Total Mass of a Patch of a Matter-Dominated Friedmann-Robertson-Walker Universe

by

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in partial fulfillment of the requirements for the degree of

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Abstract

In this thesis, I have addressed the question of how to calculate the total relativistic mass for a patch of a spherically-symmetric matter-dominated spacetime of negative curvature. This calculation provides the open-universe analogue to a similar calculation first proposed by Zel'dovich in 1962. I consider a finite, spherically-symmetric ($\mathcal{SO}(3)$) spatial region of a Friedmann–Robertson–Walker (FRW) universe surrounded with a vacuum described by the Schwarzschild metric. Provided that the patch of FRW spacetime is glued along its boundary to a Schwarzschild spacetime in a sufficiently smooth manner, the result is a spatial region of FRW which transitions smoothly to an asymptotically flat exterior region such that spherical symmetry is preserved throughout. I demonstrate that this mass diverges as the size of the patch is taken to include the entire universe, and discuss the intuition provided by a classical approximation to the total mass using the formalism of Newtonian Cosmology.

Thesis Supervisor: Alan H. Guth
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Contents

1 Background and Motivation

1.1 Notations and Conventions

Notation	Definition
M^n	n-dimensional manifold
Σ	(n-1)-dimensional hypersurface in M^n
M^+	The manifold M "exterior" to Σ, described by the Schwarzschild metric
M^-	The manifold M "interior" to Σ, described by the FRW metric
Σ^+	$M^+\|_\Sigma$, The Schwarzschild side of the hypersurface
Σ^-	$M^-\|_\Sigma$, The FRW side of the hypersurface
$\{\xi^\mu\} = (t, \chi, \theta, \phi)$	Hyperspherical coordinates on M^-
$\{x^{\mu'}\} = (t', r', \theta', \phi')$	Spherical coordinates on M^+
$y^a = (\tau, \theta, \phi)$	Coordinates on Σ
e^μ_a	The holonomic basis of M^+, $e^\mu_a = \frac{\partial x^\mu}{\partial y^a}$
u^μ	$u^\mu = e^\mu_\tau$ Tangent vectors to Σ, expressed in the holonomic basis of M^+
(n_μ) n^μ	A normal (co)vector field to a hypersurface Σ

Notation	Definition
$g^{\pm}_{\mu\nu}$	The metric in spherical/hyperspherical coordinates on $M^{\pm}$
$\gamma^{\pm}_{ab}$	The metric induced on a hypersurface Σ from the embedding space $M^{\pm}$, via the pullback map
$[A^{\mu_1,\dots,\mu_k}_{\nu_1,\dots,\nu_l}]$	The *jump* in a rank (k,l) distribution-valued tensor A across a hypersurface Σ. $[A] = A\vert_{\Sigma^+} - A\vert_{\Sigma^-}$
$\alpha(t)$	The scale factor as a function of coordinate time t
σ	$\sigma = n^{\mu}n_{\mu}$, the square of the unit normal vector on a hypersurface Σ, $\sigma = \begin{cases} +1, & \text{if } \Sigma \text{ is timelike} \\ -1, & \text{if } \Sigma \text{ is spacelike} \end{cases}$
$K^{\pm}_{\mu\nu}$	The four-dimensional extrinsic curvature tensor of a hypersurface Σ in the manifold $M^{\pm}$
$K^{\pm}_{ab}$	The three-dimensional extrinsic curvature tensor of a hypersurface Σ, defined by taking the projection of $K^{\pm}_{\mu\nu}$ to $\Sigma^{\pm}$

1.2 Motivation

General Relativists have long debated the possibility of defining a meaningful expression for the total relativistic energy of an arbitrary curved spacetime (1). Existing formalisms for calculating total relativistic energy, such as the Arnowitt-Deser-Misner (ADM) energy, are applicable only to spacetime geometries which are asymptotically flat. In 1962, the famous Russian astrophysicist Yakov Borisovich Zel'dovich devised a method for computing the total relativistic mass of a closed universe described by the Friedmann-Robertson-Walker (FRW) metric (2)(3). Zel'dovich considered a finite spherically symmetric ($O(3)$) spatial region of an FRW universe surrounded with a vacuum described by the Schwarzschild metric. Provided that the patch of FRW spacetime is glued along its boundary to a Schwarzschild spacetime in a sufficiently smooth manner, the result is a spatial region of FRW which transitions smoothly to an asymptotically flat exterior region such that spherical symmetry is preserved throughout. Some four years after Zel'dovich's initial calculation, Israel first published his well-known Junction Conditions, a set of conditions that describe the discontinuity, or junction, between two spacetimes separated by a boundary region (4). In this paper we adopt these formalisms laid out by Zel'dovich and Israel to calculate an expression for the total relativistic mass of a spatial patch of an open Friedmann-Robertson-Walker universe dominated by matter. We will show that this mass diverges as the size of the patch is taken to include the entire universe. Sec I will introduce some concepts and definitions of central importance to our calculation. We will fully characterize the geometry of an FRW patch in Sec. II and perform the embedding of its boundary in Sec III. In Sec IV we will apply the Israel Junction Conditions to derive an expression for the total relativistic mass of an FRW patch. We end with a discussion, in Sec V, of the physical interpretation of the total relativistic mass and compare our result at small circumferential radius to the classical prediction of Newtonian cosmology.

1.3 Background Theory

The Einstein Field Equations in General Relativity are a set of equations of motion which relate the Stress-Energy tensor, which specifies the distribution of energy and momentum in a spacetime, with the Einstein tensor, which describes spacetime curvature. Beginning with a spacetime metric $g_{\mu\nu}$ and Stress-Energy tensor $T_{\mu\nu}$, it is necessary to solve the Field Equations to fully characterize the evolution of a spacetime. Although we will assume a familiarity with the basic tenets of General Relativity, a short summary of the Field Equations is provided for convenience. In this paper we will work in natural units, in which Newton's gravitational constant G has been set to unity. The content of this section follows, at various times, the discussions in (5), (6), and (7).

Given a metric $g_{\mu\nu}$, the spacetime interval ds^2 is given by

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{1.1}$$

The Christoffel connection is the unique connection derived from the metric tensor. It is defined by

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\lambda} (\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu}) \tag{1.2}$$

All the information about the intrinsic curvature of a spacetime is encoded in the Riemann tensor, which is given, in terms of the metric and Christoffel connection, by

$$\mathrm{R}^\alpha{}_{\mu\nu\lambda} = -\partial_\lambda \Gamma^\alpha_{\mu\nu} + \partial_\nu \Gamma^\alpha_{\mu\lambda} - \Gamma^\alpha_{\lambda\delta} \Gamma^\delta_{\mu\nu} + \Gamma^\alpha_{\nu\delta} \Gamma^\delta_{\lambda\mu} \tag{1.3}$$

Taking the trace of the Riemann tensor, we obtain the Ricci tensor $R_{\mu\nu}$. A further contraction produces the Ricci scalar R.

$$\mathrm{R}_{\mu\nu} = \mathrm{R}^\lambda{}_{\mu\lambda\nu} \tag{1.4}$$

$$\mathrm{R} = \mathrm{R}^\mu{}_\mu \tag{1.5}$$

The Einstein tensor is constructed from the metric, along with Eqs (1.4) and (1.5)

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \tag{1.6}$$

The stress-energy tensor $T_{\mu\nu}$ specifies the energy and momentum density and distribution in the spacetime that is the source of the gravitational field. It may also be thought of, in the context of Noether's theorem, as the conserved current associated with Lorentz translation symmetry. The Einstein Field Equations relate $T_{\mu\nu}$ to Eq (1.6),

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{1.7}$$

1.3.1 The Friedmann-Robertson-Walker Universe

Note: In this section, derivatives with respect to coordinate time t will be denoted by dots. Later we will use dots to denote derivatives with respect to proper time on the hypersurface, τ; we will indicate when this notation is to be adopted.

In the next section we will construct a spherically-symmetric patch of a Friedmann-Robertson-Walker (FRW) universe, thus it will be important to have a thorough understanding of the FRW geometry. The FRW metric describes an isotropic, homogenous, universe whose expansion is governed by a scale factor $\alpha(t)$. In (hyper)spherical coordinates, the line element is given by

$$ds^2 = -dt^2 + \alpha^2(t) \left(d\chi^2 + \zeta_k^2(\chi) d\Omega^2 \right) \tag{1.8}$$

Where

$$d\Omega^2 = (\theta) d\theta^2 + \sin^2\theta d\phi^2 \tag{1.9}$$

is the metric on the two-sphere, $\mathbb{S}^2$, and

$$\zeta_\kappa(\chi) = \begin{cases} \sin(\chi), & \text{if } \kappa = +1 \text{ (closed)} \\ \chi, & \text{if } \kappa = 0 \text{ (flat)} \\ \sinh(\chi) & \text{if } \kappa = -1 \text{(open)} \end{cases} \tag{1.10}$$

We will work throughout with the open FRW metric, in which $\kappa = -1$. The FRW solution has the following non-vanishing components of the Christoffel connection in the holonomic basis, up to symmetries. [1]

$$\begin{aligned}
\Gamma^t_{\phi\phi} &= \sin^2(\theta)\Gamma^t_{\theta\theta} = \sin^2(\theta)\sinh^2(\chi)\Gamma^t_{\chi\chi} = \sin^2(\theta)\sinh^2(\chi)\alpha(t)\dot{\alpha}(t) \\
\Gamma^\chi_{t\chi} &= \Gamma^\theta_{t\theta} = \Gamma^\phi_{t\phi} = \frac{\dot{\alpha}(t)}{\alpha(t)} \\
\Gamma^\chi_{\phi\phi} &= \sin^2(\theta)\Gamma^\chi_{\theta\theta} = -\sin^2(\theta)\sinh(\chi)\cosh(\chi) \\
\Gamma^\theta_{\chi\theta} &= \Gamma^\phi_{\chi\phi} = \coth(\chi) \\
\Gamma^\theta_{\phi\phi} &= -\cos(\theta)\sin(\theta) \\
\Gamma^\phi_{\theta\phi} &= \cot(\theta)
\end{aligned} \tag{1.11}$$

Non-vanishing components of the Riemann tensor, again up to symmetries [2], are given by

$$\begin{aligned}
R^t{}_{\phi t\phi} &= \sin^2(\theta)R^t{}_{\theta t\theta} = \sin^2(\theta)\sinh^2(\chi)R^t{}_{\chi t\chi} = \sin^2(\theta)\sinh^2(\chi)\alpha(t)\ddot{\alpha}(t) \\
R^\chi{}_{t\chi t} &= R^\theta{}_{t\theta t} = R^\phi{}_{t\phi t} = -\frac{\ddot{\alpha}(t)}{\alpha(t)} \\
R^\phi{}_{\chi\phi\chi} &= R^\theta{}_{\chi\theta\chi} = \sinh^2(\chi)R^\chi{}_{\theta\chi\theta} = \sinh^2(\chi)R^\phi{}_{\theta\phi\theta} = \dot{\alpha}(t)^2 - 1 \\
\sin^2(\theta)\sinh^2(\chi)R^\theta{}_{\phi\theta\phi} &= \sin^2(\theta)\sinh^2(\chi)R^\chi{}_{\phi\chi\phi} = \dot{\alpha}(t)^2 - 1
\end{aligned} \tag{1.12}$$

From the Riemann tensor, we may compute the Ricci tensor.

$$\begin{aligned}
R_{tt} &= \frac{-3\ddot{\alpha}(t)}{\alpha(t)} \\
R_{\chi\chi} &= \sinh^{-2}(\chi)R_{\theta\theta} = \sin^{-2}(\theta)\sinh^{-2}(\chi)R_{\phi\phi} = -2 + 2\dot{\alpha}(t)^2 + \alpha(t)\ddot{\alpha}(t)
\end{aligned} \tag{1.13}$$

[1] The Christoffel connection is symmetric under interchange of the two lower indices, $\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}$.

[2] When the Riemann tensor has all lower indices, $R_{\mu\nu\lambda\sigma}$, it is antisymmetric in its first and last pair of indices under interchange of indices, $R_{\mu\nu\lambda\sigma} = -R_{\nu\mu\lambda\sigma} = R_{\mu\nu\sigma\lambda}$. It is also symmetric under exchange of the first pair of indices with the last, $R_{\mu\nu\lambda\sigma} = R_{\lambda\sigma\mu\nu}$. The Riemann tensor has additional symmetries but those described here are sufficient for our purposes.

Finally, the Ricci curvature scalar is

$$R^{\mu}{}_{\mu} = R = \frac{6[\alpha(t)\ddot{\alpha}(t) + \dot{\alpha}(t)^2 - 1]}{\alpha^2(t)} \tag{1.14}$$

1.3.2 The Friedmann Equations

The Friedmann equations are a pair of independent equations relating the energy-momentum of an FRW universe to the scale factor, $\alpha(t)$. The form of the Friedmann equations will depend on our choice of Stress-Energy tensor; we consider the $T_{\mu\nu}$ for a perfect fluid with mass density ρ and pressure P.

$$T_{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & & & \\ 0 & & g_{ij}P & \\ 0 & & & \end{pmatrix} \tag{1.15}$$

The equation of state relates mass density with pressure via

$$P = w\rho \tag{1.16}$$

where the covariant conservation of the Stress-Energy tensor, $\nabla_\mu T^{\mu\nu} = 0$, will determine the dependence of the mass density on the scale factor. Henceforth we will refer to the following two equations as the first and second Friedmann Equations, respectively.

$$\left(\frac{\dot{\alpha}(t)}{\alpha(t)}\right)^2 = \frac{8\pi\rho(\alpha)}{3} + \frac{1}{\alpha^2(t)} \tag{1.17}$$

$$\frac{\ddot{\alpha}(t)}{\alpha(t)} = -\frac{4\pi}{3}(\rho(\alpha) + 3P) \tag{1.18}$$

A little context may be helpful here. Astrophysical observations have suggested our universe has gone through three periods of domination by different forms of energy-momentum. For the first 5×10^4 years, our universe was dominated primarily by radiation. In the following 9×10^9 years, our universe was dominated by matter, and for the most recent 5×10^9 years it has been dominated by dark energy. We have considered a matter-dominated FRW universe, corresponding to the second energy-momentum phase of our own universe, in which the pressure P can be neglected. A perfect fluid with zero pressure is called dust, and the corresponding stress energy tensor takes a particularly simple form

$$T_{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{1.19}$$

We will derive the Friedmann equations by manipulating the Einstein equations, using (1.19), (1.13), and (1.14), for a universe dominated by dust. Beginning with the Einstein tensor,

$$G_{tt} = \frac{3}{\alpha(t)^2}(-1 + \dot{\alpha}(t)^2) \tag{1.20}$$

$$G_{\chi\chi} = \sinh^{-2}(\chi) G_{\theta\theta} = \sin^{-2}(\theta) \sinh^{-2} G_{\phi\phi} = 1 - \dot{\alpha}(t)^2 - 2\alpha(t)\ddot{\alpha}(t) \tag{1.21}$$

the field equations produce two independent equations,

$$\frac{3}{\alpha(t)^2}(-1 + \dot{\alpha}(t)^2) = 8\pi\rho \tag{1.22}$$

$$1 - \dot{\alpha}(t)^2 - 2\alpha(t)\ddot{\alpha}(t) = 0 \tag{1.23}$$

Manipulating Eq (1.22) yields the first Friedmann equation, (1.17). Substituting for $\dot{\alpha}(t)$ from Eq (1.22) and rearranging, we arrive at the second Friedmann equation, (1.18).

1.3.3 The Schwarzschild Solution

The Schwarzschild metric provides the unique spherically-symmetric vacuum solution to the Einstein Field equations. In spherical coordinates, with the coordinate chart

$x^\mu = \{t', r', \theta', \phi'\}$, the metric is written

$$ds^2 = -f(r')dt'^2 + \frac{1}{f(r')}dr'^2 + r'^2 d\Omega'^2 \tag{1.24}$$

where, as in the FRW metric, $d\Omega^2$ is the metric on a two-sphere $\mathbb{S}^2$. The non-vanishing components of the Christoffel connection are

$$\begin{aligned}
\Gamma^{r'}_{r'r'} &= -\Gamma^{t'}_{r't'} = \frac{M}{(2M-r')r'} & \Gamma^{r'}_{t't'} &= \frac{(-2M+r')M}{r'^3} \\
\Gamma^{\theta'}_{\phi'\phi'} &= -\cos(\theta')\sin(\theta') & \Gamma^{r'}_{\phi'\phi'} &= \sin^2(\theta')\Gamma^{r'}_{\theta'\theta'} = \sin^2(\theta')(2M-r') \\
\Gamma^{\phi'}_{r'\phi'} &= \Gamma^{\theta'}_{r'\theta'} = \frac{1}{r'} & \Gamma^{\phi'}_{\theta'\phi'} &= \cot(\theta')
\end{aligned} \tag{1.25}$$

The non-vanishing components of the Riemann Curvature tensor are given by

$$\begin{aligned}
R^{t'}{}_{r't'r'} &= 2R^{\theta'}{}_{r'r'\theta'} = 2R^{\phi'}{}_{r'r'\phi'} = \frac{2M}{(-2M+r')r'^2} \\
R^{t'}{}_{\theta't'\theta'} &= R^{r'}{}_{\theta'r'\theta'} = \frac{1}{2}R^{\phi'}{}_{\theta'\theta'\phi'} = \frac{-M}{r'} \\
R^{t'}{}_{\phi't'\phi'} &= R^{r'}{}_{\phi'r'\phi'} = \frac{1}{2}R^{\theta'}{}_{\phi'\phi'\theta'} = \frac{-M\sin^2(\theta')}{r'} \\
R^{r'}{}_{t't'r'} &= 2R^{\phi'}{}_{t'\phi't'} = 2R^{\theta'}{}_{t'\theta't'} = \frac{2M(-2M+r')}{r'^4}
\end{aligned} \tag{1.26}$$

The Schwarzschild solution describes spacetime outside of the *Schwarzschild radius*, R_*, and it can also describe the spacetime inside the Schwarzschild radius, but one must keep in mind that for $r' < R_*$, r' is the timelike variable and t' is spacelike. The Schwarzschild coordinates do not give a clear description of how these two regions join, since the metric in these coordinates is singular at $r = R_*$. The constant M is identified as the mass of the gravitating body.The uniqueness of the Schwarzschild metric as the only spherically symmetric vacuum solution to the field equations is the subject of Birkhoff's theorem. It is a consequence of Birkhoff's theorem that the Schwarzschild solution is also static for $r > R_*$. [3] A crucial consequence of the

[3]A proof of Birkhoff's theorem consists of showing that a three-manifold that is spherically symmetric, in the sense that is has three Killing vectors which are the generators of the Lie Algebra $\mathfrak{so}(3)$, may be foliated by two-spheres. By extending the coordinates on these spherical submanifolds by the coordinates on the two dimensional orthogonal subspace generated by geodesics that intersect

properties of Eq (1.24) is that the Schwarzschild metric is asymptotically flat, making it an appropriate choice of spacetime with which to surround our open FRW patch.

1.4 Hypersurfaces

In this subsection, we present a collection of definitions and concepts which will be important for our calculations in the following sections. We attempt to present the necessary background theory with an appropriate degree of formality whilst indicating the relevance of each definition to our eventual computation.

1.4.1 Definitions

Definition 1.1. *Let Σ^m and M^n be differentiable manifolds, where dim(Σ) $= m <$* dim$(M) = n$. *The differentiable mapping $\varphi : \Sigma \to M$ is called an* immersion *if the differential map between tangent spaces, $d\varphi_p : T_p\Sigma \to T_{\varphi(p)}M$ is injective for all points $p \in \Sigma$. When the immersed manifold has codimension one, $n - m$=1, Σ^{n-1} is called a* hypersurface.

Definition 1.2. *A parameterization of a surface $\Sigma^{n-1} \subset M^n$ at a point p is a differentiable homeomorphism*

$$\xi^\mu : U \subset \mathbb{R}^{n-1} \mapsto N \cap \Sigma \subset \mathbb{R}^n \tag{1.27}$$

where $U \subset \mathbb{R}^{n-1}$ is an open subset and $N \cap \Sigma \subset \mathbb{R}^n$ is the intersection of a neighborhood $N \subset \mathbb{R}^n$ about the point p and the hypersurface Σ. Then ξ^μ is called the parameterization *or* coordinate chart *of the hypersurface Σ at p and the hypersurface is* regular.

A consequence of these definitions is that a function between parameterizations (or a change of coordinate chart) is itself a diffeomorphism. Therefore, a hypersurface may be thought of as the union of open subsets of $\mathbb{R}^{n-1}$ which overlap in a smooth way.

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each sphere normally at every point, one can construct a full set of coordinates for the spacetime and thus show the resultant metric, with an appropriate choice of coordinates, may be written in the form of (1.24).

In the special case that the immersion mapping, $\varphi : \Sigma \to M$ is also a homeomorphism onto its image $\varphi(\Sigma) \subset M$ then we call the mapping φ an embedding *and Σ^{n-1} a* submanifold.[4]

Returning to our open FRW patch, we will see that the boundary of the patch as we will define it constitutes a regular submanifold of codimension one. Although the boundary is a true $n-1$-dimensional submanifold, we will consistently refer to it as a hypersurface, as is conventional.

Definition 1.3. *Let M^n be a differentiable manifold. The* interior *of M , M°, is the set of all points $p \in M$ such that there exists a homeomorphism $\phi : N_p \to U_p \subset \mathbb{R}^n$ mapping a neighborhood of p to an open subset U of $\mathbb{R}^n$. Define the* boundary *of M^n as the relative complement of the interior, $\partial M = (M^\circ)^C$.*

Definition 1.4. *Given an n-dimensional manifold M^n and an atlas, or differentiable structure (ϕ_a, U_α), consisting of differentiable homeomorphisms ϕ_α that map open subsets U_α of M to $\mathbb{R}^n$, we define a* transition map *$\nu_{\alpha\beta}$ to be the composition map that takes points from one open set in the manifold to another via $\mathbb{R}^n$*

$$\nu_{\alpha\beta} = \phi_\beta \circ \phi_\alpha^{-1} \tag{1.28}$$

When all the transition maps $\nu_{\alpha\beta}$ preserve orientation then the manifold as a whole is said to be orientable.

One way to construct a manifold with boundary, which we will use in the next section, is to define an orientable hypersurface [5] of a manifold and discard the portion of the manifold that lies to one side, so that the hypersurface forms the boundary and the remaining portion the interior.We will define an open FRW patch as a manifold with boundary, where we will treat the boundary as a hypersurface to be embedded

[4]A submanifold need not be of codimension one, it may have codimension anywhere from zero to n. A hypersurface whose immersion map is homeomorphic onto its image is a submanifold of codimension one. Informally, an immersed hypersurface which is not a submanifold may be self intersecting or, even if the immersion is injective, the immersed hypersurface may not be compact.

[5]Here the distinction between hypersurface and submanifold is important. Although we use the term hypersurface here, this procedure requires a submanifold structure.

in a manifold equipped with the Schwarzschild metric. Thus we will use the terms boundary and hypersurface interchangeably.

Although it will be helpful to have a formal understanding of a hypersurface, in practice we will construct a hypersurface by specifying a restriction on the set of coordinates. Given coordinates ξ^μ on a manifold M, a hypersurface may be defined by the function

$$f(\xi^\mu) = f_0 \tag{1.29}$$

for a constant f_0.

It follows that if a manifold with boundary is orientable and connected, as our open FRW patch will be, then there exist only two possible orientations on the manifold. Orientation on a manifold with boundary induces an orientation on the boundary in the natural way. If (ϕ_α, U_α) is an oriented atlas on M, then the restriction of open sets in the atlas to the boundary, $U_\alpha|_{\partial M} = (U_\alpha)_\Sigma$ defines an oriented atlas for the subspace $\partial M = \Sigma$. Under this construction, the tangent space of the manifold at any point $q \in \partial M$, $T_q M|_{\partial M}$, is isomorphic to $T_q\Sigma \oplus \mathbb{R}$, where $\mathbb{R}$ represents the choice of a normal vector field. Thus, fully specifying the properties of an orientable boundary requires the direction of the normal vector field be fixed.

1.4.2 Pullback Map and the Induced Metric

Definition 1.5. *Let M and N be two smooth* [6] *manifolds, and consider the smooth mappings $\varphi : M \to N$ and $f : N \to \mathbb{R}$. The* pullback *of f by φ is defined to be*

$$\varphi^* f = (f \circ \varphi) : M \to \mathbb{R} \tag{1.30}$$

Definition 1.6. *Let $\varphi : M \to N$ be a smooth map between two smooth manifolds. The* pushforward *of φ at a point $p \in M$ is defined to be the differential map*

$$d\varphi : T_p M \to T_{\varphi(p)} N \tag{1.31}$$

[6] a smooth n-dimensional manifold is one whose atlas consists of infinitely differentiable mappings of open sets into $\mathbb{R}^n$.

There is a natural way to pull back one-forms, or elements of the cotangent space of N, $\omega_p \in T_p^*(N)$, and a natural way to push forward vectors, or elements of the tangent space of M, $v_q \in T_qM$, but the converse is not true.[7] The pullback of a metric tensor, however, is well defined, as a pullback can be constructed for tensors of rank $(0, l)$, since they only have covariant indices.

Definition 1.7. *Let M and N be two smooth manifolds, and consider a metric tensor g at a point $p \in N$, $g : T_pN \times T_pN \to \mathbb{R}$, and a smooth embedding map $\varphi : M \to N$. The pullback of g to M, γ, is called the* induced metric *on M*

$$\varphi^* g = \gamma : T_qM \times T_qM \to \mathbb{R} \tag{1.32}$$

Given coordinate charts $\{\xi^\mu\}$ on N and $\{y^a\}$ on Σ, the induced metric is given by

$$\gamma_{ab} = \frac{\partial \xi^\mu}{\partial y^a} \frac{\partial \xi^\nu}{\partial y^b} g_{\mu\nu} \tag{1.33}$$

1.4.3 Extrinsic Curvature on Σ^-

Definition 1.8. *The Projection Tensor $P_{\mu\nu}$ for a hypersurface Σ with normal vector field n^μ is given by*

$$P_{\mu\nu} = g_{\mu\nu} - \sigma n_\mu n_\nu \tag{1.34}$$

where $\sigma = n_\mu n^\mu$.

Eq (1.34) implies that, for objects already tangent to Σ, $P_{\mu\nu}$ behaves simply as the metric tensor.

Definition 1.9. *The* Lie derivative *along a vector field v^α of a vector u^β is given by the Lie bracket,*

$$\mathcal{L}_v(u^\beta) = [v, u]^\beta = v^\alpha(\partial_\alpha u^\beta) - u^\beta(\partial_\beta v^\alpha) \tag{1.35}$$

[7] A more careful but, for our purposes, needlessly complex definition of the pullback map states that the codomain of the pullback map is actually the pullback bundle. This subtlety is notable when the image of the pullback map is a proper subset of its codomain.

The Lie derivative acts on one-forms as

$$\mathcal{L}_v(w_\beta) = v^\alpha(\partial_\alpha w_\beta) + (\partial_\beta v^\alpha)w_\alpha \tag{1.36}$$

One can show that the Lie derivative along the normal vector field n^μ of the metric $g_{\mu\nu}$ takes a particularly simply form,

$$\mathcal{L}_n g_{\mu\nu} = 2\nabla_{(\mu}n_{\nu)} \tag{1.37}$$

Definition 1.10. *The* extrinsic curvature *on a four-dimensional manifold is defined as*

$$K_{\mu\nu} = \frac{1}{2}\mathcal{L}_n P_{\mu\nu} \tag{1.38}$$

Thus, substituting the definition (1.34) *for* $P_{\mu\nu}$ *into our definition for extrinsic curvature,* (1.38),

$$\begin{aligned} K_{\mu\nu} &= \frac{1}{2}\mathcal{L}_n P_{\mu\nu} \\ &= \frac{1}{2}\mathcal{L}_n(g_{\mu\nu} - \sigma n_\mu n_\nu) \\ &= \nabla_{(\alpha}n_{\beta)} - \sigma n_{(\mu}a_{\nu)} \end{aligned} \tag{1.39}$$

Evidently, the component of the extrinsic curvature which lies tangent to Σ is given by $\nabla_{(\mu}n_{\nu)}$, thus a projection of the extrinsic curvature to Σ admits the decomposition

$$K_{ab} = \frac{\partial\xi^\mu}{\partial y^a}\frac{\partial\xi^\nu}{\partial y^b}K_{\mu\nu} = \frac{\partial\xi^\mu}{\partial y^a}\frac{\partial\xi^\nu}{\partial y^b}\nabla_\mu n_\nu \tag{1.40}$$

1.4.4 Gaussian Normal Coordinates

In this subsection we will develop the useful notion of Gaussian normal coordinates, which significantly reduces the amount of work required to calculate the induced metric and extrinsic curvature of a given hypersurface. Begin with a manifold M^- equipped with the FRW metric, as in Eq (1.8), where $\kappa = -1$, expressed in coordinates $\{t, \chi, \theta, \phi\}$. We define a boundary in terms of a hypersurface $\Sigma \subset M^-$ in the manner described in Eq (1.29) by taking

$$\chi = \chi_0 \tag{1.41}$$

where χ_0 is a constant. Then Σ retains the spherical symmetry of the the spacetime in which it is embedded and the coordinates θ, ϕ on M^- map identically to the angular coordinates on Σ. Now consider a point on the hypersurface, $p \in \Sigma$, and take a neighborhood $B \subset M^-$ about p. For any point $q \in B$, q lies on precisely one geodesic $x_q^\alpha(z)$ (up to scaling) which intersects Σ orthogonally. This construction is unique up to a choice of orientation of Σ. We choose the positive orientation to be that in which the normal vector field n^α points outwards from the FRW spacetime (in the direction of increasing χ). The Gaussian Normal coordinates at $q \in B$ are given by $\xi_q^{\mu'} = \{\tau, z, \theta, \phi\}$, where τ is the value of coordinate time t at the intersection of $x_q^\mu(z)$ with Σ, and z is the proper distance along the geodesic from q to Σ in the positive direction. If we label the intersection of $x_q^\alpha(z)$ with Σ as $x_q^\mu(0) = q' = (\tau, 0, \theta, \phi)$, then the natural choice for the orientation of the normal vector is to take

$$n_{q'}^\alpha = fracdx_q^\alpha dz(0) \tag{1.42}$$

1.4.5 The Israel Junction Conditions

In the special case that a hypersurface Σ divides a spacetime into two distinct regions M^+ and M^- with different metrics, one may ask the conditions which the metrics must satisfy so that the two spacetimes are smoothly joined at the boundary and the Einstein Field equations are satisfied at the boundary.

A familiar analogue from which we may gain some intuition for the junction conditions is provided by the boundary conditions for an electric field at a sheet of charge. Consider a charge distribution with density

$$\rho(x^\mu) = \sigma(x^i)\delta(z) \tag{1.43}$$

where the coordinates $x^\mu = \{z, x^i\}$ are Gaussian Normal coordinates and z is the coordinate normal to the surface Σ. $\sigma(x^i)$ is the surface charge density, which describes a delta-function contribution to the charge distribution at Σ. From the Maxwell

Equations,

$$\nabla \cdot E = \frac{\rho(x^\mu)}{\epsilon_0} \tag{1.44}$$

the discontinuity in the electric field across the surface may be calculated

$$\int_{\Sigma-\epsilon}^{\Sigma+\epsilon} \nabla \cdot E = \int_{\Sigma-\epsilon}^{\Sigma+\epsilon} \frac{\rho(x^\mu)}{\epsilon_0} \tag{1.45}$$

$$= \int_{\Sigma-\epsilon}^{\Sigma+\epsilon} \frac{1}{\epsilon_0}\sigma(x^i)\delta(z) \tag{1.46}$$

$$E_z^+ - E_z^- = \frac{1}{\epsilon_0}\sigma(x^i) \tag{1.47}$$

Eq (1.47) says that discontinuity in the normal component of the electric field is accounted for by the charge density on the surface in order for Maxwell's Equations to be satisfied across the surface.

We will see a similar dynamic in the following derivation of the Israel Junction conditions. Discontinuity in the covariant derivative of the normal vector field will come with a physical meaning: the existence of a layer of energy-momentum along the hypersurface. [8]

Definition 1.11. *The* Heaviside Distribution $\Theta(z)$ *is the distribution such that*

$$\Theta(z) = \begin{cases} +1, & \text{if } z > 0 \\ 0 & \text{if } z < 0 \\ \text{indeterminate} & \text{if } z = 0 \end{cases} \tag{1.48}$$

The Heaviside distribution has the following properties:

- It is idempotent

$$\Theta(z)^2 = \Theta(z)$$

- It is antisymmetric under multiplication

$$\Theta(z)\Theta(-z) = 0$$

[8] The derivation put forth in this subsection will closely follow the approach of Poisson (6).

- Its derivative is the Dirac Delta distribution

$$\frac{d\Theta}{dz} = \delta(z)$$

Definition 1.12. *The* jump *in a tensor A across a hypersurface Σ is the difference*

$$[A_{\alpha\beta}] = A^{+}_{\alpha\beta} - A^{-}_{\alpha\beta} \tag{1.49}$$

where

$$\begin{aligned} A^{+}_{\alpha\beta} &= A(M^{+})\big|_{\Sigma} \\ A^{-}_{\alpha\beta} &= A(M^{-})\big|_{\Sigma} \end{aligned} \tag{1.50}$$

Let Σ be a hypersurface that partitions two spacetimes on which are given the coordinate charts $\{x^{\mu}_{+}\}$ and $\{x^{\mu}_{-}\}$ and metrics $g^{+}_{\mu\nu}$ and $g^{-}_{\mu\nu}$, respectively. Define the metric $g_{\mu\nu}$ by

$$g_{\mu\nu} = \Theta(z) g^{+}_{\mu\nu} + \Theta(-z) g^{-}_{\mu\nu} \tag{1.51}$$

We will derive the Riemann tensor from this distributional metric and determine the conditions under which it is a valid solution to the field equations by explaining the meaning of any singular terms. Taking partial derivatives of 1.51,

$$\partial_{\lambda} g_{\mu\nu} = \Theta(z)\partial_{\lambda} g^{+}_{\mu\nu} + \Theta(-z)\partial_{\lambda} g^{-}_{\mu\nu} + \delta(z)(g^{+}_{\mu\nu} - g^{-}_{\mu\nu})n_{\lambda} \tag{1.52}$$

Where in the last line we have made use of the fact that, in Gaussian Normal coordinates, the (normalized) normal vector field is given by

$$n_{\lambda} = n^{\mu} n_{\mu} \partial_{\lambda} z = \partial_{\lambda} z \tag{1.53}$$

In computing the Christoffel connection from (1.52), we will clearly generate terms proportional to $\Theta(z)\delta(z)$, which are not well defined tensor distributions. Thus a condition on the metric to be well defined is that this singular term vanish. In Gaussian Normal coordinates, this is equivalent to the requirement that the jump in the metric itself vanish (Eq (1.54)). A slightly weaker requirement, that the jump in the induced metric be vanishing, is applicable in any coordinate system (Eq (1.56)).

$$(g^{+}_{\mu\nu} - g^{-}_{\mu\nu}) = [g_{\mu\nu}] = 0 \tag{1.54}$$

Note that, while this expression for the jump in $g_{\mu\nu}$ is coordinate dependent, it is possible to amend this condition to an equivalent one

$$\left[g_{\mu\nu}\frac{\partial x^{\mu}}{\partial y^{a}}\frac{\partial x^{\nu}}{\partial y^{b}}\right] = 0 \tag{1.55}$$

by noting that the jump in the holonomic basis vectors across Σ must also vanish. Note that the left hand side of Eq (1.55) is precisely the jump in the induced metric. This gives the first junction condition,

$$[\gamma_{ab}] = (\gamma^{+}_{ab} - \gamma^{-}_{ab}) = 0 \tag{1.56}$$

The Christoffel connections on either side of the hypersurface are

$$\Gamma^{\pm\mu}_{\nu\lambda} = \frac{1}{2}g^{\pm\mu\rho}\left(\partial_{\lambda}g^{\pm}_{\rho\nu} + \partial_{\nu}g^{\pm}_{\rho\lambda} - \partial_{\rho}g^{\pm}_{\nu\lambda}\right) \tag{1.57}$$

And the distribution-valued Christoffel connection is then

$$\Gamma^{\mu}_{\nu\lambda} = \Theta(z)\Gamma^{+\mu}_{\nu\lambda} + \Theta(-z)\Gamma^{-\mu}_{\nu\lambda} \tag{1.58}$$

We may derive the distribution-valued Riemann Tensor,

$$R^{\mu}{}_{\nu\lambda\sigma} = \Theta(z)R^{+\mu}{}_{\nu\lambda\sigma} + \Theta(-z)R^{-\mu}{}_{\nu\lambda\sigma} + \delta(z)\left([\Gamma^{\mu}_{\nu\sigma}]n_{\lambda} - [\Gamma^{\mu}_{\nu\lambda}]n_{\sigma}\right) \tag{1.59}$$

Since the metric and its tangential derivative are continuous across the hypersurface, any discontinuity in the derivative of the metric is along the normal direction, z. Define a tensor $h_{\mu\nu}$ proportional to the normal vector,

$$\partial_{\lambda}g^{+}_{\mu\nu} - \partial_{\lambda}g^{-}_{\mu\nu} = h_{\mu\nu}n_{\lambda} \tag{1.60}$$

We may solve for $h_{\mu\nu}$ using

$$h_{\mu\nu} = (n^\rho n_\rho)[\partial_\lambda g_{\mu\nu}]n^\lambda = [\partial_\lambda g_{\mu\nu}]n^\lambda \tag{1.61}$$

Together with the definition for $\Gamma^\mu_{\nu\lambda}$, we may write

$$\begin{aligned} [\Gamma^\mu_{\nu\lambda}] &= \frac{1}{2}g^{\mu\rho}\left([\partial_\nu g_{\lambda\rho}] + [\partial_\lambda g_{\rho\nu}] - [\partial_\rho g_{\nu\lambda}]\right) \\ &= \frac{1}{2}\left(h^\mu{}_\rho n_\nu + h^\mu{}_\nu n_\lambda - h_{\nu\lambda}n^\mu\right) \end{aligned} \tag{1.62}$$

Now consider the δ-function term in the Riemann tensor, Eq 1.59. Define a tensor $\mathscr{R}$ such that

$$R^\mu{}_{\nu\lambda\sigma} = \Theta(z)R^{+\mu}_{\nu\lambda\sigma} + \Theta(-z)R^{-\mu}_{\nu\lambda\sigma} + \delta(z)\mathscr{R}^\mu{}_{\nu\lambda\sigma} \tag{1.63}$$

$$\mathscr{R}^\mu{}_{\nu\lambda\sigma} = ([\Gamma^\mu_{\nu\sigma}]n_\lambda - [\Gamma^\mu_{\nu\lambda}]n_\sigma) \tag{1.64}$$

So

$$\mathscr{R}^\mu{}_{\nu\lambda\sigma} = \frac{1}{2}\left(h^\mu{}_\sigma n_\nu n_\lambda - h^\mu{}_\lambda n_\nu n_\sigma - h_{\nu\lambda}n^\mu n_\sigma + h_{\nu\lambda}n^\mu n_\sigma\right) \tag{1.65}$$

Since $\mathscr{R}$ represents the δ-function part in the Riemann tensor, it is natural to define the tensor $\mathscr{R}_{\mu\nu}$,

$$\mathscr{R}_{\mu\nu} = \mathscr{R}^\lambda{}_{\mu\lambda\nu} \tag{1.66}$$

for the δ-function part of the Ricci tensor,

$$\mathscr{R}_{\mu\nu} = \frac{1}{2}\left(h_{\alpha\mu}n^\alpha n_\nu + h_{\alpha\nu}n^\alpha n_\mu - hn_\mu n_\nu - h_{\mu\nu}\right) \tag{1.67}$$

and the Ricci scalar,

$$\mathscr{R} = \mathscr{R}^\mu{}_\mu = (h_{\mu\nu}n^\mu n^\nu - h) \tag{1.68}$$

where $h = h^\mu_\mu$. With equations (1.66) and (1.68), the δ-function part of the Einstein tensor is

$$\mathscr{G}_{\mu\nu} = \mathscr{R}_{\mu\nu} - \frac{1}{2}\mathscr{R}g_{\mu\nu} \tag{1.69}$$

Now that we have the full form of the Einstein tensor, we may use it to solve the Einstein Field equations, and find the stress-energy tensor of the form

$$T_{\mu\nu} = \Theta(z)T^{+}_{\mu\nu} + \Theta(-z)T^{-}_{\mu\nu} + \delta(z)S_{\mu\nu} \tag{1.70}$$

where the δ-function part of the stress-energy tensor, as expected, comes from Eq (1.69),

$$8\pi S_{\mu\nu} = \mathscr{G}_{\mu\nu} = \mathscr{R}_{\mu\nu} - \frac{1}{2}\mathscr{R}g_{\mu\nu} \tag{1.71}$$

$S_{\mu\nu}$ represents the *surface stress-energy tensor*, which describes the existence of a surface layer of energy-momentum on the hypersurface.

With this interpretation, it is clear that if a curvature singularity exists, the associated δ-function term in the Riemann-Curvature is due to this shell of energy-momentum. By examining the explicit form of $S_{\mu\nu}$, we may show that there is a meaningful tensorial object whose continuity, or lack thereof, fully characterizes the second Junction condition. From Eqs (1.66) and (1.68),

$$16\pi S_{\mu\nu} = -h_{\mu\nu}e^{\mu}_{a}e^{\nu}_{b} + \gamma^{ij}h_{\mu\nu}e^{\mu}_{i}e^{\nu}_{j}\gamma_{ab} \tag{1.72}$$

Note however that the covariant derivative of the normal is

$$\nabla_{\mu}n_{\nu} = \frac{1}{2}\left(h_{\mu\nu} - h_{\lambda\mu}n_{\nu}n^{\lambda} - h_{\lambda\nu}n_{\mu}n^{\lambda}\right) \tag{1.73}$$

Recalling from Eq (1.40) the projection of extrinsic curvature tensor, we may note that

$$[K_{ab}] = e^{\mu}_{a}e^{\nu}_{b}[\nabla_{\mu}n_{\nu}] = \frac{1}{2}h_{\mu\nu}e^{\mu}_{a}e^{\nu}_{b} \tag{1.74}$$

Which, together with Eq (1.72) enables us to relate the surface stress-energy tensor $S_{\mu\nu}$ to the discontinuity of the extrinsic curvature across the hypersurface, $[K_{ab}]$. Thus we have arrived at the second junction condition,

$$S_{ab} = \frac{-1}{8\pi}\left([K_{ab}] - \gamma_{ab}[K]\right) \tag{1.75}$$

2 A Patch of an Open Friedmann-Robertson-Walker Universe

Note: Beginning in this section, we will adopt the convention that derivatives with respect to proper time $\tau = t|_\Sigma$ will be denoted by dots, and derivatives with respect to the conformal time η by primes.

We will now construct a patch of an open Friedmann-Robertson-Walker universe, M^-, by specifying a boundary $\Sigma^+ = M^-|_\Sigma$ just as in Sec 1.4.4. Beginning with the FRW metric as in Eq (1.8) in the coordinate basis, a closed form solution to the Einstein field equations is given by the set of parametric equations for the scale factor α and the coordinate time t in terms of a new timelike variable η.

$$\alpha(\eta) = \frac{\alpha_0}{2}(-1 + \cosh\eta) \tag{2.1}$$

$$t(\eta) = \frac{\alpha_0}{2}(-\eta + \sinh\eta) \tag{2.2}$$

$$\tag{2.3}$$

Notice the relationship between Eqs (2.1) and (2.2),

$$\frac{dt(\eta)}{d\eta} = \alpha(\eta) \tag{2.4}$$

which identifies the parameter η as the *conformal time.* In terms of the conformal time, the metric of FRW may be written

$$ds^2 = \alpha^2(\eta)(-d\eta^2 + d\chi^2 + \sinh^2\chi d\Omega^2) \tag{2.5}$$

(2.4) implies that

$$\frac{d\alpha}{dt} = \dot{\alpha} = \frac{d\alpha}{d\eta}\left(\frac{1}{\frac{dt}{d\eta}}\right) = \frac{\alpha'(\eta)}{\alpha(\eta)} \tag{2.6}$$

Rewriting the first Friedmann Equation, (1.17) in terms of $\alpha(\eta)$ and $\alpha'(\eta)$,

$$\begin{aligned} 8\pi\rho &= \frac{3}{\alpha^2(\eta)}\left(\left(\frac{\alpha'(\eta)}{\alpha(\eta)}\right)^2 - 1\right) \\ &= \frac{3}{\alpha^2(\eta)}\left(\frac{\alpha_0^2\sinh^2(\eta)}{4\alpha^2(\eta)} - 1\right) \\ &= \frac{3}{\alpha^2(\eta)}\frac{1}{\sinh^2\left(\frac{\eta}{2}\right)} \end{aligned} \tag{2.7}$$

Where in the last line of 2.7 I have used the fact that $\cosh^2(\eta) - \sinh^2(\eta) = 1$ to write $\cosh^2(\eta) - 1$ as a product of differences $(1 - \cosh(\eta))(1 + \cosh(\eta))$. Letting $\rho(\alpha)$ depend on $\alpha(\eta)$, we arrive at

$$\rho(\alpha) = \frac{3}{\alpha_0^2\pi(-1 + \cosh\eta)^3} \tag{2.8}$$

$$\rho(\alpha) = \frac{3\alpha_0}{8\pi}\frac{1}{\alpha^3} \tag{2.9}$$

2.1 Gaussian Normal Coordinates

Note: Throughout this subsection only, the notation $x^\mu(z)$ will denote the normal geodesic to the boundary parameterized by z. In later sections, a similar notation, x^μ, will be used for the coordinate chart on a Schwarzschild spacetime. As was discussed in section 1.4.4, Gaussian Normal coordinates are a convenient way to express the metric near to the hypersurface Σ and make the necessary computation of both the induced metric and extrinsic curvatures on Σ much simpler. As z is the unique coordinate along which a particle might move off the hypersurface, a restriction of the metric to Σ, (at $\chi = \chi_0$) implies that the spacetime interval has no variation along z, and the coordinates of Σ are clearly just $y^{a'} = \{\tau, \theta, \phi\}$, where χ is restricted to χ_0,

$$ds^2 = -d\tau^2 + \sinh^2(\chi_0)d\Omega^2 \tag{2.10}$$

And the pullback of the metric $g_{\mu\nu}$ onto Σ in Gaussian Normal coordinates, i.e. the induced metric on Σ, is given by

$$\gamma_{\alpha\beta}^{GN} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \sinh^2\chi_0 & 0 \\ 0 & 0 & \sin^2\theta\sinh^2\chi_0 \end{pmatrix} \tag{2.11}$$

The extrinsic curvature on Σ also has a nice and simple form in Gaussian Normal coordinates (8). Because the unique normal vector field to the hypersurface, expressed in Gaussian Normal coordinates, is just $n^\mu = (0, 1, 0, 0)$, Eq (1.38) reduces to

$$K_{\alpha\beta} = -\Gamma^z_{\alpha\beta} = \frac{1}{2}\partial_z g_{\alpha\beta} \tag{2.12}$$

Where $g_{\alpha\beta}$ is given by Eq (2.11) and the indices α, β take values in $\{y^{a'}\} = (\tau, \theta, \phi)$. Our project now is to find the precise form of the metric $g_{\mu'\nu'}^{GN}$ for the entire four-dimensional FRW spacetime, from which we may deduce both the induced metric and extrinsic curvatures on Σ with the help of the definitions above. We will once again use the coordinate charts $\{\xi^\mu\} = (t, \chi, \theta, \phi)$ and $\{\xi^{m'}\} = (\tau, z, \theta, \phi)$ for the original FRW coordinates and Gaussian normal coordinates on M^-, respectively.

In order to find the metric on M^- in Gaussian normal coordinates, $g_{\mu\nu}^{GN}$, we must solve the geodesic equation to determine the normal geodesics to the boundary. The form of the geodesic equation best suited to our calculation is

$$\frac{d}{dz}\left(g_{\mu\rho}\frac{d\xi^\rho}{dz}\right) = -\frac{1}{2}\frac{\partial g_{\nu\sigma}}{\partial x^\mu}\frac{\partial \xi^\nu}{\partial z}\frac{\partial \xi^\sigma}{\partial z} \tag{2.13}$$

where z is an affine parameter. Now, expanding the coordinate time t around the point $z = 0$ (i.e. on the boundary, where $t|_{z=0} = t(0) = \tau$ and $\chi|_{z=0} = \chi(0) = \chi_0$),

$$t(z) = t(0) + z\left.\frac{dt}{d\tau}\right|_{z=0} + \frac{1}{2}z^2\left.\frac{d^2t}{d\tau^2}\right|_{z=0} + \mathcal{O}(z^3) \tag{2.14}$$

As we are interested in regions close to the boundary, we consider only the terms in Eq (2.14) up to second order in z, and note that we must have $\frac{dt}{d\tau}\big|_{z=0} = 0$ in order that the geodesic $x^\mu(z)$ be normal to Σ^- within a neighborhood of Σ. Additionally, to lowest order in z, $t \approx \tau$, thus we may reasonably take the argument of the scale factor to be τ for small values of z.

Let us examine Eq (2.13) by components. We note that both χ and t are functions of τ and z by writing explicitly $\chi(\tau, z)$ and $t(\tau, z)$. Neither θ nor ϕ depend on τ and z.

When $\mu = 0$, (2.13) becomes

$$-\frac{d^2t}{dz^2} = \alpha(\tau)\dot{\alpha}(\tau)\left(\frac{d\chi}{dz}\right)^2 \tag{2.15}$$

$$\frac{d}{dz}\left[\alpha^2(\tau)\frac{d\chi}{dz}\right] = 0 \tag{2.16}$$

From Eq (2.16), we have that

$$\alpha^2(\tau)\frac{d\chi}{dz} = f_0(\chi_0, \tau_0) \tag{2.17}$$

for an undetermined constant $f_0(\chi_0, \tau_0)$ depending only on χ_0 and τ_0, where τ_0 is the value of τ at the point where the geodesic intersects Σ^-. We will surpress the arguments of f_0 from now on. Substituting for $\frac{d\chi}{dz}$ in Eq (2.15), we obtain

$$-\frac{d^2t}{dz^2} = \frac{\dot{\alpha}(\tau)}{\alpha^3(\tau)}f_0^2 \tag{2.18}$$

Using the formula for a change of coordinates from the original coordinates on FRW, $\{\xi^\mu\} = (t, \chi, \theta, \phi)$, to the Gaussian normal coordinates $\{\xi^{\mu'}\} = (\tau, z, \theta, \phi)$, the metric in Gaussian normal coordinates is given by

$$g^{GN}_{\mu'\nu'} = \frac{\partial\xi^\mu}{\partial\xi^{\mu'}}\frac{\partial\xi^\nu}{\partial\xi^{\nu'}}g_{\mu\nu} \tag{2.19}$$

Since the θ and ϕ coordinates on M^- map identically to θ, ϕ on Σ^-,

$$g_{\theta\theta}^{GN} = g_{\theta\theta} \tag{2.20}$$

$$g_{\phi\phi}^{GN} = g_{\phi\phi} \tag{2.21}$$

it only remains to find $g_{\tau\tau}^{GN}$ and g_{zz}^{GN}. Furthermore, we will require that the coefficient of dz^2 in the line element in Gaussian normal coordinates be normalized, consistent with the definition of Gaussian normal coordinates, so that

$$g_{zz}^{GN} = 1 \tag{2.22}$$

From Eq (2.19),

$$g_{\tau\tau}^{GN} = \alpha^2(\tau)\left(\frac{\partial\chi}{\partial\tau}\right)^2 - \left(\frac{\partial t}{\partial\tau}\right)^2 \tag{2.23}$$

$$g_{zz}^{GN} = \alpha^2(\tau)\left(\frac{\partial\chi}{\partial z}\right)^2 - \left(\frac{\partial t}{\partial z}\right)^2 \tag{2.24}$$

Within a neighborhood of the boundary, $\chi(\tau, z)$ may be expanded about $z = 0$ such that

$$\begin{aligned} \left.\frac{\partial\chi}{\partial z}\right|_{z=0} &= \frac{\partial}{\partial z}\left(\chi(0) + z\left.\frac{d\chi}{dz}\right|_{z=0} + \mathcal{O}(z^2)\right) \\ &= \frac{\partial}{\partial z}\left(\chi_0 + zc_1\right) \end{aligned} \tag{2.25}$$

where we have defined $c_1 = \left.\frac{d\chi}{dz}\right|_{z=0}$. Equation (2.25) implies that, since $\frac{\partial\chi}{\partial z}$ is approximately constant close to the boundary,

$$c_1 \approx \frac{\partial\chi}{\partial z} \tag{2.26}$$

We can exploit the requirement that $g_{zz}^{GN} = 1$ to solve for c_1 by noting that

$$\begin{aligned} \frac{d}{dz}\left[g_{zz}^{GN}\right]\Big|_{z=0} &= \frac{d}{dz}\left[\alpha^2(\tau)\left.\left(\frac{\partial\chi}{\partial\tau}\right)^2\right|_{z=0} - \left.\left(\frac{\partial t}{\partial\tau}\right)^2\right|_{z=0}\right] \\ &= \frac{d}{dz}\left[\alpha^2(\tau)c_1^2\right] \end{aligned} \tag{2.27}$$

Thus

$$c_1 = \frac{1}{\alpha(\tau)} \tag{2.28}$$

Substituting $c_1 = \frac{\partial \chi}{\partial z}$ in Eq (2.17), we find that the constant function f_0 is

$$f_0 = \alpha(\tau_0) \tag{2.29}$$

We now have the information we need to determine the right hand side of Eq (2.18) near to the boundary,

$$\frac{d^2 t}{dz^2} = -\frac{\dot{\alpha}(\tau)}{\alpha(\tau)} \tag{2.30}$$

Thus $t(z, \tau)$ is completely determined to second non-vanishing order in z,

$$t(z) = \tau - \frac{z^2}{2}\left(\frac{\dot{\alpha}(\tau)}{\alpha(\tau)}\right) \tag{2.31}$$

We may now calculate the $(\tau\tau)$ component of the induced metric, $\gamma_{\tau\tau}$ on Σ. Using Eqs (2.31), (2.28), and (1.33),

$$\gamma^{GN}_{\tau\tau} = -1 \tag{2.32}$$

as expected.

2.2 Extrinsic Curvature

Recall that our objective is to calculate the extrinsic curvature tensor, (2.12), and consider the $K_{\tau\tau}$ component

$$K_{\tau\tau} = \frac{1}{2}\frac{\partial}{\partial z} g^{GN}_{\tau\tau}\Big|_{z=0} \tag{2.33}$$

$$= \frac{1}{2}\frac{\partial}{\partial z}\left[\left(\frac{\partial \chi}{\partial z}\right)^2 \alpha^2(\tau) - \left(\frac{\partial t}{\partial \tau}\right)^2\right]\Bigg|_{z=0} \tag{2.34}$$

Making use of the results from the previous section, it is evident that both terms of Eq (2.34) vanish independently, so that we may conclude

$$K_{\tau\tau} = 0 \tag{2.35}$$

The angular components of the extrinsic curvature, $K_{\theta\theta}$ and $K_{\phi\phi}$ are relatively simple to obtain from Eqs (2.20), (2.21) and Eq (2.12)

$$K_{\theta\theta} = \frac{1}{2}\frac{\partial g^{GN}_{\theta\theta}}{\partial z} = \alpha(\tau)\sinh(\chi)\cosh(\chi) \qquad K^{\theta}{}_{\theta} = \frac{\coth(\chi)}{\alpha} \tag{2.36}$$

$$K_{\phi\phi} = \frac{1}{2}\frac{\partial g^{GN}_{\phi\phi}}{\partial z} = \sin^2(\theta)K_{\theta\theta} \qquad K^{\phi}{}_{\phi} = K^{\theta}{}_{\theta} \tag{2.37}$$

3 Gluing an FRW Patch to Schwarzschild Spacetime

3.1 Embedding Equations

In order to glue the patch of FRW to Schwarzschild spacetime along the boundary defined by $\chi_0 = \chi$, it is necessary to find a set of equations which specify the way the boundary may be embedded into Schwarzschild spacetime as a hypersurface. Recall from our discussion in Section 1.4 that the embedding is specified by giving a set of parametric equations for coordinates in the Schwarzschild spacetime as a function of coordinates on Σ. In our notation, the coordinates on Σ are $\{y^a\} = (\tau, \theta, \phi)$ and the coordinates on M^+ are $\{x^\mu\} = (t', r', \theta', \phi')$.

$$\begin{aligned} x^0(y^a) &= t'(y^a) \\ x^1(y^a) &= r'(y^a) \\ x^2(y^a) &= \theta'(y^a) \\ x^3(y^a) &= \phi'(y^a) \end{aligned} \tag{3.1}$$

Our calculation is made easier by taking into account what we already know about the hypersurface. Since Σ must have a spherical symmetry consistent with that of both the FRW and Schwarzschild spacetimes, we know that $\theta'(y^a) = \theta$ and $\phi'(y^a) = \phi$.

Additionally, there can be no angular dependence in t' or r', so we may write both as functions purely of the proper time on the hypersurface, τ.

$$t'(y^a) = T(\tau) \tag{3.2}$$

$$r'(y^a) = R(\tau) \tag{3.3}$$

$$\theta'(y^a) = \theta \tag{3.4}$$

$$\phi'(y^a) = \phi \tag{3.5}$$

To solve for explicit forms of (3.2) and (3.3), we now make use of the first of the Israel Junction conditions, Eq (1.56), which requires that the induced metric on Σ^+ be equivalent to that on Σ^-. As we know, the induced metric on Σ^- was computed in Section 2. Since Eq (1.56) is manifestly coordinate invariant, we use the most convenient form of γ^-_{ab}. In Gaussian normal coordinates,

$$\gamma_{ab} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \alpha^2(\tau)\sinh^2\chi_0 & 0 \\ 0 & 0 & \alpha^2(\tau)\sin^2\theta\sinh^2\chi_0 \end{pmatrix} \tag{3.6}$$

and the spacetime interval on Σ^-

$$d\sigma^2_- = \gamma^-_{ab}dy^a dy^b \tag{3.7}$$

is

$$d\sigma^2_- = -d\tau^2 + \alpha^2(\tau)\sinh^2(\chi_0)d\Omega^2 \tag{3.8}$$

On Σ^+, we may calculate the induced metric by plugging in the embedding equations and changing the time coordinate to proper time on the hypersurface, τ,

$$ds_+^2 = -f(r')dt'^2 + \frac{1}{f(r')}dr'^2 + r'^2 d\Omega^2 \tag{3.9}$$

$$\begin{aligned} d\sigma_+^2 &= -f(R(\tau))\left(\frac{dt'}{d\tau}\right)^2 d\tau^2 + \frac{1}{f(R(\tau))}\left(\frac{dr'}{d\tau}\right)^2 d\tau^2 + R^2(\tau)d\Omega^2 \\ &= \left(-F(\tau)\dot{T}^2(\tau) + \frac{1}{F(\tau)}\dot{R}^2(\tau)\right) d\tau^2 + R^2(\tau)d\Omega^2 \end{aligned} \tag{3.10}$$

where $F(\tau) = f(R(\tau)) = 1 - \frac{2M}{R(\tau)}$. Henceforth, we will usually surpress the argument τ of the functions R, $\dot{T}$, and F unless it is instructive to include it. Comparing Eqs (3.8) and (3.10), we get two independent equations from the condition that $d\sigma_+^2 = d\sigma_-^2$,

$$\left(F\dot{T}^2(\tau) - \frac{1}{F}\dot{R}^2(\tau)\right) = 1 \tag{3.11}$$

$$R^2(\tau) = \alpha^2(\tau)\sinh^2(\chi_0) \tag{3.12}$$

Eq (3.12) immediately gives an explicit analytic function for $R(\tau)$,

$$R(\tau) = \alpha(\tau)\sinh(\chi_0) \tag{3.13}$$

With a little more work, Eq (3.11) gives the following expression for $\frac{dT}{d\tau}$,

$$\begin{aligned} \dot{T}^2(\tau) &= \frac{1}{F^2}\left(\dot{R}^2(\tau) + F\right) \\ \dot{T}(\tau) &= \frac{1}{F}\sqrt{\dot{R}^2(\tau) + F} \end{aligned} \tag{3.14}$$

Although Eq (3.14) could be integrated to determine $T(\tau)$, we do not have an analytic expression for this integral, and we won't need one. The current form is sufficient to compute the extrinsic curvature on Σ^+, our next project.

3.2 The Normal Vector Field

Recalling the definition of the holonomic basis vectors

$$e_a^\mu = \frac{\partial x^\mu}{\partial y^a} \tag{3.15}$$

we define a vector u^μ which gives the velocity of a particle travelling along a geodesic on Σ,

$$u^\mu = e_\tau^\mu = \frac{\partial x^\mu}{\partial \tau} \tag{3.16}$$

which has components

$$\begin{aligned} u^\mu &= \left(\frac{\partial t'}{\partial \tau}, \frac{\partial r'}{\partial \tau}, 0, 0\right) \\ u^\mu \partial_\mu &= \dot{T}\partial_{t'} + \dot{R}\partial_{r'} \end{aligned} \tag{3.17}$$

The normal vector field to Σ is the vector $n^\mu = g_+^{\mu\nu} n_\nu$, where the normal covector field satisfies

$$\frac{\partial x^\mu}{\partial y^a} n_\mu = 0 \tag{3.18}$$

for all a. Therefore, for $a = \tau$,

$$\begin{aligned} \frac{\partial x^\mu}{\partial y^a} n_\mu &= \dot{T}\, n_{t'} + \dot{R}\, n_{r'} = 0 \\ n_{t'} &= \frac{-\dot{R}}{\dot{T}} n_{r'} \end{aligned} \tag{3.19}$$

The $a = \theta$ and $a = \phi$ equations give $n_\theta = n_\phi = 0$. Expanding in components using the one-form basis $\{dx^\mu\}$,

$$n_\mu dx^\mu = n_{t'} dt' + n_{r'} dr' = \frac{-\dot{R}}{\dot{T}} n_{r'} dt' + n_{r'} dr' \tag{3.20}$$

Since n_μ is unique up to normalization, and both $n_{r'}$ and $n_{t'}$ are functions only of τ, there exists a normalization function $A(\tau)$ such that $n_{r'} = +A(\tau)\dot{T}$, and

$$n_\mu dx^\mu = A(\tau)\left(-\dot{R}dt' + \dot{T}dr'\right) \tag{3.21}$$

Evaluated at $r' = R(\tau)$ and $t' = T(\tau)$, $g^{t't'} = \frac{-1}{F}$ and $g^{r'r'} = F$, so

$$n^\mu \partial_\mu = A(\tau) \left(\frac{\dot{R}}{F} \partial_{t'} + F\dot{T}\partial_{r'} \right) \tag{3.22}$$

We may enforce normalization to solve for $A(\tau)$, where we choose $n^\mu n_\mu = +1$ because the hypersurface is timelike (so the normal vector field must be spacelike).

$$A^2(\tau) = \frac{F}{\dot{T}^2 F^2 - \dot{R}^2} \tag{3.23}$$

We need now consider that the normal vector field to Σ^+ must be defined consistently with that on Σ^-, as they are the same vector field. In Section 2.1, the normal vector field was constructed by taking $n_p^\mu(\tau, \theta, \phi)$ at a point $p \in \Sigma$ to be the tangent vector to the unique geodesic through p which intersects Σ orthogonally, $x_p^\mu(z, \tau)$, at $z = 0$. In this construction we specified that the normal vector field points in the direction of increasing χ, which implies it must be pointing in the direction of increasing r', the radial Schwarzschild coordinate, thus we require that the normalization factor $A(\tau)$ have the correct sign such that

$$n^\mu \partial_\mu (r') = A(\tau) F\dot{T} \partial_{r'} r' > 0 \tag{3.24}$$

Eq (3.14) implies that $F\dot{T} > 0$ for all values of $\tau > 0$, therefore $A(\tau) > 0$. Referring back to Eq (3.23) and substituting for $\dot{T}^2(\tau)$ from Eq (3.14),

$$\begin{aligned} A^2(\tau) &= \frac{F}{\dot{T}^2 F^2 - \dot{R}^2} \\ &= 1 \end{aligned} \tag{3.25}$$

So we must have $A = +1$. The normalized normal vector and covector fields are now fully determined.

$$n_\mu dx^\mu = -\dot{R}dt' + \dot{T}dr' \tag{3.26}$$

$$n^\mu \partial_\mu = \frac{\dot{R}}{F}\partial_{t'} + F\dot{T}\partial_{r'} \tag{3.27}$$

3.3 Extrinsic Curvature on Σ^+

Let $K^+_{\mu\nu}$ be the four-dimensional tensor describing the extrinsic curvature of the hypersurface Σ^+ embedded in the Schwarzschild spacetime M^+. As previously described, projecting $K^+_{\mu\nu}$ produces the three-tensor K^+_{ab} tangent to Σ^+,

$$K^+_{ab} = \frac{\partial x^\mu}{\partial y^a}\frac{\partial x^\nu}{\partial y^b}K^+_{\mu\nu} \tag{3.28}$$

$$= \frac{\partial x^\mu}{\partial y^a}\frac{\partial x^\nu}{\partial y^b}\nabla_\mu n_\nu \tag{3.29}$$

where, as before, $\{x^\mu\} = (t', r', \theta', f')$ are coordinates on M^+ and $\{y^a\} = (\tau, \theta, \phi)$ are coordinates on Σ^+. Now, Σ^+ was defined such that the angular coordinates on M^+ map identically to those on Σ^+, $\theta' = \theta$, and $\phi' = \phi$, so $K^+_{\theta\theta}$ and $K^+_{\phi\phi}$ are easily calculated with Eq (3.29).

$$\begin{aligned} K_{\theta\theta} = \nabla_\theta n_\theta &= \partial_\theta n_\theta - \Gamma^\mu_{\theta\theta} n_\mu \\ &= -\Gamma^{r'}_{\theta\theta} n_{r'} \\ &= -(2M - R(\tau))\dot{T}(\tau) \\ &= R(\tau)F(\tau)\dot{T}(\tau) \end{aligned} \tag{3.30}$$

Where $F(\tau) = f(R(\tau)) = 1 - \frac{2M}{R(\tau)}$. Using the induced inverse metric $\gamma^{\theta\theta}_+$ on Σ^+ to raise an index,

$$K^\theta{}_\theta = \gamma^{\theta\theta}_+ K_{\theta\theta} = \frac{F\dot{T}}{R} = K^\phi{}_\phi \tag{3.31}$$

Similarly,

$$
\begin{aligned}
K_{\phi\phi} = \nabla_\phi n_\phi &= \partial_\phi n_\phi - \Gamma^{\mu}_{\phi\phi} n_\mu \\
&= -\Gamma^{r'}_{\phi\phi} n_{r'} - \Gamma^{\theta}_{\phi\phi} n_\theta \\
&= -\Gamma^{r'}_{\phi\phi} n_{r'} \\
&= \sin^2(\theta) R(\tau) F(\tau) \dot{T}(\tau)
\end{aligned} \tag{3.32}
$$

And

$$
\begin{aligned}
K^{\phi}{}_{\phi} = \gamma_+^{\phi\phi} K_{\phi\phi} &= \left(\frac{1}{R^2(\tau)\sin^2(\theta)} \right) K_{\phi\phi} \\
&= \frac{F(\tau)\dot{T}(\tau)}{R(\tau)}
\end{aligned} \tag{3.33}
$$

Lastly we must determine the remaining component of extrinsic curvature on Σ^+, $K^+_{\tau\tau}$. Eq (3.29) requires we take a four-dimensional covariant derivative of the normal vector field n_μ, but, although n_μ is a four-vector field it clearly depends only upon three variables, and is defined only on Σ^+. Both $T(\tau)$ and $R(\tau)$ vary with proper time τ along Σ^+, thus a straightforward application of Eq (3.29) would require that we somehow extend the normal vector field off of the hypersurface.[9] We will instead deploy a much simpler method, bypassing the need to calculate the full four-dimensional tensor $K_{\mu\nu}$, as we are really only interested in its restriction to Σ^+. Recall the definition of a velocity vector u^μ tangent to Σ^+,

$$u^\mu = e^\mu_\tau \tag{3.34}$$

Eq (3.29), expressed in terms of these velocity vectors u^μ, is

$$K^+_{\tau\tau} = u^\mu u^\nu \nabla_\mu n_\nu \tag{3.35}$$

[9]Extending the normal vector field to the entire spacetime may be achieved by choosing a foliation of the spacetime into hypersurfaces, where each hypersurface has a normal vector field. Then any point in spacetime lies on one such hypersurface and there is a unique normal to the hypersurface at that point. Alternatively, we might simplify the calculation of extrinsic curvature by changing from coordinates on the Schwarzschild spacetime from spherical to Gaussian normal coordinates by solving the geodesic equation, as we did in Section 2. Both of these methods are computationally more complex than the one we will employ here.

Any component of the normal vector field is orthogonal to the vector u^μ, so

$$n_\mu u^\mu = 0 \tag{3.36}$$

Since the left hand side of (3.36) is vanishing, so is its covariant derivative. Using the chain rule, we may write

$$\nabla_\nu(n_\mu u^\mu) = n_\mu \nabla_\nu u^\mu + u^\mu \nabla_\nu n_\mu = 0 \tag{3.37}$$

thus,

$$-n_\mu \nabla_\nu u^\mu = u^\mu \nabla_\nu n_\mu \tag{3.38}$$

We now define the acceleration vector, a^μ, to be

$$a^\sigma = u^\mu \nabla_\mu u^\sigma \tag{3.39}$$

so that the component of interest, $K_{\tau\tau}$, can be written

$$\begin{aligned} K^+_{\tau\tau} &= u^\sigma(u^\mu \nabla_\sigma n_\mu) \\ &= -n_\mu(u^\sigma \nabla_\sigma u^\mu) \end{aligned} \tag{3.40}$$

Therefore,

$$K^+_{\tau\tau} = -n_\mu a^\mu \tag{3.41}$$

Expressing a^σ in terms of its components,

$$a^\sigma = \frac{\partial u^\sigma}{\partial \tau} + \Gamma^\sigma_{\rho\lambda} u^\rho u^\lambda \tag{3.42}$$

Eq (3.42) can be expanded, using the Christoffel symbols for the Schwarzschild metric as given in Eq (1.25)

$$a^{t'} = \ddot{T} - \frac{2M\dot{T}\dot{R}}{FR^2} \tag{3.43}$$

$$a^{r'} = \ddot{R} - \frac{M\dot{R}^2}{FR^2} + \frac{FM\dot{T}^2}{R^2} \tag{3.44}$$

Where, for neatness sake, we have suppressed the arguments of the functions $T(\tau)$, $R(\tau)$, and $F(\tau)$. Finally,

$$\begin{aligned} K^+_{\tau\tau} &= -n_{t'}a^{t'} - n_{r'}a^{r'} \\ &= -\dot{T}\left(\ddot{R} - \frac{M\dot{R}^2}{FR^2} + \frac{FM\dot{T}^2}{R^2}\right) + \dot{R}\left(\ddot{T} - \frac{2M\dot{T}\dot{R}}{FR^2}\right) \end{aligned} \tag{3.45}$$

The algebra is made easier by defining a function $\beta(\tau)$,

$$\beta(\tau) = F\dot{T} \tag{3.46}$$

By making use of the result of the first junction condition,

$$F\dot{T}^2 - \frac{\dot{R}^2}{F} = 1 \tag{3.47}$$

we can see that $\beta(\tau)$ can also be written as

$$\beta(\tau) = \sqrt{\dot{R}^2 + F} \tag{3.48}$$

The relevant derivatives with respect to proper time are then

$$\dot{T} = \frac{\beta}{F} \qquad \ddot{T} = \frac{\dot{\beta}}{F} - \frac{\beta\dot{F}}{F^2} \tag{3.49}$$

$$\dot{R} = \sqrt{\beta^2 - F} \qquad \ddot{R} = \frac{2\beta\dot{\beta} - \dot{F}}{2\dot{R}} \tag{3.50}$$

$$\dot{F} = \frac{2M\dot{R}}{R^2} \tag{3.51}$$

These expressions can be used to eliminate $\dot{R}, \dot{T}, \ddot{R}$, and $\ddot{T}$ from Eq 3.45, leaving an expression that can only depend on $\beta, \dot{\beta}$, and R. After simplification, one finds

$$K^{+}_{\tau\tau} = -\frac{\dot{\beta}}{\sqrt{\beta^2 - F}} = -\frac{\dot{\beta}}{\dot{R}} \tag{3.52}$$

Raising the first index using the induced metric of Eq (3.6), one has finally

$$K^{+\tau}{}_{\tau} = \frac{\dot{\beta}}{\dot{R}} \tag{3.53}$$

4 The Junction Conditions

Now that we have explicit forms for the extrinsic curvature for both sides of the hypersurface, we are at last in a position to apply the second junction condition and finish gluing together the FRW and Schwarzschild spacetimes. As $K^{\theta}{}_{\theta}$ and $K^{\phi}{}_{\phi}$ are equal to each other for both Σ^{+} and Σ^{-}, the second junction condition produces only two independent equations. As a consequence of the condition that $[K^{\theta}{}_{\theta}] = [K^{\phi}{}_{\phi}] = 0$, one finds

$$K^{+\theta}{}_{\theta} = K^{-\theta}{}_{\theta} \tag{4.1}$$

$$\frac{F\dot{T}}{R} = \frac{\coth \chi_0}{\alpha} \tag{4.2}$$

Also, from the condition $[K^{+\tau}{}_{\tau}] = 0$,

$$K^{+\tau}{}_{\tau} = K^{-\tau}{}_{\tau} \tag{4.3}$$

$$\frac{\dot{\beta}}{\dot{R}} = 0 \tag{4.4}$$

Recall the form of $R(\tau)$ as derived from the first junction condition, Eq (3.13), restated here for convenience

$$R(\tau) = \alpha(\tau) \sinh(\chi_0) \tag{4.5}$$

Combining Eqs (3.46) and (3.48), one finds

$$F\dot{T} = \sqrt{\dot{R}^2 + F} \tag{4.6}$$

Substituting the right hand side of (4.6) for $F\dot{T}$ in Eq (4.2) and after some manipulation, we obtain

$$F = -\dot{\alpha}^2(\tau)\sinh^2(\chi_0) + \cosh^2(\chi_0) \tag{4.7}$$

As $\alpha(\tau)$ obeys the first order Friedmann Equation, one can replace $\dot{\alpha}(\tau)$ with the appropriate expression from Eq (1.17) in Eq (4.7), and, recalling the expression for F, obtain the long sought expression for total relativistic mass of the embedded FRW patch,

$$M_{Rel} = \frac{4\pi}{3}\rho R^3(\eta) \tag{4.8}$$

Upon inserting the expression for mass density $\rho(\alpha)$ as a function of the scale factor, given in Eq (5.3),

$$\rho(\alpha) = \frac{3\alpha_0}{8\pi}\frac{1}{\alpha^3(\eta)} \tag{4.9}$$

the total relativistic mass is manifestly time independent

$$M = \frac{\alpha_0}{2}\sinh^3(\chi_0) \tag{4.10}$$

In the large χ_0 limit, which corresponds to taking the patch size to the size of the whole FRW spacetime, the total mass diverges. Furthermore, using Eq (4.2) and the definition of $\beta(\tau) = F\dot{T}$, along with Eq (3.13) for $R(\tau)$, one finds that $\beta = \cosh(\chi_0)$ is constant, thus $\dot{\beta} = 0$, and the junction conditions have been fully satisfied.

5 Conclusion: Understanding the Total Relativistic Mass with Newtonian Cosmology

5.1 Newtonian Cosmology

Note: For consistency, we will continue to work throughout this section in natural units in which $c = G = 1$ [10] The dynamics of our universe are determined, at large distance scales, by gravitation forces; the predominant theory for understanding these dynamics is Einstein's theory of General Relativity. However, much of the dynamics can be well approximated by classical Newtonian gravity; this formalism is called Newtonian Cosmology. The familiar Newtonian equation of motion for two bodies subject to a gravitational attraction may be written in the form of a gravitational Poisson's equation,

$$\nabla^2 \phi = 4\pi\rho, \qquad \vec{g} = -\vec{\nabla}\phi \tag{5.1}$$

where $\vec{g}$ is the gravitational acceleration, ρ is the mass density. Given the simplest case of a uniform mass distribution of density ρ, Eq (5.1) may be integrated over a finite region for the acceleration $\vec{g}$. At a point $\vec{p} = (x, y, z)$,

$$\vec{g}(p) = \int \rho \left(\frac{\vec{r'} - \vec{r}_p}{|\vec{r'} - \vec{r}_p|^3} \right) d^3\vec{r'} \tag{5.2}$$

Problems arise, however, when the integral in Eq (5.2) is taken over an infinite region of uniform mass distribution; in such a limit, the integral is ill-defined and the result depends upon the order of summation, that is, it is conditionally convergent. Thus one cannot measure absolute acceleration of a given particle, but must settle for the notion of relative acceleration of particles with respect to one another. The formalism of Newtonian cosmology addresses this problem by considering the dynamics of a mass distribution over a finite region and then taking the limit as the region becomes infinite careful and well-defined way.

In our discussion we will consider a spherical region of initial radius R_0 [11] Consider

[10] The discussion in this subsection will closely follow that in (9).

[11] It is not necessary that the region under consideration be spherical, in fact, the results we will

a sphere of initial radius R_i, the radius as an initial time t_i, with a uniform mass distribution of non-relativistic particles ρ_i. At a time t, the mass density is

$$\rho(t) = \frac{M(r_i)}{\frac{4\pi}{3}r(r_i,t)^3} = \frac{\frac{4\pi}{3}r_i^3\rho_i}{\frac{4\pi}{3}r(r_i,t)^3} = \frac{1}{\alpha^3(t)}\rho_i \tag{5.3}$$

where $M(r_i)$ is the mass enclosed within a spherical region of radius r_i, $r_i < R_i$ is the initial radial position of a given particle, and $\alpha(t) = \frac{r(t,r_i)}{r_i}$ is the scale factor. There are two equations analogous to the relativistic Friedmann Equations which completely determine the behavior of this spherical model universe.

$$\ddot{\alpha(t)} = -\frac{4\pi}{3}\rho(t)\alpha(t) \tag{5.4}$$

$$\left(\frac{\dot{\alpha}(t)}{\alpha(t)}\right)^2 = \frac{8\pi}{3}\rho(t) - \frac{\kappa}{\alpha^2(t)} \tag{5.5}$$

We will work with $\kappa = -1$, in agreement with our relativistic model developed in the previous sections. Equations (5.4) and (5.5) are the central results of Newtonian Cosmology. The mechanical energy of a test particle in this Newtonian formalism, E_N is given by a sum of kinetic and potential energies, K and U, which may be written in terms of the total mass M, initial radius of the sphere R_i, the initial and time dependent mass densities ρ_i and $\rho(t)$ and the scale factor and its derivative, $\alpha(t)$ and $\dot{a}(t)$.

$$K = \frac{3}{5}MR_i^2\left(\frac{1}{2}\dot{\alpha}^2(t)\right) \tag{5.6}$$

$$U = \frac{3}{5}MR_i^2\left(-\frac{4\pi}{3}\frac{\rho_i}{\alpha(t)}\right) \tag{5.7}$$

Thus the Newtonian mechanical energy is

$$E_N = K + U = \frac{3}{5}MR_i^2\left(\frac{1}{2}\dot{\alpha}^2(t) - \frac{4\pi}{3}\frac{\rho_i}{\alpha(t)}\right) \tag{5.8}$$

We are now in a position to use Eq (5.8) to compare the results of Newtonian cos-

develop here are applicable to any region whose quadrupole moment, in the limit as the size of the region is taken to infinity, is the same as that of a sphere.

mology against Eq (4.8).

5.2 Comparison to the Total Relativistic Mass

In order to compare the result in (4.8) to the total energy (5.8), we must first find the value of R_i which gives the spherical model universe of Subsec. 5.1 the same volume as the FRW patch defined in Sec 2. Consider the FRW patch at a fixed arbitrary time τ, and note that this spatial slice has a metric h_{ab}, where $\xi^a = (\chi, \theta, \phi)$ and the line element $d\sigma^2 = d\xi^a d\xi^b h_{ab}$ is

$$d\sigma^2 = -\alpha(\tau)^2(d\chi^2 + \sinh^2(\chi)d\theta^2 + \sinh^2(\chi)\sin^2(\theta)d\phi^2) \tag{5.9}$$

Using the integral over the volume element, one finds the total volume of the FRW patch is given by

$$\begin{aligned} V &= \int \sqrt{|h_{\alpha\beta}|} dV \\ &= \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \int_0^{\chi_0} d\chi |\sin\theta| \sinh^2\chi\alpha(\tau)^3 \\ &= \pi(-2\chi_0 + \sinh(2\chi_0))\alpha(\tau)^3 \\ &= \frac{4\pi}{3}\alpha^3(\tau)\chi_0^3 + \frac{4\pi}{15}\alpha^5(\tau)\chi_0^3 + \mathcal{O}(\chi_0^7) \end{aligned} \tag{5.10}$$

Thus it makes sense to choose the radius R_i to be the circumferential radius for $\chi = \chi_0$,

$$R_i = \alpha_i \sinh^2(\chi_0) \tag{5.11}$$

where $\alpha_i = \alpha(\tau)|_{t_i}$ and t_i is the same initial time as in Subsec 5.1. One can see by taking an expansion of (5.11) in powers of χ_0 that the relativistic rest mass,

$$M_{rest,rel} = \rho V \tag{5.12}$$

is equivalent to the classical rest mass,

$$M_{rest,N} = \frac{4\pi}{3} R_i^3 \rho(t_i) \tag{5.13}$$

up to third order in χ_0. Using this value of R_i, the total Newtonian energy is the sum of the rest energy and the total mechanical energy of (5.8).

$$\begin{aligned} E_{tot} &= \frac{3}{10} M \sinh^2(\chi_0) + M_{rest} \\ &= \frac{3}{10} M \sinh^2(\chi_0) + \frac{4\pi}{3} R_i^3 \rho(t_i) \end{aligned} \tag{5.14}$$

Expanding in powers of χ_0, one finds

$$E_{tot} = \frac{\alpha_0 \chi_0^3}{2} + \frac{\alpha_0 \chi_0^5}{4} + \frac{5\alpha_0 \chi_0^7}{84} + \mathcal{O}(\chi_0^8) \tag{5.15}$$

Expanding the total relativistic energy (4.8) in powers of χ_0, one finds excellent agreement in with the results of newtonian cosmology to fifth order in χ_0,

$$\begin{aligned} M_{rel} &= \frac{4}{3} \pi R^3(\chi_0) \rho(\tau) \\ &= \frac{\alpha_0 \chi_0^3}{2} + \frac{\alpha_0 \chi_0^5}{4} + \frac{13\alpha_0 \chi_0^7}{240} + \mathcal{O}(\chi_0^8) \end{aligned} \tag{5.16}$$

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