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DISCRETE-TIME PRIORITY QUEUES WITH PARTIAL INTERFERENCE*

by

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Abstract

A class of discrete time priority queueing systems with partial interference is considered. Packet-radio communication networks that use a certain mode of operation fall into this class. In these systems N nodes share a common channel to transmit their packets. One node uses random access scheme while the other nodes use the channel according to prescribed priorities. Packet arrivals are modeled as a discrete-time batch processes, and packets are forwarded through the network according to fixed prescribed probabilities.

Steady-state analysis of the class of systems under consideration is provided. In particular, we present a recursive method for the derivation of the joint generating function of the queue lengths distribution at the nodes in steady-state. The condition for steady-state is also derived. A simple example demonstrates the general analysis and provides some insights into the behavior of systems with partial interference.

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Introduction

The survey paper by Kobayashi and Konheim [1] discusses many models of discrete-time queueing systems. Such systems have been receiving increased attention in recent years, [2-4] due to their usefulness in modelling and analyzing various types of communication systems. Packet-switched communication networks with point-to-point links between the nodes and fixed length data packets motivated most of these models. The models in [2-4] are of tandem nature since in point-to-point networks the transport of a packet from its source to its destination involves the transmission of the packet over a succession of links. The fixed packet length assumption induces the discrete-time nature of the models.

In this paper we consider a class of discrete-time priority queueing systems with partial interference. Consideration of these systems have been primarily motivated by the class of packet-switched communication networks called the multi-access/broadcast networks, or packet-radio networks. In these communication networks all nodes share a common channel through which they transmit their packets and from which they extract packets destined to them, hence the multi-access nature of these networks. In addition, when a node transmits a packet through the shared channel, all nodes that are within its transmission range hear this transmission, hence the broadcast nature of the systems.

We assume that the channel time axis is slotted into intervals of size equal to the transmission time of a packet. All packets are assumed to be of fixed and equal size. The nodes are synchronized so that they may start transmission of a packet only at the beginning of a slot, hence the discrete-time nature of the system. All nodes are assumed to have infinite buffers.

One of the most crucial issues in multi-access networks is the protocol required to transmit packets on a shared channel in a distributed environment. For a survey of multi-access protocols the reader is referred to [5]. The design and analysis of multi-access protocols is not trivial. This is due to the folllowing two facts that hold for packet-radio networks: (i) If two or more nodes transmit during the same slot to the same node, then the overlap in transmission destroys all packets involved in the transmission; (ii) A transmiting node is unable to receive packets transmitted by other nodes of the system. These two facts together with the broadcast nature of the network give rise to statistical dependence between the queues of the nodes of the network. In most cases this dependence is rather complicated and there is little hope to obtain analytical results for general multi-access protocols and for general configuration of networks. The purpose of this paper is to analyze quite general network with a specific mode of operation.

One mode of operation that can be accomplished in multi-access networks is a conflict-free mode. This can be achieved if every node knows perfectly which are the nodes that have packets ready for transmission. This is possible in systems where a central scheduler schedules the transmission according to information it receives from the nodes, or in systems where the nodes exchange this information between themselves [6]. Generally, any order of transmission can be used, in particular, priority can be easily implemented. Yet, if there are some nodes that cannot exchange information with the scheduler or with other nodes, on which nodes have packets ready for transmission, then their transmissions cannot be accommodated in a conflict-free mode of operation and they must use some random access scheme [5].

The class of discrete-time queueing systems that we consider in this paper consists of systems having N-1 nodes that access the channel in a conflict-free mode according to fixed priorities that are preassigned to them. No two nodes have the same priority and a given node is allowed to use the channel in a given slot only if it has a packet ready for transmission and all nodes with higher priority have empty queues. In addition, there is an extra node in the system that cannot be accommodated in the conflict-free mode of operation and therefore is allowed to use the channel in any slot on a random basis. If the node uses the channel along with any other node then their packets are destroyed and must be retransmitted, hence the interfering feature of the systems under consideration.

To enchace the network structure of the problem we attach to each node a given probability distribution that indicates the probabilities that a packet transmitted by the node is forwarded to one of the other nodes or to the outside of the system.

Outside sources feed the nodes of the system with new packets. An important feature of this paper is that these sources are allowed to depend on each other. Thus we are able to characterize a rather general class of arrival processes.

Several discrete-time queueing systems that have been previously investigated [7-9] are related to the class of systems considered in this paper. In [7] a "loop system" in which nodes transmit packets only to the outside of the system, the arrival processes are independent and there is no interference, has been considered. In [8] only two-node systems have been analyzed and in [9] no interference was allowed.

The paper is organized as follows: In Section 2 we describe the model along with the assumptions and several definitions and notations that we use throughout the paper. In Section 3 we present the steady state analysis of the class of systems under consideration. In particular we develop a method for deriving the joint generating function of the queue lengths of the nodes and we give the ergodicity condition for the system. Moments of the queue lengths at the nodes can be derived from the generating function and average time delays can be obtained by using Little's law [10]. Finally, in Section 4 we give an example and several numerical results.

2. Model Description

We consider a discrete-time queueing system in which the time axis is divided into intervals of equal size referred to as slots. The slots correspond to the transmission time of a packet and all packets are assumed to be of the same fixed size. The system consists of N nodes and packets arrive randomly to the nodes from N sources that in general, may be correlated. Let $A_i(t)$, $i = 1,2,\ldots,N$, $t = 0,1,2,\ldots$ be the number of packets entering node i from its corresponding source during the time interval $(t,t+1)$. The input process $\{A_i(t)\}_{i=1}^{N}$ $t = 0,1,2,\ldots$ is assumed to be a sequence of independent and identically distributed random vectors with integer-valued elements. Let the corresponding probability distribution and generating function of the input processes be:

$$a(i_1,i_2,\ldots,i_N) = \text{Prob}\{\bigcup_{j=1}^{N} A_j(t) = i_j\} \quad i_j = 0,1,2,\ldots \quad 1 \le j \le N \tag{1a}$$

and

$$F(\underline{z}) = E\{\prod_{i=1}^{N} z_i^{A_i(t)}\} \tag{1b}$$

where we use the notation $\underline{z} = (z_1,z_2,\ldots,z_N)$.

All nodes share a common channel for transmission of their packets, and transmissions are started only at the beginning of a slot. No more than one packet may be transmitted in any given time slot. Using some conflict-free protocol the channel is made available to nodes $i = 1,2,\ldots,N-1$ according to a fixed priority. Namely, node i $(1 \le i \le N-1)$ transmits the packet at the head of its queue whenever the queues at nodes $1,2,\ldots,i-1$ are empty and the one at node i is nonempty. Node N is a special node that cannot participate in the conflict-free protocol and therefore apply a random access protocol. Namely, at the beginning of each slot for which the queue at node N is nonempty, a coin with probability of success p is tossed. In case of a success node N transmits the packet at the head of its queue; otherwise it remains silent. Whenever node N transmits while another node is also transmitting, then both transmissions are unsuccessful and the two nodes must retransmit the packets at the head of their queues according to the protocols described above.

In any case, when a node i ($1 \leq i \leq N$) transmits a packet successfully, then the packet joins node j ($1 \leq j \leq N$, $j \neq i$) with probability $\theta_i(j)$ or leaves the system with probability $\theta_i(0)$. We assume here $\theta_i(i) = 0$. All packets received by a node from an outside source or from other nodes, are buffered in a common outgoing queue and transmitted on a first-come first-served basis. It is assumed that packets indeed arrive at every node of the system, so that there is no node that is empty with probability 1 (in other words, empty nodes are ignored). Finally we assume that the buffers at the nodes have <u>infinite</u> length. A schematic figure of a node i in the system is depicted in Fig. 1.

3. Steady-State Analysis

To describe the evolution of the queue contents at the nodes, we need several definitions. Let $L_i(t)$ $1 \leq i \leq N$, $t = 0,1,2,\ldots$ be the number of packets at node i at time t and let $U_i(L_i(t))$ $(1 \leq i \leq N,\ t = 0,1,2,\ldots)$ be a binary-valued random variable that takes value 1 if $L_i(t) > 0$ and 0 otherwise. Let V be a binary-valued random variable that takes values 1 and 0 with probabilities p and $\bar{p} = 1-p$ respectively. Also let $D_i^j(t)$, $1 \leq i \leq N$, $0 \leq j \leq N$, $t = 0,1,2,\ldots$ be a binary-valued random variable that takes value 1 if a packet is successfully transmitted from node i to node j at time t, where j=0 corresponds to the case that the packet leaves the system.

Using these definitions it is easy to see that the system under consideration evolves for $t = 0,1,2,\ldots$ as follows:

For $1 \leq i \leq N$,

$$L_i(t+1) = L_i(t) + A_i(t) + \sum_{m=1}^{N} D_m^i(t) - V_i(t)U_i(L_i(t)) \prod_{m=1}^{i-1} [1-U_m(L_m(t))] \tag{2a}$$

where

$$V_i(t) = \begin{cases} 1 - V\, U_N(L_N(t)) & 1 \leq i \leq N-1 \\ V & i = N \end{cases} \tag{2b}$$

Notice that $V_i(t)$ is a binary valued random variable and for $1 \leq i \leq N-1$ it can be interpreted as the interference factor at time t, i.e. it indicates whether or not node N interfers with the transmission of node i. Clearly, $\{L_i(t)\}_{i=1}^{N}$ is a vector Markov chain. Assume that this Markov chain is ergodic (we shall derive the condition for this later), let us consider the steady-state joint generating function of the queue lengths distribution,

$$G(\underline{z}) = \lim_{t\to\infty} E\left\{\prod_{i=1}^{N} z_i^{L_i(t)}\right\} \tag{3}$$

For notational convenience, let us define the following boundary generating functions:

$$G_o(\underline{z}) = G(\underline{z}) \tag{4a}$$

$$G_i(\underline{z}) = G(\underline{z})\Big|_{z_1=z_2=\ldots=z_i=0} \qquad 1 \le i \le N \tag{4b}$$

$$\hat{G}_i(\underline{z}) = G_i(\underline{z})\Big|_{z_N=0} \qquad 0 \le i \le N-1 \tag{4c}$$

Notice that by our definition $G_N(\underline{z}) = \hat{G}_{N-1}(\underline{z})$ is a constant representing the steady-state probability that the system will be empty. Let us also define the following polynoms:

$$Q_i(\underline{z}) = \theta_i(0) + \sum_{m=1}^{N} \theta_i(m) z_m \qquad 1 \le i \le N \tag{5}$$

Theorem 1

With the above notations the following holds:

$$\begin{aligned} G(\underline{z}) = F(\underline{z})\{G_N(\underline{z}) &+ [G_{N-1}(\underline{z}) - G_N(\underline{z})][\bar{p} + p z_N^{-1} Q_N(\underline{z})] + \\ &+ \sum_{i=1}^{N-1} [\hat{G}_{i-1}(\underline{z}) - \hat{G}_i(\underline{z})] z_i^{-1} Q_i(\underline{z}) + \\ &+ \sum_{i=1}^{N-1} [G_{i-1}(\underline{z}) - G_i(\underline{z}) - \hat{G}_{i-1}(\underline{z}) + \hat{G}_i(\underline{z})][p + \bar{p} z_i^{-1} Q_i(\underline{z})] \end{aligned} \tag{6}$$

The formal proof of Theorem 1 appears in the Appendix. Let us give here an intuitive explanation for (6). The right-hand side of (6) is a multiplication of the generating function of the joint arrival process, that by our assumptions is independent of the state of the system, and an expression that indicates, for the various states that the system may be at, which node is transmitting and how packets are moved within the network. Specifically, $G_N(\underline{z})$ corresponds to the case that the queues at all nodes are empty. $G_{N-1}(\underline{z})-G_N(\underline{z})$ corresponds to the situation that all nodes except node N are empty, therefore with probability p a packet leaves node N and joins another node or leaves the system according to the probabilities $\theta_N(j)$ $0 \leq j \leq N$. $\hat{G}_{i-1}(\underline{z})-\hat{G}_i(\underline{z})$ for $1 \leq i \leq N-1$ corresponds to the situation that node N is empty as well as nodes 1,2,...,i-1 and node i has a packet for transmission. Then, a packet leaves node i and joins another node or leaves the system according to the probabilities $\theta_i(j)$ $0 \leq j \leq N$. Finally, the term $G_{i-1}(\underline{z})-G_i(\underline{z})-\hat{G}_{i-1}(\underline{z}) + \hat{G}_i(\underline{z})$ for $1 \leq i \leq N-1$ corresponds to the case that nodes 1, 2,..., i-1 are empty and nodes i and N have both packets for transmission. In this case with probability p the two nodes interfer and no packet is moved, while if node N remains silent (this happens with probability $\bar{p}$ = 1-p) then a packet leaves node i and joins another node or leaves the system as before. Rearranging (6) we obtain:

$$G(\underline{z}) = F(\underline{z}) \frac{\sum_{i=1}^{N} H_i(\underline{z})G_i(\underline{z}) + \sum_{i=0}^{N-1} \hat{H}_i(\underline{z})\hat{G}_i(\underline{z})}{1 - F(\underline{z})[p + \bar{p}z_1^{-1}Q_i(\underline{z})]} \qquad (7a)$$

where

$$H_i(\underline{z}) = \begin{cases} \bar{p}[z_{i+1}^{-1}\, Q_{i+1}(\underline{z}) - z_i^{-1}\, Q_i(\underline{z})] & 1 \leq i \leq N-2 \\ 1 - 2p + pz_N^{-1}Q_N(\underline{z}) - \bar{p}\, z_{N-1}^{-1}\, Q_{N-1}(\underline{z}) & i = N-1 \\ p[1 - z_N^{-1}Q_N(\underline{z})] & i = N \end{cases} \qquad (7b)$$

and

$$\hat{H}_i(\underline{z}) = \begin{cases} p[1-z_1^{-1}Q_1(\underline{z})] & i = 0 \\ p[z_{i+1}^{-1}Q_{i+1}(\underline{z}) - z_i^{-1}Q_i(\underline{z})] & i \le i \le N-2 \\ p[1 - z_{N-1}^{-1}Q_{N-1}(\underline{z})] & i = N-1 \end{cases} \quad (7c)$$

In (7) we encounter a common phenomena in interfering queues, namely that the generating function $G(\underline{z})$ is expressed in terms of several boundary functions. In order to uniquely determine $G(\underline{z})$ in our system we still have to determine 2N-1 boundary functions,[1] $G_i(z)$ $1 \le i \le N$ and $\hat{G}_i(z)$ $0 \le i \le N-2$. In what follows, we develop the method for obtaining these boundary functions. The basic idea is to first express $\hat{G}_i(\underline{z})$ $i = 0,1,\ldots,N-2$ (in this order) in terms of $\hat{G}_j(\underline{z})$ $i+1 \le j \le N-1$. Then $G_i(\underline{z})$ $i = 1,2,\ldots,N-1$ is expressed in terms of $\hat{G}_j(\underline{z})$ $0 \le j \le N-1$ and $G_j(\underline{z})$ $i+1 \le j \le N$. Finally the constant $G_N(\underline{z})$ is determined from the normalization condition and using backward substitutions all the boundary functions are determined. Along the above process we mainly use the fact that the generating function $G(\underline{z})$ is an analytic function in the poly disc $|z_i| < 1$ $1 \le i \le N$.

In order to proceed we shall need the following Lemma:

Lemma 1: Let $F(\underline{z})$ be the generating function of the joint arrival process (1b), $Q_1(\underline{z})$ the function defined in (5) and $0 \le p \le 1$. Then for given

[1]Notice that in a general system where each node can interfer with any other node we might have up to 2^N-1 boundary functions to determine. An example for such a system is a network that all nodes use random access policy.

$|z_i| < 1 \quad 2 \leq i \leq N$, the following equation in z_1,

$$F(\underline{z})[pz_1 + (1-p)Q_1(\underline{z})] = z_1 \tag{8}$$

has a unique solution $z_1 = z_1(z_2,z_3,\ldots,z_N)$ in the unit circle $|z_1| < 1$.

Proof: Here we let $|z_1| = 1$ and $|z_i| < 1$, $2 \leq i \leq N$. We distinguish between two cases: The first is the case that packets do arrive to some node ℓ, $2 \leq \ell \leq N$, from its corresponding source. The second is the case that no packets arrive to nodes $2 \leq \ell \leq n$ from their corresponding sources. Our assumption that packets indeed arrive to all nodes implies that in the latter case, packets do arrive at node 1, and it routes some of them to at least one of the nodes ℓ, $2 \leq \ell \leq N$.

Case 1. There exists some node ℓ $(2 \leq \ell \leq N)$ for which the probability that a packet will arrive to it from its corresponding source is strictly positive, i.e., there exists $a(i_1,i_2,\ldots,i_N) > 0$ for some i_1 and some $i_\ell > 0$ $(2 \leq \ell \leq N)$. Therefore,

$$\begin{aligned}
|F(\underline{z})[pz_1 + (1-p)Q_1(\underline{z})]| &\leq |F(\underline{z})| = \\
&= \left| \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} \cdots \sum_{i_N=0}^{\infty} a(i_1,i_2,\ldots,i_N) \prod_{j=1}^{N} z_j^{i_j} \right| \\
&\leq \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} \cdots \sum_{i_N=0}^{\infty} a(1_1,i_2,\ldots,i_N)\, |z_\ell^{i}| \\
&< \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} \cdots \sum_{i_N=0}^{\infty} a(i_1,i_2,\ldots,i_N) = 1 = |z_1|.
\end{aligned} \tag{9}$$

Hence, applying Rouches' theorem [11] the claim is proved in this case.

Case 2. Packets arrive at node 1 and it routes some of them to at least one of the nodes ℓ $(2 \leq \ell \leq N)$, i.e. there exist $\theta_1(\ell) > 0$ for some $2 \leq \ell \leq N$. Therefore,

$$F(\underline{z})[pz_1 + (1-p)Q_1(\underline{z})] \leq p + (1-p)Q_1(\underline{z}) =$$

$$= p + (1-p)\ \theta_1(0) + \sum_{i=2}^{N} \theta_1(i) z_i| < p + (1-p) = 1 = |Z_1| \quad (10)$$

Hence, applying Rouches' theorem the proof is completed.

Let $\sigma_1(z_2,z_3,\ldots,z_N)$ (for simplicity σ_1) denote the unique solution of (8). Let $\underline{z}^{(1)}$ denote the vector $\underline{z}$ with its first component z_1 replaced by σ_1. Using a similar proof as for Lemma 1 we can show that for $|z_i| < 1$, $3 \leq i \leq N$, the following equation in z_2,

$$F(\underline{z}^{(1)})[p\ z_2 + (1-p)Q_2(\underline{z}^{(1)})] = z_2 \quad (11)$$

has a unique solution in the unit circle $|z_2| < 1$. Let $\sigma_2(z_3,z_4,\ldots,z_N)$ denote this solution and $\underline{z}^{(2)}$ denote the vector $\underline{z}$ with its first component z_1 replaced by $\sigma_1(\sigma_2(z_3,z_4,\ldots,z_N), z_3,\ldots,z_N)$ and its second component z_2 replaced by $\sigma_2(z_3,z_4,\ldots,z_N)$. Continuing this procedure we can have the following Lemma that recursively determines the unique functions $\sigma_i(z_{i+1}, z_{i+2},\ldots,z_N)$ $2 \leq i \leq N-1$ as follows:

Lemma 2: With the notations above and for $2 \leq i \leq N-1$, the following equation in z_i,

$$F(\underline{z}^{(i-1)})[p\ z_i + (1-p)Q_i(\underline{z}^{(i-1)})] = z_i \quad (12)$$

has a unique solution in the unit circle $|z_i| < 1$ for $|z_j| < 1$, $i+1 \leq j \leq N$

and $\underline{z}^{(i-1)}$ denotes the vector $\underline{z}$ with the variables z_j replaced by σ_j for $1 \leq j \leq i-1$. This unique solution is denoted by $\sigma_i(z_{i+1}, z_{i+2}, \ldots, z_N)$.

If we let p=0 and $z_N=0$ in Lemma 1 and 2 and we use the recursions defined by (8) and (12) for this case, then the unique functions $\hat{\sigma}_i(z_{i-1}, z_{i+2}, \ldots, z_{N-1})$ $1 \leq i \leq N-2$ are obtained, i.e. $\hat{\sigma}_1$ is the unique solution in the unit circle $|z_1| < 1$ of the equation $F(\hat{\underline{z}})Q_1(\hat{\underline{z}}) = z_1$ where $\hat{\underline{z}} = (z_1, z_2, \ldots, z_{N-1}, 0)$ and $|z_i| < 1$ $2 \leq i \leq N-1$ and $\hat{\sigma}_i$ $2 \leq i \leq N-2$ is the unique solution in the unit circle $|z_i| < 1$ of the equation $F(\hat{\underline{z}}^{(i-1)})Q(\hat{\underline{z}}^{(i-1)}) = z_i$ where $\hat{\underline{z}}^{(i-1)}$ is the vector $\hat{\underline{z}}$ with $z_1 = \sigma_1$, $z_2 = \sigma_2, \ldots, z_{i-1} = \sigma_{i-1}$ and $|z_j| < 1$ $i+1 \leq j \leq N-2$. We are now armed enough to attack the problem of determination of the 2N-1 boundary functions.

Determination of the boundary functions $\hat{G}_i(z)$ $0 \leq i \leq N-2$

Letting $z_N \to 0$ in (6) we obtain:

$$\hat{G}_o(\underline{z}) = F(\hat{\underline{z}})\{\hat{G}_{N-1}(\underline{z}) + pQ_N(\hat{\underline{z}})G'_{N-1}(\underline{z}) + \sum_{i=1}^{N-1} [\hat{G}_{i-1}(\underline{z}) - \hat{G}_i(\underline{z})]z_i^{-1}Q_i(\hat{\underline{z}})\} \tag{13a}$$

where $\hat{z} = (z_1, z_2, \ldots, z_{N-1}, 0)$

and $$G'_{N-1}(\underline{z}) = \left.\frac{dG_{N-1}(\underline{z})}{d\, z_N}\right|_{z_N=0} \tag{13b}$$

Notice that $G'_{N-1}(\underline{z})$ is a constant.

Rearranging (13a) and noticing that by definition $\hat{G}_{N-1}(\underline{z}) = G_N(z)$ we obtain:

$$\hat{G}_0(\underline{z}) = F(\hat{\underline{z}}) \frac{E(\hat{\underline{z}}) + \sum_{i=1}^{N-2} [z_{i+1}^{-1} Q_{i+1}(\hat{\underline{z}}) - z_i^{-1} Q_i(\hat{\underline{z}})] \hat{G}_i(\underline{z})}{1 - F(\hat{\underline{z}}) z_1^{-1} Q_1(\hat{\underline{z}})} \tag{14a}$$

where,

$$E(\hat{\underline{z}}) = [1 - Z_{N-1}^{-1} Q_{N-1}(\hat{\underline{z}})] \hat{G}_{N-1}(\underline{z}) + p Q_N(\hat{\underline{z}}) G'_{N-1}(\underline{z}) \tag{14b}$$

Notice that in (14) the boundary function $\hat{G}_0(\underline{z})$ is expressed in terms of the boundary functions $\hat{G}_i(\underline{z})$ $1 \leq i \leq N-1$ and the constant $G'_{N-1}(\underline{z})$. Now using the analytic property of $G_0(\underline{z})$ we immediately obtain the following result:

Theorem 2: Let $\hat{\sigma}_1$ and $\hat{\underline{z}}^{(1)}$ be as defined before. Then,

$$\hat{G}_1(\underline{z}) = F(\hat{\underline{z}}^{(1)}) \frac{E(\hat{\underline{z}}^{(1)}) + \sum_{i=2}^{N-2} [z_{i+1}^{-1} Q_{i+1}(\hat{\underline{z}}^{(1)}) - z_i^{-1} Q_i(\hat{\underline{z}}^{(1)})] \hat{G}_i(\underline{z})}{1 - F(\hat{\underline{z}}^{(1)}) z_2^{-1} Q_2(\hat{\underline{z}}^{(1)})} \tag{15}$$

This is true since $\hat{G}_0(\underline{z})$ is an analytic function in the polydisk $|z_i| < 1$ $1 \leq i \leq N-1$. Then in this polydisk whenever the denominator of $\hat{G}_0(\underline{z})$ vanishes, the numerator must also vanish. Since the demoninator of $\hat{G}_0(\underline{z})$ vanishes at $\hat{\sigma}_1$, we have from (14) that:

$$E(\hat{\underline{z}}^{(1)}) + \sum_{i=2}^{N-2} [z_{i+1}^{-1} Q_{i+1}(\hat{\underline{z}}^{(1)}) - z_i Q_i(\hat{\underline{z}}^{(1)})] \hat{G}_i(\underline{z}) =$$

$$= [\hat{\sigma}_1^{-1} Q_1(\hat{\underline{z}}^{(1)}) - z_2^{-1} Q_2(\hat{\underline{z}}^{(1)})] \hat{G}_1(\underline{z}) \tag{16}$$

which together with the fact that $F(\hat{\underline{z}}^{(1)}) \sigma_1^{-1} Q_1(z^{(1)}) = 1$ and imply (15).

Now, exploiting the similarity between (14) and (15) and repeating the above procedure for $i = 2,3,\ldots,N-2$ we obtain the following result:

Theorem 3: Let $\hat{\sigma}_i$ and $\hat{\underline{z}}^{(i)}$ $2 \leq i \leq N-2$ be as defined before. Then for $2 \leq i \leq N-2$,

$$\hat{G}_i(\underline{z}) = F(\hat{\underline{z}}^{(i)}) \frac{E(\hat{\underline{z}}^{(i)}) + \sum_{j=i+1}^{N-2} [z_{j+1}^{-1} Q_{j+1}(\hat{\underline{z}}^{(i)}) - z_j^{-1} Q_j(\hat{\underline{z}}^{(i)})]\hat{G}_j(\underline{z})}{1 - F(\hat{\underline{z}}^{(i)}) z_{i+1}^{-1} Q_{i+1}(\hat{\underline{z}}^{(i)})} \tag{17}$$

The proof of (17) is similar to that of (15).

Now, using (17) for i = N-2 we have,

$$\hat{G}_{N-2}(\hat{\underline{z}}) = F(\hat{\underline{z}}^{(N-2)}) \frac{[1 - z_{N-1}^{-1} Q_{N-1}(\hat{\underline{z}}^{(N-2)})]\hat{G}_{N-1}(\underline{z}) + pQ_N(\hat{\underline{z}}^{(N-2)}) G'_{N-1}(\underline{z})}{1 - F(\hat{\underline{z}}^{(N-2)}) z_{N-1}^{-1} Q_{i+1}(\hat{\underline{z}}^{(N-2)})} \tag{18}$$

and since $\hat{G}_{N-2}(\underline{z})$ is an analytic function for $|z_{N-1}| < 1$ we obtain from (19) that

$$pG'_{N-1}(\underline{z}) = \hat{G}_{N-1}(\underline{z}) \frac{\hat{\sigma}_{N-1}^{-1} Q_{N-1}(\hat{\underline{z}}^{(N-1)}) - 1}{Q_N(\hat{\underline{z}}^{(N-1)})} \tag{19}$$

Now, substituting (19) in (18) we get $\hat{G}_{N-2}(\underline{z})$ expressed in terms of the constant $\hat{G}_{N-1}(\underline{z})$. Using (17) for i = N-3, N-4,...2, and then (15) and (14) we obtain all the functions $\hat{G}_i(\underline{z})$ $0 \leq i \leq N-2$ expressed in terms of the constant $\hat{G}_{N-1}(\underline{z}) = G_N(\underline{z})$. Specifically, as we shall need it later let us define the function $k(\hat{\underline{z}})$ as follows:

$$k(\hat{\underline{z}}) = \hat{G}_0(\underline{z})/G_N(\underline{z}) \tag{20}$$

Determination of the boundary functions $G_i(\underline{z})$ $1 \leq i \leq N-2$

To obtain the boundary functions $G_i(\underline{z})$ $1 \leq i \leq N-2$ we use a similar procedure as for $\hat{G}_i(\underline{z})$ $0 \leq i \leq N-2$. Let us first rewrite (7a) as follows:

$$G(\underline{z}) = F(\underline{z}) \frac{H(\underline{z}) + \sum_{i=1}^{N} H_i(\underline{z})G_i(\underline{z})}{1 - F(\underline{z})[p+\bar{p}Z_1^{-1}Q_1(\underline{z})]} \tag{21a}$$

where $H_i(z)$ $1 \leq i \leq N$ are defined in (7b) and $H(\underline{z})$ is a known function up to the constant $G_N(\underline{z})$ defined by:

$$H(\underline{z}) = \sum_{i=0}^{N-1} \hat{H}_i(\underline{z})\hat{G}_i(\underline{z}) \tag{21b}$$

$\hat{H}_i(\underline{z})$ are defined in (7b).

Now, using Lemma 1 and 2 we immediately obtain the following result:

Theorem 4: Let σ_i, $\underline{z}^{(i)}$ $1 \leq i \leq N-1$ be as defined in Lemma 1 and 2. Then for $1 \leq i \leq N-2$ we have:

$$G_i(\underline{z}) = F(\underline{z}^{(i)}) \frac{H(\underline{z}^{(i)}) + \sum_{j=i+1}^{N} H_j(\underline{z}^{(i)})G_j(\underline{z})}{1 - F(\underline{z}^{(i)})[p+\bar{p}z_{i+1}^{-1}Q_i(\underline{z}^{(i)})]} \tag{22a}$$

and

$$G_{N-1}(\underline{z}) = - \frac{H(\underline{z}^{(N-1)}) + H_N(\underline{z}^{(N-1)})G_N(\underline{z})}{H_{N-1}(\underline{z}^{(N-1)})} \tag{22b}$$

We will demonstrate how (22a) is proved for i=1. Then by induction one can easily obtain (22a) and (22b). Since $G(\underline{z})$ is an analytic function for $|z_i| < 1$ $1 \leq i \leq N$ and since the demoninator of $G(\underline{z})$ vanishes at σ_1, we have from (21a) that:

$$H(\underline{z}^{(1)}) + \sum_{i=2}^{N} H_i(\underline{z}^{(1)})G_i(\underline{z}) + H_1(\underline{z}^{(1)})G_1(\underline{z}) = 0 \tag{23}$$

Using the definition of $H_1(\underline{z}^{(1)})$ from (7b), i.e. $H_1(\underline{z}^{(1)}) = \bar{p}[z_2^{-1}Q_2(\underline{z}^{(1)}) - \sigma_1^{-1}Q_1(\underline{z}^{(1)})]$ and the fact that $F(\underline{z}^{(1)})[p+\bar{p}\sigma_1^{-1}Q_1(\underline{z}^{(1)})] = 1$ we get immediately (22a) for i=1.

Now in (22b) $G_{N-1}(\underline{z})$ is expressed in terms of the constant $G_N(\underline{z})$. Using (22a) for i = N-2, N-3,..., 1 we finally have all the boundary functions $G_i(\underline{z})$ $1 \leq i \leq N-1$ expressed in terms of the constant $G_N(\underline{z})$.

Now that we have already determined $\hat{G}_i(\underline{z})$ $0 \leq i \leq N-2$ and $G_i(\underline{z})$ $1 \leq i \leq N-2$ in terms of the constant $G_N(\underline{z})$ we still have to determine this constant.

Determination of the constant $G_N(\underline{z})$

To determine the constant $G_N(\underline{z})$ let us first prove the following:

Theorem 5: For $1 \leq \ell \leq N$ let,

$$r_\ell = \left.\frac{\partial F(\underline{z})}{\partial z_\ell}\right|_{z_1=z_2=\ldots=z_N=1} \tag{24a}$$

and

$$\lambda_\ell = r_\ell + \sum_{j=1}^{N} \lambda_j \, \theta_j(\ell) \tag{24b}$$

Then the following holds:

$$\lambda_i = \bar{p}[G_{i-1}(\underline{1}) - G_i(\underline{1})] + p[\hat{G}_{i-1}(\underline{1}) - \hat{G}_i(\underline{1})] \qquad 1 \leq i \leq N-1 \tag{25a}$$

$$\lambda_N = p[G_{N-1}(\underline{1}) - G_N(\underline{1})] \tag{25b}$$

where,

$$G_i(\underline{1}) = G_i(\underline{z})\Big|_{z_{i+1}=z_{i+2}=\ldots z_N=1} \qquad 0 \le i \le N-1 \tag{26a}$$

$$\hat{G}_i(\underline{1}) = \hat{G}_i(\underline{z})\Big|_{z_{i+1}=z_{i+2}=\ldots=z_{N-1}=1} \qquad 0 \le i \le N-2 \tag{26b}$$

and $G_N(\underline{1}) = \hat{G}_{N-1}(\underline{1})$ is just the constant we are looking for.

Proof: For $1 \le i \le N$, let us derive both sides eq. (6) with respect to z_i and substitute $z_1=z_2=\ldots=z_N=1$. Then for $1 \le i \le N-1$ we obtain

$$\begin{aligned}
0 = \; & r_i + [G_{N-1}(\underline{1}) - G_N(\underline{1})]p\,\theta_N(i) + \\
& + \sum_{\substack{j=1\\ j\neq i}}^{N-1} [\hat{G}_{j-1}(\underline{1}) - \hat{G}_j(\underline{1})]\theta_j(i) - [\hat{G}_{i-1}(\underline{1}) - \hat{G}_i(\underline{1})] + \\
& + \sum_{\substack{j=1\\ j\neq i}}^{N-1} [G_{j-1}(\underline{1}) - G_j(\underline{1}) - \hat{G}_{j-1}(\underline{1}) + \hat{G}_j(\underline{1})]\overline{p}\,\theta_j(i) \\
& - \overline{p}[G_{i-1}(\underline{1}) - G_i(\underline{1}) - \hat{G}_{i-1}(\underline{1}) + \hat{G}_i(\underline{1})]
\end{aligned} \tag{27a}$$

and

$$\begin{aligned}
0 = \; & r_N - p[G_{N-1}(\underline{1}) - G_N(\underline{1})] \\
& + \sum_{i=1}^{N-1} [\hat{G}_{i-1}(\underline{1}) - \hat{G}_i(\underline{1})]\theta_i(N) \\
& + \sum_{i=1}^{N-1} [G_{i-1}(\underline{1}) - G_i(\underline{1}) - \hat{G}_{i-1}(\underline{1}) + \hat{G}_i(\underline{1})]\overline{p}\theta_i(N)
\end{aligned} \tag{27b}$$

when in (27) we used the fact that $G(\underline{1}) = G_0(\underline{1}) = 1$. Rearranging (27) we get for $1 \le i \le N-1$:

$$0 = r_i + [G_{N-1}(\underline{1}) - G_N(\underline{1})]p\theta_N(i) +$$

$$+ \sum_{\substack{j=1 \\ j\neq i}}^{N-1} \{\bar{p}[G_{j-1}(\underline{1}) - G_j(\underline{1})] + p[\hat{G}_{j-1}(\underline{1}) - \hat{G}_j(\underline{1})]\}\theta_j(i)$$

$$- \{\bar{p}[G_{i-1}(\underline{1}) - G_i(\underline{1})] + p[\hat{G}_{i-1}(\underline{1}) - \hat{G}_i(\underline{1})]\} \tag{28a}$$

and

$$0 = r_N - p[G_{N-1}(\underline{1}) - G_N(\underline{1}) +$$

$$+ \sum_{i=1}^{N-1} \{\bar{p}[G_{i-1}(\underline{1}) - G_i(\underline{1})] + p[\hat{G}_{i-1}(\underline{1}) - \hat{G}_i(\underline{1})]\}\theta_i(N) \tag{28b}$$

In (28) we have N linear equations with N unknowns $\bar{p}[G_{i-1}(\underline{1}) - G_i(\underline{1})] + p[\hat{G}_{i-1}(\underline{1}) - \hat{G}_i(\underline{1})]$ for $1 \leq i \leq N-1$ and $p[G_{N-1}(\underline{1}) - G_N(\underline{1})]$. Clearly (25) solves these equations. $\square$

From (25) we obtain:

$$\sum_{i=1}^{N-1} \lambda_i = \bar{p}[1-G_{N-1}(\underline{1})] + p[\hat{G}_0(\underline{1})-\hat{G}_{N-1}(\underline{1})] =$$

$$= \bar{p}[1-\lambda_N/p-G_N(\underline{1})] + p[G_0(\underline{1}) - G_N(\underline{1})] \tag{29}$$

Therefore,

$$G_N(\underline{1}) - p\hat{G}_0(\underline{1}) = \bar{p}[1-\lambda_N/p] - \sum_{i=1}^{N-1} \lambda_i \tag{30}$$

Recalling that $\hat{G}_0(\underline{z}) = k(\hat{\underline{z}})G_N(\underline{z})$ we finally have that:

$$G_N(\underline{1}) = \frac{\bar{p}[1-\lambda_N/p] - \sum_{i=1}^{N-1} \lambda_i}{1 - pk(\hat{\underline{1}})} \tag{31}$$

where $k(\hat{\underline{1}}) = k(\hat{\underline{z}})\Big|_{z_1=z_2=\ldots=z_{N-1}=1}$. (31) implies that the condition for steady-state is:

$$\sum_{i=1}^{N-1} \lambda_i < \overline{p}(1-\lambda_N/p) \tag{32}$$

Rewriting (32) as:

$$\lambda_N < p(1 - \sum_{i=1}^{N-1} \lambda_i/\overline{p}) \tag{33}$$

we can explain the steady-state condition intuitively as follows: Clearly, node N is the bottleneck of the system. If it is heavily loaded, then the fraction of time that the channel is used by the N-1 nodes is $\sum_{i=1}^{N-1} \lambda_i/\overline{p}$, so the fraction of time that the channel is available for node N for successful transmissions is $1 - \sum_{i=1}^{N-1} \lambda_i/\overline{p}$. As node N transmits with probability p when nonempty the rate of its successful transmissions is $p(1- \sum_{i=1}^{N-1} \lambda_i/\overline{p})$ which for stability must be greater than the arrival rate to the node.

Having obtained the joint generating function $G(\underline{z})$ we can derive, at least in principle, any moment of the queue lengths at the nodes. Specifically, if we denote by $\overline{L}_i$ the average queue length at node i in steady-state, then

$$L_i = \frac{\partial G(\underline{z})}{\partial z_i}\Big|_{z_1=z_2=\ldots=z_N=1} \tag{34}$$

Assuming that packets arrive at the nodes only at the end of a slot, then using Little's law [10] we may also obtain the average time delays at node i denoted by T_i as follows:

$$T_i = L_i/\lambda_i \tag{35}$$

where λ_i is the total arrival rate at node i as defined in (24b). The total average time delay in the system is obtained by applying Little's law to the whole system and it is given by:

$$T = \sum_{i=1}^{N} L_i / \sum_{i=1}^{N} r_i \tag{36}$$

where r_i is the arrival rate at node i from its corresponding source as defined in (24a). The total average delay T is clearly a function of the transmission probability p. Obviously, as p decreases, the total average delay increases since node N transmits rather rarely. Also when p increases the total average delay also increases since there are many conflicts in the transmission. Consequently, there is some intermediate value of p (that depends on the arrival rates to the nodes) that minimizes the total average delay in the system. This will be demonstrated in the example given in Section 4.

4. Example

In this section we will use a simple example in order to show some details of the general solution method developed in the previous section. The example conists of the network of Fig. 2, where packets arrive to nodes 1, 2, 3 and node 2 forwards its packets to node 1. Consequently $Q_1(\underline{z}) = Q_3(\underline{z}) = 1$; $Q_2(\underline{z}) = z_1$ (here $\underline{z} = (z_1, z_2, z_3)$). We shall also assume that:

$$F(\underline{z}) = r_1 z_1 + r z_2 z_3 + 1 - r_1 - r$$

i.e., during each slot a packet arrives to node 1 with probability r_1, with probability r a packet arrives to both nodes 2 and 3 and with probability $1-r_1-r$ no packet arrives to the system. Then using (8), (12) for $z_3 = 0$, $p = 0$, we obtain:

$$\hat{\sigma}_1 = 1 - \frac{r}{1-r_1}$$

$$\sigma_2 \doteq \left(1 - \frac{r}{1-r_1}\right)^2$$

Using (19), (18) and (14) we have:

$$p\, G_2'(0,0,0) = \frac{r}{1-r_1-r}\, G(0,0,0)$$

$$\hat{G}(0,z_2,0) = G(0,0,0)$$

$$\hat{G}(z_1,z_2,0) = G(0,0,0)\left[1 + \frac{r_1}{1-r_1-r}\, z_1\right]$$

Using (31) we have that:

$$G(0,0,0) = \frac{\bar{p}(1-r/p)-(r_1+2r)}{1-p(1-r)/(1-r_1-r)}$$

and the condition for steady-state is:

$$\bar{p}(1-r/p) - (r_1 + 2r) > 0$$

From (8) and (11) we obtain:

$$\sigma_1(z_2,z_3) = (1-f(z_2,z_3) - r_1\bar{p} - \sqrt{\Delta})/2r_1p$$

where,

$$f(z_1,z_3) = p(rz_2z_3 + 1 - r_1 - r)$$

$$\Delta = (1 - f(z_2,z_3) - r_1\bar{p})^2 - 4r_1\bar{p}f(z_2,z_3)$$

and $\sigma_2(z_3)$ is the solution of

$$\sigma_2(z_3) = \sigma_1^2(\sigma_2(z_3),\ z_3)$$

in the unit circle $|G_2| < 1$.

From (15) and (17) we obtain:

$$G(0,0,z_3) = G(0,0,0)\,\frac{p(z_3^{-1}-1) + \dfrac{r_1p}{1-r_1-r}\,(1-\sigma_1(\sigma_2(z_3),z_3))}{1 - 2p + pz_3^{-1} - \bar{p}\sigma_1^{-1}(\sigma_2(z_3),z_3)}$$

and

$$G(0,z_2,z_3) = \{G(0,0,0)\,[p(1-z_3^{-1}) + \frac{r_1p}{1-r_1-r}\ (1-\sigma_1(z_2,z_3))$$

$$+ G(0,0,z_3)(1-2p+pz_3^{-1}-\bar{p}z_2^{-1}\sigma_1(z_2,z_3)\}/\{\bar{p}(\sigma_1^{-1}(z_2,z_3)-z_2\sigma_1(z_2,z_3)]\}$$

Finally we have that:

$$G(z_1,z_2,z_3) = F(z_1,z_2,z_3)\{G(0,0,0)[p(1-z_3^{-1}) + \frac{r_1 p}{1-r_1-r} (1-z_1)] +$$

$$+ G(0,0,z_3)(1-2p+pz_3^{-1}-\bar{p}z_2^{-1}z_1) +$$

$$+ G(0,z_2,z_3)\bar{p}(z_2^{-1}z_1-z_1^{-1})\}/[1-F(z_1,z_2,z_3)(p+\bar{p}z_1^{-1})]$$

The explicit expressions for the average delays in the system are too complicated to be given here. To give some insight into the behavior of this network we plotted these quantities in Fig. 3-5. In Fig. 3 T_1, T_2, T_3 and T are plotted as a function of $r = r_1$ for $p = 0.4$. In Fig. 4, these quantities are plotted as a function of p for $r_1 = r = 0.05$. As we can see, for small values of p, the queue is built up only at node 3 (since it is rarely transmitting) while for large values of p, queues are built up at all the nodes and this is due to the interference.

As we see, there is an optimal transmission probability p* that minimizes the total delay in the system. In Fig. 5 T_{min}--the minimal total delay in the system is plotted as a function of $r = r_1$. It is interesting to mention that $p^* \simeq 0.34$ and it is almost insensitive to the value of $r = r_1$. Also T_{min} is not very sensitive to small variations in p*.

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Appendix

Consider the evolution equation (2) and let $G_t(\underline{z}) = E\{\prod_{i=1}^{N} z_i^{L_i(t)}\}$.

Then,

$$G_{t+1}(\underline{z}) = E\{\prod_{i=1}^{N} z_i^{L_i(t+1)}\} =$$

$$= F(\underline{z})E\{\prod_{i=1}^{N} z_i^{L_i(t)+\sum_{m=1}^{N} D_m^i(t)-V_i(t)U_i(L_i(t))\prod_{m=1}^{i-1}[1-U_m(L_m(t))]}\} \qquad \text{(A1)}$$

where in (A1) we used (1) and the fact that the vector of arrival processes $\{A_i(t)\}_{i=1}^{N}$ is independent of the state of the system.

Now for $0 \leq j \leq N$ let the event that $L_i(t) = 0$ for $1 \leq i \leq j$ and $L_{j+1}(t) > 0$ be denoted by $\Omega_j(t)$. Then from (A1) we obtain:

$$\begin{aligned}
G_{t+1}(\underline{z}) &= F(\underline{z})\{\mathrm{Prob}(\Omega_N(t)) + \\
&+ \mathrm{Prob}(\Omega_{N-1}(t),\ v = 0)E[z_N^{L_N(t)}/\Omega_{N-1}(t),\ v = 0] + \\
&+ \mathrm{Prob}(\Omega_{N-1}(t),\ v = 1)z_N^{-1}E[z_N^{L_N(t)}/\Omega_{N-1}(t),\ v = 1]\, Q_N(\underline{z}) + \\
&+ \sum_{j=0}^{N-2} \mathrm{Prob}(\Omega_j(t), L_N(t) = 0)z_j^{-1}E[\prod_{m=j}^{N-1} z_m^{L_m(t)}/\Omega_j(t), L_N(t) = 0]\ Q_j(\underline{z}) + \\
&+ \sum_{j=0}^{N-2} \mathrm{Prob}(\Omega_j(t), L_N(t) > 0,\ v = 0)z_j^{-1}\ E[\prod_{m=j}^{N} z_m^{L_m(t)}/\Omega_j(t), L_N(t) > 0,\ v=0]Q_j(\underline{z}) \\
&+ \sum_{j=0}^{N-2} \mathrm{Prob}(\Omega_j(t), L_N(t) > 0,\ v = 1)E[\prod_{m=j}^{N} z_m^{L_m(t)}/\Omega_j(t), L_N(t) > 0,\ v = 1]\}
\end{aligned} \qquad \text{(A2)}$$

where in (A2) $z_0 \equiv 1$ and we used the definition of $Q_j(\underline{z})$ $1 \leq j \leq N$. Now since v is an independent random variable we obtain from (A2):

$$G_{t+1}(\underline{z}) \;=\; F(\underline{z})\{\text{Prob}(\Omega_N(t)) +$$

$$+\ \text{Prob}(\Omega_{N-1}(t))E[z_N^{L_N(t)}/\Omega_{N-1}(t)][\bar{p}+pz_N^{-1}Q_N(\underline{z})] +$$

$$+\ \sum_{j=0}^{N-2} \text{Prob}(\Omega_j(t),L_N(t)=0)E[\prod_{m=j}^{N-1} z_m^{L_m(t)}/\Omega_j(t),L_N(t)=0]z_j^{-1}Q_j(\underline{z}) +$$

$$+\ \sum_{j=0}^{N-2} \text{Prob}(\Omega_j(t),L_N(t) > 0)E[\prod_{m=j}^{N} z_m^{L_m(t)}/\Omega_j(t),L_N(t) > 0][p+\bar{p}z_j^{-1}Q_j(\underline{z})]\} \tag{A3}$$

Now it is easy to see from (3)-(4) that for $t\to\infty$ we have:

$$G_{t+1}(\underline{z}) \to G(\underline{z})$$

$$\text{Prob}(\Omega_N(t)) \to G_N(\underline{z})$$

$$\text{Prob}(\Omega_{N-1}(t))E[z_N^{L_N(t)}/\Omega_{N-1}(t)] \to G_{N-1}(\underline{z})-G_N(\underline{z})$$

$$\text{Prob}(\Omega_j(t),L_N(t) = 0)E[\prod_{m=j}^{N-1} z_m^{L_m(t)}/\Omega_j(t),L_N(t) = 0] \to \hat{G}_j(\underline{z})-\hat{G}_{j+1}(\underline{z})$$

$$\text{Prob}(\Omega_j(t),L_N(t) > 0)E[\prod_{m=j}^{N} z_m^{L_m(t)}/\Omega_j(t),L_N(t) > 0] \to$$

$$\to G_j(\underline{z})-G_{j+1}(\underline{z})-\hat{G}_j(\underline{z})+\hat{G}_{j+1}(\underline{z})$$

Therefore (6) follows and Theorem 1 is proved.

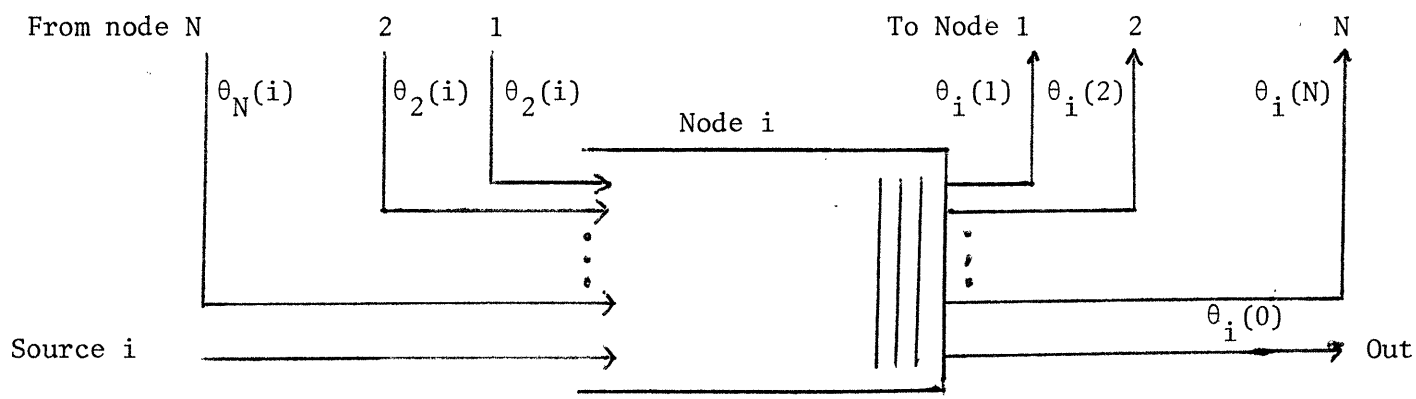

Figure 1: An example of a node i in the system

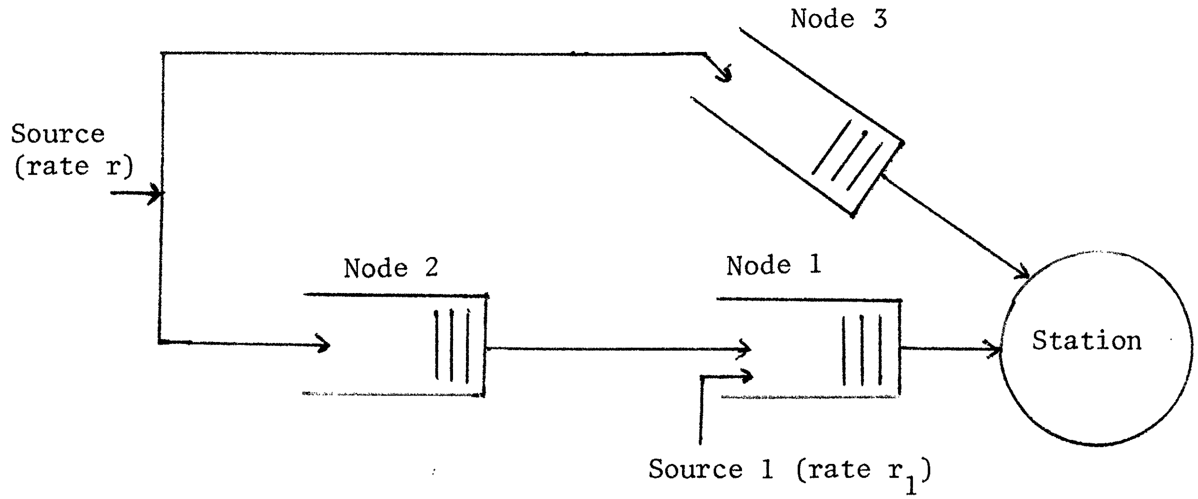

Figure 2: Example Network

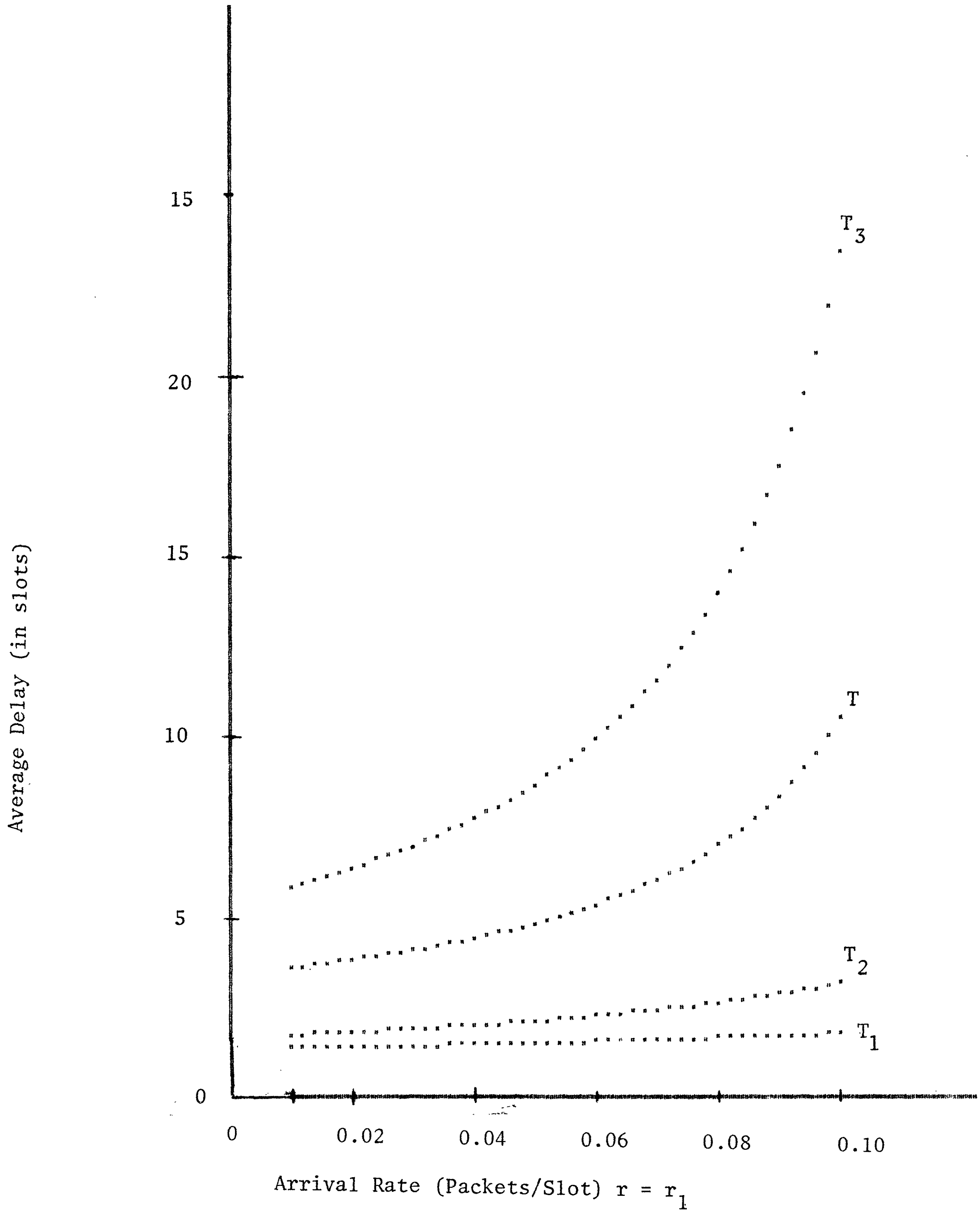

Figure 3: Average delays versus arrival rate

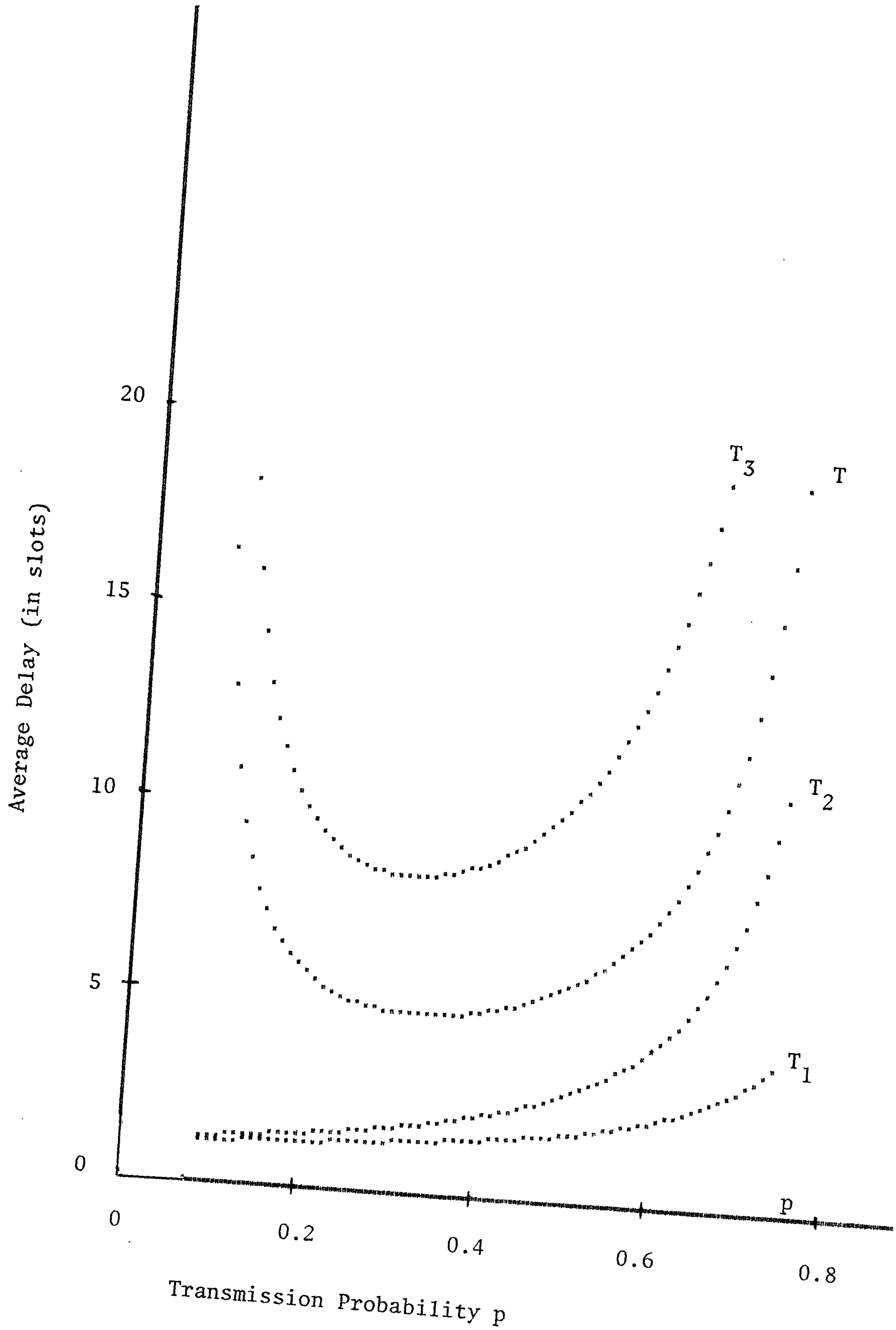

Figure 4: Average delays versus transmission probability

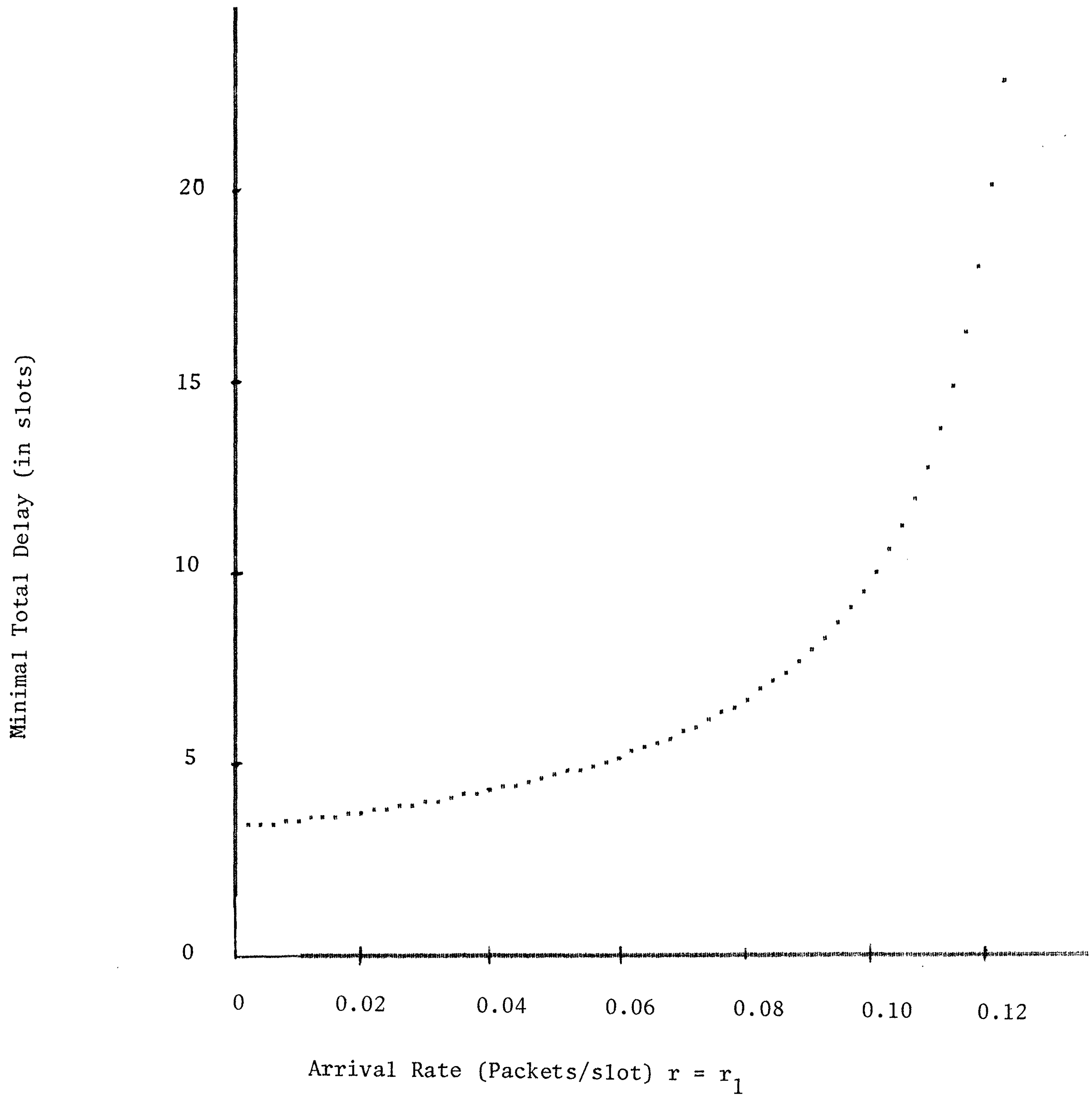

Figure 5: Minimal total delay versus the arrival rate.