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# On the power of randomization in network interdiction

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## Abstract

In this paper, we introduce the *randomized network interdiction* problem that allows the interdictor to use randomness to select arcs to be removed. We model the problem in two different ways: *arc-based* and *path-based* formulations, depending on whether flows are defined on arcs or paths, respectively. We present insights into the modeling power, complexity, and approximability of both formulations.

*Keywords:* network flows, interdiction, game theory, approximation

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## 1. Introduction

Network flows have applications in a wide variety of contexts (see, e.g., [1]). In some applications, it is useful to consider the perspective of someone who wants to restrict flows in a network. For example, law enforcement wants to inhibit the flow of illegal drugs. Water management experts want to control flows to avoid floods. Health agencies need to protect against contagion. Here, it is important to consider the problem of limiting flows in the network from the perspective of an *interdictor*, who is capable of limiting capacity in arcs or eliminating arcs. Such problems have been applied in many application areas such as military planning [22], controlling infections in a hospital [3], controlling floods [16], protecting critical infrastructures [13, 19], and drug interdiction [20].

Motivated by the above mentioned applications, *network interdiction problems* have been well studied in the literature (see, e.g., [4, 7, 9, 10, 11, 17, 18, 23]). In this paper, we focus on the basic model of network interdiction, where the interdiction of an arc requires exactly one unit of resource: a *flow player* attempts to maximize the amount of material transported through a capacitated network, while an *interdictor* tries to limit the flow player's achievable value by interdicting a certain number, say  $\Gamma$ , of arcs. This problem is also known as the  $\Gamma$ -most vital arcs problem (see, e.g., [16]). Wollmer [15] presents a polynomial time algorithm for solving this problem on planar graphs. On general networks, Wood [23] and Phillips [14] independently show that the problem is strongly NP-complete. Burch *et al.* [6] develop approximation algorithms for general instances of the network interdiction problem. In particular, they consider the case where each arc has a removal cost and its capacity can be reduced partially or completely, and there is a limited budget to attack the network and reduce the arc capacities. They provide a polynomial-time algorithm, based on a linear relaxation of an integer optimization formulation, that leads to either an approximation or pseudo-approximation result for the resulting problem.

Network interdiction can be viewed as a game between the interdictor and the flow player. This problem assumes the interdictor moves first and then the flow player determines a maximum flow in the remaining network. A closely related problem arises when a flow must be routed before arcs are removed. In this case, the flow player might be interested to find solutions which are robust against any failure of arcs. Aneja *et al.* [2] study this problem in a path-based formulation and show that the resulting problem is solvable in polynomial-time for the special case of  $\Gamma = 1$ . This problem was further expanded to an arc-based formulation by Bertsimas *et al.* [5], who introduce the concepts of robust and adaptive maximum flows. They establish structural and computational results for both the robust and adaptive maximum flow problems and their corresponding minimum cut problems.

*Our contribution.* The network interdiction problem addresses a minimax objective against a flow player, which selects adaptively a flow after observing the removed arcs. This problem requires the interdictor to choose a specific *pure strategy*. We propose a new modeling framework that permits the interdictor to use randomness to choose arcs. More precisely, the interdictor assigns a probability to each pure strategy and selects a pure strategy randomly according to these probabilities. We refer to the resulting problem as the *randomized network interdiction problem*. This provides a more realistic model for various applications such as protecting critical infrastructures against terrorism or enemy's attacks. We also consider a further modification that requires the flow player to send flow on paths, rather than the more typical arc-based model. We present results on the modeling power, complexity, and approximability of both arc-based and path-based formulations. In particular, we prove that  $Z_{NI}/Z_{RNI} \leq \Gamma + 1$ ,  $Z_{NI}/Z_{RNI}^{Path} \leq \Gamma + 1$ ,  $Z_{RNI}/Z_{RNI}^{Path} \leq \Gamma$ , where  $Z_{NI}$ ,  $Z_{RNI}$ , and  $Z_{RNI}^{Path}$  are the optimal values of the network interdiction problem and its randomized versions in arc-based and path-based formulations, respectively. We also show that these bounds are tight. Further, we provide a

$(\Gamma + 1)$ -approximation for  $Z_{\text{NI}}$ , a  $\Gamma$ -approximation for  $Z_{\text{RNI}}$ , and a  $(1 + \lfloor \Gamma/2 \rfloor \cdot \lceil \Gamma/2 \rceil / (\Gamma + 1))$ -approximation for  $Z_{\text{RNI}}^{\text{Path}}$ .

## 2. Network Interdiction

Let  $G = (V, E)$  be a directed graph with *node set*  $V$  and *arc set*  $E$ . Each arc  $e \in E$  has a *capacity*  $u_e \in \mathbb{R}_+$  setting an upper bound on the amount of flow on arc  $e$ . There are two specific nodes, a *source*  $s$  and a *sink*  $t$ .  $W$  denote an arc  $e$  from a node  $v$  to a node  $w$  by  $e := (v, w)$ . We use  $\delta^+(v) := \{(v, w) \in E \mid w \in V\}$  and  $\delta^-(v) := \{(w, v) \in E \mid w \in V\}$  to denote the sets of arcs leaving node  $v$  and entering node  $v$ , respectively. We assume without loss of generality that there are no arcs into  $s$  and no arcs out of  $t$ , that is,  $\delta^-(s) = \delta^+(t) = \emptyset$ .

### 2.1. Arc-based formulation

An *s-t-flow* (or simply a *flow*)  $x$  is a function  $x : E \rightarrow \mathbb{R}_+$  which assigns a nonnegative value to each arc so that  $x_e \leq u_e$  for each  $e \in E$ , and in addition for each node  $v \in V \setminus \{s, t\}$ , the following *flow conservation constraint* holds:

$$\sum_{e \in \delta^-(v)} x_e - \sum_{e \in \delta^+(v)} x_e = 0.$$

We refer to  $x_e$  as the *flow* on arc  $e$ . We denote the set of all *s-t-flows* by  $\mathcal{X}$ . The *value*  $\text{Val}(x)$  of an *s-t flow*  $x$  is the net flow into  $t$ , that is,  $\text{Val}(x) := \sum_{e \in \delta^-(t)} x_e$ . In the *maximum flow problem* (also referred to as the *nominal problem*), we seek an *s-t flow*  $x$  with maximum value  $\text{Val}(x)$ .

We next assume that there is an interdictor, who wants to reduce the capacity of the network. Suppose that the interdictor is able to eliminate  $\Gamma$  ( $1 \leq \Gamma \leq |E|$ ) arcs in the network. The *network interdiction problem* is to find the  $\Gamma$  arcs whose removal from the network minimizes the maximum amount of flow that can be sent to the sink. To formulate this problem, we let

$$\Omega := \left\{ \mu = (\mu_e)_{e \in E} \in \{0, 1\}^{|E|} \mid \sum_{e \in E} \mu_e = \Gamma \right\}$$

denote the set of all possible scenarios, that is, the set of all subsets of  $\Gamma$  arcs. The binary variable  $\mu_e$  indicates whether or not arc  $e$  is to be removed, depending on whether  $\mu_e = 1$  or  $\mu_e = 0$ , respectively. Given  $\mu \in \Omega$ , we denote by  $E(\mu) := \{e \in E \mid \mu_e = 1\}$  the set of removed arcs and by  $F(\mu) := \{e \in E \mid \mu_e = 0\}$  the set of available arcs after removing the arcs in the scenario  $\mu$ . We also denote by  $G(\mu) = (V, F(\mu))$  a network with arc set  $F(\mu)$ .

The network interdiction problem is formulated as

$$\begin{aligned} Z_{\text{NI}} := \min_{\mu \in \Omega} \max \quad & \text{Val}(x) \\ \text{s.t.} \quad & x \in \mathcal{X}, \\ & x_e = 0 \quad \forall e \in E(\mu). \end{aligned} \quad (1)$$

This problem determines the interdictor's best choice, assuming the flow player is in a position to select a maximum flow after observing the removed arcs. In many applications, the flow player has to make a decision before the interdictor

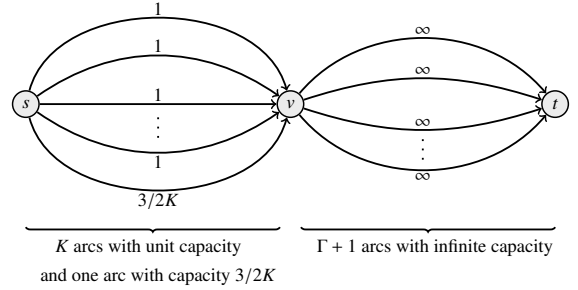


Figure 1: Illustration of the difference between the network interdiction problem and maximum adaptive flow problem. The numbers on the arcs indicate the capacities. We have  $Z_{\text{NI}} = K + 1 - \Gamma$ , while  $Z_{\text{ADP}} = \frac{5K}{2(\Gamma+1)}$ .

selects her strategy. Here, the flow player might be interested in those solutions that are robust against any possible scenario. This leads to the following problem, referred to as the *adaptive maximum flow problem*:

$$Z_{\text{ADP}} := \max_{x \in \mathcal{X}} \min_{\mu \in \Omega} f(\mu, x), \quad (2)$$

where  $f(\mu, x)$  is the maximum amount of flow that the flow player can push through the network with respect to the flow  $x$  if scenario  $\mu$  is selected. Mathematically, the function  $f$  is given by

$$\begin{aligned} f(\mu, x) := \max \quad & \text{Val}(y) \\ \text{s.t.} \quad & y \in \mathcal{X}, \\ & 0 \leq y_e \leq x_e \quad \forall e \in F(\mu) \\ & y_e = 0 \quad \forall e \in E(\mu). \end{aligned} \quad (3)$$

This problem is introduced by Bertsimas *et al.* [5], who establish structural properties and complexity results for the problem. In particular, they show that the adaptive maximum flow problem is NP-hard using a reduction from the network interdiction problem.

Note that  $Z_{\text{ADP}} \leq Z_{\text{NI}}$ . This follows from the fact that Problem (1) is equivalent to

$$Z_{\text{NI}} = \min_{\mu \in \Omega} \max_{x \in \mathcal{X}} f(\mu, x). \quad (4)$$

To compare the difference between the network interdiction problem and the adaptive maximum flow problem, we consider a network with three nodes  $s$ ,  $v$ , and  $t$  as shown in Figure 1. There are  $K$  arcs with unit capacity and one arc with capacity  $3/2K$  from  $s$  to  $t$  and there are  $\Gamma + 1$  arcs with infinite capacity from  $v$  to  $t$ . Let  $\Gamma \geq 2$  and  $K$  be enough large. It is easy to see that  $Z_{\text{NI}} = K + 1 - \Gamma$ , while  $Z_{\text{ADP}} = \frac{5K}{2(\Gamma+1)}$ . Hence,  $Z_{\text{NI}}/Z_{\text{ADP}} = \frac{2(\Gamma+1)(K+1-\Gamma)}{5K}$ , and the ratio becomes close to  $2(\Gamma + 1)/5$  when  $K$  gets large. An interesting question is: How large can  $Z_{\text{NI}}/Z_{\text{ADP}}$  be in general? We will show later that this ratio is bounded by  $\Gamma + 1$  and this bound is tight.

### 2.2. Path-based formulation

So far, we have considered flows in an arc-based formulation. We next focus on an alternative formulation of flows, in which

the flow player must specify paths on which to route the material. This leads to a different model for the adaptive problem.

Let  $\mathcal{P}$  denote the set of all  $s$ - $t$ -paths (i.e., paths from  $s$  to  $t$ ). For  $P \in \mathcal{P}$ , we write  $e \in P$  to indicate that arc  $e \in E$  lies on  $P$ . An  $s$ - $t$ -(path-based) flow is a function  $x : \mathcal{P} \rightarrow \mathbb{R}_+$  that assigns a nonnegative value to each path so that the total flow on each arc does not exceed the capacity of the arc, that is,

$$\sum_{P \in \mathcal{P}: e \in P} x_P \leq u_e \quad \forall e \in E.$$

The value of  $x$  is the sum of the flows on the paths, i.e.,  $\text{Val}(x) = \sum_{P \in \mathcal{P}} x_P$ . We use  $\mathcal{X}_P$  to denote the set of all  $s$ - $t$ -path-based flows.

Notice that the flow on path  $P$  cannot reach the sink if some arc in  $P$  is removed. In particular, if the interdictor selects the strategy  $\mu$  and the flow player chooses a flow  $x \in \mathcal{X}_P$ , then the amount of flow that can reach the sink is given by

$$g(\mu, x) := \sum_{P \in \mathcal{P}} \max \left\{ 0, 1 - \sum_{e \in P} \mu_e \right\} x_P$$

This function differs from the arc-based function  $f$ , defined in Equation (3), because flows are not permitted in this version to be routed. The value  $g(\mu, x)$  gives the amount of flow that can reach the sink if the arcs in  $\mu$  are removed. We point out that if no arc in a path  $P$  is removed, then  $\max \{0, 1 - \sum_{e \in P} \mu_e\} = 1$  and the flow on path  $P$  is counted in computing the value  $g(\mu, x)$ . Otherwise, we will have  $\max \{0, 1 - \sum_{e \in P} \mu_e\} = 0$ , and then the flow on path  $P$  does not contribute to the value  $g(\mu, x)$ .

We now present an alternative formulation of the network interdiction problem as follows:

$$Z_{\text{NI}}^{\text{Path}} := \min_{\mu \in \Omega} \max_{x \in \mathcal{X}_P} g(\mu, x). \quad (5)$$

We point out that  $Z_{\text{NI}}^{\text{Path}} = Z_{\text{NI}}$ . We next consider the case where the flow player has to choose a flow up front before the interdictor chooses her strategy. In this situation, the flow player addresses the following problem:

$$Z_{\text{ADP}}^{\text{Path}} := \max_{x \in \mathcal{X}_P} \min_{\mu \in \Omega} g(\mu, x). \quad (6)$$

This problem is introduced by Aneja *et al.* [2], who study the case where only one arc is permitted to be removed. They show that the problem is solvable in polynomial time in this special case. Later, Du and Chandrasekaran [8] claim that the problem is NP-hard if two arcs can be removed. Unfortunately, Matuschke *et al.* [12] find out that the proof that is given in [8] is wrong, and therefore the complexity of the problem is open for  $\Gamma > 1$ .

Problems (4) and (5) are equivalent; that is,  $Z_{\text{NI}} = Z_{\text{NI}}^{\text{Path}}$ , since the interdictor first chooses arcs to be removed and then the flow player solves a maximum flow problem in the remaining network. But this situation becomes different for the adaptive problem, and in general  $Z_{\text{ADP}} \neq Z_{\text{ADP}}^{\text{Path}}$ . To see the difference between the arc-based and path-based formulations, we refer to the network in Figure 1. In this network,  $Z_{\text{ADP}}^{\text{Path}} = \frac{2K}{\Gamma+1}$  and  $Z_{\text{ADP}} = \frac{5K}{2(\Gamma+1)}$  for  $\Gamma \geq 2$ . We notice that  $Z_{\text{ADP}} = K$  when  $\Gamma = 1$ . Thus,  $Z_{\text{ADP}}/Z_{\text{ADP}}^{\text{Path}} = \frac{5}{4}$  for  $\Gamma \geq 2$ . We will show that this ratio is bounded by  $\Gamma$  and this bound is tight.

### 3. Randomized network interdiction

Network interdiction can be viewed as a two-person zero-sum game between an interdictor and a flow player. The set of (pure) strategies for the interdictor is given by the scenario set  $\Omega$ . The set of (pure) strategies for the flow player is given by the feasible set  $\mathcal{X}$ . If the interdictor chooses the pure strategy  $\mu \in \Omega$  and the flow player chooses the pure strategy  $x \in \mathcal{X}$ , then  $f(\mu, x)$  is the payoff of the game. In the network interdiction problem, the interdictor goes first and determines  $\Gamma$  arcs to be removed. The flow player observes the set of removed arcs and determines a flow to be sent through the remaining network. In this case, the flow player has complete knowledge of the interdictor's behavior. Our goal is to make the interdictor more powerful and make the flow player weaker. To achieve this, we allow the interdictor to use randomness to decide which strategy to play. More precisely, the interdictor assigns a probability to each pure strategy, and then randomly selects a pure strategy according to the probabilities. The flow player does not see the interdictor's strategy, but observes a probability distribution of how the interdictor decides to select arcs. In what follows, we formally define the randomized network interdiction problem. We first focus on the arc-based formulation of flows and then turn our attention to flows on paths, instead of arcs.

A *mixed* (or *randomized*) strategy over  $\Omega$  is given by a probability distribution  $\alpha : \Omega \rightarrow [0, 1]$ , where  $\alpha(\mu)$  is the probability that strategy  $\mu$  is selected by the interdictor. We denote the set of all mixed strategies over  $\Omega$  by  $\Delta(\Omega)$ . We extend the payoff function to mixed strategies by defining

$$f(\alpha, x) := \sum_{\mu \in \Omega} \alpha(\mu) f(\mu, x) \quad \forall \alpha \in \Delta(\Omega), x \in \mathcal{X}.$$

The value  $f(\alpha, x)$  represents the *expected* payoff of the game if the interdictor chooses a mixed strategy  $\alpha \in \Delta(\Omega)$  and the flow player selects a pure strategy  $x \in \mathcal{X}$ .

Given a mixed strategy  $\alpha$ , the flow player aims to find a flow with maximum expected value. The interdictor wishes to choose a mixed strategy to minimize this value. Therefore, the interdictor deals with the following problem:

$$Z_{\text{RNI}} := \min_{\alpha \in \Delta(\Omega)} \max_{x \in \mathcal{X}} f(\alpha, x). \quad (7)$$

We refer to this problem as the *randomized network interdiction* problem.

**Theorem 1.**  $Z_{\text{RNI}} = Z_{\text{ADP}}$ .

*Proof.* The basic idea is to allow the flow player to select a flow randomly. It is well known from game theory (see, e.g., [21]) that if both players select their strategies randomly, then there exists an equilibrium; that is, no matter which player selects her strategy first, no one has an incentive to change her mixed strategy. Notice that the set of pure strategies for the flow player is an infinite set. Here, a mixed strategy is given by a finite distribution over  $\mathcal{X}$ . In fact, a *random* strategy over  $\mathcal{X}$  is a probability distribution  $\beta : \mathcal{X} \rightarrow [0, 1]$  with finite support, that is, it only assigns a non-zero value to a finite number of  $s$ - $t$ -flows. The

value  $\beta(x)$  gives the probability that the flow  $x$  is selected. We denote the set of all mixed strategies over  $\mathcal{X}$  by  $\Delta(\mathcal{X})$ .

We extend the payoff function  $f$  to mixed strategies for both players by defining

$$f(\alpha, \beta) := \sum_{\mu \in \Omega} \sum_{x \in \mathcal{X}; \beta(x) > 0} \alpha(\mu) \beta(x) f(\mu, x) \quad \forall \alpha \in \Delta(\Omega), \beta \in \Delta(\mathcal{X}).$$

The value  $f(\alpha, \beta)$  gives the expected payoff of the game if the interdictor chooses a mixed strategy  $\alpha$  and the flow player selects a mixed strategy  $\beta$ . If the interdictor chooses a pure strategy  $\mu$  and the flow player chooses a mixed strategy  $\beta$ , we denote the expected payoff by  $f(\mu, \beta) := \sum_{x \in \mathcal{X}; \beta(x) > 0} \beta(x) f(\mu, x)$ .

The result now follows from the following observations:

$$Z_{RNI} = \min_{\alpha \in \Delta(\Omega)} \max_{x \in \mathcal{X}} f(\alpha, x) = \min_{\alpha \in \Delta(\Omega)} \max_{\beta \in \Delta(\mathcal{X})} f(\alpha, \beta) \quad (8)$$

$$= \max_{\beta \in \Delta(\mathcal{X})} \min_{\alpha \in \Delta(\Omega)} f(\alpha, \beta) \quad (9)$$

$$= \max_{\beta \in \Delta(\mathcal{X})} \min_{\mu \in \Omega} f(\mu, \beta) \quad (10)$$

$$= \max_{x \in \mathcal{X}} \min_{\mu \in \Omega} f(\mu, x) = Z_{ADP}. \quad (11)$$

The second equality in Equation (8) holds since the payoff function  $f(\alpha, x)$  is concave in  $x$  and the pure strategy set  $\mathcal{X}$  is convex.

The equality in Equation (9) follows from the well-known Wald's Minimax Theorem [21] due to the fact that pure strategy set  $\Omega$  is finite.

Furthermore, the equality in Equation (10) holds since

$$\min_{\alpha \in \Delta(\Omega)} f(\alpha, \beta) = \min_{\mu \in \Omega} f(\mu, \beta)$$

for a fixed  $\beta \in \Delta(\mathcal{X})$  due to the fact that  $\Omega$  is a finite set.

It remains to prove the validity of the first equality in Equation (11). For a fixed  $x \in \mathcal{X}$ , we define  $AVal(x) := \min_{\mu \in \Omega} f(\mu, x)$ . This function is concave in  $x$ . Hence, the first equality in Equation (11) holds. This completes the proof of the theorem.  $\square$

Theorem 1 shows that randomization permits the interdictor to perform as well as when she has perfect knowledge of the flow player's choice. On the other hand, it follows from the proof of the theorem that the randomization does not help the flow player, as the randomization of a flow is just a flow.

We next consider the path-based formulation of network interdiction and assume that the interdictor uses randomness to select arcs to be deleted. This leads to the following problem, referred to as the *randomized network interdiction* problem in the *path-based formulation*:

$$Z_{RNI}^{Path} := \min_{\alpha \in \Delta(\Omega)} \max_{x \in \mathcal{X}_P} \sum_{\mu \in \Omega} \alpha(\mu) g(\mu, x). \quad (12)$$

**Theorem 2.**  $Z_{RNI}^{Path} = Z_{ADP}^{Path}$ .

*Proof.* The flow player does not benefit by choosing a mixed strategy because the payoff function  $g(\mu, x)$  is linear in  $x$  and the set of pure strategies  $\mathcal{X}_P$  is convex. The proof now follows in a similar way as in the proof of Theorem 1.  $\square$

## 4. Complexity results

In this section, we investigate computational complexity of the randomized network interdiction problem. By Theorems 1 and 2, we know  $Z_{RNI} = Z_{ADP}$  and  $Z_{RNI}^{Path} = Z_{ADP}^{Path}$ , respectively. Thus, complexity results for computing  $Z_{ADP}$  and  $Z_{ADP}^{Path}$  carry over  $Z_{RNI}$  and  $Z_{RNI}^{Path}$ , respectively.

Bertsimas *et al.* [5] formulate the adaptive maximum flow problem as a linear optimization problem with exponentially many variables and constraints. When  $\Gamma$  is fixed, the linear optimization problem has polynomial many variables and constraints, and thus can be solved in polynomial time. But, in general, they show that the adaptive maximum flow problem is strongly NP-hard by a reduction from the network interdiction problem. Thus, we have the following theorem.

**Theorem 3.** For a fixed  $\Gamma$ , the value  $Z_{RNI}$  can be computed in polynomial-time as a linear optimization problem. For a general  $\Gamma$ , it is strongly NP-hard to compute  $Z_{RNI}$ .

The next result (proved by Aneja *et al.* [2]) shows that that computing  $Z_{ADP}^{Path}$  is solvable in polynomial-time when  $\Gamma = 1$ .

**Theorem 4.** If  $\Gamma = 1$ , then  $Z_{RNI} = Z_{RNI}^{Path}$  and an optimal mixed strategy can be computed in polynomial-time.

For  $\Gamma > 1$ , the complexity of the path-based randomized network interdiction problem is open. We note that the NP-hard proof in [8] for computing  $Z_{ADP}^{Path}$  for the case that the interdictor is able to remove only two arcs is wrong [12].

## 5. On the power of randomization

In this section, we provide tight bounds on the ratio of the optimal value of the network interdiction problem to that of randomized versions. In particular, our main result is the following theorem.

**Theorem 5.** It is always true that

$$\frac{Z_{NI}}{Z_{RNI}} \leq \Gamma + 1, \quad (13)$$

$$\frac{Z_{NI}}{Z_{RNI}^{Path}} \leq \Gamma + 1, \quad (14)$$

$$\frac{Z_{RNI}}{Z_{RNI}^{Path}} \leq \Gamma. \quad (15)$$

and these bounds are tight.

To prove this theorem, we require several lemmas. The core of our analysis is based on the following parametric linear optimization problem:

$$\begin{aligned} Z_{LO}(\theta) := \max \quad & Val(x) - \Gamma \theta \\ \text{s.t.} \quad & \sum_{e \in \delta^+(v)} x_e - \sum_{e \in \delta^-(v)} x_e = 0, \quad \forall v \in V \setminus \{s, t\}, \\ & 0 \leq x_e \leq u_e, \quad \forall e \in E, \\ & x_e \leq \theta, \quad \forall e \in E. \end{aligned} \quad (16)$$

We let  $Z_{LO} := \max_{\theta \geq 0} Z_{LO}(\theta)$  and refer to the latter problem as the LO model. It is worth to point out that this model is the dual of a linear relaxation of an integer optimization formulation examined by Burch *et al.* [6] to obtain approximations for network interdiction.

We first show that the optimal value of the LO model gives a lower bound on  $Z_{RNI}^{Path}$ ,  $Z_{RNI}$ , and  $Z_{NI}$ .

**Lemma 1.** *We have*

$$Z_{LO} \leq Z_{RNI}^{Path} \leq Z_{RNI} \leq Z_{NI}. \quad (17)$$

*Proof.* By Theorems 1 and 2, we know  $Z_{RNI} = Z_{ADP}$  and  $Z_{RNI}^{Path} = Z_{ADP}^{Path}$ , respectively. Thus, it suffices to show that

$$Z_{LO} \leq Z_{ADP}^{Path} \leq Z_{ADP} \leq Z_{NI}.$$

Here, the first inequality from the right is immediate since

$$Z_{ADP} = \max_{x \in \mathcal{X}} \min_{\mu \in \Omega} f(\mu, x) \leq \min_{\mu \in \Omega} \max_{x \in \mathcal{X}} f(\mu, x) = Z_{NI}.$$

The second inequality is intuitively straightforward because the flow player in Problem (6) is more restricted than Problem (2).

Therefore, it remains to prove  $Z_{LO} \leq Z_{ADP}^{Path}$ . We assume that the optimal value of the LO model is strictly positive since otherwise the statement is trivial. Let  $(x^*, \theta^*)$  be an optimal solution for the LO model and  $(x_p^*)_{p \in \mathcal{P}}$  be an arbitrary path-decomposition of  $x^*$ . It is easy to see that

$$Z_{LO} = \text{Val}(x^*) - \Gamma\theta^* \leq \min_{\mu \in \Omega} g(\mu, x^*). \quad (18)$$

This completes the proof of the lemma.  $\square$

In what follows, we exploit structural properties of the LO model that are needed for the proof of Theorem 5. We first give some basic definitions and notation. An  $s$ - $t$ -cut is defined as a subset  $S \subseteq V$  of nodes with  $s \in S$  and  $t \in V \setminus S$ . The *capacity*  $\text{Cap}(S)$  of  $S$  is defined as the sum of the capacities of the arcs going from  $S$  to  $V \setminus S$ , that is,  $\text{Cap}(S) := \sum_{e \in \delta^+(S)} u_e$ . Here and subsequently,  $\delta^+(S)$  denotes the set of arcs  $e = (v, w)$  with  $v \in S$  and  $w \in V \setminus S$ . We use  $\mathcal{S}$  to denote the set of all  $s$ - $t$ -cuts. For a given value  $\theta \geq 0$ , we let  $u_e(\theta) := \min\{u_e, \theta\}$ , and we let  $\text{Cap}(S, \theta) := \sum_{e \in \delta^+(S)} u_e(\theta)$  denote the capacity of the cut with respect to the arc capacities  $u(\theta)$ . We let  $A(S, \theta)$  denote the set of all arcs  $e \in \delta^+(S)$  with  $\theta \leq u_e$ , and we let  $B(S, \theta)$  denote the set of all arcs  $e \in \delta^+(S)$  with  $\theta < u_e$ .

**Lemma 2.** *Suppose that  $(x^*, \theta^*)$  is an optimal solution to the LO model with maximum value  $\theta^*$  (i.e., if there are multiple optimal solutions, the one with the largest value  $\theta^*$  is selected).*

(i) *There exists an  $s$ - $t$ -cut  $S'$  so that*

$$\text{Val}(x^*) = \text{Cap}(S', \theta^*) \quad \text{and} \quad |A(S', \theta^*)| \geq \Gamma.$$

(ii) *There exists an  $s$ - $t$ -cut  $S''$  so that*

$$\text{Val}(x^*) = \text{Cap}(S'', \theta^*) \quad \text{and} \quad |B(S'', \theta^*)| < \Gamma.$$

*Proof.* For each  $\epsilon > 0$ , we have

$$Z_{LO}(\theta^*) \geq Z_{LO}(\theta^* - \epsilon), \quad (19)$$

$$Z_{LO}(\theta^*) > Z_{LO}(\theta^* + \epsilon), \quad (20)$$

since  $(x^*, \theta^*)$  is an optimal solution with maximum value  $\theta^*$ . In addition, there exists an  $s$ - $t$ -cut  $S'$  which is a minimum cut with respect to arc capacities  $u(\theta^*)$  and arc capacities  $u(\theta^* - \epsilon)$  for a very small  $\epsilon > 0$ . More precisely, it is enough to choose  $\epsilon$  as follows:

$$\epsilon := \frac{1}{|E|} \min_{S \in \mathcal{S}} \{\text{Cap}(S, \theta^*) - \text{Val}(x^*) \mid \text{Cap}(S, \theta^*) - \text{Val}(x^*) > 0\}.$$

Therefore, we can write

$$\begin{aligned} Z_{LO}(\theta^*) &= \text{Cap}(S', \theta^*) - \Gamma\theta^*, \\ Z_{LO}(\theta^* - \epsilon) &= \text{Cap}(S', \theta^* - \epsilon) - \Gamma(\theta^* - \epsilon) \\ &= \sum_{e \in S' \setminus A(S', \theta^*)} u_e + \sum_{e \in A(S', \theta^*)} (\theta^* - \epsilon) - \Gamma\theta^* + \Gamma\epsilon \\ &= \text{Cap}(S', \theta^*) - \epsilon|A(S', \theta^*)| - \Gamma\theta^* + \Gamma\epsilon. \end{aligned}$$

It then follows from Inequality (19) that  $|A(S', \theta^*)| \geq \Gamma$ .

We prove the second part of the lemma by a similar argument. There exists an  $s$ - $t$ -cut  $S''$ , which is a minimum cut with respect to arc capacities  $u(\theta^*)$  and  $u(\theta^* + \epsilon)$  for a very small  $\epsilon > 0$ . Therefore,

$$\begin{aligned} Z_{LO}(\theta^*) &= \text{Cap}(S'', \theta^*) - \Gamma\theta^*, \\ Z_{LO}(\theta^* + \epsilon) &= \text{Cap}(S'', \theta^* + \epsilon) - \Gamma(\theta^* + \epsilon) \\ &= \text{Cap}(S'', \theta^*) + \epsilon|B(S'', \theta^*)| - \Gamma\theta^* - \Gamma\epsilon. \end{aligned}$$

It now follows from Inequality (20) that  $|B(S'', \theta^*)| < \Gamma$ .  $\square$

**Lemma 3.** *Suppose that  $(x^*, \theta^*)$  is an optimal solution to the LO model with maximum value  $\theta^*$ . Then,*

$$(i) \quad Z_{NI} = Z_{LO} \text{ if } Z_{LO} < \frac{1}{\Gamma+1} \text{Val}(x^*);$$

$$(ii) \quad Z_{NI} = Z_{LO} \text{ if } Z_{LO} < \theta^*;$$

$$(iii) \quad Z_{NI} \leq \text{Val}(x^*);$$

$$(iv) \quad Z_{RNI} = Z_{LO} \text{ if } x^* \text{ is a maximum flow for the nominal problem};$$

$$(v) \quad Z_{RNI} \leq \text{Val}(x^*) - \theta^*;$$

*Proof. Part (i):* It follows from  $Z_{LO} < \frac{1}{\Gamma+1} \text{Val}(x^*)$  that  $\text{Val}(x^*) < (1 + \Gamma)\theta^*$ . In addition, by Part (i) of Lemma 2, there exists an  $s$ - $t$ -cut  $S'$  with  $\text{Val}(x^*) = \text{Cap}(S', \theta^*)$  and  $|A(S', \theta^*)| \geq \Gamma$ . Therefore,

$$\begin{aligned} (1 + \Gamma)\theta^* > \text{Val}(x^*) = \text{Cap}(S', \theta^*) &= \sum_{e \in A(S', \theta^*)} \theta^* + \sum_{e \in \delta^+(S') \setminus A(S', \theta^*)} u_e \\ &= \theta^*|A(S', \theta^*)| + \sum_{e \in \delta^+(S') \setminus A(S', \theta^*)} u_e, \end{aligned}$$

and consequently

$$|A(S', \theta^*)| = \Gamma \quad \text{and} \quad Z_{LO} = \text{Val}(x^*) - \Gamma\theta^* = \sum_{e \in \delta^+(S') \setminus A(S', \theta)} u_e.$$

If the arcs in  $A(S', \theta^*)$  are removed, then the maximum flow value in the remaining network is at most  $\sum_{e \in \delta^+(S') \setminus A(S', \theta^*)} u_e$ , which is equal to  $Z_{LO}$ . This implies that  $Z_{NI} \leq Z_{LO}$ . On the other hand, it follows from Theorem 1 that  $Z_{LO} \leq Z_{NI}$ . Hence, we must have  $Z_{LO} = Z_{NI}$ .

**Part (ii):** Notice that

$$\begin{aligned} Z_{LO} < \theta^* &\iff \text{Val}(x^*) - \Gamma\theta^* < \theta^* \\ &\iff \text{Val}(x^*) - (\Gamma + 1)\theta^* < 0 \\ &\iff (\Gamma + 1)(\text{Val}(x^*) - \Gamma\theta^*) < \text{Val}(x^*) \\ &\iff Z_{LO} < \frac{1}{\Gamma + 1} \text{Val}(x^*). \end{aligned}$$

Therefore, it follows from the previous part that  $Z_{LO} = Z_{NI}$ .

**Part (iii):** By Part (ii) of Lemma 2, there exists an  $s$ - $t$ -cut  $S''$  with  $\text{Val}(x^*) = \text{Cap}(S'', \theta)$  so that  $|B(S'', \theta^*)| < \Gamma$ . For each  $e \in \delta^+(S'') \setminus B(S'', \theta^*)$ , we have  $u_e \leq \theta^*$ . Therefore, if the arcs in  $B(S'', \theta^*)$  are removed, the maximum amount of flow that can be sent from  $s$  to  $t$  is at most  $\sum_{e \in \delta^+(S'') \setminus B(S'', \theta^*)} u_e$ . This means that  $Z_{NI} \leq \sum_{e \in \delta^+(S'') \setminus B(S'', \theta^*)} u_e$ . Hence, we can write

$$\begin{aligned} \text{Val}(x^*) = \text{Cap}(S'', \theta^*) &= \sum_{e \in B(S'', \theta^*)} \theta^* + \sum_{e \in \delta^+(S'') \setminus B(S'', \theta^*)} u_e \quad (21) \\ &\geq \theta^* |B(S'', \theta^*)| + Z_{NI}, \quad (22) \end{aligned}$$

and consequently  $Z_{NI} \leq \text{Val}(x^*)$ .

**Part (iv):** For an  $s$ - $t$ -cut  $S$  and an  $s$ - $t$ -flow  $x$ , we define

$$R(x, S) := \min_{\mu \in \Omega} \sum_{e \in \delta^+(S)} (1 - \mu_e) x_e.$$

It follows from Lemma 8 in [5] that

$$\min_{\mu \in \Omega} f(\mu, x) = \min_{S \in \mathcal{S}} R(x, S).$$

Therefore, we can write

$$Z_{ADP} = \max_{x \in \mathcal{X}} \min_{S \in \mathcal{S}} R(x, S) \leq \min_{S \in \mathcal{S}} \max_{x \in \mathcal{X}} R(x, S).$$

Furthermore, we know from Part (i) of Lemma 2 that there exists a cut  $S'$  with  $|A(S', \theta^*)| \geq \Gamma$  so that

$$\text{Val}(x^*) = \text{Cap}(S', \theta^*) = \sum_{e \in A(S', \theta^*)} \theta^* + \sum_{e \in \delta^+(S') \setminus A(S', \theta^*)} u_e.$$

Since  $x^*$  is a maximum flow for the nominal problem,  $\text{Val}(x^*)$  is the maximum amount of flow that can be pushed through the cut  $S'$ . The cut  $S'$  has  $|A(S', \theta^*)|$  arcs with capacity  $u_e \geq \theta^*$  and the remaining arcs have capacity  $u_e < \theta^*$ . Hence, the maximization of  $R(x, S')$  is attained by sending  $u_e$  units of flow on arcs  $e \in \delta^+(S') \setminus A(S', \theta^*)$  and  $\theta^*$  units on arcs  $e \in A(S', \theta^*)$  as in flow  $x^*$ . This implies that  $\max_{x \in \mathcal{X}} R(x, S') = R(x^*, S')$ , and we can write

$$\begin{aligned} Z_{ADP} &\leq \max_{x \in \mathcal{X}} R(x, S') = R(x^*, S') = \min_{\mu \in \Omega} \sum_{e \in \delta^+(S')} (1 - \mu_e) x_e^* \\ &= \text{Val}(x^*) - \Gamma\theta^* = Z_{LO}. \end{aligned}$$

Moreover, we have  $Z_{LO} \leq Z_{ADP}$  by Lemma 1. This proves  $Z_{LO} = Z_{ADP}$ . In addition, we know  $Z_{ADP} = Z_{RNI}$  because of Theorem 1. Hence, we must have  $Z_{LO} = Z_{RNI}$ .

**Part (v):** If  $x^*$  is a maximum flow for the nominal problem, then it follows from previous part that  $Z_{RNI} = Z_{LO} = \text{Val}(x^*) - \Gamma\theta^* \leq \text{Val}(x^*) - \theta^*$  and we are done. Hence, we assume that  $x^*$  is not a maximum flow. Let  $S''$  as in Lemma 2 be an  $s$ - $t$ -cut with  $\text{Val}(x^*) = \text{Cap}(S'', \theta^*)$  and  $|B(S'', \theta^*)| < \Gamma$ . We must have  $1 \leq |B(S'', \theta^*)|$  since otherwise it follows from  $\text{Val}(x^*) = \text{Cap}(S'')$  that  $x^*$  is a maximum flow. Furthermore, it follows from Inequality (21) that  $Z_{NI} \leq \text{Val}(x^*) - \theta |B(S'', \theta^*)|$ . This shows that  $Z_{NI} \leq \text{Val}(x^*) - \theta^*$  since  $|B(S'', \theta^*)| \geq 1$ .  $\square$

**Lemma 4.** *It is always true that*

$$\frac{Z_{NI}}{Z_{LO}} \leq \Gamma + 1, \quad (23)$$

$$\frac{Z_{RNI}}{Z_{LO}} \leq \Gamma. \quad (24)$$

*Proof.* Suppose that  $(x^*, \theta^*)$  is an optimal to the LO model with maximum value  $\theta^*$ . If  $Z_{LO} \geq \frac{1}{\Gamma+1} \text{Val}(x^*)$ , then  $Z_{LO} \geq \frac{1}{\Gamma+1} Z_{NI}$  because of Part (iii) of Lemma 3, and consequently Inequality (23) holds. If  $Z_{LO} < \frac{1}{\Gamma+1} \text{Val}(x^*)$ , then  $Z_{LO} = Z_{NI}$  by Part (i) of Lemma 3. This establishes Inequality (23).

We proceed to prove the validity of Inequality (24). If  $Z_{LO} < \theta^*$ , then by Part (ii) of Lemma 3 we must have  $Z_{NI} = Z_{LO}$ . This implies that  $Z_{RNI} = Z_{LO}$  since  $Z_{LO} \leq Z_{RNI} \leq Z_{NI}$  by Lemma 1, and consequently Inequality (24) holds. Thus, we assume that  $Z_{LO} \geq \theta^*$ . We can write

$$Z_{RNI} \leq \text{Val}(x^*) - \theta^* = Z_{LO} + (\Gamma - 1)\theta^* \leq \Gamma \cdot Z_{LO},$$

where the first inequality follows from Part (v) of Lemma 3 and the second inequality follows from the fact that  $Z_{LO} \geq \theta^*$ . This shows that Inequality (24) always holds.  $\square$

*Proof of Theorem 5.* The validity of the bounds in (13), (14), and (15) immediately follows from Lemmas 1 and 4. We next provide two examples to show these bounds are all tight.

In the first example, we consider a network with three nodes  $s$ ,  $v$ , and  $t$ , and parallel arcs from  $s$  to  $v$  and  $v$  to  $t$ . There are  $K$  parallel arcs with unit capacity from  $s$  to  $v$  and  $\Gamma + 1$  parallel arcs with infinite capacity from  $v$  to  $t$  (see the network in Figure 2(a)). We let  $K \geq \Gamma + 1$ . In this network, we have  $Z_{NI} = K - \Gamma$ , whereas  $Z_{RNI} = Z_{RNI}^{\text{Path}} = K/(\Gamma + 1)$ . Therefore,  $\frac{Z_{NI}}{Z_{RNI}} = \frac{Z_{NI}}{Z_{RNI}^{\text{Path}}} = \frac{(\Gamma+1)(K-\Gamma)}{K}$ . When  $K$  gets enough large, the bound becomes enough close to  $\Gamma + 1$ . This shows the bounds in Inequalities (13) and (14) are tight.

In the second example, we consider a network with four nodes  $s$ ,  $v$ ,  $w$ , and  $t$  as shown in Figure 2(b). There are  $K$  parallel arcs with unit capacity from  $s$  to  $v$  and  $\Gamma$  parallel arcs with infinite capacity from  $v$  to  $t$ . In addition, there is one arc from  $s$  to  $v$  with capacity  $\Gamma \cdot K$ , one arc from  $w$  to  $v$  with capacity  $K$ , and one arc from  $w$  to  $t$  with infinite capacity. In this network, we have  $Z_{RNI} = K - \Gamma + 1$ , whereas  $Z_{RNI}^{\text{Path}} = K/\Gamma$ . Therefore,  $\frac{Z_{RNI}}{Z_{RNI}^{\text{Path}}} = \frac{\Gamma(K-\Gamma+1)}{K}$ . When  $K$  gets enough large, the bound becomes enough close to  $\Gamma$ . This shows the bound in Inequality (15) is tight.  $\square$

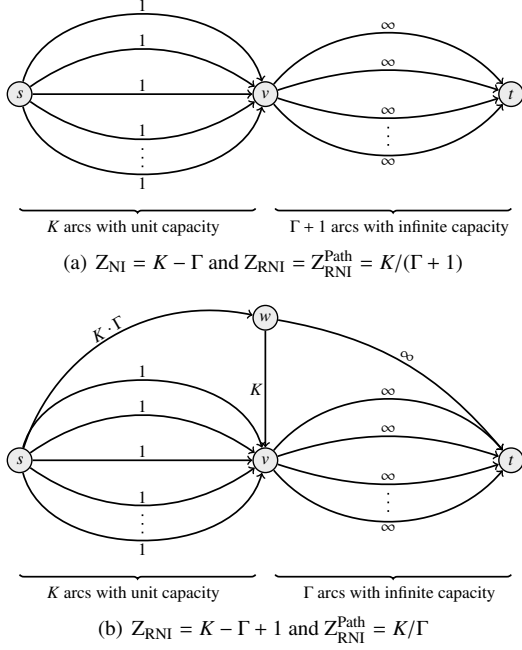


Figure 2: Networks for the proof of Theorem 5. The numbers on the arcs indicate the capacities.

## 6. Approximation bounds

As mentioned before, a pseudoapproximation algorithm is developed by Burch *et al.* [6] for the network interdiction problem. For a given  $\epsilon > 0$ , their algorithm generates either a  $(1 + \epsilon)$ -approximation or a solution whose value is at most that of an optimal solution for network interdiction, but it requires to delete  $\Gamma(1 + 1/\epsilon)$  arcs. When  $\epsilon = \Gamma$ , their algorithms returns a  $(\Gamma + 1)$ -approximation or a solution whose value is at most  $Z_{\text{NI}}$ , but requires to delete  $\Gamma(1 + 1/\Gamma) = \Gamma + 1$  arcs. However, it is not known which is *a priori*. If the latter case happens, the solution is not technically feasible. In contrast, it follows from Lemma 4 that the optimal value of the LO model is a  $(\Gamma + 1)$ -approximation for  $Z_{\text{NI}}$  and a  $\Gamma$ -approximation for  $Z_{\text{RNI}}$ . We next show that the optimal value of the LO model also provides a good approximation for  $Z_{\text{RNI}}^{\text{Path}}$ .

**Theorem 6.** *We have*

$$\frac{Z_{\text{RNI}}^{\text{Path}}}{Z_{\text{LO}}} \leq 1 + \frac{\lceil \Gamma/2 \rceil \cdot \lceil \Gamma/2 \rceil}{\Gamma + 1}, \quad (25)$$

and this bound is tight.

*Proof.* By Theorem 2, it is enough to show that

$$\frac{Z_{\text{ADP}}^{\text{Path}}}{Z_{\text{LO}}} \leq 1 + \frac{\lceil \Gamma/2 \rceil \cdot \lceil \Gamma/2 \rceil}{\Gamma + 1}. \quad (26)$$

Suppose that  $(x^*, \theta^*)$  is an optimal to the LO model with maximum value  $\theta^*$ . By Parts (i) and Parts (ii) of Lemma 2, there are  $s$ - $t$ -cuts  $S'$  and  $S''$  so that  $|A(S', \theta^*)| \geq \Gamma$ ,  $|B(S'', \theta^*)| < \Gamma$ , and

$$\text{Val}(x^*) = \sum_{e \in \delta^+(S')} u_e(\theta^*) = \sum_{e \in A(S', \theta^*)} \theta^* + \sum_{e \in \delta^+(S') \setminus A(S', \theta^*)} u_e, \quad (27)$$

$$\text{Val}(x^*) = \sum_{e \in \delta^+(S'')} u_e(\theta^*) = \sum_{e \in B(S'', \theta^*)} \theta^* + \sum_{e \in \delta^+(S'') \setminus B(S'', \theta^*)} u_e. \quad (28)$$

We let  $a$  and  $b$  denote the number of arcs in  $A(S', \theta^*)$  and  $B(S'', \theta^*)$ , respectively. Further, we let  $p := \sum_{e \in \delta^+(S') \setminus A(S', \theta^*)} u_e$  and  $q := \sum_{e \in \delta^+(S'') \setminus B(S'', \theta^*)} u_e$ . Then, we can rewrite (27) and (28), respectively, as

$$\text{Val}(x^*) = a\theta^* + p, \quad \text{and} \quad \text{Val}(x^*) = b\theta^* + q. \quad (29)$$

Note that  $a \geq \Gamma$ . If  $a = \Gamma$ , then

$$Z_{\text{LO}} = \text{Val}(x^*) - \Gamma\theta^* = p.$$

Further, we have  $Z_{\text{NI}} \leq p$  since if the interdicator deletes the  $\Gamma$  arcs in  $A(S', \theta^*)$ , then the capacity of the cut  $S'$  will be  $p$ , and consequently the flow player can push at most  $p$  units of flow from  $s$  to  $t$ . This implies that  $Z_{\text{ADP}}^{\text{Path}} \leq p$  since  $Z_{\text{ADP}}^{\text{Path}} \leq Z_{\text{NI}}$ . Hence, Inequality (26) holds for  $a = \Gamma$ . In what follows, we assume that  $a \geq \Gamma + 1$  and derive an upper bound on  $Z_{\text{ADP}}^{\text{Path}}$  by computing the maximum amount of flow that can be sent from  $s$  to  $t$  under the following restriction on the set  $\Omega$ : The interdicator deletes all the arcs in  $B(S'', \theta^*)$  and is restricted to select the remaining  $\Gamma - b$  arcs from  $A(S', \theta^*)$ .

We define

$$\Omega' := \left\{ \mu \in \{0, 1\}^{|E|} \mid \mu_e = 0 \quad \forall e \in E \setminus (A(S', \theta^*) \cup B(S'', \theta^*)) \right. \\ \left. \mu_e = 1 \quad \forall e \in B(S'', \theta^*), \quad \sum_{e \in A(S', \theta^*)} \mu_e = \Gamma \right\}.$$

It is obvious that

$$Z_{\text{ADP}}^{\text{Path}} \leq Z := \max_{x \in \mathcal{X}_p} \min_{\mu \in \Omega'} g(\mu, x).$$

since  $\Omega' \subseteq \Omega$ . We next provide an upper bound on  $Z$ .

After deleting the arcs in  $B(S'', \theta^*)$ , the cut  $S''$  has a capacity of  $q$ . It follows from (29) that

$$q = \text{Val}(x^*) - b\theta^* = p + (a - b)\theta^*.$$

Therefore, at most  $p + (a - b)\theta^*$  units of flow that can be sent through the arcs in the cut  $S'$ .

The flow player should send as much flow as possible through the arcs in the  $\delta^+(S') \setminus A(S', \theta^*)$  since these arcs are not subject to removal. Let  $r$  be the maximum amount of flow that can be sent through the arcs in  $\delta^+(S') \setminus A(S', \theta^*)$  among all  $s$ - $t$ -flows. Note that  $r \leq p$ . Then, at most  $p - r + (a - b)\theta^*$  units of flow that can be sent through the arcs in  $A(S', \theta^*)$ . On the other hand, the interdicator is allowed to remove  $\Gamma - b$  arcs from this set. Thus, at most

$$\frac{(p - r + (a - b)\theta^*)(a - \Gamma + b)}{a}$$

units of flow can be pushed through the arcs in  $A(S', \theta^*)$  using a path-based formulation. This upper bound is obtained from the fact that in the best case the flow player distributes  $(p - r + (a - b)\theta^*)(a - \Gamma + b)$  units of flow equally among the  $a$  arcs in  $A(S', \theta^*)$  and the interdicator is only allowed to delete  $\Gamma - b$  arcs in this set.

Following the above discussion, in total, the flow player can send at most  $r + \frac{(p - r + (a - b)\theta^*)(a - \Gamma + b)}{a}$  units of flow from the source



s to the sink  $t$  even if the interdicator is restricted to remove the  $b$  arcs from  $B(S'', \theta^*)$  and the remaining  $\Gamma - b$  arcs from  $A(S', \theta^*)$ . This implies that

$$\begin{aligned} Z_{\text{ADP}}^{\text{Path}} &\leq Z \leq r + \frac{(p - r + (a - b)\theta^*)(a - \Gamma + b)}{a} \\ &\leq p + \frac{(a - \Gamma + b)(a - b)\theta^*}{a}. \end{aligned} \quad (30)$$

where Inequality (30) is satisfied as equality when  $p = r$  and as strict inequality when  $r \neq p$  since  $b < \Gamma$ .

On the other hand, we have

$$Z_{\text{LO}} = \text{Val}(x^*) - \Gamma\theta^* = p + (a - \Gamma)\theta^*.$$

Therefore,

$$\frac{Z_{\text{ADP}}^{\text{Path}}}{Z_{\text{LO}}} \leq \frac{p + \frac{(a - \Gamma + b)(a - b)\theta^*}{a}}{p + (a - \Gamma)\theta^*} \leq \frac{(a - \Gamma + b)(a - b)}{(a - \Gamma)a}. \quad (31)$$

It is easy to verify that the right hand side of Inequality (31) attains its maximum when  $a = \Gamma + 1$  and  $b = \lfloor \Gamma/2 \rfloor$ . By substituting  $a = \Gamma + 1$  and  $b = \lfloor \Gamma/2 \rfloor$  in the the right hand side of Inequality (31), we obtain

$$\frac{Z_{\text{ADP}}^{\text{Path}}}{Z_{\text{LO}}} \leq 1 + \frac{\lfloor \Gamma/2 \rfloor \cdot \lfloor \Gamma/2 \rfloor}{\Gamma + 1}.$$

This establishes the validity of Inequality (26).

We next show that the bound is tight. Consider a network with three nodes  $s$ ,  $v$ , and  $t$ . There are  $\Gamma$  parallel arcs with unit capacity from  $s$  to  $v$  and  $\Gamma + 1$  parallel arcs with infinite capacity from  $v$  to  $t$ . In addition, there are  $\lfloor \Gamma/2 \rfloor$  parallel arcs from  $s$  to  $v$  with capacity  $K$ . In this network, we have  $Z_{\text{ADP}}^{\text{Path}} = \frac{(\lfloor \Gamma/2 \rfloor + 1)K}{\Gamma + 1}$ , whereas  $Z_{\text{LO}} = \frac{K}{\lfloor \Gamma/2 \rfloor + 1}$ . Therefore,

$$\frac{Z_{\text{ADP}}^{\text{Path}}}{Z_{\text{LO}}} = \frac{(\lfloor \Gamma/2 \rfloor + 1) \cdot \lfloor \Gamma/2 \rfloor + 1}{\Gamma + 1} = 1 + \frac{\lfloor \Gamma/2 \rfloor \cdot \lfloor \Gamma/2 \rfloor}{\Gamma + 1}.$$

This shows that the bound in Inequality (26) is tight.  $\square$

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