenvalue problems [13,14].

\*\* Note that these limitations are precisely the same as the limitations on SISO  $H^\infty$ -designs [16].

### 4.2.3. Option 3: Giving up $H^2$ -optimality

A heuristic argument which explains why equations (17) or (25) cannot hold for non-minimum phase problems proceeds as follows: No stabilizing compensator  $K(s)$  can use unstable poles to cancel unstable zeros of the plant  $G(s)$ . Hence, the products  $GK$  or  $KG$  must retain the non-minimum phase structure of  $G$ . The Kalman filter and LQ-regulator loop transfers ( $C\Phi K_f$  or  $K_c\Phi B$ ), however, are known to be minimum-phase (else they could not exhibit infinite positive gain margin [17]). Thus, neither function can be recovered.

This reasoning naturally leads to the question "What class of functions can be recovered?". More precisely, if we modify the LQG/LTR process by replacing the Kalman filter design step (14a) with an arbitrary choice of filter gains, say  $K_f = F$ , but retain the sequence of LQ-regulators (14b), for what choices of  $F$  will a convergence result such as (17) apply? The answer to this question is not surprising. Any function which shares the right half plane pole/zero structure of  $G(s)$ , can be recovered by the modified process. Specifically:

Let  $G(s)$  be square. Choose filter gain  $F$  such that

$$C\Phi(s)F = B_z(s)B_p(s)^{-1}W_s(s) \quad (29a)$$

where  $W_s(s)$  is any stable, strictly proper, full rank function, and where  $B_z(s)$  and  $B_p(s)$  are all-pass factors of right half plane zeros and poles of  $G(s)$ , respectively, such that

$$G(s) = B_z(s)B_p(s)^{-1}G_{ms}(s) \quad (29b)$$

with  $G_{ms}(s)$  minimum phase and stable. Then the sequence of compensators generated by (14b)

$$K_F(s) = K_c(sI - A - BK_c - FC)^{-1}F \quad (30)$$

satisfies

$$\begin{aligned} G(s)K_F(s) &\rightarrow C\Phi(s)F \quad \textit{pointwise in } s \\ \textit{as } \rho &\rightarrow 0 \end{aligned} \quad (31)$$

This result follows from a minor variation of the original recovery derivation for (17). The modified derivation is included in Appendix A for completeness. A formal augmentation procedure which implements the result is shown in Figure 6. This procedure is analogous to Figure 3 for minimum phase plants. Hence, the same caveats apply. More importantly however, since  $F$  in (29) is no longer a Kalman filter gain, no LQG guarantees apply -- not  $H^2$ -optimality and not even stability. Separate test must be performed on the function  $C\Phi F$  in order to assure that it has desirable feedback properties.

## 5. LQG/LTR FOR NON-MINIMUM PHASE PROBLEMS

As discussed above, the relationships between weights and sensitivities in non-minimum phase  $H^2$ -problems is currently available only for SISO and certain MIMO cases. This section provides an informal description of the SISO relationships and illustrates them with a simple example.

### 5.1. SISO Relationships

The key feature which makes minimum and non-minimum phase problems different is that the latter impose certain global constraints on achievable sensitivity functions. These constraints are a consequence of the analyticity of  $s(j\omega)$  and have recently been expressed in the following Poisson Integral form [15]:

Let  $z_i, i=1, \dots, m$ , be right half plane zeros of the plant  $g(s)$ , distinct for simplicity, let  $b_p(s)$  denote the plant's all-pass pole factor (if any), and let  $\bar{s}(\omega)$  denote the magnitude of the sensitivity function of any stabilizing feedback loop for  $g(s)$ . Then\*

$$\int_0^{\infty} \log[\bar{s}(\omega)] \bar{v}_i^2(\omega) d\omega = \frac{\pi}{2} \log[b_p^{-1}(z_i)] \quad \text{for each } i \quad (32a)$$

where  $\bar{v}_i(\omega)$  are "constraint weighting functions" defined by

$$\bar{v}_i^2 = \frac{\text{Re } z_i}{(\omega - \text{Im } z_i)^2 + (\text{Re } z_i)^2} \quad (32b)$$

These constraint expressions can be used with Lagrange multipliers to turn our original constrained  $H^2$ -optimization problem in (8) (viewed as an optimization over  $\bar{s}$ , with  $\bar{s}$  constrained to the class of sensitivities produced by stabilizing compensators for  $g$ ) into an unconstrained problem, i.e.

$$J_u = \int_0^{\infty} \bar{s}^2 \bar{w}^2 d\omega + \sum_1^m \lambda_i \left[ \frac{\pi}{2} \log(b_p^{-1}(z_i)) - \int_0^{\infty} \log(\bar{s}) \bar{v}_i^2 d\omega \right] \quad (33)$$

where  $\bar{w}(\omega)$  is the magnitude of the usual weight on sensitivity, and the complementary sensitivity term in (8) has been dropped for simplicity. Since  $\bar{s}(\omega)$  in (33) is now unconstrained\*\*, simple differentiation at each frequency produces the following optimal  $\bar{s}$ :

$$\bar{s}(\omega) = \frac{\left[ \sum_1^m \lambda_i \bar{v}_i(\omega)^2 \right]^{1/2}}{\sqrt{2} \bar{w}(\omega)} \quad (34)$$

This last equation demonstrates that the optimal non-minimum phase sensitivity must be a linear combination of pre-determined constraint weighting functions divided by the designer's chosen sensitivity weighting function. Note however, that the Lagrange multiplier values  $\lambda_i$  remain unknown. Hence,  $\bar{s}(\omega)$  is not actually known until we solve the  $H^2$ -problem. Nevertheless, the equation lends substantial information about what sensitivity weights to choose. In particular, systems with a single right half plane zero will have sensitivities proportional to  $\frac{\bar{v}}{\bar{w}}$ . We can therefore compute  $\bar{w}(\omega)$  explicitly to produce an entire desired

\* For stable systems, the right hand side of (32a) is replaced by zero. For minimum phase systems, all weightings in (32a) reduce to unity and the right hand side becomes a sum of real parts of the unstable poles [15].

\*\* This simple argument depends upon an assumption that no constraints other than (32) apply to  $\bar{s}$ . Though not formally proven in [15] or here, this assumption is apparently true.

shape of  $\bar{s}(\omega)$ , with only a scale factor left unknown. An example which illustrates this possibility is given shortly.

More generally, if several right half plane constraints apply, it is possible to use equation (34) to parameterize the weights in terms of the Lagrange multiplier values, and then to search over these Lagrange parameters until the desired shape of  $\bar{s}(\omega)$  is achieved. This process is also illustrated below.

### 5.2. An Example

Consider the following simple SISO system with a non-minimum phase zero at  $z = 1 + j0$  and an unstable pole at  $p = 0.2 + j0$ :

$$g(s) = b_p(s)^{-1}g_s(s)$$

with

$$b_p(s) = \frac{s-0.2}{s+0.2} \quad g_s(s) = \frac{0.2(-s+1)}{(s+1)(s+0.2)} \quad (35)$$

According to (33), sensitivities for this plant must satisfy a single integral constraint with weighting given by

$$\bar{v}_1(\omega)^2 = \frac{1}{\omega^2+1} \quad (36)$$

Hence,  $H^2$ -optimal sensitivities will be related to the chosen sensitivity weights by

$$\bar{s}(\omega) = \frac{1}{\bar{w}(\omega)} \frac{1}{abs(j\omega+1)} \quad (37)$$

This last formula was used to define several candidate sensitivity weightings for the example. Each candidate was augmented to the plant model according to the procedure of Figure 3, and the two-step LQG/LTR calculations were carried out. Some resulting recovered sensitivity functions are shown in Figure 7.

In Figure 7a, for example, the weight  $\bar{w}(\omega) = \bar{v}_1(\omega)$  is used to produce a flat sensitivity function\*. The unknown constant level of this function turns out to be  $\bar{s} = 1.5$  which is precisely the minimum  $H^\infty$ -norm achievable via  $H^\infty$ -optimization [18]. Note that this example verifies that  $H^2$ -weights exist to solve  $H^\infty$ -problems [7]. Moreover, equation (37) provides a constructive way to find such weights for the case at hand.

Figure 7b shows an alternate choice of  $\bar{w}(\omega)$  calculated according to (37) to produce a ten-fold improvement in low frequency sensitivity compared to high frequency sensitivity. This improvement is indeed achieved, but at the expense of deterioration from the minimum achievable flat sensitivity. Figure 7c shows a choice in which the improvement is sought at high frequencies instead of low, and Figure 7d shows a choice where improvements are sought at mid-frequencies.

Finally, to illustrate the general case with more than one non-minimum phase constraint, consider a modified version of our plant with one additional right half plane zero, i.e.

$$g_s(s) = \frac{0.2(-s+1)(-s+5)}{(s+0.2)(s+1)(s+5)} \quad (38)$$

\* We have also retained a small weight ( $\mu = .001$ ) on complementary sensitivity in order to keep compensators realizable. Hence, all  $\bar{s}(\omega)$ 's in Figure 7 return eventually to unity.

This new plant must satisfy two non-minimum phase constraints, one with weighting function  $\bar{v}_1$  from (36) and the other with weighting

$$\bar{v}_2(\omega)^2 = \frac{.5}{\omega^2 + .25} \quad (39)$$

Now suppose that we still want to achieve a flat sensitivity function. According to (34), the sensitivity weighting required to do so must satisfy

$$\bar{w}(\omega) = \left[ \frac{\lambda_1}{\omega^2 + 1} + \frac{0.5\lambda_2}{\omega^2 + .25} \right]^{1/2} \quad (40)$$

This magnitude corresponds to

$$w(j\omega) = \frac{j\omega + z(\lambda_1, \lambda_2)}{(j\omega + 1)(j\omega + .5)} \quad (41)$$

where the zero location  $z(\lambda_1, \lambda_2)$  is indexed by the multipliers and remains unknown. A brute force search to determine its value is summarized in Figure 8. The desired flat sensitivity is attained for  $z \approx 2.55$  and has an  $H^\infty$ -level  $\bar{s} \approx 3.54$ . As expected, this is higher than our previous level due to the extra non-minimum phase constraint.

All of the above results confirm the basic non-minimum phase behavior identified in [15] and [16], namely that sensitivities cannot be shaped arbitrarily. They are constrained to satisfy certain integrals which translate into a minimum  $H^\infty$ -bound achievable by a flat sensitivity function. Improvements over this bound are possible in any finite range of frequencies but must be paid for with deteriorations in other ranges. We have seen that the ranges in which the improvements and deteriorations occur can be effectively manipulated with the weights  $W(s)$  in SISO  $H^2$ -problems. While no details are given here, this capability also generalizes to the class of MIMO problems with orthogonal non-minimum phase zero directions (including, for example, systems with a single real non-minimum phase zero or a single non-minimum phase complex conjugate pair). This follows because the all-pass zero factor  $B_z(s)$  for such problems can be diagonalized by orthogonal transformations, and the SISO results can then be applied separately in each direction. More general MIMO cases remain to be resolved.



## 6. CONCLUSION

This paper has presented a new perspective on the formal LQG/LTR design method for linear multivariable feedback systems. We have interpreted the design problem as an essential trade-off between sensitivity and complementary sensitivity functions in the frequency domain. This trade-off was posed as a weighted  $H^2$ -optimization problem and solved with a modified form of the standard LQG method.

Solutions of the  $H^2$ -problem have very desirable properties for minimum phase systems. For this case, a two-step design process is appropriate, beginning with a Kalman filter (or LQ-regulator) design to achieve desirable transfer functions, and followed by a sequence of LQ-regulator (or Kalman filter) designs to recover these functions to any needed accuracy. A formal weight augmentation procedure was described to define  $H^2$ -weightings which achieve essentially arbitrary shapes for sensitivity functions in the first step.

While this two-step design approach does not carry over directly to non-minimum phase problems, several options for dealing with such problems were discussed. It turns out that  $H^2$ -solutions for SISO and some MIMO cases can still be manipulated effectively through the choice of weights, and that a large class of loop transfer functions can still be recovered with the LTR approach.

These various features make LQG/LTR a very effective design tool for linear multivariable feedback systems. The major weakness of the method appears to be its restriction to design trade-offs at only one loop-breaking point. That is, the method can trade-off  $S(j\omega)$  against  $T(j\omega)$  with both defined at the output or both defined at the input. However, it cannot easily trade-off these functions when they are defined at different points.\* This means that the method currently obligates designers to reflect all feedback design requirements to one of the two loop-breaking points. While such reflections cause no difficulty in SISO problems, it is easy to construct MIMO examples where they are arbitrarily conservative. Only further research and applications experience can determine whether this will remain an important shortcoming.

### Acknowledgements

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\* This is possible within the LQG/LTR framework only for very special weights (e.g.  $W(s) = w(s)G(s)$  in equation (15)).

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## APPENDIX A

This appendix documents a modification of a derivation in [2] to show that the class of functions in Section 4.2.3 can be recovered in non-minimum phase cases. First note that the factored plant in equation (29) has the following (non-minimal) state space representation:

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ B_2 C_1 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u + \begin{bmatrix} F_1 \\ 0 \end{bmatrix} \xi \quad (\text{A1a})$$

$$y = \begin{bmatrix} D_2 C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad (\text{A1b})$$

where the first group of states represents the minimum phase (but possibly unstable) factor of the plant, and the second group represents the non-minimum phase all-pass zero factor,  $B_z$ . (For unstable plants, note that  $F$  must be selected to keep the system stabilizable from  $\xi(t)$ . Then  $C\Phi B$  and  $C\Phi F$  share a common pole factor as well as a common zero factor, as required in equation (29)).

It follows from this structure and from known asymptotic properties of LQ-regulator gains [18] that

$$\begin{aligned} \rho K_c(\rho) &\rightarrow U [C_1 \ 0] \equiv U C_m \\ \text{as } \rho &\rightarrow 0 \end{aligned} \quad (\text{A2})$$

where  $U$  is an orthonormal matrix and  $C_m$  is a minimum phase version of the plant's output matrix, such that

$$C\Phi(s)B = B_z(s) C_m \Phi(s)B \quad \text{and} \quad C\Phi(s)F = B_z(s) C_m \Phi(s)F \quad (\text{A3})$$

One application of the matrix inversion lemma to equation (30) plus the limiting behavior (A2) now shows that the compensator  $K_F(s)$  has the limit

$$K_F(s) \rightarrow (C_m \Xi(s)B)^{-1} C_m \Xi(s)F \quad (\text{A4})$$

where

$$\Xi(s) = (sI - A - FC)^{-1}$$

One more application of the inversion lemma applied to  $C_m \Xi$  together with (A3) yields

$$C_m \Xi(s) = [B_z(s)^{-1} (I - C\Phi(s)F)^{-1} B_z(s)] C_m \Phi(s) \quad (\text{A5})$$

Finally, substituting this into (A3) and (A4) gives the desired result:

$$\begin{aligned} G(s) K_F(s) &= B_z(s) C_m \Phi(s)B K_F(s) \\ &\rightarrow B_z(s) C_m \Phi(s)B (C_m \Phi(s)B)^{-1} C_m \Phi(s)F \\ &= B_z(s) C_m \Phi(s)F = C\Phi(s)F \end{aligned} \quad (\text{A6})$$

Figure 1: STANDARD FREQUENCY DOMAIN DESIGN PROBLEM

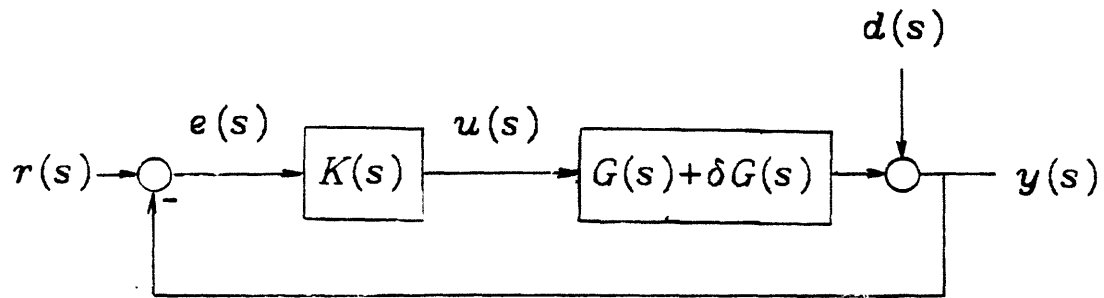
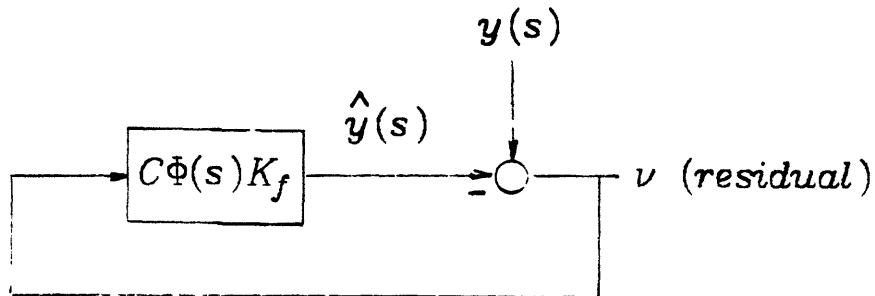
Figure 2: LIMITING SOLUTION OF  $H^2$ -PROBLEM (8)

Figure 3: FORMAL WEIGHT AUGMENTATION

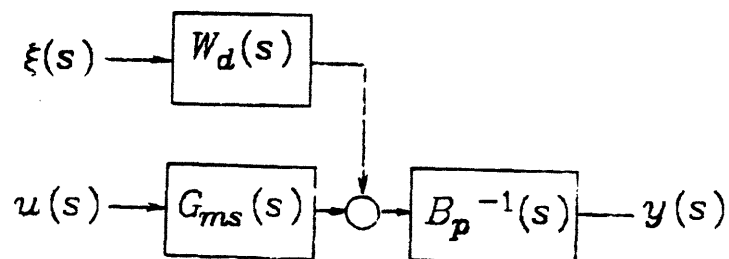


Figure 4: ALL-PASS POLE FACTORS

Let  $K_{fp}$  satisfy

$$0 = A\Sigma + \Sigma A^T + q^2 BB^T - \Sigma C^T C \Sigma$$

$$K_{fp} = -\Sigma C^T \quad \text{as } q \rightarrow 0$$

Then one possible all-pass pole factorization is given by:

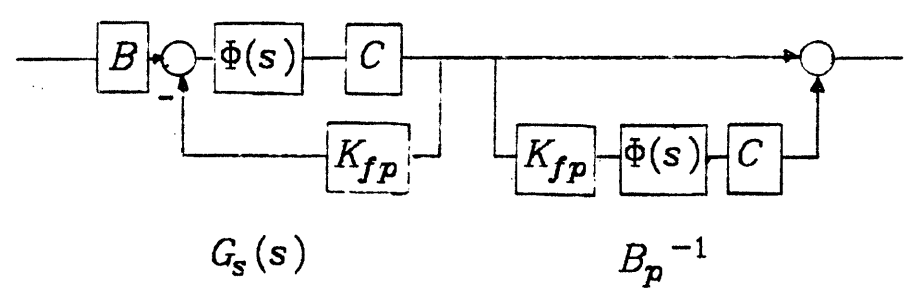


Figure 5: ALL-PASS ZERO FACTORS

Let  $K_{cz}$  satisfy

$$0 = PA + A^T P + q^2 C^T C - PBB^T P$$

$$K_{cz} = -B^T P \quad \text{as } q \rightarrow \infty$$

Then one possible all-pass zero factorization is given by:

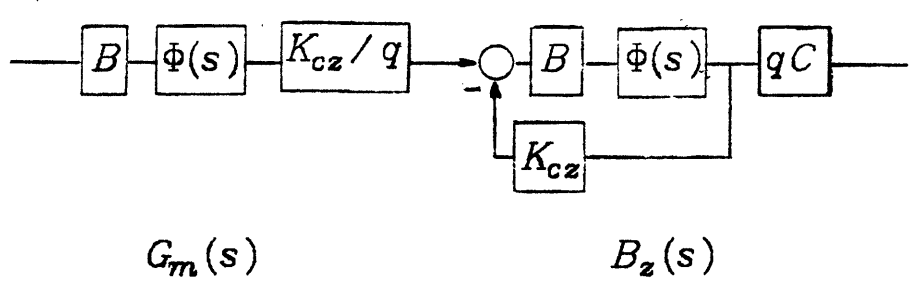


Figure 6: FORMAL WEIGHT AUGMENTATION FOR RECOVERABLE FUNCTIONS

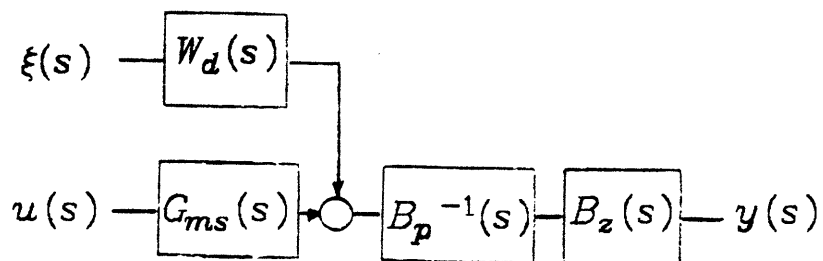


Figure 7: SENSITIVITIES AND ASSOCIATED WEIGHTS  
Part A

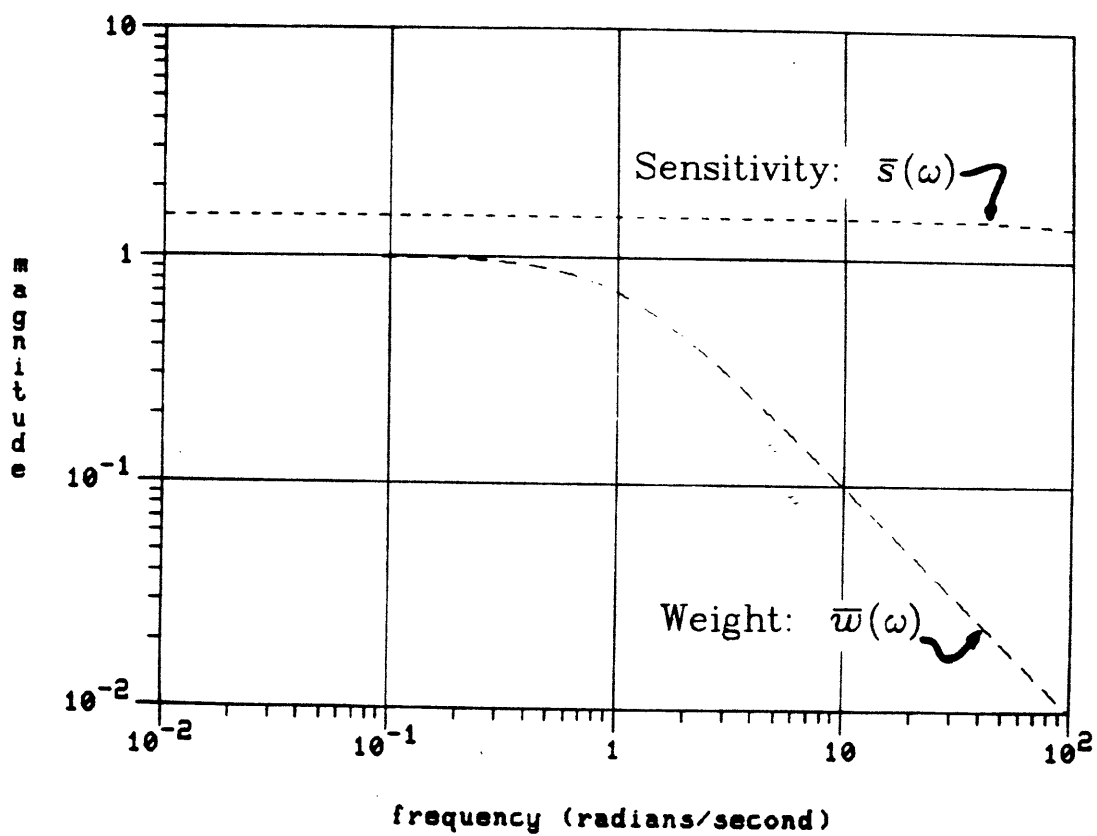


Figure 7: SENSITIVITIES AND ASSOCIATED WEIGHTS  
Part B

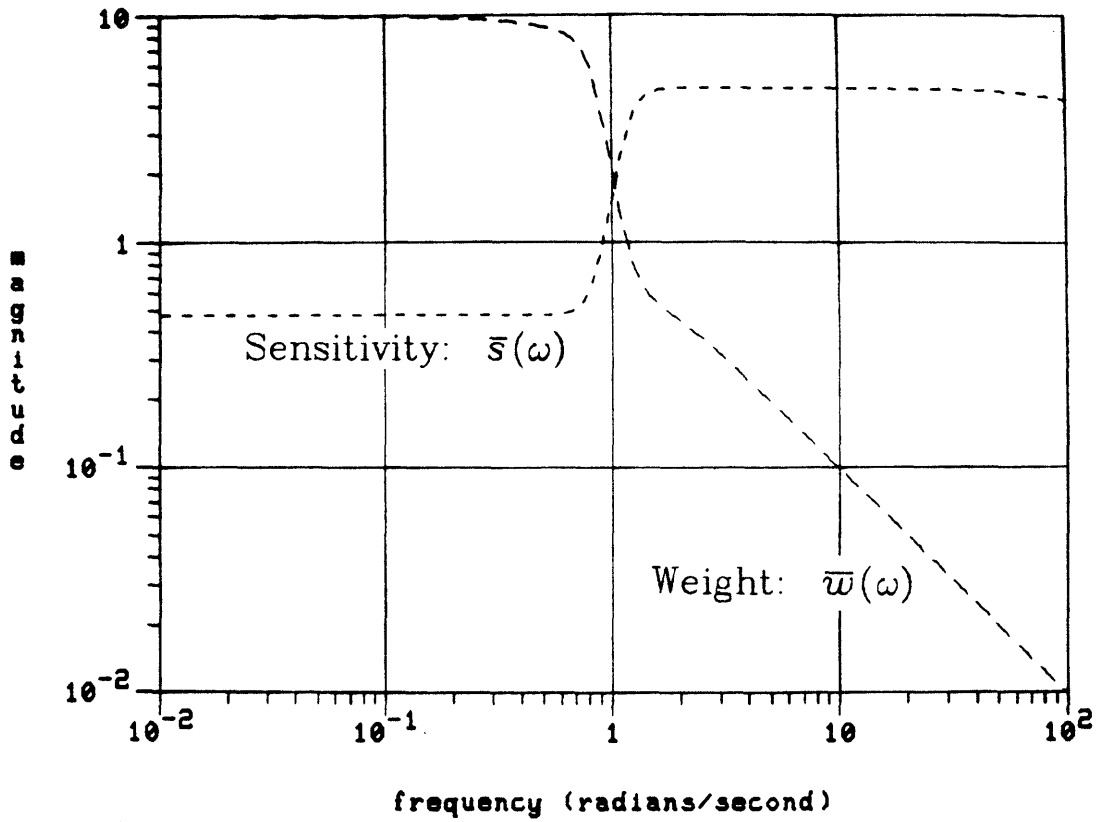


Figure 7: SENSITIVITIES AND ASSOCIATED WEIGHTS  
Part C

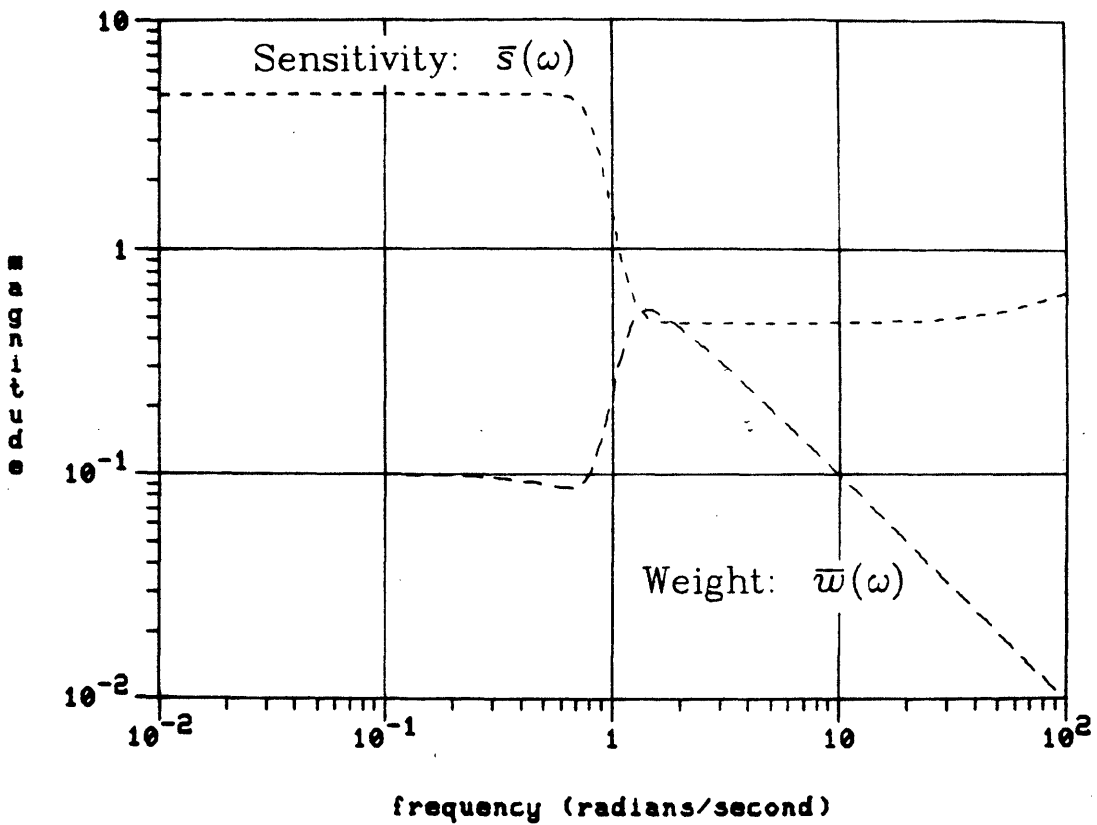


Figure 7: SENSITIVITIES AND ASSOCIATED WEIGHTS

Part D

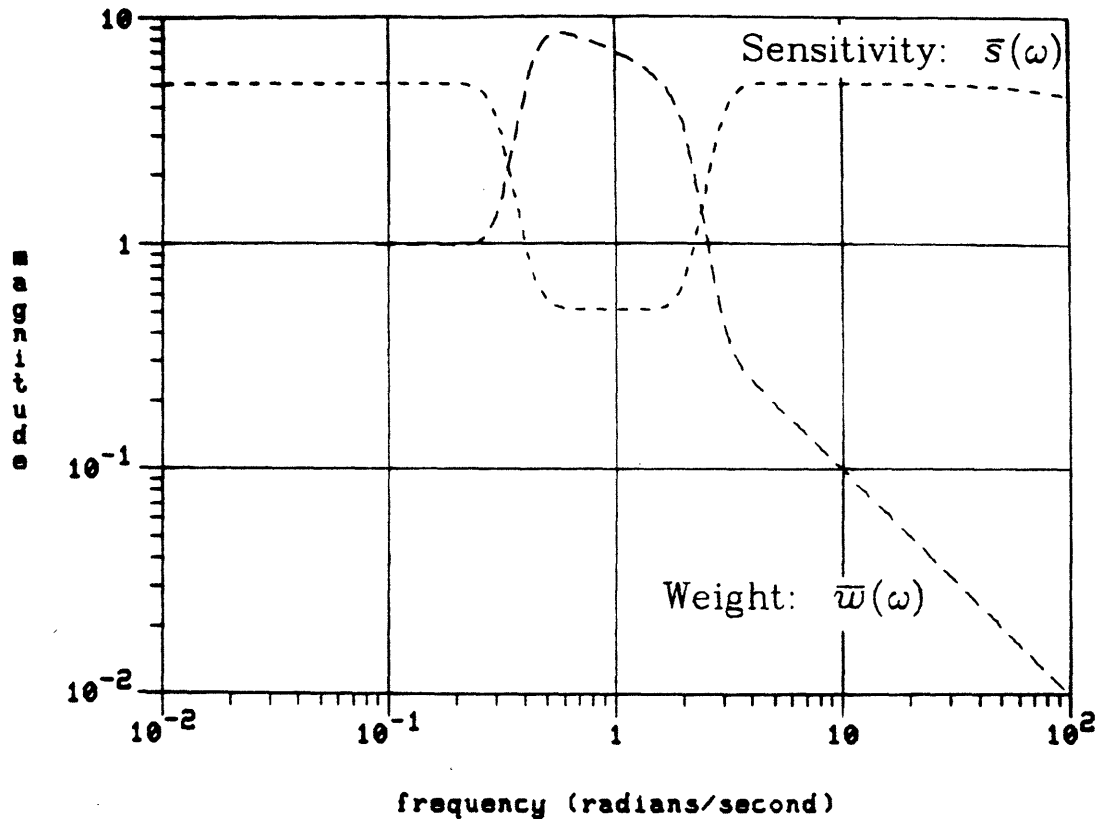


Figure 8: SEARCH OVER LAGRANGE PARAMETERS

