High-Fidelity Simulations of Transverse Electric Waves Propagating through Alcator C-Mod

By

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Abstract

This project represents an attempt to model the propagation of microwaves into Alcator C-Mod’s plasma in high fidelity and with a reduced number of degrees of freedom. The success of this endeavor would accelerate progress within the field of fusion energy, as simulations of C-Mod’s plasmas, or other plasmas in general, can be run more quickly while still maintaining their accuracy. The main procedure involves producing simulations within COMSOL that use mode numbers based on a power spectrum of waves at 4.6 GHz. These simulations are then overlaid to model how the waves will propagate as a function of position, plasma density, and local flux. Future work could focus on verifying the accuracy of the simulations when compared to data acquired from C-Mod as well as ensuring the run-time of the simulations is indeed faster than other methods.
1 Introduction

In order to reach fusion conditions, the plasma within a reactor must be confined by magnetic fields with a “rotational transform” to average out particle drifts. In a tokamak reactor, the toroidal current flowing through the plasma provides the rotational transform. To sustain the plasma in steady state, scientists have developed several types of non-inductive current drive, one of which involves launching Low Hybrid Range Frequency (LHRF) waves into plasma to add energy and momentum to electrons traveling parallel to the magnetic field. Within Alcator C-Mod, the tokamak fusion reactor at MIT, microwaves at 4.6 GHz are used for current drive experiments. Currently simulation software such as COMSOL can represent these waves as they travel through plasma in 2-D [1].

These simulations are essential to physicists and engineers designing fusion reactors because these models can reveal design flaws that otherwise would only be revealed after spending millions of dollars. However, many physical processes that occur within plasma, such as the propagation of microwaves, require 3 dimensions to accurately model. Simulations of these processes use many degrees of freedom for each particle produced, requiring significant portions of a limited available pool of memory.

The simulations of this work would make it possible to simulate with higher fidelity by separating one of the dimensions into multiple independent simulations. To do this, the 2-dimensional software capabilities within COMSOL, a multi-physics finite element modeling software package, will be used to superimpose multiple 2-D simulations on top of each other to create a virtual 3-D image of microwaves propagating within the plasma. This will enable researchers to better understand the processes that occur within plasma along with the mechanisms underlying the heating of plasma. Previous researchers running C-Mod have performed numerous experiments with the fusion reactor. These researchers need a model that can simulate the details of how the plasma will behave as a function of the various magnetic fields and particle density for comparison with experimental measurements of the LHRF wave fields near the antenna.
2 Background

2.1 Plasma Physics and Tokamak Operation

Fusion energy is produced when certain isotopes of light elements combine together into heavier elements. The mass difference between the fusion products and reactants manifests itself as kinetic energy in the products, and a reactor can use this kinetic energy to heat up a coolant such as a molten salt. This coolant can then be made to spin a turbine and produce electricity. When engineers design a reactor to produce energy, they assume that the reactor will fuse deuterium and tritium, which are isotopes of hydrogen. The reaction between these isotopes has a very high probability of interaction at a relatively low temperature of 50 keV, which makes the reaction ideal for a power plant. The high temperature of the plasma is necessary to achieve nuclear fusion due to the strong Coulomb repulsion between positive nuclei at close distances. Once two nuclei are sufficiently close together, the strong nuclear force is able to overpower the Coulomb repulsion, and the nuclei fuse together. Further, the high temperature is necessary to ionize fuel in the plasma in order for the Coulomb repulsion interaction to occur [2].

Plasma within a fusion reactor must be confined in order to ensure the heat necessary for fusion stays within. But no material in existence can withstand the enormous heat of burning plasma without incurring significant damage. To solve this, engineers have used magnetic fields to ensure that the plasma within a device does not touch the walls of its container. Because plasmas are composed of electrons and ions, the magnetic forces that do not affect neutral atoms will redirect the charged particles. The particles follow magnetic field lines, traveling in a spiral pattern along the field line and around what is known as a guiding center. The motion of a charged particle is described by the following equation:

\[ m \frac{d\vec{v}}{dt} = q(\vec{E} + \vec{v} \times \vec{B}) \]  

In this equation, \( m \) is the particle's mass, \( q \) is the particle's charge, \( \vec{v} \) is the particle's velocity vector, \( E \) is the electric field, and \( B \) is the magnetic field [2]. However, depending on the specific design of the fusion device, charged particles exposed to magnetic fields can undergo guiding center drifts that cause loss of confinement [2]. The first guiding center drift to be discussed is curvature drift. As its name implies, this drift is caused when the magnetic field that confines plasma is curved. The equation describing curvature drift is shown below:

\[ V_k = \pm \frac{v_{||}^2 R_c \times B}{\omega_c R_c^2 B} \]  

In this equation, \( V_k \) is the velocity vector describing the direction that particles travel when subjected to a curvature drift. \( v_{||} \) is the velocity of a particle's guiding center parallel to the magnetic field, \( \omega_c = \frac{qB}{m} \) is the cyclotron frequency of the particle, and \( R_c \) is the vector quantity describing the radius of curvature. The \( \pm \) symbol in front of the equation describes the polarity of the particle in question; in other words, electrons and ions drift in up and down opposite directions when the curvature is in the x-y plane. This leads to an electric potential difference
between clusters of ions and electrons and an electric field, which creates a second type of particle drift called $E \times B$ drift. This is defined by the following equation:

$$ V_E = \frac{\vec{E} \times \vec{B}}{B^2} $$

In Equation 2, $E$ is the electric field vector. If particles are separated by their charge, the $E \times B$ drift will cause particles to produce an instability that grows in magnitude with time and eventually collides with the plasma walls [2]. In addition to the first two guiding center drifts, particles within plasma can undergo two more drifts known as $VB$ drift and polarization drift. They are described by the following equations:

$$ V_{VB} = \pm \frac{v_{\perp}^2}{2 \omega_c} \frac{\vec{B} \times \nabla B}{B^2} $$

$$ V_p = \pm \frac{1}{\omega_c B} \left( \frac{dE_\perp}{dt} \right) $$

In equations 3 and 4, $v_{\perp}$ is the velocity of the particle perpendicular to the magnetic field. Often these drifts cause significant problems to reactors, as they can lead to particles colliding with the reactor’s walls, ruining confinement and damaging the device. However, these drift equations can also be manipulated to confine plasma, in the case of $VB$ drift in a mirror machine.

A tokamak is one such example of a fusion device that uses the various guiding center drifts to confine plasma. Figure 2.1 contains a basic schematic of a tokamak, including the toroidal and poloidal magnetic fields generated that stabilize plasma.

Figure 2.1: Basic tokamak diagram showing the magnetic field lines and their path around the device. The inner poloidal field coils produce a magnetic field pointing in the theta direction. This field combines with the poloidal magnetic field produced by the plasma current to produce the helical magnetic field seen above [3].
When a tokamak is operating, current is driven in the theta direction and around the cross section of the device through the toroidal field coils. This current produces a toroidal magnetic field pointing in the phi direction. Particles within the tokamak follow the magnetic field lines and cycle around the device. Simultaneously, several smaller coils that wind around the tokamak in the phi direction work together with the inner poloidal field coils to produce magnetic fields that shape the plasma within the reactor [4]. However, the magnetic fields produced by these poloidal coils are insufficient to confine the plasma.

When scientists first attempted to confine a plasma within a toroidal device, they found that the electrons and ions would drift apart due to curvature drift, creating an $E \times B$ instability that would grow in strength until the plasma collided with the tokamak walls [4]. Thus scientists determined that a magnetic field in the theta direction was necessary to confine the plasma for an extended period of time. In order to produce this magnetic field, tokamaks function by driving current through the plasma. This current produces a magnetic field described by Ampere’s circuital law:

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 \varepsilon_0 \frac{d}{dt} \iint \mathbf{E} \cdot d\mathbf{S} + \mu_0 \iint \mathbf{j} \cdot d\mathbf{S}$$

(6)

Where $E$ is the electric field within that wire, $B$ is the magnetic field produced, $S$ is the surface enclosed by the closed loop, and $J$ is the current density traveling through the closed loop. Current can be driven through several methods, including Lower Hybrid range of Frequency (LHRF), Ion Cyclotron range of Frequency (ICRF), and Neutral Beam Heating. Current drive is essential to the operation of a tokamak, as without the stabilization effect that current drive brings, tokamaks will remain essentially pulsed devices [1]. This study focuses on LHRF waves and their use within tokamaks to drive current and extend confinement time of the plasma within.

2.2 LHRF Waves

Lower Hybrid range of frequency (LHRF) current drive operates by launching Lower Hybrid (LH) waves into the plasma in one toroidal direction. Alcator C-Mod creates these waves by using Klystron vacuum tubes that amplify radio waves into the microwave range [5]. These waves transfer their energy into electrons within the plasma when launched at a critical resonant frequency [2]. When the LH waves are oriented correctly, the resulting current produced travels in the phi direction and stabilizes the plasma. LHRF has a higher efficiency than other methods, which has motivated its use in tokamaks such as Alcator C-Mod [1,2].

In order to properly orient the waves sent into the plasma and ensure the net velocity of the waves points in the toroidal direction, the LH waveguides are set to run with a phase offset of $\frac{\pi}{2}$ with respect to each other; thus, the offsets of the waveguides are $0$, $\frac{\pi}{2}$, $\pi$, and $\frac{3\pi}{2}$. The offset in phase creates constructive and destructive interference that causes the waves to travel primarily in one toroidal direction. In addition to the waveguide phase offset, a second phase offset of $\frac{5\pi}{4}$ is used to align the waves so the values of the electric field in the phi direction are roughly equivalent at the boundary of the plasma. This offset is applied at the top and bottom waveguides, or the first and fourth counting from top to bottom. Figure 2.2 shows why the waves
produced within COMSOL require this second offset as it demonstrates the electric fields produced by the waveguides when the offset is present versus when it is not.

![Wavefield plots](image)

Figure 2.2: COMSOL Wave propagation simulation at a sample mode number. (a) does not contain an offset between the inner and outer waveguides, while (b) has an offset of $\frac{5\pi}{4}$. As a result of the offset, waves in (b) are significantly more in phase when they begin to travel through the plasma.

2.3 LH Background Physics

In order to understand the LH waves that are being launched into the plasma, background physics are covered here. The dispersion relation for the LH waves is derived, obtained from Freidberg [2]. First, Maxwell’s equations are written with Ohm’s Law:

\[ \nabla \times \vec{E} = -\frac{d\vec{B}}{dt} \]  

(7)

\[ \nabla \times \vec{B} = \mu_0 \vec{J} + \frac{1}{c^2} \frac{d^2 \vec{E}}{dt^2} \]  

(8)
\[
\vec{j} = \vec{\sigma} \cdot \vec{E}
\]  

(9)

where \(\vec{\sigma}\) is the conductivity tensor for the plasma. These three equations can be combined to form the following:

\[
\nabla \times \nabla \times \vec{E} = -\frac{1}{c^2} \frac{d^2 \vec{E}}{dt^2} - \frac{1}{c} \frac{d}{dt} (\mu_0 \vec{\sigma} \cdot \vec{E})
\]

(10)

The waves that follow these equations can be assumed to take the form \(e^{i \vec{k} \cdot \vec{x} - \omega t}\), which allows the combined equation above to be Fourier analyzed and defining \(\nabla = i \vec{k}\) and \(-\frac{d}{dt} = i \omega\), where \(\vec{k}\) is the propagation vector and \(\omega\) is the angular frequency of the waves. This leads to the following:

\[
\vec{k} \times \vec{k} \times \vec{E} + i \omega \mu_0 \vec{\sigma} \cdot \vec{E} + \frac{\omega^2}{c^2} \vec{E} = 0
\]

(11)

One can easily substitute \(\vec{n} = \frac{\vec{e} \kappa}{\omega}\), where \(\vec{n}\) is the index of refraction, into this to find

\[
\vec{n} \times \vec{n} \times \vec{E} + \vec{\varepsilon} \cdot \vec{E} = 0
\]

(12)

Where

\[
\vec{\varepsilon} = \vec{I} + \frac{i c^2 \mu_0}{\omega} \vec{\sigma} = \begin{bmatrix}
\varepsilon_\perp & -i \varepsilon_x & 0 \\
 i \varepsilon_x & \varepsilon_\perp & 0 \\
 0 & 0 & \varepsilon_\parallel
\end{bmatrix}
\]

(13)

This then simplifies to

\[
(\vec{n} \vec{n} - n^2 I) \cdot \vec{E} + \vec{\varepsilon} \cdot \vec{E} = 0
\]

(14)

If \(\vec{n} = n_\parallel \hat{x} + n_\perp \hat{z}\), with a parallel and perpendicular index of refraction, this can be written in a matrix form:

\[
\begin{bmatrix}
\varepsilon_\perp - n_\parallel^2 & -i \varepsilon_x & n_\perp n_\parallel \\
i \varepsilon_x & \varepsilon_\perp - n_\parallel^2 & 0 \\
n_\perp n_\parallel & 0 & \varepsilon_\parallel - n_\perp^2
\end{bmatrix} \cdot \vec{E} = 0
\]

(15)

In order to find the dispersion relation for the LH waves, the conductivity tensor \(\vec{\sigma}\) must be determined first. The derivation is relatively simple and starts with the momentum conservation of a charged particle in a magnetic field:

\[
q(\vec{E} + \vec{v} \times \vec{B}) = m \frac{d\vec{v}}{dt}
\]

(16)
In this equation, $q$ is the charge of the particle, $m$ is its mass, $v$ is the particle’s velocity, and $E$ and $B$ are the electric and magnetic fields. It can be assumed that $\vec{B} = \vec{B}_0 + \vec{B}_1$, $\vec{E} = \vec{E}_1$, and $\vec{v} = \vec{v}_1$ with the 0 subscript indicating steady-state background quantity and the 1 subscript indicating a perturbation in the background. In addition, the equation is Fourier analyzed to produce the following, assuming only first-order quantities are conserved:

$$q(\vec{E}_1 + \vec{v}_1 \times \vec{B}_0) = -i\omega m \vec{v}_1$$

The magnetic field is assumed to propagate only in the $\hat{z}$ direction, or $\vec{B}_0 = B_0 \hat{z}$. When the components of $\vec{v}_1$ are solved for, the following is produced:

$$v_{1x} = \frac{iq}{m\omega}(E_{1x} + v_{1y}B_0)$$

(18)

$$v_{1y} = \frac{iq}{m\omega}(E_{1y} - v_{1x}B_0)$$

(19)

$$v_{1z} = \frac{iq}{m\omega}E_{1z}$$

(20)

Solving this system of equation for the components of $\vec{v}$ yields:

$$v_{1x} = \frac{1}{1 - \frac{\omega_c^2}{\omega^2}} \left( \frac{iq}{m\omega}E_{1x} - \frac{q^2B_0}{m^2\omega^2}E_{1y} \right)$$

(21)

$$v_{1y} = \frac{1}{1 - \frac{\omega_c^2}{\omega^2}} \left( \frac{iq}{m\omega}E_{1y} + \frac{q^2B_0}{m^2\omega^2}E_{1x} \right)$$

(22)

$$v_{1z} = \frac{iq}{m\omega}E_{1z}$$

(23)

In these equations, $\omega_c = \frac{qB}{m}$ is the cyclotron frequency of the species. Using the above components in conjunction with the identity $\vec{J} = \sum nq\vec{v}$, the matrix $\tilde{\sigma}$ can be found:

$$\sigma = \sum \frac{q^2n}{m} \begin{pmatrix} i\omega & \omega_c & 0 \\ \omega_c^2 - \omega_c^2 & \omega^2 - \omega_c^2 & 0 \\ 0 & 0 & \frac{i}{\omega} \end{pmatrix}$$

(24)

This sum is performed over all possible species in the plasma, including electrons and all ions. Using this expression for $\tilde{\sigma}$, $\tilde{\epsilon}$ can be expressed through by the following:
where $\omega_p^2 = \frac{nq^2}{m\varepsilon_0}$. Now non-trivial solutions of equation 15 must be found by setting the determinant of the matrix to 0.

$$\det \begin{pmatrix} \varepsilon_{\perp} - n_{\|}^2 & -i\varepsilon_x & n_{\perp} n_{\|} \\ i\varepsilon_x & \varepsilon_{\perp}^2 - n_{\|}^2 - n_{\perp}^2 & 0 \\ n_{\perp} n_{\|} & 0 & \varepsilon_{\|} - n_{\perp}^2 \end{pmatrix} = 0$$

If $n_{\|}$ is set constant, this equation can be solved for $n_{\perp}$. This leads to a fourth-order quadratic that can be solved with the quadratic formula to produce the following final dispersion relation of the LH waves being sent into the plasma:

$$n_{\perp}^2 = \frac{-C_2 \pm \sqrt{C_2^2 - 4C_4 C_0}}{C_4}$$

In equation 26:

$$C_0 = \varepsilon_{\|} \left( (n_{\|}^2 - \varepsilon_{\perp})^2 - \varepsilon_x^2 \right)$$

$$C_2 = (n_{\|}^2 - \varepsilon_{\perp})(\varepsilon_{\|} + \varepsilon_{\perp}) + \varepsilon_x^2$$

$$C_4 = \varepsilon_{\|}$$

This equation details the indices of refraction and dielectric values that will lead to propagation of waves of a given frequency within a plasma of a given density. The positive root of the equation will be used, as this represents the "slow" wave traveling through the plasma. Details of the derivation can be found in Freidberg [2].

3 Methods

3.1 Alcator C-Mod Geometry

In order to verify the accuracy of the simulations run in COMSOL, the simulations were run using the basic geometry of Alcator C-Mod's waveguide system. The geometry for the waveguides was taken from a SolidWorks model previously developed that preserved the core geometric aspects of C-Mod's waveguides. Figure 3.1 contains the final geometry used in the simulations, consisting of the waveguides and the plasma itself. The elliptical geometry was rotated around the $z$-axis at $r=0$ to produce a toroidal plasma, while the waveguides on the side were left as 2-dimensional components. The geometry shown in Figure 3.1 accurately models the
shape and positions of the waveguides as well as the interface between the waveguides and the plasma. While the rectangular shape of the plasma region at first glance appears inefficient and inaccurate, the density profile embedded within and calculated in section 3.4 ensures that the plasma physics of Alcator C-Mod is preserved because waves can only propagate when density is nonzero. Future iterations on this work could confirm this and improve runtime by importing the geometry of the first wall of C-Mod, though the core results would remain unchanged.

![Image of the Alcator C-Mod geometry](image)

**Figure 3.1:** Final geometry of Alcator C-Mod used in COMSOL simulations, where the x-axis is radius and the y-axis is height. The geometry shown was rotated around the y-axis at r=0 to produce a toroidal plasma shape. The first image comprises the plasma and vacuum vessel connected to the waveguides, while the second contains only the waveguide geometry.

### 3.2 Power Spectrum

In order to create multiple 2-dimensional layers to overlay, multiple simulations were run in COMSOL with each computational run using a different toroidal mode number. In order to determine which mode numbers were to be used, a power spectrum of the electric field of the waves was calculated and graphed, shown in Figure 3.2. The equation for the power spectrum comes from the Fourier transform of a square wave function representing the electric field emitted by the waveguides in real-space. The purpose of the power spectrum was to determine over what mode numbers the waves in C-Mod would transfer energy. The power spectrum conformed to the following equation:
\[ P(n_\parallel) = P_0 \frac{\sin^2(n_\parallel \frac{w\omega}{2c})}{n_\parallel^2} \frac{\sin^2\left(\frac{N\alpha}{2}\right)}{\sin^2\left(\frac{\alpha}{2}\right)} \]  

(27)

Where \( P_0 \) is a normalization constant, \( n_\parallel \) is the index of refraction of the plasma in the direction parallel to the magnetic field, \( w \) is the width of the waveguide, \( \omega \) is the wave frequency, \( c \) is the speed of light in a vacuum, and \( N \) is the number of waveguides. \( \alpha \) is defined by the following:

\[ \alpha = \Delta \Phi + n_\parallel \frac{\omega}{c} (w + \delta) \]

(28)

Where \( \Delta \Phi \) is the change in phase between waveguides, which in this case is 90 degrees, and \( \delta \) is the space between waveguides. In the simulations of this study, \( w=0.007 \) meters, \( \omega=28.9*10^9 \) Hz, \( \delta=0.0015 \) meters, and \( N=16 \) [6]. The portions of Figure 3.2 that form a peak in the power spectrum were studied more heavily in the COMSOL simulations, while the less-prominent portions were studied less; this was done to improve efficiency in the simulations, as the mode numbers that correspond to less energy transferred into the plasma are less interesting and useful from a physics perspective. In order to convert between mode number and the index of refraction, the following equation was used:

\[ n_\parallel = \frac{mc}{R\omega} \]

(29)

In this equation, \( R \) is the distance along the \( r \) axis from the center of the torus to the center of the plasma. In this study, \( R=0.68m \). The negative values of \( m \) in Figure 3.2 represent the waves travelling counterclockwise around the torus instead of clockwise. To select mode numbers, the power spectrum was analyzed such that if the power at a particular integer mode number had a value greater than 0.2, that mode number was used in the final COMSOL simulation.

The final results of this study should show that the plasma travels mostly in the counterclockwise direction with some fringe elements traveling in the clockwise direction. This reflects the magnitude of the power spectrum at positive and negative values of mode number.
3.3 Dielectric Tensor

Because plasmas are not homogeneous by nature, the simple use of a single dielectric constant misrepresents the characteristics of the plasma and the different electromagnetic effects it can exhibit in each direction [7]. Thus, a dielectric tensor was used to represent this difference in the Materials tab of the COMSOL model [8]. The equation for the dielectric tensor can be found in Equation 30. For use in Equation 25, the elements of the matrix are defined as variables for simplicity:

\[
K_\perp = 1 - \sum_j \frac{\omega_{pj}^2}{\omega^2 - \omega_{cj}^2},
\]
\[
K_A = \sum_j \frac{\omega_j}{\omega} \frac{\omega_{pj}^2}{\omega^2 - \omega_{cj}^2},
\]
\[
K_\parallel = 1 - \sum_j \frac{\omega_{cj}^2}{\omega^2},
\]

and \(\omega_{pj}^2 = \frac{n_0 q_j^2}{m_j \varepsilon_0}\) is the plasma frequency squared of the jth particle, where \(n_0\) is the number density of particle j, \(q_j\) is its charge, and \(m_j\) is its mass. Additionally, \(\omega_{cj} = \frac{qB}{m}\) is the cyclotron frequency of element j in the plasma with mass m, charge q, in a magnetic field B. This element could be a species of ion or an electron, though in this paper only electrons and protons were considered within the plasma.

This dielectric tensor assumes the plasma is “cold” and has a temperature of 0 Kelvin for simplicity, though incorporating temperature leads to similar results [2]. This tensor was also only used within the plasma section of the geometry in Figure 3.1a; within the waveguide, the medium through which the waves traveled was assumed to be air at 298K and 1 atm.
However, this tensor could not be directly used within COMSOL, as the derivation assumed the magnetic field propagated in the z direction while in the tokamak geometry, magnetic field travels in the phi direction. Thus a transformation was performed on the dielectric tensor:

\[
K_{\text{transform}} = Q\bar{K}Q^{-1} = \begin{bmatrix}
K_\perp & 0 & -iK_A \\
0 & K_\parallel & 0 \\
iK_A & 0 & K_\perp
\end{bmatrix}
\]  
(30)

In this transformation, \( Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \) is a rotation matrix used to account for the change in the definition of the magnetic field’s direction in the dielectric tensor [9]. To derive \( Q \), it was noted that in contrast to the initial conditions of the original dielectric tensor, the geometry in this paper required that magnetic field point in the phi direction and not the z direction. This required that the dielectric tensor be changed so the components conformed to the following condition:

\[
\begin{bmatrix}
\varepsilon_r' \\
\varepsilon_\phi' \\
\varepsilon_z'
\end{bmatrix} = Q 
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\varepsilon_z
\end{bmatrix}
\]  
(31)

In Equation 31, \( Q \) represents the rotation matrix that determines the orientation of the dielectric components in \( K \). The first matrix is the final result that corresponds to the transformation desired, while the second matrix of Eq. 27 is the original dielectric tensor. This matrix equation was solved, along with the conditions that \( \varepsilon_\phi \) should correspond to \( \varepsilon_z \) and \( \varepsilon_z' \) to \( \varepsilon_y \), to find the value of \( Q \) shown below Equation 26.

The form of Eq. 30 transforms the dielectric tensor in a similar manner. The following equation describes the relation of the electric field to the electric displacement field: \( D = KE \), where \( E \) is the electric field and \( D \) is the electric displacement field [10]. The same transformation that changes \( K \) must be applied to the displacement field and the electric field. In other words, \( D_{\text{transform}} = QD \) and \( E_{\text{transform}} = QE \). This leads to the following equation:

\[
Q^{-1}D_{\text{transform}} = KQ^{-1}E_{\text{transform}}
\]  
(32)

Where the subscript “transform” denotes the transformed coordinate system. This can be rewritten to form the equation that defines the dielectric tensor as a function of the rotation matrix:

\[
D_{tf} = QKQ^{-1}E_{\text{transform}}
\]  
(33)

Or

\[
D_{\text{transform}} = K_{\text{transform}}E_{\text{transform}}
\]  
(34)

14
The COMSOL model created used the geometry discussed in section 3.1 in conjunction with the dielectric tensor in 3.3. The model assumed that the edges of the model were perfect electrical conductors, which defined the tangential electric field at these edges to be zero. Similarly, the initial values of the electric fields within the plasma in all directions were assumed to be zero.

The electrical conductivity of the plasma was assumed to be 0.01 Siemens/meter. At higher values of conductivity, waves reflected back into the waveguides from which they launched, which does not represent reality. At lower values of conductivity, waves propagated all the way to the inner edge of the plasma geometry, again contrasting too strongly with reality. Thus the value of 0.01 was chosen through significant trial and error.

In order to send waves into the plasma, COMSOL required the definition of ports through which waves would propagate. The right-side ends of the waveguides in Figure 3.1b were used to represent the ports, and the power sent through each port was 1 Watt. This value can easily be changed within COMSOL to enable more energy to be sent into the plasma; regardless of the power, the core physics remains unchanged. The ports were arranged to send waves set as Transverse Electric in order to ensure that the electric field is always perpendicular to the direction of wave propagation [11].

After the ports were properly set up, the waveguides needed to be linked directly to the plasma geometry. This was done within COMSOL by creating an Identity Mapping between the two components. The waveguides and plasma were created separately because the plasma needed to be rotated around the y-axis at r=0, while the waveguides needed to only exist in one location in 2D. The Identity Mapping linked the output of the waveguides to the wall on the right side of the plasma component. The Identity Mapping also compensates for the different geometries between the two components; while the plasma use cylindrical coordinates, the waveguides use Cartesian coordinates because they are not rotated around any axis. As discussed in section 2.2, the phase of the waves launched from these ports was offset. However, this offset was accomplished in post-processing and not within the COMSOL model because the waveguides shown in Figure 3.1b all have the same phase in COMSOL.

In order for the Identity Mapping to work properly, the meshes of the plasma and waveguide components needed to line up exactly. The meshes at these two boundaries were made to line up. Only after this condition was met were the two components allowed to create meshes that filled in the rest of the geometry of the components. The link between the two components is shown in Figure 3.4.
It should be noted that due to limits in the amount of computation that could be performed, the density needed to be reduced in order to ensure that results could compute within a reasonable time frame. Specifically, the density in all sections of the plasma was reduced by a factor of 20. While this diminishes the model’s direct use, this enables the model to be run significantly faster by allowing the user to use fewer degrees of freedom in COMSOL. The decrease in density enables the user to utilize larger mesh sizes because the wavelengths of the waves traveling through the plasma were larger at lower densities. Thus, because simulations generally want 5 or 6 triangular meshes between each wave peak in order to properly model the physics, the density decrease improved computational runtimes significantly. Figure 3.5 shows the size of the meshes relative to the size of the waves propagating through the plasma. Tests were run, and it was determined that the important physics of LH wave propagation remained consistent despite the density change.
Once the COMSOL model was created including the geometry of Alcator C-Mod, a MATLAB script was written to enable the user to incorporate any plasma shot. While the COMSOL script thus far used an elliptical definition of density, the MATLAB script enables the user to use all density data gathered by the sensors in C-Mod to produce a density profile encompassing the entire plasma. The data collected by C-Mod includes a series of coils that gather flux data throughout the plasma’s geometry and a Thomson Scattering diagnostic that gathers density data along the z-axis. The Thomson Scattering diagnostic works by firing a laser along the z-axis at the center of the plasma from the r-z plane. A lens then focuses on several points in the plasma that the laser crosses and records Thomson Scattering events. The lens then focuses the photons collected into a detector that can then determine the density at each of these points [12].

The algorithm works by assuming that flux, and by extension density, both arrange into contours along which both density and flux are constant. The program then interpolates these values to produce a large matrix describing the particle density throughout the plasma. COMSOL can then interpolate based on this matrix to produce a continuous density profile for use in wave equations. In order for a researcher to use this tool and produce a density profile of any shot in C-Mod, they only need to provide the density along the z-axis, the flux throughout the plasma, and the density as a function of flux. Figure 3.3a shows an example of how the density from a sample shot was incorporated into the COMSOL model.

After the runs were finished, COMSOL was then connected to MATLAB to enable information to be quickly transferred between programs. This allows the user to analyze an arbitrary number of shots and mode numbers at one time. After each COMSOL run of a given mode number and shot, the information from the model was saved, and the electric field information was transferred for immediate use to MATLAB. Figure 3.6 contains several sample diagrams of the
information gathered by the COMSOL runs, including the density profiles and the effects of the waves on the plasma at the given mode numbers.

Figure 3.6: Density profile of sample shot (a) along with wave propagation at mode numbers -99, -120, -131, and 380 (b, c, d, e). The colors of images b, c, d, and e represent the electric fields in the phi direction. The information from these figures and many others will be superimposed in section 3.5 to determine how waves propagate as a function of phi.
3.5 Post-processing of COMSOL Simulations

The results produced by COMSOL contained the electric field in the phi direction of the plasma directly in front of the waveguides at various wave mode numbers. Once this was completed, the simulation needed to be generalized to encompass the entire toroidal geometry of the plasma. To do this, the results of each mode number were overlaid on top of each other through the use of the following formula:

\[
E_m(r,z,\Phi) = Re(E_m(r,z)A_m e^{im\Phi})
\]

(35)

In this formula, \(E_m(r,z)\) represents the complex electric field in the phi direction acquired in COMSOL. \(A_m\) is defined by the following:

\[
A_m = \sum_{j=0}^{N-1} \frac{\sin\left(\frac{n_j w \omega}{2c}\right)}{\frac{n_j \omega}{2c} \sqrt{2\pi}} e^{-i(\beta + \phi_j)}
\]

(36)

In this equation, \(\phi_j = j\Delta\phi\) with \(\Delta\phi = 0.5\pi\) and

\[
\beta = \frac{n_j \omega}{c} \left(j(w + \delta) + \frac{w}{2}\right)
\]

A MATLAB script was written that calculated \(E_m(r,z,\Phi)\) at every position within the plasma and at all toroidal angles between \(\Phi = -\pi\) and \(\Phi = \pi\) for each mode number. The electric fields at each point for all mode numbers were then added together to create a transform that corresponded to the electric field in the phi direction at all points in the 2-D portion of C-Mod corresponding to the area outside the waveguides. Finally, the resulting array of electric fields was used to interpolate the electric fields throughout the entire 3-D geometry of the tokamak’s torus. Figure 4.1 shows the resulting electric fields as they vary along the toroidal direction.

3.6 Line-of-sight electric field simulation

Alcator C-Mod has a Stark Effect Lower Hybrid Field (SELHF) diagnostic that measures the time-averaged absolute value of the electric field along the line of sight of the detector. Figure 3.8 shows how the diagnostic operates [13]. A spectrometer points at an antenna within the plasma, shown in Figure 3.8. In this paper, the spectrometer points at spot 4C in Figure 3.9. When C-Mod is running, the edges of the hydrogen plasma do not completely ionize; thus they emit atomic emission photons. When these photons travel through the plasma, they experience a shift and split in wavelength known as the Stark Effect. The Stark Effect describes the change in the energy, and the orbits, of electrons traveling around their nuclei when exposed to an electric field. The electric field causes the electrons to increase and decrease in energy, depending on their relative position in their orbit. This energy change leads to splitting of energy levels that are normally degenerate, and this splitting causes the wavelengths of the spectral lines of these partially ionized atoms to split [14]. The spectrometer measures this wavelength change, and researchers can determine the electric field of an area near the antenna using this shift.
A final MATLAB script was written that used the data gathered from section 3.5 to interpolate the electric field along the line between the spectrometer and the antenna. This enabled the calculation of the electric field in the area around the antenna that serves to confirm the results of the physical detector in C-Mod. The script is robust enough that the coordinates of the antenna target can be changed easily. For this paper, the coordinates for point 4C in Figure 3.9 were used.

4 Results

The final results of this paper are composed of two parts. First, the simulation of the post-processing performed in section 3.5 must be shown to verify the accuracy of the model. The images from this post-processing will show the plasma and waves traveling through it. These waves will travel predominantly in the counterclockwise direction; however, a significant
clockwise presence will be shown due to the secondary peak in the power spectrum in Figure 3.2. Second, the electric field along the line-of-sight between the antenna and spectrometer must be shown to be accurate and consistent with previous calculations.

Figure 4.1 shows the first portion of the results. The first image within Figure 4.1 shows the entire circular geometry of the plasma at 0.04 meters, which corresponds to the second waveguide counting from top to bottom. As is visible in the figure, the majority of the waves travel in the counterclockwise direction and a minority travels along the periphery of the plasma and in the clockwise direction. The second image in the figure shows how the waves would propagate when only the negative mode numbers around the power spectrum peak are included, while the subsequent image shows only the waves with positive mode numbers propagating. The fourth and fifth images zoom in on the plasma geometry and use a higher resolution to show how the waves propagate within the plasma when all mode numbers are included.
(b)

(c)

22
Figure 4.1: Final images demonstrating wave propagation due to the 4.6 GHz waves launched into the plasma from the waveguides. 4.1a shows the entire geometry of the plasma and the results of adding all mode numbers together. 4.1b contains an image of a higher resolution view of a quarter of the plasma with only negative mode numbers around the negative peak of the power spectrum of Figure 3.2, while 4.1c contains the positive mode number counterparts. 4.1d shows a higher resolution view of a quarter of the plasma with all mode numbers combined together. 4.1e contains a zoomed-in view of 4.1d. All figures show the plasma at z=0.04 meters, which corresponds to the waves propagating from the upper middle waveguide.
One interesting facet of Figure 4.1 lies in the number of high and low-energy “peaks” along the outer edge of the plasma, shown as dark blue and vibrant red in the figure. When a single mode number is plotted, these mode numbers will repeat all along the periphery of the torus, giving the impression that the waveguides repeat all along the torus. This effect is shown in Figure 4.1b and 4.1c. However, overlaying many mode numbers on top of each other causes these peaks to cancel out everywhere except the locations of the waveguides near y=0. This leads to the results shown in Figures 4.1d and 4.1e seen above.

Once this aspect of the simulations was finalized, the full 360° geometry was simulated and drawn to show how the line-of-sight sensor would function. Figure 4.2a shows the full 3-dimensional diagram of Alcator C-Mod’s first wall along with the line along which the electric field was calculated. For clarity, Figure 4.2b shows this same geometry and the line-of-sight viewed in the x-y plane, and Figure 4.2c shows this geometry in the x-z plane. This line starts at the detector and ends at the location of the antenna. One end of the line appears to be exiting the first wall, and this is absolutely accurate because the detector is located on the outer edges of the tokamak and thus outside of the first wall.
Using the data interpolated in section 3.5, the code then determines the electric field at every point along the line between the detector and the antenna. This electric field is depicted in Figure 4.3 as a function of the $x$, $y$, and $z$ coordinates of Figure 4.1a. The blank areas of the figure represent portions of the line-of-sight that exist outside of the domain of the plasma and thus could not be interpolated. As the line continues, near $x=0.9$, the electric field starts at a value of 0 and increases dramatically. This indicates that the line is in a region containing plasma and
detecting waves propagating through the plasma in this area. Because the simulation is able to
detect the changes in electric field due to the plasma and the waves traveling through it, this is
proof that the simulation functions correctly.

![Graph of Electric Field](image)

**Figure 4.3:** Graphs of the electric field along the line-of-sight as a function of distance along the Line-of-Sight.
Blank areas are outside of the domain of the plasma. The fluctuations in Electric field around x=0.85 represent the
large changes expected as the line-of-sight interacts with the dense plasma region near the antenna.

5 **Discussion, Conclusions, and Future Work**

The results shown in Section 4 demonstrate the viability of the tool developed through this
project. Figure 4.1 shows that the toroidal geometry of the plasma could be accurately modeled
in high fidelity, and Figures 4.2 thru 4.3 show how the line-of-sight diagnostic tool could be
modeled with adjustable precision. The robust nature of the code developed enables researchers
to model virtually any plasma discharge in Alcator C-Mod quickly and correctly. Any researcher
can easily modify values of magnetic field, magnetic flux, the dielectric tensor, or any other
parameter to accommodate the experiment they wish to run.

Future work can focus on verifying the accuracy of the data collected and comparing
measurements created in the model to those acquired within C-Mod. Researchers could
additionally augment the model to investigate other diagnostic tools and compare the results of simulations with reality. Finally, additional work can be performed to verify that the scripts written for this project produce accurate results more quickly than simply modeling everything directly in three dimensions. This work could test exactly how accurate the simulation results are and at what resolution this tool loses its advantage in speed.
References


