Attitude Control via Structural Vibration: an application of compliant robotics

by

Nathan S. Tyrell

Submitted to the Department of Mechanical Engineering on May 12, 2017, in partial fulfillment of the requirements for the degree of Master of Science in Mechanical Engineering

Abstract

We review and present techniques for effecting and controlling the reorientation of structures "floating" in angular-momentum-conserving environments, applicable to both space robotics and small satellite attitude control. Conventional orientation control methods require either the usage of continuously rotating structures (e.g. momentum wheels) or the jettisoning of system mass (e.g. hydrazine thrusters). However, the systems proposed herein require neither rotating structures nor mass ejection; instead, orientation is controlled by the imposition of a bounded cyclic shape change—the canonical example of such a system is a cat righting herself while falling, thereby always landing on her feet—coupled with the conservation of angular momentum, which acts analogously to a nonholonomic constraint on the system dynamics.

Further, by considering the reduced system dynamics, we extend the concept to consider the class of structures where the requisite cyclic shape change is attainable via dynamical effects, such as the normal modes of structural vibration for structures with finite stiffness. This is the central novel result of this thesis and has implications for the design of space structures where the attitude control hardware is integrated directly into the preexisting structure, the development of orientation control techniques for soft robots in space and underwater, and the design of MEMS attitude control actuators for very tiny satellites.

We apply mathematical tools drawn from differential geometry and geometric mechanics, which can be intimidating but which provide a comprehensive and powerful framework for understanding a wide range of locomotion problems fundamental to robotics and control theory. These tools allow us to make succinct statements regarding gait design, controllability, and optimality that would be otherwise inaccessible.
Acknowledgments

I acknowledge anyone who has supported my wending path through Princeton and MIT, up to wherever exactly it is I am today. In no particular order: Jim Papadopoulos (thanks for responding to an email out of the blue when I was an undergrad); Jon Prévost (thanks for teaching me most of what I know about microcontrollers); Howard Stone (you have undoubtedly shaped my academic trajectory more than any other individual; I cannot thank you enough for your kindness, self-effacing genius, and willingness to listen); David Reinfurt (thanks for taking me seriously; thanks for getting coffee in the late spring of 2014; thanks for the watch); Joe Scanlan (for introducing me to some good music; insisting that some day I’ll be an artist; please keep insisting); Clancy Rowley (especially for proof-reading a draft of the first chapter); Dave Trumper (for being the consummate engineer and academic; for teaching me about teaching; but above all for being kind); Allan Adams (for the audacity of your projects and your superhuman ability to realize them by convincing me and everyone else that they’re beautiful and necessary); Thery Mislick (for your dedication to craft, seriousness, quirkiness; in retrospect, you are exactly the person I hoped I’d meet at MIT before I came here); Peko Hosoi (for endless enthusiasm and having faith in my ideas); Andrea Chu (for helping me become a better human being; for confiding and being confided in); Abby Tyrell (for, for the past 25 years, providing the example of the better person I could try to be myself); Emily Tyrell (for always believing in me and always believing in yourself); David Tyrell (for your incessant dedication to your children; for your endless interest in my work, even when I couldn’t see the point myself); Jacqueline Steele (for your, and your belief in, goodness and for your pride in your children); some other close friends: Javier, Matt, Zach, Simon, Cam, Avneesh, Buse, Cara, Katherine (Sherlock), Andrew, Katie, Scott, Anthony; Burne Holiday: Javier (again), Joey, Cory; my extended family; my lab group: Alice, Josh, José, Ben, Sarah, Youzhi; Susan Brown; Gee Hoon; my students in 2.14/140; Google Play Music for providing such inspiring playlists as “Concrete Cool-Down”, “Suburban Ennui”, and “Looking at Pictures of Your Ex” to fuel me while I wrote this large document.
To define oneself in a word: not yet dead.

Juliette Janson
Yet this is deceptive. A place can never be betwixt and between, but it is only a hiatus on a continuum.

Charlotte Moth
The future is unwritten.

Joe Strummer
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Chapter 1

Introduction

Imagine you are floating in space, with nothing to push off of—and no rockets or thrusters at your disposal. (This is a somewhat scary hypothetical situation.) How would you move around? At least as regards translational motion, you are, sadly, out of luck, thanks to Newton's third law; no matter how much you flail, your center of mass is guaranteed to remain in exactly the same position for all time, given your aforementioned constraints. But you could change the direction you are facing, also known as your orientation or your attitude. This might be important if you want to signal to your fellow astronauts, or point your solar panels or solar sails towards a star.

We will address this orientation control problem herein, with particular interest in developing theory relevant to the design of attitude control actuators for very small satellites. This will lead to a demonstration of the storage of nonzero time-averaged angular momentum in a vibrating mechanical system, harnessing the finite stiffness of the structure to achieve useful control authority—an example of compliant robotics.

More than a century ago, George Stokes and James Clerk Maxwell evidently also took an interest in compliant robotics:

He was much interested, as also was Prof. Clerk Maxwell about the same time, in cat-turning, a word invented to describe the way in which a cat manages to fall upon her feet if you hold her by the four feet and drop
The attempts of photographic and scientific pioneer Étienne-Jules Marey, a fascinating figure in his own right, to explain the phenomenon were chronicled in *Nature* in 1894:

M. Marey thinks that it is the inertia of its own mass that the cat uses to right itself. The torsion couple which produces the action of the muscles of the vertebra acts at first on the forelegs, which have a very small motion of inertia on account of the front feet being foreshortened and pressed against the neck. The hind legs, however, being stretched out and almost perpendicular to the axis of the body, possesses a moment of inertia which opposes motion in the opposite direction to that which the torsion couple tends to produce. In the second phase of the action, the attitude of the feet is reversed, and it is the inertia of the forepart that furnishes a fulcrum for the rotation of the rear.

Though his dynamical approach was controversial at the time, with Marey's beautiful photographs (See Figure 1-1) inciting impassioned debate at the October 29, 1894 meeting of the Académie des Sciences in Paris[13], the photographer was on the right track! It is indeed an inertial phenomenon that leads to cat-turning, and the physics necessary to describe the phenomenon are directly relevant to the attitude control problem that this thesis seeks to investigate.

### 1.0.1 Motivation

Theoretical understanding of the control of an object's orientation is necessary in fields ranging from robotics to satellite design. We consider strategies for the reorientation, via cyclic shape changes, of a free deformable body in 3-space, or $SE(3)$, acted on by no external forces or constraints except conservation of angular momentum.

As I see it, there are three main areas of application of these ideas:

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1. More scientists subscribed to the belief that the cat was using air resistance or pushing off of its handler.
Figure 1-1: Marey’s falling cat chronophotographs.
MEMS-type (chip-printable) attitude control hardware that relies neither on external interaction nor continuously-rotating mechanisms

Attitude control of soft robots in space, underwater, and in other challenging environments

Design of space structures where attitude control is directly integrated as pre-existing hardware, to minimize necessary mass

We specifically consider the case where the conserved angular momentum of the deformable body is always zero; in practice, this would be the case for certain satellites and free-floating space robots. Systems of this type have been described in the literature as drift-free[32] and purely mechanical[47] (in fact, purely mechanical implies drift-free), meaning that they possess no momentum and that orientation is only a function of the time-history of internal shape.

For brevity, to pay homage to past work in the field, and to celebrate the real inspiration of this work, we refer to this dynamics and control problem simply as the Cat’s Problem. Somewhat confusingly, at least taxonomically, there is a subset of the Cat’s Problem that we will also give some time to: the Snake’s Problem, which is really just the Cat’s Problem but constrained to the plane (SE(2)).

Because of the purely mechanical nature of the system, the Cat’s Problem is equivalent to formulating a control problem on a Lie group (a mathematical ob-

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Figure 1-2: Three textbook nonholonomic systems, borrowed with great thanks from Hatton and Choset[21].
ject describing the Cat’s position and orientation), where the system’s Lagrangian is invariant under the group action\cite{48}—intuitively, the Cat dynamics are always the same, no matter the position or orientation of the Cat. There exist mathematical analogies to other canonical robotics models with nonholonomic constraints, chief among them the three-link planar snake and the differential drive vehicle. However, the Cat’s Problem differs importantly in that the “nonholonomic constraint” (conservation of linear and angular momentum) falls out of the dynamics, so to speak, and is not externally imposed on the system (as is a rolling constraint, for example, which is the textbook nonholonomic constraint). This has the interesting consequence of ensuring the fully kinematic\cite{47} nature of the angular-momentum-conserving system with zero momentum.

1.1 Conventional control strategies

![Three orthogonally-mounted reaction wheels](image)

Figure 1-3: Three orthogonally-mounted reaction wheels.

Spacecraft attitude control is provided by two possible means: shape change or interaction with the external environment. Thrusters that eject exhaust mass or ions are examples of attitude control methods that interact with the external environment (in that they transfer mass from the spacecraft to its environs), as is a magnetorquer, which relies on interaction with an external magnetic field to apply control torques.
to the spacecraft, or an electrodynamic tether, which relies on interaction with an external ionic field. Thrusters are infeasible to build on very tiny scales, while electrodynamic tethers have indeed been proposed for use on chip-based satellites[55]. We are interested in considering a theoretical basis for attitude control, implementable on very tiny scales, that does not rely on external interaction; therefore, we must rely on shape change.

A standard means of attitude control via shape change is the use of reaction wheels or momentum wheels[53]. A reaction wheel is predicated on continuous rotation: by exerting a torque on an attached flywheel, a spacecraft feels an equal-and-opposite torque applied to itself. By mounting several reaction wheels, as in Figure 1-3, full control of spacecraft orientation is achieved.

However, continuously-rotating mechanisms are subject to relatively rapid wear, often require some sort of biasing to counteract stiction, and are challenging to construct and maintain on the microscale. Therefore, we are interested in systems that do not exhibit continuous rotation, but rather have bounded control inputs or shape degrees of freedom. This boundedness might be due to design constraints—it’s hard to make a soft robot with continuously rotating parts, for example, or to make MEMS chips with rotating parts—or for other reasons (such as making a satellite with fully integrated attitude control hardware to save mass).

1.2 Nonholonomic systems

This is a difficult subject area! We give a very, very brief overview here; interested readers are strongly encouraged to consult the references.

We consider the configuration $q$ of a dynamical system evolving on some configuration manifold $Q$ (we can write $q \in Q$), subject to some constraints, generically of the form $f(q, \dot{q}) = 0$. Constraints of the form $f(q) = 0$ are termed holonomic and only serve to reduce the effective dimension of the configuration space; i.e. we could use the constraint to define a constant submanifold of $Q$ on which the dynamics are constrained to evolve. In this sense, holonomic constraints preclude controllability:
they prevent our system from ever touching certain parts of its configuration space \( Q \).

Figure 1-4: An illustration of (a) nonholonomic versus (b) holonomic constraints.

Constraints of the form \( f(q, \dot{q}) = 0 \) are termed nonholonomic. A subset of nonholonomic constraints are Pfaffian constraints[38], which are of the form \( \omega(q)\dot{q} = 0 \). Importantly, nonholonomic constraints do not reduce the effective dimensionality of the configuration space; rather, they define what is termed a distribution, or a restriction on the tangent bundle \( TQ \) of the configuration space. The tangent bundle contains all the possible directions the system could move in at every configuration; it is defined as a unique vector space, \( i.e. \) set of directions, at each point \( q \) on the configuration manifold. Intuitively, nonholonomic constraints define restrictions on the allowable directions the system can move in at a given configuration \( q \). Because nonholonomic constraints do not reduce the dimension of the configuration space, they inherently allow a higher degree of controllability than holonomic constraints.

One might then attempt constructing equations of motion using a Lagrangian formulation or Hamiltonian formulation. Entire books have been written on the subject; the reader is referred to Bloch[5] for a very comprehensive overview of nonholonomic
Symmetry and reduction

Now, imagine that we have a system subject to nonholonomic constraints such that the configuration splits naturally into two parts $q = (g, r)$, where $g$ is an element of the orientation space $G$ and $r$ an element of the shape space $M$, such that $Q = G \times M$ is a product space; we have some map $\pi$ that takes an element $q \in Q$ and returns the corresponding orientation $g \in G$. The group $G$ is special: it is chosen exactly because the nonholonomic constraints are independent of its action, such that they depend only on the body velocity $\xi$ and not the world velocity $\dot{g}$ and thus can be written $f(\xi, r, \dot{r}) = 0$. This assigns the system the overall structure of a trivial fiber bundle,\(^2\) where we might term $G$ the fiber space and $M$ the base space. Intuitively, the shape $r$ parameterizes the “shape” of the system.

\[ G = SE(3) \]

\[ g = \begin{bmatrix} x \\ y \\ \psi \end{bmatrix} \]

\[ r = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} \]

\[ \xi = \frac{R}{2} (\dot{\theta}_1 + \dot{\theta}_2) b_1 + (\dot{\theta}_1 - \dot{\theta}_2) b_3 \]

Figure 1-5: An illustration of the relevant groups and velocities for the differential-drive vehicle. Note that due to the nonholonomic constraints on the wheels, the vehicle can only move forward in the $b_1$ direction, or turn about the axis $b_3$.

These are somewhat confusing and abstract ideas, especially if the reader is not group-theoretically inclined (please see Hatton and Choset’s excellent coursebook[20] for an accessible introduction to this brand of mathematics), but some clarity might be

\(^2\) trivial referring to the fact that the map $\pi$ just returns the projection of $q$ onto $G$. 

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gleaned by considering the example case of *locomotion*, where *g* is taken to be the position and orientation of a locomoting mechanical system, such as a differential-drive vehicle. In this case, it is clear that the nonholonomic constraints (the no-slip condition applied to each vehicle wheel) are independent of position and orientation—in short, the vehicle should “drive” the same no matter what its position and orientation are.

In the symmetrical nonholonomic case, it is possible to formulate the equations of motions of the shape *r* without reference to *g* (one can formulate the *reduced Lagrangian* *l*(ξ, *r*, *ṙ*) and proceed from there); this process is called *nonholonomic reduction*[6]. Conversely, it is also true that the shape trajectory *r*(t) uniquely determines the group trajectory *g*(t)—we speak, rather poetically, of *lifting* from the base to the fiber—which is a very useful property when considering the control theory of nonholonomic systems; this process is called nonholonomic *reconstruction*. A rigorous overview is given in [33].

**Lie groups and algebra**

Given the aforementioned mathematics, the group *G* also inherits the structure of a *Lie group*. Every Lie group also has a corresponding *Lie algebra*, which is defined as the tangent space at the identity element of the Lie group. For more in-depth discussion, the reader is referred to several references: Again, Hatton and Choset[20] provide an accessible introduction; while Boothby[7] or Abraham[1] provide a more rigorous treatment of the fundamentals.

**Drift**

The term *drift* refers to “dynamics in the absence of controls”;³ intuitively, imagine the ability of the mechanical system to “coast”, or to acquire momentum *p* (*generalized nonholonomic momentum* in the terminology of Bloch[5]). We are concerned with *drift free* systems, in that they have no nontrivial dynamics without control inputs (some kind of activation of the shape *r*)—as an example, remember the differential- ³http://www.mat.uc.pt/~mmc/courses/PhD_Seminar/Fatima_Leite.pdf
drive vehicle and note that it will only move if the wheels (controls inputs) are turning. This feature implies that drift-free systems have the same number of nonholonomic constraints as dimensions of the fiber space $G$.

Figure 1-6: Ostrowski[42] provides the example of the snakeboard: a symmetric nonholonomic system with nonzero drift.

![Diagram of a snakeboard](image)

Drift-free also implies that the relationship between the shape $r$ and the orientation $g$ is determined only by the nonholonomic constraints, and not by the evolution of a dynamical equation; therefore, it is independent of time parameterization.

**Mechanical vs. dynamical nonholonomic constraints**

In the literature, there is the further distinction between *purely mechanical*[47] and *principally kinematic*[41] systems. These systems are characterized by the specific type of nonholonomic constraint that defines their dynamics: mechanical or dynamical ([5] p.185). Mechanical constraints are defined as nonholonomic constraints that are externally-imposed on the system, such as rolling constraints. Dynamical constraints are those that “fall out of” a momentum conservation law; importantly, they are only invariant under the group action if $p = 0$.

Principally kinematic systems cannot ever have $p \neq 0$ (strictly speaking, $p$ is not even defined) and arise when the number of mechanical nonholonomic constraints is exactly equal to the dimension of the group space; examples include the differential drive vehicle and the three-link kinematic snake[18]. It is important to note that the principally kinematic nature of a system with mechanical nonholonomic constraints
is contingent on design alone; as in, the designer must specifically choose to have the same number of mechanical nonholonomic constraints as group dimensions if he wants to create a principally kinematic system (for example, the kinematic snake must have exactly three links in order to be a fully kinematic system).

Purely mechanical systems differ in that \( p \) exists but the dynamics are such that \( \dot{p} = 0 \) and thus \( p \) stays constant for all time; these are simply systems that conserve momentum. If we pick a purely mechanical system and “start” it such that \( p(0) = 0 \rightarrow p(t) = 0 \), it will evolve exactly as a principally kinematic system: Per Noether’s Theorem (CITE something? Goldstein), the number of conserved quantities (the momenta \( p \)) is guaranteed to be exactly the dimension of the Lie group \( G \), since the system is symmetric with respect to \( G \). In general, the geometric phase and dynamic phase[5] decouple for purely mechanical systems. Examples of purely mechanical systems include the floating snake, planar snake robot[18], and of course the falling cat.

1.2.1 Reconstruction

The analysis of nonholonomic systems is concerned with reduction, as aforementioned, and reconstruction[6]. The reconstruction process aims to reconstruct the fiber trajectory \( g(t) \) given the base trajectory \( r(t) \) determined via reduction. We will be considering systems where the reduction process is simple: either the shape \( r \) is fully actuated and no nontrivial reduction is necessary since \( r(t) \) can be specified directly, or \( r \) is underactuated but since \( p = 0 \) the reduced dynamics are straightforward to derive.

The reconstruction equation[19] applies generally to all nonholonomic systems with symmetry:

\[
\xi(r, \dot{r}, p) = -A(r)\dot{r} + \Gamma(r)p \quad 0, \text{if } p = 0
\]

(1.1)

Where \( \xi \) is identified as the body velocity:
\[ \xi = (T_e L_g)^{-1} \dot{g} = T_g L_{g^{-1}} \dot{g} \] (1.2)

Where the operator \( T_e L_g \) represents a left lifted action (see Hatton and Choset's coursebook[20] for an accessible explanation). Suffice to say here that, in the case of matrix Lie groups, \( T_e L_g = g \) and thus \( \xi = g^{-1} \dot{g} \). In a locomotion context, this is equivalent to an object's velocity vector expressed in its body-fixed frame.

Therefore, for principally kinematic systems with matrix Lie groups only:

\[ \dot{g} = -g A(r) \dot{r} \] (1.3)

The matrix \( A(r) \) is referred to as the local connection, and it encodes the relationship between the shape \( r \) and the orientation \( g \).

Equation 1.3 is ubiquitous in the literature. It is the underlying equation that almost all nonholonomic control systems are based off of; we might consider the control input to be \( u \triangleq \dot{r} \). Our task, now, as control system designer, is to put our particular systems of interest into the appropriate mathematical form, and then to determine the control input trajectories \( u(t) \) that yield desirable orientation trajectories \( g(t) \).

In general, it is difficult to invert Equation 1.3 (i.e. to solve for the \( r(t) \) that produces a specific \( g(t) \), instead of vice versa), and so a variety of mathematical tools have been developed to aid in the design process. The concept of "desirability" also must be quantitatively defined, and, indeed, it is in the field of optimal control theory, which seeks to identify the "best" (perhaps the minimum energy, for example) trajectories that achieve a desired result (often, a desired holonomy, which will be defined below).

**Geometric phase**

Given a system with nonholonomic constraints and symmetry, it is possible to generate a base trajectory \( r(t) \) that is cyclic—such that \( r(t+\tau) = r(t) \)—but that results in a nonzero change in orientation—such that \( g(t+\tau) \neq g(t) \). We will later define these trajectories more formally as gait s. For a principally kinematic system, this fact can
Figure 1-7: Bloch[5] provides a useful visualization of lifting from the base space to the fiber space. This also visualizes the geometric phase or holonomy associated with a closed curve in the base space.

be verified by directly integrating the reconstruction equation.

As a very simple example borrowed from Bloch[5], consider the idea of parallel transport on the sphere: a vector traverses a closed loop on the surface of a sphere, always pointing in the same local direction; by the time it reaches its starting point, its pointing angle has changed. The change in pointing angle is proportional to the area enclosed by the closed loop.

In the literature, this change in orientation \( h \triangleq g(t+\tau) - g(t) \) due to a closed curve in the base space has been termed the geometric phase[29][25][30], or holonomy[5] associated with the base curve (in quantum physics, it is also called the Berry phase). Correspondingly, any change in orientation due to drift (nonzero momentum \( p \)), if it exists, is termed dynamic phase[5]. These concepts solidify the connection between holonomy and path dependence.

1.2.2 Rectification

In the context of this work, we define mechanical rectification, using terms borrowed from electrical engineering, as some kind of DC (i.e. secular, non-zero time-average) motion resulting from AC (i.e. periodic, cyclic) motion. This framework is in large
Figure 1-8: Bloch[5] provides another useful visualization of geometric phase in the example of parallel transport on the sphere.

part due to Brockett[9]. One can imagine the internal combustion engine as perhaps the most ubiquitous example of mechanical rectification.

Connection to the Lie bracket

Mechanical rectification is related to the Lie bracket: the Lie bracket is, in some sense, the mathematical encoding of a system's ability to exhibit mechanical rectification. Therefore, it is also related to path dependence, which is fundamentally related to [non]holonomy.

An intuitive physical explanation of the Lie bracket is parallel parking the textbook differential-drive vehicle from earlier: The differential-drive vehicle can only steer, or move ahead in the direction defined by its steering angle; it cannot move sideways, due to the nonholonomic rolling constraint on the wheels. Thus, it can directly access only a subset of its tangent bundle—it is not free to move in all possible directions at any given time (recall that this is called a distribution). However, through a parallel-parking-sort-of-maneuver (gait; sequence of motions), which in-
Figure 1-9: Parallel parking a differential-drive vehicle is a simple explanation of the Lie bracket.

volves a combination of steering and driving, the car can indeed move sideways, even if we shrink the steer/drive-wiggle to a differentially small size—this wiggle is in fact a physical explanation of the Lie bracket of this particular system.

Intuitively, the Lie bracket \([X, Y]\) involves flowing the system differentially along a vector field \(X\) (let’s call this moving forward a tiny bit), then along another vector field \(Y\) (this is turning just a little counterclockwise), then along \(-X\) (moving backward a tiny bit), then along \(-Y\) (turning just a little clockwise). We can imagine that as these motions in sequence get very small, the car ends up moving slightly sideways, directly to its right. Thus, the car is able to move in the \(b_2\) direction, a direction that is not part of the subspace of the tangent space that the nonholonomic constraint distribution dictates (remember that it stipulates that a car may only move forward or steer)! Therefore the distribution is not closed under the Lie bracket—i.e. the Lie bracket yields a vector that is not a member of the space spanned by \(X\) and \(Y\). And we see, now, that the Lie bracket is a kind of differential operator, and that the Lie bracket of two vector fields is also a vector field.

If we define the vector fields \(X\) and \(Y\) using arbitrary functions—\(X = f(q)\) and \(Y = g(q)\) for \(q \in Q\)—then the Lie bracket is[38]:

\[
[f, g](q) = \frac{\partial g}{\partial q} f(q) - \frac{\partial f}{\partial q} g(q)
\]  

(1.4)
If we define the vector fields $X$ and $Y$ using matrices $X = Aq$, and $Y = Bq$—then the Lie bracket is the matrix commutator:

$$[A, B] = BA - AB \quad (1.5)$$

This demonstrates that the Lie bracket is fundamentally an expression of the noncommutativity of the underlying vector fields. Noncommutativity is connected to the idea of path dependence (or ordering, if we are still thinking in terms of matrices), a concept that we have seen before in the context of holonomy!

To be technical, the Lie bracket is an operator that is applied to elements of the Lie algebra $\mathfrak{g}$ corresponding to the Lie group $G$. In many contexts, the Lie algebra can be thought of as a generalization of velocity. For example, if our Lie group is $SO(3)$, which is the space of all 3D rotations, then the corresponding Lie algebra is $\mathfrak{so}(3)$, which is the space of all 3D angular velocities. Recall that, in general, the Lie algebra is identified with the tangent space at the identity element of the Lie group (meaning that, more technically, $\mathfrak{so}(3)$ is the space of all body-fixed angular velocities).

It can be proven that the Lie bracket of two angular velocity vectors $x, y \in \mathfrak{so}(3)$ is the standard vector cross-product: $[x, y] = x \times y$.

Note that the Lie bracket defines a differential sequence of motions that is com-
prised of periodic (AC) motions (in our differential-drive vehicle example, a combination of moving forwards/backwards and steering left/right) that results in a net (DC) motion (in our example, moving to the right).

[Non]involutivity

A distribution is called involutive if it is closed under the Lie bracket. Via the Frobenius theorem, we know that a distribution is integrable iff it is involutive[38]. therefore, a nonholonomic constraint is by definition not integrable (this actually follows from the fact that nonholonomic constraints do not reduce the dimensionality of the system configuration).

Any system with nonholonomic constraint(s) has some kind of “parallel-parking ability”—which means that one can use a clever periodic sequence of motions (gaits) to produce meaningful motion that would otherwise be unavailable. One might imagine that this property is useful in a robotics context.

Further, there is a deep relationship between underactuated and nonholonomic systems—nonholonomic systems are necessarily underactuated: We don’t control $g$ directly, but indirectly through the manipulation of $r$. This segues into the next subsection.

1.2.3 Controllability

Controllability of nonholonomic systems seeks to answer the following: by manipulating the available control inputs (usually, the shape variables $r$, assuming that the shape space is fully actuated), is it possible to drive the system to any point in the full configuration space (both the shape space and the orientation space)?

Rigorous treatment of the accessibility and controllability$^4$ of nonholonomic systems is given by Kelly and Murray[25]. In particular, they demonstrate that if the underlying (fiber) Lie group is abelian (i.e. the Lie group is commutative: the Lie bracket is identically zero), the net holonomy (i.e. the net result of the lifted tra-

$^4$These are subtly different concepts... technically, accessibility and controllability only differ for symmetric nonholonomic systems with drift.
jectory through the fiber space) associated with a closed trajectory $r(t)$ in the base space (the shape space) is directly proportional to an integral of some vector field (related to the curvature\cite{1} of the local connection) over the area enclosed by $r(t)$. This result is termed the area rule and is also discussed by Leonard\cite{31}\cite{32}. An example of a system in which the underlying Lie group is abelian is in fact the floating “space snake” proposed by Hatton and Choset\cite{21}, where $G = SO(2)$ (See Figure 1-2).

This result emphasizes the fact that “non-abelian” and “nonholonomic” are not synonymous! Just because the underlying Lie group is commutative, the system can still exhibit path dependence (holonomy)—the entire configuration group $Q$ can still be noncommutative.

The central theorem of noholonomic controllability is due to Chow; sometimes called Chow’s Theorem or the Chow-Rashevsky Theorem.\textsuperscript{5} Chow’s theorem is essentially a statement about the “directions” that a system can move in at a given configuration: if at every configuration $q \in Q$ these directions span the entire tangent space, then the system is fully controllable. Some of these directions may have to be generated with Lie brackets, or even Lie brackets of Lie brackets (these would be termed higher-order brackets). This process is often referred to as iterated Lie bracketing\cite{44}.

For non-abelian Lie groups—including $SO(3)$ which is the focus of this thesis, and $SE(2)$, which is one of the most common spaces for simple locomoting robots, such as the differential-drive vehicle, to live in—the area rule does not apply. However, Hatton and Choset\cite{21}, building upon the work of Radford and Burdick\cite{44}, have developed an approximate area rule, which they have termed the corrected body velocity integral, or $cBVI$.

Feedback control

Reyhanoglu and McClamroch\cite{46} conclude that a certain class of nonholonomic systems (a subset of the Snake’s Problem, actually) cannot be feedback stabilized about

\textsuperscript{5}The earliest reference I can find in the literature is to Rashevsky’s “Any two points of a totally nonholonomic space may be connected by an admissible line” which is in Russian and from 1938.
arbitrary equilibria using a smooth \((C^1)\) feedback control law. This makes some intu-
itive sense—a nonholonomic system can be fully controlled but has to do a “parallel-
parking wiggle” in order to move in certain directions; this wiggle cannot be captured
by standard feedback control laws. As a consequence, the development of robust feed-
back control laws for nonholonomic systems is complicated; fundamentally, a degree
of open-loop control is necessary. This is embodied in the control strategy proposed
by Leonard[32], which is comprised of open-loop control interspersed with periodic
closed-loop “checkups”.

1.2.4 Steering

Determining theoretical controllability is not the same thing as ascertaining the actual
control input trajectories that would provide useful motions, which we would term
open-loop motion planning or steering.

Brockett has shown that, for systems that require only one “level” of Lie bracketing,
sinusoidal inputs are optimal[8], in the sense that they minimize energy.

Murray and Sastry[39] expand upon Brockett’s result to demonstrate motion plan-
ning for systems that require higher order Lie brackets, showing that sinusoidal inputs
at integrally-related frequencies can be used to steer the system.

1.3 Attitude control via shape change

A mechanical system that conserves net angular and linear momentum, as aforemen-
tioned, has the mathematical structure of a symmetrical nonholonomic system with
symmetry. The underlying Lie group is \(G = SO(3) \subset SE(3)\), which is the space of
all 3D rotations.\(^6\) The corresponding Lie algebra is \(\mathfrak{so}(3)\), which is the space of all
3D angular velocities.

Because of this mathematical structure, we expect these sorts of mechanical sys-
tems to have a “parallel parking ability” which can be used to control their orientation

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\(^6\)We do not have to concern ourselves with linear translations because conservation of linear
momentum is always an integrable constraint, meaning that it reduces the dimensionality of the
underlying Lie group from \(SE(3)\) to \(SO(3)\).
1.3.1 Group structure

This representation (Equation 1.3) makes it clear that, for angular-momentum-conserving systems, the columns of the local connection $A(r)$ are themselves elements of the “angular velocity space” $\mathfrak{so}(3)$. The Lie bracket is the vector cross product (one can note that, for groups that are matrix spaces, the Lie bracket is equal to the matrix commutator, and the matrix commutator of two skew-symmetric matrices is equal to the cross product of the corresponding vectors). This sets up a natural isomorphism between $\mathfrak{so}(3)$ and $\mathbb{R}^3$—meaning that the skew-symmetric-matrix-representation with matrix commutator Lie bracket is equivalent (isomorphic) to the representation with angular velocity vectors $\Omega \in \mathbb{R}^3$ with vector cross product Lie bracket.

For these systems, the reconstruction equation (Equation 1.3) is equivalent to transforming the angular velocity vector from the affixed frame $C$ to the inertial frame $I$. If we represent $g$ with rotation matrices, Equation 1.3 becomes:

\[
\frac{d}{dt} [^I R^C] = ^I R^C \mathcal{I} \omega^C
\]

1.3.2 Prior work

The fields of geometric control theory and nohononomic mechanics are rich and well-developed. Marsden, Montgomery, and Ratiu[33] provide a comprehensive treatment of classical mechanical systems with symmetry; Bloch[5] provides a good overview of nonholonomic mechanics; Bullo and Lewis[10] provide a nice overview of the control theoretic aspect. We do not attempt to fully summarize the entire subject in this section, but rather to summarize the connection to shape change attitude control.

There exists a substantial body of literature dealing specifically with attitude control via shape change of a deformable body. The field was debatably initiated (though references to the Cat’s Problem existed in the literature before this, with the earliest theoretical treatment given by Rademaker and ter Braak[43]) by NASA, providing
Figure 1-11: Rademaker and ter Braak[43] did not have as good of a photographer as Marey.

Abb. 1.
Freier Fall in Rückenlage.
I. Normale Katze mit verschlossenen Augen.
Das Tier dreht sich in der Luft.
funding to researchers in the late 1960s (exemplified by Kane and Scher[22][23]). NASA was interested in obvious practical application to spaceflight. The work of Kane was then picked up in the early 1990s by physicists like Montgomery[36][37], Enos[15], and Shapere[50] who realized the connection between the Cat’s Problem and what physicists would call gauge theory (i.e. studying dynamical systems with useful symmetries, often in the context of particle physics). Shapere, with Wilczek who would go on to win a Nobel Prize in physics, had realized this gauge theoretic connection first for the low Reynolds number locomotion of deformable bodies[51]—in certain scenarios, the Stokes equations can be reduced to a set of linear nonholonomic constraints—and went on to write his PhD thesis on the *Gauge Mechanics of Deformable Bodies*[49].

This work united the Cat’s Problem with ideas in geometric mechanics and differential geometry (which had been applied to other gauge theoretic problems; particularly ideas about Lie algebras and controllability, developed in large part by Brockett[8]) and opened the field up to engineers—see the work of Krishnaprasad on geometric phase and the Cat’s Problem[29][30], McClamroch on the shape control of spacecraft[34][52] and Chen[11][12], who considered the Cat’s Problem as applied to satellites.

Leonard[31][32] proposes the model system of a spacecraft with a point-mass oscillator (which provides the ability to change shape) as an example of motion planning and control on Lie groups.

Ananthasuresh and Koh published some work in the early 2000s on what they term “pseudo-wheels”, which are in essence shape change attitude control actuators that are a specific instantiation of the Cat’s Problem[26][27][40].

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7 McClamroch and his students built the first real experiments dealing with shape change attitude control[4], as far as I know
Chapter 2

Background

In this chapter, we formalize some notions about angular-momentum-conserving systems that have been touched upon. Please refer to Appendix A for an overview of the notational formalisms employed in this thesis. Three dimensional kinematics can get very complicated, and it is useful to have a standardized notation for vectors, rotations, and other transformations.

2.1 Free deformable bodies

Free deformable bodies, as defined in this body of work, are simply dynamical systems comprised of elements that are constrained to always behave a certain way in relation to each other, that exhibit a certain important symmetry, and that conserve angular momentum.

2.1.1 Free systems

This important symmetry is translational and rotational invariance; in mathematical terms, free systems are invariant under the action of the group $SE(3)$, which is the group of all non-mirroring (direct isometries, orientation preserving) 3D translations and rotations. By $SE(3)$ invariance, we just mean that the dynamics of the free system do not care what the position or orientation in 3-space of the system is. A
free system also conserves angular momentum; this means that the free system does not interact at all (does not apply any forces or torques to) with anything else in 3-space—it is free in this sense.

2.1.2 Deformable bodies

*Deformable* is somewhat trickier to quantitatively define. It is not hard to intuitively define: a deformable body can change its shape, like a folding chair, or a water balloon, or a pair of scissors, or even a house cat. Some deformable bodies can change shape internally, using only internal forces and torques (*e.g.* the house cat), but some deformable bodies can only change shape passively, due either to applied external forces (particular solutions; *e.g.* unfolding the folding chair) or to dynamical effects (homogeneous solutions; *e.g.* the water balloon vibrates if you poke it). It is clear that the notion of deformability is intrinsically linked to the ability of a system to qualitatively change shape.

So perhaps we can define deformable bodies by first defining their dual: rigid bodies. A rigid body can be thought of as a system of infinitesimal particles that are related by a very specific constraint: that the relative positions between all the infinitesimal particles are always the same; intuitively, the shape of a rigid body cannot change. This allows us to assign lumped parameters to rigid bodies such as mass and moment of inertia. Similarly, we might consider linear springs, dashpots, *etc.* as *lumped parameter* constraints.

Deformable bodies account for all the other possible mass-conserving systems of particles. The slightly vague stipulation that deformable bodies must still be “whole” or “cohesive” seems to me to be a kind of heuristic based on the types of deformable bodies that we tend to experience: Is a system of particles that interact only via electric charge a deformable body? Maybe not in the practical sense, but maybe yes in the mathematical sense. It also might help to somewhat-arbitrarily impose some sense of mass conservation: deformable bodies cannot lose mass. In this sense, we could consider a half-full water-bottle to be a deformable system (if capped), but a control-volume section of a pipe conveying water not to be. Similarly, if we
take a rocket ship to be only the rocket-with-onboard-fuel, it is not a deformable body as defined, as it is constantly ejecting exhaust mass, but we might consider the rocket-fuel-exhaust system to be a deformable body, since mass conservation is always retained.

Obviously, the particular nature of a deformable body is very much determined by the nature of the relationship (or constraint) between constituent infinitesimal particles. Perhaps the archetypal deformable body is a rubber band. If we model a rubber band using linear elasticity theory, we can think of it as a collection of infinitesimal particles joined by linear springs. Similarly, an Euler-Bernoulli beam can be thought of as a series of infinitesimal particles joined by torsional springs; a fluid can be thought of as a collection of infinitesimal particles that transmit normal and shear stresses to neighboring particles. These are all examples of continuums, in that one typically analyzes the behavior not of the individual infinitesimal particles that comprise the system but of the bulk.

But a deformable body need not be a continuum body in the strict sense: consider a pair of scissors, or two rigid rods joined by a revolute joint. This is obviously not a continuum in the sense described above, but is still a deformable body.
Shape of deformable bodies

Deformable bodies are characterized by their *shape*, which intuitively prescribes the relative position between all the component particles \( P_i \) that comprise the deformable body (we see that the shape of a rigid body is always constant, since for a rigid body this set of relative positions is constrained to always be constant). In general, we represent the shape with a vector \( r \) of variables that parameterizes the shape; in the example of the scissors, we could say \( r \triangleq \theta \). Also in general, the shape representation \( r \) is not unique.

We say that a point \( Q \) is fixed to the body if its position \( r_{Q/P} \), relative to all points \( P_i \) in the body is a function of the shape representation \( r \) only.

The distinction between the *shape* itself and the corresponding *shape representation* is subtle, both semantically and mathematically. It is something like the distinction between a vector, as a more abstract object that lives somewhere on a manifold and has unambiguous direction and magnitude, and the set of coordinates that parameterize the same vector. Often, we are able to make statements and do some calculations without parameterizing, but, at least in engineering, we need to plug in values eventually and to do so we need to parameterize. The subset of mathematics that is best situated to resolve these sorts of ambiguities and confusions is group theory, which explains the prevalence of group theoretic notation and rhetoric in the relevant literature.

In the example case of scissors, we need just one parameter to define the shape: the opening angle \( \theta \) of the scissors. In the case of the rubber band, the shape is defined by the position \( r_{P_i/G} \) of each infinitesimal particle \( P_i \), relative to some reference point \( Q \) fixed to the band, and so we need an infinite number of variables to define the shape: Continuum bodies are infinite dimensional. In general, the dynamics of \( n \)-dimensional deformable bodies are determined by \( 2n^{th} \) order ordinary differential equations, while the dynamics of continuum bodies are determined by partial differential equations (*e.g.* the Navier-Stokes equations for a fluid, or the Euler-Bernoulli equations for a beam).
One can also envision deformable bodies that are combinations of both continuum and rigid lumped-parameter elements: for example, consider an Euler-Bernoulli beam with a rigid body attached to the end via a linear spring. In this case, the system dynamics are determined by coupled ODEs and PDEs.

**Position and orientation of deformable bodies**

The *global position* of a deformable body can be defined by keeping track of the position $r_{G/O}$ of one infinitesimal point $G$ fixed to the body; it sometimes makes sense to define the center of mass of the deformable body to be this reference point, just as is the case with a rigid body.

But while it is straightforward to define one orientation for a rigid body $\mathcal{R}$—represented by the rotation transformation $^T\mathbf{R}^\mathcal{R}$ necessary to rotate the body from a reference orientation (which is identified with the fixed inertial frame $\mathcal{I}$) to the real orientation as the reference point stays fixed—it is nontrivial to define the same for a deformable body, for the simple reason that the relative position of each constituent element of the deformable body changes as the shape changes. In a simple case—the scissors, again, for example—we might keep track of the orientation of just one of the rigid elements in the standard rigid-body way, and in this way define an orientation for the entire deformable body; but it is not immediately clear how we would define the orientation of a rubber band, which has no rigid elements that are "easy" to keep track of!

To try and tackle this issue quantitatively, we define the *global orientation* of the deformable body as the orientation of some reference frame $\mathcal{C}$ relative to the inertial frame $\mathcal{I}$. However, in order to do useful engineering mathematics, we also need to pick a suitable representation $g$ for the orientation of $\mathcal{C}$, much in the same way that we chose a representation for the shape $r$. This amounts to choosing a set of parameters $g$ that defines the rotation transformation $^T\mathbf{R}^\mathcal{C}$. This could directly be a rotation matrix ($g \triangleq ^T\mathbf{R}^\mathcal{C}$)—or it could be some other set of parameters, such as a quaternion or set of Euler angles ($^T\mathbf{R}^\mathcal{C}(\phi, \theta, \psi) \rightarrow g \triangleq [\phi, \theta, \psi]^\top$) that defines the same rotation.

The only criteria for $\mathcal{C}$ is that it is *fixed* to the deformable system; that is, $\mathcal{C}$ can be
defined only by the position $\mathbf{r}_{P_i/Q}$ of points $P_i$ on the body relative to a point $Q$ fixed to the body in the aforementioned sense. Therefore, if we know the global position and orientation as well as the system shape $\mathbf{r}$, the position of any point on the system can be determined. The simplest example of this sort of frame is the body-fixed frame $\mathbf{A}_i$ of any component body in a deformable system comprised of multiple rigid bodies joined by kinematic constraints: $\mathbf{T}_\mathbf{C} \triangleq \mathbf{T}_\mathbf{A_i}$. Another appropriate candidate for a $\mathbf{C}$ frame is any of the $\mathbf{A}_i$ rotated or translated by an amount that is a function of the system shape: $\mathbf{T}_\mathbf{C} \triangleq \mathbf{T}(\mathbf{r}) \mathbf{T}_\mathbf{A_i}$. We refer to $\mathbf{C}$ as the orienting frame.

2.1.3 Actuation and reduced dynamics

Because we are interested in control, we are interested in deformable bodies that are internally actuated: they have some means of altering their shape via internal forces or torques, termed control torques. We are also interested in investigating systems where useful changes in shape can be sustained or effected by dynamical effects, specifically stiffness.

Therefore, we consider the case where the shape space of the system is not fully actuated: i.e. where we do not directly actuate all the parameters in $\mathbf{r}$, but instead let some evolve passively via dynamic constraints (e.g. springs and dashpots inside the joints). In this case, we refer to the shape space as underactuated. Note that the kinematic case and dynamic case are equivalent for fully actuated shape spaces, since one could always use the inverse system dynamics to calculate the joint torque trajectory necessary to create a desired joint trajectory.

2.1.4 Boundedness of the shape space

As alluded to before, we would like to consider the case where the shape space is bounded. This corresponds to many real deformable bodies: no joint in the human body allows for truly continuous rotation, for example.
2.2 Free kinematic chains

We consider the kinematics of free kinematic chains, which are a subset of free deformable bodies. That is, we would like to describe the orientation of a set of rigid bodies connected internally (as in, within the set) by particular rotational kinematic constraints (*e.g.* spherical joints, revolute joints, universal joints, constant-velocity joints, etc.), floating in a torque-free environment (*e.g.* an articulated space satellite or robot). It is not too difficult to extend this paradigm to include translational joints (*e.g.* slider joints) as well, but we do not focus on that case here, as it further complicates the—already complex—algebra, and is not necessary to derive interesting results. We would like to apply control inputs, which might, in the kinematic case, be the time-trajectory of the shape \(r(t)\), or could also, in the dynamic case, be the time-trajectory of the generalized forces \(\Xi(t)\) associated with the shape, and ascertain the resultant system performance, quantified as a performance metric on the ability of the system to change orientation in inertial space.

In this work, we consider several model systems, all examples of free kinematic chains: The No-Twist Cat, the Universal Joint Cat, and the Dual Seesaw.

**No-Twist Cat**

First proposed by Kane and Scher[22], and thus also known as the Kane-Scher Cat. This is the simplest mathematical model that captures the salient dynamics of the overturning housecat.

**Universal Joint Cat**

First proposed by Enos[15]. This is an extension of the No-Twist Cat, interesting primarily because it the inertially-symmetric version is fully controllable.

**Dual Seesaw**

This system is inspired by the work of Ananthasuresh, Koh, and Ostrowski[28][26][27][40]. With this system in particular, we are interested in the small-amplitude (linearized)
kinematics and dynamics.
Figure 2-2: Kane's No-Twist Cat. [22]
Figure 2-3: Enos’ Universal Joint Cat.[15]

Figure 2-4: Pseudo-wheels are an example of a bounded shape change attitude control actuator[40].
Chapter 3

Kinematics

We derive the kinematic equations for describing free kinematic chains. We do not formalize in terms of geometric mechanics, yet. We also present the kinematics for several example systems.

3.1 Equations for free kinematic chains

There exists some fixed inertial reference frame $\mathcal{I}$.

We can generally state the system angular momentum about the center of mass $G$, expressed in the $\mathcal{A}_1$ frame:

$$h_G = \sum_{i}^{N} \mathcal{A}_i \mathbf{R}^{\mathcal{A}_i} \mathbf{I}_{\mathcal{A}_i} \mathcal{I} \mathbf{\omega}^{\mathcal{A}_i} + \mathcal{A}_i \{ \mathbf{r}_{\mathcal{A}_i/G} \times m_i \mathbf{v}_{\mathcal{A}_i/G} \} \quad (3.1)$$

Where $A_i$ is the center of mass point of body $\mathcal{A}_i$ and we have used the notation $\mathcal{F}\{\mathbf{v}\}$ to denote vector $\mathbf{v}$ expressed in frame $\mathcal{F}$. The inertia tensor of body $\mathcal{A}_i$, expressed in the body-fixed frame, is denoted $\mathbf{I}_{\mathcal{A}_i}$.

We can use conservation of linear momentum to solve for the position of some known reference point $P$ fixed to the system, such that any point $Q$ on the system bodies, relative to $G$, can be expressed as: $\mathbf{r}_{Q/G} = \mathbf{r}_{Q/P} + \mathbf{r}_{P/G}$, where $\mathbf{r}_{Q/P}$ is some known function of the system shape. In the parallel chain case, the most obvious choice is the pivot point, while in the serial chain case any pivot point is a suitable
This allows us to solve directly for the position of each body’s mass-center relative to the system mass-center, using the known system shape:

\[
\mathbf{r}_{Ai/G} = \mathbf{r}_{Ai/P} - \frac{1}{m} \sum_{j}^{N} m_{i} \mathbf{r}_{Aj/P}
\]  

(3.3)

Via Galilean invariance and conservation of linear momentum, we know that we can always choose an inertial reference frame \( I \) (origin \( O \)) such that \( \mathbf{v}_{G/O} = 0 \) without any loss of generality. We can therefore write the linear velocity of each component mass-center as:

\[
\mathbf{v}_{Ai/G} = \mathbf{v}_{P/G} + \mathbf{\omega}^{A_{i}} \times \mathbf{r}_{Ai/P}
\]  

(3.4)

Where the velocity of \( P \) is:

\[
\mathbf{v}_{P/G} = -\frac{1}{m} \sum_{i}^{N} m_{i} \mathbf{v}_{Ai/P}
\]  

(3.5)

The relative velocity between the component mass-centers and the pivot point \( P \) can be calculated from the shape \( \mathbf{A}_{i} \mathbf{R}^{A_{i}} \) and shape velocity \( \mathbf{\omega}^{A_{i}} \).

To simplify algebra, most studies in the literature assume that all mass-centers are coincident, such that \( P = G \) and thus \( \mathbf{r}_{Ai/G} = 0 \). We will also make this assumption in this thesis.

### 3.1.1 Parallel chain

Our system is comprised of \( N \) bodies \( A_{i} \) joined at their mass-center \( G \). We orient the system relative to some frame \( C \) that is fixed to the free kinematic chain. For the purpose of notation, we consider the frame \( A_{0} \) to be equivalent to the inertial reference frame \( I \).
Conservation of total angular momentum in the $A_k$ frame (a nonholonomic constraint):

$$h_G = \sum_{i=1}^{N} A_k R A_i I_{A_i} \omega^{A_i} = 0 \quad (3.6)$$

Via the angular velocity addition theorem[3]:

$$\omega^{A_i} = [A_k R A_i]^T \omega^{A_k} A_k \omega^{A_i} + A_k \omega^{A_i} \quad (3.7)$$

Define the shape $r \in M$ to be the vector of parameters that defines the orientation between each body and the $k^{th}$ body (arbitrarily chosen):

$$A_k R A_i = A_k R A_i(r); \ i \neq k \quad (3.8)$$

We can, without loss of generality, choose $k = 1$ such that the orientation of all bodies $A_i$ are referenced back to the first body $A_1$.

Define the shape velocity $\dot{r} \in TM$ as the time-derivative of the parameters $r$. (For more complicated shape manifolds, this must be amended slightly.)
The angular velocity of $A_i$ in $A_j$ can be calculated directly from the rotation matrix from $A_j$ to $A_i$:

$$\begin{bmatrix} A_j \omega^{A_i} \times \end{bmatrix} = \left[ A_j R^{A_i} \right]^T \frac{d}{dt} \left[ A_j R^{A_i} \right]$$  \hspace{1cm} (3.9)

This implies that we can write a "Jacobian" of sorts, which relates the angular velocity of $A_i$ in $A_1$ to the shape velocity:

$$A_i \omega^{A_i} = J_i(r) \dot{r}$$  \hspace{1cm} (3.10)

Use the constraint equation (conservation of angular momentum) to solve for the body velocity $\tau \omega^{A_1} \in so(3) = \mathbb{R}^3$ as a function of the shape velocity.

$$\tau \omega^{A_1} = -\left( I_{A_1} + \sum_{i=2}^{N} A_i R^{A_i} I_{A_i} \left[ A_i R^{A_i} \right]^T \right)^{-1} \left( \sum_{i=2}^{N} A_i R^{A_i} I_{A_i} J_i \right) \dot{r}$$  \hspace{1cm} (3.11)

Since the $C$ frame is fixed to the free kinematic chain, we can write:

$$\tau \omega^C = \left[ A_1 R^C \right]^T \tau \omega^{A_1} + A_1 \omega^C$$  \hspace{1cm} (3.12)

Where $A_1 R^C = A_1 R^C(r)$ and the angular velocity can be derived directly from the rotation matrix. Thus, it is possible to write the body-fixed angular velocity of the $C$ frame in the following form, in terms of the body velocity $\xi$:

$$\xi \triangleq \tau \omega^C$$  \hspace{1cm} (3.13)

$$\xi = -A^C(r) \dot{r}$$

This is equivalent to determining the local connection $A^C(r)$ for the $C$ frame.

Define a matrix parameterization of the world orientation $g \in SO(3)$:

$$g \triangleq \tau R^C$$  \hspace{1cm} (3.14)
Define the world velocity $\dot{g} \in \mathfrak{so}(3) = \mathbb{R}^3$:

$$
\dot{g} \triangleq g\xi = {}^T R_c T_\omega A_1
$$

(3.15)

In fact, this definition, combined with the expression for the body velocity, produces the complete equation of motion for a free deformable chain whose orientation is specified by a rotation matrix:

$$
\dot{g} = -gA(r)\dot{r}
$$

(3.16)

We have seen this equation before (in fact, it is the reconstruction equation for matrix Lie groups, given in Equation 4.4)! This equation can always be integrated numerically (and sometimes can be solved exactly), given a specific shape trajectory $r(t)$, thought of as a control input. The whole field of study of control on Lie groups is really concerned with solutions to this particular equation. Equations of analogous form show up in other (fully kinematic) robotics contexts: snakes, swimmers, differential-drive wheeled vehicles, etc.

### 3.1.2 Serial chain

Figure 3-2: A serial free kinematic chain.

We also might want to define our system shape as if the bodies were linked in serial, rather than in parallel to the the first body. This would greatly simplify the modeling of a snake-like object, for example. In this case, it is natural to take the
shape to be defined by the relative orientation between sequential bodies:

\[ A_{i-1} R_{A_i} = A_{i-1} R_{A_i}(\tau) \quad (3.17) \]

We can calculate the relative angular velocities between bodies using the same formula as before (Equation 3.9). Again appealing to the angular velocity addition theorem, a compact notation for the inertial angular velocity of body \( A_i \) is:

\[
I \omega^{A_i} = \sum_{j=0}^{i-1} [A_{i-j} R_{A_i}]^T A_{i-j-1} \omega^{A_{i-j}} \\
I \omega^{A_i} = \sum_{j=0}^{i-2} [A_{i-j} R_{A_i}]^T A_{i-j-1} \omega^{A_{i-j}} + [A_1 R_{A_i}]^T I \omega^{A_1} \quad (3.18)
\]

Where we keep in mind that \( A_k R_{A_k} = 1 \), the \( 3 \times 3 \) identity matrix, and that \( A_0 \triangleq I \).

Using the properties of rotation and transformation matrices, we can also construct a compact notation for the rotation matrices:

\[
A_{i-j} R_{A_i} = \prod_{k=i-j}^{i-1} A_k R_{A_{k+1}} \quad (3.19)
\]

Where we are very careful to *always multiply the terms on the right side* due to the noncommutativity of 3D rotation matrices.

Conservation of angular momentum still holds. We plug the expression for the inertial angular velocities of the \( A_i \) (Equation 3.18) into Equation 3.6:

\[
h_G = \sum_{i=1}^{N} A_1 R_{A_i} I_{A_i} \left( \sum_{j=0}^{i-2} [A_{i-j} R_{A_i}]^T A_{i-j-1} \omega^{A_{i-j}} + [A_1 R_{A_i}]^T I \omega^{A_1} \right) = 0 \quad (3.20)
\]

Be careful about negative indices, but for now just recognize that anything with a negative index is zero (the \( i = 1 \) case corresponds to calculating the angular momentum of \( A_1 \) in the \( A_1 \) frame, so it’s clear that the second sum should be zero). We
can rearrange the conservation law:

\[
\left( \sum_{i=1}^{N} A_i R^{A_i} I_{A_i} \left[ A_i R^{A_i} \right]^T \right) I_{A_i} = - \sum_{i=2}^{N} A_i R^{A_i} I_{A_i} \left( \sum_{j=0}^{i-2} A_{i-j} R^{A_{i-j}} \right) A_{i-j-1} \left[ A_{i-j} R^{A_{i-j}} \right]^T \omega^{A_{i-j}}
\]

The algebra is starting to get to be too much, but we now have enough to calculate the local connection for the serial case.

3.1.3 \( N = 2 \)

We consider the free kinematic chain where just two rigid bodies \( A_1 \triangleq B \) and \( A_2 \triangleq A \) are coupled by any type of spherical joint. Both bodies are assumed to be inertially-axisymmetric, but \( not \) identical, such that we can write their inertia tensors as:

\[
I_B = \begin{bmatrix}
    a & 0 & 0 \\
    0 & a & 0 \\
    0 & 0 & b \\
\end{bmatrix}
\]

\[
I_A = \begin{bmatrix}
    c & 0 & 0 \\
    0 & c & 0 \\
    0 & 0 & d \\
\end{bmatrix}
\]

Note that this is more general than the configuration considered by Kane[22] or subsequent authors in the literature (Montgomery[37] and Enos[16] chief among them), which assumes that both bodies are both axisymmetric and identical. The entries of the inertia matrix are constrained by what is physically-realizable: \( a, b, c, d > 0 \), while \( b < 2a \) and \( d < 2c \).

In this case, the serial and parallel forms converge, and the local connection for the \( B \) frame reduces to:

\[
A^B(r) = (I_B + g R^A I_A \left[ g R^A \right]^T)^{-1} g R^A I_A J(r)
\]
Where \( r \) is the set of variables that parameterizes \( B^A R^A \) and \( \dot{r} \) their time-derivatives.

### 3.1.4 \( n = 2 \)

We now ask what the minimum necessary dimension \( n \) of the shape space \( M \) is, in order to achieve controllability in \( SO(3) \); Koh\cite{28} says that it is 2. For this reason, and also since Radford and Burdick's area rule approximation\cite{44} has the nicest geometric interpretation when \( n = 2 \), we focus primarily on systems with two-dimensional shape spaces. The shape representation becomes \( r = [r_1, r_2]^T \).

### 3.2 Model systems

We derive the kinematics of the model systems considered in this thesis. All are free kinematic chains.

Figure 3-3: The model systems. All mass-centers are assumed coincident but are depicted exploded for clarity.
3.2.1 Ball Joint Cat

This is comprised of two rigid bodies connected at their mass-center by a 3-DOF spherical joint. Both the Universal Joint Cat and the No-Twist Cat are subsets. Fernandes considers this system, with the system shape parameterized by Cayley parameters\[17\].

We could parameterize the shape with a normalized quaternion $r \triangleq \hat{\mathbf{q}} = [q_r, q_i, q_j, q_k]^{\top}$, which can be expressed in terms of the Euler axis of rotation $\hat{\mathbf{r}} = ^{B}\{[b_1, b_2, b_3]^{\top}\}$ and the angle of rotation $\theta$:

\[
\begin{align*}
q_r &= \cos \frac{\theta}{2} \\
q_i &= b_1 \sin \frac{\theta}{2} \\
q_j &= b_2 \sin \frac{\theta}{2} \\
q_k &= b_3 \sin \frac{\theta}{2}
\end{align*}
\] (3.24)

Such that $\hat{\mathbf{q}} = [\cos \frac{\theta}{2}, \sin \frac{\theta}{2}\hat{\mathbf{r}}]^{\top}$. When a unit quaternion is expressed in this form, its components are referred to as the *Euler parameters*\[3\] of the rotation.

Alternatively, we could parameterize the shape with a rotation matrix, which can also be expressed in terms of $\hat{\mathbf{r}}$ and $\theta$ using Rodrigues’ rotation formula\[3\]:

\[
r \triangleq ^{B}\mathbf{R}^{A} = \mathbf{I} + \sin \theta [\hat{\mathbf{r}} \times] + (1 - \cos \theta)[\hat{\mathbf{r}} \times]^{2}
\] (3.25)

In this case, we would say that $r$ is equal to the 9 elements of the *direction cosine matrix*.

Finally, we could also parameterize the shape by parameterizing $^{B}\mathbf{R}^{A}$: If we let $\hat{\mathbf{r}} = [\cos \psi \cos \phi, \cos \psi \sin \phi, \sin \psi]^{\top}$, we can take the shape representation to be $r \triangleq [\theta, \phi, \psi]^{\top}$. This leads to a parameter Jacobian of:
\[
J = \begin{bmatrix}
    (\cos \theta - 1) \sin \phi - \cos \phi \sin \theta \sin \psi & - \cos \psi (\sin \theta \sin \phi + (\cos \theta - 1) \cos \phi \sin \psi) & \cos \phi \cos \psi \\
    - \cos \theta \cos \phi + \cos \phi - \sin \theta \sin \phi \sin \psi & \cos \psi (\cos \phi \sin \theta - (\cos \theta - 1) \sin \phi \sin \psi) & \cos \psi \sin \phi \\
    \cos \psi \sin \theta & -2 \cos^2 \psi \sin^2 \frac{\theta}{2} & \sin \psi 
\end{bmatrix}
\]

Such that \( \delta \omega^A = J(r) \dot{r} \). We note that \( \det(J) = -4 \cos \psi \sin^2 \frac{\theta}{2} \), which indicates that this parameterization of the shape is not adequate if \( \cos \psi = 0 \) or \( \sin \frac{\theta}{2} = 0 \). If either of these conditions is met, we run into a singularity (much like gimbal lock with Euler angles). This shows that, if we take the shape parameters to be control inputs, arbitrary relative angular velocity \( \delta \omega^A \) is not achievable if \( \cos \psi = 0 \) or \( \sin \frac{\theta}{2} = 0 \).

However, despite the drawbacks of this particular parameterization of the ball joint, we will be considering shape space trajectories that do not intersect with the singular points, and therefore this parameterization should be adequate for our purposes.

### 3.2.2 No-Twist Cat

This kinematic chain is illustrated in Figure 3-3a.

The joint between bodies is implemented as a no-twist joint, which is, intuitively enough, a joint that allows no relative twisting between the two bodies. This type of joint was proposed by Kane[22] so as to constrain the "backbone" of the model cat in an anatomically-correct manner (or to not allow it to "break its back" in the words of Montgomery[37]). Kane provides a somewhat complex overview of the kinematical constraints that define the no-twist joint; as an engineer, I find it more intuitively useful to recognize that the no-twist joint is in fact identical to the constant-velocity

---

1 Note that if we take our shape to be the 4 Euler parameters or the 9 DCM entries, no singularities appear and arbitrary relatively angular velocities can be represented at arbitrary shapes. However, it wouldn't be appropriate to think of the Euler parameters or DCM entries as control inputs, since they cannot be specified independently. Therefore, if we would like to fully actuate the shape space without any singularities, we must take the control input to be the relative angular velocity directly (seems hard to achieve in practice).
joint (alternatively called the CV joint) that is often found on rotational driveshafts. That this type of joint already existed in engineering before Kane and Scher’s paper was apparently lost on Kane, Montgomery, Enos, and a host of other physicists that attempted the Cat’s Problem post-1969, but in a well-deserved homage, I will continue to refer to this particular mechanical coupling as a no-twist joint.

We provide an intuitive interpretation of the no-twist joint: The instantaneous axis of rotation between $A$ and $B$ is constrained to always lie in both the $b_1$-$b_2$ plane plane and the $a_1$-$a_2$ plane. Even though it is not correct to think of 3D rotations adding like vectors, any relative rotation between $A$ and $B$ that has an Euler axis with a nonzero $b_3$ (or $a_3$) component intuitively involves some twisting about the $b_3$ axis (or the $a_3$ axis). It is straightforward to verify that this interpretation of the no-twist joint is mechanically identical to Kane’s.

We can therefore parameterize the no-twist joint with two parameters: $\phi$ defines the angle between the normalized axis of rotation $\hat{t}$ (such that $\hat{t} = \cos \phi b_1 + \sin \phi b_2$) and the axis $b_1$, while $\theta$ is the amount $A$ is rotated about $\hat{t}$. This allows us to succinctly write the rotation matrix between frames, using Rodrigues’ rotation theorem[3]:

$$B^A = I + \sin \theta [\hat{t} \times] + (1 - \cos \theta) [\hat{t} \times]^2$$

$$B^A(\phi, \theta) = \begin{bmatrix}
(c\theta - 1) \sin^2 \phi + 1 & -(c\theta - 1) \cos \phi \sin \phi & \sin \theta \sin \phi \\
-(c\theta - 1) \cos \phi \sin \phi & (c\theta - 1) \cos^2 \phi + 1 & -\cos \phi \sin \theta \\
-\sin \theta \sin \phi & \cos \phi \sin \theta & \cos \theta
\end{bmatrix}$$

(3.27)

When $\theta = 0$, bodies $A$ and $B$ are aligned such that their principle axes coincide.

The relative angular velocity between $A$ and $B$ is calculated in the usual manner:

$$[B^A \times] = [B^A]^T \frac{d}{dt} [B^A]$$

(3.28)

Remember that, per our notational convention, this angular velocity is expressed
with components in the \( \mathcal{A} \) frame. We can express the angular velocity in the following form:

\[
\mathbf{B_\omega^A} = J(\phi, \theta)[\dot{\phi}, \dot{\theta}]^T
\]

(3.29)

Where the "Jacobian" is:

\[
J(\phi, \theta) = \begin{bmatrix}
-\sin \theta \sin \phi & \cos \phi \\
\cos \phi \sin \theta & \sin \phi \\
\cos \theta - 1 & 0
\end{bmatrix}
\]

(3.30)

And we can calculate the local connection for the \( \mathcal{B} \) frame, using the expression for two-body, free, kinematic chains (Equation 3.23):

\[
\mathbf{A^B}(\phi, \theta) = (I_B + \mathbf{S}^{\mathcal{A}} I_{\mathcal{A}} \left[ \mathbf{S}^{\mathcal{A}} \right]^T)^{-1} \mathbf{S}^{\mathcal{A}} I_{\mathcal{A}} J(\phi, \theta)
\]

\[
= \begin{bmatrix}
-2 \frac{s(\theta)}{r(\theta)} \sin \phi \sin \theta & \mu \cos \phi \\
\frac{2s(\theta)}{r(\theta)} \cos \phi \sin \theta & \mu \sin \phi \\
\frac{4s(\theta)}{r(\theta)} \sin^2 \frac{\theta}{2} & 0
\end{bmatrix}
\]

(3.31)

Such that \( \mathbf{T_\omega^B} = -\mathbf{A^B}(\phi, \theta)[\dot{\phi}, \dot{\theta}]^T \), and where:

\[
\begin{align*}
    r(\theta) &= bc + bd + 2cd + a(2b + c + d) - (a - b)(c - d) \cos 2\theta \\
    s(\theta) &= d(b + c) + b(c - d) \cos \theta \\
    t(\theta) &= c(a + d) + a(c - d) \cos \theta \\
    \mu &= \frac{c}{a + c}
\end{align*}
\]

(3.32)

In the inertially-symmetric case (bodies \( \mathcal{A} \) and \( \mathcal{B} \) are equivalent), Enos has shown that the No-Twist Cat is not fully controllable. This motivates the examination of another joint configuration:
3.2.3 Universal Joint Cat

If we allow that the two constituent bodies of the Cat are connected by a universal joint, as depicted in Figure 3-3b, Enos has shown that this free, deformable system is, in fact, fully controllable[15]. As far as we know, no work has been done to fully exploit this feature; the only work besides Enos' which explicitly considers the Universal Joint Cat does generate open loop control trajectories, but does so only for the problem of overturning, and not for arbitrary changes in orientation[17].

We consider the Universal Joint Cat, which is the model proposed by Enos and is comprised of two bodies \( B \) and \( A \) joined by a universal joint at the center of mass. The universal joint is a 2-DOF joint, which allows body-fixed (intrinsic) rotation of \( A \) about \( b_1 \) through angle \( \gamma_1 \) and about \( a_2 \) through angle \( \gamma_2 \). This defines the shape representation \( r \triangleq [\gamma_1, \gamma_2]^T \).

Geometrically nonlinear kinematics

Further, we consider the case where both bodies \( B \) and \( A \) are radially symmetric, and thus characterized by just one inertia each: \( I \) and \( J \), respectively:

\[
\begin{align*}
I_B &= I_1 \\
I_A &= J_1
\end{align*}
\]  

(3.33)

We can write the rotation matrix that rotates an intermediate frame \( F \) from being coincident with frame \( B \) to being coincident with frame \( A \) as:

\[
^B R^A = \begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \gamma_1 & -\sin \gamma_1 \\
0 & \sin \gamma_1 & \cos \gamma_1
\end{bmatrix} \begin{bmatrix}
\cos \gamma_2 & 0 & \sin \gamma_2 \\
0 & 1 & 0 \\
-\sin \gamma_2 & 0 & \cos \gamma_2
\end{bmatrix}
\]  

(3.34)

This yields a Jacobian.
\[ J(\gamma_1, \gamma_2) = \begin{bmatrix} \cos \gamma_2 & 0 \\ 0 & 1 \\ \sin \gamma_2 & 0 \end{bmatrix} \]  

(3.35)

We take the orienting frame \( B \) and calculate the local connection, using Equation 3.23, as:

\[ A^B(\gamma_1, \gamma_2) = \mu \begin{bmatrix} 1 & 0 \\ 0 & \cos \gamma_1 \\ 0 & \sin \gamma_2 \end{bmatrix} \]  

(3.36)

Where \( \mu = \frac{I}{J+I} \). Such that \( T^B = \omega^B = -A^B(\gamma_1, \gamma_2)[\gamma_1, \gamma_2]^T \).

**Small-angle kinematics**

We also consider the case where the amplitude of shape deformation always remains small \( (r = \mathcal{O}(\epsilon)) \), but the inertias of bodies \( A \) and \( B \) are not assumed radially symmetric, only axially symmetric, such that we can compare more readily to the No-Twist Cat and the Dual Seesaw.

We calculate an approximate local connection by discarding \( \mathcal{O}(\epsilon^3) \) terms and higher:

\[ A^B(\gamma_1, \gamma_2) = \mu \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \gamma_2 \left( \frac{1}{\mu} - 1 \right) & \frac{a+c}{b+d} \end{bmatrix} \]  

(3.37)

**3.2.4 Dual Seesaw**

This free kinematic chain is comprised of three bodies—\( B, S_1, \) and \( S_2 \)—constrained in parallel. An illustration is presented in Figure 3-3c.

The “payload” \( B \) is axisymmetric about its principle 3-axis.
While the first "seesaw" $S_1$ is axisymmetric about its principle 2-axis:

$$I_S = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{bmatrix}$$ (3.38)

And the second $S_2$, identical to $S_1$ but axisymmetric about its principle 1-axis:

$$I_{S_2} = \begin{bmatrix} e & 0 & 0 \\ 0 & \delta & 0 \\ 0 & 0 & e \end{bmatrix}$$ (3.39)

Each accessory body $S_i$ is constrained to rotate only about an axis $b_i$ by an angle $\theta_i$. We can easily write the rotation matrices in terms of the shape $r \triangleq [\theta_1, \theta_2]^T$:

$$^gR^{S_1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta_1 & -\sin \theta_1 \\ 0 & \sin \theta_1 & \cos \theta_1 \end{bmatrix}$$ (3.40)

$$^gR^{S_2} = \begin{bmatrix} \cos \theta_2 & 0 & \sin \theta_2 \\ 0 & 1 & 0 \\ -\sin \theta_1 & 0 & \cos \theta_2 \end{bmatrix}$$

We calculate a Jacobian matrix for each accessory body:
\[
^B J_{S_1} = \begin{bmatrix}
1 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0
\end{bmatrix}
\]

\[
^B J_{S_2} = \begin{bmatrix}
0 & 0 \\
0 & 1 \\
0 & 0
\end{bmatrix}
\]

(3.42)

Such that \(^B \omega_{S_i} = ^B J_{S_i} [\dot{\theta}_1, \dot{\theta}_2]^T\).

We calculate the local connection using Equation 3.11, referenced to the \(B\) frame:

\[
A^B = (I_B + ^B R_{S_1} I_{S_1} [^B R_{S_1}]^T + ^B R_{S_2} I_{S_2} [^B R_{S_2}]^T)^{-1} (^B R_{S_1} I_{S_1} ^B J_{S_1} + ^B R_{S_2} I_{S_2} ^B J_{S_2})
\]

(3.43)

Keeping only \(O(\epsilon^2)\) terms:

\[
A^B(\theta_1, \theta_2) = \mu_S \begin{bmatrix}
1 & 0 \\
0 & 1 \\
-\frac{e-\delta}{b+2e} \theta_2 & \frac{e-\delta}{b+2e} \theta_1
\end{bmatrix}
\]

(3.44)

Where \(\mu_S = \frac{e}{a+e+\delta}\).
Chapter 4

Geometry

We would like to describe the kinematics of free kinematic chains from a geometric mechanics perspective, with the aim of understanding the types of useful motion that the system can achieve, particularly motions related inherently to the nonholonomic nature of the system. Geometric mechanics provides a natural mathematical framework for understanding locomotion problems such as the Cat’s Problem, especially in terms of open-loop motion planning. A theoretical underpinning for the geometrical understanding of locomotion is presented by Hatton and Choset[21], and we borrow extensively here from their valuable contributions.

We encode the system kinematics in the local connection, which we reiterate below:

\[ \xi = -A(r)\dot{r} \]  

(4.1)

Where the body velocity \( \xi \) is equal to the inertial angular velocity of some frame \( C \) that is affixed to the system, expressed with components in the same frame \( C \): \( \xi \triangleq \tau \omega^C \). The geometric interpretation is made stronger by the symmetric nature of this type of system: it implies that trajectories \( r(t) \) in the shape space uniquely determine trajectories \( g(t) \) in the orientation space (Lie group trajectories); the geometric interpretation is made even stronger by the fully kinematic nature of the system, which implies that the net result (holonomy) of a shape trajectory \( r(t) \) is independent of time parameterization.
4.1 Gait analysis

We consider the shape $r$ of the idealized cat to be our control input.

We follow the work of Choset and define a \textit{shape change}, which is, simply put, a continuous, physically-realizable, change in the shape $r$ of the system over time. Parameterized by time, it is a mapping $\psi$ from a time variable $t$ to a shape $r \in M$ (in the shape space), on the interval from $t = 0$ to $t = \tau$ (however long the shape change takes to perform). We require only that the first time-derivative of the trajectory exists, such that we can calculate $\dot{r}$, which is the highest derivative required for analyzing purely kinematic systems. In math parlance, we can define the set of all shape changes as:

$$\Psi = \{\psi \in C^1 \mid \psi : [0, \tau] \to M\}$$  \hspace{1cm} (4.2)

A \textit{gait}, then, is defined as a shape change where $\psi$ is a closed curve in the shape space—at the end of one gait cycle the system ends up with the same shape as in the beginning—or a so-called \textit{cyclic} shape change. In set-theory-talk this can be specified as:

$$\Phi = \{\phi \in \Psi \mid \phi(0) = \phi(\tau)\}$$  \hspace{1cm} (4.3)

Where $\Phi$ is the set of all gaits $\phi$.

We consider the case where the shape (control input) $r$ is two-dimensional ($n = 2$), and so the kinematic reconstruction equation can be written as:

$$\begin{bmatrix}
\dot{\Omega}_1 \\
\dot{\Omega}_2 \\
\dot{\Omega}_3
\end{bmatrix} = \begin{bmatrix}
A_1(r) & A_2(r) & \dot{r}_1 \\
A_1(r) & A_2(r) & \dot{r}_2
\end{bmatrix} \begin{bmatrix}
\dot{r}_1 \\
\dot{r}_2
\end{bmatrix}$$  \hspace{1cm} (4.4)

Where we have written out the components of the body angular velocity in the fixed frame $C$. We choose $n = 2$ because it is the minimum number of control inputs necessary to fully control the orientation of a system in $SO(3)$. 

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Figure 4-1: A hypothetical gait defined on a two dimensional shape space. Note the directionality in time, indicated by arrows. The area $A$ is enclosed.

4.1.1 Corrected Body Velocity Integral

For a fully kinematic system, the local connection $A(r)$ provides a relationship between the vector components of the tangent space. Thus, for matrix Lie groups, we can write:

$$\dot{q} = \begin{pmatrix} \dot{r} \\ \dot{g} \end{pmatrix} = \begin{pmatrix} \dot{r} \\ -gA(r)\dot{r} \end{pmatrix}$$  \hspace{1cm} (4.5)

The result $\Delta q$ of a differentially-small gait centered about the shape $r_0$ and with two different control inputs $\dot{r}_A$ and $\dot{r}_B$ (i.e. apply $\dot{r}_A$ for a differentially-small time, then apply $\dot{r}_B$, then $-\dot{r}_A$, then $-\dot{r}_B$) is given by the Lie bracket:

$$\Delta q = \left[ \begin{pmatrix} \dot{r}_A \\ -gA(r_0)\dot{r}_A \end{pmatrix}, \begin{pmatrix} \dot{r}_B \\ -gA(r_0)\dot{r}_B \end{pmatrix} \right]$$  \hspace{1cm} (4.6)

Hatton and Choset[21] show that, for two-dimensional shape spaces and if $r_A = [1, 0]^T$ and $r_B = [0, 1]^T$, Equation 4.6 can be rewritten in a more compact form:

$$\Delta q = \left. \begin{pmatrix} 0 \\ -dA + [A_1, A_2] \end{pmatrix} \right|_{r=r_0} = \begin{pmatrix} 0 \\ \kappa(r_0) \end{pmatrix}$$  \hspace{1cm} (4.7)

Where $d(\cdot)$ denotes the exterior derivative of a $k$-form (the explication of this
topic goes beyond the scope of this thesis, but the intrepid reader is referred to [1]).
However, for 2-dimensional shape spaces:

\[
dA = \frac{\partial A_2}{\partial r_1} - \frac{\partial A_1}{\partial r_2}
\]

(4.8)

Which is just the curl of the corresponding vector field.

Here, we have to be careful and note that the Lie bracket in Equation 4.6 and the Lie bracket in Equation 4.7 are not the same operators, since they refer to different Lie algebras (\(q\) and \(g = so(3)\), respectively). In this expression, \(\kappa\) is equivalent to the local curvature\[1\] of \(A\) at \(r_0\).

Further, Hatton and Choset\[21\] as well as Radford and Burdick\[44\] have shown, with the help of Green’s theorem, that we can approximate the exponential coordinates of one cycle of a small-amplitude gait as an integral of the curvature over the area of the shape space \(A\) enclosed by the gait \(\phi\):

\[
z(\tau) = \iint_A \kappa \, dr_1 dr_2 + O(\epsilon^3)
\]

(4.9)

Where \(r \sim O(\epsilon)\). For \(SO(3)\), the exponential coordinates \(z\) live in the angular velocity space \(so(3)\) and can be interpreted as such: A body \(B\) with exponential coordinate \(z\) relative to some reference orientation \(^{\mathcal{C}}R^B\) has had angular velocity \(^{\mathcal{C}}\omega^B = z\) applied for one unit of time. This is equivalent to a geodesic flow \[1\]. It is important to note that the exponential coordinate \(z(\tau)\) derived here represents the orientation of the \(C'\) frame at time \(t = \tau\) relative to its orientation at time \(t = 0\). Therefore, the exponential coordinates \(z\) represent a sort of “average body-fixed angular velocity” generated by the gait: \(\bar{\xi} = z/\tau\).

This result is a generalization of the area rule introduced by Leonard\[32\]. Importantly, this approximation becomes exact \(iff\) the underlying Lie group \(G\) is abelian (meaning that its Lie bracket is identically zero—it is commutative). Hatton and Choset refer to this formula as the corrected body velocity integral, or cBVI. If the underlying Lie group is nonabelian (such as \(SO(3)\), which is the focus of this thesis), the cBVI is inexact and relies on the “linearization” inherent to Lie bracketing.
While we are indeed able to fully simulate the system by integrating the reconstruction equation (Equation 1.3)—and therefore obtain the net result of any gait without going into all of this complicated analysis of the local connection—Equation 4.9 is a very useful result, in that it allows us to intuitively identify gaits with desirable properties purely geometrically, and without needing to simulate the system. Further, it allows us to make some statements about gait optimality. For example: If our control input is confined to some subspace of the shape space, what is the optimal gait given some performance metric?—in the context of of the Cat’s Problem, it could be the gait that minimizes energy expenditure given a desired attitude change.

For free kinematic chains, $\mathcal{I} \omega^B \in \mathfrak{so}(3)$, and thus the task of determining the net result of applying a gait relies on the noncommutativity (as outlined by Choset) of the underlying Lie group $SO(3)$, as well as the nonconservativity of shifting inertias, as the shape of the system changes[21].

### 4.1.2 No-Twist Joint parameterization

In order to use the cBVI to evaluate candidate gaits for the No-Twist Cat, we must parameterize the shape in terms of variables that are cyclic: If the joint is parameterized by $r \triangleq [\phi, \theta]^T$, a gait corresponds to $\phi(t + \tau) = 2\pi n + \phi(t)$ and $\theta(t + \tau) = \theta(t)$, whereas we would prefer the starting/ending points of the gait to be unambiguous, such that $r(t + \tau) = r(t)$ is the only way to define a gait. This is related to the topological structure of the space defined by the no-twist joint, which is shown by Montgomery[37] to be isomorphic to the real projective plane $\mathbb{R}P^2$.$^1$

To disambiguate gaits, we thus consider a change of variables, from $r \triangleq [\phi, \theta]^T$ to $r \triangleq [\alpha, \beta]^T$ such that:

$^1$As a very hand-wavy proof that the no-twist joint space is isomorphic, consider the fact that the axis $a_3$ can be pointed in any direction relative to the $B$ frame, and further that there is exactly one $(\phi, \theta)$ pair which corresponds to each pointing direction, which is basically the definition of the real projective plane. The Universal Joint Cat, first described by Enos, can also point $a_3$ in any direction relative to $B$, but in that case the shape space is isomorphic to the 2-torus ($S^1 \times S^1$).
\begin{align*}
\theta &= \sqrt{\alpha^2 + \beta^2} \\
\dot{\theta} &= \frac{\alpha \dot{\alpha} + \beta \dot{\beta}}{\sqrt{\alpha^2 + \beta^2}} \\
\dot{k} &= \frac{\alpha b_1 + \beta b_2}{\theta} \rightarrow \tan \phi = \frac{\beta}{\alpha} \\
\dot{\phi} &= \frac{\alpha \dot{\beta} - \beta \dot{\alpha}}{\beta^2 + \alpha^2} \\
\cos \phi &= \frac{\alpha}{\sqrt{\alpha^2 + \beta^2}} \\
\sin \phi &= \frac{\beta}{\sqrt{\alpha^2 + \beta^2}} \tag{4.10}
\end{align*}

Now, \(\alpha\) and \(\beta\) have no direct physical interpretation beyond that they define \(\theta\) and \(\phi\), which do have a clear physical interpretation, but exist solely to let us put the subsequent kinematical equations into an amenable form—\(\phi\) and \(\theta\) could be thought of as the polar form of \(\alpha\) and \(\beta\).

![Diagram](image)

Figure 4-2: A change of coordinates for the No-Twist Cat.

### 4.2 Small-angle gaits

In this section, we consider the result of small-angle \(O(\epsilon)\) gaits applied to the model systems. This type of analysis is appealing for several reasons: one, it is a natural
framework to analyze gaits that are sustainable by structural stiffness; two, the cBVI is guaranteed to provide a good approximation; three, we can make some interesting statements about gait optimality; four, it provides a sort of "sanity check" allowing us to compare each of the model systems, which one might guess are roughly equivalent under small-angle approximation.

**No-Twist Cat**

Keeping only $O(\epsilon^2)$ terms in Equation 3.31:

$$
\mathbf{A}^B(\phi, \theta) \approx \begin{bmatrix}
-\mu \theta \sin \phi & \mu \cos \phi \\
\mu \theta \cos \phi & \mu \sin \phi \\
\frac{2ac-ad+cd}{2(a+c)(b+d)} \theta^2 & 0
\end{bmatrix}
$$  \hspace{1cm} (4.11)

Transforming from $(\phi, \theta)$ to $(\alpha, \beta)$ according to Equation 4.10, and again keeping only $O(\epsilon^2)$ terms in the angular velocity:

$$
\mathbf{A}^B(\alpha, \beta) = \begin{bmatrix}
\mu & 0 \\
0 & \mu \\
\frac{2ac-ad+cd}{2(a+c)(b+d)} \beta & \frac{2ac-ad+cd}{2(a+c)(b+d)} \alpha
\end{bmatrix}
$$  \hspace{1cm} (4.12)

From the local connection, we calculate the curvature and keep only the $O(\epsilon^2)$ terms:

$$
\kappa \approx \begin{bmatrix}
0 \\
0 \\
-\frac{2ac^2+bc^2+a^2(d-2c)}{(a+c)^2(b+d)}
\end{bmatrix}
$$  \hspace{1cm} (4.13)

In all, this emphasizes the fact that we can only use this small-angle approximation to evaluate $O(\epsilon)$ gaits that produce $O(\epsilon^2)$ results. If we discard higher order terms, we see that small-angle gaits applied to the No-Twist Cat can only effect orientation change about the axis $b_3$. 

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Universal Joint Cat

We calculate the curvature from Equation 3.37:

\[
\kappa = \begin{bmatrix}
  0 \\
  0 \\
  \frac{-2a^2 - 2ac^2 + bc^2 + a^2d}{(a+c)^2(b+d)}
\end{bmatrix}
\]  

(4.14)

The 3-component is identical to that of the curvature of the No-Twist Cat, even though the respective local connections differ!

Dual Seesaw

The local curvature, calculated from Equation 3.44 is:

\[
\kappa = \begin{bmatrix}
  0 \\
  0 \\
  \frac{c(-2ac + bc + 2a\delta + 2a^2)}{(b+2c)(a+c+\delta)^2}
\end{bmatrix}
\]  

(4.15)

4.2.1 Comparison between models

To compare small-angle solutions for all three models (No-Twist Cat, Universal Joint Cat, and Dual Seesaw), we allow that \( d = 2c \) and that \( \delta = 0 \). In this particular case, each model is “inertially equivalent” (if we consider the seesaws to be locked together as one rigid body).

The third component of the local curvature for each model reduces to:

\[
\kappa_3 = \mu b - 2a
\]  

(4.16)

To third order, the first two components of curvature are all zero. This indicates that each model system can only use small-angle gaits to effect rotation about the axis \( b_3 \). Further, since \( \kappa_3 \) is not a function of the shape variables, evaluating the cBVI is exceedingly simple: it evaluates to \( \Delta \psi = \kappa_3 A \), where \( A \) is the area enclosed by the small-angle gait in the cartesian shape space, and \( \Delta \psi \) is the net angular change.
effected about the axis $b_3$ per gait cycle.

### 4.3 Optimal gaits

We might also like to use this geometric mechanics construction to identify “optimal” gaits. This amounts to defining some performance metric on the shape space and ascertaining gaits that minimize this metric (given a desired starting and ending orientation\(^2\)). For example, we might choose to define a performance metric as the length of the gait’s curve in the cartesian shape-space (this might correspond with the energy expenditure required for a given gait cycle). In this case, the problem of finding the optimal gait reduces to an *isoholonomic problem*\([35]\), which is the problem of finding the shortest curve with a given holonomy.

This provides intuition as to why sinusoidal inputs are “optimal” in the small-angle case: sinusoids can be used to create circular trajectories in the shape space, which maximize enclosed area for a given perimeter. This is consistent with the work of Brockett\([8]\).

\(^{2}\) Because of bounds on the shape variables, the end orientation might only be reachable by a repeated series of cyclic motions, instead of just one motion. This is still a viable gait.
Chapter 5

Coordinate frames

Hatton and Choset have recently introduced the idea of minimum perturbation coordinates, in the context of the dynamics and control of systems which exhibit Lie group symmetry\cite{21}\cite{19}. This concept is fundamentally connected to the cBVI: the accuracy of the cBVI approximation is a function of the coordinates used, a somewhat counterintuitive concept in geometric mechanics, which strives to be coordinate free.

In terms of our systems, which live in $SO(3)$, choosing the minimum perturbation coordinates amounts to the selection of a $C$ frame for which the cBVI yields an approximation to the exponential coordinates of a gait that is the closest possible to the actual exponential coordinates of the gait.

In this chapter, we demonstrate a coordinate frame choice for the No-Twist Cat, inspired by Hatton and Choset’s work on minimum perturbation coordinates. This choice of $C$ frame in fact leads to an an easily-computable exact solution for the net result of arbitrary gaits, and is a generalization of the exact result presented by Kane and Scher\cite{23}. Because the solution is exact and invertible, we do not even need to compute the cBVI in order to design open-loop gaits! Further, our solution demonstrates the full controllability of the No-Twist Cat with broken inertial symmetry, which can also be proven geometrically using Chow’s theorem.

The discovery of this “optimal” frame for the No-Twist Cat motivates a similar change of frame for the Universal Joint Cat. While it is not proven in this work that this $C$ frame is “optimal” in the style of Hatton and Choset’s minimum pertur-
bation coordinates, it vastly simplifies the task of open-loop motion planning for the Universal Joint Cat.

5.1 No-Twist Cat

Hatton and Choset[21] note that the minimum perturbation coordinates tend to be fixed reference frames that “move” the least during the course of a gait cycle. For free kinematic chains, this ends up being something like the “average” reference frame, weighted by the component inertias \( \mathbf{I}_A \). For the No-Twist Cat, we note that if we fix \( \phi = \phi_0 \) while allowing \( \theta \) to vary in time, the axis \( \mathbf{b}_3 \) rotated an angle \( \mu \theta \) about the twist axis \( \mathbf{t} \) will remain inertially-fixed—the \( \mathbf{n} \) axis, as defined, will always point up. In fact, Kane states a priori the fact that if \( \mathbf{B} \) and \( \mathbf{A} \) are identical (and, of course, axially-symmetric) this axis will remain inertially-fixed even if \( \phi \) is not kept constant; this fact is integral to Kane’s derivation, but does not seem to me to be obvious—in the subsequent analysis we will prove it from dynamical principles.

Using the above observations, we posit a candidate change of orientation frame, from \( \mathbf{B} \) to \( \mathbf{C}' \), such that \( \mathbf{c}'_3 = \mathbf{n} \):

\[
^{\mathbf{B}} \mathbf{R}^{\mathbf{C}'}(\phi, \theta) = I + \sin \mu \theta [\mathbf{t} \times] + (1 - \cos \mu \theta) [\mathbf{t} \times]^2 \quad (5.1)
\]

We can calculate the frame-fixed angular velocity:

\[
^\mathbf{T} \omega^{\mathbf{C}'} = \left[^{\mathbf{B}} \mathbf{R}^{\mathbf{C}'} \right]^T \tau \omega^{\mathbf{B}} + ^{\mathbf{B}} \omega^{\mathbf{C}'} = \begin{bmatrix}
\dot{\phi} \\
p(\theta) \sin \phi \\
-p(\theta) \cos \phi \\
r(\theta) - q(\theta)
\end{bmatrix} \frac{\dot{\phi}}{r(\theta)} \quad (5.2)
\]

Where:
\[ p(\theta) = (a + b + 2c)d \sin(1 - \mu)\theta - b(2a + c + d) \sin \mu \theta + b(c - d) \sin(2 - \mu)\theta + d(b - a) \sin(1 + \mu)\theta \]
\[ q(\theta) = (a + b + 2c)d \cos(1 - \mu)\theta + b(2a + c + d) \cos \mu \theta + b(c - d) \cos(2 - \mu)\theta - d(b - a) \cos(1 + \mu)\theta \]
\[ r(\theta) = bc + d(b + 2c) + a(2b + c + d) - (a - b)(c - d) \cos 2\theta \]

(5.3)

If bodies \( \mathcal{A} \) and \( \mathcal{B} \) are identical (\( c = a \) and \( d = b \)), \( p(\theta) = 0 \) and this intermediate frame \( C' \) is, by inspection, a minimum perturbation frame, since \( \mathcal{T} \omega^C \) is restricted to an abelian subgroup \( \mathfrak{s}\mathfrak{o}(2) \subset \mathfrak{s}\mathfrak{o}(3) \) (since the first two components of frame-fixed angular velocity are zero). Thus, the cBVI will be exact, and we can use it to exactly recover Kane’s result[23]. Further, the fact that \( p = 0 \) when the bodies are identical proves the inertial-fixity of the \( \hat{n} \) axis from dynamical principles, rather than being stated \textit{a priori}.

In this inertially-symmetric case, if we parameterize a simple circular gait with \( \theta = \theta_0 \) and \( \phi = t \), the net result of one gait cycle is thus:

\[ \Delta \psi = 2\pi \left( 1 - \frac{q(\theta_0)}{r(\theta_0)} \right) \]

(5.4)

Where \( \Delta \psi \) is the change in angular position about the \( \mathbf{c}_3' = \hat{n} \) axis per gait cycle. This demonstrates that the inertially-symmetric No-Twist Cat can only reorient about one axis, and therefore is not fully controllable. Note that we chose \( \phi = t \) purely out of convenience; we know, due to the fully kinematic nature of the system, that the net result of a gait cycle will be \textit{independent of time parameterization}.

If \( \mathcal{A} \) and \( \mathcal{B} \) are not identical, Equation 5.2 describes \textit{nutation} of the \( C' \) frame, in that the angular velocity vector describes a cone-like object in the \( C' \) coordinate frame throughout the course of a gait cycle. In this case, the cBVI is no longer an exact solution, since the angular velocity is no longer restricted to an abelian subgroup of \( \mathfrak{s}\mathfrak{o}(3) \).
5.1.1 Gait analysis

Using the change of shape parameterization detailed in Equation 4.10, we re-express the angular velocity of the $C'$ frame:

$$
\mathbf{\omega}^C (\alpha, \beta) = \begin{bmatrix}
P \left( \sqrt{\alpha^2 + \beta^2} \right) \frac{\alpha \beta - \beta \alpha}{\beta^2 + \alpha^2} \frac{\beta}{\sqrt{\alpha^2 + \beta^2}} \\
-P \left( \sqrt{\alpha^2 + \beta^2} \right) \frac{\alpha \beta - \beta \alpha}{\beta^2 + \alpha^2} \frac{\alpha}{\sqrt{\alpha^2 + \beta^2}} \\
Q \left( \sqrt{\alpha^2 + \beta^2} \right) \frac{\alpha \beta - \beta \alpha}{\beta^2 + \alpha^2} \frac{\alpha}{\sqrt{\alpha^2 + \beta^2}}
\end{bmatrix}
$$

(5.5)

Where $P(x) = \frac{v(x)}{r(x)}$ and $Q(x) = 1 - \frac{q(x)}{r(x)}$ to keep the algebra clean.

Note that, under this change of parameterization, a singularity exists when $\theta = 0$, indicating that $\phi$ is no longer defined. This might present a real problem when using this parameterization as a design tool: we cannot easily investigate gaits which pass through $\theta = 0$. However, the limit as $\theta \to 0$ does still exist (corroborated by the fact that no singularities exist when parameterizing the shape as $r \triangleq [\phi, \theta]^T$).

From Equation 5.5, we can write a local connection for the $C'$ frame:

$$
A^C (\alpha, \beta) = \begin{bmatrix}
\frac{P \beta^2}{(\alpha^2 + \beta^2)^{3/2}} & \frac{P \alpha \beta}{(\alpha^2 + \beta^2)^{3/2}} \\
\frac{P \alpha \beta}{(\alpha^2 + \beta^2)^{3/2}} & \frac{P \alpha^2}{(\alpha^2 + \beta^2)^{3/2}} \\
\frac{Q \beta}{\alpha^2 + \beta^2} & \frac{Q \alpha}{\alpha^2 + \beta^2}
\end{bmatrix}
$$

(5.6)

By inspection, the Lie bracket of the columns vanishes! Since the Lie bracket in $so(3)$ is the vector cross product, the columns of $A^C (\alpha, \beta)$ are parallel. This indicates that, by changing to the $C'$ frame, we have chosen a coordinate system in which the system anholonomy is encoded entirely as nonconservativity[21].

From the local connection we can calculate the curvature $\kappa$ and formulate the cBVI (Equation 4.9):
\[ z(\tau) \approx \int \int _A \kappa \, dr_1 \, dr_2 = - \int \int _A \mathbf{dA}' \, d\alpha \, d\beta + \int \int _A \left[ \mathbf{A}_1' \times \mathbf{A}_2' \right] \, d\alpha \, d\beta = \mathbf{A}_1' \times \mathbf{A}_2' = 0 \]

\[ \mathbf{dA}' = \frac{\partial \mathbf{A}_5'}{\partial \alpha} - \frac{\partial \mathbf{A}_6'}{\partial \beta} \]

\[ = \frac{1}{\alpha^2 + \beta^2} \begin{bmatrix} -\beta \frac{\partial P}{\partial \theta} \\ \alpha \frac{\partial P}{\partial \theta} \\ \frac{\partial Q}{\partial \theta} \sqrt{\alpha^2 + \beta^2} \end{bmatrix} = \begin{bmatrix} \kappa_1 \\ \kappa_2 \\ \kappa_3 \end{bmatrix} (\alpha, \beta) \quad (5.7) \]

Where \( A \) is the area enclosed by the gait in \( \alpha-\beta \) space, \( \tau \) is the time required to execute one cycle of the gait, and \( z(\tau) \) are the exponential coordinates of the system at the end of the gait cycle, again, interpreted as a sort of average angular velocity over the course of the gait.

There are several interesting takeaways here. One is that if the bodies are identical, \( \frac{\partial P}{\partial \theta} = 0 \) and the orientation can change only about the axis \( c_3 \); this is obviously uncontrollable, as has been noted by Enos[15], and corroborates the same conclusion drawn from Equation 5.2. Thus, Equation 5.7 also gives us an exact solution to the inertially-symmetric No-Twist Cat’s Problem:

\[ \Delta \psi = - \int \int _A \frac{\partial Q}{\partial \theta} \frac{1}{\sqrt{\alpha^2 + \beta^2}} d\alpha \, d\beta \quad (5.8) \]

Where \( \Delta \psi \) is the change in angular position about the \( c_3 \) axis per gait cycle. Presumably, this expression is equivalent to Kane’s analytical solution[23], though is expressed in a very different form. For simple circular gaits \( (\theta = \theta_0) \), this integral can be expressed in closed-form:

\[ \Delta \psi = - \int \int _{\alpha^2 + \beta^2 < \theta_0^2} \frac{\partial Q}{\partial \theta} \frac{1}{\sqrt{\alpha^2 + \beta^2}} d\alpha \, d\beta \quad (5.9) \]

Which should reduce to both Kane’s solution and to the closed-form solution presented in Equation 5.4.
Now consider the case where the bodies are not identical. We assert that \( t(\alpha, \beta) = \frac{1}{\alpha^2 + \beta^2} \frac{\partial \phi}{\partial \theta} \) is an even function in both \( \alpha \) and \( \beta \) (i.e. \( t(-\alpha, -\beta) = (\alpha, -\beta) = t(\alpha, \beta) \)); this means that both \( \kappa_1 \) and \( \kappa_2 \) are odd in \( \alpha \) and \( \beta \) as well, as visualized in Figure 5-1. If the gait is symmetric about the origin in \( \alpha-\beta \) space (meaning that this is at least one line of symmetry that passes through the origin), then this implies geometrically (think of the area integral!) that the 1 and 2 components of the exponential coordinates \( z(\tau) \) are zero! If the gait is not symmetric, it is clear that the 1 and 2 components are nonzero; this indicates, but does not elegantly prove, that the system is fully controllable. To prove the full controllability, we should use iterated Lie brackets (Chow’s theorem): it is enough to show that the set \( \{A_1, A_2, [A_1, dA], [A_2, dA]\} \) spans the space \( \mathfrak{so}(3) = \mathbb{R}^3 \) (indeed, it does, but we will leave the detailed calculation as an exercise for the reader).

We should also note here that, just because the Lie bracket in Equation 5.7 vanishes, the approximation to the exponential coordinates is still just that: it does not
represent an exact solution. This fact is not clear from Choset's work on optimal coordinates, where the magnitude of the Lie bracket is used a proxy to ascertain the degree of path-dependence experienced by the system. Further, when simulated, the kinematics of the $C'$ frame still clearly exhibit anholonomy in the form of path dependence, as depicted in Figure 5-2, indicating the presence of some kind of higher-order (than one Lie bracket, or even iterated Lie brackets) anholonomy. Additionally, while it can be seen that the approximate solution often matches the full solution (determined numerically) in an intuitively averaged sense, it is clear that for certain combinations of parameter values this agreement falters. Determining the exact error bounds, and therefore the conditions under which Equation 5.7 is a valid approximation, is an ongoing topic of research. In general, we can say that the approximation is valid up to second order in the shape variables[44].

5.1.2 Another choice of coordinate frame

Motivated by Kane's choice of coordinate frames[23], we might try another candidate for an optimal coordinate frame. In particular, we would like to get rid of the aforementioned higher-order anholonomy seen in the $C'$ frame, if possible. To do so, we recognize that, while the system is inertially symmetric about the axis $\mathbf{n}$ (in the sense that the system looks inertially identical no matter the choice of $\phi$), it is not symmetric if we affix the body frames (in particular orientations) and consider their location. Therefore, the system does not look totally symmetric to the $C'$ frame, which is just one of the body frames $B$ rotated $\mu \theta$ about the twist axis $\mathbf{t}$.\(^1\)

However, if we align a frame $C''$ to both axes $\mathbf{n}$ and $\mathbf{t}$, we might expect the $\phi$-dependence to drop out of the angular velocity $\mathbf{T}_c \mathbf{\omega}^{C''}$ via a symmetry argument. To construct such a frame, we begin with $C''$ aligned with one of the body frames $B$, rotate it an angle $\phi$ about $c''_3$ such that $c''_1$ aligns with $\mathbf{t}$, and then rotate it an angle $\mu \theta$ about $\mathbf{t}$ such that $c''_3$ aligns with $\mathbf{n}$. The requisite transformation matrix is thus:

\(^1\)This also intuitively explains why the $C'$ frame exhibits path-dependence, since the starting and ending point of the gait should matter; hence the $\phi$-dependence in Equation 5.2.

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Figure 5-2: Comparison of two different circular gaits for the No-Twist Cat oriented using the $C'$ frame with different starting/ending points. Inertia parameters are $a = 2$, $b = 0.1$, $c = 1$, and $d = 2$.

(a) Components of the reconstructed Direction Cosine Matrix $^{T}R_{C'}$.

(b) Starting/ending point $(1,0)$.

(c) Starting/ending point $(0,1)$. 
\[
\mathbf{B}R_{c''} = \begin{bmatrix}
\cos \phi & -\sin \phi & 0 \\
\sin \phi & \cos \phi & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \mu \theta & -\sin \mu \theta \\
0 & \sin \mu \theta & \cos \mu \theta
\end{bmatrix}
\] (5.10)

Calculating the angular velocity yields:

\[
\mathbf{I}_w_{c''} = \left[\mathbf{B}R_{c''}\right]^T \mathbf{I}_w + \mathbf{B}w_{c''} = \begin{bmatrix}
0 \\
-p(\theta) \\
q(\theta)
\end{bmatrix}
\begin{bmatrix}
\phi \\
\frac{\phi}{r(\theta)} \\
\frac{\phi}{r(\theta)}
\end{bmatrix}
\] (5.11)

We see that the angular velocity is not a function of \( \phi \) (or \( \dot{\theta} \)). Importantly, we can directly solve for an entire class of solutions, corresponding to symmetric, circular gaits with \( \theta(t) = \theta_0 \) and \( \phi(t + T) = \phi(t) + 2\pi \): For these gaits, the angular velocity of the \( C'' \) frame is constant in direction!

\[
\mathbf{I}_w_{c''} = \begin{bmatrix}
0 \\
-p(\theta_0) \\
q(\theta_0)
\end{bmatrix}
\begin{bmatrix}
\frac{\Omega}{r(\theta_0)} \\
\frac{\Omega}{r(\theta_0)} \\
\frac{\Omega}{r(\theta_0)}
\end{bmatrix}
\] (5.12)

Where \( \Omega \) is the time-average of \( \dot{\phi} \) during the gait. Because we can change the direction of the angular velocity by varying the amplitude of these circular gaits, it is intuitively clear that the No-Twist Cat with broken inertial symmetry is fully controllable.

5.1.3 Controllability

We would like to rigorously assess the controllability of the No-Twist Cat with broken inertial symmetry. To do so, we calculate a local connection for the \( C'' \) frame in terms of \((\alpha, \beta)\):
\[
\tau \omega'' = \begin{bmatrix}
0 \\
-p(\theta) \\
q(\theta)
\end{bmatrix} \frac{\dot{\phi}}{r(\theta)} = \begin{bmatrix}
0 \\
-p(\sqrt{\alpha^2 + \beta^2}) \\
q(\sqrt{\alpha^2 + \beta^2})
\end{bmatrix} \frac{1}{r(\sqrt{\alpha^2 + \beta^2})} \frac{\alpha \dot{\beta} - \beta \dot{\alpha}}{\alpha^2 + \beta^2}
\]

\[
A''(\alpha, \beta) = \begin{bmatrix}
0 & 0 & 0 \\
p(\theta)\beta & -p(\theta)\alpha & \frac{1}{(\alpha^2 + \beta^2)r(\theta)}
\end{bmatrix}
\]

Note that the cross product (Lie bracket for \(so(3)\)) of the rows of the local connection vanishes, exactly as for \(C'\). Thus the local curvature reduces to:

\[
-\kappa = dA'' = \frac{\partial A''_2}{\partial \alpha} - \frac{\partial A''_1}{\partial \beta} = \begin{bmatrix}
0 \\
-r \frac{\partial p}{\partial \theta} + p \frac{\partial r}{\partial \theta} \\
r \frac{\partial q}{\partial \theta} - q \frac{\partial r}{\partial \theta}
\end{bmatrix} \frac{1}{r^2 \sqrt{\alpha^2 + \beta^2}}
\]

Remember that the local curvature was derived from the Lie bracket (on the Lie algebra \(q\) of the entire configuration space) of the vector fields generated by activating \(r_1 \triangleq \alpha\) and \(r_2 \triangleq \beta\). Chow's theorem asserts that the system is controllable iff the vectors generated by iterated Lie brackets of the columns of the local connection span the Lie algebra \(so(3)\). Therefore, if the rank of the following controllability matrix is 3, this system is controllable:

\[
C = (A_1, A_2, \kappa, [A_1, \kappa], [A_2, \kappa]) = (A_1, A_2, \kappa, A_1 \times \kappa, A_2 \times \kappa)
\]

Intuitively, \(A_1\) is the direction the \(C''\) frame moves in if only \(\alpha\) varies; \(A_2\) is the direction the frame moves in if only \(\beta\) varies; \(\kappa\) is the direction the frame moves in if a Lie bracket (differentially small) gait is generated using \(A_1\) and \(A_2\); and the next two brackets are the direction the frame moves in if a Lie bracket gait is generated using \(\kappa\) and the \(A_i\).
We rearrange the controllability matrix a bit, discarding parallel columns and ignoring common factors, such that this modified controllability matrix still has the same rank:

\[
C' = \begin{bmatrix}
0 & 0 & p \frac{\partial q}{\partial \theta} - q \frac{\partial p}{\partial \theta} \\
p & -r \frac{\partial p}{\partial \theta} + p \frac{\partial r}{\partial \theta} & 0 \\
-q & r \frac{\partial q}{\partial \theta} - q \frac{\partial r}{\partial \theta} & 0 \\
\end{bmatrix}
\]  

(5.16)

We see that \( \text{rank}(C') = 3 \) if \( p \neq 0 \), and thus the system is controllable iff the bodies are not inertially identical.

Technically, we have not yet addressed the controllability of the shape space (base space), just the controllability of the orientation space (fiber space), and so we have only proven that the No-Twist Cat is fiber controllable in the parlance of Kelly and Murray[25]. However, if we take our control inputs to be \( u \triangleq [\dot{\alpha}, \dot{\beta}]^T \) it is clear that the shape space is fully actuated and therefore controllable.

### 5.2 Universal Joint Cat

We propose a coordinate frame transformation that should improve the accuracy of the cBVI approximation. The key insight is this: the Universal Joint Cat and the No-Twist Cat are not very different; indeed, we have seen that for small amplitude oscillations, similar gaits produce identical results.

It therefore makes some sense that we might choose an analogous \( C' \) frame for the Universal Joint Cat. To do so, we note that we can define a one-to-one mapping between No-Twist Cat shapes and Universal Joint Cat shapes: we identify shapes based on the orientation of the axis \( a_3 \) in the \( B \) frame. In this way, we associate a rotation axis angle \( \phi^* \) and a rotation angle \( \theta^* \) with any given shape \((\gamma_1, \gamma_2)\) of the Universal Joint Cat.

To generate this mapping, we examine the rotation matrix \( B R^A \) which encodes the system shape. The third column of this matrix expresses the axis \( a_3 \) with components in the \( B \) frame.
We determine this axis for both the No-Twist Cat and the Universal Joint Cat:

\[ \mathbf{a}_{3}^{\text{NT}} = \begin{bmatrix} 
\sin \theta \sin \phi \\
-\sin \theta \cos \phi \\
\cos \theta \\
\end{bmatrix} \]

\[ \mathbf{a}_{3}^{\text{UJ}} = \begin{bmatrix} 
\sin \gamma_2 \\
-\sin \gamma_1 \cos \gamma_2 \\
\cos \gamma_1 \cos \gamma_2 \\
\end{bmatrix} \]

(5.17)

If we equate the two axes and solve for \( \phi \) and \( \theta \) (we have three equations in two unknowns, but they are not independent):

\[ \cos \theta = \cos \gamma_1 \cos \gamma_2 \]

\[ \sin \phi = \frac{\sin \gamma_2}{\sqrt{1 - \cos^2 \gamma_1 \cos^2 \gamma_2}} \]

(5.18)

This allows us to solve for \( \theta^* = \theta^*(\gamma_1, \gamma_2) \) and \( \phi^* = \phi^*(\gamma_1, \gamma_2) \), which gives us an axis angle and rotation angle for each shape \((\gamma_1, \gamma_2)\) of the Universal Joint Cat.

We also note a few things: if we transform from \((\phi, \theta)\) to \((\alpha, \beta)\) (as defined in the coordinate change for the No-Twist Cat), these equalities hold to second order, meaning that \( \alpha = \gamma_1 + \mathcal{O}(\epsilon^3) \) and \( \beta = \gamma_2 + \mathcal{O}(\epsilon^3) \).

We can therefore define an “optimal” orienting frame \( C'_{\text{UJ}} \) for the Universal Joint Cat:

\[ \mathcal{B} \mathbf{R}_{\text{UJ}}^C = \begin{bmatrix} 
\cos \phi^* & -\sin \phi^* & 0 \\
\sin \phi^* & \cos \phi^* & 0 \\
0 & 0 & 1 \\
\end{bmatrix} \cdot \begin{bmatrix} 
1 & 0 & 0 \\
0 & \cos \mu \theta^* & -\sin \mu \theta^* \\
0 & \sin \mu \theta^* & \cos \mu \theta^* \\
\end{bmatrix} \]

(5.19)

We calculate the curvature \( \kappa \) for the \( C'_{\text{UJ}} \) frame, using exactly the procedure outlined in §5.1.1. This is an algebraically-intensive process and was accomplished using the software package Mathematica. We do not print the symbolic results here since
they are so unwieldy.

5.2.1 Simulation results

To test the results, we propose three candidate gaits:

$$
\begin{align*}
\phi_2 & \rightarrow \begin{cases} 
\gamma_1(t) = \cos t \\
\gamma_2(t) = \sin t 
\end{cases} \\
\phi_1 & \rightarrow \begin{cases} 
\gamma_1(t) = \cos t \\
\gamma_2(t) = \sin 2t 
\end{cases} \\
\phi_3 & \rightarrow \begin{cases} 
\gamma_1(t) = \sin 2t \\
\gamma_2(t) = \cos t 
\end{cases}
\end{align*}
$$

(5.20)

Figure 5-3: Three candidate gaits for the Universal Join Cat.

$\phi_2$ describes a circle in the shape space and is intended to reorient the Universal Joint Cat about the axis $c_3$; $\phi_1$ describes a “figure-eight” in the shape space and is intended to reorient the Cat about the axis $c_2$; $\phi_3$ describes a different “figure-eight” in the shape space and is intended to reorient the Cat about the axis $c_1$. All gaits are visualized as overlays on the surfaces prescribed by the components of $\kappa$ in Figure 5-4. Note that all gaits are of relatively large amplitude.

Importantly, these gaits were generated simply by inspection of the curvature $\kappa$. 

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Figure 5-5 demonstrates the marked improvement in cBVI accuracy afforded by the $C''$ frame. Figure 5-6 demonstrates that the cBVI provides a qualitatively accurate prediction for the reorientation effected by a more complex gait, demonstrating that the Universal Joint Cat is able to reorient about the $c_2$ axis. Figure 5-7 tries to visualize the “filtering” effect that the appropriate choice of $C$ frame has on the motion of the orienting frame: we see that, by choosing to orient the system using the $C''$ frame rather than the $B$ frame, the resulting motions are much more intelligible (and deviations from the time-averaged path are of smaller amplitude).
Figure 5-4: Components of the curvature $\kappa$ for both the $C''$ and the $B$ frame affixed to the Universal Joint Cat, with $a = b = c = d = 1$. Candidate gaits are overlaid on the surfaces: $\phi_3$ is detailed in blue, $\phi_1$ is detailed in yellow, and $\phi_2$ is detailed in green.
Figure 5-5: Comparison of the quality of the cBVI approximation for the $B$ and $C'$ frames affixed to the Universal Joint Cat with $a = b = c = d = 1$. The applied gait is $\phi_1$.

Figure 5-6: The reconstructed result of the $\phi_2$ "figure-eight" candidate gait applied to the $C'$ frame affixed to the Universal Joint Cat demonstrates reorientation about the axis $c'_2$. However, the accuracy of the cBVI approximation is clearly lacking.
Figure 5-7: Reconstructed trajectories of the orienting frame unit vectors for each of the candidate gaits, through five gait cycles. The gaits $\phi_i$ are numbered such that the numbers in the figure correspond both to gaits and to unit vector directions.

(a) $\mathcal{B}$ frame.

(b) $\mathcal{C}'$ frame. Shaded regions are the angular swaths predicted by the cBVI.
Chapter 6

Dynamics

In the kinematical consideration, we assumed that the shape variables \( r \) were control inputs: we could specify any shape change \( r(t) \) that we liked and then solve for the subsequent time-evolution of the system's orientation \( g(t) \); this ability follows directly if the shape space is fully actuated. A gait \( \phi \) was defined as a periodic shape change. Further, we were able to make some statements about gait performance and optimality using a geometric mechanics analysis.

We consider in this chapter the dynamics of the Cat's Problem, where we do not have full freedom to assign \( r(t) \) arbitrarily but instead allow the shape to evolve dynamically in time according to physical laws. In this case, the shape space is necessarily underactuated.

We will consider several examples of underactuated, free, deformable systems. Via a small-angle analysis, we will demonstrate that placing a torsional spring at the pivot point of the Cat allows for the sustaining of productive gaits via dynamical effects. Further, we will show that the optimal (in the sense that they maximize both momentum efficiency and energy efficiency) gaits achievable via structural stiffness are also optimal in the sense that they solve an isoholonomic problem[35], which can be viewed as the solution to an optimal control problem.

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6.1 Reduced dynamics

For both serial and parallel kinematic chains, we have shown that the body-fixed inertial angular velocity of one of the component bodies \( A_1 \) can be expressed only as a function of the shape \( r \) and shape velocity \( \dot{r} \). This relationship is equivalent to the local connection for the frame \( A_1 \).

Thus, we see that we can express the inertial angular velocity of any body \( A_i \) in the chain, as a function of \( r \) and \( \dot{r} \); since the transformation from frame \( A_1 \) to \( A_i \) is, by definition, a function of \( r \) and \( \dot{r} \) alone (this is just equivalent to saying that any frame \( A_i \) is fixed to the deformable system). This allows us to write the total kinetic energy \( T \) as a function of just the same quantities: \( T = T(r, \dot{r}) \). Further, assuming that the potential energy \( U \) of the system is only a function of \( r \) (as would be the case if energy storage elements such as springs are incorporated as part of the deformable system), the Lagrangian \( \mathcal{L} = T - U \) can be written as the reduced Lagrangian \( l(r, \dot{r}) \).

Importantly, this implies that we can derive the dynamics of the shape space independently, while the nonholonomic constraint (which encodes interaction with the external world, so to speak) is built into the kinetic energy metric. Via the Euler-Lagrange equations, the equations of motion should look something like:

\[
\frac{d}{dt} \left( \frac{\partial l(r, \dot{r})}{\partial \dot{r}} \right) - \frac{\partial l(r, \dot{r})}{\partial r} = \Xi_r
\]

(6.1)

Where \( \Xi_r \) is the generalized force associated with the shape. In our model systems, the shape variables are all angles, and so the elements of \( \Xi_r \) are torques. If viscous damping torques were to be incorporated into the model, they would also appear as generalized forces.

Once we have written the equations of motion for the shape variables, we can solve them to find specific trajectories \( r(t) \); this is a straightforward example of nonholonomic reduction, made simpler by the fact that zero-angular-momentum-conserving systems are principally kinematic by virtue of their purely mechanical nature.\(^1\) These

\(^1\)Remember, if this were not the case, we would have to consider nontrivial dynamical evolution equations for the generalized nonholonomic momentum as well.
trajectories can then be plugged into the kinematic and geometrical equations that we have already derived, in order to ascertain the effect that the shape dynamics have on the orientation trajectory \( g(t) \) of the system; recall that this process is termed \textit{reconstruction}.

\section*{6.2 Model systems}

We provide a derivation of the reduced dynamics for the model systems considered in this thesis. For each, we place torsional springs at the rotational joints between \( B \) and its accessory body (or bodies)—either \( A \) or \( S_1 \) and \( S_2 \). We will demonstrate that, for small-amplitude oscillations, these energy-storage elements allow for the sustaining of the type of gaits that we have been analyzing; namely, sinusoidal gaits of the form \( \tau_i = a_i \sin(\omega t + \Delta_i) \). The frequency of oscillation \( \omega \) will be a function off the stiffness \( k \) of a particular joint; we assume that, as designers, we have the ability to “tune” the stiffness such that the dynamics of the system produces useful gaits.

\subsection*{6.2.1 No-Twist Cat}

For the geometrically nonlinear (large angle) case, we consider a simplified system: the two component bodies \( A \) and \( B \) are identical, such that \( c = a \) and \( d = b \). This vastly simplifies the algebra, which would otherwise be practically unintelligible.

We take the shape \( r \triangleq [\phi, \theta]^T \). Given this, we can write the angular velocity of each component body as a function of \( r \) and \( \dot{r} \):
\[ \mathcal{T}_\omega^B = -\mathbf{A}^B(\phi, \theta)[\dot{\phi}, \dot{\theta}]^\top = \begin{bmatrix} -\frac{1}{2} \dot{\theta} \cos \phi + \frac{b \sin \theta \sin \phi}{a + b + (b - a) \cos \theta} \\
-\frac{b \cos \phi \sin \theta}{a + b + (b - a) \cos \theta} - \frac{1}{2} \dot{\phi} \sin \phi \\
-2\dot{\phi} \frac{a}{a + b + (b - a) \cos \theta} \sin^2 \frac{\theta}{2} \end{bmatrix} \]

\[ \mathcal{T}_\omega^A = [B^R A]^\top \mathcal{T}_\omega^B + \mathcal{B}^\omega^A = -\mathbf{A}^A(\phi, \theta)[\dot{\phi}, \dot{\theta}]^\top \quad (6.2) \]

Note that the 1 and 2 components of these expressions are equal and of opposite sign and that the 3 components are identical. This follows rather intuitively from the inertial and geometrical symmetry of the system.

The kinetic energy of the system is:

\[ T = \frac{1}{2} (\mathcal{T}_\omega^B)^\top \cdot \mathbf{I}_B \cdot \mathcal{T}_\omega^B + (\mathcal{T}_\omega^A)^\top \cdot \mathbf{I}_A \cdot \mathcal{T}_\omega^A \]

\[ = \frac{a}{4} \dot{\theta}^2 + 2\dot{\phi}^2 \frac{ab}{a + b + (b - a) \cos \theta} \sin^2 \frac{\theta}{2} \quad (6.3) \]

We place a torsional spring with stiffness \( k \) inside the no-twist joint, such that the potential energy of the system is:

\[ U = \frac{1}{2} k \theta^2 \quad (6.4) \]

The reduced Lagrangian is therefore:

\[ l(r, \dot{r}) = T - U = \frac{a}{4} \dot{\theta}^2 + 2\dot{\phi}^2 \frac{ab}{a + b + (b - a) \cos \theta} \sin^2 \frac{\theta}{2} - \frac{1}{2} k \theta^2 \quad (6.5) \]

From the reduced Lagrangian, we derive the equations of motion. We consider the homogeneous solutions (i.e. generalized forces are zero) which are representative of the normal vibrational modes of the system. Applying the Euler-Lagrange equations yields two coupled nonlinear differential equations:
\[
\frac{4ab}{a + b + (b - a) \cos \theta} \left( \frac{\dot{\theta} \sin \theta + \dot{\phi} \sin^2 \frac{\theta}{2}}{a + b + (b - a) \cos \theta} \right) = 0
\]

\[
a\ddot{\theta} + 2k\theta - \frac{\dot{\phi}^2}{a + b + (b - a) \cos \theta} \sin \theta = 0
\]  

(6.6)

Things seem like they break down when \( b = 0 \), which is interesting. The second equation shows that \( \theta \) behaves like a nonlinearly-forced harmonic oscillator (the forcing kind of looks like a centripetal acceleration).

We can recover two classes of solutions by inspection: For the first, assume that \( \dot{\phi} = \ddot{\phi} = 0 \), and the equations reduce to:

\[
\ddot{\theta} + \frac{2k}{a} \theta = 0
\]  

(6.7)

This is a simple harmonic oscillator with natural frequency \( \omega = \sqrt{\frac{2k}{a}} \), which is to be expected—imagine two planar bodies with rotational inertia \( a \) connected by a revolute joint with torsional stiffness \( k \) at their mass-center.

For the second class of solutions, assume that \( \dot{\phi} = \Omega, \ddot{\phi} = 0, \theta = \theta_c, \dot{\theta} = \ddot{\theta} = 0 \). The equations reduce to:

\[
2k\theta_c - \Omega^2 \frac{4ab^2}{(a + b + (b - a) \cos \theta_c)^2} \sin \theta_c = 0
\]  

(6.8)

This demonstrates that circular gaits are possible for some combinations of \( \Omega \) and \( \theta_c \) (best to think of solving for the equilibrium angle \( \theta_c \) that corresponds to a given \( \Omega \)). Besides the trivial solution, the above equation is not guaranteed to have a solution: there is a minimum necessary \( \Omega \).

We can solve for \( \Omega(\theta_c) \); this allows us to evaluate the result of any circular gait sustainable via stiffness.

\( B \neq A \)

If the two bodies are not equivalent, it is still useful to consider the case when they are both inertially symmetric but not identical; \( B \) is characterized by an inertia \( I \) and
\( A \) by an inertia \( J \).

The equations of motion are very similar:

\[
\begin{align*}
\frac{IJ}{I+J} \left( \dot{\phi} \sin \theta + \ddot{\phi} \sin^2 \frac{\theta}{2} \right) &= 0 \\
\frac{IJ}{I+J} \left( \dddot{\phi} \sin \theta \right) + k \theta &= 0
\end{align*}
\] (6.9)

**Small-angle solution**

For small angles, \( \Omega = \sqrt{\frac{2k}{a}} \) is independent of \( \theta_0 \). This is the same as the natural frequency of vibration when \( \dot{\phi} = 0 \); this is not coincidental.

If we reparameterize in terms of \((\alpha, \beta)\), we know that \((\alpha, \beta) = (\gamma_1, \gamma_2) + \mathcal{O}(\epsilon^3)\); thus know that the small-angle dynamics of \((\alpha, \beta)\) will be equivalent to the small-angle dynamics of \((\gamma_1, \gamma_2)\) with \( k_1 = k_2 = k \).

### 6.2.2 Dual Seesaw

The reduced dynamics show that, to first order, the dynamics of each seesaw decouple:

\[
\frac{e(a + \delta)}{a + e + \delta} \ddot{\theta}_i + k_i \theta_i = 0
\] (6.10)

Assuming that each joint has some stiffness \( k_i \).

Therefore, arbitrary sinusoidal gaits are possible

\[
\theta_i(t) = a_i \sin(\omega t + \Delta_i)
\] (6.11)

Where \( \omega = \sqrt{\frac{k(a+e+\delta)}{e(a+\delta)}} \) and the \( \Delta_i \) and \( a_i \) are arbitrary and determined by the initial conditions.

### 6.2.3 Universal Joint Cat

For small angles, oscillations in \( \gamma_1 \) decouple from oscillations in \( \gamma_2 \), indicating, yet again, that arbitrary sinusoidal gaits are possible.
\[ \frac{ac}{a+c} \dot{\gamma}_i + k_i \gamma_i = 0 \]  

(6.12)

**Large angle solution**

If we consider the case where bodies \( B \) and \( A \) are inertially symmetric but not identical—characterized by inertias \( I \) and \( J \), respectively—then the dynamics also decouple!

\[ \frac{IJ}{I+J} \ddot{\gamma}_i + k_i \gamma_i = 0 \]  

(6.13)

The solutions are still sinusoids! Importantly, this demonstrates that we can reproduce via structural stiffness the gaits defined in Equation 5.20 if \( a = b \) and \( c = d \) and if we are allowed to tune the stiffnesses \( k_i \). We will refer to this symmetric version of the Universal Joint Cat as the *inertially-simplified Universal Joint Cat.*

### 6.3 Reconstruction

#### 6.3.1 Small-amplitude

For all the small-amplitude gaits, the shape space trajectories achievable via dynamics are generically of the form:

\[ r_i(t) = a_i \sin(\omega t + \Delta_i) \text{ for } i = 1, 2 \]  

(6.14)

These gaits trace out ellipses in the cartesian shape space, with eccentricity determined by both the phase difference \( \Delta_1 - \Delta_2 \) between oscillators as well as the relative magnitude \( a_1/a_2 \). Because the curvature component \( \kappa_3 \) is constant for each of the small-amplitude model systems, the net angular change per gait cycle is equal to \( \kappa_3 A \). Bounding the shape space by stipulating that \( a_i \leq a_o \), the maximum possible value of \( A \) (and thus the maximum value of \( \Delta \psi \)) occurs when \( \Delta_1 - \Delta_2 = \frac{\pi}{2} \) and when \( a_1 = a_2 = a_o \), in which case \( A \) is simply a circle of radius \( a_o \).
For these gaits, the change in orientation angle about the axis \( b_3 \) per gait cycle is:

\[
\Delta \psi = \pi a_c^2 \kappa_3
\]  

(6.15)

The average angular velocity about the axis \( b_3 \) is simply the change in orientation divided by the period of oscillation \( \tau = \frac{2\pi}{\omega} \):

\[
\langle \dot{\psi} \rangle = \frac{\omega a_c^2}{2} \kappa_3
\]  

(6.16)

Note that this average angular velocity scales with the amplitude of oscillation squared, which is very small, but also with the frequency of vibration, which for a stiff system could be large.

### 6.3.2 Large-amplitude

For the large-amplitude solutions, we can only easily investigate circular shape space trajectories. The inertially-symmetric No-Twist Cat has an exact solution since the underlying Lie group is abelian (therefore it is uncontrollable via dynamical effects, since it can reorient only about the \( c_3 \) axis); the inertially-simplified Universal Joint Cat does not have an easily-computable exact solution (since the underlying Lie group is non-abelian), but it is fully controllable.

### 6.4 Performance metrics

To evaluate the ability of these systems to reorient, as compared to more traditional shape change actuators such as momentum wheels, we introduce two nondimensional metrics: momentum efficiency and energy efficiency.

The momentum efficiency is the ratio of the change in orientation per cycle to the maximum amplitude of vibration:

\[
\eta_m = \frac{\Delta \psi}{a_o}
\]  

(6.17)
The energy efficiency is the ratio of the useful “kinetic energy” stored in the time-averaged rotational motion of the system to the potential energy stored in the spring element:

\[
\eta_e = \frac{\ddot{\xi} I_{\text{tot}} \ddot{\xi}}{ka_2^2}
\]  

(6.18)

Where \( I_{\text{tot}} \) is the total, “locked” moment of inertia of the system at the beginning/end of the gait cycle.

### 6.4.1 Small-amplitude

The performance metrics are relatively simple:

\[
\eta_m = \pi a_c \kappa_3 \\
\eta_e = \frac{1}{4} I_{3,\text{tot}} J_{\text{red}} a_c^2 \kappa_3^2
\]

(6.19)

Where \( I_{3,\text{tot}} \) is the total inertia about the reorientation axis, and \( J_{\text{red}} \) is the reduced inertia about the axes of vibration.
Chapter 7

Conclusion: Engineering Applications

I hope that this thesis has served as both a useful and interesting, though necessarily incomplete, introduction to nonholonomic mechanics and geometric control and an explanation and investigation of a particular nonholonomic phenomenon with engineering application. To that end, this thesis has largely been a theoretical endeavor; though this is not for lack of experimental attempts. Thus, I would like to conclude by quickly going over the design and construction of a robotic device implemented using the dynamical principles developed herein. This should also place this work firmly in the category of compliant robotics, with the goal of harnessing structural properties of materials to design novel mechanical systems.

I would also like to summarize two important results of this thesis—full $SO(3)$ controllability via shape change and the sustaining of useful gaits via structural stiffness—in the context of potential application to the design of attitude control actuators for spacecraft and space robots.

7.1 Structural attitude control

In §5.2.1 we demonstrated large-amplitude gait design for the Universal Joint Cat that is able to effect orientation change independently about three orthogonal axes. Further, in §6.2.3 we demonstrated that for certain inertial and stiffness configurations, these gaits are sustainable via structural stiffness. Attractively, this system
only requires two actuators.

One might imagine incorporating this general structure into the design of a satellite, where the inertia required for reorientation is not contained in dedicated hardware (e.g. reaction wheels) but is instead comprised of the satellite’s superstructure and other required hardware (maybe cameras or other instruments). Further, by incorporating stiffness at the joints, some of the energy required to effect orientation change could be stored in springs or other energy-storage elements, reducing the power required to change orientation. Also, as a consequence of the bounded motion required to change orientation, no fragile bearings would be required to sustain continuous rotational motion, as in a reaction wheel setup.

A slight complication is the fact that, in order to be fully controllable, the stiffnesses of the two joints would have to be dynamically tunable (so as to allow generation of all the gaits in Figure 5-3).

7.2 MEMS attitude control

Both Reiter and Campbell[45] and Koh, Ostrowski, and Ananthasuresh[40] propose bounded shape change MEMS actuators for microsatellites. However, Reiter’s analysis is somewhat flawed—he ends up concluding that the bounded microactuator can apply a torque of nonzero time-average to a satellite, which violates conservation of angular momentum. Koh provides a very thorough dynamical analysis but does not incorporate structural stiffness.

In §6.3.1 we demonstrated that productive small-amplitude gaits for each of the model systems can be sustained via structural stiffness. This might motivate the development of MEMS resonator devices, analogous to the vibrating structure gyroscopes that have become ubiquitous in modern technology, which can effect (instead of sense, as does a gyroscope) orientation change about one axis. The performance of these devices would necessarily be coupled to the inertia parameters of the spacecraft that they are affixed to; however, one might improve performance by implementing arrays of such devices, effectively increasing actuator inertia.
This design is particularly attractive since it could be chip-printable, for use in so-called “satellites-on-a-chip”[2].

7.3 Catstronaut II

This small robot was envisaged as a proof-of-concept for attitude control via shape change. A tiny custom PCB was designed around the Atmel ATtiny167 microcontroller, capable of driving (via a TI DRV8835 motor driver) precisely phased oscillatory signals through a set of four linear resonant actuators (Precision Microdrives C10-100), which provide the requisite change of shape. Orientation data is sensed via a 6-axis IMU (Invensense MPU-6000) and logged to flash memory (Winbond W25Q128FV). Regulated power is supplied by a pair of small LiPo batteries, and wireless communication is achieved via infrared.

![Figure 7-1: Catstronaut II.](image)

Eulogy

Unfortunately, Catstronaut II did not provide useful experimental data. This was due to the immense challenge of providing a suitable torque-free environment for
the robot to operate in. The first iteration attempted to use a steel-on-steel pin support, which had the interesting consequence of transmitting angular momentum to the robot when the LRAs were activated.\footnote{This phenomenon was rationalized to occur due to nonzero lateral damping at the pivot point, which evinces some complex contact mechanics that would be better avoided.} The second iteration used a custom spherical air bearing (cast from epoxy and with one airflow perforation drilled with a 0.35mm bit) that supported a 5mm stainless steel ball bearing affixed to the robot. This also proved insufficient to demonstrate attitude control via shape change, as the bearing performance (though qualitatively impressive) was dominated at low angular velocities by nonuniform airflow through the bearing. Amusingly, Catstronaut II would spin happily in circles if activated while resting on a flat surface, much like a BristleBot.\footnote{http://www.bristlebots.org/}

### 7.4 Future work

Many and myriad. The extension of these ideas to continuum mechanics might be interesting theoretically—\emph{i.e.} the attitude control of a continuum-deformable soft space robot. Experimentally, a lot is left to be done; the implementation of a practical device using these ideas is still lacking.

In my opinion, the keys to demonstrating the phenomenon experimentally are both scale (Catstronaut II was designed around tiny actuators, which meant that the observed attitude change was too small to be distinguished from disturbances without a precision air-bearing support) and, relatedly, the construction of an appropriate torque-free experimental environment. Since the Cats considered in this thesis rely on fully three-dimensional rotational kinematics, either a sophisticated spherical support is necessitated, or a truly torque-free environment.
Appendix A

Note on Notation

We have decided to adopt an unconventional notation for keeping track of reference frames, reference points, vectors, and tensors. This notation is, in general, adapted from Kasdin and Paley[24]. While it is perhaps not the prettiest, we think that it does a good job.

We adopt the following notational conventions:

- Reference frames are distinct from coordinate systems: a reference frame is best thought of as a set of orthogonal "directions" (unit vectors) that is "fixed" to a body or some similar kind of reference in space, while a coordinate system is often constructed as a set of measurements against a reference frame's "directions". A reference frame has its origin at some point \(P\).

- In general, we refer to a reference frame with an uppercase, script letter, such as \(A\). The set of three orthogonal unit vectors that define the reference frame are denoted by lowercase, boldface type: \(a_1, a_2, \text{ and } a_3\), for example.

- A position vector that connects points \(P\) and \(Q\) is denoted \(r_{P/Q}\) where the vector is taken to point from \(Q\) to \(P\).

- A rigid body has an implicitly-defined reference frame, aligned with the principle axes of inertia and fixed to the center of mass \(G\). In this thesis, we generally refer to rigid bodies and their respective reference frames interchangeably.
• Moment of inertia tensors $I_A$ and angular momentum vectors $h_A$ are expressed with components in the body-fixed reference frame of body $A$, as defined above.

• The angular velocity of reference frame $A$ in reference frame $F$ is denoted $\omega^A$ and is taken to be expressed in frame $A$.

• We locate the orientation of frame $A$ relative to frame $F$ using the rotation matrix $R^A_F$; conceptually, frames $A$ and $F$ start exactly aligned and then rotation $R^A_F$ is applied to $A$.

• Rotation matrices can also be thought of as coordinate transformations: A vector $u$ is expressed with components in frame $F$; then, that same vector expressed with components in frame $A$ is given by $[R^A_F]^T u = A^F_R u$; equivalently, a vector $v$ written in the frame $F$ and transformed to frame $A$ is given by $R^A_F v$.

• We introduce the following notation to denote the skew-symmetric matrix that corresponds to a vector $a = [a_1, a_2, a_3]^T$:

$$[a \times] = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$$ (A.1)
Bibliography


