Dynamic Global Game Coordination Risks

by

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Abstract

We study a dynamic global game model of coordination risk among a group of agents invested in a project over a finite time horizon. Once every round each agent gets a private noisy signal of the health of the project. The agent must then decide to continue participating in the project in anticipation of receiving a full return on their investment upon the projects successful completion or foreclose on the project early and receive a reduced payout. This model extends the debt global game models of coordination risk by Morris and Shin to a multi period model similar to Dynamic Global Game Models of Angeletos, Hellwig and Pavan. This extended model allows us to study coordination risk over a finite time horizon and introduce new information structures of the the agents invested in the project.

Our main results come from extensions to the dynamic global game model. First, we model public signals of the health of the project between all agents invested in it and show under certain conditions that positive public information of the project can decrease the projects chances of success. Second, we allow for agents to receive private and public noisy signals of past actions, introducing herd behaviour. We then show how this herd behaviour can increase the fragility of the system to external shocks of public or private information concerning the fundamentals of the project. Last, we introduce feedback into the reserve price that agents receive upon leaving the project early. We show that this feedback can be a positive or negative force on the health of the project. We conclude with an interpretation of the model to real world bond yields and numerical examples.

Thesis Supervisor: Mardavij Roozbehani
Title: Principle Research Scientist

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Chapter 1

Introduction

1.1 Problem Motivation

Game theory provides a rich modelling framework to explain real world interactions between agents. A classic example is the prisoner's dilemma, where two criminals are caught stealing from a bank but the authorities do not have enough proof of their crime, only that they were trespassing [13]. The authorities propose a plea bargain to each of the criminals: if you confess, you can walk away with no jail time but if you do not confess and your partner does, you will be much worse off. If both confess they will be in jail for a longer sentence, but if neither do, they will both only be charged with the minor crime of trespassing. The Nash Equilibrium for each criminal with these pay-offs is for each to confess, since they cannot coordinate or enforce what the other criminal will say to the authorities. The decision matrix looks like the below.

	Confess	Lie
Confess	$(-7,-7)$	$(-10,0)$
Lie	$(0,-10))$	$(-1,-1)$

We imagine that before these two criminals were caught "trespassing" in the bank, they faced a different coordination problem on whether to rob the bank. The two bank robbers already had a plan of action for the night of robbery, but it involved trusting that the other robber would show up that night to rob the bank. If only one robber showed up, all of his efforts would be in vain. Each robber only wants to go forward with the plan if they believe that the bank has enough cash inside its vault to make the heist worthwhile. The pay-off to each robber, assuming that there is a cost of 1 associated with robbing the bank can be seen below.

	Rob	Bail
Rob	$(V_\$ - 1, V_\$ - 1)$	$(-1, 0)$
Bail	$(0, -1))$	$(0, 0)$

This game can have two Nash Equilibriums and no way to break the tie between them without introducing agent preferences. We introduce noise into the underlying state of the system to break the tie, while introducing the possibility of making a sub-optimal decision. We make the amount of money in the vault, $V_\$$, unknown to each robber and sampled from a normal random variable. Each robber gets their own private estimate of this signal, $\hat{V_{i\$}} = V_\$ + \eta_i$ where η_i is some i.i.d. noise. This new, but similar, problem set up introduces the concept of a global game. A global game introduces uncertainty about the underlying state of the system a single Nash Equilibrium can be chosen versus a system with perfect information and multiple Nash Equilibria.

Global Games can be used to study various phenomena that have a coordination problem between agents trying to make a decision such as currency attacks against a central bank [22], depositors starting bank runs [10], or bond holders in a project [2]. Many models of these phenomena assume that each agent has perfect information regarding the fundamentals of the system, and categorize phase transitions that occur when the fundamentals change, as well as how fragile the system is to outside shocks. Introducing a global games framework allows to measure not just the phase transitions in these systems but also how much of the phase transition is due to the inability to coordinate between agents versus worsening fundamentals of the system.

1.2 Contributions

In this section we summarize the main contributions of this thesis to coordination risk in dynamic global game models and financial simulation of debt contract prices. The main results are stated in the Dynamic Model Extensions section while numerical examples of the phenomena discussed can be found in the appendix.

The dynamic global game literature has various models of games of repeated attacks against a regime [4, 5, 6, 9, 14]. Our main contribution is to extend the dynamic global game framework to model games of coordination risk between investors in a project over multiple rounds. In this formulation, we have a continuum of creditors invested in a single risky project. At each round, the creditors have the choice to continue investing in the project or to foreclose and take a reserve price for leaving the project before completion. If too many investors foreclose early, the project does not have enough liquidity on hand and fails. The risk to the project becomes the failure to coordinate among the projects creditor's; if all of them were able to share their private information and make decisions jointly no early investor withdrawals would occur unless the project was guaranteed to fail. In studying and interpreting the information structure of this new type of dynamic global game, we submit three main contributions:

Positive information regarding the state of the project can induce investor withdrawals. This result is counter-intuitive to how positive information affects an asset's price and doesn't occur in standard debt models found in the finance literature [2, 21]. The information structure that causes this to occur can be found in other dynamic global game models [6, 18]. Intuitively, the increased certainty of the public information among all investors in the project can overpower their private information estimate of the project, which among all the investors may have been higher than the public information signal. In this scenario, the public information emboldens investors to foreclose on the project.

Private and public feedback of the state of the project can make the project more likely to fail. Financial and game theory models incorporate feedback to show path dependency [6, 7, 19]. We extend these feedback information structures into our dynamic model of coordination risk to come to our conclusion of the increased fragility of the system in two ways. First, a positive feedback loop can occur for sufficiently strong public signals or if the information gained in the private feedback is sufficiently strong. Second, a system with feedback becomes more sensitive to outside shocks to its fundamentals than a system without feedback.

We introduce the concept of liquidity in the reserve price payouts if an investor of the project decides to foreclose on the project. This type of feedback is common in financial models [10, 16] but uncommon in the global games literature [20]. This is accomplished by allowing the reserve price to be a function of the number of investors that will foreclose on the project in any given round, introducing positive or negative feedback. In the positive feedback case we assume that the project incurs a liquidation penalty that grows in proportion with number of investors foreclosing on a given round. For the negative feedback case we assume there is a robust market that the project can access for liquidity so large withdrawals are easier to make.

This formulation of a dynamic global game model lends itself to interpreting bond interest rates over time and how the information structure of debt contracts affect the interest rates of the bond. In the real world data comparison section we discuss how to interpret the model in terms of bond yields and compare to other models in the literature. We also discuss possible ways to calibrate a dynamic global game model to a set of data points.

1.3 Related Literature

The global game model on which the dynamic model is built upon was introduced by Carrlsson and Damme [8]. From this base model, a rich assortment of strategic behavior involving coordination risk has been explained involving currency crises, bank runs, debt holders, and political organization [20]. We start by an overview of the literature in global game currency crises that introduce various ways to interact with the underlying state of the global game.

Metz models public information in self fulfilling currency crises to show how agents behavior differ in a game with and without public information and how in most cases there will always be multiple equilibria possible with public information shocks [17]. Further characterizations of the multiplicity in global games with public and private information is studied in Hellwig [20]. The model most similar to this thesis can be found in Dynamic Global games of Regime change, which provides a global game structure of attacking a regime over time [6]. This paper provides key insights the the structure of equilibrium in a global game that is played over repeated rounds with different information structures given to the players. These information structures determine whether there are multiple equilibriums or a unique one, whether the regime gets attacked on any given round, and whether the regime will get overthrown at some round of the repeated dynamic global game. As the game is played over multiple rounds, different amounts of information flow to the participants in different scenarios. Dasgupta takes a different approach and studies a global game structure where agents try to coordinate to make the same decision optimally over a finite horizon [9].

An approach with positive feedback built into the base model is taken by Dilp Abrea and Markus Brunnermeier where they show how asset bubbles can exist within a rational market framework instead of currency attacks [1]. Their model has a group of investors buying a stock and causing a price increase because they believe "its different this time." There is another group of informed investors that over time start to short the stock due to its poor fundamental value. This model shows coordination risk in terms of leaving an inflated bubble regime and how this bubble can form in terms of positive feedback. We contrast this to periods of negative and positive feedback when the reserve price in our dynamic global game model can vary to the size of price movements. The study of how at each stage in the dynamic global game a sequential equilibrium must be reached was studied by Kreps and Wilson [15].

1.4 Organization

Section 2 starts with an introduction to a one round static global game model describing coordination risk in a debt project. Section 3 extends the one round scenario to a simple dynamic model that covers a debt project where investors can withdraw from the project over a finite time horizon N. Section 4 extends the information structure of the simple dynamic model to include public information, publicly known shocks to the health of the project, private and public feedback of the current state of the project and liquidity in terms of the reserve price of the project. Section 5 discusses the interpretation of the model in terms of real world bond prices. Section 6 concludes the thesis. In the appendixes there are numerical examples of the model.

Chapter 2

Static Debt Model

We model a debt contract between a single owner of a risky project or bond requesting debt from a continuum of small investors as a global game over two rounds, similar to the Morris and Shin financial model for debt [21]. We expand on their initial model by allowing positive or negative feedback in the reserve price of the debt contract, which is the amount of cash an investor receives if they decide to pull money from the project before it is completed.

2.1 Overview

Here we explain the mechanics of a single round debt contract global game. We start with a project that takes two rounds to realize any returns to its investors. The project's owner issues debt to a number of small investors that have the option to pull out their money at a reserve price before the projects completion at the end of round two where full repayment of the debt with interest is issued if the project is successful.

In round one, each investor receives a private, noisy signal of how much liquidity is available for early withdrawals of the project. Each investor decides to continue financing the project into round two or foreclose on the loan at the end of round one based on the value of this signal and shared prior information regarding the risk of the project. Any signal where liquidity is estimated to be less than 0 means that the project is predicted to fail. For the project to succeed it must have on hand enough liquidity to survive early withdrawals at the end of round one from investors who, due to the noise in their signal, decide to withdraw.

In round two, the fate of the project is determined when the number of investors who pulled their money out of the project in round one is revealed as well as the amount of liquidity on hand for the project. If more investors pulled out their money that there was liquidity on hand for the project, the project fails and investors get a fraction of their expected return. If the project has enough liquidity on hand to pay the investors that pulled out early the project succeeds and only the investors that stayed in receive a full payout.

The investors decision to foreclose on the loan or continue financing the project is dependant on the private signal they receive as well as common knowledge about the viability of the project. Factors that influence the investors are the payout of the project upon success or failure, how noisy the private signal was and the prior probability distribution of the project's success.

2.2 Model Parameters

The projects success at the end of round two are based on two factors: the underlying state θ of the project and the proportion of investors that foreclosed. The underlying state θ of the project can be thought of as the amount of short term liquidity the project has on hand at the end of round two to satisfy demand for early withdrawals from the project. It is assumed that if the project survives the liquidity withdrawals it will be successful in round two. The underlying state of the project θ is modelled as a random variable sampled from a normal distribution $N(z, \frac{1}{\alpha})$ which is common knowledge to all investors before investing in the project. This distribution can be thought of as the expected amount of liquidity that projects of this type have based on historical examples or common industry models. For example, if investors are deciding to invest in a widget factory, $N(z, \frac{1}{\alpha})$ represents the amount of liquidity on hand halfway through building the factories similar to the current project. In standard practice we denote the number of investors staying in the project and not foreclosing as $\hat{\theta}$, which is always in the range $[0, 1]$.

We allow for not all of the money on hand to be available for early investor withdrawals at the end of round one. In this case we introduce $w \geq 1$ which represents a sensitivity that the debtor has to investors calling in their loans early. w can be thought of as a liquidation friction upon the project when they have to liquidate assets early. (If $w = .8$ for example, then for every dollar of liquidity the project has on hand only .8 can be used to pay off investors calling in the debt early.)

Contract payout terms for the debt financing are agreed upon before investors choose to engage in the global game. Every investor is sufficiently enticed by the estimated returns at the beginning of round one, even if they decide to pull funding for the project early after receiving their private signal. There are three different possible payouts for the investor outlined in the debt financing contract.

1. L: Full repayment of the debt with interest if the investor does not foreclose on the project at the end of round one and the project is successful.

2. $K^*(\hat{\theta})$: The reserve price, which is a monotonically increasing or decreasing function of θ. Amount received if the investor decide to foreclose at the end of round one, earned regardless of the success or failure of the project.

3. K_*: Amount received upon project failure and the investor not foreclosing on the project.

The following relation between the debt contract payouts always hold for any value of θ

$$K_* < K^*(\hat{\theta}) < L$$

The three possible payouts of this static debt game can be normalized to the following if we let $\lambda(\hat{\theta}) = (K^*(\hat{\theta}) - K_*)/(L - K_*)$,

	Project Success	Project Failure
Roll-over Loan	1	0
Foreclose Loan	$\lambda(\hat{\theta})$	$\lambda(\hat{\theta})$

If the θ sampled from this distribution is less than 0 or larger than l , the project will always fail or succeed regardless of how many investors foreclose. Additionally, if the exact value of θ is known in round one, the model would degenerate to a continuous game with one or two possible Nash Equilibrium in the interesting case where the outcome of the project is uncertain, global game dynamics come into play.

Each investor of the continuum, denoted by i, observes the noisy signal x_i

$$x_i = \theta + e_i$$

where e_i is an IID and drawn from the normal distribution $N(0, \frac{1}{\beta})$. This represents each investors private information regarding the underlying state of the project which is combined with the prior information of the distribution of θ to get each investors best estimation of the state of the project.

2.3 Equilibrium Dynamics

A strategy for an individual investor i consists of the decision to either foreclose or roll over the loan given the investors private observation of x_i. Given the continuity of the signal, we look for a Bayesian Nash Equilibrium based on a switching rule where we assume there is a single threshold $\hat{x}$ such that each investor only rolls over the loan if the signal they observe is $x \geq \hat{x}$. With this switching strategy assumption we construct a monotonic BNE solution to the global game. The intuition behind this approach is that for sufficiently low signals of x_i, always foreclosing on the loan is a strictly dominant strategy and that for sufficiently high signals of x_i, always rolling over the loan is a strictly dominant strategy.

The BNE is characterized by two variables: $\hat{x}$ determines the switching strategy for each investor and $\hat{\theta}$ is the proportion of investors in the range $[0, 1]$ that decide to foreclose early. The value of these two variables are only dependent on the common knowledge between all investors in the game, not the realization of θ for any game that is played. In this sense, the solution of $\hat{\theta}$ represents the lowest possible realized value of θ the project can have and still be successful. Using two logical constraints placed on the investors behavior we can solve for the BNE switching strategy and minimum viable $\hat{\theta}$.

First, there must exist a critical point where between the number of investors calling in the loan early due to seeing a low signal and the number of investors that see a signal in that range are equal to each other. This condition is the point where the project has exactly the right amount of liquidity to fulfill early investor withdrawals.

$$P(x_i \leq \hat{x}|\theta) = w\Phi(\sqrt{\beta}(\hat{x} - \theta))$$

The status-quo is only abandoned if $\theta \leq \hat{\theta}$, where $\hat{\theta}$ becomes

$$\hat{\theta} = w\Phi(\sqrt{\beta}(\hat{x} - \hat{\theta}))$$

Second, there must exist an investor that upon receiving their signal is indifferent (in the sense of maximizing their expected pay-off) between foreclosing on the loan and continuing to invest in the project. Intuitively, this investor must exist since we have continuum of private signals received by the investors. After normalizing the payout of the project to be between 0 and 1, the expected payout of a successful project becomes the probability that the project succeeds while the expected payout of foreclosing is always $\lambda(\hat{\theta})$.

$$P(\theta \leq \hat{\theta}|x) = 1 - \Phi((\sqrt{\beta + \alpha})(\hat{\theta} - \frac{\beta}{\beta + \alpha}\hat{x} - \frac{\alpha}{\beta + \alpha}z)) = \lambda(\hat{\theta})$$

With these two equations we can solve for a $\hat{x}$ that each investor will use to decide to stay in the game or not as well as the number of investors that will decide to foreclose early, $\hat{\theta}$.

$$\Phi^{-1}(1 - \lambda(\hat{\theta}))\sqrt{\beta + \alpha} = -\beta\hat{x} - \alpha z + (\beta + \alpha)\hat{\theta}$$

$$\frac{-\sqrt{\beta+\alpha}\Phi^{-1}(1-\lambda(\hat{\theta}))-\hat{\theta}(\beta+\alpha)+\alpha z}{\beta}=\hat{x}$$

$$\hat{\theta}=w\Phi(\frac{\sqrt{\beta+\alpha}}{\sqrt{\beta}}(\alpha z-\Phi^{-1}(1-\lambda(\hat{\theta})))-\hat{\theta}(\frac{\beta+\alpha}{\sqrt{\beta}}+1)) \tag{1}$$

The only valid values of $\hat{\theta}$ are between 0 and 1 and a bisection algorithm between $[0,1]$ can be used to solve for valid values. As long as

$$\frac{\alpha}{\sqrt{\beta}}\leq\frac{\sqrt{2\pi}}{w}$$

there is a unique solution to the above equation as found in the literature.

2.4 Limiting Behavior

2.4.1 Reserve Price Feedback

Here we discuss the limiting behavior in terms of feedback to the reserve price earned by the investor for foreclosing on the project earlier.

By allowing the reserve payout price to depend on the proportion of investors leaving, we introduce the concept of feedback based on the decision to foreclose or stay in. In the final equilibrium equation we can see the direct impact of the reserve price depending on the equilibrium of $\hat{\theta}$.

Stabilizing feedback to the project owner occurs when $\lambda(\hat{\theta}$ is monotonically decreasing as $\hat{\theta}$ decreases. This means that compared to the static case of constant λ, this feedback function will make the equilibrium solution of $\hat{\theta}$ lower, so that the project a priori has a higher chance of success. The intuition behind this behavior is that as more investors decide to leave early, the pool of liquidity split between early withdrawals gets smaller. This is acts as a deterrent to leaving early, as the reserve price you will receive will be less while the expected value of staying in the project increases. In the opposite case, where $\lambda(\hat{\theta}$ is monotonically increasing as $\hat{\theta}$ decreases, provides positive feedback to the project owner and will increase the equilibrium solution of $\hat{\theta}$.

This interplay between lambda and the health of the project can be intuitively explained by looking how equilibrium values of $\hat{\theta}$ over a range of λ and z. As λ increases for fixed z, the "cost", in terms of amount of money left on the table if the project succeeds and the investor pulled out becomes less. Since its cheaper to leave early, $\hat{\theta}$ decreases as λ increases. In the case of fixed λ and increasing z, we have the expected response of decreasing $\hat{\theta}$, which means that as the prior distribution of θ increases, the lowest minimum viable $\hat{\theta}$ decreases as pulling out of the project requires a sufficiently higher negative signal to overcome the weight of the prior distribution in the agents decision.

2.4.2 Static Limiting Behavior: Fundamental and Strategic Uncertainty

Here we discuss how to categorize the changes in $\hat{\theta}$ according to a change in the fundamental value of θ or due to increased coordination risks.

We can categorize the risk that the investors face when choosing to foreclose or roll-over their loan in terms of the information available to them. The reason behind an investors choice in the global game can be due to the fundamental uncertainty in the value of θ, the state of the project and its probability of success, or in terms of the the strategic uncertainty in not knowing what other investors will choose based on their private signals.

When there is no fundamental uncertainty in the project for the investor it means that the private information received completely reveals the state of the debtor project. By letting β approach infinity we see that the foreclosure pay-off of the game determines the limiting behavior.

$$\lim_{\beta \to \infty} \hat{\theta} = w\lambda$$

For a global game with perfect information, upon receiving their private signals investors will all make the same choice, as everyone will know the true realization of θ.

This gives the interpretation that the success of the project has a uniform expected pay-off between 0 and 1. The rate of foreclosure therefore perfectly matches the expected value of playing the game based on the pay out and not what other creditors decide to do in the game. Therefore the θ distribution, the actual distribution of outcomes in the game, only matters in the edge cases of where θ is larger than 1 or smaller than 0.

The second interesting case is what occurs in the limit of of private information while still having the unique equilibrium hold. If we look at a set of (α, β) that increases to infinity but always with the constraint that for a given constant c

$$\lim_{\beta \to \infty, \alpha \to \infty} \hat{\theta} \quad where \quad \frac{\alpha}{\sqrt{\beta}} = c \leq \frac{\sqrt{2\pi}}{w}$$

Solving the global game equations with this constraint.

$$\hat{\theta} = w\Phi(c(\hat{\theta} - u + \Phi^{-1}(\lambda)/c))$$

This shows that even in the limit of no strategic uncertainty, in terms of knowing what your fellow investors will do with their given private signal, the equilibrium is still determined by the prior mean of the θ distribution as well as the foreclosure value of the project.

2.4.3 Summary of Limiting Behavior

Here we provide two tables illustrating the effects that initial parameters have on the equilibrium solution (θ, x). The first column represents the the effect on the equilibrium parameter as that initial condition for the given row approaches its minimum possible value, while the second column is the effect on the equilibrium parameter as the initial condition for the given row approaches its maximum possible value.

Debt Global Game Initial Value Effect on θ

	Min	Max
$z = [-\infty, \infty]$	-	+
$\alpha = [0, \infty]$	$[0, 1]$	M
$\beta = [0, \infty]$	$[0, 1]$	F^*
$\lambda = [0, 1]$	-	+

Debt Global Game Initial Value Effect on x

	Min	Max
$z = [-\infty, \infty]$	-	+
$\alpha = [0, \infty]$	$[-\infty, \infty]$	M
$\beta = [0, \infty]$	$[-\infty, \infty]$	F^*
$\lambda = [0, 1]$	-	+

Where in the case of diminishing α or β, whether θ approaches $[0, 1]$ (or x approaching $[-\infty, \infty]$) is dependent on the initial conditions of u, λ. M represents that multiple solutions become possible as α grows larger, holding other variables constant. F^* represents some fixed value is approached in the limit.

Chapter 3

Dynamic Model Equilibrium

We extend the static debt model to a dynamic debt model similar to the literature of Dynamic Global Games of Regime Change [6], which consist of a series of global games over time. Each subsequent round is modelled as global game but with additional information that the project is still successful, altering the equilibrium dynamics and solution set.

3.1 Overview

Extending the motivation from the static round, we allow investors to withdraw from the project at a finite number of rounds before completion. Payouts in each round represent the value of the project at that time, while we assume the investor only cashes out of the project if they take the reserve price of a given round or stay until project completion. We again assume that if a project has enough liquidity to survive withdrawals that it is successful and all remaining investors are paid out in full.

The main difference of this model to the literature is the coordination risk is in terms of a debt model instead of an attack model. This means that the payout structure can now vary over time and that there is a finite number of rounds. The interpretation of the model solution set is also interpreted backwards, taking θ as a measure of risk in the project, so at the beginning of the dynamic game the investor has the least information and is solving for a risk estimate of the riskiest part of the project. As time progresses, depending on the parameters of the dynamic game, the project will get less riskier for the investors and as such should be able to survive larger investor withdrawals.

Another addition to the standard model in the literature is allowing the reserve price to be a function of the total number of investor withdrawals up that point. With this it is possible to introduce positive or negative feedback into the dynamic global game, which can model liquidity providing market makers or a withdrawal penalty similar to a w that worsens the health of the project upon large investor withdrawals.

3.2 Model Parameters

We expand the notation from the one round case to specify which round the game is being played. There is a finite number of rounds denoted by $t = 1, 2, 3, ...N$. The prior normal distribution of θ is unchanged from the one round case, $N(z, \frac{1}{\alpha})$. The liquidity conversion cost, represented by w in the one round case is assumed to be constant throughout the global game and always in the range $0 < w \leq 1$

Each round, every investor receives a noisy signal estimating the true value of the liquidity on hand for the project that is independent of signals in given to the investor in previous rounds and independent to the signal received by the other investors . We denote the private noisy signal received by investor i in round t as n_{it}, which is sampled as

$$n_{it} = \theta + N(0, \frac{1}{\eta_t})$$

where η_t is always positive, as new information only makes estimate of the state of the debtor project more precise. We define β_t as

$$\beta_t = \sum_{n=1}^{t=n} \eta_n$$

and note that for each round t, we the same multiplicity condition as the static round case. Due to previous information, multiple solutions are still possible, but this multiplicity condition does reduce them.

$$\frac{\alpha}{\sqrt{\beta_t}} \leq \frac{\sqrt{2\pi}}{w} \quad \forall t$$

At each round we will want to represent the best available private information that an individual investor can use to make a decision, which we denote as x_{it}. Given standard normal updating procedure we can solve for a general accumulated private information metric x_t, which is the best available information at time t

$$x_t = \frac{\beta_{t-1}x_{t-1}}{\beta_t} + n_t\frac{n_t}{\beta_t}$$
$$\beta_t = \beta_{t-1} + \eta_t$$

Now we can specify a normal distribution distribution $p_{x_t|x_{t-1}}$ that represents the best estimate of θ given all previous rounds of signals.

$$N(\frac{\beta_{t-1}x_{t-1}}{(\beta_t + \alpha)} + \frac{\alpha z}{(\beta_t + \alpha)}, \frac{1}{(\beta_t + \alpha)})$$

This shows that for each round, given that the project is still viable, future information regarding the project can be categorized as a sufficient statistic.

Similar to the one round case, there are three possible payout valuations in each round t for the investor. We assume that if an investor stays in the project, they do not receive a payout that round and only receive a final payout when the project is completed. The reserve price each round can vary based on the number of investors leaving $K^*(\hat{\theta}_{1,2,3...N})$. We assume once an investor has left the project he will not return in the base model scenario. If the project fails the investor receives $K_{*1,2,3...N}$ while if the project has survived all withdrawal rounds a sum L will be earned by the investors still invested in the project. We assume any discount factors over time are already incorporated into these values and normalize the payout each round to be between 0 and 1 as in the static case. This produces a $\lambda(\hat{\theta}_{1,2,3...N})$ time series which helps determine whether investors leave at any point during the project.

3.3 Notion of Equilibrium

Here we categorize the notion of equilibrium over multiple global game rounds. Since the dynamic game has the possibility of multiple equilibria over time that cannot be reduced with constraints on the initial conditions of the debt investment game, we allow multiple characterizations of equilibrium conditions. At the beginning of each period t we denote whether the project still has enough liquidity on hand as $P_t = 1$. If the project during a round t suffers from too many withdrawals and must terminate early, we denote this with $P_t = 0$. The options available to the investor each round are $a_{it} = \{0, 1\}$ which denote whether the investor is staying in the project or leaving early that round while $A_t = [0, 1]$ denotes the total number of investors that decided to stay invested in the project.

Equilibrium in the dynamic game is characterized by a Perfect Bayesian Equilibrium during each round of the game. Every investor has a unique switching strategy for each round t denoted by $a_t(x_t) = [0, 1]$ with $a_t = 0$ indicating the investor chose to foreclose and $a_t = 1$ the investor chose to roll over the loan. $a^t = (a_\tau)_{\tau=1}^t$ denotes a sequence of actions the investor took until period t. The switching rule $x_i t$ are i.i.d across periods conditional on θ thus any fully characterized strategy for the dynamic global game depends on whether the regime made it to that round and each investors private signal up to that point.

The switching rule each round uses $p_t(\theta; a_t)$ to denote the probability distribution that the project fails round t given all of the investors are using the same threshold strategy $a_t(x_t)$ for that round. Let $\Psi_1(\theta|x_1)$ be the c.d.f of the posterior beliefs for the investors in period 1 and for $t \geq 2$, $\Psi_t(\theta|x_t; a^{t-1})$ be the c.d.f of the posterior beliefs conditional that the project has not failed, which must be true past round one. We will later see this conditional knowledge introduces multiple equilibriums into the dynamic game.

Using this notation we characterize the equilibrium strategies of the dynamic game a^∞ if for every t

$$a_1(x_1) = argmax_{a=[0,1]}(a[\int p_1(\theta; a_1)d\Psi_1(\theta|x_1)] + [1-a]\lambda_1)$$

$$a_t(x_t) = argmax_{a=[0,1]}(a[\int p_t(\theta; a_t)d\Psi_t(\theta|x_t; a^{t-1}] + [1-a]\lambda_t$$

In the dynamic game, the first round plays out exactly the same as the static game for each investor. Following periods can be characterized by a sequence $(x_t^*, \theta_t^*)_{t=1}^{\infty})$ following a valid switching rule sequence a^t.

Valid switching rule sequences can be explained by the notion of a monotone equilibrium set of strategies for the players. These strategies are monotonic because once a proportion of investors choose to foreclose on the project but the project continues, all investors know that the project's health is at least θ_{t-1}. All further investor withdrawals will only occur if private information induces more than θ_{t-1} proportion of the investors to withdraw their money early.

Formally, the first round solution set of the dynamic global game follows from the static case, producing a solution set (x_1, θ_1). We categorize a valid solution set for all times t as set of thresholds that produce a monotone equilibrium for a given sequence $(x_t^*, \theta_t^*)_{t=1}^{\infty})$. A monotone equilibrium $\theta_t^* = (0, 1)$ and $\theta_t^* \leq \theta_{t-1}^*$ for all $t \geq 2$ such that

1. An investor only forecloses early on the project on any round t if $x_t < x_t^*$

2. The project is in place at time $t \geq 2$ iff $\theta *_t < \theta_{t-1}^*$, the number of investors staying in is larger than the minimum amount of liquidity in the project.

For the project to be successful at each subsequent round, the cut-off for what the agents assume to be the valid θ_t^* must be non increasing, as the cash on hand as the project progresses should only decrease. In the subsequent sections we describe how to find a monotonic equilibrium solution set for the dynamic game.

3.4 Equilibrium Constraint Equations

The first round of the dynamic case plays out the same as the one round game. For subsequent rounds, the two equations that determine whether the debt funding will be valid at each round are similar as the static case, with the added constraint that θ_t^* must be non increasing. Unlike the first round case, there may exist a scenario where after the project succeeds the first round, no investor receives new private information that would ever change their mind to foreclose on the project early. This makes $(\theta_t^* = 0, x_t^* = -\infty)$ a possible valid solution set for any time $t > 1$.

The first condition, that there must exist a critical point for a given (θ_t^*, x_t^*) where the proportion of investors that recall the loan early must match the number of investors who received a signal to leave early. This equation's dynamics are unchanged from the one round case.

$$\theta_t^* = w\Phi(\sqrt{\beta_t}(x_t^* - \theta_t^*)$$

The second equation, that there must exist an investor that upon receiving their signal is indifferent between rolling over the loan and foreclosing, exists with the added condition that $\theta_t^* < \theta_{t-1}^*$ if the investors choose to withdraw.

$$P(\theta_t^* \geq \hat{\theta}_t | x_t, \theta_t < \theta_{t-1}^*) = \lambda_t$$

$$\frac{P(\theta_{t-1}^* \geq \theta_t^* \geq \hat{\theta}_t | x_t)}{P(\theta_{t-1}^* \geq \theta_t | x_t)} = \lambda_t$$

$$\frac{1 - P(\theta_{t-1}^* \leq \theta_t^* | x_t) - P(\hat{\theta}_t \geq \theta_t^* | x_t)}{1 - P(\theta_{t-1}^* \leq \theta_t^* | x_t)} = \lambda_t$$

$$1 - \frac{1 - P(\hat{\theta}_t \leq \theta_t^* | x_t)}{1 - P(\theta_{t-1}^* \leq \theta_t^* | x_t)} = \lambda_t$$

$$1 - \frac{\Phi(\sqrt{\beta_t + \alpha}(\frac{\beta_t}{\beta_t+\alpha} x_t^* + \frac{\alpha}{\beta_t+\alpha} z - \theta_t^*))}{\Phi(\sqrt{\beta_t + \alpha}(\frac{\beta_t}{\beta_t+\alpha} x_t^* + \frac{\alpha}{\beta_t+\alpha} z - \theta_{t-1}^*))} = \lambda_t$$

We introduce shorthand notation for analyzing the behavior of the dynamic game after round one.

$$u(x,\theta^*,\theta^*_{-1},\beta,\alpha,z) = 1 - \frac{\Phi(\sqrt{\beta_t+\alpha}(\frac{\beta_t}{\beta_t+\alpha}x^*_t + \frac{\alpha}{\beta_t+\alpha}z - \theta^*_t))}{\Phi(\sqrt{\beta_t+\alpha}(\frac{\beta_t}{\beta_t+\alpha}x^*_t + \frac{\alpha}{\beta_t+\alpha}z - \theta^*_{t-1}))} - \lambda_t \quad if \quad \theta^* > \theta^*_{-1}$$

$$u(x,\theta^*,\theta^*_{-1},\beta,\alpha,z) = -\lambda_t \quad if \quad \theta^* \leq \theta^*_{-1}$$

$u(x,\theta^*,\theta^*_{-1},\beta,\alpha,z)$ represents the marginal pay-off that an investor i receives for any given round t given the previous information that the project is still successful.

$$X(\theta^*,\beta) = \theta^* + \frac{1}{\sqrt{\beta}}\Phi^{-1}(\theta^*)$$

$X(\theta^*,\beta)$ represents the critical point for a round where the number of investors pulling their money from the project equal the number of investors receiving signals to leave.

$$U(X(\theta^*,\beta),\theta^*,\theta^*_{-1},\beta,\alpha,z) = u(x,\theta^*,\theta^*_{-1},\beta,\alpha,z) \quad for \quad \theta^* \quad in \quad (0,1)$$

$$U(x,\theta^*,\theta^*_{-1},\beta,\alpha,z) = 0 \quad otherwise$$

$U(x,\theta^*,\theta^*_{-1},\beta,\alpha,z) = 0$ for some $((\hat{\theta_t}),\hat{x}_t)$ represents a solution set that satisfies both of the constraints outlined in the constraints section.

Continuing the notion of monotone equilibrium, a sequence $((\hat{\theta_t}),\hat{x}_t)_{t=1}^{t=N}$ is a monotone equilibrium for a given action set $(a_t([0,1]))_{t=1}^{t=N}$ if the following conditions hold:

1) During every round t of the global game, the investor chooses to stay in the project, $a_t = 1$, if their private signal is above the threshold signal for that round, $x_t > x^*_t$. Similarly, the investor chooses to leave, $a_t = 1$, if their private signal is below the threshold signal for that round, $x_t < x^*_t$.

2) The first round solution set is the solution set to the static round game. This is the θ^*_1 that solves $U(\theta^*_1,\theta^*_{-1},\beta_1,\alpha,z) = 0$ and $x^*_t = X(\theta^*_1,\beta_1)$

3) For all subsequent rounds there are two scenarios. The first is that there is no new θ_t^* solution to $U(x_t, \theta_t^*, \theta_{t-1}^*, \beta_t, \alpha, z) = 0$ and the equilibrium solution for the next round is $\theta_t^* = \theta_{t-1}^*$ and $x_t = \infty$. The second is there exists a new $0 < \theta_t < \theta_{t-1}$ that is a solution to $U(x_t, \theta_t^*, \theta_{t-1}^*, \beta_t, \alpha, z) = 0$ and $x_t^* = X(\theta_t^*, \beta_t)$

This formulation of the monotone equilibrium categorizes the actions that an individual investor takes each round given a valid solution set to the dynamic global game. Before we explain how this equilibrium can be constructed, we note some key properties $U(x_t, \theta_t^*, \theta_{t-1}^*, \beta_t, \alpha, z)$ used to analyze the solution set.

There always exist a solution set to the dynamic game. In the simplest case, there is always a solution to the first round since it is static global game. In all subsequent rounds, if no solution to $U(x_t, \theta_t^*, \theta_{t-1}^*, \beta_t, \alpha, z) = 0$ can be found, then the initial θ_1^* carries over to all subsequent rounds and the threshold for switching stays at $+\infty$.

In cases where there will be different values of θ_t than the initial round, the new values will always be monotonically decreasing but above 0. This behavior represents that the project, once under way, can only get less risky as time progresses, which means that less of a liquidity buffer should be needed for completion. This is represented in the constraint equations by the truncation of the probability in the indifference constraint.

Now we categorize $U(x_t, \theta_t^*, \theta_{t-1}^*, \beta_t, \alpha, z)$ as a non-monotonic function that has three different value regimes. For values in the range $\theta_t^* = [\theta_{t-1}^*, 1]$ the function takes the constant negative value $-\lambda_t$. This is due to a a combination of the truncation of the previous rounds beliefs making sure no new value of θ can be found as well as the indifference constraint defaulting to the opposite of λ_t. A $\theta_t^* < \theta_{t-1}^*$, $U(x_t, \theta_t^*, \theta_{t-1}^*, \beta_t, \alpha, z)$ begins to increase. This can be seen from the decreasing numerator in $U()$ that is not initially counterbalanced by the lower threshold in x_t^*. The intuition being that a lower threshold can only occur if more agents switch. As $\theta_t^* \to 0$ however, this behavior switches and starts to approach the marginal pay-off, $1 - \lambda_t$. This can be seen that the numerator of the indifference constraint approaches 0 faster for small values of x_t^* than the denominator since $\theta_t^* < \theta^* t_{t-1}$. Combining these observations we note that $U()$ will have a single peak and that this peak may or may not be above 0 or occur in the range $[0, \theta^*_{t-1}]$.

The last observation on $U()$ we observe is in the limit of perfect information. While we only model scenarios with a finite number of rounds, we can allow almost perfect private information in the last round of the global game.

$$lim_{\beta \to \infty} U(\theta^*_N, \theta^*_{N-1}, \beta_t, \alpha, z) = 1 - \lambda_N - \theta^*_N$$

We now let $1-\lambda_N$ act as a lower bound to the final value that θ_t can take in dynamic global game, which can only occur when investors receive almost perfect information about the underlying state of the project. This means given a $\theta^*_{t-1} > 1 - \lambda_N$ with a large enough new information signal β_t, a new equilibrium solution set could be reached. Similarly the converse, if $\theta^*_{t-1} < 1 - \lambda_N$, no amount of new information received by an investor can induce a new solution set.

3.5 Constructing an Equilibrium

To construct an equilibrium for any given dynamic game, we first solve the one round case. Given this solution set and the above notation we outline an iterative algorithm to compute valid solution sets.

1. Start with θ_{t-1} and a project that is still viable at time t.
2. Let $X_t(\theta_t, \beta_t)$ be a function that returns the value of x_t that exists to satisfy the first critical point constraint, using θ_t as a free parameter.
3. Use $u(x_t, \theta_t, \theta_{t-1}, \beta_t, \alpha, z)$ to calculate the value of the marginal indifference constraint for a given value of θ_t. We can now calculate a value for $U(x_t, \theta_t, \theta_{t-1}, \beta_t, \alpha, z)$ by using $X_t(\theta_t, \beta_t)$ to get the only satisfactory value of x_t that will satisfy the critical point constraint for a given value of θ_t.
4. Find $U^* = argmax(U(X_t(\theta_t, \beta_t), \theta_t, \theta_{t-1}, \beta_t, \alpha, z)) \quad in \quad [0, \theta_{t-1}]$.
5. If $U^* < 0$, this round admits no new value of θ and therefore no investors pull out money this round, $\theta_t = \theta_{t-1}$. This is because $U(...)$ is a concave function.

6. If $U^* > 0$ there could be valid solutions. Check the endpoints of the range for $U(X_t(\theta_t = 0, \beta_t), \theta_t = 0, \theta_{t-1}, \beta_t, \alpha, z) < 0$ and $U(X_t(\theta_t = \theta_{t-1}, \beta_t), \theta_t = \theta_{t-1}, \theta_{t-1}, \beta_t, \alpha, z) < 0$ and then use a bisection search algorithm to find new valid values of θ_t

Each round t when solving for $U(x_t, \theta_t, \theta_{t-1}, \beta_t, \alpha, z)$ there are three different scenarios. $U(x_t, \theta_t, \theta_{t-1}, \beta_t, \alpha, z)$ can admit no solution so that we use the previous θ_{t-1} with no new investors foreclosing on the project. The project can admit one solution to $U(x_t, \theta_t, \theta_{t-1}, \beta_t, \alpha, z)$. In the third case $U(x_t, \theta_t, \theta_{t-1}, \beta_t, \alpha, z)$ can admit two solutions for round t which we denote as θ^L, θ^H representing the low and high solution for the new θ_t respectively.

There can be up to two solutions for each round after the first, producing up to 2^{N-1} different valid solutions for a multi-period game. We observe that solution sets over time can be categorized into two different regimes, a low solution set and a high solution set. Once a low solution set has been played in a round of the dynamic game, a high solution set for the remaining rounds can never occur again At any point there is a high solution set, with enough additional private information in future rounds its possible to remain on the high solution set path but also at any point drop down to the low solution path. If there is not enough new private information, the last round status quo can stay in place so no new path, high or low is taken. Using this information we reduce the total number of possible paths from being exponential in N to being upper bounded by $2N - 1$ in the worst case. When analyzing a given dynamic global game with multiple solution sets, we will only look at the best case path, which is the highest time series value of θ_t over time.

3.6 Equilibrium Behavior

Here we summarize the type of monotone equilibrium that occurs in the dynamic global game. We start by observing the behavior that the mean of the prior distribution of the success of the project, z, has on the equilibrium values of θ.

Sufficiently large values of z are interpreted by investors in the project that there is little chance that the project will fail. If the project survives withdrawals the first round, on all subsequent rounds the new information gained in the private signals of the investors is not enough to discount the the large value of z combined with knowledge that the project has survived the first round. While in future rounds there may be perfect information regarding the state of the project, no investor will leave the project given that they know that all other investors stayed in with the project the first round. This means that for sufficiently high enough $\overline{\overline{z}}$ there is a unique monotone equilibrium solution set no matter the number of rounds or new information.

Since we are only looking at a finite number of rounds, there may not be enough new information gained in these rounds to find a second monotone equilibrium. In this case, $z < \overline{\overline{z}}$, but there is only still one unique monotone equilibrium. We denote this region with $\overline{z}$

In the final case if $z < \overline{z}$ there exist multiple monotone equilibrium within the given finite round. We therefore categorize z for a given global game with parameters (N, β_{max}) as being a part of these three above regimes. The final observation is that when $\lambda_t < 1/2$ for all t, $\overline{z} = \overline{\overline{z}}$ and there will either be multiple solution sets the second round or none at all.

Chapter 4

Dynamic Game Extensions

In this section we explore extensions to the base dynamic game through enhanced information structures. We begin with the base dynamic model and add additional parameters to the set of equilibrium equations to incorporate public information, public shocks to the health of the project, feedback in terms of knowledge of previous rounds and feedback in terms of the reserve price of the project. Each section begins with a motivating example and an explanation how to update base dynamic model procedure to find a monotone equilibrium. Each section concludes with remarks on the equilibrium behavior of the expanded model. All of these additional information structures can be made independently of each other besides the feedback of the reserve price, which requires private or public feedback of the value of θ in previous rounds.

4.1 Public Information

Public information in global games is a natural first extension studied in the literature with currency attacks over multiple rounds [6] and in idealized two round global games [2]. We continue this here by extending our debt model to include public information that is shared between all participants. This public information can be thought of as a quarterly earnings report on the status of the project or some other periodic information that over time paints a clearer picture of how the company operates. This additional information gained over time does not reflect a change in the projects fundamentals, but better knowledge concerning the prior distribution that the project liquidity was sampled from. Additional information concerning a change in the underlying liquidity will be treated as a "shock" to the project, in terms of a sudden change of financial straights, will be explored in the next section.

4.1.1 Model Parameters

After the first round we assume all investors receive a noisy public signal of the true θ in the form of

$$\tilde{z}_t = \theta + N(0, \frac{1}{\eta_t^z})$$

where η_t is always positive, as new information only makes estimate of the state of the debtor project more precise. Similarly this source of noise is IID between round and IID from the noise given in the private investor estimate of θ between all investors. We use standard normal updating procedure to solve for $(z_t,\ \alpha_t)$, the best available public information of the debtor at time t based on the prior information of the debtor and all public announcements.

$$z_t = \frac{\alpha_{t-1} z_{t-1}}{\alpha_t} + \eta_t^z \frac{\tilde{z}_t}{\alpha_t}$$

$$\alpha_t = \alpha_{t-1} + \eta_t^z$$

The simple interpretation of the public signal each round is a shared update to the prior distribution of θ between all investors. When solving for a monotone equilibrium this added information is only used in the indifference constraint equation, represented by z_t in $u(x, \theta_t^*, \theta_{t-1}^*, \beta_t, \alpha_t, z_t)$. Each round z_t is path dependant on the realized noise when a public signal is sampled each round.

4.1.2 Equilibrium Behavior: Positive Public Information Can Cause Investor Withdrawals

The most significant change with public information s not that investor withdrawals on any round become possible regardless of θ or β if the public information increases precision α and is of a large enough negative value $\tilde{z}_t$. This model now captures some of the outside uncertainty that can occur when noisy public signals are received by all participants, even if the true value is well above the public information value.

A not intuitive result of including public information shocks is that positive news of the state of project, or an update $\tilde{z}_t > z_{t-1}$, is not always beneficial. Intuitively, we can see that if a project's prior distribution of success is sampled from an expected value positive but large variance distribution, knowledge that the project has survived previous rounds of investors leaving is more valuable to the investor. Upon a public information announcement the expected prior distribution can be larger than the previous round but with less variance in possible project outcomes, making the previous knowledge of surviving investor withdrawals less valuable versus the investors private signal.

This behavior occurs in the indifference constraint $u(x, \theta^*_t, \theta^*_{t-1}, \beta_t, \alpha_t, z_t)$. Increasing α has the effect of increasing the weight of public information versus the agents private signal, as well as making the knowledge of the distribution of θ more precise. The cases where this information decreases θ_t is best compared with the solution set $(\theta^*_{t\not{p}}, x^*_{t\not{p}}$ when there is no public information. In the case there is public information, and the new public information is $z_t > z_{t\not{p}}$ but also $z_t < x^*_{t\not{p}}$, which indicates that the private information is not as positive as the public information concerning the project. As long as the increase in $\alpha_t > \alpha_{t-1}$ is small enough, the weight of the negative private information overshadows the increase in public information.

4.2 Public Shocks

In this section we introduce the concept of a public shock the project's fundamentals. The shock, negative or positive, can occur any time after the the first round and its exact affect is known to all investors of the project. The possible values of θ_t, x_t are directly altered by these shocks, since there is no uncertainty in the shock value. The ability to introduce shocks into the model help incorporate real world scenarios where external events to the project, such as government regulation or natural disasters, provide a noticeable shift in the project fundamentals.

4.2.1 Model Parameters

We modify our equilibrium solution set, $\{\theta_t^*, x_t^*\}_{t=1}^{t=N}$, to include a dependency on a series of external shocks over the dynamic global game rounds which we denote by $(h_1, h_2, h_3, ..., h_N)$. These external shocks are independent of the value of θ and i.i.d every round. When calculating a new equilibrium, these shocks directly influence the value of θ when computing both equilibrium constraints. Due to this dependency, we each of the solution set variables are a function of the shock that occurred in that round, $\{\theta_t^*(h_t), x_t^*(h_t)\}_{t=1}^{t=N}$

To determine and construct a monotone equilibrium, we modify the inputs to our shorthand notion for finding an equilibrium. The indifference constraint becomes $u(x_t, \theta_t^* + h_t, \theta_{-1}^* + h_t, \beta_t, \alpha, z_t + h_t)$, where we add the current round shock to θ_{t-1}^* to represent that the previous rounds threshold of the state of the system is affected by the current shock since historical shocks do not carry over into the future. The critical point constraint becomes $X(\theta_t^* + h_t, \beta_t)$. Since shocks are linear in their effect on θ_t^*, merely using the modified $\theta_t^* + h_t$ in the equilibrium constraints and adjusting the prior mean of the distribution is enough to solve for a new equilibrium.

4.2.2 Equilibrium Behavior: Multiple Solution Sets Are Always Possible

Public shock to fundamentals do not affect how individual investors process private or public information regarding the state of the project, the shocks only provide a momentary shift to the equilibrium. This lack of new information to process is important, as previous results in the dynamic model equilibrium behavior are still valid. However, the case of only a single monotone equilibrium for a set of parameters no longer holds as a future negative shock could cause a loss of confidence in the project and induce investor withdrawals that would not have occurred without the shock, similar to the public information case. One can also imagine the opposite case, where investor withdrawals are occurring every round but a large positive shock occurs and convinces the remaining investors to stay invested in the project.

We contrast these results with Frankel, which show how a continuum of investors choosing between projects to invest in will have multiple equilibria solutions [12]. The introduction of shocks in this case however help differentiate possible investments to investors and resolves the indeterminacy of multiple equilibriums in his model.

4.3 Private and Public Feedback

The only information from the actions of individual investors in previous rounds received is whether the project is still viable. This information is manifested in the monotonically decreasing θ as the rounds progress, as well as the conditional probability in the indifference constraint after round one. In this section, we allow investors to learn a noisy estimate of how many investors chose to leave the project and take the reserve price. This information can be distributed to each investor or can be a public signal shared among all investors.

This feedback mechanism introduces a path dependency on whether the noisy feedback in the beginning of the global game is positive or negative. The resulting herd behavior was initially studied in Banerjee [7] and we elaborate on the effects that this extra information has in terms of the sensitivity of the system to public information shocks.

4.3.1 Model Parameters

To incorporate previous information into the current agents' beliefs about the project we must make sure that the normal information structure is preserved. Dasgupta provides a method to incorporate this information in a global games model over multiple periods [9]. We use the same approach here in updating our dynamic model.

$$\hat{X}_{it} = S(\theta_{t-1}, \Xi_{it}) \quad \text{and} \quad \hat{Z}_t = S(\theta_{t-1}, \xi_{it})$$

$$\Xi_{it} = N(0, \frac{1}{\gamma_t^x}) \qquad \xi_{it} = N(0, \frac{1}{\gamma_t^z})$$

$$S(\theta, v) = \Phi^{-1}(\theta) + v \quad \text{for} \quad \theta \text{ in } (0,1)$$

If there are no investor withdrawals the previous round, there is no private or public signal distributed to investors. If there is new information, it can never reveal the exact number of investors leaving the previous round, or the exact value of θ would be revealed to the participants and the global game aspect would be over and we would settle into an indeterminate Nash equilibrium. We keep the sufficient statistic property of the investors estimate of private and public information by taking the inverse normal of the value ($\theta = (0, 1)$ so this is always valid) and adding IID noise.

Now we outline how each investor updates their private and public information with these feedback signals over time.

$$x_t = \frac{\beta_{t-1} x_{t-1}}{\beta_t} + \tilde{x}_t \frac{n_t}{\beta_t} + \mathbf{1}_{t-1} \frac{\beta_{t-1}\gamma_t^x}{\beta_t}(x_{t-1}^* - \frac{\Xi_{it}}{\sqrt{\beta_{t-1}}})$$

$$z_t = \frac{\alpha_{t-1} z_{t-1}}{\alpha_t} + \tilde{z}_t \frac{n_t^z}{\alpha_t} + \mathbf{1}_{t-1} \frac{\beta_{t-1}\gamma_t^z}{\beta_t}(x_{t-1}^* - \frac{\xi_{it}}{\sqrt{\beta_{t-1}}})$$

$$\beta_t = \beta_{t-1} + n_t^x + \mathbf{1}_{t-1}\beta_{t-1}\gamma_t^x$$

$$\alpha_t = \alpha_{t-1} + n_t^z + \mathbf{1}_{t-1}\beta_{t-1}\gamma_t^z$$

Where $\mathbf{1}_{t-1}$ is an indicator function signifying whether there were investor withdrawals from the project the previous round. While $(x_t, z_t, \alpha_t, \beta_t)$ retain their normal distributions, their values are now dependant on the actions of previous rounds, making each equilibrium solution path dependant.

4.3.2 Equilibrium Behavior: Increased Sensitivity to Public Shocks

We compare a dynamic game with and without private feedback and note the differences in equilibrium behavior. The private feedback's the main effect is to provide more private information to the investor over time, allowing more opportunities for them to withdraw from the project if they become confident that the project is doing badly. In this sense, the private feedback is almost equivalent to a dynamic global game without feedback but with better precision in the private information for investors over time.

The herding effect is most prominent when we introduce public feedback, since this affects the precision as well as the mean of the prior distribution. To illustrate, imagine there is just one public signal after the first round. If this signal is sufficiently positive, no further withdrawals can occur as the positive signal will prevent investor withdrawals the second round and will have positive feedback into future rounds increasing the positive signals of the health of the project. In the opposite scenario, more investor's may leave upon a bad signal, which leads to even more investors leaving on the second round given the outflow on the first round, causing a cascading effect based on a spurious bad signal the first round.

In the same way that the cascading effect occurs when an extremely bad or good public signal is seen, a public shock has the same effect on the outcome of the dynamic global game. Given a dynamic global game subject to shocks with feedback versus one without it, the variance of possible outcomes, good and bad, will be larger in the model with feedback due to the cascading effect described above.

4.4 Reserve Price Feedback

In this section, we introduce feedback into the reserve price received by an investor that leaves the project before completion. This feedback allows us to model outside forces that could affect the project's ability to tap into its liquidity pool. We imagine the scenario, good for the project and its investors, that there is an robust market for the intermediate goods produced by the project. This would allow the project easy access to liquidate its goods on any round with a large investor withdrawal. We can also imagine the opposite, a market scarce of resell opportunity, where the effects of large investor withdrawals from the project are magnified due to poor liquidity. Both of these examined in the equilibrium behavior section.

4.4.1 Model Parameters

Any reserve price feedback mechanism must be monotonically increasing or decreasing over $(0, 1)$, the range of valid θ values to guarantee that there is a uniquely dominant BNE switching strategy. We denote this reserve price function as in the static case $K^*(\theta_t)$, which we assume has the same behavior to investor outflows on every round. The normalized reserve price for any payout becomes $\lambda(\hat{\theta}_t) = (K^*(\hat{\theta}_t) - K_{*t})/(L_t - K_{*t})$.

This update only affects the indifference constraint when solving for a possible equilibrium solution set $u(x_t, \theta_t^* + h_t, \theta_{-1}^* + h_t, \beta_t, \alpha, z_t + h_t)$ and indirectly effects the equilibrium switching value for any round. For the investors to calculate the reserve price for any given round would require the exact value of θ_t, which means if an exact reserve price was known by the investors it would be equivalent to knowing the true value of θ. To prevent this from occurring we only allow reserve price feedback to dynamic global games where there is private or/and public information about the size of past investor withdrawals, which serves as an estimate of the reserve signal that an investor would receive on any given round.

4.4.2 Equilibrium Behavior: Positive and Negative Feedback Possible

These reinforcing feedback effects are studied in a two round context for risky assets by Morris and Shin [19]. The results here differ in that by adding multiple rounds, negative feedback does not make the equilibrium worse on every given round, only those with withdrawals. There are scenarios where a large withdrawal from the project and knowledge that the project survived the withdrawal makes the project more resilient to negative information in the following rounds. This contrasts with the feedback effects for risky assets and the negative feedback spiral that can occur with small but consistent withdrawals over time.

Chapter 5

Model Interpretation and Data Analysis

In this section we outline how to use the dynamic global game model structure to price bond yields. We explain how the dynamic global game model allows us to incorporate coordination risk between the bond holders decision to foreclose on the project as well as the inherit risk in lending money to a venture that might not succeed.

5.1 Static Bond Pricing Model

To compare and contrast bond prices produced from the static global game model, we price bonds as if there was one creditor that will realize the the project if $\theta > 0$ or take the reserve price if $\theta < 0$, known as the Merton Model [16]. The model also produces the same results if formulated as a multitude of creditors behaving independently in choosing to stay in the project. In this case, there is no coordination risk and the yield of the bond corresponds directly to the probability of failure of the project.

A quick review of how bonds are typically quoted and priced. The par value of a bond is how much is returned to the investor at the end of the bond term. The bond yield is typically the yearly interest accrued by the bond divided by its par value. Finally, the bond price is the final discounted value that an investor would receive over the lifetime of the bond. We are interested in the bond yield as a measure of value and riskiness in lending money.

$$Yield = \frac{Par - Price}{Price}$$

To price a bond in the Merton model, we assume common knowledge of a prior distribution of the bond's ability to repay itself, equal to the $N(u, 1/\alpha)$ in the global games model. The health of a company, or θ, is sampled from this prior distribution and the $P(\theta < 0) = Yield$. With perfect coordination between the investors in the bond, the creditors would only try to foreclose on the company if they believe the company is going to fail for sure. This contrasts to the global game solution, where $\theta > 0$ introduces coordination risk to the lenders, increasing bond yields. We illustrate this with an example. In the case of an ex ante mean $u = 2, \alpha = 1$ we can calculate the yield as

$$\Phi(\sqrt{\alpha}(\theta_i - u)) = \Phi(-2) \approx .023$$

In the global games approach, we allow coordination risk, which introduces a higher θ threshold when calculating the bond yield. To do so, we have some prior values for the global game model,

$$z = 1, \lambda = .5$$

and can now solve for the equilibrium value of $\hat{\theta}$ which will be used to calculate a corresponding bond yield of

$$\Phi(\sqrt{\alpha}(\theta_i - u)) = \Phi(.212u - 2) \approx .039$$

which is higher than the Merton model because of the added coordination risk that is priced in.

5.2 Dynamic Bond Pricing Model

We extend the bond pricing model introduced in the static case to the multiple period case. Here, we interpret a valid solution set of θ_t as a yield curve computed from the longest dated bond. The first θ_1 represents the bond price for the farthest dated yield curve from the end of the project and we would expected this to have the highest yield as the investors have the most uncertainty regarding the outcome of the project. As $\theta_t^* \leq \theta_{t-1}^*$ holds for all subsequent t, as the game progresses θ_t is decreasing which means that the bond yields are getting cheaper due to the passage of time and that more information is known of the riskiness of the project. Even in the case of no new private information, the passage of time should decrease the price of the bond as there is less inherent risk, which matches the behavior of the θ equilibrium equations [11]. This comparison is not perfect as there are solutions where a lowest θ^* is found on the first round or in a round before the end of the game and no other investor withdrawals occur. In these cases it is important that the prior health of the project is uncertain enough to introduce possible withdrawals in the future as well as β over time increases enough to allow some investors to be confident that they should leave.

Next, within the context of the dynamic bond pricing model we examine what the ideal parameters and information structure would be for the project to maximize its chance of success. The goal of the project owner and the investors that do not foreclose are to maximize the chance of the project getting to completion and therefore maximizing θ_n over the horizon of the game [3].

For this analysis, we assume that investors are always optimistic and when presented with multiple equilibrium in a round they always take the equilibrium with the higher θ_t. If the project must switch to a lower equilibrium $theta_t^L$ during one of the rounds, it is always preferable to delay this switch as long as possible. One can see this from how θ^H path must be bounded by its previous value, and that lower values of θ^H are due to increased information about the health of the system. Due to the single peak of the indifference constraint $U()$, as θ^H decreases θ_L increases as the time progresses, so a later switch is always preferable.

If the debtor can control the terms of λ_t, the best case scenario is to have a very low λ_1, as this will allow for a high θ_1 and then in subsequent rounds choose $\lambda_t > .5$ to disincentive future withdrawals given the information that project survived any withdrawals in the first round.

Increasingly accurate β_t updates to the private information of the investors is a double edge sword depending on the previous value of θ and the amount of information already revealed. If there is little new information and a bad prior, subsequent rounds decrease the strategic uncertainty and allow for more investor withdrawals. If there is a good prior, surviving previous rounds will decrease fundamental uncertainty and strategic uncertainty. If the precision of information revealed is significant over previous rounds, this will decrease θ^H but increases θ^L for any given round. Therefore depending on which equilibrium that the debtor believes will occur the next round will determine whether the debtor would want to give out more private information on the project.

5.3 Fitting a Dynamic Game To Data

In this section we describe how we fit a bond yield curve to the dynamic global game parameters. We pose this as a constrained minimization problem and discuss how to reduce the dimensions of the model.

We start with a time series of bond yields, $Y_t = (Y_1, Y_2, ...Y_N)$, which represent bonds of different maturities for the same underlying company. The value of time between each bond maturity will be represented by the increase in private information that the investor receives in each round of the game. Also note that Y_N maps to the first round of the dynamic global game, θ_1. To produce this time series of bond yields from a solution set to a dynamic global game, we calculate each round the bond yield given the $theta_t$ for that round and the (u, α) that describe the prior probability as described in the static bond pricing section. In the case that α changes over time due to public shocks, we update accordingly.

For any given time series of bond yields, there will be more free parameters than data points to be fitted by the four parameters (α, λ, u, w) since we have a variable for the amount of new information received each round, β_N. We can assume some prior knowledge of (Z) as a parameter that represents an estimates of the immediate liquidation value of the company. In practice, we find that having less than three free parameters and solving for an approximate solution oftentimes lead to bad approximations. For now we leave (α, u, λ) as free parameters to be found in the optimization

Next we define a score function to be minimized that will determine how far a guess of initial parameters for our dynamic game is from being representative of a bond yield curve.

$$f(\alpha, u, \lambda, \beta_1, ..., \beta_N | w, Y_1, ...Y_N) = \sum_{i=1}^{N} (Y_{N+1-i} - \Phi(\sqrt{\alpha}(\theta_i - u)))^2$$

where $\theta_1 ... \theta_N$ represents the best case scenario solution for a dynamic global game with the given function parameters. We now formulate the minimization problem below.

$$\begin{aligned}
&\underset{\alpha, u, \lambda, \beta_1, \ldots, \beta_N}{\text{minimize}} && f(\alpha, u, \lambda, \beta_1, ..., \beta_N | w, Y_1, ...Y_N) \\
&\text{subject to} && \beta_i > 0, \; i = 1, \ldots, N \qquad \beta_i \geq \frac{(w\alpha)^2}{2\pi}, \; i = 1, \ldots, N \\
& && 0 < \lambda < .5 \qquad \alpha > 0
\end{aligned}$$

This minimization function is non-linear in its input parameters and when using a standard optimization library, a local instead of global minimum is found depending on the initial parameters and accuracy tolerance levels.

Chapter 6

Conclusions

In this thesis we create a new dynamic global game model that describes coordination risk among a group of agents over time. We present a method to construct an equilibrium solution set of the base dynamic model, a characterization of different equilibrium regimes in the model and a financial interpretation to the model parameters. We expand the base dynamic model to include public information, public shocks to the health of the project, information feedback of previous investor withdrawals from the project, and reserve price feedback received by investors. These additional information structures show how positive public information is not always beneficial to the health of the project, how introducing feedback on previous rounds withdrawals makes the project more fragile with respect to public information shocks and how to incorporate liquidity into a global games framework through feedback in the reserve price. We formulate an optimization problem over the free parameters of the dynamic model to fit data to the model. Finally, we finish with fitting a hypothetical companyâĂŹs bond yield curve to provide an estimate of the health of the company.

6.1 Future Work

This thesis provided a modeling framework for dynamic global game debt models over time, similar to the literature on dynamic global game models of currency attacks [6]. Future work could be done to provide a more rigorous analysis of fragility of the system to external shocks when feedback is introduced, similar to the work on liquidity currency crises[19]. In the same vein analytical instead of simulated solutions on how additional public information can hurt the prospects of the project over time is a logical next step [4]. Introducing a large investor that controls a significant portion of the projects funding would provide another interesting avenue to reducing coordination risk among investors, as explored in currency crises attack models [20]. Finally, richer financial debt contracts could be fitted to the extended dynamic model by introducing estimates of public information and expanding our optimization formulation to include these free parameters.

Appendix A

Numerical Examples

A.1 Static Model

We start with the following parameters

$$\alpha = 1 \quad , \quad \beta = 5 \quad u = .8, \quad \lambda = .3 \quad , \quad w = .95$$

First, check that there is a single unique solution to the game,

$$\frac{\alpha}{\sqrt{\beta}} \leq \frac{\sqrt{2\pi}}{w}$$

$$\frac{1}{\sqrt{5}} \leq \frac{\sqrt{2\pi}}{.3}$$

$$.447 \leq 2.639$$

Now, we solve for $\hat{\theta}$ by taking the combined indifference constraint and using a bisection algorithm to search for a valid value in range $[0, 1]$

$$0 = w\Phi(\frac{\sqrt{\beta+\alpha}}{\sqrt{\beta}}(\alpha z - \Phi^{-1}(1-\lambda(\hat{\theta}))) - \hat{\theta}(\frac{\beta+\alpha}{\sqrt{\beta}}+1)) - \hat{\theta}$$

$$\hat{\theta} = .188$$

We can now solve for $\hat{x}$ using the constraint that there must exist a critical point of the investors foreclosing on the project and receiving a signal to do so

$$\hat{\theta} = w\Phi(\sqrt{\beta}(\hat{x} - \hat{\theta}))$$

$$\hat{x} = \frac{\Phi^{-1}(\frac{\hat{\theta}}{w})}{\sqrt{\beta}} + \hat{\theta}$$

$$\hat{x} = -.191$$

We interpret these results so that an investor will choose to foreclose on their share of the investment at the end of round one if the private signal they see is below $-.191$. For the project, if the project's sampled $\theta > .188$, the project will survive early investor withdrawals.

A.2 Dynamic Model

We use the dynamic game model structure to analyze a project that will payout in full over three rounds. The primary factor that determines whether the dynamic game will be interesting to analyze rests on the payout structure of the game. Let us take a \$1000 initial investment in the project that will be worth 10% more each round. In the case of early withdrawal the \$600 is recouped and only \$300 is recovered if the project fails.

	t_1	t_2	t_3
L_t	\$300	\$300	\$300
K_{*t}	\$600	\$600	\$600
K_t^*	\$1100	\$1200	\$1300

Normalizing each round we get

$$\lambda_{(1,2,3)} = (3/8, 3/9, 3/10)$$

For this example we set the rest of the parameters to

$$\alpha = 1 \quad , \quad u = .3 \quad , \quad w = 1 \quad , \quad \beta_{(1,2,3)} = (1, 3, 10)$$

Using the iterative optimization method each round we obtain the following solutions sets, where $(\hat{\theta}, \hat{x})$ represent a valid solution that round.

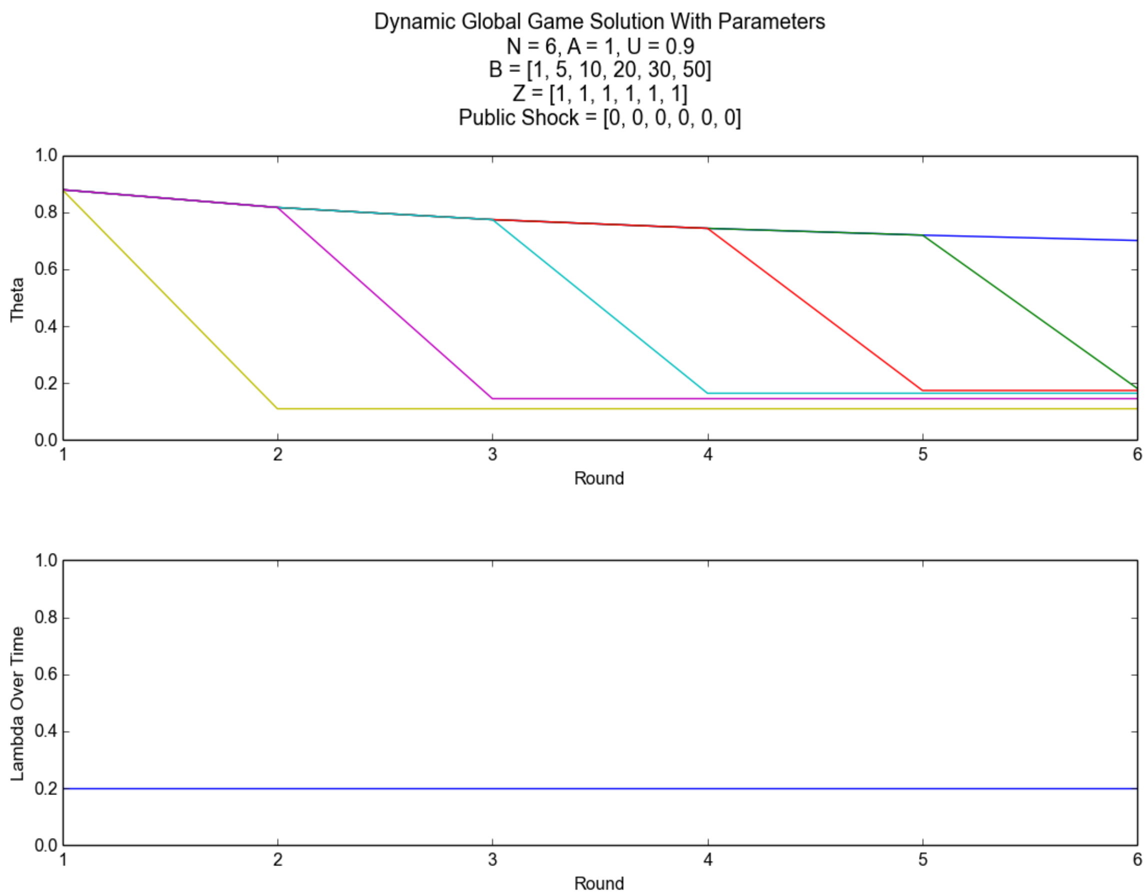
Round 1 $= (.839, 1.828)$

Round 2 $= (.839, \infty)$

Round 3 $= (.751, .932), (.297, .154)$

This example shows in round two that there is not enough new information to cause any more withdrawals from the project and the previous equilibrium is maintained. In Round 3 enough new information is introduced and there are now two valid solution sets, a high θ and low θ.

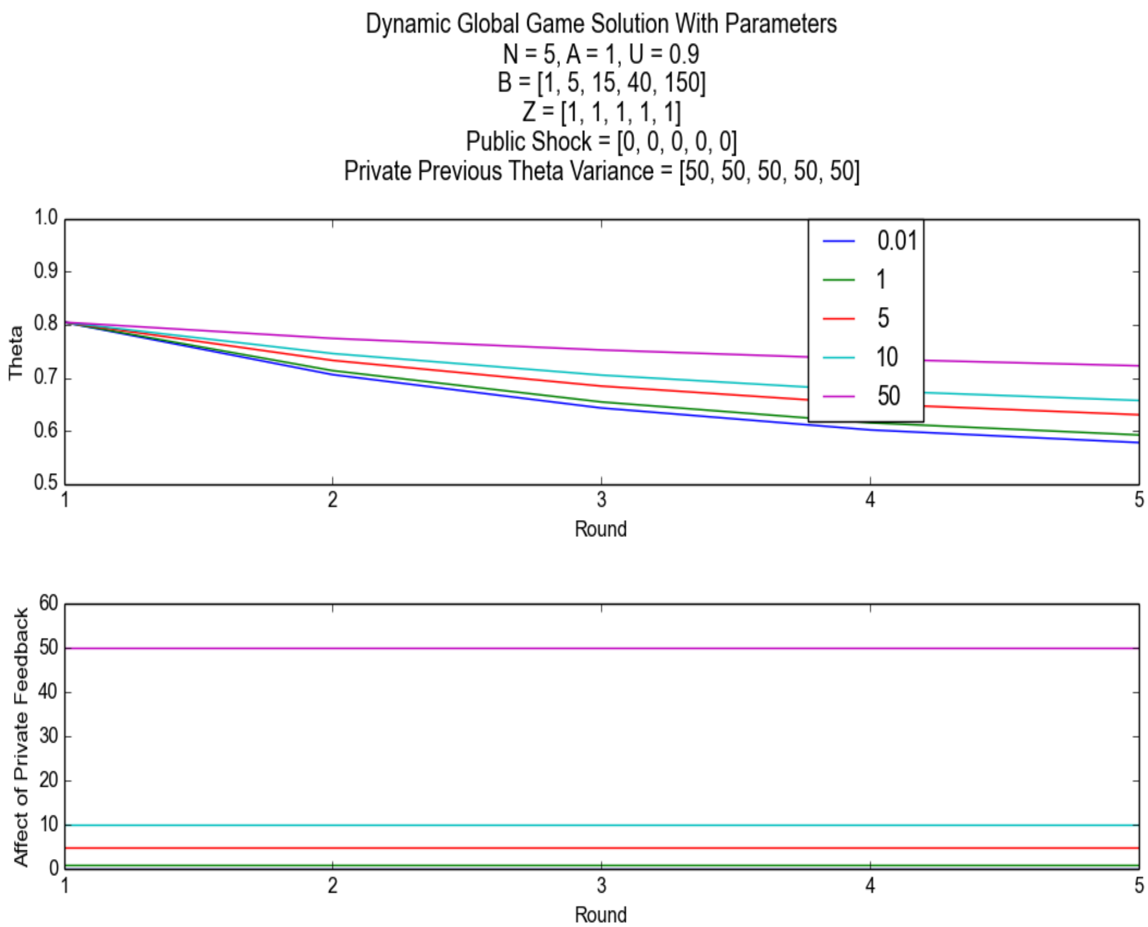
Below we provide an example of a six round dynamic game that further illustrates the branching between the high path and the low path of the valid solution sets.



A.3 Dynamic Model with Public Shocks and Information Feedback

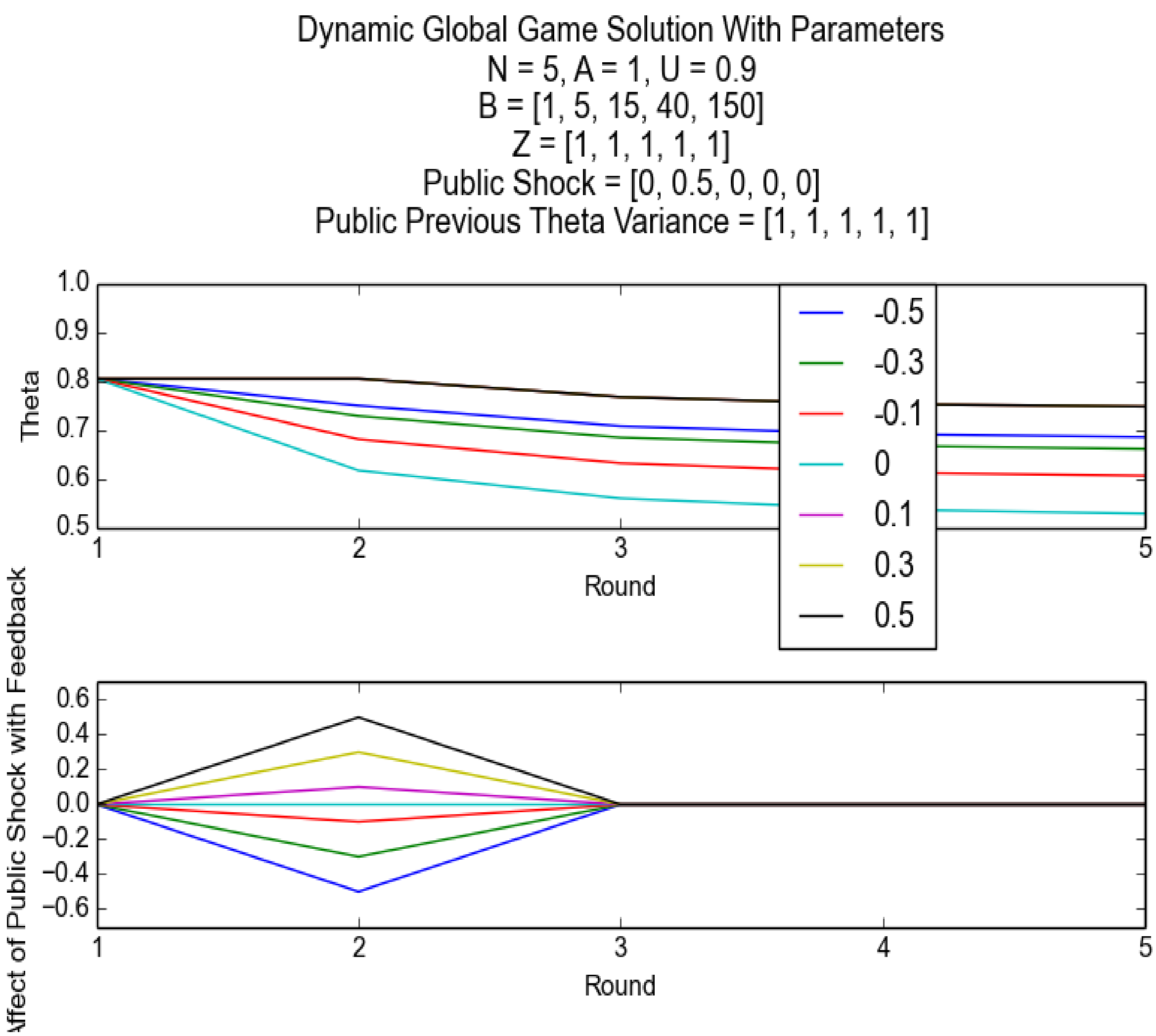
Here we examine the combined effects of public shocks and information feedback between rounds. We start by examining the effect of private feedback on the dynamics of the game, where investors learn how many investors left the project in previous rounds. The legend describes the prevision of which the investors learn of the previous rounds value, ranging from $.01 - 50$. The higher the precision of the information, the higher the best case scenario for that project.

Private Feedback



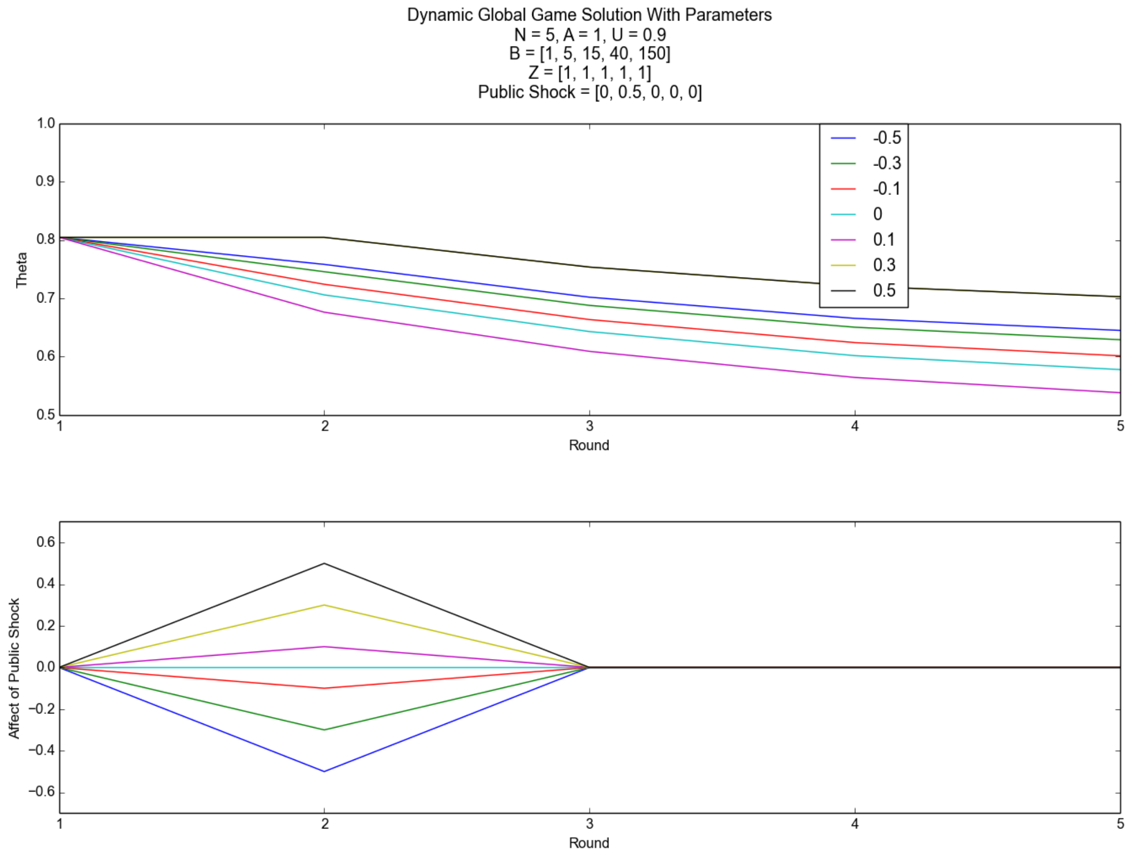
Next we examine the case of introducing public shocks and public feedback on the stability of the project over time. First, we examine the case of introducing varying shocks into the system only on the second round. We see that positive shocks have a positive effect on the equilibrium over time, while negative shock lowers the equilibrium over time.

Public Shocks In the Second Round



Now, we introduce a slight public feedback precision of 1 each round and note that the system is now worse off due to the shocks.

Public Shocks in the Second Round with Feedback



A.4 Dynamic Model with Reserve Price Feedback

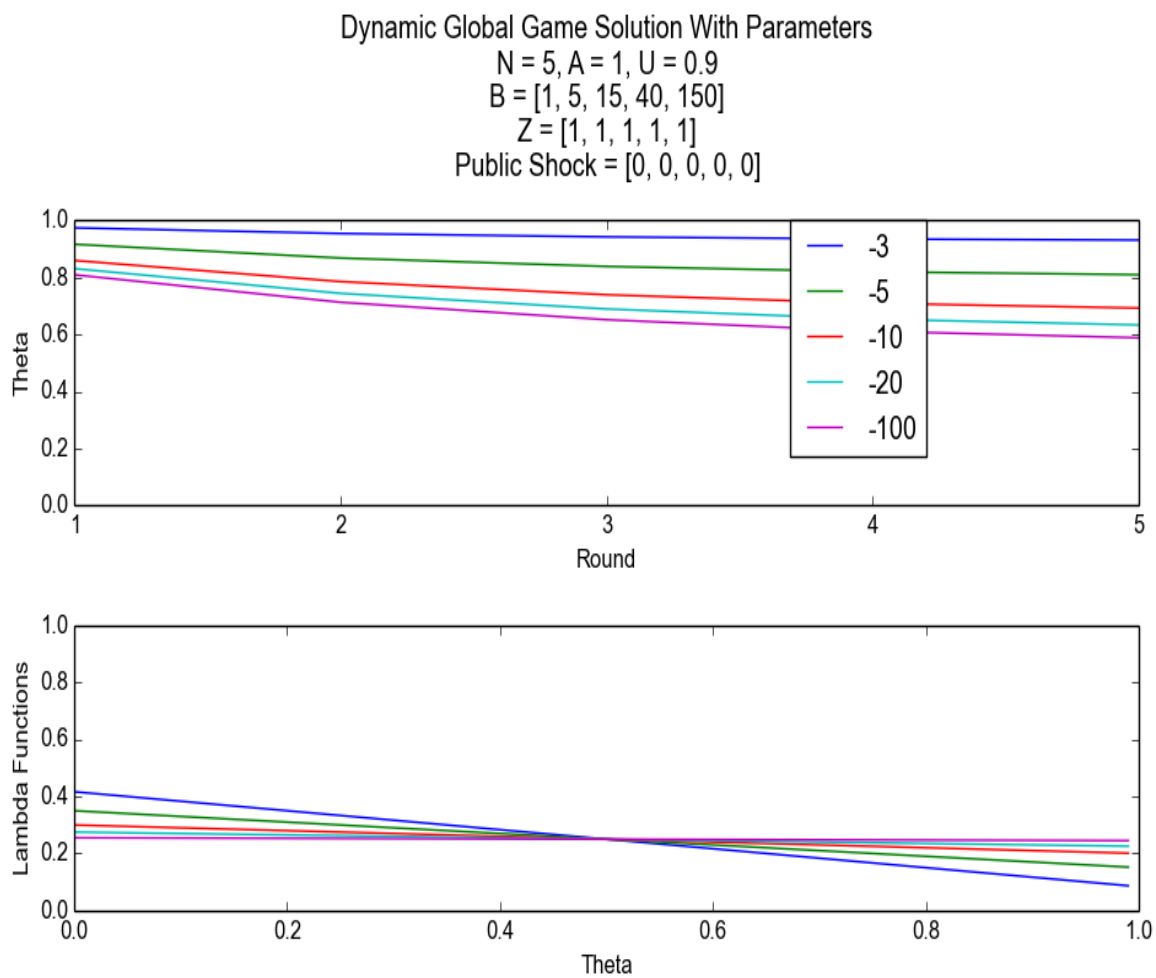
Here we examine the effect of having the reserve price λ be dependant on the number of investors who leave the project in any given round. Instead of earning a constant λ, we determine the reserve price of any round as a function of how many investors leave that round according to the linear function

$$\lambda(\theta) = (\theta - .5)/x + .25$$

where x represents the legend in each of the below graphs. This means large values of x translate to nearly constant values of λ over any possible range of θ.

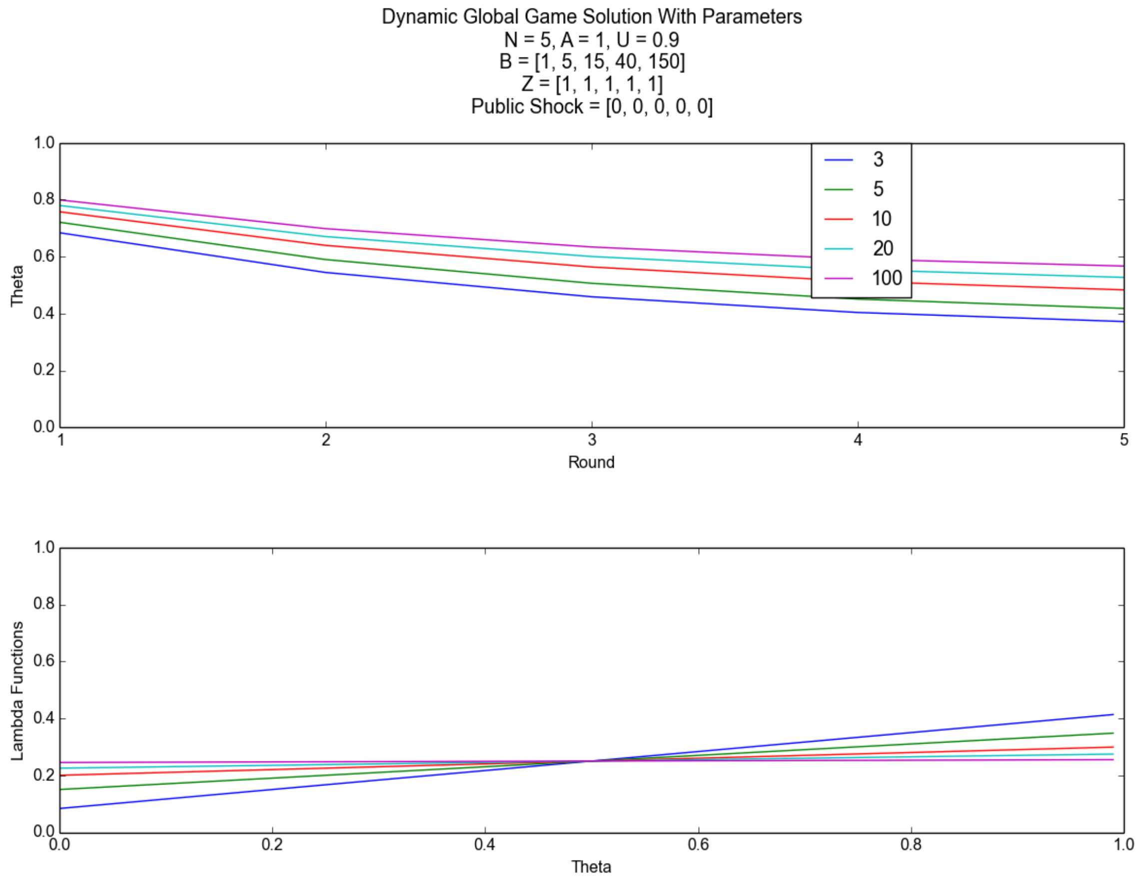
First we examine the case of positive feedback, which means that a lower reserve price is earned by all investors the more that leave, so there is a less of an incentive to leave. We note that the more aggressive the loss in value over time, the higher the the chance of success of the project by the end of round three.

Positive Reserve Price Feedback Shifts curve up



Next we examine the case of negative feedback, which means that a higher reserve price is earned by all investors the more that leave, so there is a more of an incentive to leave when others leave.

Negative Reserve Price Feedback Shifts curve down



A.5 Fitting Data Example

Let us start with a yield curve of $Y = (.15, .19, .22, .25)$ which represent 1, 2, 3 and 4 year bond maturity yields for a hypothetical company. We take $w = 1$ and an initial guess.

$$\alpha = 1, u = .9, \lambda = .25, \beta_1 = 1, \beta_2 = 1.5, \beta_3 = 2.25, \beta_4 = 3.375$$

A solution to the minimization problem:

$$\alpha = .684, u = 1.5, \lambda = .224, \beta_1 = 1.93, \beta_2 = 1.72, \beta_3 = 2.25, \beta_4 = 3.35$$

$$\theta_1 = .684, \theta_2 = .565, \theta_3 = .439, \theta_4 = .245$$

Checking the four constraints on β_i in the form $\beta_i - \frac{(w\alpha)^2}{2\pi} \geq 0$

$$1.8 > 0, 1.6 > 0, 2.2 > 0, 3.3 > 0$$

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