ON THE FORMULA OF de JONQUIÈRES FOR MULTIPLE CONTACTS

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ABSTRACT

We prove a formula for the homology class of the scheme parametrizing the members of a family of divisors possessing points with specified multiplicities. The formula includes as special cases, (1) the formula of de Jonquières for multiple contacts of curves of given degree with a fixed plane curve; (2) the formulas for multiple contacts of lines with hypersurfaces; (3) formulas for tangent planes to a surface in projective 3-space. Our method also yields a formula for the curves of a family in a family of surfaces displaying an m-fold point with assigned coincidences of tangents. This generalizes the classical formula for the cuspidal members of a net on a surface, and that for the cusp-nodes of a web.

We also study the questions of finiteness and of multiplicity one for the solutions of the proposed contact problems.

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TABLE OF CONTENTS

Introduction.

The aim of this thesis is to obtain formulas for the number of divisors of a family which possess points with specified multiplicities. For instance, if the family is that of the hyperplane sections of some embedding of a variety Y in a projective space, we wish to count those hyperplanes which satisfy specified contact conditions with Y.

About a century ago, Jean Phillipe Ernest de Fauque de Jonquières published his Memoire [de Jonquières], exhibiting a formula for the number of plane curves of given degree having prescribed contacts with a fixed plane curve. The formula per se has generated a lot of interest (cf. (5.1.8) below). Shortly afterwards, the flourishing school of enumerative geometry produced a wealth of formulas for contacts of lines and planes with surfaces in 3-space. Cayley, Clebsch and Salmon obtained the degree of the curves traced on a surface by the points of contacts of lines satisfying 3 conditions (e.g. triple tangent lines, or lines inflexional at one point and simply tangent at some other) [Salmon, pp. 277 and ff]. Schubert found the number of tangent lines which satisfy 4 conditions (e.g., five-point tangents, ordinary fourfold tangents, etc...) [Schubert, Math. Ann., 1876, X, p. 102; 1877, XI pp 348-78, or Kalkül der abzählenden Geometrie (1879), pp. 236-7, 246]. Zeuthen [Math. Ann., 1876, X, p. 446] obtained several formulas relating the singularities of a surface and its plane sections to their dual counterparts.

The next generation of algebraic geometers changed somewhat the emphasis, from the counting of singularities per se, to the discovery of numerical invariants definable in terms of those singularities. Thus, one finds in C. Segre [Annali di Matematica 1894, XXII, p. 75] the definition of the genus of a curve in terms of the invariant $\nu-2n$, where ν is the number of double points of a g_n^1 on the curve. The Zeuthen-Segre invariant (cf. [Enriques, Le Superficie Algebriche, Bologna 1949, p. 167] and the very definition of the canonical system [loc. cit. p. 49] are further examples of the "new" trend.

Possessing now as we do, a well-developed intersection theory, it is natural to try and go back to the origins, and vindicate (to today's taste and sense of rigor) those classical formulas. In fact, Hilbert's 15th problem calls for "...the actual carrying out of the process of elimination in the case of equations of special form in such a way that the degree of the final equations and the multiplicity of their solutions may be foreseen" [Hilbert]. However, the real test still is whether you can "beat them on their own ground", and provide answers to questions the enumerative geometers might have asked themselves. We hope this work will provide a step in this direction.

Our main result is the description of a homology class, in a rather general setting (see 8.3.4). We are given a family of divisors $\{D_s\}_{s\varepsilon S}$ in a family $\{Y_s\}_{s\varepsilon S}$ of smooth ambient spaces. Our homology class expresses in terms of $\underline{m}$ and basic invariants of the families, the class of the set of points $(s,y_1,\ldots,y_t)$ such that y_i is an m_i-fold point of D_s. We retrieve, as special cases, (i) the formula of de Jonquières (5.1) (here the family of divisors is a linear system on a projective curve Y and the family of ambient spaces is the trivial family $Y_s = Y$); (ii) the formulas for contacts of lines with a hypersurface (7.1) (the family of ambient spaces is a family of lines, and the family of divisors is that cut out by the hypersurface on each line); (iii) the formulas for contacts of planes with smooth surfaces in 3-space (8.5.4) (the ambient spaces can be chosen as the planes or as the fixed surface, and the family of divisors as the plane sections. The answers are the same by a general result (8.2.8)). In fact, we can get formulas for contacts of a smooth hypersurface with linear spaces of arbitrary dimensions, (8.6) as well as for contacts of hyperplanes with a smooth variety of arbitrary dimension. A little manipulation with that general homology class enables us to get also a formula for the number of curves of a family on a surface which display an m-fold point with specified tangent coincidences (9.5). This generalizes the classical formula for the cuspidal members of a net.

To give substance to these formulas one must check, as pointed out by S. L. Kleiman in his address [Kleiman 15], that, for general values of the parameters, (a) there are indeed only finitely many solutions and, (b) each of these appears with multiplicity one.

With regard to the first point, we have verified its validity for the case that the family of ambient spaces has relative dimension 1, e.g. for a linear system on a fixed curve, contact of lines with a hypersurface and also the case of curves with m-fold point and specified tangent coincidences. However, for the case of relative dimensions ≥ 2, we could handle the question only under the additional assumption that the sequence $\underline{m}$ of contacts satisfy a relaxed version of the classical proximity inequalities. These, we recall, are

$$m_i \geq m_{i+1} + \cdots + m_t ,$$

and they constitute the n.s.c. for the existence of plane curves of sufficiently high degree displaying a group of points of multiplicities $m_{i+1}, \ldots, m_t$ and infinitely near to an m_i-fold point. But we are convinced that this additional assumption will eventually be proven superfluous. Further, the same method should produce formulas for the homology class for singularities with specified Dynkin diagram.

As to the multiplicity one question, we give a somewhat detailed answer in the specific case envisaged by the formula of de Jonquières. Here, the multiplicity is one iff the characteristic of the ground field does not divide any of the m_i. In char. 0, as expected, and always implicitly believed in the classical literature, the multiplicity is also one for each of the contact formulas mentioned before. In fact, this is a consequence of Sard's theorem, or rather its algebraic version, which comes down to the fact that all field extensions are separable in char. 0.

The contents are as follows.

In section 1 we recall the definition of the scheme of zeroes of a map of sheaves . We then introduce the incidence correspondence of a pair of families of subschemes.

In section 2 we describe the basic set up to treat the case of relative dimension 1 or just 1 assigned multiplicity. The $\underline{m}$-Jacobian scheme $J(\underline{m};D)$ is defined and shown to be the scheme of zeroes of a section of a certain locally free sheaf $\mathcal{E}(\underline{m};L)$.

In section 3 we compute the class of $\mathcal{E}(\underline{m};L)$ in $K^{\bullet}X[t]$ and establish a recursive relation which is instrumental in deriving the formula of de Jonquières.

Section 4 is devoted to linear systems. In this case, we prove $J(\underline{m};D)$ is a certain projective bundle. This enables us to state a regularity criterion. This criterion, when applied to a smooth projective curve of genus g, implies that the formula of de Jonquières holds for all sufficiently general linear subsystems of a complete system of degree $> 2g-2 + \Sigma m_i$.

Section 5 contains our proof of the classical formula of de Jonquières.

In section 6 we consider the question of whether, for a given smooth, projective curve C, each of the solutions counted by the formula of de Jonquières appears with multiplicity 1. The answer is affirmative in general, for char. 0, and depends on whether p divides some of the m_i's in char. $p > 0$.

In section 7, we explain how the general construction of $J(\underline{m};D)$ can be used to solve problems of contacts of lines with hypersurfaces.

In section 8 we discuss the situation for relative dimensions >1 and arbitrary number of assigned multiplicities. We explain our failure with a first, direct approach, and then go on to introduce a remedy of sorts, the step by step construction of the scheme $J(\underline{m};D)$. The generic homology class of $J(\underline{m};D)$ is computed. Next, we show that $J(\underline{m};D)$ satisfies the necessary regularity assumptions whenever D moves in a sufficiently ample linear system <u>and</u> $\underline{m}$ satisfies the relaxed proximity inequalities. In the ensuing examples, we retrieve the Zeuthen-Segre invariant, the formula for the number of bitangent planes of a general surface in P^3 which go through a general point, and the number of tritangent planes.

In the last section, we study the curves with an m-fold point with specified tangent coincidences.

§1

<u>All schemes are of finite type over an algebraically closed field</u> k.

(1.1). <u>Scheme of zeros</u>. Let $f : X \to S$ be a map of schemes. Let $u : A \to B$ denote a map of $\mathcal{O}_X$-modules. For each map $t : T \to S$, let $u(t) : A(t) \to B(t)$ denote the pullback of u to $X_T = T \underset{S}{\times} X$.

(1.1.1). <u>Definition</u>: (cf. [F.S.], (2.2), p. 20). A closed subscheme of S is called the <u>scheme of zeros</u> of u if it has the universal property that $t : T \to S$ factors through it if and only if $u(t)$ is zero. If it exists, the scheme of zeros is denoted $Z_S(u)$. If $X = S$ and $f = \mathrm{id}$, we set $Z_S(u) = Z(u)$.

The result below is a central tool. It tells us how to get the equations of $Z_S(u)$ in the parameter space.

(1.1.2). <u>Proposition</u>: Suppose A is of the form f^*C for some quasi-coherent $\mathcal{O}_S$-module C. Assume f_*B is locally free and its formation commutes with base change. Then $Z_S(u)$ exists and is equal to $Z(u')$, where $u' : C \to f_*B$ is the adjoint of u.

<u>Proof</u>. See ([F.S.], (2.3), p. 21).

(1.1.3). Proposition. Let s be a section of a locally free $\mathcal{O}_S$-module A. Then the scheme of zeros of s is defined by the Ideal $\check{s}(A^\vee)$, that is, the image of the dual map $\check{s} : \check{A} \to \mathcal{O}$.

Proof: The proof is easy and will be omitted. (cf. [EGA I], proof of (9.7.9.1).

The next proposition shows that a projective subbundle of a projective bundle is naturally a scheme of zeros.

(1.1.4). Proposition. Let S be a scheme and let

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$$

be an exact sequence of quasi-coherent $\mathcal{O}_S$-modules. Set

$$X = P(B), \qquad X' = P(C).$$

Let $u : A_X \to \mathcal{O}_X(1)$ denote the composition of α_X with the universal 1-quotient $\gamma : B_X \to \mathcal{O}_X(1)$. Then, we have

(i) $Z_X(u) = X'$.

(ii) If A and C are locally free and α is injective, then the section $u^\vee \otimes \mathcal{O}_X(1)$ of $\check{A}_X$ is regular and its scheme of zeros is equal to X'.

Proof: (i) Let $t : T \to X$ be a T-valued point of X. We have the equivalence:

$$t^*u = 0 \quad \text{iff} \quad (t^*\alpha)(A_T) \subset \ker(t^*\gamma)$$
$$\text{iff} \quad t^*\gamma \text{ factors through } t^*\beta$$
$$\text{iff} \quad t \text{ factors through } X'.$$

This proves (i). For the proof of (ii), see ([F.S.], (2.6), p. 22).

(1.2). Incidence correspondences. Consider the diagram of maps of schemes,

$$\begin{array}{ccccccc} W \subset Y_{S'} & \longrightarrow & Y & \longleftarrow & Y_S & \supset & D \\ \downarrow & & \downarrow & & \downarrow & & \\ S' & \longrightarrow & Z & \longleftarrow & S & & . \end{array}$$

Suppose the squares are cartesian. For each $t : T \to S' \underset{Z}{\times} S$, denote by $D(t)$, $W(t)$ the pullbacks of D and W to $Y(t) = Y \underset{Z}{\times} T$.

(1.2.1). Definition. A closed subscheme of $S' \underset{Z}{\times} S$ is called the incidence correspondence of W in D if it has the universal property that $t : T \to S' \underset{Z}{\times} S$ factors through it if and only if $W(t) \subseteq D(t)$ holds (as subschemes of $Y(t)$).

(1.2.2). <u>Proposition</u>. Let $f : X \to S$ be a map of schemes, let $W \subset X$ be a closed subscheme, and let $D \subset X$ be the scheme of zeros of a section s of an invertible $\mathcal{O}_X$-Module L. Suppose $f_*(L \otimes \mathcal{O}_W)$ is locally free and that its formation commutes with base change (e.g. if W is flat and proper $/S$ and $R^1f_*(L \otimes \mathcal{O}_W) = 0$ holds). Then the incidence correspondence of W in D exists and is equal to the scheme of zeros of a section of the locally free $\mathcal{O}_S$-Module $f_*(\mathcal{O}_W \otimes L)$.

<u>Proof</u>: There is a natural diagram of maps of $\mathcal{O}_X$-Modules,

$$\begin{array}{ccccccc} & & & \mathcal{O}_X & & & \\ & & s & \downarrow & \searrow u & & \\ 0 \longrightarrow & L \otimes I(W) & \longrightarrow & L & \longrightarrow & L \otimes \mathcal{O}_W & \longrightarrow 0. \end{array}$$

Now it is clear that, for each $t : T \to S$,

$$D(t) \supseteq W(t)$$

holds if and only if $u(t) = 0$. Since $\mathcal{O}_X$ is just $f^*\mathcal{O}_S$, we may apply the proposition (1.1.2).

The next lemma is only needed for the proof of (8.2.8). Let S be a scheme and M a coherent $\mathcal{O}_S$-Module. Let

r be a nonnegative integer. We recall the definition of the r^{th} Fitting scheme $F_r(M)$ of M. Given a local presentation of M over an open subset $U \subset S$,

$$K \xrightarrow{u} L \twoheadrightarrow M_{|U} ,$$

where L is a locally free $\mathcal{O}_U$-Module of rank s, we have that $F_r(M) \cap U$ is equal to the scheme of zeros of the exterior power $\overset{s-r}{\wedge} u$. (It is well known that this scheme of zeros is independent of the presentation, (cf. [R] p. 232, [Ka] p. 145).)

(1.2.3). <u>Lemma</u>. Let $f : \mathcal{X} \to \mathcal{S}$ be smooth. Let $W \subset \mathcal{D} \subset \mathcal{X}$ be closed subschemes. Assume $\mathcal{D}$ is a Cartier divisor and W is transversally regularly embedded in $\mathcal{X}$ relatively to $\mathcal{S}$ ([EGA IV_4], 19.2.2). Suppose $f|W$ is an isomorphism of W onto $\mathcal{S}$. Let J and I denote the Ideals of W in $\mathcal{X}$ and in $\mathcal{D}$. Let mW denote the scheme with Ideal J^m. Then, for each $m \geq 2$, the incidence correspondence $\mathcal{J}_m \subset \mathcal{S}$ of mW in $\mathcal{D}$ is equal to the $r_m{}^{th}$ Fitting scheme of the image of $(I^{m-1}/I^m)|_{\mathcal{J}_{m-1}}$ in $(\mathcal{O}_{\mathcal{D}}/I^m)|_{\mathcal{J}_{m-1}}$, where $r_m = \text{rank}(J^{m-1}/J^m) - 1$ and we view I^{m-1}/I^m (which is naturally a $\mathcal{O}_W$-Module) as a $\mathcal{O}_{\mathcal{S}}$-Module via the given isomorphism $f|W$.

<u>Proof</u>: The proof is divided in several steps. Let H

denote the (invertible!) Ideal of $\mathcal{D}$ in $\mathcal{X}$. Thus, we have the exact sequence

$$0 \longrightarrow H \longrightarrow J \longrightarrow I \longrightarrow 0.$$

Step 1. We claim that, because J is regular and H is invertible, the sequence

(*) $$0 \longrightarrow H \otimes J^{m-1} \longrightarrow J^m \longrightarrow I^m \longrightarrow 0$$

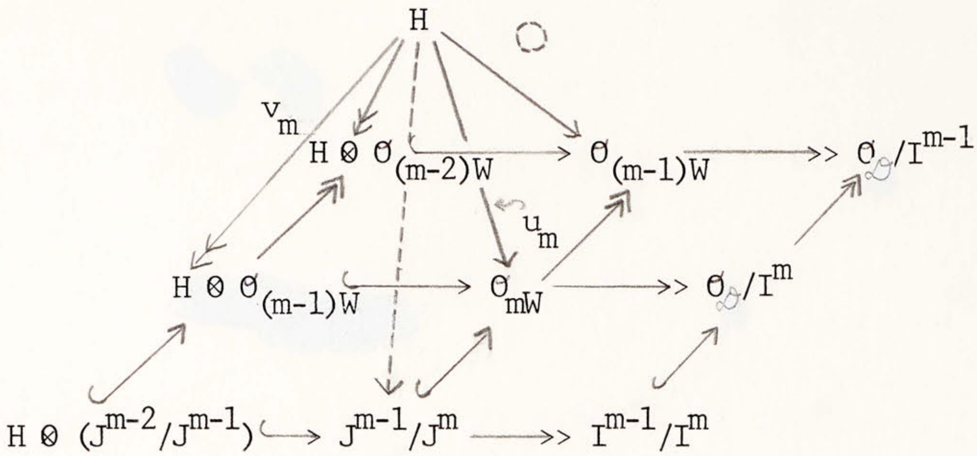
is exact. Indeed, this follows from the equality

(**) $$H \cap J^m = HJ^{m-1}.$$

To verify the latter, we may "go affine". In fact, we may assume J is an ideal generated by the regular sequence $h, j_1, \ldots, j_n$ in a local ring A, where h is a generator of H. Since the graded rings $\oplus J^\nu/J^{\nu+1}$ and $\mathrm{Sym}(J/J^2)$ are naturally isomorphic, and since the image of h in the latter is a nonzero divisor, the equality follows. For, if $a \neq 0$ is in A and ah is in J^m, then a must lie in J^{m-1}. Indeed, let i be the smallest nonnegative integer such that $a \in J^{m-i}$ holds. Let $\bar{a}$ denote class in J^{m-i}/J^{m-i+1}. If i is ≥ 2, then $J^m \subset J^{m-i+2}$ holds, whence $\bar{a}\bar{h}$ is zero in J^{m-i+1}/J^{m-i+2}, which forces $\bar{a} = 0$, that is, $a \in J^{m-(i-1)}$. Thus $i \leq 1$ holds, completing step 1.

Step 2. Having proved the exactness of (*), we may construct the marvelous diagram, where all sequences are exact,

(***)



and the dotted figures correspond to simultaneous events. To see the middle horizontal sequence is exact, observe the kernel of the natural surjection

$$\mathcal{O}_{mW} = \mathcal{O}_{\mathcal{X}}/J^m \twoheadrightarrow (\mathcal{O}_{\mathcal{X}}/H)/I^m = \mathcal{O}_{\mathcal{D}}/I^m$$

is clearly $H + J^m/J^m = H/J^m \cap H$. The latter is precisely $H \otimes \mathcal{O}_{(m-1)W}$ in view of (**). The map u_m is defined by the composition,

$$\begin{array}{ccc} H & \hookrightarrow & \mathcal{O}_{\mathcal{X}} \\ & {\scriptstyle u_m} \searrow & \downarrow \\ & & \mathcal{O}_{\mathcal{X}}/J^m, \end{array}$$

whence it clearly factors through the surjection

$v_m : H \to H + J^m/J^m = H \otimes \mathcal{O}_{(m-1)W}$.

Step 3. Recall that $\mathcal{J}_m$ is the scheme of zeros of u_m in $\mathcal{S}$. Restrict the diagram over $\mathcal{J}_{m-1}$. Denoting the restrictions by a prime (e.g. u'_m, etc...), and setting M = image of $(I^{m-1}/I^m)'$ in $(\mathcal{O}_{\mathcal{D}}/I^m)'$, we obtain the sequence

$$H' \xrightarrow{\omega} (J^{m-1}/J^m)' \twoheadrightarrow M.$$

The map ω corresponds to the dotted arrow in (***). A routine diagram chase shows this last sequence is exact.

Step 4. Denote by $i : W' \hookrightarrow \mathcal{X}'$ the inclusion. Observe that $M = i_* i^* M$ holds, and similarly for $(J^{m-1}/J^m)'$. Consequently, ω factors through the surjection,

$$H' \twoheadrightarrow i_* i^* H' \xrightarrow{\tilde{\omega}} (J^{m-1}/J^m)'.$$

We may regard $(J^{m-1}/J^m)'$ as a locally free $\mathcal{O}_{\mathcal{J}_{m-1}}$-Module via the identification $W' \cong \mathcal{J}_{m-1}$ induced by $W \cong \mathcal{S}$. Thus, $\tilde{\omega}$ (or rather $i^*\tilde{\omega}$) is a presentation of M. By definition, the zeros of $\tilde{\omega}$ in $\mathcal{J}_{m-1}$ is precisely the r_m^{th} Fitting scheme of M.

Finally, for any map $t : T \to \mathcal{J}_{m-1}$, the assertions $\omega(t) = 0$ and $\tilde{\omega}(t) = 0$ are equivalent. The former defines $\mathcal{J}_m$, whereas the latter is also equivalent to $t^*\tilde{\omega} = 0$, that is, t factors through the Fitting scheme of M.

§2

(2.1). Notation. Fix a smooth map $f : X \to S$. Let D denote the scheme of zeros of a section of an invertible $\mathcal{O}_X$-Module L. Let $\underline{m} = m_1,\ldots,m_t$ be a sequence of non-negative integers. Unless stated otherwise, we will assume $t = 1$ if the relative dimension of f is > 1. We denote the t-fold cartesian product of X over S by $X[t]$ or $X_S[t]$. The projections onto or omitting the i^{th} factor will be denoted by p_i and $p_{\hat{i}}$. The diagonal subscheme of $X[2]$ is denoted by Δ or Δ_X. We denote by $m\Delta$ the subscheme with Ideal $I(m\Delta) = I(\Delta)^m$. The pullback of Δ to $X[t]$ via the projection onto the i,j factors is denoted Δ_{ij}.

Our object of study is the set of singular points of the fibres of D. Since each fibre $D(s)$ is (locally) defined by one equation in $X(s)$, a point is of multiplicity $\geq m$ on $D(s)$ if and only if the local equation of $D(s)$ lies in the m^{th} power of the maximal ideal there. (If $D(s) = X(s)$, each point of $X(s)$ is considered to be of multiplicity $\geq m$ for any m). In order to globalize this observation, as well as to treat the case of several mult-iplicities, we are lead to consider the subscheme

$$\underline{m}\Delta \subset X \underset{S}{\times} X[t]$$

defined by the Ideal

$$I(\underline{m}\Delta) = I(\Delta_{01})^{m_1} \ldots I(\Delta_{0t})^{m_t} \qquad \text{(product)}.$$

(2.1.1). Lemma. $\underline{m}\Delta$ is finite and flat over $X[t]$.

Proof: The subscheme $m\Delta$ of $X[2]$ is flat over X (via, say, p_1). Indeed, this assertion is trivial for $m = 1$. For $m \geq 2$, we consider the exact sequence

$$0 \longrightarrow I^{m-1}/I^m \longrightarrow \mathcal{O}_{m\Delta} \longrightarrow \mathcal{O}_{(m-1)\Delta} \longrightarrow 0$$

where I is short for $I(\Delta)$. Because X is smooth $/S$, each I^{m-1}/I^m is a locally free $\mathcal{O}_X$-Module (in fact isomorphic to the symmetric power $S_{m-1}\Omega^1_{X/S}$). Thus $m\Delta$ is flat over X as claimed. Now, for $t \geq 2$, (hence rel. dim = 1), each $m_i\Delta_{0i}$ is a relative divisor of $X \underset{S}{\times} X[t]$ over $X[t]$. Since $p_{\hat{0}}$ is flat, $\underline{m}\Delta$ is a relative divisor because it induces a divisor on each fibre. The proof of the finiteness assertion is easy and will be omitted.

(2.2). $\underline{m}$-Jacobians.

(2.2.1). Definition. The $\underline{m}$-Jacobian of D is the incidence correspondence of $\underline{m}\Delta$ in D (which exists by (1.2.2), because $\underline{m}\Delta$ is finite and flat over $X[t]$). It will be denoted by $J(\underline{m};D)$. The $\underline{m}$-incidence sheaf of L is

$$\mathcal{E}_{X/S}(\underline{m};L) = (p_{\hat{o}})_*(\mathcal{O}_{\underline{m}\Delta} \otimes p_o^*L).$$

This will also be written simply $\mathcal{E}(\underline{m};L)$ or $\mathcal{E}(\underline{m})$ if no confusion is likely. The $\underline{m}$-incidence section is the section of $\mathcal{E}_{X/S}(\underline{m};L)$ which is the adjoint (= direct image via $p_{\hat{o}}$) of the composition,

$$\mathcal{O} \xrightarrow{p_o^*s} p_o^*L \xrightarrow{r} L \otimes \mathcal{O}_{\underline{m}\Delta},$$

where s is the section of L defining D and r is induced by the restriction $\mathcal{O} \twoheadrightarrow \mathcal{O}_{\underline{m}\Delta}$.

<u>Remark</u>. The $\underline{m}$-incidence sheaf is a secant sheaf, in the sense of [Schwarzenberger].

(2.2.2). <u>Proposition</u>. (1) $J(\underline{m};D)$ is the scheme of zeros of the $\underline{m}$-incidence section.

(2) Suppose the $\underline{m}$-incidence section is regular. Then $J(\underline{m};D)$ represents the top Chern class of $\mathcal{E}_{X/S}(\underline{m};D)$ in any decent intersection theory.

<u>Proof</u>: (1) The assertion follows from (1.2.2).

(2) The assertion is well known for nonsingular S ([TCC], p. 153). For the general case, cf. [Fulton].

(2.2.3). Proposition. The formation of $J(\underline{m};D)$ commutes with base change. Precisely, given a cartesian diagram,

$$\begin{array}{ccccccc} D' & \subset & X' & \longrightarrow & X & \supset & D \\ & & {\scriptstyle f'}\big\downarrow & \square & \big\downarrow{\scriptstyle f} & & \\ & & S' & \longrightarrow & S & & \end{array}$$

where $D' = S' \underset{S}{\times} D$, we have $J(\underline{m};D') = J(\underline{m};D) \underset{S}{\times} S'$.

Proof: The assertion follows immediately from the definition.

§3

(3.1). <u>The class of the $\underline{m}$-incidence sheaves</u>. Preserve the notation of §2.

We start with the case $t = 1$ (and arbitrary relative dimension).

(3.1.1). <u>Proposition</u>. We have the formula,

$$\mathcal{E}_{X/S}(m;L) = L \sum_{0}^{m-1} S_i(\Omega^1_{X/S}) \qquad \text{in } K^{\cdot}[X].$$

($S_i = i^{th}$ symmetric power).

<u>Proof</u>: We have the canonical exact sequence,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I^{m-1}/I^m & \longrightarrow & \mathcal{O}_{m\Delta} & \longrightarrow & \mathcal{O}_{(m-1)\Delta} & \longrightarrow & 0 \\ & & \| & & & & & & \\ & & S_{m-1}\Omega^1_{X/S}, & & & & & & \end{array}$$

where I is short for $I(\Delta)$. The equality holds because X is smooth over S. Tensoring this exact sequence with p_0^*L and pushing down via $p_{\hat{0}}$, yields

$$0 \longrightarrow L \otimes S_{m-1}(\Omega^1_{X/S}) \longrightarrow E(m) \longrightarrow E(m-1) \longrightarrow 0,$$

where $E(m)$ is short for $\mathcal{E}_{X/S}(m;L)$. The formula now follows by induction on m.

(3.1.2). Remark. Our $\mathcal{E}_{X/S}(m;L)$ is the sheaf of principal parts of order m-1 of ([EGA IV_4], (16.3.1)).

(3.1.3). Proposition. The restriction of the $\underline{m}$-incidence sheaf $\mathcal{E}_{X/S}(\underline{m};L)$ to the complement U of the union of the diagonals in X[t] is equal to the direct sum

$$\bigoplus_{1}^{t} p_i^*\mathcal{E}_{X/S}(m_i;L)|_U.$$

Proof: The restriction of $\underline{m}\Delta$ over U is obviously equal to the disjoint union of the divisors $(m_i\Delta_i)|_U$. Thus, $\mathcal{O}_{\underline{m}\Delta|U}$ is equal to $\bigoplus \mathcal{O}_{m_i\Delta_i|U}$. Since

$$\mathcal{E}_{X/S}(\underline{m};L)|_U = (p_{\hat{o}}|_U)_*[(\mathcal{O}_{\underline{m}\Delta} \otimes p_o^*L)|_U]$$

holds (either by flat base change or because $\underline{m}\Delta$ is finite over X[t]), we are reduced to verifying (again by flat base change) that we have,

$$(p_{\hat{o}})_*(\mathcal{O}_{m_i\Delta_i} \otimes p_o^*L) = p_i^*\mathcal{E}_{X/S}(m_i;L).$$

This is easily seen to be true by the Principle of Exchange applyed to a diagram we would rather omit.

(3.1.4). <u>Theorem</u>. We have in the Grothendieck ring $K^{\bullet}(X[t])$ the formula,

$$\mathcal{E}_{X/S}(\underline{m};L) = \sum_{i=1}^{t} p_i^{*}\mathcal{E}_{X/S}(m_i;L)\Big(-\sum_{h>i} m_h\Delta_{hi}\Big).$$

<u>Proof</u>: Let W denote the subscheme of $X \underset{S}{\times} X[t]$ with Ideal

$$I(W) = I(m_2\Delta_{02}) \cdot \ldots \cdot I(m_t\Delta_{0t}) \qquad \text{(products).}$$

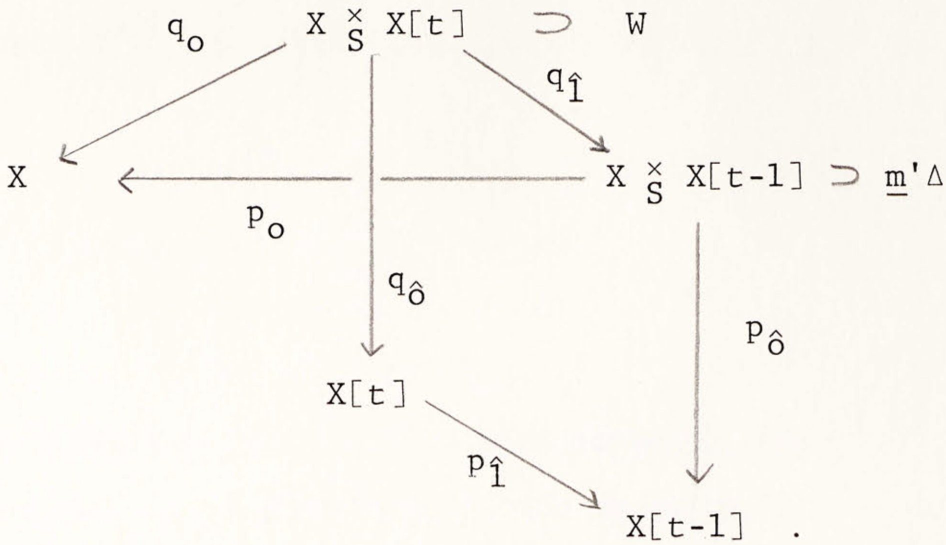
Thus, $I(\underline{m}\Delta) = I(m_1\Delta_{01})I(W)$ holds. Because $I(W)$ is invertible, we get an exact sequence,

$$0 \longrightarrow \mathcal{O}_{m_1\Delta_{01}} \otimes I(W) \longrightarrow \mathcal{O}_{\underline{m}\Delta} \longrightarrow \mathcal{O}_W \longrightarrow 0.$$

Tensoring it with $q_{\hat{0}}^{*}L$ and applying $(q_{\hat{0}})_{*}$, yields

$$\begin{array}{ccccccc}
0 \longrightarrow & (q_{\hat{0}})_{*}(\mathcal{O}_{m_1\Delta_{01}} \otimes I(W) \otimes q_{\hat{0}}^{*}L) & \longrightarrow & \mathcal{E}_{X/S}(\underline{m};L) & \longrightarrow & (q_{\hat{0}})_{*}(\mathcal{O}_W \otimes q_{\hat{0}}^{*}L) & \longrightarrow 0 \\
 & \Big\| \text{ (def'n).} & & & & \Big\| & \\
 & A & & & & p_{\hat{1}}^{*}\mathcal{E}_{X/S}(\underline{m};L) &
\end{array}$$

where $\underline{m}'$ denotes the sequence $m_2,\ldots,m_t$. The latter equality holds by the Principle of Exchange (or flat base change). Here is the relevant diagram:



By induction on t, it follows that $p_{\hat{1}}^{*}\mathcal{E}_{X/S}(\underline{m}';L)$ is the sum of the $(t-1)$-last terms of the proposed formula.

Now it remains to identify A with
$p_1^{*}\mathcal{E}_{X/S}(m_1;L)(-\sum_{h>1} m_h\Delta_{hi})$.

For this, we observe that $X \underset{S}{\times} X[t]$ is equal to the fiber product $X[t] \underset{X[t-1]}{\times} X[t]$, where we regard $X[t]$ as a scheme over $X[t-1]$ via $p_{\hat{1}}$. Further, with this identification, the projections onto the 1[st] and 2[nd] factors are equal to $q_{\hat{o}}$ and $q_{\hat{1}}$ (see diagram). And Δ_{01} is precisely $\Delta_{X[t]}$. Thus, applying (3.1.1) to $X[t] \to X[t-1]$ (in place of $X \to S$) and $p_1^{*}L \otimes I(W)$, we get

$$A = \mathcal{E}_{X[t]/X[t-1]}(m_1;p_1^{*}L \otimes I(W))$$

$$= p_1^{*}L \otimes I(W) \sum_{0}^{m_1-1} S_i(\Omega^1_{X[t]/X[t-1]}).$$

Finally, since $p_{\hat{1}}:X[t] \to X[t-1]$ is the pullback of $X \to S$,

$$\begin{array}{ccc} X[t] & \xrightarrow{p_1} & X \\ {\scriptstyle p_{\hat{1}}}\downarrow & & \downarrow \\ X[t-1] & \longrightarrow & S, \end{array}$$

and since the formation of relative differentials and of symmetric powers of a locally free Module commutes with base change, we see that A is indeed equal to the 1st term of our formula . This completes the proof of the theorem.

(3.2). <u>A recursive relation</u>.

(3.2.1). <u>Proposition</u>. The incidence sheaves satisfy the relation,

$$\mathcal{E}(\underline{m};u+1;L) = \mathcal{E}(\underline{m},u;L) + p^{*}_{t+1}(L(\Omega^1_{X/S})^u)(-\Sigma m_h \Delta_{h,t+1}) \quad \text{in} \quad K^{\bullet}X[t+1].$$

<u>Proof</u>: Consider the fibre square,

$$\begin{array}{ccc} X \times_S X[t+1] & \overset{j}{\hookrightarrow} & X \times_S X[t+2] \\ \downarrow {\scriptstyle q'_{\hat{0}}} & & \downarrow {\scriptstyle q_{\hat{0}}} \\ X[t+1] & \overset{i}{\hookrightarrow} & X[t+2] \end{array}$$

$$(x_1,\ldots,x_{t+1}) \longmapsto (x_{t+1},x_1,\ldots,x_t,x_{t+1}),$$

where the horizontal maps are diagonal embeddings. Set $W = (1,\underline{m},u)\Delta$, that is, the subscheme of $X \underset{S}{\times} X[t+2]$ with ideal $I(\Delta_{01})\, I(\Delta_{02})^{m_1} \ldots I(\Delta_{0t+1})^{m_t} I(\Delta_{0,t+2})^u$. Since W is flat over $X[t+2]$, we have the formula,

$$I(j^{-1}(W)) = j^*(W).$$

By the construction of j, we have

$$j^*I(W) = I(\Delta_{0,t+1})^{u+1} I(\Delta_{01})^{m_1} \ldots I(\Delta_{0t})^{m_t} .$$

Thus, $j^{-1}(W)$ is just $(\underline{m},u+1)\Delta$. Consequently, we may write,

$$\begin{aligned}
\mathcal{E}_{X/S}(\underline{m},u+1;L) &= (q'_{\hat{o}})_*(\mathcal{O}_{(\underline{m},u+1)\Delta} \otimes (q'_o)^*L) \quad \text{(by def'n)} \\
&= (q'_{\hat{o}})_*(j^*(\mathcal{O}_W \otimes q_o^*L)) \\
&= i^*(q_{\hat{o}})_*(\mathcal{O}_W \otimes q_o^*L) \quad \text{(by the Principle of Exchange)} \\
&= i^*\mathcal{E}_{X/S}(1,\underline{m},u;L)
\end{aligned}$$

$$= [i^*\, q_1^*L(-u\Delta_{t+2,1} - \Sigma m_h \Delta_{t+2,h+1}) + \Sigma q^*_{\ell+1} E_\ell + q^*_{t+2}\mathcal{E}_{X/S}(u;L)] \quad \text{in } K^{\bullet}(X[t+1])$$

(by (3.1.4)), where E_ℓ is short for

$$\mathcal{E}_{X/S}(m_\ell;L)(-u\Delta_{t+2,\ell+1} - \sum_{h>\ell} m_h\Delta_{h+1,\ell+1}).$$

By definition of i, we have the formulas,

$$q_\ell i = \begin{cases} p_{t+1} & \text{for } \ell = 1 \text{ or } t+2, \\ p_{\ell-1} & \text{for } 2 \le \ell \le t+1, \end{cases}$$

and

$$i^*I(\Delta_{h+1,\ell+1}) = \begin{cases} p^*_{t+1}\Omega^1_{X/S} & \text{for } (h+1,\ell+1) = (t+2,1) \\ I(\Delta_{h,\ell}) & \text{for } 1 \le \ell < h \le t+1. \end{cases}$$

Therefore we get,

$$\mathcal{E}_{X/S}(\underline{m},u+1;L) = p^*_{t+1}L(\Omega^1_{X/S})^u(-\Sigma m_h\Delta_{t+1,h})$$
$$+ \Sigma p^*_\ell\mathcal{E}_{X/S}(m_\ell;L)(-\sum_{h>\ell} m'_h\Delta_{h\ell}),$$

where we put $m'_h = m_h$ for $h \le t$ and $m'_{t+1} = u$. By (3.1.4), the last term in the expression above is precisely $\mathcal{E}(\underline{m},u;L)$ in $K^\bullet X[t+1]$, thus completing the proof.

§4

(4.1). <u>Linear systems</u>. Let $g : Y \to Z$ be a map of schemes. Let M denote an invertible $\mathcal{O}_Y$-Module, and suppose V is a locally free $\mathcal{O}_Z$-submodule of g_*M. Let

$$a : g^*V \to M$$

denote the adjoint of the inclusion $V \subseteq g_*M$. Set $S = P(V^\vee)$ and consider the fibre square,

$$\begin{array}{ccc} X = P(V_Y^\vee) & \longrightarrow & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ S & \longrightarrow & Z. \end{array}$$

We have, on X, the following diagram,

$$\begin{array}{ccc} V_X^\vee & \overset{a^\vee}{\longleftarrow} & M_X^\vee \\ \downarrow & \swarrow & \\ \mathcal{O}_X(1) & & \end{array}$$

from which, after dualizing and tensoring with $\mathcal{O}_X(1)$, we get the key section,

(4.1.1) $$s : \mathcal{O}_X \to \mathcal{O}_X(1) \otimes M_X.$$

(4.1.2). *Definition*. We say S is a *linear system of* M, and call the scheme of zeros of s the *universal divisor of* S. If $V = g_*M$, we say S is *complete*. The *$\underline{m}$-Jacobian scheme of* S is the $\underline{m}$-Jacobian scheme of its universal divisor, and is denoted by $J(\underline{m};S)$. If S is complete, we also call $J(\underline{m};S)$ the *$\underline{m}$-Jacobian scheme of* M, and write $J(\underline{m};M)$.

(4.1.3). *Remark*. We do not insist that the universal divisor of M be either flat over S or even a Cartier divisor on X. In fact, s need not be a regular section.

We examine next the relationship of a linear system with its trace on a subscheme of the ambient space. Given the diagram of maps of schemes proper and flat over Z,

$$\begin{array}{ccc} W & \overset{i}{\hookrightarrow} & Y \\ & {}_{h}\searrow \quad \swarrow_{g} & \\ & Z & \end{array},$$

where i is a closed immersion, set $I = I(W)$, set $A = g_*(I \otimes M)$, set $B = g_*M$ and set $C = h_*(i^*M)$. Suppose the natural sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is also exact on the right (e.g., $R^1g_*(I \otimes M) = 0$), that each of these three $\mathcal{O}_Z$-Modules is locally free, and that their formation commutes with base change.

(4.1.4). Proposition. (1) The subbundle $P(A^\vee)$ of $P(B^\vee)$ is equal to the incidence correspondence (1.2.1) of W in the universal divisor D of M.

(2) The natural map of bundles over Z,

$$\pi : U = P(B^\vee) - P(A^\vee) \longrightarrow P(C^\vee)$$

is smooth and surjective.

(3) The restriction of D over U is equal to the pullback of the universal divisor of i^*M.

Proof: The first assertion follows from (1.2.2) and (1.1.4).

The 2^{nd} assertion is local on Z. Thus, we may assume the sequence

$$0 \longrightarrow C^\vee \xrightarrow{c} B^\vee \xrightarrow{b} A^\vee \longrightarrow 0$$

splits. Let $r : B^\vee \to C^\vee$ be a retraction, that is, $rc = 1_{C^\vee}$ holds. By functoriality of Proj., this retraction yields a section $P(C^\vee) \hookrightarrow P(B^\vee)$ of π. Consequently, π is smooth and surjective.

The proof of the last assertion is easy and will be omitted.

(4.2). <u>Regularity of the $\underline{m}$-incidence section</u>. Preserve the notation and assumptions of (4.1). Assume $g : Y \to Z$ is smooth. We will show that for all "sufficiently general" linear systems, the corresponding universal divisor gives rise to a <u>regular</u> $\underline{m}$-incidence section. The basic fact here is that $X[t]$ is the projective bundle $P(V^{\vee}{}_{Y[t]})$ over $Y[t]$ and, as we will see, that $J(\underline{m};D)$ is a certain subbundle.

(4.2.1.) <u>Lemma</u>. (1) There exists a natural exact sequence of $\mathcal{O}_{Y[t]}$-Modules,

$$\begin{array}{ccccccccc} & & & & V_{Y[t]} & & & & \\ & & & \swarrow & & \searrow{\scriptstyle v} & & & \\ 0 \to (q_{\hat{o}})_*(q_o^*M \otimes I(\underline{m}\Delta_Y)) & \to & (g^*g_*M)_{Y[t]} & & \to & & \mathcal{E}_{Y/Z}(\underline{m};M) & \to & (R^1q_{\hat{o}})_*(q_o^*M \otimes I(\underline{m}\Delta_Y)) \end{array}$$

(2) Set $L = M \otimes O_S(1)$. Then we have

$$\mathcal{E}_{X/S}(\underline{m};L) = \mathcal{E}_{Y/Z}(\underline{m};M) \otimes \mathcal{O}_S(1)$$

(3) The $\underline{m}$-incidence section of $\mathcal{E}_{X/S}(\underline{m};L)$ (2.2) factors through $w = v \otimes \mathcal{O}_S(1)$,

$$\begin{array}{ccc} \mathcal{O} & & \\ {\scriptstyle u}\downarrow & \searrow & \\ \mathcal{O}_S(1) \otimes V_{X[t]} & \xrightarrow{w} & \mathcal{E}_{X/S}(\underline{m};L), \end{array}$$

Where $u^{\vee} \otimes \mathcal{O}_S(1)$ is the universal 1-quotient of $P(V^{\vee}_{Y[t]})$.

<u>Proof</u>. (1) Applying $(q_{\hat{o}})_*$ to the exact sequence

$$0 \longrightarrow q_o^*M \otimes I(\underline{m}\Delta_Y) \longrightarrow q_o^*M \longrightarrow q_o^*M \otimes \mathcal{O}_{\underline{m}\Delta_Y} \longrightarrow 0,$$

and using the formula,

$$(q_{\hat{o}})_* q_o^*M = g^*(g_*M)_{Y[t]} \qquad \text{(flat base change)},$$

the assertion follows.

(2) The assertion follows by the Principle of Exchange (or flat base change) plus the projection formula, applied to the fibre square,

$$\begin{array}{ccc} X \times_S X[t] & \xrightarrow{h'} & Y \times_Z Y[t] \\ \downarrow {\scriptstyle p_{\hat{o}}} & & \downarrow {\scriptstyle q_{\hat{o}}} \\ X[t] & \xrightarrow{h} & Y[t] \end{array} .$$

Indeed, since we clearly have $\underline{m}\Delta_X = (\underline{m}\Delta_Y) \times_Z S$, we may write,

$$\mathcal{E}_{X/Y}(\underline{m};L) = (p_{\hat{o}})_*[(h')^*(\mathcal{O}_{\underline{m}\Delta_Y} \otimes q_o^*M) \otimes \mathcal{O}_S(1)]$$

$$= \mathcal{O}_S(1) \otimes h^*[(q_{\hat{o}})_*(\mathcal{O}_{\underline{m}\Delta_Y} \otimes q_o^*,)]$$

$$= \mathcal{O}_S(1) \otimes \mathcal{E}_{Y/Z}(\underline{m};M).$$

(3) By definition, the $\underline{m}$-incidence section e is the adjoint of rp_o^*s:

$$\begin{array}{ccc} \mathcal{O} & & \\ \downarrow{\scriptstyle p_o^*s} & \searrow & \\ L_{X[t+1]} & \xrightarrow{\ r\ } & L \otimes \mathcal{O}_{\underline{m}\Delta_X} \end{array} .$$

On the other hand, s factors, by construction, as follows:

$$\begin{array}{ccc} \mathcal{O}_X & \xrightarrow{\ s\ } & L = \mathcal{O}_X(1) \otimes M \\ & \searrow & \uparrow \\ & & \mathcal{O}_X(1) \otimes V \end{array}$$

Thus, we get the diagram,

$$\begin{array}{ccccc} \mathcal{O}_{X[t]} & \longrightarrow & (p_{\hat{o}})_*L = \mathcal{O}_S(1) \otimes (g_*M)_{X[t]} & \xrightarrow{(p_{\hat{o}})_*r} & \mathcal{E}_{X/S}(\underline{m};L), \\ & \searrow{\scriptstyle u} & \uparrow & & \\ & & \mathcal{O}_S(1) \otimes V_{X[t]} & & \end{array}$$

completing the proof of (3).

(4.2.3). Theorem. Denote the cokernel of
$v^{\vee} : \mathcal{E}_{Y/Z}(\underline{m};M)^{\vee} \to V^{\vee}_{Y[t]}$ by F.

(1) The $\underline{m}$-Jacobian scheme of S (4.1.2) is equal to the projective subbundle $P(F)$ of $X[t] = P(V^{\vee}_{Y[t]})$.

(2) If v ((4.2.1), (1)) is surjective (e.g., if $(R^1 q_{\hat{o}})_*(q_o^* M \otimes I(\underline{m}\Delta))$ vanishes and S is the complete system of M) then the $\underline{m}$-incidence section is regular (that is, its Koszul complex is exact).

Proof: Denote the $\underline{m}$-incidence section by e. We clearly have the equalities,

$$Z(e) = Z(e^{\vee}) = Z(e^{\vee} \otimes \mathcal{O}_S)).$$

In view of (3) of the lemma, the assertion (1) follows from ((1.1.4), (i), with $\alpha = v^{\vee}$ and $u = e^{\vee} \otimes O_S(1)$).

Finally, if v is surjective, we have that $\ker(v)$ is locally free, $v^{\vee}$ is injective, and F is locally free. Consequently, we may conclude with the help of ((1.1.4), (ii)).

§5

(5.1). <u>The formula of de Jonquières</u>. We assume throughout this section Y is a smooth projective curve over k of genus g. We will be consistent with the notation of the previous sections. Thus, M denotes an invertible O_Y-Module, V denotes a k-vector subspace of $H^o(Y,M)$, etc... We can now rephrase and sharpen some of the results of the previous sections.

(5.1.1). <u>Theorem</u>. (1) There exist a locally free sheaf $E = \mathcal{E}_Y(\underline{m};M)$ on $Y[t]$, and a section e of $E \otimes O_S(1)$ on $S \times Y[t]$ such that

(i) The scheme of zeros $J(\underline{m};S)$ of e parametrizes the points $(D,y_1,\ldots,y_t)$ in $S \times Y[t]$ such that

$$D \geq \Sigma m_i y_i$$

holds;

(ii) The class of E in $K^{\bullet}(Y[t])$ is

$$\Sigma q_i^* \mathcal{E}(m_i;M)(-\sum_{h>i} m_h \Delta_{hi})$$

(2) If $J(\underline{m};S)$ is either empty or has codimension rank E $(= \Sigma m_i)$ then its class in $A(S \times C[t])$ is equal to

the top Chern class of $E \otimes O_S(1)$.

(3) Suppose S is the complete linear system of M. Then $J(\underline{m};S)$ is either empty or has the right codimension, provided we have

$$\deg M - \Sigma m_i > 2g - 2 - \nu,$$

where ν is the number of m_i's equal to 1.

(4) Suppose $J(\underline{m};S)$ has the correct codimension. Then so does $J(\underline{m};S')$ for all sufficiently general subsystems S' of S.

(5) Suppose $J = J(\underline{m};S)$ is finite and that

$$\dim S + t = \sum_{1}^{t} m_i$$

holds. Then the degree of the zero cycle of J is equal to the degree of the t^{th} Chern class $c_t E$.

(6) (<u>The formula of de Jonquières</u>.) Set $n = \deg M$ and assume $n = \Sigma m_i$. Then the degree of $c_t E$ is

$$(\Pi m_i) \sum_{j=0}^{t} (t-j)!j!\binom{g}{j}\sigma_j(m_1-1,\ldots,m_t-1)$$

(where $\sigma_0 \equiv 1$ and $\sigma_1,\ldots,\sigma_t$ are the elementary symmetric functions in t-variables).

Proof: The two assertions in (1) are just restatements of (2.2.1 (1) and 3.1.4). The description of the k-points of $J(\underline{m};S)$ is straightforward from (2.2.2).

Assertion (2) is a well-known fact of all decent intersection theories.

To prove (3), we apply the criterion for regularity given in (4.2.3, (2)). In fact, $(R^1 q_{\hat{o}})_*(q_o^* M \otimes I(\underline{m}'\Delta_Y))$ vanishes by the Principle of Exchange, because $H^1(Y, M \otimes I(\Sigma m_i' y_i))$ is zero by the hypothesis on deg M. Here, we have replaced $\underline{m}$ by the sequence $\underline{m}'$ obtained by deleting all the 1's. The assertion is now a consequence of the following.

(5.1.2). Observation: $J(\underline{m},1;S)$ and $J(\underline{m};S)$ have the same dimension, unless $J(\underline{m},1;S)$ is empty.

Proof: Assume $J(\underline{m},1;S) \neq \emptyset$. Then the projection of $S \times Y[t+1]$ onto $S \times Y[t]$ via the first t factors restricts to a finite surjective map

$$p : J(\underline{m},1;S) \to J(\underline{m};S).$$

Indeed, over each point $(D, y_1, \ldots, y_t)$ in $J(\underline{m};S)$, the fibre of p is (at least set theoretically) isomorphic to $D - \Sigma m_i y_i$.

Proof of the theorem, continued. The assertion (4) is an immediate consequence of the theorem on the transversality of a general translate ([K], Theorem 2, (i), p. 290).

To prove (5), we observe that we have

$$[J] = c_r(E \otimes O_S(1)) \qquad \text{in } A(S \times Y[t]),$$

by (2) of the theorem. Here r is short for Σm_i. By standard properties of Chern classes, setting $h = c_1 O_S(1)$, we have

$$\begin{aligned}[J] &= \sum_{i=0}^{r} c_i(E)h^{r-i} \\ &= c_t(E)h^{r-t}\end{aligned}$$

because $c_{t+i}(E) = 0$ for $i > 0$ (as E comes from the t-dimensional variety $Y[t]$), and $h^{r-t+i} = 0$ for $i > 0$ because $\dim S$ is $r-t$. Since the degree of a zero cycle remains unchanged under push down, and since the push down of $[J]$ to $Y[t]$ is $c_t(E)$, the assertion is proved.

The proof of the formula in (6) is a little tricky. The rest of this section will be devoted to it.

Denote the degree of a zero cycle Z by $|Z|$.

Set

$$|\underline{m};M| = |c_t\mathcal{E}_Y(\underline{m};M)| .$$

Set $n = \deg M$.

(5.1.3). Lemma. For each nonnegative integer u, the following recursive formula holds:

$$|\underline{m},u+1;M| = |\underline{m},u;M| + (n + u(2g-2))|\underline{m};M(-uy)| - \Sigma m_i|m_1,\ldots,m_i+u,\ldots,m_t;M|,$$

where y is a point of Y and $M(-uy)$ is $M \otimes I(y)^u$.

Proof: Set $E(u) = \mathcal{E}_Y(\underline{m},u;M)$ for short. Recall the relation (3.2.1),

$$E(u+1) = E(u) + q^*_{t+1}(M(\Omega^1_Y)^u)(-\Sigma m_h\Delta_{h,t+1}) \quad \text{in} \quad K^\bullet(Y[t+1]).$$

Let $\underline{y}$ and $\underline{h}$ denote the embeddings of $Y[t]$ onto $Y[t] \times y$ and $\Delta_{h,t+1}$ in $Y[t+1]$. We compute Chern classes modulo algebraic equivalence. Thus, we get,

$$c_1(M) = ny,$$

and

$$|\underline{m},u+1;M| = |\underline{m},u;M| + (n+u(2g-2))|c_t(\underline{y}^*E(u))| - \Sigma m_h|c_t(\underline{h}^*E(u))|.$$

We have used the projection formula and the invariance of degree under lower star:

$$c_t(E(u)) \cdot q_{t+1}^*(y) = c_t(E(u))\underline{y}_*(1)$$

$$= \underline{y}_*(\underline{y}^* c_t(E(u)))$$

$$= \underline{y}_*(c_t(\underline{y}^* E(u))),$$

and similarly for $\Delta_{h,t+1}$.

Recalling the formula for the class of $E(u)$, (3.1.4),

$$E(u) = q_{t+1}^* \mathcal{E}_Y(u;M) + \sum_1^t q_i^* \mathcal{E}_Y(m_i;M)(-u\Delta_{t+1,i} - \sum_{h>i} m_h \Delta_{hi}),$$

and observing the formulas,

$$q_{t+1}\underline{y} : Y[t] \to \mathrm{Spec}(k),$$

$$\underline{y}^*(\Delta_{t+1,i}) = p_i^*(y),$$

$$\underline{y}^*(\Delta_{h,i}) = \Delta_{h,i} \quad , \qquad \text{for} \quad i < h \leq t,$$

we get,

$$\underline{y}^* E(u) = (\text{trivial}) + \Sigma p_i^* \mathcal{E}_Y(m_i;M(-uy))(-\sum_{h>i} m_h \Delta_{hi})$$

$$\text{in} \quad K^\bullet(Y[t]).$$

Thus, we get

$$|c_t \underline{y}^* E(u)| = |\underline{m};M(-uy)|.$$

Similarly, we get, for $h = t$,

$$\underline{t}^*E(u) = p_t^*\mathcal{E}_Y(u;M) + p_t^*\mathcal{E}_Y(m_t;M)(\Omega_Y')^u +$$

$$+ \sum_{1}^{t-1} p_i^*\mathcal{E}(m_i;M)(-u\Delta_{t,i} - m_t\Delta_{t,i} - \sum_{i+1}^{t-1} m_h\Delta_{hi}).$$

However, the 1st two terms in the r.h.s. add up precisely to

$$p_t^*\mathcal{E}_Y(u + m_t;M) \quad !!!$$

Hence, we have proved

$$\underline{t}^*E(u) = \mathcal{E}_Y(m_1,\ldots,m_{t-1},m_t+u;M).$$

Finally, with the help of obvious permutations on $Y[t]$ and $Y[t+1]$ (see also proof of lemma below), we get

$$\underline{h}^*E(u) = {}_Y(m_1,\ldots,m_h+u,\ldots,m_t;M)$$

which completes the proof of (5.1.3).

(5.1.4). <u>Lemma</u>. The symbol $|\underline{m};M|$ is symmetric in $\underline{m}$.

<u>Proof</u>: Each permutation τ of $\{1,\ldots,t\}$ induces an automorphism of $Y[t]$, still denoted τ, such that, with a self

evident notation, we have,

$$(1 \times \tau)^{-1}(\underline{m}\Delta) = (\tau\underline{m})\Delta \qquad \text{in} \quad Y \times Y[t].$$

Consequently, recalling the definition of $\mathcal{E}_Y(\underline{m};M)$, we clearly get

$$\tau^*\mathcal{E}_Y(\underline{m};M) = \mathcal{E}_Y(\tau\underline{m};M)$$

whence

$$|\underline{m};M| = |\tau\underline{m};M|$$

holds.

(5.1.5). Lemma. There are formulas,

(0) $|\underline{m},0;M| = 0$

(1) $|\underline{m},1;M| = (n-\Sigma m_i)|\underline{m};M|$

Proof: The 2^{nd} of these follows from the 1^{st}, in view of (5.1.3). The 1^{st}, in turn, follows from the obvious formula,

$$\mathcal{E}_Y(\underline{m},0;M) = q_{\widehat{t+1}}\mathcal{E}_Y(\underline{m};M)$$

because the r.h.s has zero $(t+1)^{st}$ Chern class.

(5.1.6). Lemma. Assume $1 + u + \Sigma m_i = n$ holds. Then we have,

$$|m,u+1;M| = |m,u,1;M| + (n+u(2g-2))|\underline{m},1;M(-uy)| - \Sigma m_i|m_1,\ldots,m_i+u,\ldots,m_t,1;M|.$$

Proof: The formula is just a restatement of (5.1.3), in view of (1) of the preceding lemma and the hypothesis.

End of proof of (6) of the theorem. We proceed by induction on the number of indices i such that $m_i \neq 1$. If each m_i is 1, the formula in (6) gives $t!$, which is just fine by (5.1.5, (1)). To finish, it suffices to verify that the proposed formula satisfies the recursive relation in (5.1.6).

For this, set $a_j = (t+1-j)!j!\binom{g}{j}$, and set

$$\underline{m}_i = (m_1,\ldots,m_{i-1},m_{i+1},\ldots,m_t) \qquad \text{(omit } m_i\text{)}.$$

Cancelling πm_i and replacing each m_i by $1+m_i$, we are reduced to verifying the identity,

$$\begin{aligned}(u+1)\sum_0^{t+1} a_j\sigma_j(\underline{m},u) \overset{?}{=} {} & u\sum_0^{t+1} a_j(t+2-j)\sigma_j(\underline{m},u-1) + \\ & + (1+t+u(2g-1) + \Sigma m_i)\sum_0^t a_j\sigma_j(\underline{m}) - \\ & - \sum_0^t a_j \sum_1^t (1+u+m_i)\sigma_j(m_1,\ldots,m_i+u,\ldots,m_t).\end{aligned}$$

Using, as needed, the relations,

$\sigma_j(\underline{m},u) = \sigma_j(\underline{m}) + u\sigma_{j-1}(\underline{m})$; $\Sigma\sigma_{j-1}(\underline{m}_i) = (t-j+1)\sigma_{j-1}(\underline{m})$;

$j\sigma_j(\underline{m}) = \Sigma m_i\sigma_{j-1}(\underline{m}_i)$,

we arrive at

$$2u \sum_0^{t+1} a_j[(g-j)\sigma_j(\underline{m}) - (t+2-j)\sigma_{j-1}(\underline{m})] \overset{?}{=} 0,$$

with the convention that $\sigma_{t+1}(\underline{m}) = \sigma_{-1}(\underline{m}) \equiv 0$. At this stage, we recall what a_j stands for, and happily conclude that the question mark can be erased.

(5.1.7). Remarks. (i) The formula in (6) of the theorem is the one at the bottom of p. 286 of J.L. Coolidge's Treastise on Algebraic Plane Curves, Dover, (1959), N.Y. However, the recursion formula he establishes (cf. formula (6), p. 286. loc. cit.) is apparently different from ours.

(ii) If we just assume $n \geq \Sigma m_i$, a formula for $|\underline{m};M|$ may be easily derived from (6). It suffices to replace $\underline{m}$ by the sequence $\underline{m},1,\ldots,1$, and use (5.1.5, (1)).

(5.1.8). Historical note. The formula of de Jonquières has been one of the most repeatedly proved enumerative formulas. De Jonquières himself offered two proofs [J.F. Math. 1866, p. 289]. R. Torelli gave another proof based on the correspondence principle [Rendic. Circ. Mat. di Palermo, 1906].

Ms. Vittoria Notari, in her dissertation at the Universita di Bologna, in 1920, also addressed to the question. Several particular cases were treated by Castelnuovo [Circ. Math. di Palermo, 1888]; Zeuthen [Lehrbuch, 1914, p. 246]; Brill [Math. Ann. VI 1873, p. 47]; Cayley [Papers, VII, p. 41]. More recently, I. MacDonald extended the formula to the case of several linear systems [Proc. Camb. Ph. Soc. (54), 1958 p. 399], and, in a later paper, rederived the formula employing his computation of the cohomology ring of the symmetric product of a Riemann surface [Topology, I (1962), p. 319]. Schwarzenberger, in an earlier version of his paper on Secant Bundles, reportedly tried to extend the formula to positive characteristics. A. Mattuck [A.J.M., (87), no. 4 (1965), p. 779] obtained a new proof of the formula, employing several intersection formulas for the Chow ring of the symmetric product of a curve and its Jacobean variety. Our proof requires only the rudiments of intersection theory. The deepest intersection relation we need is the formula "$|\Delta^2| = 2-2g$". Also our approach enables us to give the criterion for finiteness (cf. (3) and (4) of the theorem), and is also instrumental in the analysis of the multiplicities of the solutions (see next section). Finally, the method lends itself to generalizations to higher relative dimensions, as well as to the case of families.

§6

(6.1). Conditions for multiplicity one. In char. 0, as expected (and always implicitly believed in the classical literature), each of the solutions counted by the formula in (5.1.1, (6)) does appear, in general, with multiplicity 1. In char. $p > 0$, however, this is only so if p does not divide any of the m_i. Here is the precise assertion.

(6.1.1). Theorem. Let S denote a linear system of dimension d on a smooth projective curve Y. Suppose $J = J(\underline{m};S)$ is integral and has the dimension $\delta = t + d - \Sigma m_i$. Let $\pi : J \to S$ denote the map induced by the projection $S \times Y[t] \to S$.

(1) If π is generically unramified (resp. everywhere ramified), then for every sufficiently general subsystem S' of S of dimension $d' = -t + \Sigma m_i$, the scheme $J(\underline{m};S')$ is finite and reduced. (resp. there exists a positive integer e such that the length of each Artin local ring of $J(\underline{m};S')$ is p^e, where $p = \text{char. } k > 0$).

(2) Set $J_i = J(m_1,\ldots,m_i+1,\ldots,m_t;S)$.

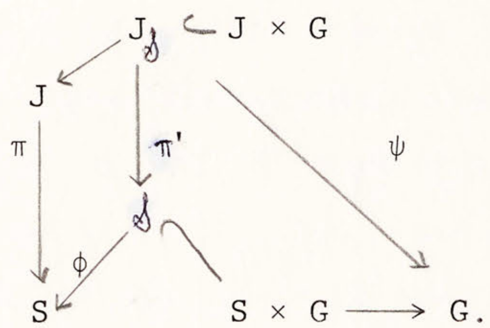
Then π is unramified on the restriction U of $J - UJ_i$ over $Y[t] - \cup\Delta_{ij}$ if and only if $p \nmid m_1,\ldots,m_t$.

(3) Suppose each J_i has the right dimension $(\delta-1)$.

Then π is generically unramified if and only if $p \nmid m_1, \ldots, m_t$.

<u>Proof</u>: Assertion (3) follows from (2) because the open set U is dense.

We now prove (1). Let G denote the Grassmannian of subspaces of S of dimension d' as above. Let $\mathcal{S} \to G$ denote the universal family. We form the diagram,



By construction, the fibre of ψ over a (say, rational point representing the) subspace S' is precisely $J(\underline{m};S) \times_S S'$, which is $J(\underline{m};S')$ by (2.2.3). Clearly, $J_{\mathcal{S}}$ is integral. Also, $\dim J_{\mathcal{S}} = \dim G$ holds. Because ϕ is faithfully flat (in fact smooth), π and π' are generically ramified or unramified together.

We claim that π and ψ are generically ramified or unramified together. Obviously, if ψ is generically unramified, then so is π' and hence also π. Conversely, if π is generically unramified, by ([K], cor. 11, p. 296) the fibre $\psi^{-1}(S') = J(\underline{m};S')$ is finite and reduced for all sufficiently general S' in G. (One must be careful to require that S' miss the ramification locus of π). Consequently, ψ is generically unramified by the lemma below. This proves the claim. Now the assertion (1) follows by applying to ψ the lemma below. (We recall that a map of integral algebraic schemes of the same dimension is generically unramified if and only if the induced function fields extension is separable).

(6.1.2). Lemma. Let $f : X' \to X$ be a finite surjective map of integral algebraic schemes. Then there exists an open dense subset X_0 of X such that the geometric fibre of f over each x in X_0 has s = separable degree of f distinct points and the length of each of its Artin local rings is i = inseparable degree of f.

Proof: Replacing X by an open dense subset (e.g. the complement of the image of the singular laws of X') we may assume X' is normal. Now, let X_s denote the normalization of X in the separable closure of its function field $R(X)$ in $R(X')$. Since X' is the normalization of X in $R(X')$, we get a factorization for f,

$$X' \xrightarrow{i} X_s \xrightarrow{s} X,$$

where i and s are purely inseparable and separable. Thus, we may assume f is either i or s. In both cases, by factoring f through the normalization of X in some intermediate field, the result follows easily by induction on the degree.

Proof of (2) of the theorem. The assertion will result from an explicit computation of the tangent space of $J(m;S)$. I learned this from secret notes of Dan Laksov.

Let R be a local ring of Y. Let x denote a uniformizer of R. Set $Y_0 = \operatorname{Spec} R$. We will describe explicit equations for the subbundle $J|_{Y_0}$ of $S \times Y_0$.

Fix a basis $f_0,\dots,f_d$ for the vector space $V \subset H^0(Y,M)$ defining S (i.e., $S = P(V^\vee)$ holds). Let $z_0,\dots,z_d$ be the dual basis. Set $E = \mathcal{E}_Y(m;M)$. There is a basic identification of $E|_{Y_0}$ with the R-module $R[y]/(y^m)$ ([Roberts], (3.7) p. 236). Let

$$d_m : R \to R[y]/(y^m)$$

be the map which computes "truncated Taylor expansions". Precisely, this is the unique homomorphism of k-algebras that sends the uniformizer x to the class of $x+y$. Fix an identification of M with R. Thus, we may think of the f_i as elements of R. It can be checked that the map $v : V_Y \to E$ of (4.2.1, (1)) is identified with

$$V \otimes R \longrightarrow R[y]/(y^m).$$

$$f_i \longmapsto d_m f_i$$

By (4.2.1, (3)), the m-incidence section becomes

$$\mathcal{O}_{S\times Y_0} \longrightarrow \mathcal{O}_S(1) \otimes V_{Y_0} \longrightarrow \mathcal{O}_S(1) \otimes E_{|Y_0}$$

$$1 \longmapsto \Sigma z_i f_i \longmapsto \Sigma z_i d_m f_i \ .$$

Denote by $\partial^0, \partial^1, \ldots, \partial^{m-1}$ the dual basis of the R-basis $1, y, \ldots, y^{m-1}$ of $R[y]/(y^m)$. Set $D^{(j)} = \partial^j d_m$. Now, it follows from (1.1.3) that the homogeneous equations of $J_{|Y_0}$ in $S \times Y_0$ are

$$(*) \qquad \sum_{i=0}^{d} z_i D^{(j)}(f_i) = 0, \qquad j=0,\ldots,m-1.$$

To get the equations for the tangent spaces of J, we dehomogenize (*) and apply the chain rule to compute the differentials of the local equations. Choosing coordinates on S so that, say, f_0 is $(1,0,\ldots,0)$, the tangent space to S at f_0 can be identified with the set of h with coordinates $(1,h_1,\ldots,h_d)$. With the above conventions, it follows that a tangent vector (h,w) of $S \times Y$ is tangent to J at (f_0,y_0) (y_0 = closed point of Y_0) if and only if we have

$$(**)\qquad \sum_{i=1}^{d} h_i(D^{(j)}(f_i))(y_0) + w\left[\frac{d}{dx}(D^{(j)}f_0)\right](y_0) = 0,$$

$$j=0,\ldots,m-1.$$

However, expanding $(x+y)^N$ by the binomial formula, we get the relations

$$D^{(j)}(x^N) = \binom{N}{j}x^{N-j},$$

whence

$$\frac{d}{dx}D^{(j)}(x^N) = (N-j)\binom{N}{j}x^{N-j-1}$$

$$= (j+1)D^{(j+1)}(x^N).$$

Since $(D^j(f_0))(y_0) = 0$ holds (because (f_0,y_0) must satisfy (*)!), the equations (**) become simply

$$h(y_0) = D^{(1)}(h)(y_0) = \ldots = D^{(m-1)}(h)(y_0) + mwD^{(m)}(h)(y_0) = 0,$$

where we put

$$D^{(j)}(h) \equiv \Sigma h_i D^{(j)}(f_i).$$

Summarizing, we have proved the following

(6.1.3). <u>Proposition</u>. Preserve the notation above.

(i) The restriction $J|_{Y_0}$ is the subbundle of $S \times Y_0$ defined by the homogeneous equations

$$D^{(j)}(f) = 0 , \qquad j=0,\ldots,m-1.$$

(ii) The tangent space of $J(m;S)$ at a point (f_0,y_0) is the subspace of the tangent vectors (h,w) of $S \times Y$ at (f_0,y_0) satisfying

$$D^{(j)}(h)(y_0) = 0 \qquad 0 \leq j \leq m-2$$

$$D^{(m-1)}(h)(y_0) + mD^{(m)}(h)(y_0)w = 0.$$

(6.1.4). Corollary. (i) If $p|m$ then $\pi : J(m;S) \to S$ is everywhere ramified.

(ii) If $p \nmid m$ then $J(m+1;S)$ is the ramification locus of $J(m;S) \to S$.

Proof. Both assertions follow from the equations for the tangent space, once we observe that the tangent map of π is just $(h,w) \mapsto h$.

(6.1.4). Remarks. (a) Usually, one gives the ramification locus of a map the structure of scheme defined by a Fitting Ideal of the Module of relative differentials. It can be shown that, if $p \nmid m$, then $J(m+1;S)$ is scheme theoretically equal to the ramification locus of π. This is the approach of A. Lascoux [L].

(b) The proposition enables us to give a description of the (embedded) tangent space to the image of $J(m;S)$ in S similar to the usual statements on duality (which is the case $m = 2$). Indeed, it is easy to show that π is a local isomorphism at each point (f,y) in $J(m;S)$ such that (f,y) is not in $J(m+1;S)$ and (f,y) is the only point in the fibre $\pi^{-1}(f)$ (provided, of course, $p \nmid m$). Further, for any such point, the embedded tangent space to $\pi(J(m;S))$ at f is the fibre of $J(m-1;S)$ over y. Thus, loosely speaking, the tangent space at a divisor that passes m times through a point is the subsystem of divisors passing $(m-1)$-times through that point.

Unfortunately, we don't know how this statement generalizes to dimensions higher than 1, except, of course, for the wellknown case of $m = 2$.

<u>End of the proof of (2) of the theorem</u>. Having dealt with the case $t = 1$, we observe that, for $t \geq 2$, the restriction of $J = J(\underline{m};S)$ over $W = Y[t] - \cup\Delta_{ij}$ is equal to the intersection of the pullbacks of the $J(m_i,S)$ to $S \times W$ via the projections $S \times W \to S \times Y$. (This assertion follows from (3.1.3)). Consequently, the tangent space to J at a point $y = (f_0, y_1, \ldots, y_t)$ is likewise an intersection of tangent

spaces. These are the equations we get, in view of (6.1.3, (ii)):

$$\begin{aligned} &(D^{(j)}h)(y_i) = 0 && 0 \le j \le m_i - 2 \\ &(D^{(m_i-1)}h)(y_i) + m_i(D^{(m_i)}f_0)(y_i)w_i = 0 && i=1,\dots,t \end{aligned}$$

Thus, if p divides, say m_1, or if y is in $J(m_1+1,\dots;S)$, the tangent vector $(h,w_1,\dots,w_t)$ with $h = 0$, $w_1 \neq 0$, $w_2=\dots=w_t = 0$ is tangent to J, and is killed by the tangent map of π. Conversely, if y is not in any of the J_i, and $p \nmid m_1,\dots,m_t$, the condition $h = 0$ implies $w_1=\dots=w_t = 0$ for each tangent vector $(h,w_1,\dots,w_t)$ in T_yJ. This finishes the proof of the theorem.

(6.2). Examples. (1) In char. 2, the number of distinct lines through a general point and tangent to a smooth conic is just 1, as everyone knows. (In fact, you just have to join the given point to the strange point of the conic). This agrees with (6.1.1), because, setting S = pencil of lines through a point, the degree of $J(2;S)$ is 2 and each of its points must appear with multiplicity $2^e > 1$.

(2) How about the flexes of a smooth cubic, in char. 3? Their number, counted with multiplicity, is $9 = |J(3;S)|$, where we put S = net of line sections. However, (6.1.1) does not apply, because $J = J(3;S)$ is not integral. But we

do know, by (6.1.3), (ii), that J is <u>not</u> reduced at any of its points. In principle, the lengths could differ from point to point. But we can save the situation by reversing the roles of the cubic and the lines, and look instead, at the family of divisors cut out by a fixed cubic in the universal line $\underset{\sim}{L} \to \check{P}^2$ (cf. Prop. 8.2.9). First, we look at $J = J(3; O_{\underset{\sim}{L}}(3))$, and happily verify it is integral (in fact isomorphic to $\underset{\sim}{L}$ over $\check{P}^2$). Then we take the (smooth) pullback of $\underset{\sim}{J}$ to the open U in $\check{P}^2 \times \check{P}^9$ (complement of the incidence correspondence "line in a cubic"). Well, the fibers of $\underset{\sim}{J}|_U$ over an open dense subset of $\check{P}^9$ (= parameter space for the cubics) are precisely the various $J(3;S)$. The upshot is that, since $J(3;S)$ is never reduced, the map $\underset{\sim}{J}|_U \to \check{P}^9$ is not separable (by (6.1.2)). Consequently, the inseparable degree, 3^e, is the uniform multiplicity of each of the points of $J(3;S)$, at least for an open dense subset of $\check{P}^9$. Since the cubic $y^2 = x(x^2-x-1)$ has precisely 3 flexes, and since this number is the maximum possible, it follows that <u>the generic cubic has precisely 3 flexes</u>. (We are using the fact that the number of geometric components of the fibres is lower semicontinuous (cf. [EGA IV_3], 15.5.2). We observe that cubics with just one flex exist, e.g., $y^2 = x^3 - x$. Since the points of inflexion are the points of order 3 for the group law of our cubic (provided we pick one of them for the zero), these results agree with the general statement on the number of points of a given order on an abelian variety (cf. [M.A.V.], (4) of prop. on p. 64).

(6.3) Problem. Suppose Y is a reduced plane curve of degree n. Let Y' denote its normalization. Let n' be a positive integer. Let $S_{n'}$ be the linear system on Y' cut out by the plane curves of degree n'. When does $J(\underline{m};S_{n'})$ have the right dimension?

§7

(7.1). Lines with prescribed contacts with a hypersurface.

Fix a projective space $P^r = P(V)$ of dimension r over a field k.

(7.1.1). Definition. Let $\underline{m} = m_1, \ldots, m_t$ be a sequence of positive integers. We say that a hypersurface h (resp. a line ℓ) has $\underline{m}$-contact with ℓ (resp. h) if the intersection $h \cap \ell$ contains a divisor of ℓ of the form $\Sigma m_i x_i$.

Let G denote the Grassmann variety of lines in P^r. Denote by Q the universal 2-quotient of V_G. Thus, $\underset{\sim}{L} = P(Q) \to G$ is the universal family of lines.

Fix a positive integer d, and set $W = H^0(P, \mathcal{O}_P(d))$. Set $T = P(W^\vee)$. Thus T parametrizes the hypersurfaces of P^r of degree d. We recall that the universal hypersurface $\underset{\sim}{H} \to T$ is the scheme of zeros in $T \times P^r$ of a section of $O_T(1) \otimes O_{P^r}(d)$. Pulling back to $T \times P^r \times G$ and restricting to $T \times \underset{\sim}{L}$, we get a subscheme D of $T \times \underset{\sim}{L}$ defined by a section of $O_T(1) \otimes O_{\underset{\sim}{L}}(d)$. Clearly, the rational points of D are the triplets (h, ℓ, x) where h is a hypersurface of degree d, ℓ is a line and x lies in $h \cap \ell$.

(7.1.2). Definition. The scheme of $\underline{m}$-contacts of lines and hypersurfaces is the $\underline{m}$-Jacobian scheme $J(\underline{m}; D)$, with

D as above. The fibre of $J(\underline{m};D)$ over a point h in T is called the scheme of $\underline{m}$-contacts of lines with h.

Thus, the rational points of $J(\underline{m};D)$ are the points $(h,x_1,\ldots,x_t,\ell)$ of $T \times \underset{\sim}{L}[t]$ such that the intersection $h \cap \ell$ contains the divisor $\Sigma m_i x_i$ of ℓ.

(7.1.3). Theorem. (1) There exists a locally free sheaf $E = E(\underline{m};d)$ of rank $\rho = \Sigma m_i$ on $\underset{\sim}{L}[t]$, and for each hypersurface h of degree d in P^r there exists a section s_h of E whose scheme of zeros is the scheme of $\underline{m}$-contacts of lines with h.

(2) The class of E in $K(\underset{\sim}{L}[t])$ is

$$\Sigma p_i^* \mathcal{E}(m_i;\mathcal{O}_{\underset{\sim}{L}}(d))(-\sum_{h>i} m_h \Delta_{hi}).$$

(3) Suppose Z is a Cohen-Macaulay closed subscheme of G of pure codimension c. Suppose the restriction over Z of the scheme of $\underline{m}$-contacts of lines with h is empty or has codimension $c + \rho$ in $\underset{\sim}{L}[t]$. Then the class $J(\underline{m};D)(h) \times_G Z$ in $A(\underset{\sim}{L}[t])$ is

$$[Z] \; c_\rho(E).$$

(4) Suppose the codimension c of the subscheme Z as in (3) satisfies,

$$c > 2(r-1) - (d+1) .$$

Then for most hypersurfaces h of degree d, the scheme $J(\underline{m};D)(h) \underset{G}{\times} Z$ has the correct codimension $c + \rho$ in $\underset{\sim}{L}[t]$, if nonempty.

Proof: Assertions (1) and (2) are special cases of (2.2.2) and (3.1.4) taking into account that the formation of $J(\underline{m};D)$ commutes with base change. By the Cohen-Macaulay assumption in (3), the codimension requirement ensures that the section of $E|_Z$ defining $J(\underline{m};D)(h)|_Z$ is regular. This yields the desired class first in $A_*(\underset{\sim}{L}[t]|_Z)$ and hence, by the projection formula, the assertion follows.

To prove the last assertion, we consider the subscheme X of $T \times G$ of pairs (h,ℓ) such that h contains ℓ. It is easy to see that X is in fact a $\mathbb{P}^n$-bundle over G, where $n = \dim T - (d+1)$. One then checks that there is a smooth, surjective map of schemes over G,

$$\begin{array}{ccc} T \times G - X & \longrightarrow & P(S_dQ)^{\vee} \\ & \searrow \quad \swarrow & \\ & G & \end{array} \qquad (S_dQ = \text{symmetric power})$$

which, fibrewise, sends a hypersurface h (such that $h \not\supset \ell$) to the divisor $h \cap \ell$. Further, the restriction of D over $T \times G - X$ is the pullback of the universal divisor $\mathbb{E}$ of $O_{\underset{\sim}{L}}(d)$ (see 4.1.4). Counting dimensions, one sees that there is an open dense subset U of T such that $U \times G$ is disjoint from X_Z. Now, if ρ is bigger than d, any line with an $\underline{m}$-contact with a hypersurface ℓ of degree d is confined in h, by Bézout's theorem. In other words, $J(\underline{m};D) \to T \times G$ factors (at least set-theoretically) through X, and therefore, $J(\underline{m};D)(h) \underset{G}{\times} Z$ is empty for each h in U. Finally, if $\rho \leq d$ holds, we know that $J(\underline{m};\mathbb{E})_Z$ is regularly embedded in $\underset{\sim}{L} \underset{G}{\times} P(S_dQ)^{\vee}|_Z$ with the codimension ρ (by 4.2.3, (2). Notice that $(R^1p_{\hat{o}})_*(I(\underline{m}\Delta) \otimes p_o^*O(d))$ vanishes by the Principle of Exchange and because the fibers are lines and the invertible sheaf induced on each of these has the nonnegative degree $d-\rho$). Therefore, its smooth pullback to $Z \times U$,

$$J(\underline{m};D)_{Z\times U}$$

has codimension ρ in $(L[t] \times U)|_{Z\times U}$. By the theorem on the dimensions of the fibres, ([Sh], Thm. 7, (2), p. 60). There exists an open dense subset U' of U over which the fibres of $J(\underline{m};D)_{Z\times U}$ have the right dimension. However, for each h in U, we have

$$[J(m;D)(h)] \underset{G}{\times} Z = [J(\underline{m};D)|_{Z\times U}](h).$$

This finishes the proof of the theorem.

(7.2). Examples. Let us compute a few cases in P^3.

Preserve the notation of (7.1).

Set $\lambda = c_1(Q)$ and set $\pi = c_2Q$. Thus, λ is the "condition" that a line meet a fixed other, and π that a line pass through a fixed point. There are formulas

$\lambda^4 = 2$ (= # of lines meeting 4 others)

$\pi^2 = 1$ (= # of lines through 2 points...)

$\lambda^2\pi = 1$ (= # of lines meeting 2 others and passing through a point).

Set $h = c_1\mathcal{O}_{\underset{\sim}{L}}(1)$. We need the Chern class of $\Omega = \Omega^1_{\underset{\sim}{L}/G}$.
In view of the canonical sequence on $\underset{\sim}{L} = P(Q)$,

$$0 \longrightarrow \Omega(1) \longrightarrow Q \longrightarrow \mathcal{O}(1) \longrightarrow 0$$

we can compute $w = c_1\Omega$ as

$$w = \lambda - 2h.$$

We also have the formulas $h^2 = \lambda h - \pi$; $h^3 = (\lambda^2 - \pi)h - \lambda\pi$.
(The 1st holds by Grothendieck's construction of Chern classes, cf. [TCC]; the 2nd then follows).

(7.2.1). Five-fold contact with a surface F of degree d.

(t=1, m=5). The sought for number is (the degree of)

$$c_5 \, \underset{\sim}{L}/G(5;\mathcal{O}_{\underset{\sim}{L}}(d)).$$

By (3.1.1), this is the term of top degree in

$$(1 + dh)(1 + dh + w) \;\ldots\; (1 + dh + 4w).$$

That term is

$$dh((d-2)h+\lambda)((d-4)h+2\lambda)((d-6)h+3\lambda)((d-8)h+4\lambda) =$$

$$= dh[2\lambda^2+(3d-8)h\lambda+(d-2)(d-4)h^2][12\lambda^2+(7d-48)h\lambda+(d-6)(d-8)h^2]$$

(because $h^4 = 0$)

$$= dh[24\lambda^4+(50d-192)\lambda^3h+(35d^2-300d+576)\lambda^2h^2]$$

Pushing down to G, we get,

$$d[24\lambda^4+(50d-192)\lambda^4+(35d^2-300d+576)(\lambda^4-\pi\lambda^2)]$$

which has degree

$$d[35d^2-200d+240]$$

$$= 5d[7d^2-40d+48]$$ (cf. [Baker], formula (4) p.90.)

(7.2.2). <u>Double inflexional lines</u> $(t=2;\ m_1 = m_2 = 3)$.

Their number is one half of the degree of

$$z = c_6[\mathcal{E}(3)_1(-3\Delta)+\mathcal{E}(3)_2] = (dh_1-3\Delta)[(d-2)h_1-3\Delta+\lambda]$$

$$[(d-4)h_1-3\Delta+2\lambda]\cdot dh_2[(d-2)h_2+\lambda][(d-4)h_2+2\lambda].$$

(The indices mean pullback to $\underset{\sim}{L}[2]$ via the corresponding projection .)

Since we have $\Delta^2 = c_1(T_{\underset{\sim}{L}/G}) = -w\Delta = (2h-\lambda)\Delta$, we may compute

$$(dh_1-3\Delta)[(d-2)(d-4)h_1^2+(3d-8)h_1\lambda-(6d-36)h\Delta-18\Delta\lambda+2\lambda^2] =$$

$$= d(d-2)(d-4)[(\lambda^2-\pi)h_1-\lambda\pi]+d(3d-8)(\lambda^2h_1-\lambda\pi)-6d(d-6)(\lambda h-\pi)\Delta -$$

$$- 18d\lambda h\Delta+2d\lambda^2h_1-3(d-2)(d-4)(\lambda h-\pi)\Delta-3(3d-8)\lambda h\Delta+18(d-6)(\lambda h-2\pi)\Delta +$$

$$+ 54\lambda(2h-\lambda)\Delta-6\lambda^2\Delta$$

$$= ((d^3-3d^2+2d)\lambda^2-(d^3-6d^2+8d)\pi)h_1-(9d^2-45d)\lambda h\Delta+[(9d^2-90d+240)\pi -$$

$$- 60\lambda^2]\Delta-(d^3-3d^2)\lambda\pi$$

Using the projection formula together with the well known formulas,

$$p_{2*}(\Delta) = p_{2*}(h_1) = 1 \qquad \text{in} \qquad A(\underset{\sim}{L})$$

and $p_{2*}(1) = 0$, we may compute $p_{2*}(z)$ as

$$\{[(d^3-3d^2+2d-60)\lambda^2-(d^3-15d^2+98d-240)\pi]-(9d^2-45d)\lambda h\} \,.$$

$$\{-(d^3-3d^2)\lambda\pi+[(d^3-3d^2+2d)\lambda^2-(d^3-6d^2+8d)\pi]h\}$$

$$= (9d^2-45d)(d^3-3d^2)\lambda^2\pi h-(9d^2-45d)[(d^3-3d^2+2d)\lambda^2 -$$
$$- (d^3-6d^2+8d)\pi]\lambda(\lambda h-\pi) + (d^3-3d^2+2d-60)(d^3-3d^2+2d)\lambda^4 h +$$
$$+ (d^3-15d^2+98d-240)(d^3-6d^2+8d)\pi^2 h - [(d^2-3d^2+2d-60)(d^3-6d^2+8d) +$$
$$+ (d^3-15d^2+98d-240)(d^3-3d^2+2d)]\lambda^2\pi h$$

which has the degree...

$$d^6 - 6d^5 + 22d^4 = 261d^3 + 1120d^2 = 1200d =$$

$$= d(d-4)(d-5)(d^3 + 3d^2 + 29d - 60)$$

(cf. [Baker], formula (5), p. 91).

§8 Higher relative dimensions

(8.1) What goes wrong. For simplicity, let's consider first the case of just two assigned multiplicities, m_1 and m_2. Suppose Y is a smooth, projective variety. As before, denote by $m_i\Delta_{io}$ the subscheme of $Y \times Y[2]$ defined by the power $I(\Delta_{io})^{m_i}$. If $\dim Y$ is >1, the scheme

$$\underline{m}\Delta = m_1\Delta_{10} + m_2\Delta_{20} ,$$

defined by

$$I(\underline{m}\Delta) = I(m_1\Delta_{10}) \cap I(m_2\Delta_{20})$$

is flat only over the open subset $U = Y[2] - \Delta$. Now, if S is, say, a linear system on Y, and if $D \subset S \times Y$ denotes the universal divisor, we already know that the restriction over U of the incidence correspondence $J = J(\underline{m}\Delta \subset D)$ (1.2.1) is the scheme of zeros of a section of a certain locally free sheaf $\mathcal{E}$. However, if J is finite and $\mathrm{rank}(\mathcal{E})$ is equal to $\dim(S \times U)$, we can't compute the number of elements in J as the degree of the top Chern class of $\mathcal{E}$, because the numerical equivalence ring of a non-complete variety is trivial.

What one needs to do, is to get a "computable" compactification of U. By this we mean a complete variety B , together with an open dense immersion $U \subset B$, plus a subscheme $\underline{m}\Delta' \subset Y \times B$ finite and flat over B , and such that $\underline{m}\Delta'\big|_U$ is equal to $\underline{m}\Delta\big|_U$. Grant the existence of such a pair $(\underline{m}\Delta', B)$ (one may take, for instance, the closure in $Y[2] \times \mathrm{Hilb}_Y$ of the graph of the map defined by $\underline{m}\Delta\big|_U$). It is very easy to show that $J(\underline{m}\Delta' \subset D)$ is a B-projective subbundle of $S \times B$, and that its restriction over $B - U$ is "negligible", at least for a sufficiently general linear system S . The catch, however, is in the computability of the relevant Chern-classes.

(8.1.1) Remark. We don't know whether the map

$$\underline{m}\colon U \longrightarrow \mathrm{Hilb}_Y$$

(defined by $(\underline{m}\Delta)\big|_U \subset Y \times U)$ extends to the blow-up of Δ in $Y \times Y$, except when either m_1 or m_2 is 1 . By general results on flattening by blowing up (cf. [Raynaud]), one does know that $\underline{m}$ extends to the blow-up of $Y \times Y$ along some subscheme.

(8.2) The step by step construction. Let $f\colon X \to S$ denote a smooth map. We construct inductively smooth maps of schemes

$$p_{t,2}, p_{t,1}: X\{t\} \to X\{t-1\} \quad .$$

We set $X\{1\} = X$ and $X\{0\} = S$ and $p_{1,1} = p_{1,2} = f$. For $t \geq 2$, we view $X\{t-1\}$ as a scheme/$X\{t-2\}$ via $p_{t-1,1}$, and we let

$$b_t: X\{t\} \to X\{t-1\} \underset{X\{t-2\}}{\times} X\{t-1\}$$

denote the blow-up of $\Delta_{X\{t-1\}}$. Then set $p_{t,i} = p_i b_t$, where p_i denotes the projection onto $X\{t-1\}$. $p_{t,i}$ is smooth by ([EGA IV_4] 19.4.8). We denote the exceptional divisor $b_t^{-1}(\Delta_{X\{t-1\}})$ by E_t, and the mth power of its Ideal by $\mathcal{O}_t(m)$. For each $\mathcal{O}_X$-Module M, we define inductively

$$M(\underline{m}) = \mathcal{O}_{t+1}(m_t) \otimes p^*_{t+1,2}M(m_1,\ldots,m_{t-1}) \quad .$$

(We put $M(\emptyset) = M$ ...).

Fix an invertible $\mathcal{O}_X$-Module L and let D denote the scheme of zeros of a section s of L. We will define inductively a closed subscheme $J(\underline{m};D)$ of $X\{t\}$. We need a few preliminaries. For $t = 1$, we consider the diagram

$$\begin{array}{ccccccc} & & \mathcal{O} & & & & \\ & & {\scriptstyle p^*_{2,2}s}\downarrow & \searrow{\scriptstyle u_{m;D}} & & & \\ L(m) & \longrightarrow & p^*_{2,2}L & \xrightarrow{r} & (p^*_{2,2}L)\otimes\mathcal{O}_{mE_2} & \longrightarrow & 0 \end{array} \quad (8.2.1)$$

<u>(8.2.2) Proposition</u>. The scheme of zeros of $u_{m;D}$ in X exists and is equal to $J(m;D)$, the m-Jacobian of D (2.2.1).

We first need the following lemma

<u>(8.2.3) Lemma</u>. There are formulas,

(i) $$(R^i p_{2,1})_*(L(m)) = (R^i p_1)_*(p_2^* L \otimes I(\Delta)^m) \; ;$$

(ii) $$(R^i p_{2,1})_*(p^*_{2,2}L \otimes \mathcal{O}_{mE}) = (R^i p_1)\;(p_2^* L \otimes \mathcal{O}_{m\Delta}) \quad ,$$

and the latter is zero for $i \neq 0$ and $m \geq 1$.

<u>Proof</u>. Since X/S is smooth, Δ is regularly embedded in $X[2]$. Consequently, there are formulas (cf. [Manin], p.62)

$$(R^i b_2)_*\mathcal{O}_2(m) = \begin{cases} 0 & \text{for } i \neq 0 \\ I(\Delta)^m & \text{for } i = 0 \end{cases}$$

and

$$(R^i b_2)_*\mathcal{O}_{mE} = \begin{cases} 0 & \text{for } i \neq 0 \\ \mathcal{O}_{m\Delta} & \text{for } i = 0 \end{cases} \quad .$$

Setting $A_m = \mathcal{O}_2(m)$ or $\mathcal{O}_{mE}$, the spectral sequences for

$$R^i(p_1b_2)_*(p^*_{2,2}L \otimes A_m)$$

degerate and yield the formulas (i) and (ii). The last assertion holds because $\mathcal{O}_{m\Delta}$ is finite / X .

Proof of the Proposition. First we recall that $E = E_2$ is flat / X (in fact smooth, because it is equal to $P(\Omega^1_{X/S})$). Further, we have exact sequences,

$$0 \longrightarrow \mathcal{O}_E(m-1) \longrightarrow \mathcal{O}_{mE} \longrightarrow \mathcal{O}_{(m-1)E} \longrightarrow 0 \ .$$

Thus, by induction, $\mathcal{O}_{mE}$ is flat / X . In view of (ii) of the lemma, it follows by the Principle of Exchange that the formation of $\mathcal{E} = (p_{2,1})_*(p^*_{2,2}L \otimes \mathcal{O}_{mE})$ commutes with base change. By (1.1.2), the scheme of zeros of $u_{m;D}$ in X exists and is in fact equal to the zeros of the section u' of $\mathcal{E}$, adjoint to $u_{m;D}$. By (ii) of the lemma, (applied to i = 0), $\mathcal{E}$ is precisely the m-incidence sheaf of L (2.2.1). It remains to be shown that u' coincides with the m-incidence section e (the zero of which, we recall, defines J(m;D)). But this is an immediate consequence of the formulas $u' = (p_{2,1})_*(u_{m;D}) = (p_1)_*(b_2)_*(u_{m;D})$, once we remark that $(b_2)_*(u_{m;D})$ equals the section of $L \otimes \mathcal{O}_{m\Delta}$ of which e is the direct image via p_1 (cf. (2.2.1)).

Continuing with our program of defining the $\underset{\sim}{J}(\underline{m};D)$, we observe that, because mE_2 is flat / X , the exact sequence in (10.2.1) remains so after restriction over $J(m;D)$. Hence, $p^*_{2,2}s$ induces a section,

$$s_{D(m)}: \mathcal{O} \longrightarrow L(m)\Big|_{J(m;D)} .$$

(8.2.4) Definition. We call the scheme of zeros of $s_{D(m)}$ the m-virtual transform of D , and denote it by $D(m)$. We set $\underset{\sim}{J}(m;D) = J(m;D)$, and define, by induction,

$$\underset{\sim}{J}(\underline{m};D) = \underset{\sim}{J}(m_2,\ldots,m_t;D(m_1)) .$$

We call $\underset{\sim}{J}(\underline{m};D)$ the $\underline{m}$-Jacobian scheme of D . The $\underline{m}$-Jacobian of a linear system S on a scheme Y/Z (4.1) is the $\underline{m}$-Jacobian of its universal divisor, and is denoted by $\underset{\sim}{J}(\underline{m};S)$. If S is the complete linear system of an invertible $\mathcal{O}_Y$-Module M , we set $\underset{\sim}{J}(\underline{m};M) = \underset{\sim}{J}(\underline{m};S)$ and refer to it as the $\underline{m}$-Jacobian of M .

(8.2.5) Remarks. (i) If the relative dimension of X/S is 1 then $\underset{\sim}{J}(\underline{m};D)$ and $J(\underline{m};D)$ agree. (Thus, we will henceforth write J instead of $\underset{\sim}{J}$.)

Proof. First observe that $X\{t\} = X[t]$ holds. Further, $p_{t+1,1}$ can be identified with $p_{\hat{0}}: X \underset{S}{\times} X[t] \to X[t]$. Now, it suffices to prove that $J(\underline{m};D)$ is equal to $J(m_2,\ldots,m_t;D(m_1))$. This can be easily verified on T-points, and it essentially means that, for a divisor d in a smooth curve, the assertions

$$d \geq m_1x_1 + \ldots + m_tx_t$$

and

$$d - m_1x_1 \geq m_2x_2 + \ldots + m_tx_t$$

are equivalent (where x_i are points on the curve).

(ii) Clarly, a point x_t in $X\{t\}$ lying over x_i in $X\{i\}$ is in $J(\underline{m};D)$ iff x_1 is an m_1-fold point of the fibre $D(x_0) \otimes k(x_1)$, and then x_2 is an m_2-fold point of $(D(m_1))(x_1) \otimes k(x_2)$ (the m_1-virtual transform of $D(x_0) \otimes k(x_1)$ at x_1) , and so on.

(8.2.6) Example. Suppose S consists of a single point and X is, say, $\mathbb{A}^2$. Suppose $D \subset X$ is a curve with a triple point together with an infinitely near triple point as the only singularity (e.g. $y^3 = x^7$). Then $J(2;D)$ is supported at the singuler point. The 2-virtual transform of D is the proper transform $\tilde{D}$ plus the exceptional line counted once. Since $\tilde{D}$ has a triple point, therefore $D(2)$ has at least a 4-fold point. Thus, $J(2,4;D) \neq \emptyset = J(4,2;D)$.

<u>(8.2.7) Proposition</u>. The formation of $J(\underline{m};D)$ commutes with base change through S .

<u>Proof</u>. Because X is smooth/S , for each base change $S' \to S$, we have that

$$X\{2\} \times_S S' = X'\{2\}$$

holds, where we put $X' = X \times_S S'$. The smoothness is required to ensure that the formation of powers of $I(\Delta_X)$ commutes with base change. The assertion now follows easily by induction on t .

The next Proposition draws its interest from the following situation. Suppose Y ia a smooth hypersurface in some projective space P^r . Then one can consider the hyperplane sections of Y as a family of Cartier divisors in two ways: either as divisors in the trivial family $\check{P}^r \times Y$ or in the family of hyperplanes of P^r parametrized by $\check{P}^r$. Accordingly, there are two possible definitions for the $\underline{m}$-Jacobian, and of course one should expect them to yield the same thing. In other words, the $\underline{m}$-Jacobian of a good family of divisors should be intrinsic.

<u>(8.2.8) Proposition</u>. Suppose D is a relative Cartier divisor of X over S . Then $J(\underline{m};D)$ depends only on the

structure map $D \to S$ (and not on the embedding $D \subset X$).

Proof. Suppose $t = 1$. We will apply lemma (1.2.3) to $\mathcal{X} = D \times_S X$, $\mathcal{S} = D$, f = projection , $\mathcal{D} = D \times_S D$ and $W = D \times_X \Delta_X = \Delta_D$. Notice that W is transversally regularly embedding in $\mathcal{X}$ relatively to $\mathcal{S}$ because Δ_X is so in $X \times_S X$ relatively to X ([EGA IV_4] 19.2.3). By the same token, $\mathcal{D}$ is a relative Cartier divisor of $\mathcal{X}/\mathcal{S}$. Now observe that, for each $m \geq 2$, the incidence correspondence of mW in $\mathcal{D}$ is precisely $J(m;D)$. By the lemma, $J(m;D)$ depends only on $J(m-1;D)$ and the Ideal of Δ_D in $D \times_S D$. Since $J(1;D)$ is obviously equal to D , the proof for $t = 1$ is complete. The proposition now follows by induction, once we remark that the m-virtual transform of D is also a relative Cartier divisor of $J \times_X X\{2\}$ over $J = J(m;D)$. This assertion holds because the fibre of $D(m)$ over a point x of J is obviously a divisor on the fibre of $X\{2\}$ over x .

(8.3) The generic class of $J(\underline{m};D)$. Preserve the notation of (8.2).

(8.3.1) Definition. We say D is $\underline{m}$-generic (or $\underline{m}$-regular) if, for each $i = 1,\ldots,t$, the Koszul complex of the m_i-incidence section (which defines $J(m_1,\ldots,m_i;D)$ in $J(m_1,\ldots,m_{i-1};D) \underset{X\{i-1\}}{\times} X\{i\})$ is exact. If D is the universal divisor of a linear system S (4.1.2), we also say S is *m-generic* if D is *m*-generic. Finally, if S is the complete linear system of the invertible $\mathcal{O}_Y$-module M, we say M is *m-generic* if S is so.

(8.3.2) Definition. The *m-incidence sheaf* of L is the element of the Grothendieck ring $K^{\bullet}X\{t\}$, defined inductively by

$$\mathcal{E}_{X/S}(\underline{m};L) = \mathcal{E}_{X\{t\}/X\{t-1\}}(m_t;L(m_1,\ldots,m_{t-1})) + p^*_{t,2}\mathcal{E}_{X/S}(m_1,\ldots,m_{t-1};L) \quad .$$

where the first term on the r.h.s. is the (class of the) m_t-incidence sheaf (2.2.1).

(8.3.3) Remarks. (i) One verifies without pain that, if the relative dimension of X/S is 1, then $\mathcal{E}_{X/S}(\underline{m};L)$ is precisely the class of the old $\underline{m}$-incidence sheaf (2.2.1).

(ii) If S is Cohen-Macaulay, then $\underline{m}$-regularity is equivalent to *each* $J(m_1,\ldots,m_i;D)$ being either empty or of the right codimension $\left(= \binom{m_i+d-1}{d}\right.$, $d = \dim X/S)$ in the restriction of $X\{i\}$ over $J(m_1,\ldots,m_{i-1};D)$.

Warning. The statement of the next theorem requires that X and S lie in a category of schemes closed under fibre products and under the formation of scheme of zero of a regular section of a locally free sheaf, and which is endowed with a Fulton homology-cohomology intersection theory. One such category consists of the quasi-projective schemes over a field. (cf. [Fulton].)

(8.3.4) Theorem. Suppose D is $\underline{m}$-generic. Then the class of $J(\underline{m};D)$ in $A_{\bullet}(X\{t\})$ is the Poincaré dual of the top Chern class of the $\underline{m}$-incidence sheaf of L. In symbols,

$$[J(\underline{m};D)] = c_{top}(\mathcal{E}_{X/S}(\underline{m};L)) \cap [X\{t\}] .$$

Proof. The basic fact is that, on account of the warning preceding the theorem, the scheme of zeros of a regular section of a locally free sheaf represents the Poincaré dual of the top Chern class of that sheaf. Thus, we have, to start with, the equality

$$[J(m_1;D)] = c_{top}(\mathcal{E}_{X/S}(m_1;L)) \cap [X] .$$

Set $\underline{m}' = m_2,\ldots,m_t$, and replace $X \to S$ by

$$X' = X\{2\} \underset{X}{\times} J(m_1;D) \to S' = J(m_1;D) \quad .$$

and D by $D' = D(m_1)$ (8.2.5). By induction, the class of

$$J(\underline{m};D) \quad = \quad J(\underline{m}';D')$$

in $A.(X'\{t-1\})$ is dual to the top Chern class of $\mathscr{E}_{X'/S'}(\underline{m}';L(m_1))$. Now $X'\{t-1\}$ is obviously equal to $X\{t\} \underset{X}{\times} S'$. Since $X\{t\}$ is smooth (hence flat) / X , the operations of pulling back the homology class of a subscheme and taking the homology class of the pull back of that subscheme are interchangeable. Thus, we may write, in $A.(X\{t\})$,

$$\begin{aligned} [X'\{t-1\}] &= (p^*_{t,1}\cdots p^*_{2,1})[J(m_1;D)] \\ &= ((p^*_{t,1}\cdots p^*_{2,1})c_{top}(\mathscr{E}_{X/S}(m_1;L))) \cap [X\{t\}] \ . \end{aligned}$$

Finally, denoting by i the inclusion $X'\{t-1\} \subset X\{t\}$, we have,

$$\begin{aligned} [J(\underline{m};D)] &= i_*[J(\underline{m}';D(m_1)] \\ &= i_*(c_{top}\underset{\sim}{\mathscr{E}}_{X'/S'}(\underline{m}';L(m_1)) \cap [X'\{t-1\}]) \\ &= c_{top}\underset{\sim}{\mathscr{E}}_{X\{2\}/X}(\underline{m}';L(m_1)) \cap i_*[X'\{t-1\}] \qquad \text{(proj. formula)} \\ &= c_{top}(\underset{\sim}{\mathscr{E}}_{X/S}(\underline{m};L)) \cap [X_t] \end{aligned}$$

(because the top Chern class of a sum is the product of the top Chern classes).

(8.3.5) Problem. Find the universal polynomials $p(m_1,\ldots,m_t,c_1,\ldots,c_d,\lambda)$ which express the push down of $c_{top}(\mathcal{E}_{X/S}(\underline{m};L))$ to $A.(X)$ in terms of the Chern classes of $\Omega^1_{X/S}$ and L .

De Jonquières did this for $X=$ curve, $S=$ point. For $t=1$ and $S=$ linear system on a curve or a surface, A. Lascoux related $p(m,c_\bullet,\lambda)$ to Thom polynomials ([L]).

(8.4) Conditions for $\underline{m}$-regularity. Fix a smooth and proper map $g\colon Y\to Z$, and let M denote an invertible $\mathcal{O}_Y$-module. We will transport the notations and constructions introduced at the beginning of (8.2) to $Y\to Z$. However, we will denote the maps $Y\{t\}\to Y\{t-1\}$ by $q_{t,1}$ and $q_{t,2}$. Further, throughout this section, $X\to S$ will denote the pullback of $Y\to Z$ by the map $S\to Z$, where S is the complete linear system of M .

We establish in this section sufficient conditions for M to be $\underline{m}$-generic (8.3.1).

(8.4.1) Theorem. (1) Suppose $(R^1q_{t+1,1})_*M(\underline{m})=0$ holds. Then M is $\underline{m}$-generic.

(2) Suppose $\underline{m}$ satisfies the relaxed proximity

inequalities

$$m_i \geq m_{i+1} + \dots + m_t - 1 \quad \text{for each} \quad i = 1,\dots,t-1.$$

Then $(R^1 q_{t+1,1})_* M(\underline{m}) = 0$ holds for all sufficiently high multiples M of an invertible $\mathcal{O}_Y$-module N ample / Z .

Proof. There are several steps. The first part of the theorem will follow from (1.1.4, (ii)) in view of the following sharper statement.

(8.4.2) Proposition. Suppose $(R^1 q_{t+1,1})_* M(\underline{m}) = 0$ holds. Let $\underline{m}_i$ denote the truncated sequence $m_1,\dots,m_i$. Then there are exact sequences of locally free $\mathcal{O}_{Y\{i\}}$-modules,

$$\begin{array}{l} 0 \longrightarrow V_{\underline{m}_i} \longrightarrow q^*_{i,1}(V_{\underline{m}_{i-1}}) \to \mathcal{E}_{Y\{i\}/Y\{i-1\}}(m_i; M(\underline{m}_{i-1})) \to 0 \\ \qquad\quad \| \\ (q_{i+1,1})_* M(\underline{m}_i) \end{array}$$

such that the dual surjection $q^*_{i,1}(V^\vee_{\underline{m}_{i-1}}) \to V^\vee_{\underline{m}_i}$ identifies $P(V^\vee_{\underline{m}_i})$ with the subscheme $J(\underline{m}_i;D)$ of $P(V_{\underline{m}_{i-1}}) \underset{Y\{i-1\}}{\times} Y\{i\} = J(\underline{m}_{i-1};D) \underset{Y\{i-1\}}{\times} Y\{i\}$.

Proof. The assertion follows by an iterative application of the lemma below. (Do it first for $M(\underline{m}_{t-1})$ and m_t , then to $M(\underline{m}_{t-2})$ and m_{t-1} , etc., down to M and m_1 .

You get that $J(m_1,M)$ and $P(V_{m_1}^{\vee})$ are equal. Replace $Y \to Z$ by $q_{2,1}: Y\{2\} \to Y$, M by $M(m_1)$ and $\underline{m}$ by $\underline{m}' = m_2,\ldots,m_t$ and conclude by induction on t .)

<u>(8.4.3) Lemma</u>. Assume $(R^1q_{2,1})_*M(m) = 0$ for some positive integer m . Then we have:

(i) $R^1g_*M = 0$;

(ii) $V = g_*M$ and $V_m = (q_{2,1})_*M(m)$ are locally free;

(iii) there is a canonical exact sequence,

$$0 \to V_m \to g^*V \to \mathcal{E}_{Y/Z}(m;M) \to 0$$

such that the dual surjection $g^*V^{\vee} \to V_m^{\vee}$ induces an identification of $P(V_m^{\vee})$ with the subscheme $J(m;M)$ of $P(g^*V^{\vee}) = S \times_Z Y$.

(iv) The m-virtual transform of the universal divisor of M , is equal to the universal divisor of $M(m)$.

<u>Proof</u>: We apply $(q_{2,1})_*$ to the canonical sequence,

$$0 \to M(m) \to q_{2,2}^*M \to q_{2,2}^*M \otimes \mathcal{O}_{mE} \to 0 \quad .$$

Invoking Lemma (8.2.3), and using the hypothesis, the resulting long exact sequence gives,

$$0 \to V_m \to (q_{2,1})_*q_{2,2}^*M \to \mathcal{E}_{Y/Z}(m;M) \to 0$$

and

$$(R^1 q_{2,1})_* q^*_{2,2} M = 0 .$$

By the same Lemma, we have the formulas

$$(R^i q_{2,1})_* q^*_{2,2} M = (R^i q_1)_* q_2^* M .$$

By flat base change, the latter is equal to $g^* R^i g_* M$.
By faithfully flat descent, $g^* R^1 g_* M = 0$ implies $R^1 g_* M = 0$.

The last assertion of (iii) is a special case of (4.2.3).

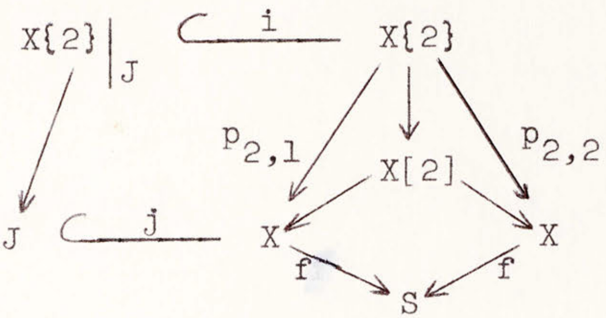
To prove (iv), we recall the definition of the m-virtual transform $D(m)$ of the universal divisor D of M . We put $S = P(V^{\vee})$, $X = S \times_Z Y$, $L = \mathcal{O}_S(1) \otimes M$. Observe $X\{2\}$ is just $S \times_Z Y\{2\}$. Now $D(m)$ is defined by the section $s_{D(m)}$ of $L(m)$ over $J = J(m;D)$ which factors $p^*_{2,2} s_D$:

$$\begin{array}{ccccccccc} & & & & \mathcal{O} & & & & \\ & & & \swarrow^{s_{D(m)}} & \downarrow {\scriptstyle p^*_{2,2} s_D} & & & & \\ (*) \quad 0 & \longrightarrow & L(m) & \longrightarrow & p^*_{2,2} L & \longrightarrow & p^*_{2,2} \otimes \mathcal{O}_{mE} & \longrightarrow & 0 \quad (\text{over } J). \end{array}$$

Next observe that $f_*(s_D)$ is the section u of $V_S \otimes \mathcal{O}_S(1)$ corresponding to the universal quotient $V_S^{\vee} \to \mathcal{O}_S(1)$ under the natural isomorphism

$$V_S \otimes \mathcal{O}_S(1) \;=\; \underline{\mathrm{Hom}}_S(V_S, \mathcal{O}_S(1)) \quad .$$

Since g_*M commutes with base change, this implies that $(p_{2,1})_*(p_{2,2})^*(s_D)$ is equal to f^*u .



Also, we have

$$[(p_{2,1})\big|_J]_* \, i^*(p_{2,2}^* s_D) \;=\; j^*f^*u \; .$$

Because j is a linear embedding of projective bundles / Y , j^*f^*u is precisely the section of $V_J \otimes \mathcal{O}_J(1)$ which is the image of that of $V_m \otimes \mathcal{O}_J(1)$ corresponding to the universal quotient u_m: $(V_m)^\vee_J \to \mathcal{O}_J(1)$. Since $[(p_{2,1})\big|_J]_*(s_{D(m)})$ is also a section of $V_m \otimes \mathcal{O}_J(1)$ mapped to j^*f^*u (in view of (*)), it must be equal to u_m . Since u_m is the direct image of the section of $L(m)\big|_J$ defining the universal divisor of $M(m)$, that section must coincide with $s_{D(m)}$, q.e.d.

The main ingrediant of the proof of (2) of the theorem is the following

<u>(8.4.4) Lemma</u> (of proximity inequalities). Let B denote a regular scheme. Set $B_1 = B$, and for $i \geq 2$ let $b_i: B_i \to B_{i-1}$ denote the blowing up of B_{i-1} at a closed point y_{i-1} with algebraically closed residue field. Set

$$\mathcal{O}(\underline{m}) = \mathcal{O}_{B_{t+1}}(m_t) \otimes b_{t+1}\mathcal{O}_{B_t}(m_{t-1}) \otimes \ldots \otimes (b^*_{t+1} \ldots b^*_2)\mathcal{O}_{B_2}(m_1) .$$

Then we have

$$R^1(b_2 \ldots b_{t+1})_* \mathcal{O}(\underline{m}) = 0 ,$$

provided $\underline{m}$ satisfies the inequalities

$$m_i \geq m_{i+1} + \ldots + m_t - 1 \quad \text{for} \quad i = 1, \ldots, t-1 .$$

<u>Proof</u>. The lemma is a consequence of the following (apparently stronger) statement. Let $\mathfrak{m}_j$ denote the Ideal of a closed point z_j in B_{t+1} . For each sequence of nonnegative integers $\underline{n} = n_1, \ldots, n_s$, set $\mathfrak{m}^{\underline{n}} = \mathfrak{m}_1^{n_1} \ldots \mathfrak{m}_s^{n_s}$ (product of ideals). Then we have

$$R^1(b_2 \ldots b_{t+1})_* (\mathcal{O}(\underline{m})\mathfrak{m}^{\underline{n}}) = 0$$

provided the new sequence $m_1,\ldots,m_t,n_1+\ldots+n_s$ satisfies the inequalities.

We proceed by induction on t.

Suppose $t = 1$. Observe that, for each j such that z_j is not in the exceptional divisor $E_2 = b_2^{-1}(y_1)$, we have

$$R^i b_{2*}(\mathcal{O}(m)\mathfrak{m}^{\underline{n}}) = \mathfrak{m}_j^{n_j} R^i(\mathcal{O}(m)\mathfrak{m}^{\underline{n}_j}) \quad \text{(where } \underline{n}_j = \underline{n} \text{ with } n_j \text{ replaced by } 0\text{)}.$$

This can be seen by restricting to $B_1 - \{y_1\}$ and $B_1 - \{b_2(z_j)\}$, because the formation of R^i commutes with flat base change. Thus we may assume each z_j is in E_2. Now, there is an exact sequence,

$$(*) \qquad 0 \to \mathcal{O}_{B_2}(1)\mathfrak{m}^{\underline{n}-\underline{1}} \to \mathfrak{m}^{\underline{n}} \to \mathfrak{m}^{\underline{n}}\mathcal{O}_{E_2} \to 0 ,$$

where $\underline{1}$ denotes the sequence which is 1 on each slot where n_j is > 0 and is zero otherwise. To verify the exactness, notice that on the complement of $\{z_2,\ldots,z_s\}$ in B_2, the above sequence is just

$$0 \to \mathcal{O}_{B_2}(1)\mathfrak{m}_1^{n_1-1} \to \mathfrak{m}_1^{n_1} \to \mathfrak{m}_1^{n_1}\mathcal{O}_{E_2} \to 0 .$$

The latter is exact because B_2 being regular implies the equality

$$\mathfrak{m}_1^{n_1} \cap \mathcal{O}_{B_2}(1) = \mathcal{O}_{B_2}(1)\mathfrak{m}^{n_1-1}$$

(Cf. proof of (1.2.3)).

Tensoring (*) with $\mathcal{O}_{B_2}(m)$ and applying b_{2*}, we get the exact sequence,

$$(R^1b_2)_*(\mathcal{O}_{B_2}(m+1)\mathfrak{m}^{\underline{n}-\underline{1}}) \to (R^1b_2)_*(\mathcal{O}_{B_2}(m)\mathfrak{m}^{\underline{n}}) \to (R^1b_2)_*(\mathfrak{m}^{\underline{n}}\mathcal{O}_{E_2}(m)) .$$

By induction on $\max\{n_1,\ldots,n_s\}$, the first term is zero. Now the last term is just $H^1(E_2,\mathfrak{m}^{\underline{n}}\mathcal{O}_{E_2}(m))$, which is zero by the lemma below. This finishes the proof for $t = 1$.

Assume $t \geq 2$. We study the exact sequence (derived from the spectral sequence of composite functors),

$$R^1(b_2\ldots b_t)_*[(b_{t+1})_*(\mathcal{O}(\underline{m})\mathfrak{m}^{\underline{n}})] \to R^1(b_2\ldots b_{t+1})_*(\mathcal{O}(\underline{m})\mathfrak{m}^{\underline{n}})$$
$$\to (b_2\ldots b_t)[(R^1b_{t+1})_*(\mathcal{O}(\underline{m})\mathfrak{m}^{\underline{n}})].$$

Set $\underline{m}' = m_1\ldots m_{t-1}$. By the projection formula, we have

$$(R^ib_{t+1})_*(\mathcal{O}(\underline{m})\mathfrak{m}^{\underline{n}}) = \mathcal{O}(\underline{m}')(R^ib_{t+1})_*(\mathcal{O}(m_t)\mathfrak{m}^{\underline{n}}) .$$

The latter is zero for $i = 1$ by the case $t = 1$ already presented.

Now renumber the z_j so that

$$z_j \text{ is in } E_{t+1} \text{ iff } s' \leq j \leq s .$$

Set $\underline{n}' = n_1,\dots,n_{s'-1}$ and set $\underline{n}'' = n_{s'},\dots,n_s$. We have the exact sequence

$$0 \to \mathfrak{m}^{\underline{n}'}\mathcal{O}(m_t+1)\mathfrak{m}^{\underline{n}''-\underline{1}} \to \mathfrak{m}^{\underline{n}'}\mathcal{O}(m_t)\mathfrak{m}^{\underline{n}''} \to \mathfrak{m}^{\underline{n}''}\mathcal{O}_{E_{t+1}}(m_t) \to 0 .$$

(Check over $B_{t+1} - E_{t+1}$ and $B_{t+1} - \{z_1,\dots,z_{s'-1}\}$.) By the case $t = 1$, applying $(b_{t+1})_*$ yields a short exact sequence. Tensoring this with $\mathcal{O}(\underline{m}')$ and applying $(b_2\dots b_t)_*$ finally yields the exact sequence

$$R^1(b_2\dots b_t)_*[\mathcal{O}(\underline{m}')(b_{t+1})_*(\mathfrak{m}^{\underline{n}'}\mathcal{O}(m_t+1)\mathfrak{m}^{\underline{n}''-\underline{1}})]$$
$$\to R^1(b_2\dots b_t)_*[\mathcal{O}(\underline{m}')(b_{t+1})_*(\mathfrak{m}^{\underline{n}'}\mathcal{O}(m_t)\mathfrak{m}^{\underline{n}''})] \to 0 .$$

The zero is right because $(b_{t+1})_*(\mathfrak{m}^{\underline{n}''}\mathcal{O}_{E_{t+1}}(m_t))$ is supported at one point. We now argue by induction on $\max(\underline{n}'')$ and conclude that the last relevant term in the above sequence is zero. The zero*th* step in this argument is true by the induction hypothesis (of the induction on t). Indeed, when $\underline{n}''$ is zero, the assertion is

$$R^1(b_2\dots b_t)_*[\mathcal{O}(\underline{m}')(b_{t+1})_*(\mathfrak{m}^{\underline{n}'}\mathcal{O}(m_t))] = 0 .$$

This equality holds because we have,

$$(b_{t+1})_*(\mathfrak{m}^{\underline{n}'}\mathcal{O}(m_t)) = \mathfrak{m}^{\underline{n}'}(b_{t+1})_*(\mathcal{O}(m_t))$$
$$= \mathfrak{m}^{\underline{n}'}\mathfrak{m}_t^{m_t}$$

where $\mathfrak{m}_t$ denotes the Ideal of y_t . We have also abused the notation by writing $\mathfrak{m}^{\underline{n}'}$ both for the product of Ideals in B_{t+1} and in B_t . This completes the proof of the lemma of proximity inequalities, modulo the result needed for the case $t = 1$ and which we will now take care of.

<u>(8.4.5) Lemma</u>. Let $\mathfrak{m}_1,\ldots,\mathfrak{m}_s$ denote the Ideals of s distinct closed points $z_1,\ldots,z_s$ in a projective space P . For each sequence of nonnegative integers $\underline{n} = n_1,\ldots,n_s$, set $\mathfrak{m}^{\underline{n}} = \mathfrak{m}_1^{n_1}\ldots\mathfrak{m}_s^{n_s}$. Let m be an integer ≥ -1 . The following are equivalent:

(i) The natural map $H^0(\mathcal{O}_P(m)) \to \mathcal{O}/\mathfrak{m}^{\underline{n}}$ is surjective;

(ii) $H^1(\mathfrak{m}^{\underline{n}}\mathcal{O}_P(m)) = 0$.

Moreover, both hold provided m is at least $n_1 + \ldots + n_s - 1$.

<u>Proof</u>. The equivalence follows from the cohomology exact sequence derived from

$$0 \to \mathfrak{m}^{\underline{n}}\mathcal{O}_P(m) \to \mathcal{O}_P(m) \to \mathcal{O}/\mathfrak{m}^{\underline{n}} \to 0 .$$

Now we prove (i) holds for $m \geq n_1 + \ldots + n_s - 1$ by induction on $\max|\underline{n}|$, where $|\underline{n}| = \Sigma n_i$. When this is zero, the assertion is trivial. So, assume $n_1 \geq 1$. Set $\underline{n}' = n_1-1 , n_2,\ldots,n_s$. Consider the diagram

$$\begin{array}{ccc}
K_m & \longrightarrow & \mathfrak{m}_1^{n_1-1}/\mathfrak{m}_1^{n_1} \\
\cap & & \cap \\
H^0(\mathcal{O}_P(m)) & \longrightarrow & \mathcal{O}/\mathfrak{m}^{\underline{n}} = \oplus\, \mathcal{O}/\mathfrak{m}_i^{n_i} \\
\Big| & & \Big| \\
\mathcal{O}/\mathfrak{m}^{\underline{n}'} & = & (\mathcal{O}/\mathfrak{m}_1^{n_1-1}) \oplus (\bigoplus_{i \neq 1} \mathcal{O}/\mathfrak{m}_i^{n_i})
\end{array}$$

The equalities hold by the Chinese Remainder Theorem. The right vertical sequence is clearly exact. The left one is also exact by definition of K_m and by the induction hypothesis. It follows that the middle horizontal map is surjective iff the top one is so. To verify this surjectivity, it suffices to produce liftings of the generators of $\mathfrak{m}_1^{n_1-1}/\mathfrak{m}_1^{n_1}$ in K_m . For this, choose sections $h_2,\ldots,h_s$ of $\mathcal{O}(1)$ such that $h_i(z_i) = 0 \neq h_i(z_1)$ holds. Set

$$h = h_2^{\otimes n_2} \otimes \ldots \otimes h_s^{\otimes n_s} ,$$

and let g be a product of (n_1-1)-global sections of $\mathcal{O}(1)$ which vanish at z_1 . Then $g \otimes h$ is a section of

$\mathcal{O}(n_1-1+n_2+\ldots+n_s)$, and $g\otimes h$ lies in K_m . Moreover, its image in $\mathfrak{m}^{n_1-1}/\mathfrak{m}^{n_1}$ is a scalar multiple of g . Since $\mathfrak{m}^{n_1-1}/\mathfrak{m}^{n_1}$ is generated by products like g , the lemma is proved.

<u>Proof of (2) of the Theorem.</u> Let y be a rational point of $Y\{t\}$, and let y_i denote its image in $Y\{i\}$. Let B_i denote the fibre of $Y\{t\}$ over y_{i-1} . Thus, we have the sequence of blowing ups

$$B_{t+1} \xrightarrow{b_{t+1}} B_t \xrightarrow{b_t} \ldots \xrightarrow{b_2} B_1 = Y(y_o) .$$

Set $b = b_2\ldots b_{t+1}$. Set $N_1 = N|_{B_1}$.

By the Principle of Exchange, it suffices to prove that

$$H^1(B_{t+1}, N_1^{\otimes n}(\underline{m})) = 0$$

holds for $n >> 0$. For this, we use again the first terms of the exact sequence derived from Leray's spectral sequence,

$$H^1(B_1, N_1^{\otimes n}\otimes b_*\mathcal{O}(\underline{m})) \to H^1(B_{t+1}, N_1^{\otimes n}(\underline{m})) \to H^o(B_1, N^{\otimes n}\otimes \underbrace{R^1 b_*\mathcal{O}(\underline{m})}_{=0}) .$$

The last term vanishes (for all n) by the lemma on the proximity inequalities. On the other hand, by ampleness, the first term is zero for all $n >> 0$. This finishes the proof.

(8.4.6) Remarks. (i) Whenever $J = J(\underline{m};S)$ is faithfully flat / $Y\{t\}$ (e.g. if J is a projective bundle as in (8.4.1)), it equals the proper transform of $J' = J(m_1;S) \times_S \dots \times_S J(m_t;S)$ in the sequence of blowing ups

$$X\{t\} \longrightarrow X\{t-1\} \times_{X\{t-2\}} X\{t-1\} \to \dots \to X \times_S \dots \times_S X = X[t] .$$

Indeed, denoting by U the complement of the diagonals in $Y[t]$, the above composition yields an isomorphism over U which clearly identifies $J|_U$ with $J'|_U$. Since U is scheme theoretically dense in $Y[t]$ (provided rel. dim Y/Z is ≥ 1) it follows that $J|_U$ is also scheme theoretically dense in J.

(ii) As an amusing consequence of the observation above, we get that $J(m,m-1;S)$ is mapped isomorphically onto $J(m-1,m;S)$ under the natural involution of $X\{2\}$ (induced by the factor switching in $X \times_S X$). When Y is the projective plane, this is a very special case of the classical "Principio di Scaricamento dei Punti Prossimi" (Cf. [Enriques] p. 431). The moral is the following. If one imposes on a sufficiently general linear system the condition that a member have a point of multiplicity $m-1$, with a neighboring point of multiplicity m, the generic solution will actually have a point of effective multiplicity m, which we "pretend" to be virtually just $m-1$. The catch is that the $(m-1)$-virtual transform will contain the exceptional divisor once, thus adding one to the multiplicity of the singular neighboring point.

(8.5) Applications. The theorem below summarizes and sharpens some of the results of the preceding sections in a form more suitable for "practical" purposes.

(8.5.1) Theorem. Let Y be a smooth projective variety of dimension d. Let S denote a linear system on Y of dimensions s. Set $J = J(\underline{m};S)$ and set $r = \sum\binom{m_i+d-1}{d}$.

(1) If S is $\underline{m}$-generic (8.3.1), then there exists an open dense subset U of the Grassmann variety G parametrizing the subsystems of S of the codimension

$$c = td + s - r$$

such that each S' is U is $\underline{m}$-generic (and in particular $J(\underline{m};S')$ is finite).

(2) If J is smooth and of the right dimension ($= c$), and if $\mathrm{char}.k = 0$, then $J(\underline{m};S')$ is finite and reduced for each S' in an open dense subset of G.

(3) For each subsystem S' of S of the codimension c (as above) and such that S' is $\underline{m}$-generic, we have the formula

$$|J(\underline{m};S')| = |c_{td}\mathcal{E}_Y(\underline{m};M)| \qquad \text{(see 8.3.2)},$$

where M denotes the invertible $\mathcal{O}_Y$-module associated to S.

(4) If S is the complete system of a sufficiently high multiple of an ample invertible $\mathcal{O}_Y$-module and $\underline{m}$ satisfies the proximity inequalities, then J is the projective bundle of a locally free $\mathcal{O}_{Y\{t\}}$-module whose class in $K^\bullet(Y\{t\})$ is

$$-(\mathcal{E}_Y(\underline{m};M))^\vee \qquad \text{(see 8.3.2)}.$$

(5) If S and $\underline{m}$ are as in (4) above, then $J(\underline{m};S')$ lies over the complement of the diagonals in $Y[t]$ for each S' in an open dense subset of G.

(6) If Y is a projective space, then the ν^{th} power of its canonical ample sheaf is $\underline{m}$-generic for $\nu \geq -1 + \sum m_i$, provided $\underline{m}$ satisfies the proximity inequalities.

Proof. The first two assertions follow from the theorem on the transversality of a general translate, once we recall that we have

$$J \underset{S}{\times} S' = J(\underline{m};S') \ .$$

To prove (3), we use the expression for the generic homology class z of $J(\underline{m};S')$,

$$z = c_r(\underset{\sim}{\mathcal{E}}) \ ,$$

where $\underset{\sim}{\mathcal{E}}$ is short for $\underset{\sim}{\mathcal{E}}_{Y \times S'/S'}(\underline{m};M \otimes \mathcal{O}_{S'}(1))$ (8.3.2). Setting $\mathcal{E} = \underset{\sim}{\mathcal{E}}_Y(\underline{m};M)$, and observing that

$$\underset{\sim}{\mathcal{E}} = \mathcal{E} \otimes \mathcal{O}_{S'}(1)$$

holds, we get

$$z = \sum_{0}^{r} c_{r-i}(\mathcal{E})h^i \quad \text{(by a standard property of Chern classes)}$$

$$= c_{td}(\mathcal{E})h^{r-td} \ ,$$

where h denotes the $1^{\underline{st}}$ Chern class of $\mathcal{O}_{S'}(1)$. The latter equality holds because we have $h^i = 0$ for $i > \dim S' = r - td$, and, on the other hand, $c_i(\mathcal{E})$ is zero for $i > \dim Y\{t\} = td$. Finally, since the degree of a zero cycle is invariant under pushing forward, (3) follows.

Assertion (4) is merely a restatement of (8.4.1). Assertion (5) then follows because the restriction of J

over a codimension 1 subvariety or $Y\{t\}$ (such as the union of the pullbacks of the diagonals) has codimension 1 in J. A fortiori, its image in S misses most subspaces of S of codimension c .

We now work on the last assertion. Referring back to the proof of (2) of (8.4.1) (see p. 63), we see that it suffices to prove the result below. (Indeed, recalling the exact sequence on p. 63, we see that the middle term is killed as soon as the first one vanishes.)

(8.5.2) Lemma. Let

$$B_{t+1} \xrightarrow{b_{t+1}} B_t \longrightarrow \cdots \longrightarrow B_2 \xrightarrow{b_2} B_1 = P$$

be a sequence of blowing ups of closed points, where P is a projective space. Set

$$\mathcal{O}(\underline{m}) = \mathcal{O}_{B_{t+1}}(m_t) \otimes b_{t+1}^* \mathcal{O}_{B_t}(m_{t-1}) \otimes \cdots \otimes (b_{t+1}^* \cdots b_3^*) \mathcal{O}_{B_2}(m_1) .$$

and set $b = b_2 \ldots b_{t+1}$. Then we have

$$H^1(P, \mathcal{O}_P(\nu) \otimes b_* \mathcal{O}(\underline{m})) = 0 \quad \text{for} \quad \nu \geq \Sigma m_i - 1 .$$

Proof. The lemma is a consequence of the following

Assertion. Let $\tilde{\mathcal{M}}_1,\ldots,\tilde{\mathcal{M}}_s$ denote the Ideals of the distinct closed points $\tilde{z}_1,\ldots,\tilde{z}_s$ in B_{t+1} . Let $\underline{n} = n_1,\ldots,n_s$ denote a sequence of nonnegative integers. Set $\tilde{\mathcal{M}}^{\underline{n}} = \tilde{\mathcal{M}}_1^{n_1}\ldots\tilde{\mathcal{M}}_1^{n_s}$. Then there exist distinct closed points $z_1,\ldots,z_n$ in P with Ideals denoted by $\mathcal{M}_1,\ldots,\mathcal{M}_n$ and there exists a sequence of nonnegative integers $\underline{r} = r_1,\ldots,r_n$ such that

$$\mathcal{M}^{\underline{r}} \subset b_*(\tilde{\mathcal{M}}^{\underline{n}}\mathcal{O}(\underline{m})) \qquad \text{and} \qquad |\underline{r}| = |\underline{m}| + |\underline{n}|$$

(where we put $|\underline{m}| = \Sigma m_i$) hold and the cokernel has finite support.

Granting the assertion, the lemma follows by taking $\underline{n}$ to be the zero sequence. Indeed, we get an exact sequence

$$H^1(P,\mathcal{O}_P(\nu)\otimes\mathcal{M}^{\underline{r}}) \to H^1(P,\mathcal{O}_P(\nu)\otimes b_*\mathcal{O}(\underline{m})) \to 0 ,$$

because a finitely supported sheaf has zero positive cohomology. Now, by (10.3.4) the first term also vanishes whenever $\nu \geq |\underline{r}| - 1$ holds. This proves the lemma.

Now we prove the assertion by induction on t . (Notice we no longer require that B_1 be a proj. space.) Suppose $t = 1$. As in the proof of (8.4.3) we may assume each z_i lies in the exceptional locus. Using the exact sequence (*) of p.88 , we obtain the inclusion

$$b_*(\mathcal{O}_{B_2}(m+1)\widetilde{\mathfrak{m}}^{\underline{n}-1}) \subset b_*(\mathcal{O}_{B_2}(m) \otimes \widetilde{\mathfrak{m}}^{\underline{n}}) = A \quad .$$

By induction on $\max(\underline{n})$, it follows that the Ideal A above contains $b_*(\mathcal{O}_{B_2}(m + |\underline{n}|))$, which is just $\mathfrak{m}_1^r$, where we put $r = m + |\underline{n}|$ and $\mathfrak{m}_1$ = Ideal of the blow up center z_1. Since the support of $A/\mathfrak{m}_1^r$ is z_1, the case $t = 1$ is proved.

For $t \geq 2$, write $b = b_2 b'$, where b' is short for $b_3 \ldots b_{t+1}$. Also, set $\underline{m}' = m_2, \ldots, m_t$. We have,

$$b_*(\widetilde{\mathfrak{m}}^{\underline{n}}\mathcal{O}(\underline{m})) = b_{2*}[\mathcal{O}_{B_2}(m_1) b'_*(\widetilde{\mathfrak{m}}^{\underline{n}}\mathcal{O}(m'))] \quad .$$

Applying the induction hypothesis to $A_1 = b'_*(\widetilde{\mathfrak{m}}^{\underline{n}}\mathcal{O}(\underline{m}'))$, we get a product of powers of maximal Ideals,

$$A_2 = (\mathfrak{m}')^{\underline{r}'} \subset A_1 \; ,$$

such that A_1/A_1 is finitely supported and $|\underline{r}'| = |\underline{n}| + |\underline{m}'|$ holds. By the case $t = 1$, there exists a product of powers of maximal Ideals

$$A_4 = \mathfrak{m}^{\underline{r}} \subset b_{2*}(\mathcal{O}_{B_2}(m_1) A_2) = A_3$$

such that $\mathrm{supp}(A_3/A_4)$ is finite and $|\underline{r}| = m_1 + |r'|$ holds. Set $A_0 = b_*(\widetilde{\mathfrak{m}}^{\underline{n}}\mathcal{O}(\underline{m}))$. Thus, we have the inclusions,

$$A_4 \subset A_3 \subset A_0$$

and A_3/A_4 and A_0/A_3 have both finite support. Therefore so does A_0/A_4 , q.e.d.

(8.5.3) Remarks. (1) With the notation of (8.5.1), (3), we don't know whether the finiteness of $J = J(\underline{m};S')$ implies S' is $\underline{m}$-generic. This is obviously true if one also assumes $J' \neq \emptyset$, for it then follows (by a trivial inductive argument) that each of the preceding $J'_i = J(m_1,\ldots,m_{i-1};S')$ is regularly embedded in $J'_{i-1} \underset{Y\{i-1\}}{\times} Y\{i\}$.

(2) The special cases $t = 1$ and $d = 1$ or 2 have been considered by A. Lascoux [L].

(8.5.4) Examples. Suppose Y is a surface and S is a sufficiently general linear system of dimension s , associated to the invertible $\mathcal{O}_Y$-module M . Set

$$\chi = c_2(\Omega^1_Y) \ , \quad K = c_1(\Omega^1_Y)$$

(and by abuse) $\quad M = c_1(M) \quad .$

(i) Assume $s = 1$. Then the formula for $|J(2;S)|$ is equivalent to that expressing the classical Zeuthen-Segre invariant I in terms of χ . Indeed, set

β = # base points

γ = # singular members, i.e., $|J(2;S)|$

g = (arithmetic) genus of a member of S .

Then I is defined classically by

$$I = \gamma - \beta - 4g \quad . \qquad \text{(cf. [Baker] p. 185)}$$

On the other hand, we may compute,

$$\begin{aligned}\gamma &= |c_2\mathcal{E}_Y(2;M)| \\ &= \chi + 2KM + 3M^2 \qquad \text{(cf. (9.5, (2), with m = 2).} \\ &= \chi + 2(KM + M^2) + M^2 \\ &= \chi + 4(g-1) + \beta \quad .\end{aligned}$$

(The assertion $\beta = M^2$ holds because the base points are the zeros of the map

$$\mathcal{O}_Y^{\oplus 2} \to M$$

defined by a choice of 2 members of S).

Thus, we get the well-known formula $I + 4 = \chi$ (cf. [Iversen] p. 974).

(ii) Assume $s = 2$. Then the formula for $|J(2,2;S)|$ is

$$(\chi + 2KM + 2M^2)^2 - (7\chi + 6K^2 + 39KM + 42M^2) \quad .$$

This result is gotten from (3) of the theorem (8.5.1), by pushing down $c_4 \mathcal{E}_Y(2,2;M)$ from $A(Y\{2\})$ to $A(Y[2])$ and then to $A(Y)$. The main ingredients of the computation are:

(a) The formula

$$\Omega^1_{Y\{2\}/Y} = p^*_{2,2}\Omega^1_Y \otimes \mathcal{O}(E) + \mathcal{O}(-E) - \mathcal{O} \quad \text{in} \quad K(Y\{2\})$$

for the class of the cotangent sheaf of $Y\{2\}/Y$ (via $p_{2,1}$, we recall). The formula follows from the two standard exact sequences,

$$b_2^*\Omega^1_{Y\times Y/Y} \hookrightarrow \Omega^1_{Y\{2\}/Y} \twoheadrightarrow \Omega^1_{Y\{2\}/Y\times Y} = j_*\Omega^1_{E/\Delta} \quad ,$$

and

$$\Omega^1_{E/\Delta} \hookrightarrow \Omega^1_Y\,(-1) \twoheadrightarrow \mathcal{O}_E \quad ,$$

where $j\colon E \hookrightarrow Y\{2\}$ is the inclusion of the exceptional divisor. (It is worth recalling the well known facts of a blowup such as b_2 : first, $E = P(\Omega^1_Y)$; second, the tautological ample sheaf $\mathcal{O}_E(1)$ on E is equal to $j^*\mathcal{O}_B(-E)$.)

(b) The intersection relations

$$E^i = j_*((-e)^{i-1}) \quad \text{in } A(Y\{2\})$$

where we put

$$e = c_1\mathcal{O}_E(1) \quad .$$

We also have

$$e^2 = Ke - \chi \, , \quad (b_2 j)_*(1) = 0 \quad \text{and} \quad (b_2 j)_*(e) = 1_{A(Y)} .$$

In particular, if Y is a surface in P^3 of degree n and S is a net of plane sections, we have

$$\chi = (n^2 - 4n + 6)n$$

$$K^2 = (n-4)^2 n \; ; \quad M^2 = n \; ; \quad KM = n(n-4) \quad .$$

Substituting these in the formula above, we get the expression

$$n(n-1)(n-2)(n^3 - n^2 + n - 12)$$

which is twice the number of bitangent planes through a general point (cf. [Baker] p. 153).

If Y is the projective plane and S is a set of cubic curves, the number $|J(2,2;S)|$ is 42. This can be seen directly to be twice the degree of the subvariety of ∞^7 reducible cubics. Indeed, given 7 general points in P^2 , each of the $\binom{7}{2} = 21$ pairs determines a unique line and a unique conic containing all 7 points.

(iii) Assume $s = 3$. A rather lengthy calculation yields the following expression for $|J(2,2,2;S)|$:

$$(\chi + 2KM + 3M^2)^3 - 14(\chi + 2KM + 3M^2)^2$$
$$- (\chi + 2KM + 3M^2)(7\chi + 18K^2 + 84M^2 + 89KM)$$
$$+ 138\chi + 376K^2 + 1380M^2 + 1576KM \quad .$$

If Y is P^2 and S is a general web of cubics, so that we have $\chi = 3$, $M^2 = K^2 = 9$, $MK = -9$, the formula gives the number $6 \cdot 15$. One can check directly that 15 is precisely the number of triangles containing 6 general points.

If Y is a surface in P^3 of degree n and S is the complete system of plane sections, substituting in the values for χ, K^2, etc. computed in (ii), we get the formula

$$n^9 - 6n^8 + 15n^7 - 59n^6 + 204n^5 - 339n^4 + 770n^3 - 2056n^2 + 1920n$$

which is 6 times the classical formula for the number of tritangent planes (cf. Salmon], formula (vi), p. 292).

To properly justify the formula, we will sketch a proof of the following

<u>(8.5.5) Proposition.</u> There exists an open dense subset A

of the projective space S parametrizing the surfaces of degree $n \geq 2$ in P^3 such that, for each F in A, $\mathcal{O}_F(1)$ is $(2,2,2)$-generic and, moreover, $J(2,2,2;\mathcal{O}_F(1))$ is reduced and lies off the diagonals.

Proof. Let $D \subset S \times P^3$ and $H \subset \check{P}^3 \times P^3$ denote the universal divisors of S and $\check{P}^3$. Let D', $\mathbf{H}$ denote their pullbacks to $S \times \check{P}^3 \times P^3$. Set $\mathbf{D} = D' \cap \mathbf{H}$. Thus, $\mathbf{D}$ is the scheme of zeros of an invertible $\mathcal{O}_{\mathbf{H}}$-module (namely $\mathcal{O}_S(1) \otimes \mathcal{O}_{\mathbf{H}}(n)$). Set Z = subset of (F,h) in $S \times \check{P}^3$ such that F contains h. (This is the incidence correspondence of H in D.) Set $U = S \times \check{P}^3 - Z$. Set $W = p_*\mathcal{O}_{\mathbf{H}}(d)$, where $p\colon \mathbf{H} \to \check{P}^3$ is the structure map. There is a smooth, surjective map of schemes $/\check{P}^3$,

$$\begin{array}{ccc} U & \longrightarrow & P(W^\vee) \\ & \searrow \quad \swarrow & \\ & \check{P}^3 & \end{array}$$

such that the restriction $\mathbf{D}|_U$ is the pullback of the universal divisor $\mathbf{C}$ of $P(W^\vee)$ in H. Consequently, we have $J(\underline{m};\mathbf{D}|_U) = J(\underline{m};\mathbf{C}) \times_{P(W^\vee)} U$. Set (provisorily) A = the complement of the image of Z in S. An easy dimension counting shows A is dense in S. Clearly, for each F in A, we have that the fibres over F of $J(2,2,2;\mathbf{D})$ and $J(2,2,2;\mathbf{D}|_U)$ are both equal to

$J(2,2,2;\mathcal{O}_F(1))$. Now let T denote the set of plane $n^{\underline{ics}}$ that are bad, namely, those containg either

(a) a point of multiplicity ≥ 3 ;

(b) at least 2 double points, one of which also posses an infinitely near double point;

(c) at least one double point with 2 successive inf. near double points;

(d) at least 3 double point, one of which is cuspidal;

(e) at least 4 double points.

One checks that T has codimension ≥ 4 . Since T is clearly invariant under $PGL(3)$, it induces a subset T' of $P(W^{\vee})$ of codimension ≥ 4 , whose fibres over $\check{P}^3$ are equal to T. The pullback of T' to U also has codimension ≥ 4 . Thus, shrinking A , we may assume $T' \times_S A$ empty. Since the family of plane $n^{\underline{ics}}$ with exactly three nodes has codimension three, the same argument as above shows we may assume that, for each F in A , there are only finitely many plane sections of F with three nodes. Since $J(2,2;\underset{\sim}{C})$ has the right codimension (for $n \geq 3$, by an easy extension of (6) of 8.5.1 to families of projective spaces), therefore so does its pullback $J(2,2;\underset{\sim}{D}|_U)$. Counting dimensions, we see that we may also assume the fibres of $J(2,2;\underset{\sim}{D}|_U)$ over each F in A is one dimensional. Finally, we also assume each F in A is smooth.

Now suppose F is in the open dense set A constructed above. Let (c,b) be a point of $J(2,2,2;\mathcal{O}_F(1))$. Thus, c is a plane section of F, and b is a point of $F\{3\}$. Let b_2 and b_1 denote the images of b in $F\{2\}$ and F. Thus, b_1 is a double point of c, b_1 is a double point of the 2-virtual transform c(2) of c, and b is a double point of the 2-virtual transform of c(2). Since c has no triple points, the 2-virtual and proper transforms are one and the same. Further, since c is not in T, it follows that b_2 (resp. b) is not on the exceptional line over b_1 (resp. b_2). The upshot is that c must be a plane curve with exactly 3 nodes, and b is precisely any of the permutations of these.

It remains to verify that $J(2,2,2;\mathcal{O}_F(1))$ is reduced. We show its tangent spaces are 0-dimensional. First, the tangent space to $J(2;\mathcal{O}_F(1))$ at a point representing a curve with a node can be identified with the set of planes through the node ([SGA VII] p. 229. Cf. also [Severi] p. 19). Next, recall $J(2,2,2;\mathcal{O}_F(1))$ is equal to the intersection $J(2;\mathcal{O}_F(1)) \times_{P^3} J(2;\mathcal{O}_F(1)) \times_{P^3} J(2;\mathcal{O}_F(1))$ off the diagonals. Thus, the tangent space at $(c,\varkappa_1,\varkappa_2,\varkappa_3)$, where $\varkappa_i$ are the 3 nodes of c, is the set of planes through these 3 points. Hence, it suffices to show that the set of plane $n^{\underline{ics}}$ with 3 <u>collinear</u> nodes has codimension ≥ 4. This is easily seen to be true for $n = 3$ and 5. When n is 6 or bigger, we also win because the family of $n^{\underline{ics}}$ with 3 double points is now irreducible. For

$n = 4$, however, the statement is false, on account of the ∞^{11} reducible quartics. But here we may invoke Lefsch tz-Noether's theorem, for the effect that a general surface of P^3 of degree ≥ 4 contains only curves that are complete intersections. Actually, since all we need is that our quartic contain no line, the result follows elementarily, anyway.

(8.6) Contacts of higher dimensional linear spaces with a hypersurface.

The difficulties with the explicit computations of formulas increase rapidly. Conceptually, however, this is a special case of the situation for divisors with specified singularities on a smooth family. One takes $X \to S$ to be the universal family of n-subspaces in a fixed projective r-space P , or its restriction to a suitable subvariety of the Grassmannian of n-spaces in P . Then, each hypersurface h in P of degree d induces a subscheme $D_h \subset X$, which is the zeros of a section of $L = \mathcal{O}_X(d)$. The fibre of D_h over each s in S is the intersection $h \cap s$. In this situation, we can play again the game of successive blowing ups and compute the generic class of $\underset{\sim}{J}(\underline{m};D_h)$. The complementary result we need is that, for a sufficiently general hypersurface h , D_h is $\underline{m}$-generic.

For $n \geq 2$, we can show this is true for $d \geq (\Sigma m_i) - 1$, provided $\underline{m}$ satisfies the proximity inequalities. The proof is essentially the same as the one given for (7.1.3, (4)), by pulling back $J(\underline{m};\mathcal{O}_X(d))$.

§9 Curves with specified coincidences of tangents at a singularity

Let $f: X \to S$ be proper and smooth, of relative dimension 2. Let $D \subset X$ be the scheme of zeros of a section s_D of an invertible $\mathcal{O}_X$-module L.

We have defined, for each positive integer m, a closed subscheme $J_m = J(m;D)$ of X, which parametrizes the points x of X such that the fibre $D(f(x))$ contains x as an m-fold point. "In general", there should be m distinct tangent directions to $D(f(x))$ at x. Now, for each subpartition of m.

$$n_1 + \dots + n_s \leq m$$

where the n_i are positive integers, one may ask for the generic homology class of the set of x for which there are tangent directions $\tau_1,\dots,\tau_s$ at x (in the surface $X(f(x))$) such that n_1 of the tangents to $D(f(x))$ at x coincide with τ_1, n_2 with τ_2, etc. We discuss here the general set-up, then "compute" the generic homology class we sought, and finally compute it explicitly in a few cases.

(9.1) Definition. The m-virtual (projectivized) tangent cone of D is the intersection

$$T_m(D) = D(m) \cap E$$

of the m-virtual transform of D (8.2.5) with the exceptional divisor of $X\{2\}$. Thus, $T_m(D)$ is the scheme of zeros of the restriction of the section $s_{D(m)}$ of $L(m)$ to $E_m = E|_{J_m}$.

So we are back to the happy situation where we have got a family of curves (lines, to be more precise) $E_m \to J_m$, together with the scheme of zeros of a section of an invertible $\mathcal{O}_{E_m}$-module.

<u>(9.2) Definition</u>. The <u>scheme (resp. sheaf) of tangent coincidences of type</u> $\underline{n}$ <u>of</u> D <u>(resp.</u> L) is

$$T(m;\underline{n};D) = J(\underline{n};T_mD) ,$$

(resp.

$$\mathcal{E}(m;\underline{n};L) = \mathcal{E}_{E/X}(\underline{n};L(m)|_E) \qquad \text{(see 2.2.1)).}$$

One defines similarly $T(m;\underline{n};S)$ and $T(m;\underline{n};M)$ for a linear system S and an invertible module M on a scheme Y proper and smooth, of rel. dim. 2 over a base scheme Z. We say that D (resp. S, resp. M) is $(m;\underline{n})$-<u>generic</u> if D (resp...) is m-generic and T_mD is $\underline{n}$-generic.

(9.3) Proposition. (1) $T = T(m;\underline{n};D)$ is the scheme of zeros in $E_m[s]$ of a section of the restriction of $\mathcal{E}(m;\underline{n};L)$.

(2) If D is $(m;\underline{n})$-generic, the class of T in $A.(E[s])$ is the Poincaré dual of the top Chern class of

$$\mathcal{E}_{X/S}(m;L) + \mathcal{E}_{E/X}(\underline{n};L\otimes\mathcal{O}_E(m)) \quad .$$

Proof. The first assertion is a special case of (2.2.2,(1)). The second assertion follows by computing first the class in $A.(E_m[s])$ and then substituting in the class of J_m in X .

(9.4) Proposition. Let $g: Y \to Z$ be proper and smooth of rel. dim. 2. Let M be an invertible $\mathcal{O}_Y$-module. Assume (with the notation of (8.4)) $(R^1q_{2,1})_*M(m+1) = 0$. Then

(1) $T(m;\underline{n};M)$ has codimension Σn_i in $E_m[s]$ if non empty.

(2) $T(m;\underline{n};M)$ is regularly embedded in $E_m[s]$ if Z is Cohen-Macaulay.

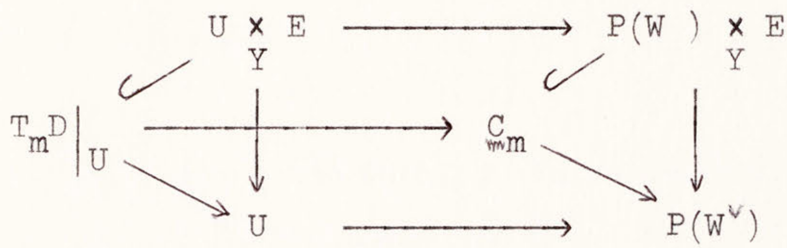
Proof. Set $V_m = (q_{2,1})_*M(m)$. Set $W = (q_{2,1})_*(M\otimes\mathcal{O}_E(m))$, where E denotes the exceptional divisor of $Y\{2\}$. There is a natural exact sequence of locally free sheaves,

$$0 \longrightarrow V_{m+1} \longrightarrow V_m \longrightarrow W \longrightarrow 0 \quad .$$

Set $U = P(V_m^\vee) - P(V_{m+1}^\vee)$. There is a smooth surjective map of Y-schemes, (4.1.2)

$$U \longrightarrow P(W^\vee) \quad .$$

Let D denote the universal divisor of M . One can easily check that $T_m D\big|_U$ is the pullback of the universal divisor $\underset{\sim}{C}$ of $M \otimes \mathcal{O}_E(m)$,



Consequently (2.2.3), we have

$$J(\underline{n};T_m D\big|_U) = U \underset{P(W^\vee)}{\times} J(\underline{n};C) \quad .$$

Since $E \to Y$ is a family of lines, by the Principle of Exchange we get

$$(R^1 q_{\hat{0}})_*(q_{\hat{0}}^* M \otimes \mathcal{O}_E(m) \otimes I(\underline{n}\Delta)) = 0 \ .$$

Therefore, by (4.2.3, (2)), $J(\underline{n};\underset{\sim}{C})$ is regularly embedded in $P(W^\vee) \times_Y E[s]$ with codimension Σn_i . Hence $J(\underline{n};T_nD|_U)$ is regularly embedded in $U \times_Y E[s]$ with the same codimension.

It remains to consider the restriction of $T(m;\underline{n};M)$ over $J_{m+1} = P(V^\vee_{m+1})$. Here, we have that $(T_mD)|_{J_{m+1}}$ is equal to $J_{m+1} \times_Y E[s]$, which has codimension $\binom{m+2}{2} - \binom{m+1}{2} = m+1$. Since each component of $J(\underline{n};T_mD)$ has codimension $\leq \Sigma n_i < m+1$, it follows that $J(\underline{n};T_mD)$ has the correct codimension. Since $J_{m+1} \times_Y E[s]$ is Cohen-Macaulay if Z is so, therefore $J(\underline{n};T_mD)$ is regularly embedded.

<u>(9.5) Corollary</u>. Suppose Y is a surface and let S denote the complete system of M .

(1) There exists an open dense subset G' of the Grassmannian of subsystems of dimension

$$d = \binom{m+1}{2} - 2 - s + \sum_{1}^{s} n_i$$

such that, for each S' in G', $T(m;\underline{n};S')$ is finite and S' is m-generic.

(2) For each linear system S' of the dimension d (as above) and such that $T(m;\underline{n};S')$ is finite and S' is m-generic, the degree of the associated zero cycle is the degree of the (s+2)<u>nd</u> Chern class of

$$\mathcal{E}_Y(m;M) + M\mathcal{E}_{E/Y}(\underline{n};\mathcal{O}_E(m)) \qquad \text{in} \quad K^\cdot(E[s]) .$$

Proof. The first assertion is an immediate consequence of the theorem on the transversality of a general translate (applied to $T(m;\underline{n};S) \to S$).

To prove (2), we compute the class of $T = T(m;\underline{n};S)$. Set $J' = J(m;S')$. By (8.3.4), we may write

$$[J'] = c_{top}(\mathcal{E}_Y(m;M) \otimes \mathcal{O}_{S'}(1)) \quad \text{in} \quad A(S' \times Y) .$$

Since J' is Cohen-Macaulay (in fact a l.c.i. in $S' \times Y$), so is $E' = J' \times_Y E[s]$. Since T is the scheme of zeros of a section of $\mathcal{E}(m;\underline{n};L)\big|_{E'}$ $(L = M \otimes \mathcal{O}_{S'}(1))$ and has the right codimension, therefore T is regularly embedded in E'. Hence we have,

$$[T] = c_{top}(\mathcal{E}(m;\underline{n};L)\big|_{E'}) \cap [E'] \quad \text{in} \quad A.(E') .$$

Using the projection formula and the formula for $[E']$ in $A(S' \times Y)$ derived from that for $[J']$ above, we arrive at

$$[T] = c_{top}(\mathcal{O}_{S'}(1) \otimes \mathcal{J}) \quad \text{in} \quad A(S' \times E[s]) ,$$

where we put for short $\mathcal{J} = \mathcal{E}_Y(m;M) + M\mathcal{E}_{E/Y}(\underline{n};\mathcal{O}_E(m))$.

Using standard properties of Chern classes, we finally find that the push down of [T] to A(E[s]) is indeed $c_{s+2}(\mathcal{J})$. Since taking the degree of a zero cycle commutes with push down, the assertion (2) is proven.

(9.5.1) Remark. With the notation of (2) of the Corollary, we do not know whether $T = T(m,\underline{n};S')$ finite implies S' is m-generic. In fact, the question is whether J(m;S') can have too big dimension and T be empty. For, if T is non empty, one can easily show J(m;S') must be of the right dimension.

(9.6) Formulary and examples. We list first a few formulas. The notation is this:

Y = smooth, projective variety.

M = invertible $\mathcal{O}_Y$-module or its first Chern class.

$\Omega = \Omega^1_{Y/k}$; $K = c_1\Omega$; $\chi = c_2\Omega$.

$\mathcal{E}(m) = \mathcal{E}_Y(m;M)$; $c_\bullet\mathcal{E}(m)$ = total Chern class.

(1) dim Y = 1 : $c_\bullet(\mathcal{E}(m)) = 1 + mM + \binom{m}{2}K$.

(2) dim Y = 2 : $c_\bullet(\mathcal{E}(m)) = 1 + \binom{m+1}{2}M + \binom{m+1}{3}K + {}$

$$+ 2\binom{m+2}{4}(m-1)MK + 3\binom{m+2}{4}M^2 + \binom{m+2}{4}\chi + \frac{5}{3}m\binom{m+2}{5}K^2 \quad .$$

(3) $Y = P^2$; $M = \mathcal{O}(d)$; $h = c_1\mathcal{O}(1)$

$$c(\mathcal{E}(m)) = 1 + [d\binom{m+1}{2} - 3\binom{m+1}{3}]h + [15m\binom{m+2}{5} + (3(d^2+1) - 6d(m-1))\binom{m+2}{4}]h^2$$

(4) Chern classes of $\mathcal{E}_{P/Y}(n;N)$ for a P^1-bundle $P = P(F) \to Y$.

F = rank-2, locally free $\mathcal{O}_Y$-module; N = invertible $\mathcal{O}_P$-module.

$$\omega = \Omega^1_{P/Y} = (\Lambda^2 F) \otimes \mathcal{O}_P(-2) \ ; \quad \epsilon = c_1\mathcal{O}_P(1) \ ;$$

$$\epsilon^2 = (c_1F)\epsilon - c_2(F) \ ; \quad \omega = c_1(F) - 2\epsilon$$

$$\begin{aligned} c(\mathcal{E}_{P/Y}(n;N)) &= (1+N)(1+N+\omega) \ \ldots \ (1+N+(n-1)\omega) \\ &= 1 + nN + \binom{n}{2}\omega + \binom{n}{2}N^2 + [3\binom{n}{3} + \binom{n}{2}]N\omega \\ &+ [3\binom{n}{4} + 2\binom{n}{3}]\omega^2 + \binom{n}{3}N^3 \\ &+ [15\binom{n}{6} + 20\binom{n}{5} + 6\binom{n}{4}]\omega^3 + [6\binom{n}{4} + \binom{n}{3}]N^2\omega \\ &+ [15\binom{n}{5} + 14\binom{n}{4} + 2\binom{n}{3}]N\omega^2 + \binom{n}{4}N^4 \\ &+ (1465\binom{n}{8} + 210\binom{n}{7} + 130\binom{n}{6} + 24\binom{n}{5}]\omega^4 \\ &+ [105\binom{n}{7} + 165\binom{n}{6} + 70\binom{n}{5} + 6\binom{n}{4}]N\omega^3 \\ &+ [10\binom{n}{5} + 6\binom{n}{4}]N^3\omega \\ &+ [45\binom{n}{6} + 50\binom{n}{5} + 11\binom{n}{4}]N^2\omega^2 + \text{higher order.} \end{aligned}$$

One computes the coefficients by solving difference equations.

(9.5.1) Examples:

(i) Cusps of a general net on a surface. We must compute the third Chern class

$$z = c_3(\mathcal{E}_Y(2;M) + M\mathcal{E}_{E/Y}(2;\mathcal{O}_E(2))).$$

That is the term of degree 3 in

$$(1 + 3M + K + 2MK + 3M^2 + \chi)[1 + 2(2\epsilon + M) + \omega + (2\epsilon + M)^2 + (2\epsilon + M)\omega].$$

Recalling $E = P(\Omega)$, we get

$$\omega = K - 2\epsilon \quad \text{and} \quad \epsilon^2 = K\epsilon - \chi \quad .$$

Thus, we have

$$\begin{aligned} z &= (3M + K)(4\epsilon^2 + 4\epsilon M + 2\epsilon K - 4\epsilon^2 - 2\epsilon M) + (2MK + 3M^2 + \chi)(2\epsilon) \\ &= 2\epsilon(6M^2 + 6MK + K^2 + \chi) \quad . \end{aligned}$$

Pushing down to Y , yields

$$2(6M^2 + 6MK + K^2 + \chi)$$

(cf. [L], p. 19; [E], p. 537).

(ii) <u>Curves of a web possessing two double points, one of which is cuspidal</u>.

We start with our surface Y and a general web S (= 3 dimensional linear system). Form $Z = J(2;S) \subset X = S \times Y$. Then look at the restriction $B \to Z$ of $X\{2\} = S \times Y\{2\}$ over Z, together with the 2-virtual transform D of the universal divisor of S. Set $J = J(2;D)$. Finally, take $T = T(2;2;D)$. This gives us the cuspidal points of D, which is what we were after. The regularity of T (i.e., $\dim J(2;D) = 1$ and T finite) is assured by (9.4) (with $Y = B$ and $\widetilde{M} = M(2)$ in place of M, and $m = 2$). The class of T is computed as follows.

Set $\widetilde{E} = P(\Omega^1_{Y\{2\}/Y})$. We have, by (9.3,(2)),

$$[T] = c_5(\mathcal{E}_{B/Z}(2;\mathcal{O}_S(1)\otimes\widetilde{M}) + \mathcal{E}_{\widetilde{E}_{B/B}}(2;\mathcal{O}_S(1)\otimes\widetilde{M}\otimes\mathcal{O}_{\widetilde{E}_B}(2))) \cap [\widetilde{E}_B] \quad \text{in } A.[\widetilde{E}_B] .$$

$$= c_8(\mathcal{E}_{X/S}(2;\mathcal{O}_S(1)\otimes M) + \mathcal{E}_{X\{2\}/X}(2;\mathcal{O}_S(1)\otimes\widetilde{M}) + \mathcal{E}_{\widetilde{E}_{X/X}}(2;\mathcal{O}_S(1)\otimes\widetilde{M}\otimes\mathcal{O}_{\widetilde{E}_X}(2))) \quad \text{in } A[\widetilde{E}_X] .$$

Pushing down to $\widetilde{E}$, we get the class

$$z = c_5(\underbrace{\mathcal{E}_Y(2;M) + \mathcal{E}_{Y\{2\}/Y}(2;\widetilde{M})}_{A} + \underbrace{\mathcal{E}_{E/Y}(2;\widetilde{M}\otimes\mathcal{O}_{\widetilde{E}}(2))}_{B}) \quad \text{in} \quad A[\widetilde{E}] .$$

$$= c_4(A)c_1(B) + c_3(A)c_2(B) \quad (\text{because } \dim Y\{2\} = 4 \text{ and } \operatorname{rank} B = 2).$$

We have (putting $\widetilde{e} = c_1\mathcal{O}_{\widetilde{E}}(1)$, $\widetilde{K} = c_1\Omega'_{Y\{2\}/Y}$)

$$c_1(B) = 2(\widetilde{M} + 2\widetilde{e}) + \widetilde{K} - 2\widetilde{e} = 2\widetilde{M} + \widetilde{K} + 2\widetilde{e}$$

$$c_2(B) = (\widetilde{M} + 2\widetilde{e})^2 + (\widetilde{M} + 2\widetilde{e})(\widetilde{K} - 2\widetilde{e}) \qquad (\text{by } (9.6), (4))$$

$$= 2(\widetilde{M} + \widetilde{K})\widetilde{e} + \widetilde{M}^2 + \widetilde{M}\widetilde{K} .$$

Pushing down z to $A[Y\{2\}]$, we find the class

$$\omega = 2c_4(A) + 2(\widetilde{M} + \widetilde{K})c_3(A) .$$

Note that $|c_4(A)|$ was computed in (8.5.4, (ii)).
Set $E = P(\Omega^1_Y)$ (= exceptional divisor in $Y\{2\}$) , and recall

$$\widetilde{M} = M_2 - 2E$$

$$\widetilde{K} = K_2 + E \qquad (\text{by } 8.5.4, (ii), a).$$

One finds

$$c_3(A) = (K+3M)_1(\chi+2KM+3M^2)_2 + (3M)_2(2KM)_1$$
$$- E(5\chi+4K^2+36KM+50M^2) + E^2(28M+13K) .$$

(The indices mean pullback via $q_{2,i}$.)

Hence, pushing down ω to $Y \times Y$ and computing degrees, we get

$$|\omega| = 2\{|c_4(A)| + |K^2+4KM+3M^2||\chi+2KM+3M^2|$$
$$- |5\chi+30K^2+105KM+78M^2|\}$$
$$= 2\{|\chi+2KM+3M^2|^2 + |K^2+4KM+3M^2||\chi+2KM+3M^2|$$
$$- |12\chi+36K^2+144KM+120M^2|\} .$$

Computing for $Y = P^2$ and $M = \mathcal{O}_{P^2}(3)$ one finds zero. This can be checked directly by analyzing the possible degenerations of a cubic. The only ones with a double point and a cusp are the unions of double lines with another line. But these form a family of dimension 4, which can, therefore, be safely avoided by a general web.

Computing for Y = surface in P^3 of degree n and $M = \mathcal{O}_Y(1)$, substituting the values for χ , etc. from page104, one finds, lo and behold, precisely the number

$$4n(n-2)(n-3)(n^3+3n-16)$$

of [Salmon], formula (4) of p. 292.

NOTATIONS

(The numbers refer to pages.)

$u(t)$: restriction of u to fiber over t; 13.

$Z_S(u)$: scheme of zeros of u in S; 13.

$Z(u)$: 13.

$A^{\vee}$: dual Module.

$X_S[t]$: t-fold cartesian product/S; 22.

$X[t]$: same as above.

$\underline{m}$: sequence of pos. integers $m_1, \ldots, m_t$.

$p_i, q_i, p_{\hat{i}}, q_{\hat{i}}$: projections of cartesian pdct. onto or omitting i^{th} factor.

Δ_X:diagonal of $X[t]$.

Δ: Δ_X.

Δ_{ij}:pullback of Δvia projection onto i,j factors.

$m\Delta$: 22.

$\underline{m}\Delta$:22.

$J(\underline{m};D)$: 23, 75.

$\mathcal{E}_{X/S}(\underline{m};L)$:24.

$\mathcal{E}(\underline{m};L)$: short for the above.

$\mathcal{E}(\underline{m})$: " " " " .

$K^{\cdot}(X)$: Grothendieck ring of loc. free sheaves...

$J(\underline{m};S)$, $J(\underline{m};M)$: 34, 76.

$A(X)$: rational equivalence ring.

$|Z|$: degree of zero cycle.

$|\underline{m};M|$: 43.

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BIOGRAPHICAL NOTE

Israel Vainsencher was born in Recife, Brazil, in 1948, second of four offsprings of the happy union of Clara and Ghers Vainsencher. He married Semira Adler in 1967, and entered the Catholic University of Rio de Janeiro the same year. Another girl, Marta, came into their lives in 1972. The enlarged family was to spend most of the ensuing five years in Cambridge, Massachusetts. In 1975 Marta got a sister, Katia. In 1976 Semira received a Master's degree in Education at Boston University. The whole family will be living for the next few years at the Boa Viagem beach, back in Recife, where Israel has a position at the Federal University.