The topology of Baues complexes and flip graphs

by

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Abstract

This thesis contains several results on the connectedness of Baues complexes and flip graphs, which are topological spaces modeling certain sets coming from geometric combinatorics. These sets include the triangulations of a polytope, the tilings of a zonotope, and the extensions of an oriented matroid. Some long-standing conjectures are resolved, including the connectedness of triangulations of a product of two simplices and the sphericity of extension spaces of realizable oriented matroids. This thesis covers the main construction which is common to these proofs, but defers the details specific to each problem to other papers.

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Chapter 1

Introduction

This thesis addresses some long-standing questions involving objects known as Baues complexes and the closely related flip graphs. Vaguely speaking, these objects are topological models for how certain polyhedral complexes covering the same set in Euclidean space are related to each other. A concrete example is the associahedron, which models the triangulations of a convex polygon. The primary motivation to study these objects comes from the theories of secondary polytopes and fiber polytopes, which give vast generalizations of the associahedron. We begin by covering the motivating example of triangulations of a polytope as well as other common examples of Baues complexes and flip graphs. We then discuss some of the conjectures in the theory and, in the subsequent chapters, the (negative) resolutions for some of them.

1.1 Triangulations and subdivisions

Let \( A \) be a finite subset of \( \mathbb{R}^d \). A\ triangulation\ of \( A \) is a simplicial complex embedded in \( \mathbb{R}^d \) whose vertices are in \( A \) and which covers the convex hull of \( A \). If \( A \) is in convex position, we can more generally define a\ polyhedral subdivision, or subdivision, of \( A \) to be a polyhedral complex embedded in \( \mathbb{R}^d \) whose vertices are in \( A \) and which covers the convex hull of \( A \). If \( A \) is not in convex position, we can define a polyhedral subdivision of \( A \) in almost exactly the same way, but we will want some additional information about the interior points of \( A \) which are not vertices of the subdivision.
For now we will ignore this technicality and address this in Chapter 2 where we give a more combinatorial definition for subdivisions.

We will partially order the set of all polyhedral subdivisions of \( A \) by a relation called refinement. If \( A \) is in convex position, we can define refinement as follows: If \( \mathcal{P}, \mathcal{P}' \) are both subdivisions of \( A \), we say \( \mathcal{P} \) refines \( \mathcal{P}' \) if \( \mathcal{P} \) can be obtained by further subdividing \( \mathcal{P}' \). For general \( A \) the definition is morally the same, but it is made in such a way that any triangulation of \( A \) is a minimal element of the order. (We defer the definition for now.) Refinement gives a partial order on the set of all subdivisions of \( A \), and we denote this poset by \( Q(A) \) and call it the \textit{Baues poset} of \( A \). The maximal element of \( \Omega(A) \) is the subdivision whose unique maximal cell is the convex hull of \( A \). The minimal elements of \( \Omega(A) \) are the triangulations of \( A \).

The prototypical example of a Baues poset is when \( A \) is the set of vertices of an \( n \)-gon. In this case, it is well-known (due originally to work by Tamari [28] and Stasheff [25]) that \( \Omega(A) \) is isomorphic to the face lattice of a polytope called the \textit{associahedron}.

For general point sets \( A \), \( \Omega(A) \) is not necessarily the face lattice of a polytope, and can in fact have maximal chains of different lengths. However, a nice result is obtained when we restrict our attention to a certain subset of \( \Omega(A) \) called the regular subdivisions. Loosely speaking, a regular subdivision is a subdivision which is "defined globally." More specifically, let \( \omega : A \to \mathbb{R} \) be a function, and consider the lifted set

\[
A^\omega = \{(x, \omega(x)) : x \in A\} \subset \mathbb{R}^{d+1}.
\]

The convex hull of \( A^\omega \) is a polytope in \( \mathbb{R}^{d+1} \). If we set the \((d+1)\)-th coordinate of \( \mathbb{R}^{d+1} \) to be the "up" direction, then the lower faces of this polytope will form a subdivision of \( A \) when projected onto the first \( d \) coordinates. Any subdivision obtained in this way is said to be regular.

Let \( \Omega_{\text{reg}}(A) \) denote the restriction of \( \Omega(A) \) to the regular subdivisions of \( A \). We have the following key result by Gelfand, Kapranov, and Zelevinsky.

**Theorem 1.1** ([8]). Let \( A \) be a \( d \)-dimensional point set of size \( n \). Then \( \Omega_{\text{reg}}(A) \)
is isomorphic to the face lattice of an \((n - d - 1)\)-dimensional polytope, called the secondary polytope of \(A\).

For example, when \(A\) is the set of vertices of an \(n\)-gon, it can be shown that all subdivisions of \(A\) are regular, thus \(\Omega(A) = \Omega_{\text{reg}}(A)\) is isomorphic to the face lattice of an \((n - 3)\)-dimensional polytope.

The central question in this thesis can be phrased as whether in general \(\Omega_{\text{reg}}(A)\) is, in a sense, a good geometric representative of the entire poset \(\Omega(A)\). One can think of this as a question of how the non-regular subdivisions are related to the regular ones: are non-regular subdivisions “local” deformations of regular ones, and can the “space” of all of them be retracted onto the space of regular ones? We will make these questions more precise shortly.

We now focus on the minimal elements of \(\Omega(A)\), the triangulations. In applications, it is useful to think of triangulations as being related to each other through minimal moves called flips. In terms of the Baues poset, we can define a flip as follows: Consider an element of \(\Omega(A)\) with height one; that is, all of its children are triangulations. It can be shown that any such element has exactly two children. In this case, we say that these two children are related by a flip. The graph whose vertices are triangulations of \(A\) and whose edges are pairs of triangulations related by flips is called the flip graph of \(A\).

A simple question one can ask is: Is the flip graph of a point set \(A\) always connected? In two dimensions the answer is yes, and is not too hard to prove. In higher dimensions the question is much harder. The answer was shown to be no by Santos, who constructed a counterexample in dimension 6 [20] and later in dimension 5 [23]. The question is open for dimensions 3 and 4.

1.2 Zonotopal tilings and fiber polytopes

We now consider an alternate class of polyhedral complexes which produces a theory analogous to the one of polyhedral subdivisions. (Actually, the theory of zonotopal tilings can be considered a special case of that of polyhedral subdivisions, via the
Cayley trick. See Theorem 2.6.) Let \( V = \{v_1, \ldots, v_n\} \) be a finite set of nonzero vectors in \( \mathbb{R}^d \). For now, we will assume no element of \( V \) is a multiple of another, although we will later remove this condition. The zonotope \( Z(V) \) generated by \( V \) is defined to be

\[
Z(V) := \{\lambda_1 v_1 + \cdots + \lambda_n v_n : \lambda_i \in [0, 1] \text{ for all } i\}.
\]

A zonotopal tiling of \( Z := Z(V) \) is a polyhedral complex which covers \( Z \) and such that every maximum-dimensional cell of the complex is a translation of \( Z(W) \) for some \( W \subseteq V \). As before, the set of zonotopal tilings of \( Z \) is partially ordered by refinement. We will call the resulting poset the Baues poset of \( Z \), and denote it by \( \Omega(V) \).

There is also a notion of regularity for zonotopal tilings, although in this context is generally called coherence. Let \( \omega : V \to \mathbb{R} \) be a function, and consider the lifted set

\[
V^\omega = \{(v, \omega(v)) : v \in V\} \subseteq \mathbb{R}^{d+1}.
\]

Setting the \((d + 1)\)-th coordinate of \( \mathbb{R}^{d+1} \) to be the up direction, the lower faces of \( Z(V^\omega) \) form a zonotopal tiling of \( Z(V) \) when projected onto the first \( d \) coordinates. Any tiling obtained this way is said to be coherent.

Let \( \Omega_{coh}(V) \) to be the restriction of \( \Omega(V) \) to the coherent tilings of \( Z(V) \). We have an analogous result to Theorem 1.1 for coherent zonotopal tilings:

**Theorem 1.2** (Billera-Sturmfels [3]). Let \( V \) be a set of \( n \) nonzero vectors spanning \( \mathbb{R}^d \). Then \( \Omega_{coh}(V) \) is isomorphic to the face lattice of an \((n-d)\)-dimensional polytope.

There is also an analogous definition of a flip graph for zonotopal tilings of \( Z(V) \), and one can ask the same questions about connectivity of this graph. The flip graph is always connected when \( V \) is at most two-dimensional [10], and this thesis answers the question in higher dimensions.

Theorem 1.2 was actually proved by Billera and Sturmfels in a much broader setting, which encompasses Theorem 1.1 as well. Suppose we have an affine surjection \( \pi : P \to Q \), where \( P \) and \( Q \) are polytopes. Then \( \pi \) defines a class of \( \pi \)-induced subdivisions, which are certain polyhedral complexes covering \( Q \) whose cells are images.
of faces of $P$. Rather than give the formal definition of $\pi$-induced subdivisions, we give some examples for specific $\pi$.

If $A$ is a set of $n$ points, $P$ is an $(n-1)$-dimensional simplex, $Q$ is the convex hull of $A$, and $\pi : P \rightarrow Q$ is an affine map sending the vertices of $P$ bijectively to the points of $A$, then the $\pi$-induced subdivisions are the polyhedral subdivisions of $A$. If $V$ is a set of $n$ nonzero vectors, $P = [0, 1]^n$, $Q = Z(V)$, and $\pi : P \rightarrow Q$ is a linear map sending the standard basis vectors of $\mathbb{R}^n$ bijectively to the vectors of $V$, then the $\pi$-induces subdivisions are the zonotopal tilings of $Z(V)$. Other examples include the cellular strings of a polytope $P$, which are given when $\pi$ is a linear functional.

For any such map $\pi$, we obtain a poset $\Omega(\pi)$ of $\pi$-induced subdivisions ordered by refinement, called the Baues poset of $\pi$. There is also a notion of a coherent subdivision for any such $\pi$, and we obtain a subposet $\Omega_{coh}(\pi)$ of coherent $\pi$-induced subdivisions. The main theorem of [3] is that $\Omega_{coh}(\pi)$ is always isomorphic to the face lattice of a certain polytope of dimension $\dim(P) - \dim(Q)$, which they called the fiber polytope of $\pi$. This construction generalizes the secondary polytope of Gelfand, Kapranov, and Zelevinsky.

The actual constructions of $\pi$-induced subdivisions and fiber polytopes are beyond this paper, and we refer to the survey [17] by Reiner for more information. We mention them mostly to give historical motivation for the problems discussed.

1.3 The generalized Baues conjecture

As mentioned earlier, the main problem in this paper is whether, in each of the cases mentioned above, the poset of coherent subdivisions is "geometrically similar" to the poset of all subdivisions. We now make these questions more precise.

Given a poset $\mathcal{P}$, the order complex of $\mathcal{P}$ is the simplicial complex $\Delta(\mathcal{P})$ whose faces are the finite chains of $\mathcal{P}$. The order complex allows us to study the "topology" of a poset by studying the topology of its order complex. For example, if $\mathcal{P}$ is the face lattice of a polytope, then $\Delta(\mathcal{P})$ is isomorphic to the barycentric subdivision of the polytope. Note that for a Baues poset $\Omega(\pi)$, the order complex $\Delta(\Omega(\pi))$ is always
contractible because $\Omega(\pi)$ has a unique maximal element. We thus make the following modification: we define $\hat{\Omega}(\pi)$ to be $\Omega(\pi)$ with its top element removed. Similarly, we define $\hat{\Omega}_{\text{coh}}(\pi)$ to be $\Omega_{\text{coh}}(\pi)$ with its top element removed.

The fiber polytope construction implies $\Delta(\hat{\Omega}_{\text{coh}}(\pi))$ is the barycentric subdivision of the boundary of a polytope, and hence is homeomorphic to a sphere. However, it is known that $\Delta(\hat{\Omega}(\pi))$ is not necessarily homeomorphic to a sphere (and in fact rarely is). Nevertheless, one can ask if the homotopy type of the complex remains unchanged if one includes the non-coherent subdivisions. This leads the following conjecture.

**Conjecture 1.3.** Given $\pi$ as above, $\Delta(\hat{\Omega}(\pi))$ is homotopy equivalent to a $(\dim(P) - \dim(Q) - 1)$-sphere.

This conjecture is known as the *Generalized Baues conjecture*, and was originally posed as a problem by Billera, Kapranov, and Sturmfels in [2]. The name comes from a conjecture originally made by Baues [1], which is the above conjecture in the case where $P$ is a permutahedron and $\pi$ is a generic linear functional. In [2], the authors proved Conjecture 1.3 whenever $\pi$ is a linear functional, thus resolving the conjecture for the case of cellular strings.

A few years later, Rambau and Ziegler gave a counterexample to Conjecture 1.3 with $\dim(P) = 5$ and $\dim(Q) = 2$; in their example, $\hat{\Omega}(\pi)$ is disconnected. Afterwards, attention shifted to special cases of the Generalized Baues Conjecture, in particular the cases of polyhedral subdivisions and zonotopal tilings (i.e., when $P$ is a simplex or a cube). Eventually, Santos found a point set $A$ in 6 dimensions such that $\hat{\Omega}(A)$ is disconnected [24], resolving the generalized Baues conjecture for polyhedral subdivisions.

The main focus of this thesis will be the conjecture in the case of zonotopal tilings, which remained open. This case was of special interest primarily due to its connection with *oriented matroids*, which are combinatorial abstractions of real vector arrangements or real hyperplane arrangements. In particular, the generalized Baues conjecture for zonotopal tilings is equivalent to the *extension space conjecture* for oriented matroids. See Chapter 5.
Related to the generalized Baues conjecture is the question of connectivity of flip graphs. For example, if for some point set $A$, $\Omega(A)$ is the face lattice of a polytope (as is the case when all the subdivisions of $A$ are regular), the flip graph of $A$ is the 1-skeleton of this polytope and hence is connected. In general, however, sphericity or connectedness of $\Omega(A)$ does not imply connectedness of the associated flip graph, and vice versa. However, there are cases where information about one can lead to information about the other. In particular, as stated in [20, Cor. 4.3], if $A$ is a point set in \textit{general} position whose flip graph is disconnected, then there is a non-trivial subset $A' \subseteq A$ such that $\Omega(A')$ is disconnected. This was the strategy used by Santos to disprove the generalized Baues conjecture for polyhedral subdivisions. An analogous proposition holds for zonotopal tilings (Proposition 5.8), and this is the strategy we use to disprove the generalized Baues conjecture for zonotopal tilings.

1.4 Summary of results

We now summarize our results and outline the rest of the thesis. In Chapter 2 we develop the technical framework for the rest of the thesis. In Chapter 3 we prove the central result, which is as follows:

\textbf{Theorem 1.4.} There exists a vector arrangement $V$ in $\mathbb{R}^3$ whose associated flip graph of zonotopal tilings is disconnected.

In Chapter 4 we consider the point arrangement $A$ where $A$ is the set of vertices of $\Delta^m \times \Delta^n$, the Cartesian product of two simplices. We present the following results:

\textbf{Theorem 1.5.} The flip graph of $A$ is connected when $m \leq 3$.

\textbf{Theorem 1.6.} The flip graph of $A$ is disconnected when $m = 4$ and $n$ is large.

In Chapter 5 we cover oriented matroids and extension spaces. We describe the connection between zonotopal tilings and oriented matroids. The following is the main result of the section.
Theorem 1.7. There exists a vector arrangement \( V \) in \( \mathbb{R}^3 \) in general position whose associated flip graph of zonotopal tilings is disconnected.

Corollary 1.8. There exists a vector arrangement \( V \) in \( \mathbb{R}^3 \) such that \( \hat{\Omega}(V) \) is disconnected.

In particular, this Corollary disproves the generalized Baues conjecture for zonotopal tilings and the extension space conjecture for oriented matroids.

Theorem 1.4 will be the only result which will be proven in full in this thesis. Theorems 1.6 and 1.7 can be proven using the ideas of Theorem 1.4, although this requires additional work. We will instead give an outline of the proofs and refer to the full papers [13], [11] for the details. The proof of Theorem 1.5 is too long for this thesis, and is found in [12].
Chapter 2

Subdivisions and flips

This chapter forms the technical foundation for the thesis. We give the modern definition of polyhedral subdivisions as well as definitions for flips, regular subdivisions, and mixed subdivisions. The presentation here mostly follows the comprehensive book [15], which we recommend for further reading.

2.1 Polyhedral subdivisions

In the previous chapter we gave impromptu definitions for polyhedral subdivisions and refinement in terms of polyhedral complexes. While this is the idea we want to capture, we noted that there are issues with these definitions when the point set $A$ is not in convex position. In particular, we would like any triangulation of $A$ to be a minimal subdivision with respect to refinement, as this is what is compatible with the geometric theories of secondary polytopes and fiber polytopes. While we could modify the previous definitions to suit our purposes, it will be easier to start from scratch with more combinatorial definitions.

Let $A$ be a finite subset of $\mathbb{R}^d$. We define a cell of $A$ to be any subset of $A$. A simplex is a cell which is affinely independent. A face of a cell $C$ is a subset $F \subseteq C$ such that either $F$ is empty or there exists a linear functional $\phi \in (\mathbb{R}^d)^*$ such that $F$ is the set of all points which minimize $\phi$ on $C$. For any cell $C$, let $\text{conv}(C)$ denote the convex hull of $C$. 

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Definition 2.1. A *polyhedral subdivision*, or *subdivision*, of $A$ is a collection $\mathcal{S}$ of cells of $A$ such that

1. If $C \in \mathcal{S}$ and $F$ is a face of $C$, then $F \in \mathcal{S}$.

2. If $C, C' \in \mathcal{S}$, then $C \cap C'$ is a face of both $C$ and $C'$ and $\text{conv}(C) \cap \text{conv}(C') = \text{conv}(C \cap C')$.

3. $\bigcup_{C \in \mathcal{S}} \text{conv}(C) = \text{conv}(A)$.

The subdivision consisting of $A$ and all faces of $A$ is the *trivial subdivision*. A subdivision all of whose elements are simplices is a *triangulation*.

For subdivisions $\mathcal{S}, \mathcal{S}'$, we say that $\mathcal{S}$ is a *refinement* of $\mathcal{S}'$ if every element of $\mathcal{S}$ is a subset of an element of $\mathcal{S}'$. Refinement gives a partial order on the set of all subdivisions of $A$; we denote the resulting poset by $\Omega(A)$. The maximal element of $\Omega(A)$ is the trivial subdivision and the minimal elements are the triangulations. We let $\hat{\Omega}(A)$ denote $\Omega(A)$ minus the trivial subdivision.

Note that in these definitions, a cell may contain a point which is in the interior of the convex hull of other points in the cell. If such a cell appears in a subdivision, then removing this point from every cell in the subdivision results in another subdivision which is a refinement of the original one.

If $\mathcal{S}$ is a subdivision of $A$ and $F$ is a face of $A$, then $\mathcal{S}$ induces a subdivision $\mathcal{S}[F]$ of $F$ by

$$\mathcal{S}[F] := \{C \in \mathcal{S} : C \subseteq F\}.$$ 

2.2 Flips

In the previous chapter we defined flips in terms of the Baues poset $\Omega(A)$. In practice, it will be easier to use an equivalent alternate definition in terms of affine circuits.

A *circuit* is a minimal affinely dependent subset of $\mathbb{R}^d$. If $X = \{x_1, \ldots, x_k\}$ is a circuit, then the elements of $X$ satisfy an affine dependence equation

$$\sum_{i=1}^{k} \lambda_i x_i = 0$$
where \( \lambda_i \in \mathbb{R} \setminus \{0\} \) for all \( i \), \( \sum_i \lambda_i = 0 \), and the equation is unique up to multiplication by a constant. This gives a unique partition \( X = X^+ \cup X^- \) of \( X \) given by \( X^+ = \{ x_i : \lambda_i > 0 \} \) and \( X^- = \{ x_i : \lambda_i < 0 \} \). We will write \( X = (X^+, X^-) \) to denote a choice of which part we call \( X^+ \) and which we call \( X^- \).

A circuit \( X = (X^+, X^-) \) has exactly two non-trivial subdivisions, which are the following triangulations:

\[
\mathcal{T}_X^+ := \{ \sigma \subseteq X : \sigma \nsubseteq X^+ \} \quad \mathcal{T}_X^- := \{ \sigma \subseteq X : \sigma \nsubseteq X^- \}.
\]

Given a subdivision \( \mathcal{S} \) and a cell \( C \in \mathcal{S} \), we define the link of \( C \) in \( \mathcal{S} \) as

\[
\text{link}_\mathcal{S}(C) := \{ C' \in \mathcal{S} : C \cap C' = \emptyset, C \cup C' \in \mathcal{S} \}.
\]

We can now state the definition of a flip, in the form of a proposition.

**Proposition 2.2** (Santos [21]). Let \( \mathcal{T} \) be a triangulation of \( A \). Suppose there is a circuit \( X = (X^+, X^-) \) contained in \( A \) such that

1. \( \mathcal{T}_X^+ \subseteq \mathcal{T} \).

2. All maximal simplices of \( \mathcal{T}_X^+ \) have the same link \( \mathcal{L} \) in \( \mathcal{T} \).

Then the collection

\[
\mathcal{T}' := \mathcal{T} \setminus \{ \rho \cup \sigma : \rho \in \mathcal{L}, \sigma \in \mathcal{T}_X^+ \} \cup \{ \rho \cup \sigma : \rho \in \mathcal{L}, \sigma \in \mathcal{T}_X^- \}
\]

is a triangulation of \( A \). We say that \( \mathcal{T} \) has a flip supported on \( (X^+, X^-) \), and that \( \mathcal{T}' \) is the result of applying this flip to \( \mathcal{T} \) (or that \( \mathcal{T} \) and \( \mathcal{T}' \) are related by this flip).

In brief, a flip changes the triangulation of a single circuit within a larger triangulation. This definition is equivalent to the one given in the Introduction. We define the *flip graph* of \( A \) to be the graph whose vertices are the triangulations of \( A \) and with an edge between two triangulations if they are related by a flip.
2.3 Regular subdivisions

Let \( A \subseteq \mathbb{R}^d \) be as before. Let \( \omega : A \to \mathbb{R} \) be any function. For a cell \( C \subseteq A \), we define the \textit{lift of} \( C \) to be the set \( C^\omega \subseteq \mathbb{R}^m \times \mathbb{R} \) given by

\[
C^\omega := \{(x, \omega(x)) : x \in C\}.
\]

We call a subset \( F \subseteq A^\omega \) a \textit{lower face} of \( A^\omega \) if either \( F \) is empty or there is a linear functional \( \phi \in (\mathbb{R}^m \times \mathbb{R})^* \) such that \( \phi(0, 1) > 0 \) and \( F \) is the set of all points which minimize \( \phi \) on \( A^\omega \). Then the collection of all \( C \subseteq A \) such that \( C^\omega \) is a lower face of \( A^\omega \) is a subdivision of \( A \). We call this the \textit{regular} subdivision of \( A \) with respect to \( \omega \), and denote it by \( \mathcal{S}_A^\omega \).

If \( F \) is a face of \( A \), then \( \mathcal{S}_A^\omega[F] = \mathcal{S}_F^{\omega|_F} \).

Both triangulations of a circuit are regular, as stated below.

**Proposition 2.3.** Suppose \( \sum_{i=1}^{k} \lambda_i x_i = 0 \) is the affine dependence equation for a circuit \( X = (X^+, X^-) \) with \( X^+ = \{x_i : \lambda_i > 0\} \) and \( X^- = \{x_i : \lambda_i < 0\} \). Let \( \omega : X \to \mathbb{R} \) be a function. Then

\[
\mathcal{S}_X^\omega = \begin{cases} 
\mathcal{S}_X^+ & \text{if } \sum_{i=1}^{k} \lambda_i \omega(x_i) > 0 \\
\mathcal{S}_X^- & \text{if } \sum_{i=1}^{k} \lambda_i \omega(x_i) < 0.
\end{cases}
\]

2.4 Mixed subdivisions

We now describe a construction called a mixed subdivision. While mixed subdivisions appear to be a generalization of subdivisions, they can in fact be represented themselves as ordinary subdivisions through a technique called the Cayley Trick (see Section 2.5). The advantage of mixed subdivisions is that they are lower-dimensional than their corresponding ordinary subdivision and can thus be visualized easier. We will use mixed subdivisions to define zonotopal tilings, and later to study triangulations of the product of two simplices.

Let \( A_1, \ldots, A_n \) be finite subsets of \( \mathbb{R}^d \). The \textit{Minkowski sum} of \( A_1, \ldots, A_n \) is the
set of points

$$\sum A_i = A_1 + \cdots + A_n := \{x_1 + \cdots + x_n : x_i \in A_i \text{ for all } i\}.$$ 

In this paper, we want the phrase "\(\sum A_i\)" to identify a set of points but also retain the information of what \(A_1, \ldots, A_n\) are. In other words, \(\sum A_i\) will formally mean an ordered tuple \((A_1, \ldots, A_n)\) but by abuse of notation will also refer to the set of points in the Minkowski sum.

A mixed cell of \(\sum A_i\) is a set \(\sum B_i\) where \(B_i\) is a cell of \(A_i\) for all \(i\). A mixed cell is fine if all the \(B_i\) are simplices and lie in independent affine subspaces. A face of a mixed cell \(\sum B_i\) is a mixed cell \(\sum F_i\) of \(\sum B_i\) such that there exists a linear functional \(\phi \in (\mathbb{R}^m)^*\) such that for all \(i\), either \(F_i = \emptyset\) or \(F_i\) is the set of all points which minimize \(\phi\) on \(B_i\).

**Definition 2.4.** A mixed subdivision of \(\sum A_i\) is a collection \(\mathcal{S}\) of mixed cells of \(\sum A_i\) such that

1. If \(\sum B_i \in \mathcal{S}\) and \(\sum F_i\) is a face of \(\sum B_i\), then \(\sum F_i \in \mathcal{S}\).

2. If \(\sum B_i, \sum B'_i \in \mathcal{S}\), then \(\sum(B_i \cap B'_i)\) is a face of both \(\sum B_i\) and \(\sum B'_i\) and \(\text{conv}(\sum B_i) \cap \text{conv}(\sum B'_i) = \text{conv}(\sum(B_i \cap B'_i))\).

3. \(\bigcup_{\sum B_i \in \mathcal{S}} \text{conv}(\sum B_i) = \text{conv}(\sum A_i)\).

Geometrically, a mixed subdivision is a polyhedral complex covering the convex hull of \(\sum A_i\) whose cells are mixed cells of \(\sum A_i\). There are also "compatibility" relations between the mixed cells of a mixed subdivision; in particular, if two mixed cells \(\sum C_i\) and \(\sum C'_i\) are geometrically adjacent, then their intersection \(\sum F_i\) must have each summand \(F_i\) a subset of \(C_i\) and \(C'_i\).

The mixed subdivision consisting of \(\sum A_i\) and all faces of \(\sum A_i\) is the trivial mixed subdivision. A mixed subdivision is fine if all of its elements are fine. For mixed subdivisions \(\mathcal{S}, \mathcal{S}'\), we say that \(\mathcal{S}\) is a refinement of \(\mathcal{S}'\) if every element of \(\mathcal{S}\) is a mixed cell of an element of \(\mathcal{S}'\). Refinement gives a partial order on the set of
all mixed subdivisions of $\sum A_i$; denote this poset by $\Omega(\sum A_i)$. The maximal element of $\Omega(\sum A_i)$ is the trivial mixed subdivision and the minimal elements are the fine mixed subdivisions. Let $\hat{\Omega}(\sum A_i)$ denote $\Omega(\sum A_i)$ with the trivial mixed subdivision removed.

### 2.4.1 Coherent mixed subdivisions

Let $A_1, \ldots, A_n$ be finite subsets of $\mathbb{R}^d$. For each $i = 1, \ldots, n$, let $\omega_i : A_i \to \mathbb{R}$ be a function. For a mixed cell $\sum B_i$ of $\sum A_i$, define the lift $(\sum B_i)^\omega$ by

$$(\sum B_i)^\omega := \sum B_i^{\omega_i}.$$ 

We call a mixed cell $\sum F_i$ of $(\sum A_i)^\omega$ a lower face of $(\sum A_i)^\omega$ if there is a linear functional $\phi \in (\mathbb{R}^m \times \mathbb{R})^*$ such that $\phi(0, 1) > 0$ and for each $i$, either $F_i$ is empty or $F_i$ is the set of points which minimize $\phi$ on $A_i^{\omega_i}$. The collection of all $\sum B_i$ such that $(\sum B_i)^\omega$ is a lower face of $(\sum A_i)^\omega$ is a mixed subdivision of $\sum A_i$ called the coherent mixed subdivision of $\sum A_i$ with respect to $\omega$. We denote it by $\mathcal{S}_\sum A_i^\omega$.

In this paper we will often deal with sums $\sum A_i$ where each $A_i$ is a subset of $\{e_1, \ldots, e_d\}$, the standard basis of $\mathbb{R}^d$. In this case, we have the following description of coherent mixed subdivisions.

**Theorem 2.5** (Develin and Sturmfels [6]). Suppose $A_i \subseteq \{e_1, \ldots, e_d\}$ for all $i$. Let $\omega_i : A_i \to \mathbb{R}$ be functions. For any $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ and $A_i$, let type$(x, \omega, A_i)$ be the set of all $e_j \in A_i$ such that

$$x_j - \omega_i(e_j) = \max\{x_k - \omega_i(e_k) : e_k \in A_i\}.$$ 

Then $\sum B_i$ is an element of $\mathcal{S}_\sum A_i^\omega$ if and only if there is some $x \in \mathbb{R}^m$ such that for all $i$, either $B_i = \emptyset$ or $B_i = \text{type}(x, \omega, A_i)$.
2.4.2 Zonotopal tilings

Let $A_1, \ldots, A_n$ be as before. If $|A_i| = 2$ for all $i$, then we will call $\sum A_i$ a zonotope. The mixed subdivisions of a zonotope are called zonotopal tilings.

For example, let $e_i$ denote the $i$-the standard basis vector of $\mathbb{R}^d$, and let $A_{ij} = \{e_i, e_j\}$ for all $1 \leq i < j \leq d$. Then $\Pi^{d-1} := \sum_{1 \leq i < j \leq d} A_{ij}$ is a zonotope called the permutohedron. The convex hull of $\Pi^{d-1}$ is a $(d - 1)$-dimensional polytope (and is usually what the term permutohedron refers to).

2.5 The Cayley trick

Let $A_1, \ldots, A_n$ be as before. Let $\{f_1, \ldots, f_n\}$ be a basis of $\mathbb{R}^n$. We define the Cayley embedding of $\sum A_i$ to be the following set in $\mathbb{R}^d \times \mathbb{R}^n$:

$$C\left(\sum A_i\right) := \bigcup_{i=1}^n \{(x, f_i) : x \in A_i\}.$$ 

The Cayley trick says the following.

**Theorem 2.6** (Sturmfels [26], Huber et al. [9]). The following are true.

1. $C$ is a bijection between the mixed cells of $\sum A_i$ and the cells of $C(\sum A_i)$, and this map preserves facial relations.

2. For any mixed subdivision $\mathcal{S}$ of $\sum A_i$, the collection $C(\mathcal{S})$ is a subdivision of $C(\sum A_i)$. The map $\mathcal{S} \mapsto C(\mathcal{S})$ is a poset isomorphism from $\Omega(\sum A_i)$ to $\Omega(C(\sum A_i))$.

3. If $\omega_i : A_i \to \mathbb{R}$ are functions for $i = 1, \ldots, n$ and $C(\omega) : C(\sum A_i) \to \mathbb{R}$ is defined as $C(\omega)(x, f_i) = \omega_i(x)$, then

$$C\left(\mathcal{S}^\omega \sum A_i\right) = \mathcal{S}^{C(\omega)} C(\sum A_i).$$

We say that two fine mixed subdivisions $\mathcal{S}$ and $\mathcal{S}'$ are related by a flip if the triangulations $C(\mathcal{S})$ and $C(\mathcal{S}')$ are related by a flip. The flip graph of $\sum A_i$ is the
graph whose vertices are fine mixed subdivisions of $\sum A_i$ and with an edge between two mixed subdivisions if they are related by a flip.
Chapter 3

A zonotope with disconnected flip graph

In this chapter we describe the central construction of the thesis, a zonotope with a disconnected flip graph. Explicitly, this zonotope is the three-dimensional permutohedron, but with each of its generating vectors repeated a large number of times. The idea of the proof will be to generate sufficiently "random" zonotopal tilings of this zonotope, and then show that the most of the generated tilings will be in different components of the flip graph. We begin by developing the machinery needed to generate the random tilings.

3.1 Zonotopal tilings of the 3-permutohedron

Recall that we defined the 3-dimensional permutohedron $\Pi^3$ to be the Minkowski sum $\sum A_{i<j} A_{i,j}$ where $A_{i,j} = \{e_i, e_j\}$ and $e_i$ is the $i$-th standard basis vector of $\mathbb{R}^4$. We will now construct eight specific zonotopal tilings of $\Pi_3$.

We first set up some notation. For a set $S$, let $\Gamma^k_S$ denote the set of all ordered $k$-tuples $(i_1, \ldots, i_k)$ of distinct $i_1, \ldots, i_k \in S$ under the equivalence relation $(i_1, \ldots, i_k) \sim (i_2, \ldots, i_k, i_1)$. We will use $(i_1 \cdots i_k)$ to denote the equivalence class of $(i_1, \ldots, i_k)$ in $\Gamma^k_S$. We write $-(i_1 \cdots i_k)$ to denote $(i_k \cdots i_1)$. We abbreviate $\Gamma^k_{[n]}$ as $\Gamma^k_n$.

Recall that the Cayley embedding associates $\Pi^3$ to a point set $C(\Pi^3)$ in $\mathbb{R}^4 \times \mathbb{R}^6$.  

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Letting $P^3$ denote this Cayley embedding and letting $\{f_\alpha\}_{\alpha \in \Gamma_4^3}$ be a basis of $\mathbb{R}^6$, we can write $P^3$ as

$$P^3 := \mathcal{C}(\Pi^3) = \bigcup_{(ij) \in \Gamma_4^3} \{(e_i, f_{(ij)}), (e_j, f_{(ij)})\}.$$ 

For the rest of this chapter, we will be working with subdivisions of $P^3$ instead of tilings of $\Pi^3$, as the notation is slightly easier. The two notions are equivalent by Theorem 2.6.

For any $(ijk) \in \Gamma_3^4$, we have a circuit $X_{(ijk)} = (X_{(ijk)}^+, X_{(ijk)}^-)$ in $P^3$ with affine dependence relation

$$(e_i, f_{(ij)}) - (e_j, f_{(ij)}) + (e_j, f_{(jk)}) - (e_k, f_{(jk)}) + (e_k, f_{(ki)}) - (e_i, f_{(ki)})$$

and $X_{(ijk)}^+, X_{(ijk)}^-$ defined in terms of this affine relation (see Section 2.2). Define $\mathcal{T}_{(ijk)} := T_{X_{(ijk)}^+}$. Note that $T_{-\gamma} = T_{X_{-\gamma}}$.

We now construct eight regular triangulations of $P^3$, each indexed by a different element of $\Gamma_3^4$. Fix some $\gamma = (ijk) \in \Gamma_3^4$. Let $\omega : P^3 \to \mathbb{R}$ be the function with

$$\omega(e_i, f_{(ij)}) = \omega(e_j, f_{(jk)}) = \omega(e_k, f_{(ki)}) = 1$$

and $\omega(x) = 0$ for all other $x \in P^3$. We define $\mathcal{T}_{P^3}^\gamma := \mathcal{T}_{P^3}^\omega$. It is easy to check that $\mathcal{T}_{P^3}^\gamma$ is a triangulation.

Let us take a closer look at $\mathcal{T}_{P^3}^\gamma$ where $\gamma = (ijk)$. For each $\gamma' \in \Gamma_3^4$, the circuit $X_{\gamma'}$ is a face of $P^3$, and thus $\mathcal{T}_{P^3}^\gamma$ induces a triangulation on $X_{\gamma'}$. From the definition of $\omega$, we have

$$\omega(e_i, f_{(ij)}) - \omega(e_j, f_{(ij)}) + \omega(e_j, f_{(jk)}) - \omega(e_k, f_{(jk)}) + \omega(e_k, f_{(ki)}) - \omega(e_i, f_{(ki)}) > 0$$

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and if \( l \) is the element of \([4] \setminus \{i, j, k\}\), we have

\[
\omega(e_i, f(ij)) - \omega(e_j, f(ij)) + \omega(e_j, f(jk)) - \omega(e_l, f(jk)) + \omega(e_l, f(li)) - \omega(e_i, f(li)) > 0
\]

\[
\omega(e_j, f(jk)) - \omega(e_k, f(jk)) + \omega(e_k, f(kl)) - \omega(e_l, f(kl)) - \omega(e_j, f(kl)) - \omega(e_k, f(kl)) > 0.
\]

Thus, by Proposition 2.3, \( \mathcal{T}_{P_3}^\gamma \) induces the following triangulations on the circuits \( X_\gamma \):

\[
\mathcal{T}(ijk), \mathcal{T}(ijl), \mathcal{T}(jkl), \mathcal{T}(kil)
\]

and hence

\[
\mathcal{T}(ijk), \mathcal{T}(ijl), \mathcal{T}(jkl), \mathcal{T}(kil) \subseteq \mathcal{T}_{P_3}^\gamma. \tag{3.1}
\]

### 3.2 A group action on \( \Gamma_4^3 \)

The key property of \( \mathcal{T}_{P_3}^\gamma \) is that it only has flips on the circuits \( X(ijl), X(jkl), \) and \( X(kil) \). The idea will be to tile a larger zonotope with 3-permutohedra and then tile each 3-permutohedron with some \( \mathcal{T}_{P_3}^\gamma \) so that in the end, no circuit of size six can be flipped. To help with this construction, we will define a group action on \( \Gamma_4^3 \).

For each \( \gamma = (ijk) \in \Gamma_4^3 \), we define a function \( o_\gamma : \binom{[4]}{3} \to \Gamma_4^3 \) by

\[
o_\gamma\{(i, j, k)\} = (ijk)
\]

\[
o_\gamma\{(i, j, l)\} = (ijl)
\]

\[
o_\gamma\{(j, k, l)\} = (jkl)
\]

\[
o_\gamma\{(k, i, l)\} = (kil)
\]

where \( \{l\} = [4] \setminus \{i, j, k\} \). Hence, equation (3.1) implies \( \mathcal{T}_{o_\gamma(S)} \subseteq \mathcal{T}_{P_3}^\gamma \) for all \( S \in \binom{[4]}{3} \).

It is easy to check that \( \gamma \) is determined by \( o_\gamma \).

Now, we will map each \( \alpha \in \Gamma_4^2 \) to a permutation \( \pi_\alpha : \Gamma_4^3 \to \Gamma_4^3 \). This map is
completely determined by the following rules: For any distinct \(i, j, k, l \in [4]\), we have

\[
\begin{align*}
\pi_{(ij)}(ijk) &= (jil) \\
\pi_{(kl)}(ijk) &= (ijl).
\end{align*}
\]

Let \(G_{\Gamma_4^3}\) be the permutation group of \(\Gamma_4^3\) generated by all the \(\pi_\alpha\).

**Proposition 3.1.** The following are true.

(a) Every element of \(G_{\Gamma_4^3}\) is an involution, and \(G_{\Gamma_4^3}\) is abelian and transitive on \(\Gamma_4^3\).

(b) For \(l \in [4]\), let \(H_l\) be the subgroup of \(G_{\Gamma_4^3}\) generated by \(\pi_{(il)}\) for all \(i \in [4] \setminus \{l\}\).

Let \(i,j,k \in [4] \setminus \{l\}\) be distinct, and let \(\Gamma_4^3(ijk)\) be the set of all \(\gamma \in \Gamma_4^3\) such that \(o_\gamma(\{i,j,k\}) = (ijk)\). Then \(\Gamma_4^3(ijk)\) is an orbit of \(H_l\).

**Proof.** Since each \(\gamma\) is determined by \(o_\gamma\), we can view \(G_{\Gamma_4^3}\) as an action on the set of functions \(o_\gamma\). We check that for all distinct \(i, j, k, l \in [4]\) and \(\gamma \in \Gamma_4^3\), we have

\[
\begin{align*}
o_\pi_{(ij)}\gamma(\{i,j,k\}) &= -o_\gamma(\{i,j,k\}) \\
o_\pi_{(ij)}\gamma(\{i,j,l\}) &= -o_\gamma(\{i,j,l\}) \\
o_\pi_{(ij)}\gamma(\{j,k,l\}) &= o_\gamma(\{j,k,l\}) \\
o_\pi_{(ij)}\gamma(\{k,i,l\}) &= o_\gamma(\{k,i,l\})
\end{align*}
\]

It follows that we can embed \(G_{\Gamma_4^3}\) as a subgroup of \(Z_2^4\). This implies that every element of \(G_{\Gamma_4^3}\) is an involution and \(G_{\Gamma_4^3}\) is abelian. It is also easy to check from the above action on the \(o_\gamma\) that every element of \(\Gamma_4^3\) has orbit of size 8, and hence \(G_{\Gamma_4^3}\) is transitive.

From the above action on \(o_\gamma\), we see that \(H_l\) maps \(\Gamma_4^3(ijk)\) to itself and every element of \(\Gamma_4^3(ijk)\) has orbit of size 4 under \(H_l\). Since \(|\Gamma_4^3(ijk)| = 4\), \(\Gamma_4^3(ijk)\) is an orbit of \(H_l\). \(\square\)
3.3 A zonotope and a component of its flip graph

Let $N$ be a positive integer to be determined later. For each distinct $i, j \in [4]$ and $-N \leq r \leq N$, we create a variable $f_{ij}^r$, and we make the identification of variables

$$f_{ij}^r = f_{ji}^{-r}.$$ 

Let $\{f_{ij}^r\}_{1 \leq i < j \leq 4, -N \leq r \leq N}$ be a basis for $\mathbb{R}^{6(2N+1)}$. Let $P$ be the point set

$$P := \bigcup_{-N \leq r \leq N} \bigcup_{1 \leq i < j \leq 4} \{(e_i, f_{ij}^r), (e_j, f_{ij}^r)\}.$$  

(3.2)

Note that $P$ is the Cayley embedding of the 3-dimensional permutohedron with $2N+1$ copies of each generating vector. We will now prove the main result.

**Theorem 3.2.** For large enough $N$, the flip graph of $P$ is not connected.

Since $P$ is the Cayley embedding of a zonotope, this gives the result claimed at the beginning of this chapter.

For distinct $i, j, k \in [4]$ and for any $-N \leq r, s, t \leq N$, let $X_{ijk}^{rst} = ((X_{ijk}^{rst})^+, (X_{ijk}^{rst})^-)$ be the circuit in $P$ with affine dependence relation

$$(e_i, f_{ij}^r) - (e_j, f_{ij}^r) + (e_j, f_{jk}^s) - (e_k, f_{jk}^s) + (e_k, f_{ki}^t) - (e_i, f_{ki}^t)$$

and $(X_{ijk}^{rst})^+$, $(X_{ijk}^{rst})^-$ defined in terms of this relation. Let $\mathcal{T}_{ijk}^{rst} := \mathcal{T}_{X_{ijk}^{rst}}^+$. The following Lemma identifies a component of the flip graph of $P$.

**Lemma 3.3.** Let $C$ be a collection of triangulations of the form $\mathcal{T}_{ijk}^{rst}$ such that for all $\mathcal{T}_{ijk}^{rst} \in C$ and $\{l\} = [4] \setminus \{i, j, k\}$, there exist $1 \leq u, v, w \leq N$ such that

$$\mathcal{T}_{ijl}^{ru(-u)}, \mathcal{T}_{jkl}^{sv(-v)}, \mathcal{T}_{kil}^{tw(-w)} \in C.$$ 

Let $\mathcal{S}_C$ be the set of all triangulations of $P$ which contain every element of $C$ as a subset. Then $\mathcal{S}_C$ is closed under flips.
The proof follows immediately from the following two facts.

**Proposition 3.4.** Let $\mathcal{T}$ be a triangulation of $P$ such that $\mathcal{T}_{ijk} \subseteq \mathcal{T}$. Let $\mathcal{T}'$ be the result of a flip on $\mathcal{T}$ which is not supported on $X_{ijk}^{rst}$. Then $\mathcal{T}_{ijk}^{rst} \subseteq \mathcal{T}'$.

**Proof.** Suppose the flip from $\mathcal{T}$ to $\mathcal{T}'$ is supported on $X = (X^+, X^-)$. By Proposition 2.2, if $\sigma \in \mathcal{T}$ and $\sigma \notin \mathcal{T}'$, then $\sigma \supseteq X^-$. If $\sigma \in \mathcal{T}_{ijk}^{rst}$, it can be checked that the latter statement is only possible if $X = X_{ijk}^{rst}$. Thus $\mathcal{T}_{ijk}^{rst} \subseteq \mathcal{T}'$. □

**Proposition 3.5.** Let $\mathcal{T}$ be a triangulation of $P$, and suppose that there are distinct $i, j, k, l \in [4]$ and $1 \leq r, s, t, u, v, w \leq N$ such that

\[ \mathcal{T}_{ij}^{rv(-u)}, \mathcal{T}_{jk}^{sw(-v)}, \mathcal{T}_{kl}^{tu(-w)} \subseteq \mathcal{T}. \]

Then $\mathcal{T}$ does not have a flip supported on $X_{ijk}^{rst}$.

**Proof.** Consider the cell

\[ C := \{(e_i, f_{ij}^u), (e_t, f_{tij}^u), (e_j, f_{jij}^v), (e_l, f_{lij}^v), (e_k, f_{kij}^w), (e_t, f_{tij}^w)\}. \]

$C$ is a face of $P$, as seen by the linear functional $\phi$ such that

\[ \phi(e_a, 0) = 0 \quad \text{for all } a \]

\[ \phi(0, f_{ab}^p) = \begin{cases} 1 & \text{if } f_{ab}^p = f_{ij}^u, f_{tij}^u, \text{ or } f_{kij}^w \\ 0 & \text{otherwise.} \end{cases} \]

Since $C$ is a simplex, we must have $C \in \mathcal{T}$. Thus, there is some maximal simplex $\tau \in \mathcal{T}$ such that $C \subseteq \tau$. Since $\tau$ is maximal, it must contain one element from each of the sets

\[ \{(e_i, f_{ij}^u), (e_j, f_{jij}^v), (e_k, f_{kij}^w), (e_t, f_{tij}^w)\}. \]

Suppose $(e_i, f_{ij}^u) \in \tau$. Then since $C \subseteq \tau$, we have

\[ (X_{ijl}^{rv(-u)})^+ = \{(e_i, f_{ij}^u), (e_j, f_{jij}^v), (e_t, f_{tij}^u)\} \subseteq \tau. \]

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On the other hand, since \( \mathcal{T}_{ij}^{r(u)} \subseteq \mathcal{T} \), we have \( (X_{ij}^{r(u)})^- \in \mathcal{T} \). This is a contradiction, because the opposite parts of a circuit cannot both be cells of a triangulation (since the interiors of these cells intersect). Hence \( (e_j, f_{ij}^t) \in \tau \). Similarly, \( (e_k, f_{jk}^t) \in \tau \) and \( (e_i, f_{ki}^t) \in \tau \). Hence, \( (X_{ij}^{r(u)})^- \cup C \subseteq \tau \).

Now suppose that \( \mathcal{T} \) has a flip supported on \( X_{ijk}^{r(st)} \), and let \( \mathcal{T}' \) be the result of this flip. Since \( \tau \) is of maximum dimension and \( |\tau \setminus X_{ijk}^{r(st)}| = |\tau| - 3 \), \( \tau \setminus X_{ijk}^{r(st)} \) cannot be in the link of a maximal simplex of \( \mathcal{T}_{ijk}^{r(st)} \). Thus, \( \tau \in \mathcal{T}' \). But this is a contradiction, since \( (X_{ijk}^{r(st)})^+ \in \mathcal{T}' \) and \( (X_{ijk}^{r(st)})^- \subseteq \tau \). So there is no flip supported on \( X_{ijk}^{r(st)} \).\( \square \)

It remains to prove that there is some \( \mathcal{C} \) for which \( \mathcal{S}_C \) is neither empty nor the whole set of triangulations of \( P \). This will be done in the next section.

### 3.4 Construction of \( \mathcal{C} \) and \( \mathcal{T} \in \mathcal{S}_C \)

We first consider the regular subdivision \( \mathcal{S}_P \) where \( \omega : P \to \mathbb{R} \) is a function such that

\[
\omega(e_i, f_{ij}^t) - \omega(e_j, f_{ij}^t) = r
\]

for all distinct \( i, j \in [4] \) and \(-N \leq r \leq N\). The cells of \( \mathcal{S}_P \) are described as follows.

**Proposition 3.6.** Let \( \mathcal{X} \) be the set of \( x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \) such that \( x_1 + \cdots + x_4 = 0 \) and if \( ijkl \) is a permutation of \([4]\) such that \( x_i \geq x_j \geq x_k \geq x_l \), then \( x_i - x_j, x_j - x_k, \) and \( x_k - x_l \) are integers at most \( N \). Let \( \mathcal{X}^* \) be the set of \( x \in \mathcal{X} \) such that \( |x_i - x_j| \leq N \) for all \( i, j \in [n] \). The following are true.

(a) The map \( C(x) = \{(e_i, f_{ij}^r) : x_i - x_j \geq r\} \) is a bijection from \( \mathcal{X} \) to the maximal cells of \( \mathcal{S}_P \).

(b) If \( x \in \mathcal{X}^* \), then \( C(x) \) is the Cayley embedding of a translated 3-permutohedron.

Specifically, \( C(x) = P(x) \cup D(x) \), where

\[
P(x) := \bigcup_{1 \leq i < j \leq 4} \{(e_i, f_{ij}^{x_i-x_j}), (e_j, f_{ij}^{x_i-x_j})\}
\]
and $D(x)$ is a simplex affinely independent to $P(x)$.

(c) Let $x \in \mathcal{X} \setminus \mathcal{X}^*$. Suppose $F_1, \ldots, F_k$ are faces of $C(x)$ and $\mathcal{T}_1, \ldots, \mathcal{T}_k$ are triangulations of these faces, respectively, which agree on intersections of these faces. Then there is a triangulation of $C(x)$ which contains $\mathcal{T}_1, \ldots, \mathcal{T}_k$ as subsets.

Proof. Parts (a) and (b) follows from Theorem 2.5. The only nontrivial case of part (c) is when $x$ satisfies $x_i \geq x_j \geq x_k \geq x_l$ for some permutation $ijkl$ of $[4]$, $x_i - x_k \leq N$, $x_j - x_l \leq N$, and $x_i - x_l > N$. In this case $C(x)$ is of the form $X_{ijkl}^{rst} \cup X_{ijkl}^{uvw} \cup D$, where $D$ is a simplex affinely independent to $X_{ijkl}^{rst} \cup X_{ijkl}^{uvw}$. It is easy to check that any triangulations of $X_{ijkl}^{rst}$ and $X_{ijkl}^{uvw}$ can be extended to a triangulation of $X_{ijkl}^{rst} \cup X_{ijkl}^{uvw}$, and hence to a triangulation of $C(x)$. \hfill \Box

In terms of mixed subdivisions, the coherent mixed subdivision associated to $\omega$ is given by tiling a large permutohedron with smaller unit permutohedra (the tiles corresponding to $\mathcal{X}^*$) and pieces of permutohedra (the tiles corresponding to $\mathcal{X} \setminus \mathcal{X}^*$).

For each $x \in \mathcal{X}^*$, we have an affine isomorphism $P^3 \to P(x)$ given by $f(ij) \mapsto f_{ij}^{x_i-x_j}$. For each $\gamma \in \Gamma_4^3$, let $\mathcal{T}^\gamma_{P(x)}$ be the image of $\mathcal{T}^\gamma_{P^3}$ under this isomorphism.

We will now choose a random triangulation of every $C(x)$, $x \in \mathcal{X}^*$, as follows:

1. For each $1 \leq i < j \leq 4$ and $-N \leq r \leq N$, let $g_{ij}^r = g_{ji}^{-r}$ be an independent random element of $G_{14}^3$ which is 1 with probability $1/2$ and $\pi_{(ij)}$ with probability $1/2$.

2. For each $x \in \mathcal{X}^*$, triangulate $P(x)$ by $\mathcal{T}^\gamma_{P(x)}$, where

$$\gamma(x) := \left( \prod_{1 \leq i < j \leq 4} g_{ij}^{x_i-x_j} \right) (123).$$

3. Extend $\mathcal{T}^\gamma_{P(x)}$ uniquely to a triangulation $\mathcal{T}_{C(x)}$ of $C(x) = P(x) \cup D(x)$.

**Proposition 3.7.** For any two $x, x' \in \mathcal{X}^*$, the triangulations $\mathcal{T}_{C(x)}$ and $\mathcal{T}_{C(x')}$ agree on the common face of $C(x)$ and $C(x')$. 

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Proof. The only non-trivial case is when \( C(x) \cap C(x') \) contains a circuit \( X_{ijk}^{rst} \). We need to check that \( \mathcal{T}_{C(x)} \) and \( \mathcal{T}_{C(x')} \) agree on this circuit. If \( X_{ijk}^{rst} \subseteq C(x) \cap C(x') \), then

\[
\begin{align*}
    x_i - x_j &= x_i' - x_j' = r \\
    x_j - x_k &= x_j' - x_k' = s \\
    x_k - x_i &= x_k' - x_i' = t.
\end{align*}
\]

On the other hand, by Proposition 3.1(b), \( o_\gamma(x)(\{i, j, k\}) \) depends only on \( g_{ij}^{x_i-x_j} \), \( g_{jk}^{x_j-x_k} \), and \( g_{ki}^{x_k-x_i} \). It follows that \( o_\gamma(x)(\{i, j, k\}) = o_\gamma(x')(\{i, j, k\}) \). Thus \( \mathcal{T}_{P(x)}^s \) and \( \mathcal{T}_{P(x')}^s \) contain the same triangulation of \( X_{ijk}^{rst} \), as desired. \( \square \)

By Proposition 3.7 and Proposition 3.6(c), we can now extend the above triangulations of the \( C(x) \) to a full triangulation \( \mathcal{T} \) of \( P \).

Let \( \mathcal{C} \) be the collection of all triangulations \( \mathcal{T}_{ijk}^{rst} \subseteq \mathcal{T} \) with \( i, j, k \in [4] \) distinct, \(-N \leq r, s, t \leq N\), and \( r + s + t = 0 \). We prove that with high probability \( \mathcal{C} \) satisfies the hypotheses of Lemma 3.3.

**Proposition 3.8.** For large enough \( N \), with probability greater than 0, \( \mathcal{C} \) satisfies the hypotheses of Lemma 3.3.

**Proof.** Suppose that \( \mathcal{T}_{ijk}^{rst} \in \mathcal{C} \) (where \( i, j, k \in [4] \) distinct, \(-N \leq r, s, t \leq N\), and \( r + s + t = 0 \)) and let \( \{l\} = [4] \setminus \{i, j, k\} \). Let \( H \) be the set of all \( x \in \mathcal{T}^s \) such that \( x_i - x_j = r \), \( x_j - x_k = s \), and \( x_k - x_i = t \). Note that \( |H| \geq N \). If \( \gamma(x) = (ijk) \) for some \( x \in H \), then we have \( \mathcal{T}_{P(x)}^{(ijk)} \subseteq \mathcal{T} \), and hence

\[
\mathcal{T}_{ijl}^{(x_i-x_l)(x_i-x_l)}, \mathcal{T}_{jkl}^{(x_k-x_l)(x_l-x_j)}, \mathcal{T}_{kli}^{(x_l-x_i)(x_l-x_k)} \in \mathcal{C}
\]

as desired. We will bound the probability that this does not happen.

Since \( \mathcal{T}_{ijk}^{rst} \subseteq \mathcal{T} \), we have \( g_{ij}^r g_{jk}^s g_{ki}^t (123) \in \Gamma_4^3(ijk) \). Suppose we fix \( g_{ij}^r, g_{jk}^s, \) and \( g_{ki}^t \) such that \( g_{ij}^r g_{jk}^s g_{ki}^t (123) \in \Gamma_4^3(ijk) \). Then for each \( x \in H \), it follows from Proposition 3.1 and the definition of \( \gamma(x) \) that \( \gamma(x) = (ijk) \) with probability \( 1/4 \).
Moreover, since \((x_i - x_l, x_j - x_l, x_k - x_l)\) is different for each \(x \in H\), these probabilities are mutually independent for all \(x \in H\). Thus, the probability that there is no \(x \in H\) with \(\gamma(x) = (ijk)\) is

\[
\left(\frac{3}{4}\right)^{|H|} \leq \left(\frac{3}{4}\right)^N.
\]

Now, the probability that the above event occurs for some distinct \(i, j, k \in [4]\), \(-N \leq r, s, t \leq N\) with \(r + s + t = 0\) is, by the union bound, at most

\[
24(2N + 1)^2 \left(\frac{3}{4}\right)^N.
\]

Hence, the probability that \(\mathcal{C}\) does not satisfy the hypotheses of Lemma 3.3 is at most (3.3). For large enough \(N\) (e.g., \(N \geq 50\)), this is less than 1. \(\square\)

Thus there exists \(\mathcal{C}\) which satisfies the hypotheses of Lemma 3.3 and \(\mathcal{T} \in \mathcal{S}_{\mathcal{C}}\). There are triangulations of \(P\) which are not in \(\mathcal{S}_{\mathcal{C}}\); for example, a different choice of the \(g_{ij}\) would yield a triangulation which does not contain every element of \(\mathcal{C}\). This proves Theorem 3.2.
Chapter 4

Products of two simplices

The Cartesian product of two simplices is a polytope for which the theory of polyhedral subdivisions is particularly nice. The triangulations of this polytope have appeared in algebraic geometry, commutative algebra, optimization, and tropical geometry, and have been studied extensively for their own sake as well. See [15, Chapter 6.2] for an overview. In this chapter we describe two new results on the flip graph of this polytope, and give a brief overview of their proofs.

4.1 Subdivisions

Let \( \Delta^{m-1} := \{e_1, \ldots, e_m\} \) and \( \Delta^{n-1} := \{f_1, \ldots, f_n\} \) be the vertex sets of two simplices of dimension \( m - 1 \) and \( n - 1 \), respectively. Let

\[
A := \Delta^{m-1} \times \Delta^{n-1} = \{(e_i, f_j) : 1 \leq i \leq m, 1 \leq j \leq n\}
\]

be their Cartesian product. A key observation, made in [22], is that \( A \) is the Cayley embedding of a certain Minkowski sum, namely the sum of \( n \) copies of \( \Delta^{m-1} \) with itself. We denote this sum by \( n\Delta^{m-1} \). (Of course, \( A \) is also isomorphic to the Cayley embedding of \( m\Delta^{n-1} \).) It follows that subdivisions of \( A \) correspond to mixed subdivisions of \( n\Delta^{m-1} \).

For example, suppose \( m = 2 \), so that \( \Delta^1 \) is (the vertex set) of a line segment.
Then $\Delta^1 \times \Delta^{n-1}$ is the product of a simplex and an interval, and its triangulations correspond to mixed subdivisions of $n\Delta^1$, a line segment $n$ times the length of $\Delta^1$. The nonempty mixed cells of any mixed subdivision of $n\Delta^1$ are of the form $\Sigma \sigma_i$, where each $\sigma_i$ is either a point or a copy of $\Delta^1$. If the mixed cell is fine, then it has exactly one summand $\sigma_i$ which is a copy of $\Delta^1$, and it can be shown that the $i$ for which this occurs is different for any fine mixed cell in the subdivision. Hence, the fine mixed subdivisions of $n\Delta^1$ are the different ways to arrange $n$ labeled translates of $\Delta^1$ along $n\Delta^1$; i.e., they correspond to permutations of $[n]$. Moreover, flips correspond to the transposition of adjacent elements of the corresponding permutation. Using the classical definition of the permutohedron, it follows that the flip graph of $n\Delta^1$ (and hence of $\Delta^1 \times \Delta^{n-1}$) is isomorphic to the 1-skeleton of the $(n - 1)$-dimensional permutohedron. In fact, all subdivisions of $\Delta^1 \times \Delta^{n-1}$ are regular and the permutohedron is the secondary polytope of $\Delta^1 \times \Delta^{n-1}$.

We now consider $m = 3$. In a mixed subdivision of $n\Delta^2$, the nonempty mixed cells are of the form $\Sigma \sigma_i$ where each $\sigma_i$ is a subset of $\Delta^2$. If the mixed cell is fine, then either

(a) Two summand $\sigma_i, \sigma_j$ have size 2 and the others have size 1, and $\sigma_i \neq \sigma_j$.

(b) One summand has size 3 and the others have size 1.

In the first case, $\text{conv}(\Sigma \sigma_i)$ is a parallelogram, and in the second case, $\text{conv}(\Sigma \sigma_i)$ is a triangle. Thus, the mixed subdivisions of $n\Delta^2$ are the different ways to tile $\text{conv}(n\Delta^2)$ with these smaller parallelograms and triangles. (For a complete correspondence between mixed subdivisions of $n\Delta^2$ and such tilings, one should also label the unit triangles in the tiling from 1 to $n$.)

It should be noted that while all subdivisions of $\Delta^1 \times \Delta^{n-1}$ are regular, the subdivisions of $\Delta^{m-1} \times \Delta^{n-1}$ are not in general regular. The first such example was given by De Loera, who constructed a non-regular triangulation of $\Delta^3 \times \Delta^3$ [14]. A non-regular triangulation of $\Delta^2 \times \Delta^5$ was give by Sturmfels [27].
4.2 Flips

We noted previously that flips in triangulations of $\Delta^1 \times \Delta^{n-1}$ are given by transpositions of the corresponding permutation (which implies connectivity of the flip graph of $\Delta^1 \times \Delta^{n-1}$). This interpretation was obtained by considering how flips act upon the corresponding mixed subdivisions of $n\Delta^1$. In general, one can geometrically interpret flips on triangulations of $\Delta^{m-1} \times \Delta^{n-1}$ by considering how these flips act on the corresponding mixed subdivisions of $n\Delta^{m-1}$. In [22], Santos used this geometric interpretation to prove that the flip graph of $\Delta^2 \times \Delta^{n-1}$ is connected for all $n$.

Here, we present the following.

Theorem 4.1. The flip graph of $\Delta^3 \times \Delta^{n-1}$ is connected for all $n$.

Theorem 4.2. The flip graph of $\Delta^4 \times \Delta^{n-1}$ is disconnected for large $n$.

Flip graphs of products of two simplices, and more generally of lattice polytopes, have significance in algebraic geometry. Each lattice polytope defines a toric ideal and an associated toric Hilbert scheme. For a totally unimodular polytope (such as the product of two simplices), the associated toric Hilbert scheme is connected if and only if the flip graph of the polytope is connected; see [27, Chapter 10] or [16]. Thus, our result implies that the toric Hilbert scheme of $\Delta^4 \times \Delta^n$ is not connected for large $n$. While non-connected toric Hilbert schemes had previously been constructed [23], our proof demonstrates non-connectivity for the first time for a totally unimodular polytope. In addition, the toric ideal associated to $\Delta^m \times \Delta^n$ is the well-studied determinantal ideal generated by $2 \times 2$ minors of a $(m+1) \times (n+1)$ matrix; its zero-locus is the Segre variety.

We now give a brief description of the proofs of the previous two theorems.

4.2.1 Connectivity of $\Delta^3 \times \Delta^{n-1}$

Using the Cayley trick, triangulations of $\Delta^3 \times \Delta^{n-1}$ can be interpreted as fine mixed subdivisions of $n\Delta^3$, which (using similar reasoning as in the previous section), are given by tilings of $\text{conv}(n\Delta^3)$ with smaller parallelepipeds, triangular prisms, and
tetrahedra. One can then determine how flips act upon these mixed subdivisions and attempt to proceed similarly to Santos’s proof of flip-connectivity of $\Delta^2 \times \Delta^{n-1}$.

The issue that arises is that very complicated structures can appear in these mixed subdivisions despite being only three-dimensional. To demonstrate, we recall the construction from the previous chapter.

Previously, we showed that the flip graph of

$$P := \bigcup_{-N \leq r \leq N} \bigcup_{1 \leq i < j \leq 4} \{(e_i, f_{ij}^r), (e_j, f_{ij}^r)\}$$

is not connected for large enough $N$. For large enough $n$, $A := \Delta^3 \times \Delta^{n-1}$ contains $P$ as a subset. In fact, it can be shown that any subdivision of $P$ can be extended to a subdivision of $A$ ([13, Proposition 5.8]). Thus, there exist triangulations of $A$ which cannot be connected to each other through flips that involve only simplices in $P$.

Given this, it may be surprising that the flip graph of $\Delta^3 \times \Delta^{n-1}$ still ends up being connected. In particular, this means that Lemma 3.3 must fail when $P$ is replaced with $A$, as the construction from Section 3.4 still works in $A$. The failure occurs with Proposition 3.4: In $A$, there are flips supported on circuits of four elements which may result in $I''$ but $r$ $g$ $g'$.

It turns out that one can use these additional flips to prove flip-connectivity of $A$, but the process is very delicate. In [12], the author defines a certain quasiorder on the maximal simplices of any triangulation of $A$, and uses this quasiorder to find a specific circuit in $A$ which supports a flip of the triangulation. It is then shown that flipping on this circuit is monotonic in a sense, such that repeating this process will eventually lead to a specific triangulation, thus proving flip-connectivity. The details are very technical, and we leave the rest to the paper.

### 4.2.2 Non-connectivity of $\Delta^4 \times \Delta^{n-1}$

The delicateness of the proof of Theorem 4.1 suggested to the author that flip-connectivity was not fundamental to the product of two simplices, and that it was likely to fail in higher dimensions. This turns out to be the case even when $m = 4$. 

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The idea of the proof is to use the triangulation $\mathcal{T}$ of $P$ from Section 3.4, but take many copies of $\mathcal{T}$ in different affine subspaces. In $\Delta^3 \times \Delta^{n-1}$, we could embed a copy of a triangulation of $\mathcal{T}$, but there would be flips on circuits of size 4 in the larger triangulation which would "undo" $\mathcal{T}$. In $\Delta^4 \times \Delta^{n-1}$, we can arrange things such that these circuits of size 4 are in turn "blocked" by the other copies of $\mathcal{T}$. The result is that we obtain something similar to Lemma 3.3, but in which circuits of size 4 are also accounted for.

In addition to requiring many copies of $\mathcal{T}$, the proof requires that the size of $\mathcal{T}$ be increased (from the size in the previous Chapter) in order to ensure $\mathcal{T}$ satisfies certain additional properties. This results in the $n$ required in the proof to be around $4 \cdot 10^4$. 


Chapter 5

The extension space conjecture

Oriented matroids are abstractions of real point arrangements or real hyperplane arrangements. One of the cornerstones of the theory of oriented matroids is the topological representation theorem (Theorem 5.4), which states that all oriented matroids arise from arrangements of pseudospheres. This ties oriented matroid theory closely with topology, and the connections between the two are the subject of this chapter. Our goals will be to state the extension space conjecture, explain the connection to zonotopal tilings, and sketch the proof of the counterexample.

We have actually defined enough in the previous chapters to state the extension space conjecture now, in an equivalent form. Recall that for a zonotope $Z$, $\hat{\Omega}(Z)$ denotes the poset of zonotopal tilings of $Z$ minus the top element. The following is the generalized Baues conjecture for zonotopal tilings, which is equivalent to the extension space conjecture.

**Conjecture 5.1.** For any zonotope $Z$ with $n$ generators and of dimension $d$, the order complex of $\hat{\Omega}(Z)$ is homotopy equivalent to an $(n - d - 1)$-sphere.

On the other hand, it will be instructive to discuss the extension space conjecture from the oriented matroid perspective, as it clarifies both the geometric motivations behind the conjecture and the ideas of the counterexample.
5.1 Oriented matroids

We will give a brief overview of oriented matroids. We refer to Björner et al. [4] or Richter-Gebert and Ziegler [19] for a more comprehensive treatment.

5.1.1 Basic definitions

Throughout Section 5.1, let $E$ be a finite set. Let $\{+, -, 0\}$ be the set of signs, and let $\{+, -, 0\}^E$ be the set of sign vectors on $E$. For $\alpha \in \{+, -, 0\}$, define $-\alpha \in \{+, -, 0\}$ in the obvious way. For $X \in \{+, -, 0\}^E$, define $-X \in \{+, -, 0\}^E$ such that $(-X)(e) = -X(e)$ for all $e \in E$. Define a partial order on $\{+, -, 0\}$ by $0 < +$ and $0 < -$, and extend this to the product order on $\{+, -, 0\}^E$.

An oriented matroid is a pair $(E, L)$ where $L$ is a set of sign vectors on $E$ satisfying certain axioms. We will not use this axiomatic description in this thesis, but we include it for completeness:

**Definition 5.2.** An oriented matroid is a pair $M = (E, L)$ where $L \subseteq \{+, -, 0\}^E$ such that

1. $(0, \ldots, 0) \in L$

2. If $X \in L$, then $-X \in L$.

3. If $X, Y \in L$, then $X \circ Y \in L$, where

   $$(X \circ Y)(e) = \begin{cases} X(e) & \text{if } X(e) \neq 0 \\ Y(e) & \text{otherwise.} \end{cases}$$

4. If $X, Y \in L$ and $e \in E$ such that $\{X(e), Y(e)\} = \{+,-\}$, then there exists $Z \in L$ such that $Z(e) = 0$ and $Z(f) = (X \circ Y)(f)$ whenever $\{X(f), Y(f)\} \neq \{+,-\}$.

The set $L$ is called the set of covectors of $M$. A minimal element of $L \setminus \{0\}$ (with respect to the above product order) is called a cocircuit of $M$, and the set of cocircuits is denoted $C^*(M)$. An oriented matroid is determined by its set of cocircuits.
Given an oriented matroid \( M = (E, \mathcal{L}) \), an element \( e \in E \) is a loop of \( M \) if \( X(e) = 0 \) for all \( X \in \mathcal{L} \). An element \( e \in E \) is a coloop of \( M \) if there is some \( X \in \mathcal{L} \) with \( X(e) = + \) and \( X(f) = 0 \) for all \( f \in E \setminus \{e\} \). An independent set of \( M \) is a set \( \{e_1, \ldots, e_k\} \subseteq E \) such that there exist \( X_1, \ldots, X_k \in \mathcal{L} \) with

\[
X_i(e_j) = \begin{cases} + & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}
\]

All maximal independent sets of \( M \) have the same size, and this size is the rank of \( M \). The corank of \( M \) is \(|E| - \text{rank}(M)\). An oriented matroid of rank \( d \) is uniform if all \( d \)-element subsets of \( E \) are independent sets.

For \( X, Y \in \{+, -, 0\}^E \), we write \( X \perp Y \) if the set \( \{X(e) \cdot Y(e) : e \in E\} \) is either \( \{0\} \) or contains both \(+\) and \(-\). The next theorem defines duality of oriented matroids.

**Theorem 5.3.** For any oriented matroid \( M = (E, \mathcal{L}) \) of rank \( d \), the pair \( M^* = (E, \mathcal{L}^*) \) where

\[
\mathcal{L}^* = \{X \in \{+, -, 0\} : X \perp Y \text{ for all } Y \in \mathcal{L}\}
\]

is an oriented matroid of rank \(|E| - d\), called the dual of \( M \). We have \( M^{**} = M \).

Finally, if \( M = (E, \mathcal{L}) \) is an oriented matroid and \( A \subseteq E \), the pair \( M|_A = (A, \mathcal{L}_A) \) where

\[
\mathcal{L}_A := \{X|_A : X \in \mathcal{L}\}
\]

is an oriented matroid called the restriction of \( M \) to \( A \).

### 5.1.2 Topological representation

For each \( e \in E \), let \( v_e \in \mathbb{R}^d \) be a vector. For each point \( x \in \mathbb{R}^d \), we obtain a sign vector \( X \in \{+, -, 0\}^E \) by letting \( X(e) \) be the sign of the inner product \( \langle x, v_e \rangle \). The set of all such sign vectors is the set of covectors of an oriented matroid, which we call the oriented matroid of the vector configuration \( \{v_e\}_{e \in E} \). An oriented matroid is realizable if it is the oriented matroid of some vector configuration.
Now assume that all of the $v_e$ above are nonzero, and let $S_e$ be the intersection of the hyperplane normal to $v_e$ with the unit sphere $S^{d-1}$. The $S_e$ form a \textit{sphere arrangement} of $(d - 2)$-dimensional spheres in $S^{d-1}$, and each $S_e$ is oriented in the following way: $S_e$ separates $S^{d-1}$ into two hemispheres, exactly one of which has points with positive inner product with $v_e$. The \textit{topological representation theorem} says that all oriented matroids arise as topological deformations of such arrangements; we now describe this more precisely.

A \textit{pseudosphere} in $S^{d-1}$ is an image of $\{x \in S^{d-1} : x_d = 0\}$ under a homeomorphism $\phi : S^{d-1} \rightarrow S^{d-1}$. A pseudosphere $S$ separates $S^{d-1}$ into two regions called \textit{sides}; if we choose one side to be $S^+$ and the other to be $S^-$, then we say that $S$ is \textit{oriented}. A \textit{pseudosphere arrangement} is a collection $\mathcal{A} = \{S_e\}_{e \in E}$ of oriented pseudospheres in $S^{d-1}$ such that

1. For all $A \subseteq E$, the set $S_A := \bigcap_{e \in A} S_e$ is homeomorphic to a sphere or empty.

2. If $A \subseteq E$ and $e \in E$ such that $S_A \not\subseteq S_e$, then $S_A \cap S_e$ is a pseudosphere in $S_A$ with sides $S_A \cap S_e^+$ and $S_A \cap S_e^-$.

Let $\mathcal{A} = \{S_e\}_{e \in E}$ be a pseudosphere arrangement in $S^{d-1}$. For each $x \in S^{d-1}$, we obtain a sign vector $X \in \{+,-,0\}^E$ by setting

$$X(e) = \begin{cases} + & \text{if } x \in S_e^+ \\ - & \text{if } x \in S_e^- \\ 0 & \text{if } x \in S_e \end{cases}$$

Let $\mathcal{L}(\mathcal{A})$ be the set of all sign vectors obtained this way along with the 0 sign vector. Call $\mathcal{A}$ \textit{essential} if $\bigcap_{e \in E} S_e = \emptyset$. We can now state the topological representation theorem.

\textbf{Theorem 5.4} (Folkman-Lawrence [7]). For any essential pseudosphere arrangement $\mathcal{A}$ in $S^{d-1}$, $(E, \mathcal{L}(\mathcal{A}))$ is an oriented matroid of rank $d$. Conversely, every oriented matroid without loops is $(E, \mathcal{L}(\mathcal{A}))$ for some essential pseudosphere arrangement $\mathcal{A}$, and $\mathcal{A}$ is unique up to homeomorphisms $\phi : S^{d-1} \rightarrow S^{d-1}$. 

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For an oriented matroid \( M \), we call an essential pseudosphere arrangement \( A \) such that \( M = (E, \mathcal{L}(A)) \) a topological representation of \( M \). If \( M \) has rank \( d \) and \( A = \{S_e\}_{e \in E} \) is a topological representation of \( M \), we call any nonempty \( S_A \) (where \( A \subseteq E \)) for which \( \dim S_A > d - 1 - |A| \) a special pseudosphere of \( A \). \( M \) is uniform if and only if \( A \) has no special pseudospheres. The cocircuits of \( M \) are given by points of \( S_A \) where \( \dim S_A = 0 \).

### 5.1.3 Extensions, liftings, and weak maps

Let \( M = (E, \mathcal{L}) \) be an oriented matroid. Let \( M' = (E', \mathcal{L}') \) be another oriented matroid such that \( E' = E \cup \{f\} \) for some \( f \notin E \). We say that \( M' \) is a one-element extension, or extension, of \( M \) if \( M = M'|_E \); that is,

\[
\mathcal{L} = \{X|_E : X \in \mathcal{L}'\}.
\]

We say that \( M' \) is a one-element lifting, or lifting, of \( M \) if

\[
\mathcal{L} = \{X|_E : X \in \mathcal{L}', X(f) = 0\}.
\]

If \( M' \) is an extension (or lifting) of \( M \), we call it trivial if \( f \) is a coloop (resp., a loop) of \( M' \). The notions of extension and lifting are dual to each other: \( M' \) is a (non-trivial) extension of \( M \) if and only if \( (M')^* \) is a (non-trivial) lifting of \( M^* \). Finally, if \( M' \) is a non-trivial extension of \( M \), then \( \operatorname{rank}(M') = \operatorname{rank}(M) \), and if \( M' \) is a non-trivial lifting of \( M \), then \( \operatorname{rank}(M') = \operatorname{rank}(M) + 1 \).

We can understand liftings better using topological representation. Suppose \( M' \) is a lifting of \( M \), and assume \( \operatorname{rank}(M) = d \) and \( M \) and \( M' \) have no loops. Let \( A = \{S_e\}_{e \in E'} \) be a topological representation of \( M' \) in \( S^d \); by applying an appropriate homeomorphism \( \phi : S^d \to S^d \), we may assume \( S_f = \{x \in S^d : x_{d+1} = 0\} \) and \( S^+_f = \{x \in S^d : x_{d+1} > 0\} \). Let \( A^+ = \{S_e \cap S^+_f\}_{e \in E}. \) Consider the "gnomonic projection" which maps \( S^+_f \) to \( \mathbb{R}^d \). The image of \( A^+ \) under this map is a (not necessarily central)

\[\text{Note that } M' \text{ is determined by } A^+. \text{ In addition, we have } S_e \neq S_f \text{ for all } e \neq f, \text{ because otherwise, by the definition of a lifting, } e \text{ would be a loop of } M.\]
arrangement $B$ of oriented pseudohyperplanes in $\mathbb{R}^d$ such that the intersection of $B$ with the "sphere at infinity" (that is, $S_f$) is a pseudosphere arrangement representing $M$. Conversely, given such a pseudohyperplane arrangement $B$ (with the appropriate definition of "pseudohyperplane arrangement"), we can uniquely construct a lifting $M'$ of $M$ such that the set of covectors of $M'$ which are positive on $f$ is topologically represented by $B$. Hence, liftings of $M$ are given by pseudohyperplane arrangements in $\mathbb{R}^d$ whose intersection with the sphere at infinity are topological representations of $M$.

Given two oriented matroids $M_1 = (E, L_1)$ and $M_2 = (E, L_2)$ on the same ground set $E$, we say that there is a weak map $M_1 \rightsquigarrow M_2$ if for every $X_2 \in L_2$, there exists $X_1 \in L_1$ such that $X_1 \geq X_2$. We say that this weak map is rank-preserving if $M_1$ and $M_2$ have the same rank. If $M_1 \rightsquigarrow M_2$ is a rank-preserving weak map, then $M_1^* \rightsquigarrow M_2^*$ is also a (rank-preserving) weak map [4, Cor. 7.7.7].

For any set $S$ of oriented matroids on the same ground set, we obtain a partial order on $S$ by letting $M_1 \geq M_2$ if there is a weak map $M_1 \rightsquigarrow M_2$. We call the set of all non-trivial extensions of an oriented matroid $M$ partially ordered this way the extension poset $E(M)$ of $M$. Similarly, we call the poset of all non-trivial liftings of $M$ the lifting poset $F(M)$ of $M$. Since all non-trivial extensions of an oriented matroid $M$ have the same rank, we have $E(M) \cong F(M^*)$. The extension poset (or lifting poset) has a unique minimal element $\hat{0}$, corresponding to extension by a loop (resp., lifting by a coloop).

We can finally state the extension space conjecture.

**Conjecture 5.5.** If $M$ is a realizable oriented matroid, then $E(M) \setminus \hat{0}$ is homotopy equivalent to a sphere of dimension $\text{rank}(M) - 1$.

Since an oriented matroid is realizable if and only if its dual is, this is equivalent to the following.

**Conjecture 5.6.** If $M$ is a realizable oriented matroid, then $F(M) \setminus \hat{0}$ is homotopy equivalent to a sphere of dimension $\text{corank}(M) - 1$. 

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5.1.4 Zonotopal tilings

Recall that a zonotope \( Z \) is a Minkowski sum of a collection \( V = \{v_i\} \) of vectors. Let \( M(V) \) denote the oriented matroid of the configuration \( V \), as defined in Section 5.1.2. The *Bohne-Dress theorem* gives an elegant connection between zonotopal tilings of \( Z \) and liftings of \( M(V) \).

**Theorem 5.7** ([5], see also [18]). Let \( V \) be a vector configuration, and let \( Z = Z(V) \) and \( M = M(V) \) be the associated zonotope and oriented matroid, respectively. Then the zonotopal tilings of \( Z \) are in bijection with the liftings of \( M \), and this bijection is an order-reversing poset map between \( \Omega(Z) \) and \( \mathcal{F}(M) \).

From this we immediately obtain the equivalence of Conjecture 5.1 and Conjecture 5.6.

5.2 A counterexample

It suffices to find some zonotope \( Z = Z(V) \) such that \( \hat{\Omega}(Z) \) is disconnected and \( |V| - \dim(Z) > 1 \). To do this, we will use the following lemma. It is a zonotopal version of [17, Lem. 3.1] and [20, Cor. 4.3].

**Proposition 5.8.** Let \( Z = Z(V) \) be a zonotope where \( |V| - \dim(Z) > 1 \). If \( V \) is in general position and the flip graph of \( Z \) is disconnected, then there is a subconfiguration \( W \subseteq V \) such that \( |W| - \dim(Z(W)) > 1 \) and \( \hat{\Omega}(Z(W)) \) is disconnected.

**Proof.** For any poset \( P \), a *lower ideal* of \( P \) is a subposet \( I \subseteq P \) such that \( x \in I \) and \( y < x \) implies \( y \in I \). For any \( x \in P \), define the lower ideals \( I_{\leq x} = \{y \in P : y \leq x\} \) and \( I_{< x} = \{y \in P : y < x\} \). The following is an easy exercise.

**Proposition 5.9.** Let \( P \) be a finite connected poset, and let \( G \) be a lower ideal of \( P \) containing all the minimal elements of \( P \). Suppose that \( I_{< x} \) is connected for any \( x \in P \setminus G \). Then \( G \) is connected.

Now, let \( G \) be the lower ideal of \( \Omega(Z) \) consisting of all fine zonotopal tilings of \( Z \) and the flips between them. If \( G \) is disconnected, then by Proposition 5.8 there is
some $\mathcal{J} \in \Omega(Z) \setminus G$ such that $I_{<\mathcal{J}}$ is disconnected. Let $C_1, \ldots, C_k$ be the non-fine maximal mixed cells of $\mathcal{J}$. Since $V$ is in general position, the faces on the boundary of $C_1, \ldots, C_k$ are fine and cannot be tiled further. Hence, $I_{<\mathcal{J}}$ is isomorphic to

$$\Omega(C_1) \times \Omega(C_2) \times \cdots \times \Omega(C_k).$$

It follows that $I_{<\mathcal{J}}$ is disconnected only if $k = 1$ and $\Omega(C_1)$ is disconnected. Note that $C_1$ is a translation of $Z(W)$ for some $W \subseteq V$. Finally, since $\mathcal{J} \notin G$, we must have $|W| - \dim(Z(W)) > 1$. This completes the proof. \hfill \square

Thus, it suffices to find some zonotope $Z = Z(V)$ such that $V$ is in general position, $|V| - \dim(Z) > 1$, and the flip graph of $Z$ is disconnected.

We construct this $Z$ as follows. Let $V_N$ be the vector configuration consisting of each vector of $\{e_i - e_j : 1 \leq i < j \leq 4\}$ repeated $2N + 1$ times, where $e_i$ is the $i$-th standard basis vector. Let $\tilde{V}_N$ be the vector configuration obtained by perturbing each vector of $V_N$ by a small random displacement in the span of $V_N$. The main result is the following.

**Theorem 5.10.** For large enough $N$, with probability greater than 0, the flip graph of $Z(\tilde{V}_N)$ is disconnected.

The rest of the thesis will sketch the proof of this result.

### 5.2.1 The unperturbed case

Before describing the proof of Theorem 5.10, let us revisit the following easier statement, whose proof was the subject of Chapter 3.

**Theorem 5.11.** For large enough $N$, the flip graph of $Z(V_N)$ is disconnected.

We will recast the proof in terms of hyperplane arrangements. For $1 \leq i, j \leq 4$ and $-N \leq r \leq N$, let

$$H_{ij}^r = \{x \in \mathbb{R}^4 : x_i - x_j = r\} \subset \mathbb{R}^4$$
and consider the arrangement $\mathcal{A}_0 := \{H_{ij}\}_{1 \leq i < j \leq -N, -N \leq r \leq N}$. The arrangement $\mathcal{A}_0$ corresponds to a lifting of the oriented matroid $M(V_N)$, and by the Bohne-Dress theorem this gives a zonotopal tiling of $Z(V_N)$. This zonotopal tiling is precisely $\mathcal{F}_P$ from the beginning of Section 3.4.

Now consider the random fine refinement $\mathcal{T}$ of $\mathcal{F}_P$ that we constructed in Section 3.3. Let $A$ be the pseudohyperplane arrangement corresponding to this refinement. We now show how the existence of $A$ implies Theorem 5.11. We first set up some notation. Let $Z$ be the set of all points which are the intersection of six hyperplanes of $\mathcal{A}_0$. Now, notice that we can canonically biject the hyperplanes of $\mathcal{A}_0$ with the pseudo-hyperplanes of $A$. For $-N \leq r, s, t \leq N$, let $L_{ijk}^{rst} \subseteq A$ be the image of

$$\{H_{ij}, H_{jk}, H_{ki}\}$$

under this bijection, and for $x \in Z$, let $X^x \subseteq A$ be the image of

$$\{H \in \mathcal{A}_0 : x \in H\}$$

under this bijection. By the Bohne-Dress correspondence, the orientation of $L_{ijk}^{rst}$ corresponds to the triangulation of the circuit $X_{ijk}^{rst}$, and the arrangement of $X^x \subseteq A$ corresponds to the tiling of the cell $C(x)$. In addition, for $r + s + t = 0$, let

$$L_{ijk}^{rst} = \{x \in Z : x_i - x_j = r, x_j - x_k = s, x_k - x_i = t\}.$$ 

Now, let $r + s + t = 0$, and suppose we want to apply a flip to $A$ so that the orientation of $L_{ijk}^{rst}$ is reversed (or equivalently, a flip supported on $X_{ijk}^{rst}$ in the corresponding zonotopal tiling). For each $x \in L_{ijk}^{rst}$, we consider the arrangement $X^x$; this arrangement involves $L_{ijk}^{rst}$ along with three other pseudo-hyperplanes. If these three other pseudo-hyperplanes intersect in the interior region of $L_{ijk}^{rst}$, then the orientation of $L_{ijk}^{rst}$ cannot be immediately flipped. We say in this case that $x$ "blocks" the flip on
\( L_{ij} \). By the proof of Proposition 3.5, \( x \) blocks the flip on \( L_{ijk} \) when the arrangements

\[
L_{ij}^{r(x_j-x_i)(x_i-x_i)}, L_{jkl}^{s(x_k-x_i)(x_i-x_j)}, L_{kil}^{t(x_l-x_k)(x_i-x_l)}
\]

are oriented a certain way, where \( \{l\} = [4] \setminus \{i, j, k\} \). Thus, we would need to flip the orientation of one of these arrangements first before flipping \( L_{ijk} \). However, we could then apply the same argument to each of \( L_{ij}^{r(x_j-x_i)(x_i-x_i)}, L_{jkl}^{s(x_k-x_i)(x_i-x_j)}, L_{kil}^{t(x_l-x_k)(x_i-x_l)} \). So what we want is that for each \( L_{ijk}^{rs} \), there is some \( x \in L_{ijk} \) which would block a flip from reversing its orientation. For large enough \( N \), Proposition 3.8 shows this will happen with positive probability. This shows that none of these orientations can be reversed through flips, implying disconnectivity of the flip graph.

### 5.2.2 The perturbed case

Since Proposition 5.8 requires a vector configuration in general position to work, Theorem 5.11 cannot immediately be used to disprove the extension space conjecture. A natural approach is to perturb the vectors of \( V_N \) so that they are in general position. This works, but the proof is significantly more complicated than the unperturbed case and requires a much larger value of \( N \). We will sketch the ideas involved but we will be extremely informal; the full proof can be found in [11].

What changes in the above argument when we perturb the vectors? Consider again the random arrangement \( A \), and now "tilt" each pseudo-hyperplane by a small random vector to create an arrangement \( A' \). The tilt should be small enough so that the internal structure of \( A \) remains intact inside \( A' \) (in other words, the oriented matroid of \( A \) should be a weak map image of \( A' \)). By abuse of notation, we will now use \( L_{ijk}^{rs} \) and \( A'x \) to refer to the corresponding subarrangements inside \( A' \) instead of \( A \). A major difference in \( A' \) compared to \( A \) is that the pseudohyperplanes of \( L_{ijk}^{rs} \) now intersect. This means that we can no longer talk about the orientation of \( L_{ijk}^{rs} \) itself; rather, we need to talk about its orientation with respect to each of the other pseudohyperplanes. For each \( x \in L_{ijk}^{rs} \), we will say the "orientation of \( L_{ijk}^{rs} \) with respect to \( x \)" to refer to the orientation of \( L_{ijk}^{rs} \) with respect to each of the other three
pseudohyperplanes of $\mathcal{X}^x$.

Let $r + s + t = 0$, and let $x_{ijk}^{rst}$ be the intersection point of $L_{ijk}^{rst}$. Depending on how we tilted the pseudohyperplanes, this point will either be far in the $x_l$ direction or far in the $-x_l$ direction. First, suppose it is far in the $x_l$ direction. We can subdivide $L_{ijk}^{rst}$ into two equal intervals—one in the $x_l$ direction and one in the $-x_l$ direction. We will say that the points in the former interval are “close” to $x_{ijk}^{rst}$, whereas the points in the other interval are “far” from $x_{ijk}^{rst}$. We make analogous definitions if $x_{ijk}^{rst}$ is in the $-x_l$ direction.

Now, for $x \in L_{ijk}^{rst}$, consider the orientation of $L_{ijk}^{rst}$ with respect to $x$. Suppose we want to apply a flip $f$ to $A'$ to change this orientation. If $x$ previously blocked the flip $f_0$ in $A$ which reverses the orientation of $L_{ijk}^{rst}$, then it will also block $f$, in the sense that we must change the orientation of one of $L_{ijl}^{r(x_l-x_i)(x_l-x_i)}$, $L_{jkl}^{s(x_k-x_i)(x_l-x_j)}$, or $L_{kil}^{t(x_l-x_k)(x_l-x_i)}$ with respect to $x$ first. In fact, if there is any $y \in L_{ijk}^{rst}$ between $x$ and $x_{ijk}^{rst}$ which previously blocked the flip $f_0$, then $y$ will block $f$. What this means is that if $x$ is far from $x_{ijk}^{rst}$, then it is likely that there will be some $y$ which will block $f$. But if $x$ is close to $x_{ijk}^{rst}$, then we cannot use probability to guarantee this. In the extreme case, if $x$ is the endpoint of $L_{ijk}^{rst}$ closest to $x_{ijk}^{rst}$, then the only way this flip can be blocked is if $x$ blocks it itself, which is not likely to happen for every endpoint.

The idea of the proof is to simply to give up being able to say anything about the $x$ which are close to $x_{ijk}^{rst}$ and which do not themselves block $f$. This means that we give up not only the orientations of $L_{ijk}^{rst}$ with respect to these $x$, but also we cannot use these $x$ in our argument to block any flips. This in turn may force us to give up on some other $x$, and so on. After going through all of this, we give up a lot, but we end up having enough left over to be able to make the previous argument (almost) work. Specifically, we are left with the following: the set of points $x \in L_{ijk}^{rst}$ which block flips from reversing the orientation of $L_{ijk}^{rst}$ with respect to $x$, and which are far from $x_{ijl}^{r(x_j-x_i)(x_l-x_i)}$, $x_{jkl}^{s(x_k-x_i)(x_l-x_j)}$, and $x_{kil}^{t(x_l-x_k)(x_l-x_i)}$. If the method we used to

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2 Since there are actually three different orientations we are looking at (one for each element of $\mathcal{X}^x \setminus L_{ijk}^{rst}$), to “change” the orientation of $L_{ijk}^{rst}$ with respect to $x$ means to change one of these three orientations.

3 When we say “between”, we are viewing $L_{ijk}^{rst}$ as a line and $x_{ijk}^{rst}$ as one of the points at infinity.
perturb the pseudohyperplanes was random, this set will be dense enough that we can nearly carry out the same argument as before.

There is a hole in the above logic, however—we are assuming some structure on the lines $L_{ij}^{rst}$. In particular we are saying that some points are “between” others. This is not something we can assume, as there may be sequences of flips which disrupt this order. The way to address this is to not only consider subarrangements $L_{ij}^{rst}$ with $r + s + t = 0$, but also $L_{ij}^{rst}$ for arbitrary values of $r + s + t$. We can make similar “blocking” arguments for these arrangements, and this allows us to maintain some of the original grid structure of $A_0$. We won’t be able to keep the structure completely intact (in fact very little of it will remain), but with the correct probabilistic argument we can show that enough order will remain that we can still proceed as before.

This last argument is quite complicated and ends up being the main technical hurdle in the proof. To help with this argument, it is convenient to choose a specific way of randomly perturbing the vectors of $V_N$: Instead of choosing the perturbation directions uniformly, the proof concentrates them highly around 6 certain directions. This gives more knowledge of the locations of the $x_{ij}^{rst}$ and essentially allows the probability arguments to be “discretized”. (Of course, we could also use a uniform distribution and just pick out the appropriate vectors once we have enough of them.)

In the end, the value of $N$ required for this argument to work is around $10^5$. Note that this is not necessarily the size of the extension space conjecture counterexample, because in Proposition 5.8 we need to take some unknown subset.
Bibliography


