1.138J/2.062J/18.376J, WAVE PROPAGATION

Fall, 2006 MIT

Notes by C. C. Mei

CHAPTER FIVE

MULTIPLE SCATTERING BY AN EXTENDED REGION OF INHOMOGENEITIES

In this chapter we shall treat two types of extended inhomogeneities: (i) periodic and (ii) random.

1 Waves in a periodic medium

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Propagation of light or sound wave is of long standing interest in several branches of basic and appied physics, from old disciplines such as x ray diffraction in crytallography, to the modern science of photonic crystals. Many problems in natural environment also involve wave propagaion in periodic media. For example, nearly periodic sand bars are freqeuently found in shallow seas outside the surf zone; their presence changes the wave climate near the coast. The technology of remote-sensing, either by underwater sound or by radio waves from a satellite, depends on our understanding of scattering by the wavy sea surface.

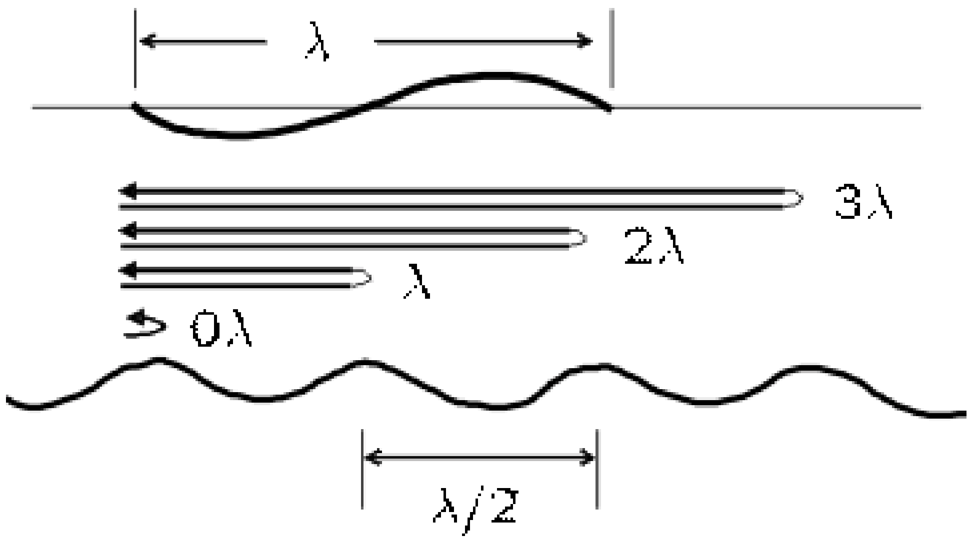

Figure 1: Bragg resonance due to contructive interference

In a periodic medium, the pheonomenon of Bragg resonance holds a special position. Let us explain it for the one-dimensional case of monochromatic water waves passing over a long stretch of parallel bars on an otherwise horizontal seabed. In nature, the bars are usually of much smaller amplitude than the wave depth. Most waves can pass over them without singificant reflection, except when the wavelength is an integral multiple of the bar period. In Figure1, the special case where $\lambda = 2\lambda_{bar}$ is sketched. Upon encountering each bar crest, every incident wave crest is mostly transmitted and slightly reflected. At a given bar crest, the height of the reflected wave crest is the sum of infintely many left-going crests. We called it the $n-$th crest if its reflection is originated at the n-th bar crest on the right. Because of the $1:2$ ratio, each crest has traveled the distance of $2n\lambda_{bar} = n\lambda$ since its first passage over the bar crest in question. As a consequence, they are all in phase with one another, hence the sum of the reflected wave intensity is very high.

We treat first waves in an elastic laminate of infinite extent. Borrowing the existing knowledge in solid-state physics, we discuss the relation between Bragg resonance and band gaps in the dispersion relation. We then treat the more practical case of finite extent in order to study the tranmission and reflection. The asymptotic method of multiple scales is applied to derive the coupled-mode equations for the slowly varying envelopes of incident and scattered waves.

The physics discussed here is representative of two and three dimensional period media, which would call for a much more elaborate theoretical analysis.

2 Waves in an infinitely thick laminate

We begin with

$$\frac{\partial}{\partial x}\left\{E(x)\frac{\partial u}{\partial x}\right\} = \rho\frac{\partial^2 u}{\partial t^2} \tag{2.1}$$

for the longitudinal displacment $u(x,t)$. We shall assume that the elasticity is periodically modulated about the mean value E_0. Defining $E(x) = E_0\mathcal{E}$ where $\mathcal{E}$ is a dimensionless periodic function of x. Equation (2.1) can be rewritten as

$$\frac{\partial}{\partial x}\left\{\mathcal{E}(x)\frac{\partial u}{\partial x}\right\} = \frac{1}{C^2}\frac{\partial^2 u}{\partial t^2} \tag{2.2}$$

where $C = \sqrt{E_0/\rho}$.

Consider a monochromatic wave in an elastic laminate so that

$$u(x,t) = \Re\left(U(x)e^{-i\omega t}\right) \tag{2.3}$$

then

$$\frac{\partial}{\partial x}\left\{\mathcal{E}(x)\frac{\partial U}{\partial x}\right\} + \frac{\omega^2}{C^2}U = 0 \tag{2.4}$$

What sort of wave can exist with period $L = Na$ (Born-von Karman condition)? Mathematically this is a homogeneous BVP, hence only eigensolutions exist for certain value of the eigen frequency ω. In the limiting case of uniform medium $\mathcal{E} = 1$, the solution is clearly

$$U = Ve^{ikx} \tag{2.5}$$

where $\omega = Ck$.

To solve the eigenvalue problem, Bloch (and Floquet) found the following theorem.

2.1 1-D Bloch's theorem

Bloch' theorem: For the one-dimensional problem governed by the ODE

$$\frac{d}{dx}\left(\mathcal{E}(x)\frac{dU}{dx}\right) + \alpha^2 U = 0, \quad \text{where } \mathcal{E}(x) = \mathcal{E}(x+a) \text{ and } \alpha = \frac{\omega}{C}, \tag{2.6}$$

subject to the condition that $U(x)$ is periodic over the period $L = Na$, the solution is of the form:

$$U(x) = \mathcal{U}(x)e^{ikx} \tag{2.7}$$

where $\mathcal{U}(x) = \mathcal{U}(x+a)$ is periodic in x and k is an eigenvalue.

We follow the reasoning in Ashcroft & Mermin in their more general proof for 3D quantum mechanics.

Since $E(x)$ is periodic with period a we can Fourier expand

$$\mathcal{E}(x) = \sum_K F_K e^{iKx} \tag{2.8}$$

where K is the short-hand notation for $K_n = 2n\pi/a$, with $n = 0, \pm1, \pm2, \pm3, ... \pm \infty$ and F_K is a function of K. Note that

$$e^{iKa} = e^{i2n\pi} = 1. \tag{2.9}$$

Let the general periodic soluton be

$$U(x) = \sum_q V_q e^{iqx} \tag{2.10}$$

where the expansion coefficient V_q is a function of q. We substitute (2.8) and (2.10) into (2.6), to get

$$\alpha^2 \sum_q V_q e^{iqx} - \sum_q \sum_K q(q+K)F_K V_q e^{i(q+K)x} = 0$$

Let us change the summation variable in the double series by letting $q + K = q'$, or, $q' = k - K$:

$$\alpha^2 \sum_q V_q e^{iqx} - \sum_{q'} \sum_K (q'-K)q' F_K V_{q'-K} e^{iq'x} = 0$$

or,

$$\sum_q \left[\alpha^2 V_q - \sum_K (q-K)qF_K V_{q-K}\right] e^{iqx} = 0$$

Becasue of the Born-von Karman condition, it can be shown that for different q, e^{iqx} are orthogonal to each other, so that

$$\alpha^2 V_q - \sum_K (q-K) q F_K V_{q-K} = 0 \tag{2.11}$$

Let us change the dummy symbol K to K' in the sum, and replace

$$q = k - K \tag{2.12}$$

$$\alpha^2 V_{k-K} - \sum_{K'} (k-K-K')(k-K) F_{K'} V_{k-K-K'} = 0 \tag{2.13}$$

Let us change again: $K'' = K + K'$ in the sum, then

$$\alpha^2 V_{k-K} - \sum_{K''} (k-K'')(k-K) F_{K''-K} V_{k-K''} = 0 \tag{2.14}$$

Because $K = K_0, \pm K_1, \pm K_2, ...$, this is an infinite set of simultaneous equations for

$$..., V_{k+K_2}, V_{k+K_1}, V_k (= V_{k-K_0}), V_{k-K_1}, V_{k-K_2}, ...$$

For a fixed k, we have by definition (2.12), $q = ...k+K_2, k+K_1, k, k-K_1, k = K_2, k-K_3, ...$ The corresponding coefficient solutions are: $...V_{k+K_2}, V_{k+K_1}, V_k, V_{k-K_1}, V_{k-K_2}, ...$ Thus the wave solution is

$$U_k(x) = \sum_K V_{k-K} e^{i(k-K)x} \tag{2.15}$$

Since for a fixed k

$$U_k(x) = \sum_K V_{k-K} e^{i(k-K)x} = \left\{ \sum_K V_{k-K} e^{-iKx} \right\} e^{ikx} \tag{2.16}$$

Denoting

$$\mathcal{U}_k(x) = \sum_K V_{k-K} e^{-iKx} \tag{2.17}$$

we get

$$U_k(x) = \mathcal{U}_k(x) e^{ikx} \tag{2.18}$$

Clearly

$$\mathcal{U}_k(x) = \sum_K V_{k-K} e^{-iKx} = \mathcal{U}_k(x+a) = \sum_K V_{k-K} e^{-iKx} e^{-iKa} \tag{2.19}$$

because of (2.9). Bloch's theorem is proven for the one dimensional problem.

We now use Bloch's theorem to solve the eigenvalue problem.

2.2 The dispersion relation- the eigenvalue condition

Consider a monochromatic wave in an elastic laminate so that

$$u(x,t) = \Re\left(U(x)e^{-i\omega t}\right) \tag{2.20}$$

then

$$\frac{\partial}{\partial x}\left\{\mathcal{E}(x)\frac{\partial U}{\partial x}\right\} + \frac{\omega^2}{c^2}U = 0 \tag{2.21}$$

where $C^2 = E_0/\rho$ and $\mathcal{E}$ is periodic in x with period L. In in general $\mathcal{E}$ can be expanded as a Fourier series

$$\mathcal{E}(x) = \sum_{m=-\infty}^{\infty} e_m e^{imKx} \tag{2.22}$$

where $K = \frac{2\pi}{L}$ is the wavenumber of the fundamental harmonic[1]. We can assume that $\mathcal{E}$ consists of the mean and the fundamental harmonic and rewrite it as

$$\mathcal{E}(x) = 1 + \sum_{m\neq 0} e_m e^{imKx} \tag{2.23}$$

Applying the Floquet-Bloch theorem we assume the solution to be of the following form:

$$U(x) = e^{ikx}\sum_n V_n(x)e^{inKx} \tag{2.24}$$

Substituting (2.23) and (2.24) and in (2.21), and noting that

$$e^{\pm iKx}\frac{\partial U}{\partial x} = \sum i(k+nK)V_n e^{i[k+(n\pm 1)K]x} = \sum i[k+(n\mp 1)K]V_{n\mp 1}e^{i[k+nK]x}$$

we get

$$\sum_n \left(\frac{\omega^2}{C^2} - (k+nK)^2\right)V_n e^{i(k+nK)x} - \sum_{n,m} e_m V_n (k+nK)(k+(n+m)K)e^{i(k+(n+m))x}$$

After changing the summation indices from $n+m$ to n', we find

$$\sum_n \left(\frac{\omega^2}{C^2} - (k+nK)^2\right)V_n e^{i(k+nK)x} - \sum_{n',m} e_m V_{n'-m}(k+(n'-m)K)(k+n'K)e^{i(k+n'K)x}$$

[1] In two or three dimensions, K is replaced by the vector $\mathbf{K}$ which is called the reciprocal lattice vector in crystallography and solid state physics.

From the coefficients of all harmonics, an infinite set of homogeneous algebraic equations for V_n:

$$\left[\frac{\omega^2}{C^2} - (k+nK)^2\right] V_n - \sum_m e_m V_{n-m}(k+(n-m)K)(k+nK) = 0 \tag{2.25}$$

In general these can be solved numerically.

As a reference, the solution for the limiting case of all $e_m = 0$ (uniform medium) is

$$U(x) = e^{ikx} V_0, \quad V_n = 0, \quad n \neq 0; \tag{2.26}$$

where V_0 is a constant. The eigenvalue condition is simply

$$\frac{\omega}{C} = \pm k \tag{2.27}$$

which are two straight lines in the upper half of $k - \omega$ plane. See Figure **??**.

We know, however, that the general Bragg resonance condition is

$$k + nK = -k, \quad \text{i.e.,} \quad k = \pm K/2, \pm K, \pm 3K/2, ... \tag{2.28}$$

Let us consider the Bragg resonance near $k - NK = -k$ or $k = NK/2$, for small $e_m = O(\epsilon)$.

We first take $n = -N$,

$$\begin{aligned} &\left[\frac{\omega^2}{C^2} - (k-NK)^2\right] V_{-N} - e_{-N} k(k-NK) V_0 \\ &- \sum_m e_m V_{-N-m}(k+(-N-m)K)(k-NK) = 0 \end{aligned} \tag{2.29}$$

and then take $n = 0$,

$$\begin{aligned} &\left[\frac{\omega^2}{C^2} - k^2\right] V_0 - e_N V_{-N}(k-NK)k \\ &- \sum_m e_m V_{-m}(k-mK)k = 0 \end{aligned} \tag{2.30}$$

For any $n \neq N$, (2.25) tells us that $V_n = O(\epsilon) V_0$. Hence in (2.29) and (2.30), we can ignore most terms except V_{-N} and V_0, so that

$$\left[\frac{\omega^2}{C^2} - (k-NK)^2\right] V_{-N} - e_{-N} k(k-NK) V_0 = 0 \tag{2.31}$$

and

$$\left[\frac{\omega^2}{C^2} - k^2\right] V_0 - e_N(k - NK)kV_{-N} = 0 \tag{2.32}$$

Let us now shift the origin:

$$k = \frac{NK}{2} + k', \quad \frac{\omega}{C} = \frac{NK}{2} + \frac{\omega'}{C}, \quad \text{where} \quad k' \sim O(e_{-N}) \ll 1, \ \frac{\omega'}{C} \sim O(e_{-N}) \ll 1. \tag{2.33}$$

By keeping only terms of the first order k', ω' we get

$$e_{-N}\frac{N^2K^2}{4}V_0 + NK\left(\frac{\omega'}{C} + k'\right)V_{-N} = 0$$

$$NK\left(\frac{\omega'}{C} - k'\right)V_0 + e_N\frac{N^2K^2}{4}V_{-N} = 0$$

Vanishing of the coefficient determinant gives the eigenvalue condition

$$\left(\frac{\omega'}{C}\right)^2 - k'^2 = \frac{1}{16}|e_N|^2N^2K^2 \tag{2.34}$$

which is represented by two branches of hyperbola in the k' vs. ω' plane. Use is made of the fact that $e_{-N} = e_N^*$. Since

$$k' = \sqrt{\left(\frac{\omega'}{C}\right)^2 - \frac{1}{16}|e_N|^2N^2K^2} \tag{2.35}$$

If,

$$\frac{\omega'}{C} > \frac{1}{4}|e_N|NK \tag{2.36}$$

k' is real; the wave is propagating. If

$$\frac{\omega'}{C} < \frac{1}{4}|e_N|NK \tag{2.37}$$

k' is imaginary; the wave is evanescent. The hyperbolic branches appear as discontinuities near the point of Bragg resonance as shown in Figure 2.2. This narrow band of frequencies in the ω/C vs. k plane is called the *forbidden band* in solid state physics.

We can assign indices to the pieces of the dispersion curve : $\omega_1(k)$ for the branch in $0 < k < K/2$, $\omega_2(k)$ for $K/2 < k < K$, $\omega_3(k)$ for $K < k < 3K/2$ etc.

Remark: If we replace k by $k + MK$, (2.2) becomes

$$U(x) = e^{ikx+iMKx}\sum_n V_n(x)e^{inKx} = e^{ikx}\sum_n V_n(x)e^{i(n+M)Kx}$$

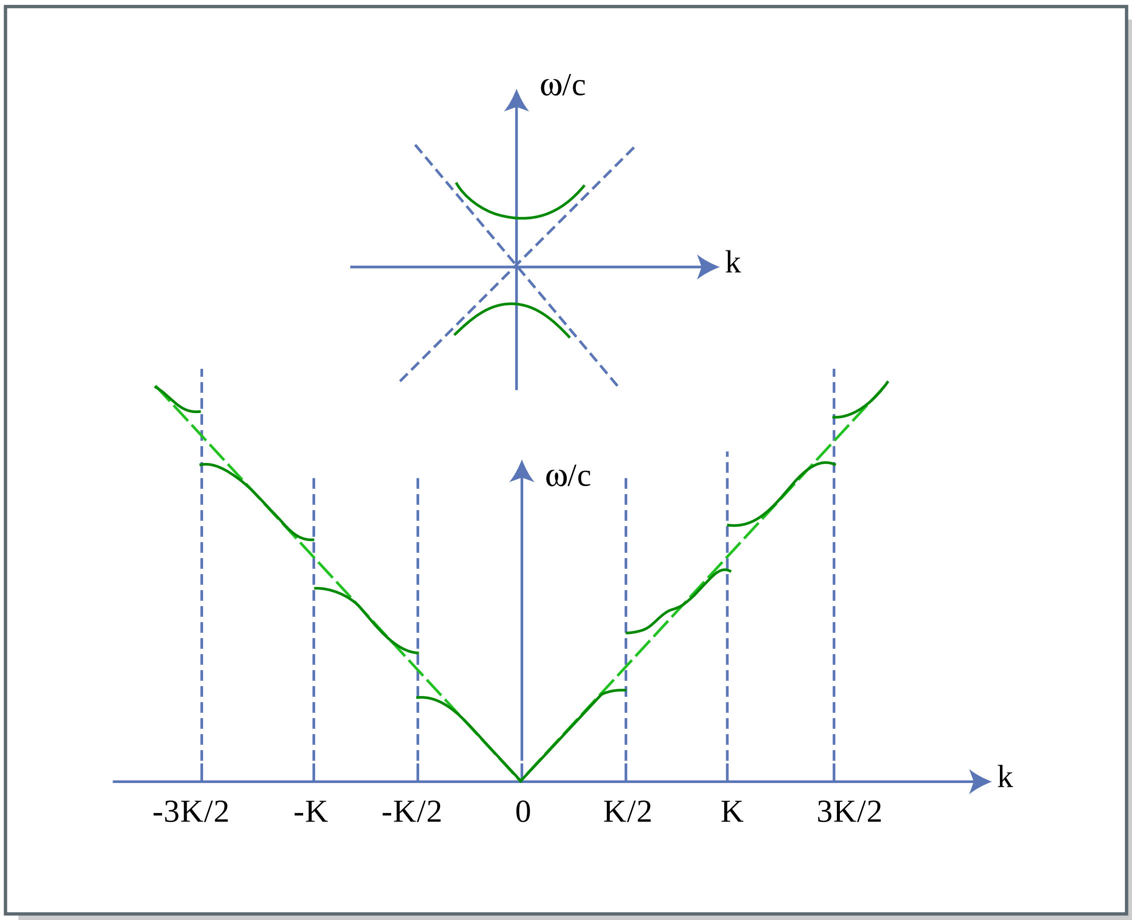

Figure by MIT OCW.

Figure 2: Dispersion relation and bandgaps: Top: Correction near the first Bragg resonance. Bottom: Dispersion curves.

or

$$U(x) = e^{ikx} \sum_{n'} V_{n'-M}(x) e^{in'Kx}$$

We can rename $V_{n'-M} = W_{n'}$, without changing the solution. Also(2.25) becomes

$$\left[\frac{\omega^2}{C^2} - (k + MK + nK)^2\right] V_n - \sum_m e_m V_{n-m}(k + MK + (n-m)K)(k + MK + nK) = 0$$

or

$$\left[\frac{\omega^2}{C^2} - (k + n'K)^2\right] V_{n'-M} - \sum_m e_m V_{n'-M-m}(k + (n'-m)K)(k + n'K) = 0$$

or

$$\left[\frac{\omega^2}{C^2} - (k + n'K)^2\right] W_{n'} - \sum_m e_m W_{n'-m}(k + (n'-m)K)(k + n'K) = 0$$

Thus the eigen-value problem remains the same.

It is therefore possible to shift all the branches $\omega_n(k)$ vs k and plot them in $-K/2 < k < K/2$, which is called the first Brillouin zone.

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3 Scattering by a finite layer of periodic laminate

Let us use a one-dimensional example to describe the phenomenon. First, since many scatters must be involved in order for this phenomenon to be appreciable, the total region of disturbances must be much greater than the typical wavelength. The perturbation method of multiple scales can be used. Second, since reflection is strong, incident and reflected wave must be allowed to be comparable in order.

Let us consider the one-dimensional scattering of elastic waves in a rod with a slightly periodic elasticity,

$$\rho = \text{constant}, \quad E = E_o(1 + \epsilon D \cos Kx), \tag{3.1}$$

where D is of order unity, i.e.,

$$E_0 \frac{\partial}{\partial x}\left[(1 + \epsilon\, D \cos\, Kx)\frac{\partial u}{\partial x}\right] = \rho \frac{\partial^2 u}{\partial t^2} \tag{3.2}$$

We now assume that the spatial period of inhomogeneity $\ell \equiv 2\pi/K$ and the elastic wavelength $\ell' \equiv 2\pi/k = 2\pi\sqrt{E_o/\rho}/\omega$ are comparable. As a consequence, wave reflection can be significant.

Let us first try a naive expansion, $u = u_0 + \epsilon u_1 + \cdots$. The crudest solution is easily found to be

$$u_0 = \frac{A}{2} e^{ikx - i\omega t} + \text{c.c.}, \tag{3.3}$$

where c.c. signifies the complex conjugate of the preceding term, and

$$\frac{2\pi}{k} \equiv \sqrt{\frac{E_o}{\rho}} \frac{2\pi}{\omega}, \quad \text{or} \quad \frac{\omega}{k} = C = \sqrt{\frac{E_0}{\rho}}. \tag{3.4}$$

At the next order the governing equation is

$$\begin{aligned} \frac{\partial}{\partial x}\left(E_o \frac{\partial u_1}{\partial x}\right) - \rho \frac{\partial^2 u_1}{\partial t^2} &= \frac{-E_o D}{2} \frac{\partial}{\partial x}\left[\left(e^{iKx} + e^{-iKx}\right)\frac{\partial u_0}{\partial x}\right] \\ &= \frac{-E_o D}{2} \frac{\partial}{\partial x}\left[\left(e^{iKx} + e^{-iKx}\right)\left(\frac{ikA}{2} e^{ikx - i\omega t} - \frac{ikA_*}{2} e^{-ikx + i\omega t}\right)\right]. \end{aligned} \tag{3.5}$$

Clearly, when

$$K = 2k + \delta, \quad \delta \ll k, \tag{3.6}$$

some of the forcing terms on the right will be close to a natural mode $\exp(\pm i(kx + \omega t))$. Resonance of the reflected waves must be expected. It sufices to illustrate the response to one of these terms,

$$E_o \frac{\partial^2 u_1}{\partial x^2} - \rho \frac{\partial^2 u_1}{\partial t^2} = A e^{i\phi_o} e^{\imath\delta x}, \quad \text{with} \quad \phi_o = kx + \omega t.$$

Combining homogeneous and inhomogeneous solutions and requiring that $u_1(0, t) = 0$, we find

$$u_1 = \frac{A e^{i\phi_o}\left(1 - e^{i\delta x}\right)}{E_o((k + \delta)^2 - k^2)}.$$

Clearly if $\delta = O(\epsilon)$, $\epsilon u_1 \sim O(\epsilon/\delta)$ and is not small compared to u_0 except for $\delta x \ll 1$. Furthermore as x increases, u_1 grows as ϵx. Thus when Bragg condition is satisfied, the reflected waves are resonated and is no longer much smaller that the incident waves in the distance $\epsilon x = O(1)$.

Let us now focus attention on the case of Bragg resonance. To render the solution uniformly valid for all x, we introduce fast and slow variables in space

$$x, \bar{x} = \epsilon x \tag{3.7}$$

To allow slight detuning from exact resonance, we assume that the incident wave frequency is $\omega + \epsilon\omega'$, where $\epsilon\omega'$ represents the small detuning and gives rise to a very slow variation in time. Therefore two time variables are needed,

$$t, \bar{t} = \epsilon t \tag{3.8}$$

The following multiple scale expansion is then proposed,

$$u = u_0(x, \bar{x}; t, \bar{t}) + \epsilon u_1(x, \bar{x}; t, \bar{t}) + \cdots . \tag{3.9}$$

After making the changes

$$\frac{\partial}{\partial x} \to \frac{\partial}{\partial x} + \epsilon \frac{\partial}{\partial \bar{x}}, \quad \frac{\partial}{\partial t} \to \frac{\partial}{\partial t} + \epsilon \frac{\partial}{\partial \bar{t}} \tag{3.10}$$

and substituting (3.9), (3.10) into (3.2), we get

$$\frac{\partial}{\partial x}\left(E_o \frac{\partial u_0}{\partial x}\right) - \rho \frac{\partial^2 u_0}{\partial t^2} = 0 \tag{3.11}$$

at $O(1)$. Anticipating strong but finite reflection, we take the solution to be

$$u_0 = \frac{A}{2} e^{ikx - i\omega t} + * + \frac{B}{2} e^{-ikx - i\omega t} + \text{c.c.}. \tag{3.12}$$

where $A(x_1, t_1)$ and $B(x_1, t_1)$ vary slowly in space and time. At the order $O(\epsilon)$ we have

$$\begin{aligned}
\frac{\partial}{\partial x}\left(E_o \frac{\partial u_1}{\partial x}\right) - \rho \frac{\partial^2 u_1}{\partial t^2} &= -2E_o \frac{\partial^2 u_0}{\partial x \partial \bar{x}} + 2\rho \frac{\partial^2 u_0}{\partial t \partial \bar{t}} \\
&\quad - \frac{E_o D}{2} \frac{\partial}{\partial x}\left[\left(e^{2ikx} + e^{-2ikx}\right) \frac{\partial u_0}{\partial x}\right] \\
= &\quad -E_o \left[\frac{\partial A}{\partial \bar{x}}(ik) e^{ikx - i\omega t} + \text{c.c.} + \frac{\partial B}{\partial \bar{x}}(-ik) e^{-ikx - i\omega t} + \text{c.c.}\right] \\
&\quad + \rho \left[\frac{\partial A}{\partial \bar{t}}(-i\omega) e^{ikx - i\omega t} + \text{c.c.} + \frac{\partial B}{\partial \bar{t}}(-i\omega) e^{-ikx - i\omega t} + \text{c.c.}\right] \\
&\quad - \frac{E_o D}{4} \frac{\partial}{\partial x}\left\{\left(e^{2ikx} + \text{c.c.}\right) \frac{\partial}{\partial x}\left[A e^{ikx - i\omega t} + \text{c.c.} + B e^{-ikx - i\omega t} + \text{c.c.}\right]\right\}
\end{aligned} \tag{3.13}$$

The last line can be reduced to

$$\begin{aligned}
-\frac{E_o D}{4} &\left(k^2 B e^{ikx - i\omega t} + \text{c.c.} + k^2 A e^{-ikx - i\omega t} + \text{c.c.}\right. \\
&\left. -3k^2 A e^{3ikx - i\omega t} + \text{c.c.} - 3k^2 B e^{-3ikx - i\omega t} + \text{c.c.}\right)
\end{aligned}$$

To avoid unbounded resonance of u_1, i.e., to ensure the solvability of u_1, we equate to zero the coefficients of terms $e^{\pm i(kx-\omega t)}$ and $e^{\pm i(kx+\omega t)}$ on the right of (3.13). The following equations are then obtained:

$$\frac{\partial A}{\partial \bar{t}} + c\frac{\partial A}{\partial \bar{x}} = \frac{ikCD}{4}B \tag{3.14}$$

$$\frac{\partial B}{\partial \bar{t}} - c\frac{\partial B}{\partial \bar{x}} = \frac{ikCD}{4}A, \tag{3.15}$$

where $\sqrt{E_o/\rho_o} = C = \omega/k$ denotes the phase speed. These equations govern the macroscale variation of the envelopes of the incident and reflected waves, and can be combined to give the Klein-Gordon equation

$$\frac{\partial^2 A}{\partial \bar{t}^2} - C^2\frac{\partial^2 A}{\partial \bar{x}^2} + \left(\frac{kCD}{4}\right)^2 A = 0. \tag{3.16}$$

Introducing the symbol Ω_o

$$\frac{kCD}{4} = \frac{\omega D}{4} \equiv \Omega_0 \tag{3.17}$$

which has the dimension of frequency, (3.16) many be written as

$$\frac{\partial^2 A}{\partial \bar{t}^2} - C^2\frac{\partial^2 A}{\partial \bar{x}^2} + \Omega_0^2 A = 0. \tag{3.18}$$

With suitable initial and boundary conditions on the macro scale, one finds the slow variation of these wave envelopes, hence the global behaviour of wave motion.

As an aside, let us first try a progressive-wave solution for the envelope in an infinite domain:

$$A = A_0 e^{iK\bar{x}-\Omega\bar{t}} \tag{3.19}$$

Physically, this amounts to a detuned wave

$$u = A(\bar{x},\bar{t})e^{ikx-i\omega t} = A_o \exp[i(k+\epsilon K)x - i(\omega+\epsilon\Omega)t] + *, \quad \bar{x} < 0\,, \tag{3.20}$$

Clearly

$$\Omega^2 - C^2K^2 = \Omega_0^2 \tag{3.21}$$

which represents two branches of hyperbola. Within the frequency gap $|\Omega| < \Omega_0$, K is imaginary and the envelope wave is evanescent. This result is equivalent to the band-gap theory before.

Now back to the finite laminate $0 < \bar{x} < L$ which is sandwiched between two semi-infinte solids ($\bar{x} < 0, \bar{x} > L$) with the same uniform elasticity E_0. Let the the incident wave train arriving from $\bar{x} \sim -\infty$ be slightly detuned from resonance To the left and to the right of the laminates, the governing equations are simply

$$A_{\bar{t}} + CA_{\bar{x}} = 0\,, \quad B_{\bar{t}} - CB_{\bar{x}} = 0\,, \quad \bar{x} < 0\,, \quad \text{and} \quad \bar{x} > L\,. \tag{3.22}$$

We shall assume further that $B = 0$ for $\bar{x} > L$ (the rediation condition). Over the bars (3.14) and (3.15), or (3.16) hold. In order that displacement and stress and horizontal velocity be continuous at $x = 0, L$, A and B must be continuous at $\bar{x} = 0, L$. Since the solutions must be of the form,

$$(A, B) = A_0(T(\bar{x}), R(\bar{x}))e^{-i\Omega\bar{t}}\,, \quad 0 < \bar{x} < L\,.$$

T and R are governed by

$$T_{\bar{x}\bar{x}} + \frac{(\Omega^2 - \Omega_0^2)}{C}T = 0\,, \quad 0 < \bar{x} < L\,.$$

Several cases can be distinguished according to the sign of $\Omega^2 - \Omega_0^2$:

Subcritical detuning: $0 < \Omega < \Omega_0$.

Let

$$Qc = (\Omega_0^2 - \Omega^2)^{1/2} \tag{3.23}$$

then

$$T(x) = \frac{iQC\cosh Q(L-\bar{x}) + \Omega\sinh Q(L-\bar{x})}{iQC\cosh QL + \Omega\sinh QL} \tag{3.24}$$

and

$$R(x) = \frac{Q\sinh Q(L-\bar{x})}{iQC\cosh QL + \Omega\sinh QL}\,. \tag{3.25}$$

On the incidence side the reflection coefficient is just $R(0)$ and on the transmission side the transmission coefficient is $T(L)$. Clearly the dependence on L and $\bar{x}$ is monotonic. In the limit of $L \to \infty$, it is easy to find that

$$T(x) = e^{-Q\bar{x}}\,, \quad R(x) = \frac{Q}{iQC + \Omega}e^{-Q\bar{x}}\,. \tag{3.26}$$

Thus all waves are localized in the range $\bar{x} < O(1/Q)$.

Supercritical detuning: $\Omega > \Omega_0$.

Let

$$PC = (\Omega^2 - \Omega_0^2)^{1/2} \tag{3.27}$$

then the transmission and reflection coifficients are:

$$T(x) = \frac{PC\cos P(L-\bar{x}) - i\Omega\sin P(L-\bar{x})}{PC\cos PL - i\Omega\sin PL} \tag{3.28}$$

and

$$R(x) = \frac{-iQ_0\sin P(L-\bar{x})}{PC\cos PL - i\Omega\sin PL}\,. \tag{3.29}$$

The dependence on L and $\bar{x}$ is clearly oscillatory. Thus Ω_0 is the cut-off frequency marking the transition of the spatial variation. For subcritical detuning complete reflection can occur for sufficiently large L. For super-critical detuning there can be windows of strong transission.

In the special case of perfect resonance, we get from (3.24) and (3.25) that

$$T(\bar{x}) = \frac{A}{A_o} = \frac{\cosh\frac{\Omega_o(L-\bar{x})}{C}}{\cosh\frac{\Omega_o L}{C}} \quad R(\bar{x}) = \frac{B}{A_o} = -\frac{i\sinh\frac{\Omega_o(L-x)}{C}}{\cosh\frac{\Omega_o L}{C}}\,. \tag{3.30}$$

A similar result is known for optical waves in layered media (Yariv & Yeh, 1984, p 197), and was also found for water waves over a wavy bed. In a laboratory experiment for water waves, Heathershaw(1982) installed 10 sinusoidal bars of amplitude $D = 5$ cm and wavelength 100 cm on the bottom of a long wave flume. Incident waves of length $2\pi/k = 200$ cm were sent from one side of the bar patch. On the transmission side, waves are essentially absorbved by breaking on a gentle beach. Sizable reflection coefficients were measured along many stations over the bar patch.

As shown in Figure 3, both $T(\bar{x})$ and $R(\bar{x})$ decrease monotonically from $\bar{x} = 0$ to $\bar{x} = L$, in good agreement with the experiments of Heathershaw. Thus enough small bars can generate strong reflection, especially in very shallow water.

Exercise 5.1: Bragg resonance by a corrugated river bank.

An infinitely long river has contant depth h and contant averaged width $2a$. In the stretch $0 < x < L$, the banks are slightly sinusoidal about the mean so that

$$y = \pm a \pm B\sin Kx\,, \quad KB \equiv \epsilon \ll 1\,. \tag{3.31}$$

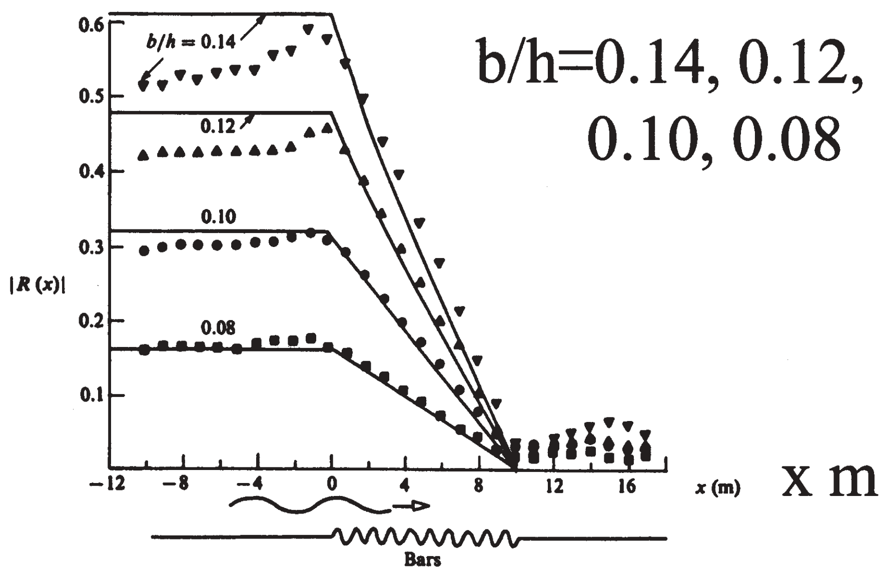

Figure 3: Bragg scattering of surface water waves by periodic bars. Comparison of theory with measurements by Heathershaw.

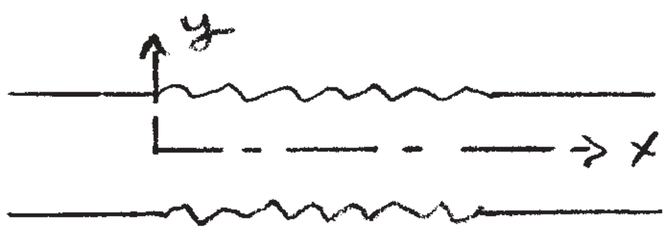

Figure 4: Can wavy banks serve as a breakwater?

See Fig. 4. Let a train of monochromatic waves be incident from $x \sim -\infty$,

$$\zeta = \frac{A}{2} e^{i(kx-\omega t)} \tag{3.32}$$

where $kh, ka = O(1)$. Develop a uniformly valid linearized theory to predict Bragg resonance. Can the corrugated boundary be used to reflect waves as a breakwater? Discuss your results for various parameters that can affect the function as a breakwater.

4 Wave localization in a random medium

[Ref]: Mei & Pihl Localization of nonlinear dispersive waves in weakly random media, *Proc. Roy. Soc. Lond.* 2002, 458, 119-134.

There are numerous situations where one needs to know how waves propagate through a medium with random impurities: light through sky with dust particles, sound through water with bubbles, elastic waves through a solid with cracks, fibers, cavities, hard or soft grains. Sea waves over a irregular topography, etc. It is known that, for one-dimensional propagation, multiple scattering yields a change in the wavenumber (or phase velocity) as well as an amplitude attenuation, if the inhomogeneities extend over a large distance. These changes amount to a shift of the complex propagation constant with the real part corresponding to the wavenumber and the imaginary part to attenuation. In particular, the spatial attenuation is a distinctive feature of randomness and is effective for a broad range of incident wave frequencies.This is in sharp contrast to periodic inhomogeneities which cause strong scattering only for certain frequency bands (Bragg scattering, see e.g., Chapter 1). Phillip W. Anderson (1958) was the first to show, in the context of solid-state physics, that a metal conductor can behave like an insulator, if the ,ir-crostructure has is disordered. This phenomoena, now called *Anderson localization*, is now known to be important in classical systems also. A survey of localization in many types of classical waves based on linearized theories can be found in Sheng (1998).

For weak inhomogeneties, the shift of propagation constant amounts to slow spatial modulations with a length scale much longer than the wavelnegth by a factor inversely proportional to the correlation of the fluctuations. In this section we apply the *method of multiple scales* to introduce the theory for the simplest exampole of one-dimensional sound.

We begin with the Helmhotz equation for sinusoidal waves,

$$\frac{d^2U}{dx^2} + k^2(1+\epsilon V(x))^2 U = 0, \quad \infty x < \infty. \tag{4.1}$$

Let $V(x)$ be a random function of x with zero mean and $V(x) \to 0$, for $x \sim -\infty$. An incident wave train

$$U_{inc} = A_0 e^{ikx} \tag{4.2}$$

arrives from the left-infinity where there is no disorder. What will happen, on the average, to waves after they enter the region of disorder?

Consider an ensemble of random media. For each realization, the wave number now fluctuates about the mean k by the amount order $O(\epsilon)$. Since $\langle V \rangle = 0$, we expect that, on the average, the wave phase is affected only by the root-mean-square, wich is of the order $O(\epsilon^2)$. With this guess, it is natural that slow variations described by $x_2 = \epsilon^2 x$ will be relevant. We assume that the disorder has two characteristc scales so that

$$V = V(x, x_2) \tag{4.3}$$

For simplicity we shall further assume that V is stationary with respect to the short scale

$$\langle V(x, x_2) V(x', x_2) \rangle = C_{vv}(|x - x'|, x_2) \tag{4.4}$$

where $\langle f \rangle$ denotes the ensemble average of f.

Let us try the following expansion,

$$U = U_0(x, x_2) + \epsilon U_1(x, x_2) + \epsilon^2 U_2(x, x_2) + \cdots \tag{4.5}$$

Subsituting (4.5) into (4.1), the following perturbation equations are found,

$$\frac{\partial^2 U_0}{\partial x^2} + k^2 U_0 = 0, \tag{4.6}$$

$$\frac{\partial^2 U_1}{\partial x^2} + k^2 U_1 = -2k^2 V U_0, \tag{4.7}$$

$$\frac{\partial^2 U_2}{\partial x^2} + k^2 U_2 = -2\frac{\partial U_0}{\partial x \partial x_2} - k^2 \left(2VU_1 + V^2 U_0\right), \tag{4.8}$$

The solution at the leading order is

$$U_0 = A(x_2) e^{ikx} \quad \text{where} \quad A(0) = A_o. \tag{4.9}$$

At the next order the inomogeneous equation is solved by Green's function $G(x, x')$ defined by

$$\frac{\partial^2 G}{\partial x^2} + k^2 G = \delta(x - x'), \tag{4.10}$$

where G is outgoing at infinities. The solution is found in Appendix A, Chapter 2 to be

$$G = -\frac{i}{2k} e^{ik|x - x'|} \tag{4.11}$$

$$(4.12)$$

The solution for U_1 is

$$\begin{aligned} U_1 &= -\int_{-\infty}^{\infty} dx' G(x,x') \left[2k^2 V(x',x_2)U(x',x_2)\right] \\ &= ik \int_{-\infty}^{\infty} dx' V(x',x_2) e^{ikx'} e^{ik|x-x'|} \end{aligned} \tag{4.13}$$

which is random with zero mean. For the $O(\epsilon^2)$ problem, we note that

$$2\frac{\partial^2 U_0}{\partial x \partial x'} = 2ike^{ikx}\frac{\partial A}{\partial x_2},$$

$$2k^2 V U_1 = 2ikA(x_2)e^{ikx} \int V(x,x_2)V(x',x_2)e^{ik|x-x'|}e^{-ik(x-x')}dx',$$

$$k^2 V^2 U_0 = k^2 e^{ikx} V(x,x_2)V(x,x_2)A(x_2).$$

We now take the ensemble average of (4.14), and get

$$\begin{aligned} \frac{\partial^2 \langle U_2 \rangle}{\partial x^2} + k^2 \langle U_2 \rangle &= -2ike^{ikx}\frac{\partial A}{\partial x_2} \\ &- 2ikk^2 A(x_2)e^{ikx} \int_{-\infty}^{\infty} \langle V(x,x_2)V(x',x_2)\rangle e^{ik|x-x'|}e^{-ik(x-x')}dx' \\ &- k^2 e^{ikx} A(x_2)\langle V^2(x,x_2)\rangle \end{aligned}$$

For $\langle U_2 \rangle$ to be solvable, we set the right-hand-side to zero,

$$\frac{\partial A}{\partial x_2} + A\left\{k^2 \int_{-\infty}^{\infty} dx' C_{vv}(|x-x'|,x_2)e^{ik|x-x'|}e^{-ik(x-x')}dx' - \frac{ik}{2}C_{vv}(0,x_2)\right\} = 0$$

Clearly the integral above is just a known function of x_2 once the correlation function is prescribed. Denoting

$$\beta(x_2) = \beta_r + i\beta_i = k^2 \int_{-\infty}^{\infty} dx' C_{vv}(|x-x'|,x_2)e^{ik|x-x'|}e^{-ik(x-x')}dx' - \frac{ik}{2}C_{vv}(0,x_2). \tag{4.14}$$

If $\beta = 0, x_2 < 0$ and $\beta =$ constant, $x_2 > 0$, then the solution is simply

$$A = A(0)e^{-i\beta_i x_2}e^{-\beta_r x_2} \tag{4.15}$$

Thus, not only the phase is changed but the amplitude decays exponentially over the distance $O(L)$ where

$$L = 1/\beta_r \epsilon^2 \tag{4.16}$$

In summary, due to scattering by disorder, an apparent damping is created. The distance L is called the localization distance.

For simple correlation functions, the integral for β can be explicitly evaluated. For example let

$$C_{vv}(|x-x'|, x_2) = \sigma^2(x_2)e^{-\alpha|x-x'|} \tag{4.17}$$

so that σ^2 is the RMS amplitude of the disorder. We leave it as an exercise to show that

$$\int_{-\infty}^{\infty} dx' e^{-\alpha|x-x'|} e^{ik|x-x'|} e^{-ik(x-x')} dx' = \frac{2(\alpha^2+2k^2)}{\alpha(\alpha^2+4k^2)} + \frac{2ik}{\alpha^2+4k^2} \tag{4.18}$$

so that

$$\beta = \beta_r + i\beta_i = 2k^2\sigma^2 \frac{\alpha^2+2k^2}{\alpha(\alpha^2+4k^2)} - \frac{ik\sigma^2}{2}\frac{\alpha^2}{\alpha^2+4k^2} \tag{4.19}$$

The leading order wave is

$$U_0 = A_0 \exp\left\{ik\left[1+\frac{\epsilon^2\sigma^2}{2}\frac{\alpha^2}{\alpha^2+4k^2}\right]x\right\} \exp -\left\{\frac{2\epsilon^2k^2\sigma^2}{\alpha}\left(\frac{\alpha^2+2k^2}{\alpha^2+4k^2}\right)x\right\}, \quad x>0 \tag{4.20}$$

As the RMS of the disorder increases, the wwavnumber increases, hence the wave length decreases. A dimensionless localization distance can be defined as

$$kL_{loc} = \frac{1}{2\epsilon^2\sigma^2}\frac{1+4k^2/\alpha^2}{(k/\alpha)(1+2k^2/\alpha^2)} \tag{4.21}$$

Note that the correlation length is $O(\alpha^{-1})$. If the waves are much longer than the correlation length, $k/\alpha \ll 1$; kL_{loc} increases without bound and localization is weak. If the waves are much shorter than the correlation length $k/\alpha \gg 1$; kL_{loc} decreases; waves cannot penetrate deeply into the disordered region.

IAP (challenge) Project : Scattering of elastic waves by random distribution of hard grains or cavities.

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