A Doublet Panel Method for Generalized Supersonic Lifting Surfaces

by

Dominique Hoskin

Submitted to the Department of Aeronautics and Astronautics
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Abstract

An improved doublet panel method for calculating the aerodynamic properties of lifting surfaces in supersonic flows is derived and implemented. The lifting surfaces are discretized into an arbitrary number of doublet panels with unknown singularity strength, whose perturbation velocity potentials satisfy the supersonic Prandtl-Glauert equation.

To prevent field singularities in the potential and velocity, the doublet strength of each panel is piecewise linear and continuous in the streamwise direction. In addition, the panels can be swept at different angles at their leading and trailing edges, allowing for generalized lifting surfaces to be analyzed.

The component of the perturbation velocity that is normal to the lifting surface at each specified control point induced by each doublet panel is calculated. Applying the flow tangency boundary condition at the control points forms a linear system that is solved for the difference in doublet strength between the trailing and leading edges of each panel.

The perturbation velocity components of this method are compared to those of the Woodward method. The numerical solutions of this method are compared to the analytical solutions of some test cases. Convergence is obtained for both the lift coefficient and the perturbation velocity component distribution.

Thesis Supervisor: Mark Drela
Title: Terry J. Kohler Professor

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I would like to start by thanking my advisor, Professor Mark Drela, for taking me in as one of his few graduate student advisees and allowing me to pursue a project that so closely matched my research interests.

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Finally, I would like to thank my parents for all of their support over the years and for believing in my abilities.

Contents

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Nomenclature

Acronyms

AIC	Aerodynamic Influence Coefficient
AVL	Athena Vortex Lattice
CFD	Computational Fluid Dynamics
MATLAB	Matrix Laboratory
PAN AIR	Panel Aerodynamics
Rubi	Rule-Based Integrator

Roman Symbols

$\dot{\mathcal{V}}$	volume flow rate
$\hat{\mathbf{n}}$	outward pointing normal unit vector
$\hat{\mathbf{r}}$	outward pointing radial unit vector
$\hat{\mathbf{x}}$	x Cartesian unit vector
$\hat{\mathbf{y}}$	y Cartesian unit vector
$\hat{\mathbf{z}}$	z Cartesian unit vector
$\hat{u}$	x perturbation velocity component per unit singularity strength parameter
$\hat{v}$	y perturbation velocity component per unit singularity strength parameter
$\hat{w}$	z perturbation velocity component per unit singularity strength parameter
$\mathbf{n}$	outward pointing normal vector
$\mathbf{V}$	velocity vector
A	area

a	geometric constant
A_i	i_{th} panel area
A_{ref}	reference area
b	geometric constant
C_D	induced drag coefficient
C_L	lift coefficient
c_n	normal force coefficient
c_p	pressure coefficient
C_x	x component of total force coefficient
c_x	x component of normal force coefficient
C_Y	sideforce coefficient
C_y	y component of total force coefficient
c_y	y component of normal force coefficient
C_z	z component of total force coefficient
c_z	z component of normal force coefficient
$c_{p,l}$	lower surface pressure coefficient
$c_{p,u}$	upper surface pressure coefficient
$c_{x,i}$	i_{th} panel x component of normal force coefficient
$c_{y,i}$	i_{th} panel y component of normal force coefficient
$c_{z,i}$	i_{th} panel z component of normal force coefficient
e	internal energy
h	average side length of panels
h	hyperbolic radius
M	Mach number
N	number of panels
n_x	x outward pointing wall normal vector component
n_y	y outward pointing wall normal vector component
n_z	z outward pointing wall normal vector component
p	pressure
r	distance from the origin

t	time
u	x perturbation velocity component
V	velocity magnitude
v	y perturbation velocity component
w	z perturbation velocity component
x	x Cartesian coordinate
x'	x integration variable
y	y Cartesian coordinate
y'	y integration variable
y^*	modified y Cartesian coordinate
z	z Cartesian coordinate

Greek Symbols

α	angle of attack
β	sideslip angle
β_{PG}	Prandtl-Glauert factor
$\boldsymbol{\gamma}$	vortex sheet strength vector
$\Delta\mu$	doublet panel singularity strength parameter
γ	ratio of specific heats
Γ_z	infinitesimal width horseshoe vortex strength
$\hat{\phi}_{\Gamma_z}$	velocity potential of a unit strength infinitesimal width horseshoe vortex
$\hat{\phi}_{\kappa_z}$	velocity potential of a unit strength z-doublet
$\hat{\phi}_{\Sigma}$	velocity potential of a unit strength point source
κ_z	z-doublet strength
μ	doublet sheet strength
Φ	full velocity potential
ϕ	perturbation velocity potential
ϕ	velocity potential
ρ	density
Σ	point source strength

Subscripts

0	leading edge
0	left edge
1	right edge
1	trailing edge
∞	freestream
μ	doublet panel
cp	control point
i	control point matrix index
j	panel matrix index
l	left edge
l, le	left leading edge
l, te	left trailing edge
r	radial component
r	right edge
r, le	right leading edge
r, te	right trailing edge
$shell$	spherical shell
x	partial derivative with respect to x
y	partial derivative with respect to y
z	partial derivative with respect to z

Operators

$\cdot$	dot product
Re	real part
∇	gradient
∂	partial derivative
$\tilde{\nabla}$	surface gradient

Chapter 1

Introduction

Solutions to the partial differential equations that govern supersonic flows are fundamentally different from those of the partial differential equations that govern subsonic flows. This presents a variety of complications to numerical methods for supersonic flows. Extensive research has been conducted on the computationally expensive grid based CFD methods, but only a few of the computationally inexpensive singularity methods codes exist for supersonic flows. Grid based CFD methods for supersonic flows solve either the compressible Euler or Navier-Stokes equations and are computationally expensive because they require a large domain to be discretized. Singularity methods only require the discretization of the aircraft surface and make use of mathematical constructs such as vortices, doublets, and sources that are made to automatically satisfy the supersonic Prandtl-Glauert equation.

Panel methods are a class of singularity methods where aircraft surfaces are discretized into a discrete number of panels, each of which having an unknown singularity strength distribution. Panel methods work by assuming the velocity field is a superposition of a freestream velocity component and a perturbation velocity component. The perturbation velocity induced by each panel depends linearly on a set of singularity strength parameters. The values of these parameters are found by enforcing the flow tangency boundary condition at a number of specified control points. The so called AIC matrix contains the perturbation velocity induced by each panel on each control point per unit of the singularity strength parameter. The AIC matrix together with the flow tangency boundary condition form a linear system that can be solved either directly or iteratively. Once the singularity strength parameters are known, the desired aerodynamic properties can be computed.

One of the first singularity methods for supersonic flows was a vortex panel code written in FORTRAN developed by F. A. Woodward [4] in the 1960s. The perturbation velocity components used give the correct results if the panels are rectangular, but not if the panels are swept at different angles at the leading and trailing edges. The Woodward code itself is outdated and difficult to read. Another singularity method for supersonic flows is a doublet panel code called PAN AIR [3] that was developed by NASA in the 1990s. PAN AIR is more accurate than Woodward, but also considerably more complex because the perturbation velocity component expressions are not analytic functions that can be easily implemented.

The present doublet panel method was developed to provide a fast, accurate, and straightforward way to compute the aerodynamic properties of supersonic lifting surfaces. Its intended use is for the conceptual design of supersonic wings and tails. The current implementation is in MATLAB, but this method can also be implemented in other coding languages. It was originally designed to be implemented in FORTRAN as an AIC routine to extend the capabilities of AVL [2], a useful code for subsonic aircraft configuration development, to handle supersonic flows.

In chapter 2, the governing supersonic Prandtl-Glauert equation and the supersonic doublet panel perturbation potential and velocity components are derived. In chapter 3, the MATLAB implementation of this code is discussed and the numerical solutions are compared with analytical solutions for various test cases. Appendix A contains the Mathematica notebooks used to calculate the perturbation potential and velocity component expressions.

Chapter 2

Derivations

Starting from the Euler equations, assumptions are made and the equations are simplified until they are reduced to the Prandtl-Glauert equation. The supersonic tapered doublet panel perturbation velocity components are then derived, starting from a supersonic point source. The perturbation velocity components are compared to those of a vortex panel as specified by Woodward [7]. A similar derivation of the Prandtl-Glauert equation and the supersonic point source is presented in [1].

2.1 Euler Equations

The Euler equations represent the conservation of mass, momentum, and energy of a fluid under the assumptions that the fluid is a continuum and is inviscid and adiabatic. The equations for mass, momentum, and energy, respectively are:

$$\begin{aligned}
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) &= 0 \\
\frac{\partial (\rho \mathbf{V})}{\partial t} + \nabla \cdot (\rho \mathbf{V} \mathbf{V}^T) + \nabla p &= \mathbf{0} \\
\frac{\partial (\rho E)}{\partial t} + \nabla \cdot (\rho \mathbf{V} E) + \nabla \cdot (p \mathbf{V}) &= 0
\end{aligned}$$

where:

$$E = e + \frac{1}{2} \mathbf{V}^T \mathbf{V}$$

These equations need to be coupled with an equation of state and appropriate boundary conditions. For solid walls, a flow tangency boundary condition, shown in Figure 2-1, is imposed:

$$\mathbf{V} \cdot \mathbf{n} = 0$$

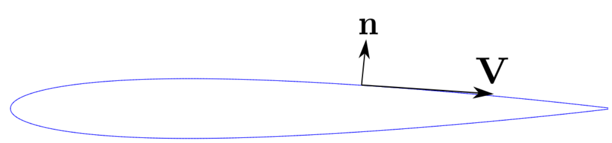

Figure 2-1: Flow Tangency Boundary Condition

Gravitational forces are insignificant compared to aerodynamic forces, so the gravitational contribution to the momentum equation has been neglected as is typical in aerodynamics.

2.2 Full Potential Equation

By taking the curl of the momentum equation, we can show that the velocity is irrotational outside of the viscous layer. The velocity vector field can then be written as the gradient of a scalar velocity potential:

$$\mathbf{V} = \nabla\Phi$$

For steady flows, the mass equation becomes the steady full potential equation:

$$\nabla \cdot (\rho \nabla \Phi) = 0$$

Irrotational flows are also isentropic, for which the density can be related to the velocity potential gradient:

$$\rho = \rho_\infty \left[1 + \frac{\gamma - 1}{2} M_\infty^2 \left(1 - \frac{\nabla \Phi \cdot \nabla \Phi}{V_\infty^2} \right) \right]^{\frac{1}{\gamma - 1}}$$

The solid wall flow tangency boundary condition becomes:

$$\nabla \Phi \cdot \mathbf{n} = 0$$

It should be noted that the full potential equation is non-linear in $\nabla \Phi$.

2.3 Second-Order Perturbation Potential Equation

The full potential equation can be simplified further with the assumption that the velocity does not deviate significantly from the freestream value anywhere in the flow field. This is valid for slender bodies at small angles of attack and sideslip. With this assumption, it is useful to decompose the velocity field into freestream and perturbation components, shown in Figure 2-2.

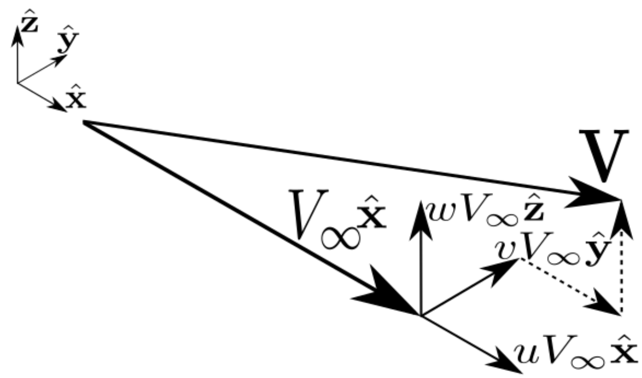

Figure 2-2: Freestream and Perturbation Velocity Components

Without any additional loss of generality, the freestream velocity will be aligned with the x-axis for convenience. The velocity field becomes:

$$\mathbf{V} = V_\infty[(1+u)\hat{\mathbf{x}} + v\hat{\mathbf{y}} + w\hat{\mathbf{z}}]$$

Substituting this expression into the isentropic density equation yields:

$$\rho = \rho_\infty \left\{1 - (\gamma - 1)M_\infty^2[u + \frac{1}{2}(u^2 + v^2 + w^2)]\right\}^{\frac{1}{\gamma-1}}$$

A useful Taylor series expansion of a small parameter ϵ is:

$$(1-\epsilon)^b = 1 - b\epsilon + \frac{1}{2}b(b-1)\epsilon^2 + \mathcal{O}(\epsilon^3)$$

For small perturbation velocities, the quadratic approximation for the density equation is:

$$\rho \approx \rho_\infty \left\{1 - M_\infty^2 \left[u + \frac{1}{2}(u^2 + v^2 + w^2)\right] + \frac{2-\gamma}{2} M_\infty^4 u^2\right\}$$

Substituting the above expression and the perturbation velocity expression into the full potential equation, dividing by $\rho_\infty V_\infty$, and simplifying yields:

$$\nabla \cdot \left(\left\{1 - M_\infty^2 \left[u + \left(\frac{1}{2} - \frac{2-\gamma}{2} M_\infty^2\right) u^2 + \frac{1}{2}(v^2 + w^2)\right]\right\} \{(1+u)\hat{\mathbf{x}} + v\hat{\mathbf{y}} + w\hat{\mathbf{z}}\}\right) = 0$$

Expanding and eliminating cubic terms yields:

$$\{(1 - M_\infty^2)u - \frac{1}{2} M_\infty^2 [(3 - (2-\gamma)M_\infty^2)u^2 + v^2 + w^2]\}_x + \{v - M_\infty^2 uv\}_y + \{w - M_\infty^2 uw\}_z = 0$$

Substituting the expression for the perturbation velocity and dividing by V_∞, the solid wall flow tangency boundary condition becomes:

$$(1+u)n_x + vn_y + wn_z = 0$$

A new perturbation potential ϕ is defined such that:

$$\nabla\phi = u\hat{\mathbf{x}} + v\hat{\mathbf{y}} + w\hat{\mathbf{z}}$$

Substituting the perturbation velocity potential into the simplified full potential equation gives the second-order perturbation potential equation and flow-tangency boundary condition:

$$\begin{aligned}\{(1 - M_\infty^2)\phi_x - \frac{1}{2} M_\infty^2 [(3 - (2-\gamma)M_\infty^2)\phi_x^2 + \phi_y^2 + \phi_z^2]\}_x \\ + \{\phi_y - M_\infty^2 \phi_x\phi_y\}_y + \{\phi_z - M_\infty^2 \phi_x\phi_z\}_z = 0\end{aligned}$$

$$(1 + \phi_x)n_x + \phi_y n_y + \phi_z n_z = 0$$

2.4 Transonic Small Disturbance Equation

For small disturbances, $M_\infty^2 \phi_x \phi_y$, which is quadratic in the perturbation potential is small compared to ϕ_y, which is linear in the perturbation potential. Likewise, $M_\infty^2 \phi_x \phi_z$ is small compared to ϕ_z. For slender bodies, n_x is much smaller than n_y and n_z, so the product $\phi_x n_x$ is also considered higher order. Thus the second-order perturbation potential equation and flow-tangency boundary condition can be simplified:

$$\begin{aligned}\{(1 - M_\infty^2)\phi_x - \frac{1}{2}M_\infty^2[(3 - (2-\gamma)M_\infty^2)\phi_x^2 + \phi_y^2 + \phi_z^2]\}_x \\ +\{\phi_y\}_y + \{\phi_z\}_z = 0\end{aligned}$$

$$n_x + \phi_y n_y + \phi_z n_z = 0$$

When M_∞ is close to unity, $\frac{1}{2}M_\infty^2[(3 - (2-\gamma)M_\infty^2)\phi_x^2 + \phi_y^2 + \phi_z^2]$ is not small compared to $(1 - M_\infty^2)\phi_x$; however, transonic flows have strong lateral dilation, which means ϕ_y^2 and ϕ_z^2 are small compared to ϕ_x^2. In addition, $3-(2-\gamma)M_\infty^2$ can be approximated as $\gamma + 1$ when M_∞ is close to unity. Applying these transonic approximations is valid because the term effected is small when the flow is not transonic. These approximations result in the transonic small disturbance equation and flow tangency boundary condition:

$$\{(1 - M_\infty^2)\phi_x - \frac{\gamma+1}{2}M_\infty^2\phi_x^2\}_x + \{\phi_y\}_y + \{\phi_z\}_z = 0$$

$$n_x + \phi_y n_y + \phi_z n_z = 0$$

2.5 Prandtl-Glauert Equation

When the flow is sufficiently far from sonic, the transonic term may be dropped to give the Prandtl-Glauert equation and flow tangency boundary condition:

$$(1 - M_\infty^2)\phi_{xx} + \phi_{yy} + \phi_{zz} = 0$$

$$n_x + \phi_y n_y + \phi_z n_z = 0$$

The Prandtl-Glauert equation is a linear partial differential equation with a single unknown ϕ. This equation is only valid when all the assumptions mentioned in the derivation above apply.

2.6 Supersonic Point Source

In the incompressible limit of $M_\infty = 0$, the Prandtl-Glauert equation simplifies to the Laplace equation. A point source of strength Σ in an incompressible flow produces a volume flow rate that is equal to its strength:

$$\dot{\mathcal{V}} = \Sigma$$

By definition, the flow travels radially outward symmetrically in all directions, such that the flow velocity is only a function of the radius:

$$\mathbf{V} = V_r \hat{\mathbf{r}} = F(r)\hat{\mathbf{r}}$$

where:

$$r = \sqrt{x^2 + y^2 + z^2}$$

To satisfy conservation of mass, the volume flux, shown in Figure 2-3, must remain constant as the flow travels away from the source. This means the radial velocity must decrease at a rate that is proportional to the rate of increase of the area of the spherical shell:

$$\Sigma = \dot{\mathcal{V}} = V_r A_{shell}$$

where:

$$A_{shell} = 4\pi r^2$$

thus:

$$V_r(r) = \frac{\Sigma}{4\pi r^2}$$

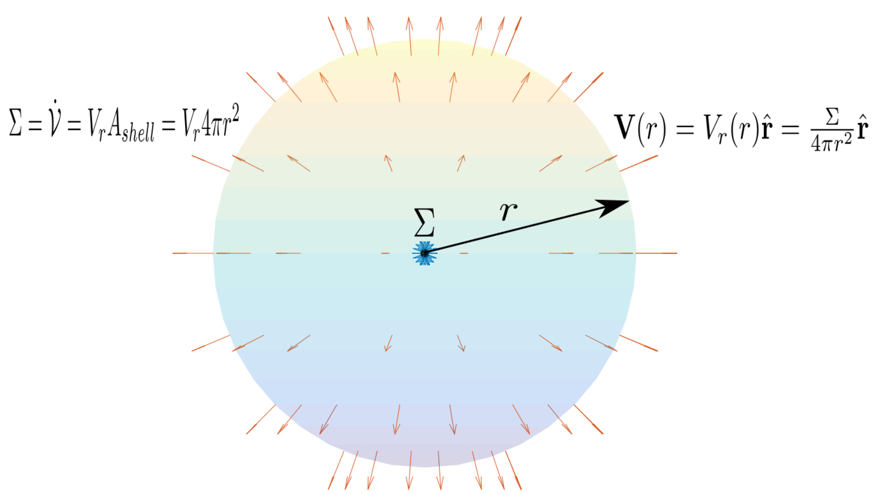

Figure 2-3: Incompressible Point Source Volume Flux

The gradient of the velocity potential must equal the velocity by definition, so the velocity potential of a point source in an incompressible flow is:

$$\phi(r) = -\frac{\Sigma}{4\pi r}$$

The point source velocity potential can be made to satisfy the Prandtl-Glauert equation with a small modification to r known as the Prandtl-Glauert transformation:

$$r(x, y, z; M_\infty) = \sqrt{x^2 + (1 - M_\infty^2)(y^2 + z^2)}$$

The velocity potential of a point source is now:

$$\phi(x, y, z; M_\infty) = -\frac{\Sigma}{4\pi\sqrt{x^2 + (1 - M_\infty^2)(y^2 + z^2)}}$$

For the subsonic case, the form of the solution to the Prandtl-Glauert equation for a point source remains the same as in the incompressible case. The Prandtl-Glauert equation is an elliptic partial differential equation, and the equations $r = constant$ represent ellipsoids, shown in Figure 2-4, on which the potential is constant.

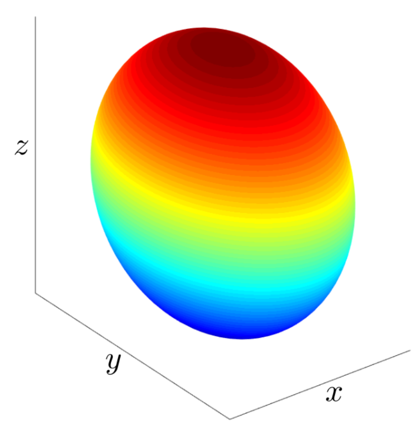

Figure 2-4: Constant r Ellipsoid in Subsonic Flow

For the supersonic case, the form of the solution to the Prandtl-Glauert equation for a point source becomes fundamentally different. The Prandtl-Glauert equation becomes a hyperbolic partial differential equation, and the equations $r = constant$ represent hyperboloids of two sheets, shown in Figure 2-5, on which the potential is constant.

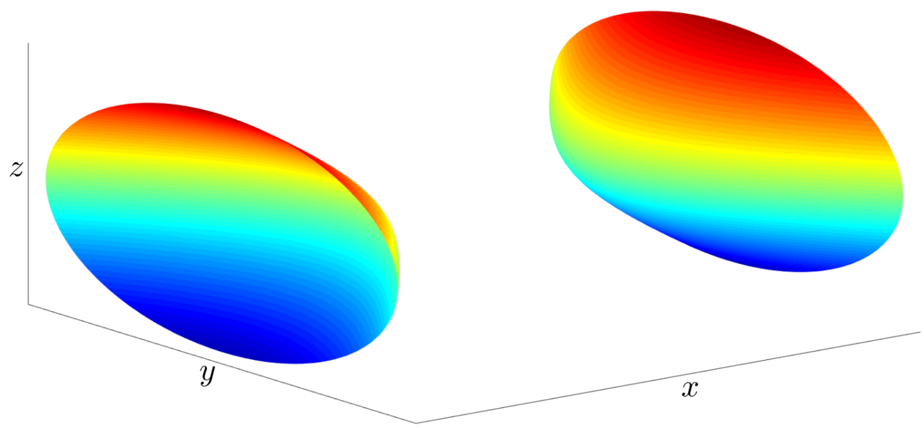

Figure 2-5: Constant r Hyperboloid in Supersonic Flow

For the supersonic Prandtl-Glauert transformation, it is convenient to use the variable:

$$\beta_{PG} = \sqrt{M_\infty^2 - 1}$$

The variable r will now become h, the so-called hyperbolic radius:

$$h(x, y, z; M_\infty) = \sqrt{x^2 - \beta_{PG}^2(y^2 + z^2)}$$

The equation $h = 0$ defines what are known as Mach cones. There are two Mach cones, shown in Figure 2-6, one extend upstream and one extending downstream from the point source.

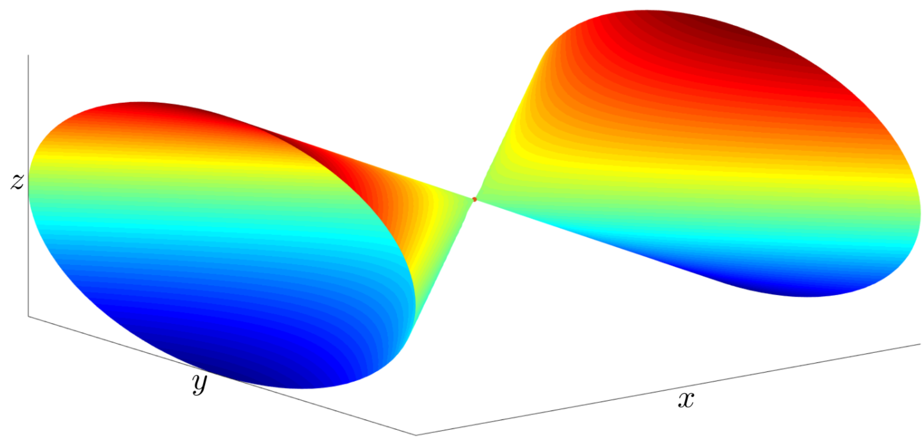

Figure 2-6: $h = 0$ Mach Cone Surfaces for Point at Apex

The Prandtl-Glauert equation makes no distinction between the upstream and downstream Mach cones, but physically disturbances cannot propagate upstream, so only the downstream Mach cone will be considered for the point source. To compensate for this, the velocity potential must be doubled to keep the volume flux the same as it is for the incompressible point source, so the velocity potential of a unit strength supersonic point source $\hat{\phi}_\Sigma$, shown in Figure 2-7, is:

$$\hat{\phi}_\Sigma(x, y, z; M_\infty) = -\frac{1}{2\pi h}$$

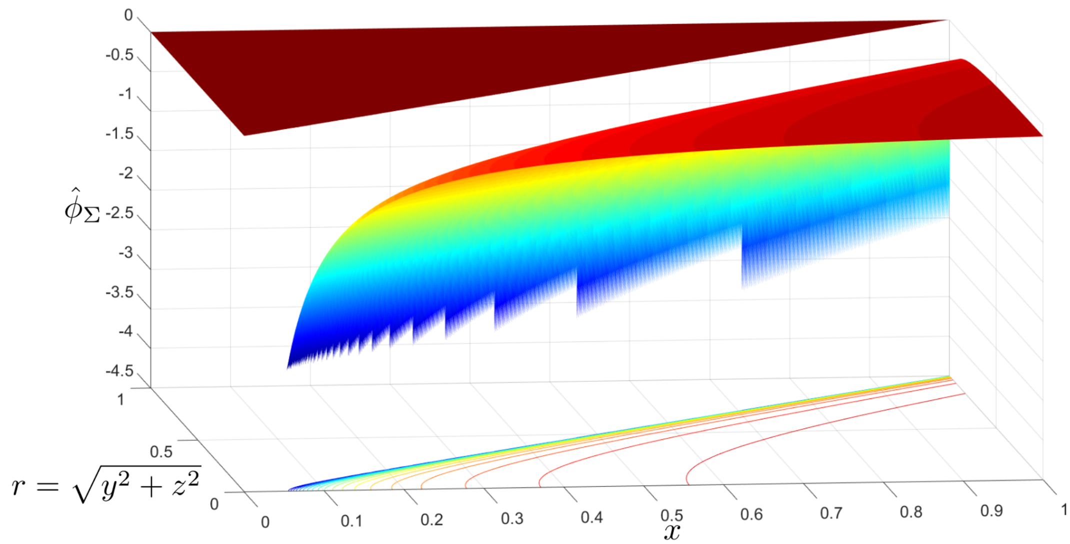

Figure 2-7: Velocity Potential of a Supersonic Point Source

It is important to note that the velocity potential of a supersonic point source is singular everywhere on its downstream Mach cone, not just at a single point as it is in the subsonic case. Also, the disturbance from the supersonic point source is restricted to inside the downstream Mach cone and does not affect the entire flow field as in the subsonic case.

The velocity potential is imaginary everywhere outside of the Mach cones, so the physical velocity potential can be obtained by taking the real part of the mathematical velocity potential; however, the velocity potential is also real inside the upstream Mach cone, so special care must be taken to discard the velocity potential for any points upstream of the point source:

$$\hat{\phi}_\Sigma(x, y, z; M_\infty) = \begin{cases} -\frac{1}{2\pi h} & , \; x > \beta_{PG}\sqrt{y^2 + z^2} \\ 0 & , \; x \leq \beta_{PG}\sqrt{y^2 + z^2} \end{cases}$$

2.7 Supersonic Z-Doublet

A supersonic z-doublet of strength κ_z can be used to represent lifting flow. The velocity potential of a unit strength supersonic z-doublet $\hat{\phi}_{\kappa_z}$, shown in Figures 2-8 and 2-9, is obtained by taking the limit as the separation distance goes to zero of a supersonic point source and sink each of strength Σ separated by a distance z along the z-axis while keeping the product $\Sigma z = 1$ constant.

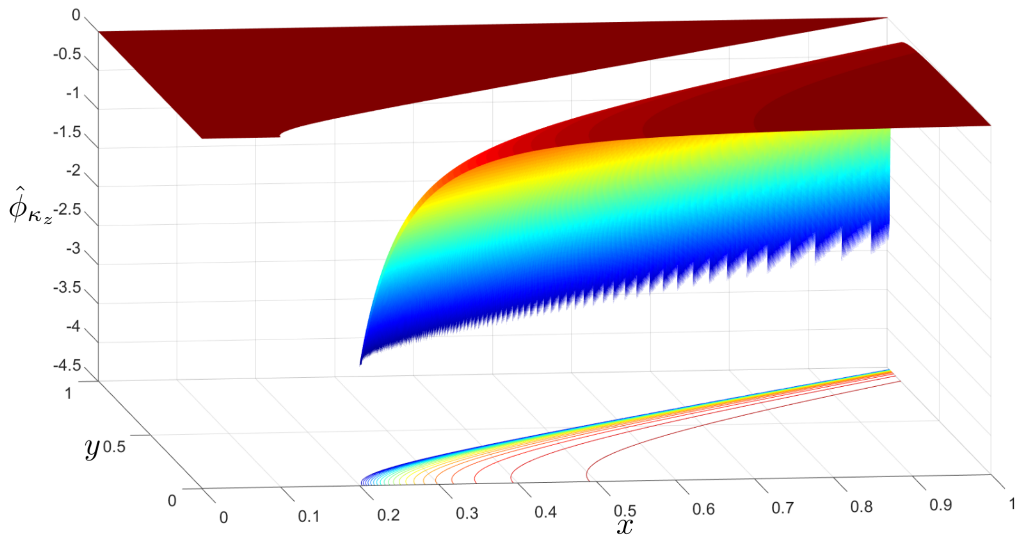

Figure 2-8: Velocity Potential of a Supersonic Z-Doublet at $z = 0.1$

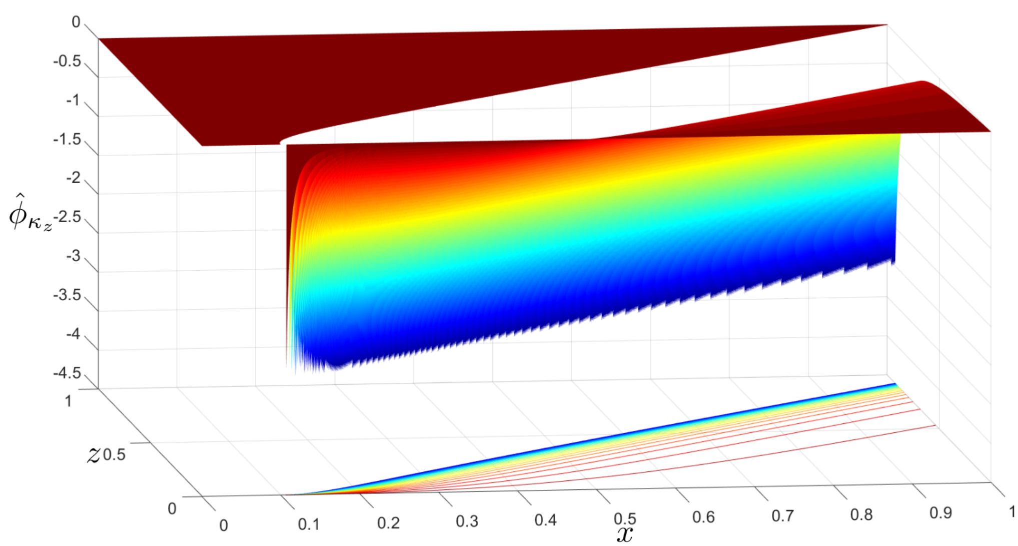

Figure 2-9: Velocity Potential of a Supersonic Z-Doublet at $y = 0.1$

The same expression can also be obtained by taking the z-derivative of the velocity potential of a unit strength supersonic point source:

$$\hat{\phi}_{\kappa_z}(x,y,z;M_\infty) = \frac{\partial \hat{\phi}_\Sigma}{\partial z} = \frac{\partial \hat{\phi}_\Sigma}{\partial h}\frac{\partial h}{\partial z} = \frac{1}{2\pi h^2}\frac{-\beta_{PG}^2 z}{h} = \frac{1}{2\pi}\frac{-\beta_{PG}^2 z}{h^3}$$

Again special care must be taken to discard the velocity potential of the supersonic z-double outside of its downstream Mach cone:

$$\hat{\phi}_{\kappa_z}(x,y,z;M_\infty) = \begin{cases} \frac{1}{2\pi}\frac{-\beta_{PG}^2 z}{h^3} & , \; x > \beta_{PG}\sqrt{y^2+z^2} \\ 0 & , \; x \leq \beta_{PG}\sqrt{y^2+z^2} \end{cases}$$

2.8 Infinitesimal Width Supersonic Horseshoe Vortex

The infinitesimal width supersonic horseshoe vortex is equivalent to a semi-infinite z-doublet line extending downstream along the x-axis from the origin. The velocity potential for the unit strength infinitesimal width supersonic horseshoe vortex $\hat{\phi}_{\Gamma_z}$ is obtained by integrating the velocity potential over the semi-infinite supersonic z-doublet. In order to perform this integration, the hyperbolic radius needs to be shifted by the integration variable x', which represents the location along the semi-infinite z-doublet line:

$$h(x-x',y,z;M_\infty) = \sqrt{(x-x')^2 - \beta_{PG}^2(y^2+z^2)}$$

A field point located at (x,y,z) only feels the influence from the portion of the horseshoe vortex that is inside the field point's upstream Mach cone. This can be accounted for by truncating the horseshoe vortex, so the limits of integration along the x-axis go from 0 to $x - \beta_{PG}\sqrt{y^2+z^2}$. The intersection of the domain of dependence of a field point with the plane of an infinitesimal width horseshoe vortex is shown in

Figure 2-10.

$$
\begin{aligned}
\hat{\phi}_{\Gamma_z}(x,y,z;M_\infty) &= \int_0^{x-\beta_{PG}\sqrt{y^2+z^2}} \hat{\phi}_{\kappa_z}(x-x',y,z;M_\infty)dx' \\
&= \frac{-\beta_{PG}^2 z}{2\pi}\int_0^{x-\beta_{PG}\sqrt{y^2+z^2}} \frac{dx'}{[(x-x')^2-\beta_{PG}^2(y^2+z^2)]^{3/2}} \\
&= \frac{-\beta_{PG}^2 z}{2\pi}\left.\frac{x-x'}{\beta_{PG}^2(y^2+z^2)\sqrt{(x-x')^2-\beta_{PG}^2(y^2+z^2)}}\right|_0^{x-\beta_{PG}\sqrt{y^2+z^2}} \\
&= \frac{1}{2\pi}\frac{z}{y^2+z^2}\frac{x}{\sqrt{x^2-\beta_{PG}^2(y^2+z^2)}}
\end{aligned}
$$

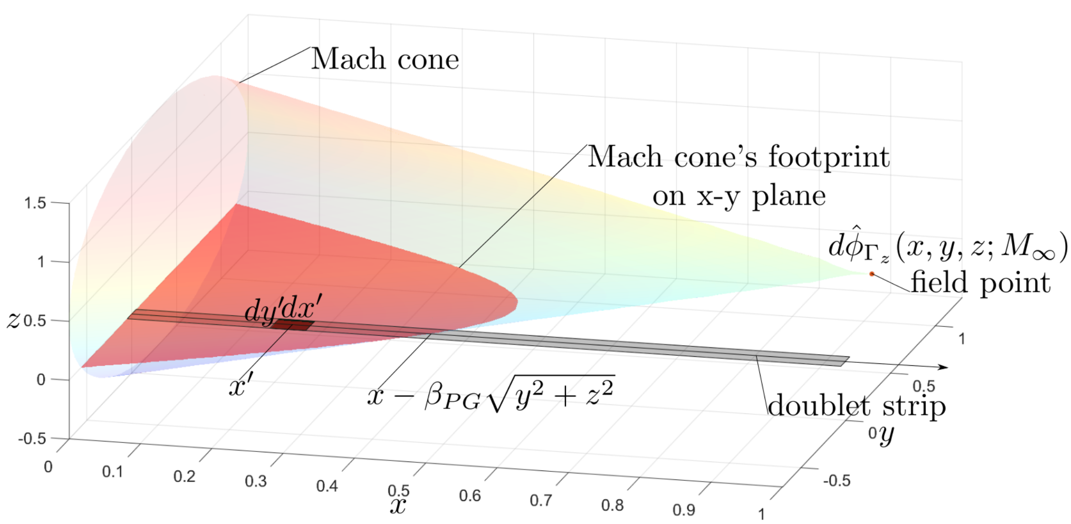

Figure 2-10: Velocity Potential Integration of an Infinitesimal Width Supersonic Horseshoe Vortex

Note that the indefinite integral evaluated at the upper integration limit is infinite, but this term is imaginary so it is discarded. The only restriction on the domain of influence of the infinitesimal width supersonic horseshoe vortex is $x > 0$. Redefining h as the hyperbolic radius from the origin again, the potential of an infinitesimal width supersonic horseshoe vortex, shown in Figures 2-11 and 2-12, becomes:

$$\hat{\phi}_{\Gamma_z}(x,y,z;M_\infty) = \begin{cases} \frac{1}{2\pi}\frac{z}{y^2+z^2}\frac{x}{h} & , \ x > 0 \\ 0 & , \ x \le 0 \end{cases}$$

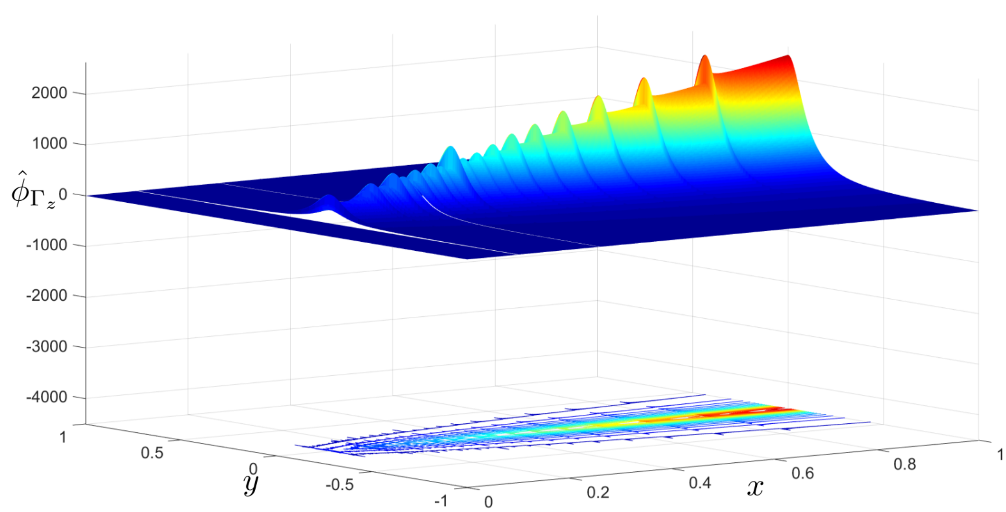

Figure 2-11: Velocity Potential of a Supersonic Infinitesimal Width Horseshoe Vortex at $z = 0.1$

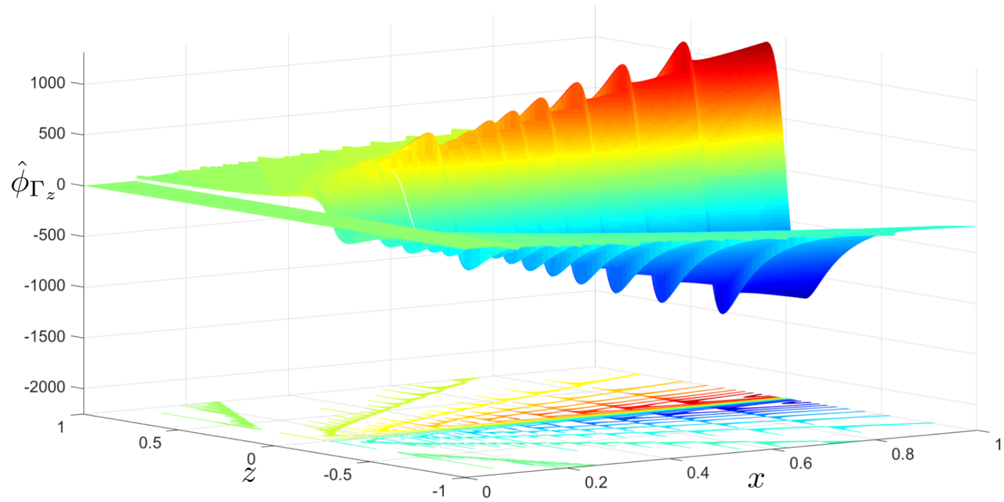

Figure 2-12: Velocity Potential of a Supersonic Infinitesimal Width Horseshoe Vortex at $y = 0.1$

2.9 Tapered Supersonic Doublet Panel

In the present formulation, lifting surfaces will be represented by their mean camber line and discretized into several doublet panels. These doublet panels will reside entirely in the x-y plane and the slope of the mean camber line will only affect the surface normal vectors of the control points used in the flow tangency boundary condition. The side edges of the panels are required to be parallel to the freestream direction, but the leading and trailing edges of the panels are allowed to be swept at different angles. The doublet sheet strength μ will be constant along the leading and trailing edge of each panel and will vary linearly in the streamwise direction. For a doublet panel of this form, the leading edge (0) and trailing edge (1) lines are expressed as:

$$x_0(y) = a_0 + b_0 y$$

$$x_1(y) = a_1 + b_1 y$$

and the doublet sheet strength, shown in Figure 2-13, as:

$$\mu(x,y) = \mu_0 + \frac{(\mu_1 - \mu_0)[x - x_0(y)]}{x_1(y) - x_0(y)}$$

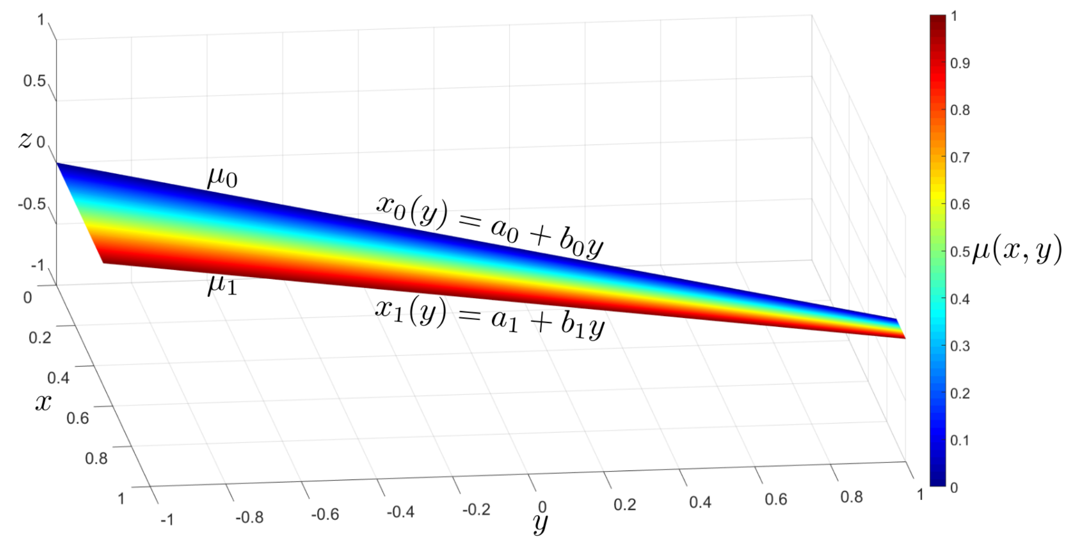

Figure 2-13: Tapered Panel Doublet Sheet Strength

The vortex sheet strength vector $\boldsymbol{\gamma}$, shown in Figure 2-14, is defined as:

$$\boldsymbol{\gamma} = \hat{\mathbf{n}} \times \tilde{\nabla}\mu$$

where $\tilde{\nabla}\mu$ is the surface gradient of μ.

Here the normal vector $\hat{\mathbf{n}}$ is required to be a unit vector, unlike in the flow tangency boundary condition. Plugging in the above definitions of μ, x_0, and x_1:

$$\begin{aligned} \tilde{\nabla}\mu(x,y) &= \frac{\partial \mu}{\partial x}\hat{\mathbf{x}} + \frac{\partial \mu}{\partial y}\hat{\mathbf{y}} \\ &= \frac{\mu_1 - \mu_0}{a_1 - a_0 + (b_1 - b_0)y}\hat{\mathbf{x}} - \left\{ \frac{(\mu_1 - \mu_0)(b_1 - b_0)[x - (a_0 + b_0 y)]}{[a_1 - a_0 + (b_1 - b_0)y]^2} + \frac{(\mu_1 - \mu_0)b_0}{a_1 - a_0 + (b_1 - b_0)y} \right\}\hat{\mathbf{y}} \end{aligned}$$

For a panel in the x-y plane:

$$\hat{\mathbf{n}} = \hat{\mathbf{z}}$$

thus:

$$\begin{aligned}\boldsymbol{\gamma}(x,y) &= \hat{\mathbf{z}} \times \tilde{\nabla}\mu = \hat{\mathbf{z}} \times \left\{\frac{\partial\mu}{\partial x}\hat{\mathbf{x}} + \frac{\partial\mu}{\partial y}\hat{\mathbf{y}}\right\} = -\frac{\partial\mu}{\partial y}\hat{\mathbf{x}} + \frac{\partial\mu}{\partial x}\hat{\mathbf{y}} \\ &= \left\{\frac{(\mu_1-\mu_0)(b_1-b_0)[x-(a_0+b_0y)]}{[a_1-a_0+(b_1-b_0)y]^2} + \frac{(\mu_1-\mu_0)b_0}{a_1-a_0+(b_1-b_0)y}\right\}\hat{\mathbf{x}} + \frac{\mu_1-\mu_0}{a_1-a_0+(b_1-b_0)y}\hat{\mathbf{y}}\end{aligned}$$

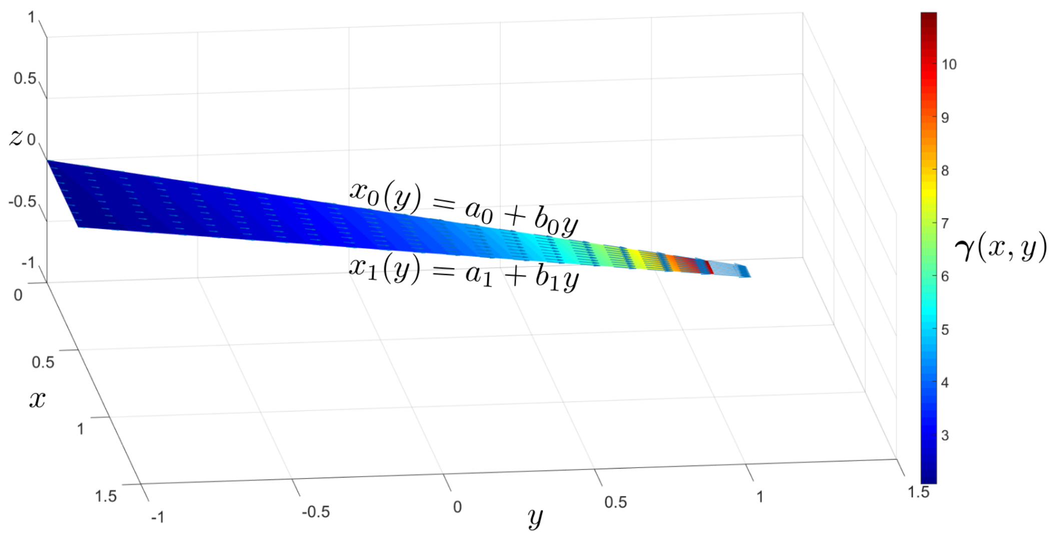

Figure 2-14: Tapered Panel Vortex Sheet Strength Vector Field

The velocity potential of a tapered supersonic doublet panel with leading edge doublet sheet strength μ_0 and trailing edge doublet sheet strength μ_1 at a field point located at (x, y, z) is calculated by integrating the expression for the potential of an infinitesimal width supersonic horseshoe vortex over the panel span with the appropriate $\Gamma_z(y)$. The infinitesimal width supersonic horseshoe vortex of strength $\Gamma_z(y)$ is equivalent to a z-doublet line of constant strength μ, but since these panels do not have constant doublet sheet strength in the streamwise direction, μ must be replaced with $\frac{\partial\mu}{\partial x}$ and integrated over the streamwise length of the panel. This expression is then integrated over the span of the panel. To perform this integration, x must be replaced with $x - x'$ and y must be replaced with $y - y'$, where (x', y') represents the location within the panel. The x limits of integration are from x_0 (leading edge) to x_1 (trailing edge) and the y limits of integration are from y_0 (left edge) to y_1 (right edge). The intersection of the domain of dependence of a field point with the plane of a tapered supersonic doublet panel is shown in Figure 2-15. Note that the x limits of integration depend on y' to account for the taper. The expression obtained from evaluating this integral is omitted from this section due to its length and can be found in appendix A.

$$\phi(x,y,z;M_\infty) = \int_{y_0}^{y_1}\int_{x_0(y')}^{x_1(y')} \frac{\partial\mu(x',y')}{\partial x'}\hat{\phi}_{\Gamma_z}(x-x',y-y',z;M_\infty)dx'dy'$$
$$= \frac{1}{2\pi}\int_{y_0}^{y_1}\int_{a_0+b_0y'}^{a_1+b_1y'} \frac{\mu_1-\mu_0}{a_1-a_0+(b_1-b_0)y'}\frac{z}{(y-y')^2+z^2}\frac{x-x'}{\sqrt{(x-x')^2-\beta_{PG}^2[(y-y')^2+z^2]}}dx'dy'$$

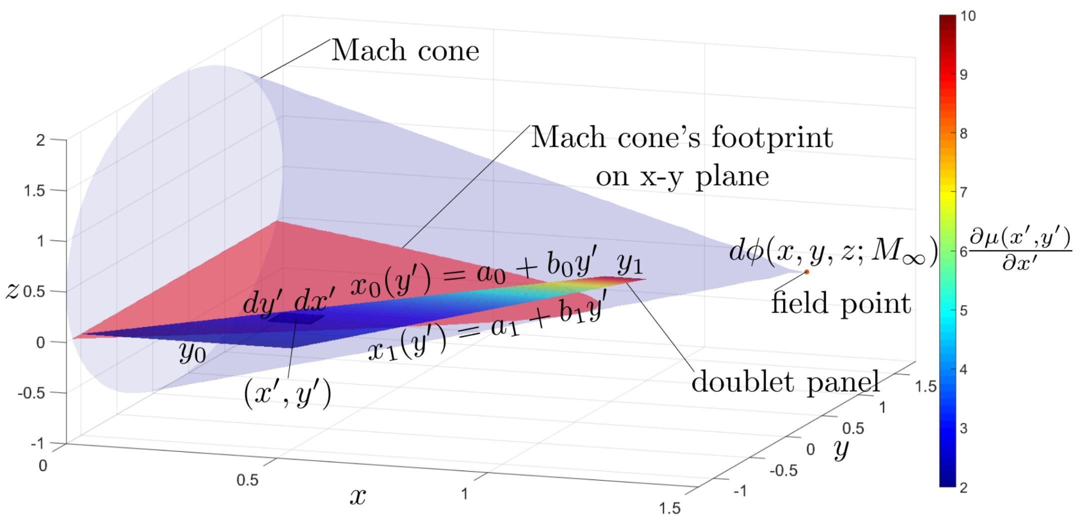

Figure 2-15: Velocity Potential Integration of a Tapered Supersonic Doublet Panel

A field point located at (x, y, z) only feels the influence from the portion of the tapered supersonic doublet panel that is inside the field point's upstream Mach cone. The velocity potential is imaginary everywhere outside of the field point's Mach cones, so the physical velocity potential can be obtained by taking the real part of the mathematical velocity potential; however, the velocity potential is also real inside the field point's downstream Mach cone, so special care must be taken to discard the velocity potential from any portions of the panel downstream of the field point. The simplest way to do this is to truncate the panel downstream of the field point before evaluating the superposition integral; however, this task is non-trivial and requires the treatment of multiple cases.

Case 1

If the field point's x location is downstream of the entire panel, the integral can be evaluated without modification.

$$\phi(x,y,z;M_\infty) = \frac{1}{2\pi}\int_{y_0}^{y_1}\int_{x_0(y')}^{x_1(y')} \frac{\mu_1-\mu_0}{a_1-a_0+(b_1-b_0)y'}\frac{z}{(y-y')^2+z^2}\frac{x-x'}{\sqrt{(x-x')^2-\beta_{PG}^2[(y-y')^2+z^2]}}dx'dy'$$

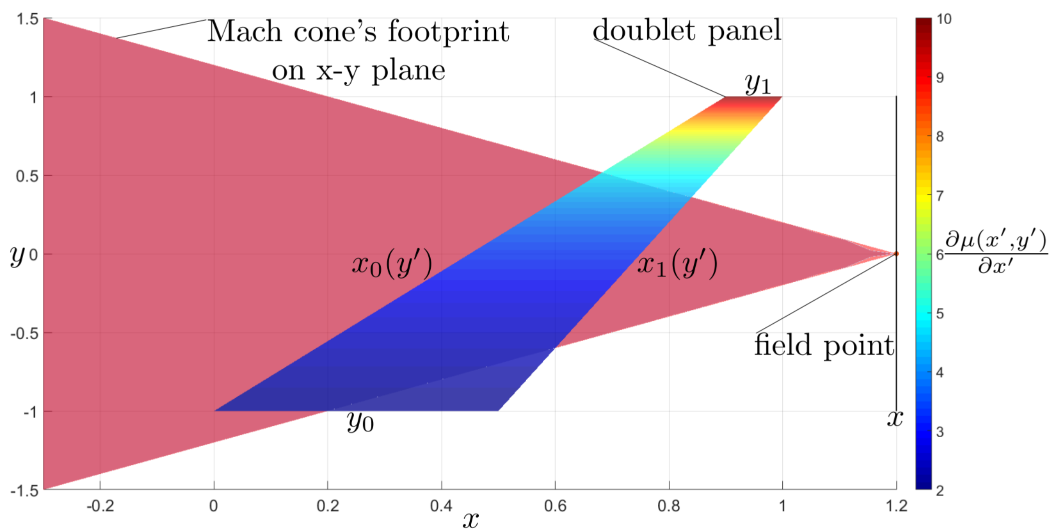

Figure 2-16: Case 1

Case 2

If x is upstream of a portion of the panel trailing edge, the panel must be divided into two sections at the span location y^* such that:

$$x_1(y^*) = x = a_1 + b_1 y^*$$

thus:

$$y^* = \frac{(x - a_1)}{b_1}$$

On the side of y^* where the trailing edge of the panel is upstream of x, the integral can be evaluated as is. On the other side, the upper x limit needs to be changed from x_1 to x to truncate the panel downstream of x. There will now be two integrals, one going from y_0 to y^* and the other from y^* to y_1. Note that the integrand goes to zero as x' approaches x.

$$\begin{aligned}
&\phi(x, y, z; M_\infty) \\
&= \frac{1}{2\pi} \int_{y_0}^{y^*} \int_{x_0(y')}^{x_1(y')} \frac{\mu_1 - \mu_0}{a_1 - a_0 + (b_1 - b_0)y'} \frac{z}{(y - y')^2 + z^2} \frac{x - x'}{\sqrt{(x - x')^2 - \beta_{PG}^2[(y - y')^2 + z^2]}} dx'dy' \\
&+ \frac{1}{2\pi} \int_{y^*}^{y_1} \int_{x_0(y')}^{x} \frac{\mu_1 - \mu_0}{a_1 - a_0 + (b_1 - b_0)y'} \frac{z}{(y - y')^2 + z^2} \frac{x - x'}{\sqrt{(x - x')^2 - \beta_{PG}^2[(y - y')^2 + z^2]}} dx'dy'
\end{aligned}$$

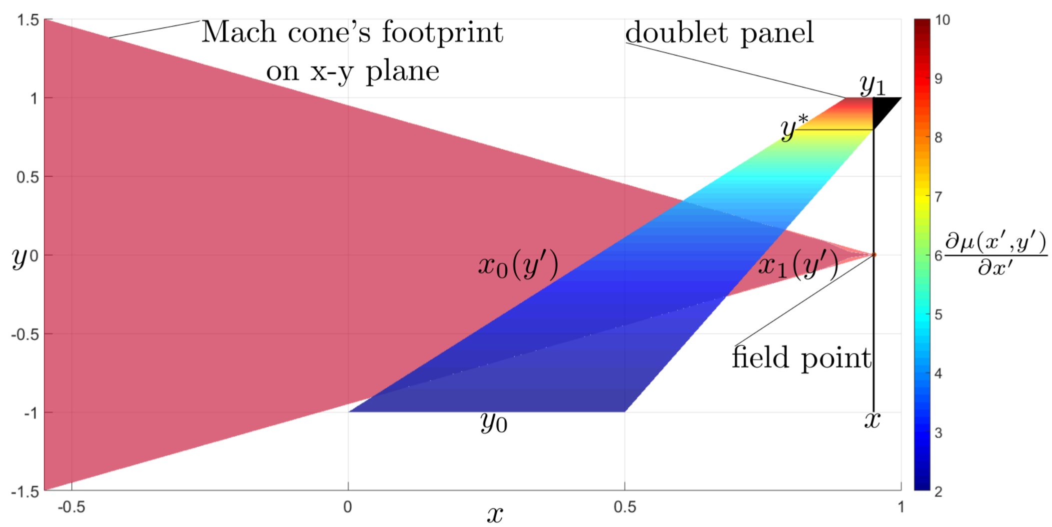

Figure 2-17: Case 2

Case 3

If x is upstream of a portion of the panel leading edge, the y limits of integration must be changed to eliminate the section of the panel that is entirely downstream of the field point. Either y_0 or y_1 needs to be changed to y_0^* or y_1^* such that:

$$x_0(y_0^*) = x = a_0 + b_0 y_0^*$$
$$x_0(y_1^*) = x = a_0 + b_0 y_1^*$$

thus:

$$y_0^* = \frac{(x - a_0)}{b_0}$$
$$y_1^* = \frac{(x - a_0)}{b_0}$$

$$\begin{aligned}
&\phi(x, y, z; M_\infty) \\
&= \frac{1}{2\pi}\int_{y_0}^{y^*}\int_{x_0(y')}^{x_1(y')} \frac{\mu_1 - \mu_0}{a_1 - a_0 + (b_1 - b_0)y'} \frac{z}{(y - y')^2 + z^2} \frac{x - x'}{\sqrt{(x - x')^2 - \beta_{PG}^2[(y - y')^2 + z^2]}} dx'dy' \\
&+ \frac{1}{2\pi}\int_{y^*}^{y_1^*}\int_{x_0(y')}^{x} \frac{\mu_1 - \mu_0}{a_1 - a_0 + (b_1 - b_0)y'} \frac{z}{(y - y')^2 + z^2} \frac{x - x'}{\sqrt{(x - x')^2 - \beta_{PG}^2[(y - y')^2 + z^2]}} dx'dy'
\end{aligned}$$

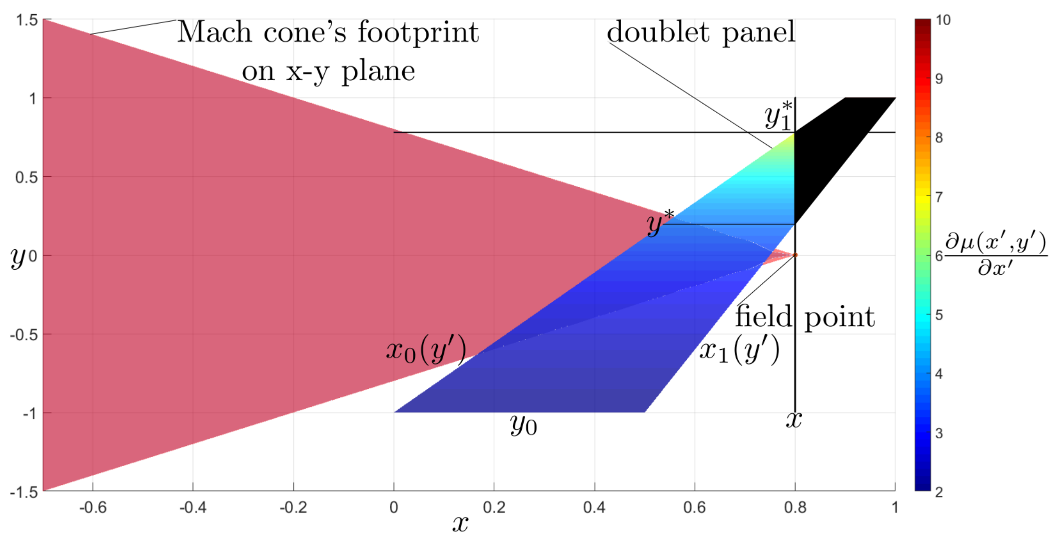

Figure 2-18: Case 3

Case 4

If x is upstream of the entire trailing edge, only one integral is needed with x limits from x_0 to x.

$$\phi(x,y,z;M_\infty) = \frac{1}{2\pi}\int_{y_0}^{y_1^*}\int_{x_0(y')}^{x}\frac{\mu_1-\mu_0}{a_1-a_0+(b_1-b_0)y'}\frac{z}{(y-y')^2+z^2}\frac{x-x'}{\sqrt{(x-x')^2-\beta_{PG}^2[(y-y')^2+z^2]}}dx'dy'$$

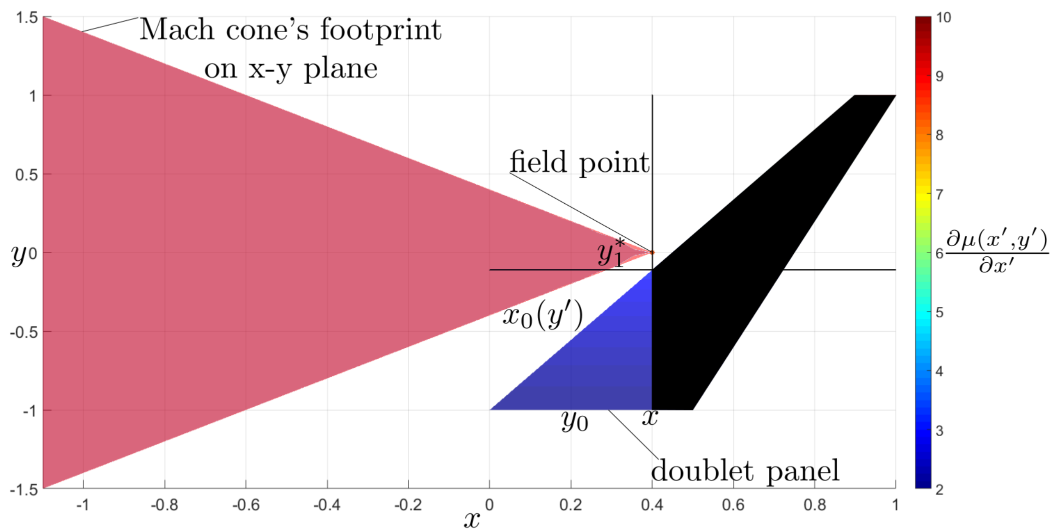

Figure 2-19: Case 4

Case 5

If x is upstream of the entire panel, the potential at the field point is zero.

$$\phi(x,y,z;M_\infty) = 0$$

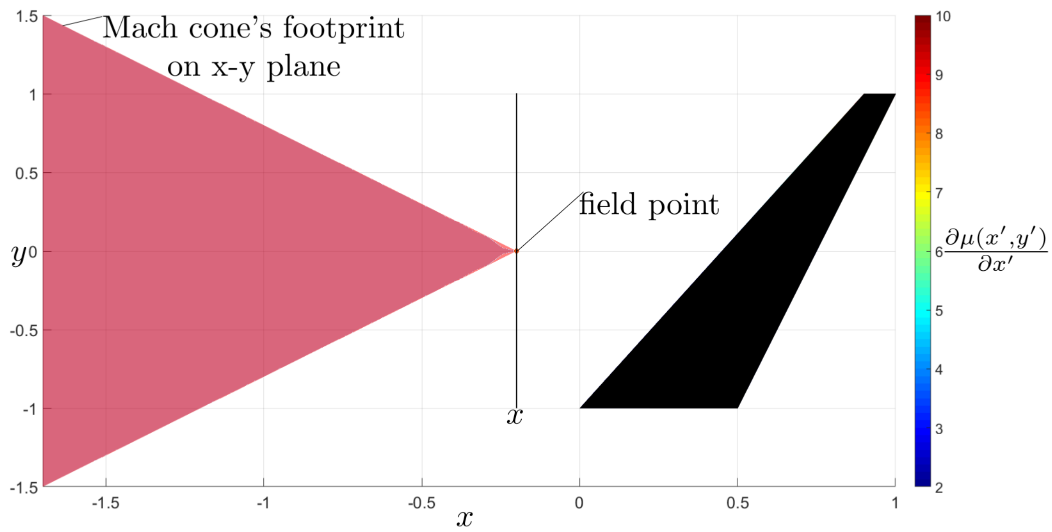

Figure 2-20: Case 5

2.10 Perturbation Velocity Components

The perturbation velocity components of a tapered supersonic doublet panel are obtained by taking the spatial derivatives of the velocity potential. These expressions are omitted from this section due to their length and can be found in appendix A:

$$u = \frac{\partial \phi}{\partial x}$$
$$v = \frac{\partial \phi}{\partial y}$$
$$w = \frac{\partial \phi}{\partial z}$$

One way to examine the validity of these expressions is to plot the perturbation velocity component transverse to the x-axis in the y-z plane far behind the doublet panel in the so called Trefftz plane. Here the panel should look like a single horseshoe vortex in an incompressible flow.

Far behind the doublet panel, only the integral vortex strength $\Gamma_z(y')$ matters. The details in the variation of the doublet sheet strength over the chord of the panel are not seen:

$$\Gamma_z(y') = \int_{x_0(y')}^{x_1(y')} \frac{\partial \mu(x', y')}{\partial x'}$$

Since $\frac{\partial \mu(x',y')}{\partial x'}$ is inversely proportional to the chord, the integral vortex strength Γ_z is constant. Because Γ_z is constant, when it is integrated across the span of the panel, all interior vortex pairs cancel leaving only a single horseshoe vortex. This holds for both non-tapered (Figure 2-21) and tapered (Figure 2-22) panels.

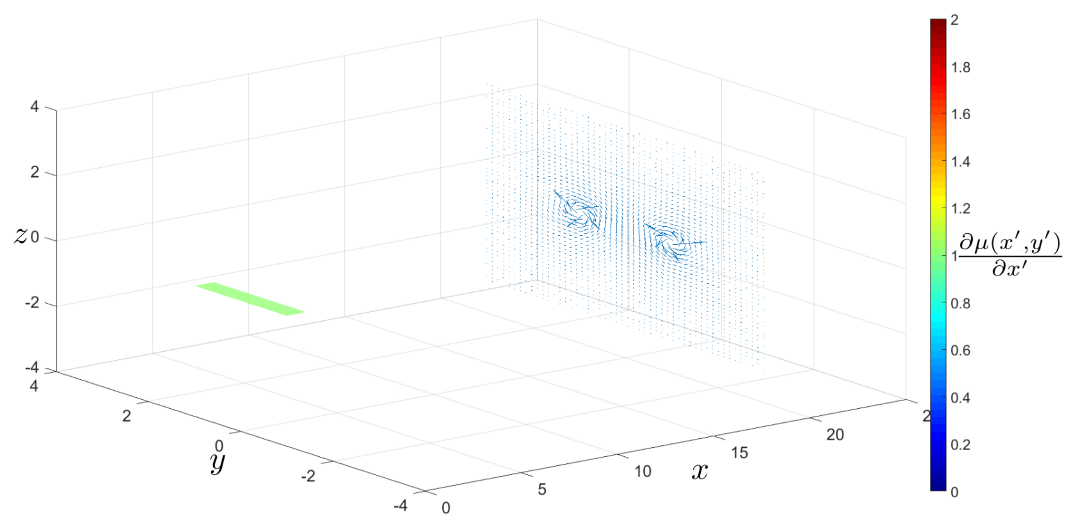

Figure 2-21: Non-tapered Panel Trefftz Plane Velocity

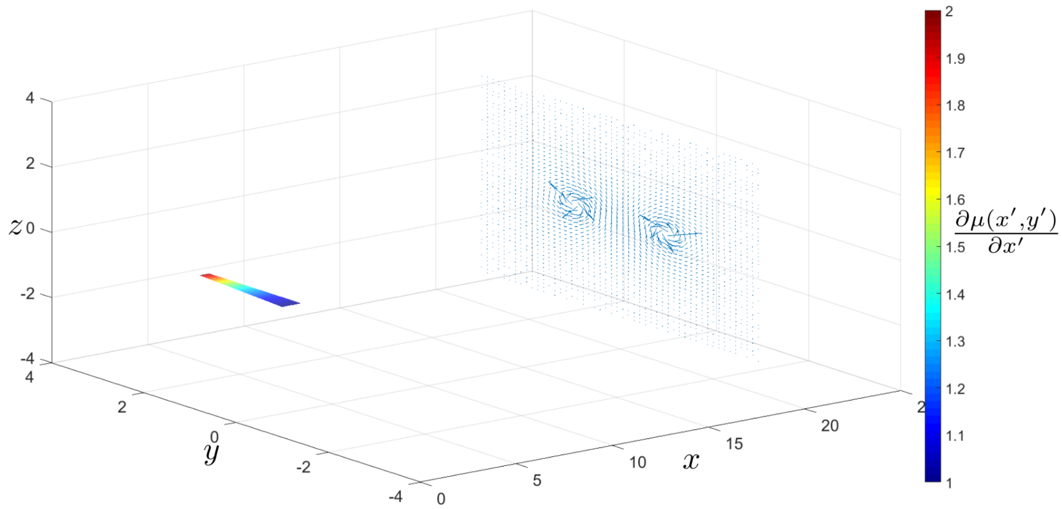

Figure 2-22: Tapered Panel Trefftz Plane Velocity

2.11 Woodward Vortex Panel Method

The Woodward method uses panels with a $\frac{\partial\mu(x',y')}{\partial x'}$ that is constant in the spanwise direction, even for tapered panels. Since $\frac{\partial\mu(x',y')}{\partial x'}$ is constant spanwise, the integral vortex strength Γ_z is proportional to the local chord length in the spanwise direction. Because Γ_z is not constant, when it is integrated across the span of the panel, interior vortex pairs do not cancel for tapered panels with different slopes at the leading and trailing edges. The resulting Trefftz plane velocity is not that of a single horseshoe vortex in this case.

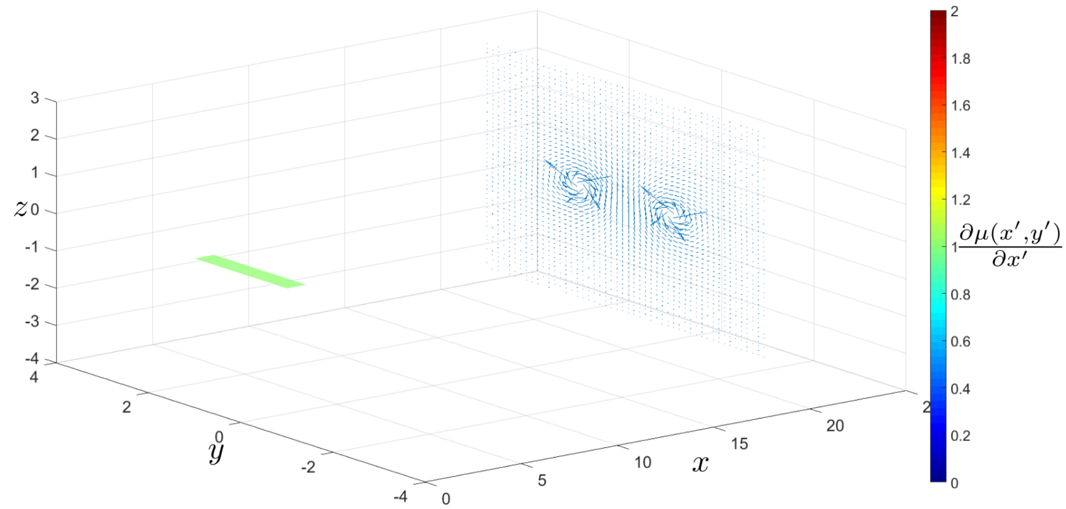

Figure 2-23: Woodward Non-tapered Panel Trefftz Plane Velocity

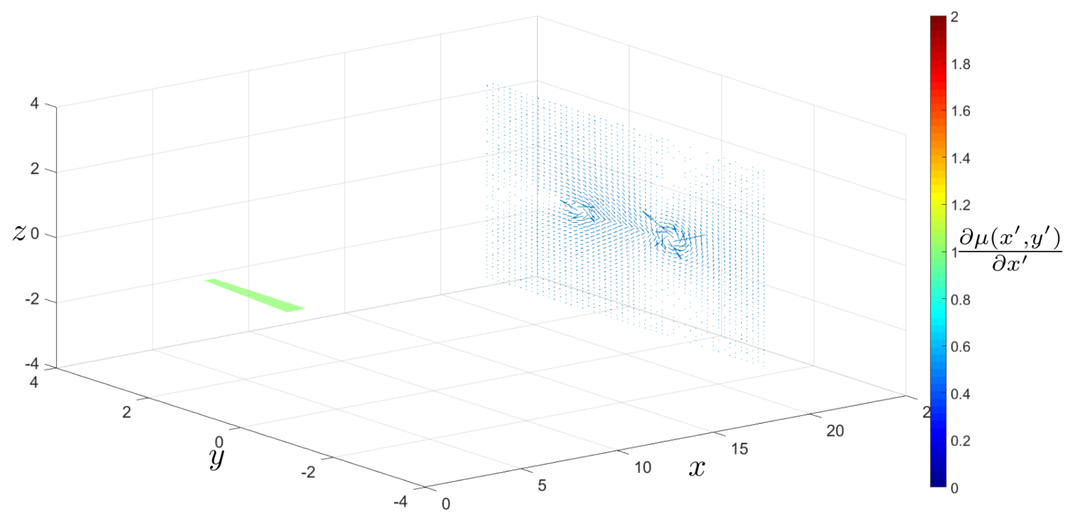

Figure 2-24: Woodward Tapered Panel Trefftz Plane Velocity

The expressions plotted above in Figures 2-23 and 2-24 are from Woodward panels having a constant $\frac{\partial \mu(x',y')}{\partial x'}$ in the streamwise direction. Woodward also uses vortex panels with a linearly varying $\frac{\partial \mu(x',y')}{\partial x'}$ in the streamwise direction, but these still have constant $\frac{\partial \mu(x',y')}{\partial x'}$ in the spanwise direction, so the results still do not represent a single horseshoe vortex in the case of tapered panels with different leading and trailing edge slopes.

Chapter 3

Implementation

The matrix system that arises from the enforcement of the flow tangency boundary conditions is derived. The supersonic doublet panel code that solves this matrix system is described in detail. A method for calculating the aerodynamic coefficients is presented. The numerical solutions of this code are compared to the analytical solutions of four test cases.

3.1 Flow Tangency Boundary Condition

To evaluate the aerodynamic properties of lifting surfaces, the supersonic doublet panel singularity strength parameters must be determined such that the flow tangency boundary condition is satisfied at each control point:

$$\mathbf{V}(x_{cp}, y_{cp}, z_{cp}) \cdot \mathbf{n}(x_{cp}, y_{cp}, z_{cp}) = 0$$

For steady flow at an angle of attack α and sideslip β:

$$\begin{aligned}
\mathbf{V}(x_{cp}, y_{cp}, z_{cp}) &= \mathbf{V}_\infty + \mathbf{V}_\mu(x_{cp}, y_{cp}, z_{cp}) \\
&= V_\infty \cos(\alpha)\cos(\beta)\hat{\mathbf{x}} - V_\infty \sin(\beta)\hat{\mathbf{y}} + V_\infty \sin(\alpha)\cos(\beta)\hat{\mathbf{z}} \\
&+ V_\infty u(x_{cp}, y_{cp}, z_{cp})\hat{\mathbf{x}} + V_\infty v(x_{cp}, y_{cp}, z_{cp})\hat{\mathbf{y}} + V_\infty w(x_{cp}, y_{cp}, z_{cp})\hat{\mathbf{z}}
\end{aligned}$$

$$
\begin{aligned}
&= V_\infty(\cos(\alpha)\cos(\beta) + u(x_{cp}, y_{cp}, z_{cp}))\hat{\mathbf{x}} + V_\infty(-\sin(\beta) + v(x_{cp}, y_{cp}, z_{cp}))\hat{\mathbf{y}} \\
&+ V_\infty(\sin(\alpha)\cos(\beta) + w(x_{cp}, y_{cp}, z_{cp}))\hat{\mathbf{z}}
\end{aligned}
$$

Plugging the above expression for $\mathbf{V}$ into the flow tangency boundary condition and dividing by V_∞ yields:

$$
\begin{aligned}
[\cos(\alpha)\cos(\beta) + u(x_{cp}, y_{cp}, z_{cp})]n_x + [-\sin(\beta) + v(x_{cp}, y_{cp}, z_{cp})]n_y \\
+[\sin(\alpha)\cos(\beta) + w(x_{cp}, y_{cp}, z_{cp})]n_z = 0
\end{aligned}
$$

Rearranging then yields:

$$
un_x + vn_y + wn_z = -\cos(\alpha)\cos(\beta)n_x + \sin(\beta)n_y - \sin(\alpha)\cos(\beta)n_z
$$

In this chapter, the freestream is no longer aligned with the x-axis. The axes will be those used to specify the geometry of the aircraft.

3.2 Matrix System

The perturbation velocity components u, v, and w depend linearly on the supersonic doublet panel singularity strength parameter $\Delta\mu = \mu_1 - \mu_0$. Each panel contributes to the perturbation velocity components of all the control points in its downstream Mach cone. Let $\hat{u}_{ij}$ be the contribution of the j_{th} panel to the u perturbation velocity component of the i_{th} control point per unit $\Delta\mu$. Using this notation, the above flow tangency boundary condition can be written in matrix form:

$$
[n_{xii}\hat{u}_{ij} + n_{y_{ii}}\hat{v}_{ij} + n_{zii}\hat{w}_{ij}]\Delta\mu_j = -\cos(\alpha)\cos(\beta)n_{xi} + \sin(\beta)n_{y_i} - \sin(\alpha)\cos(\beta)n_{zi}
$$

The matrix $[n_{xii}\hat{u}_{ij} + n_{y_{ii}}\hat{v}_{ij} + n_{zii}\hat{w}_{ij}]$ is the supersonic AIC matrix. For this system to have a unique solution, the number of control points must be equal to the number of panels.

3.3 Supersonic Doublet Panel Code

The supersonic doublet panel MATLAB function calculates the following outputs:

- supersonic doublet panel singularity strength parameters: $\Delta\mu$
- velocity potential evaluated at control points: ϕ
- perturbation velocity components evaluated at control points: u, v, and w

And requires the following inputs:

- x, y, and z coordinates of the control points: (x_{cp}, y_{cp}, z_{cp})
- x, y, and z components of the control point normal vectors: $(n_{x_{cp}}, n_{y_{cp}}, n_{z_{cp}})$
- y coordinates of left and right panel side edge: (y_l, y_r)
- x coordinates of left leading edge, left trailing edge, right leading edge, and right trailing edge panel corners: $(x_{l,le}, x_{l,te}, x_{r,le}, x_{r,te})$
- flow angles: α and β
- flow Mach number: M_∞

```
function [dm,phi,u,v,w] = supersonic_doublet_panel(
   x_cp_vec,y_cp_vec,...
    z_cp_vec,n_x_cp,n_y_cp,n_z_cp,y_l_vec,y_r_vec,
       x_l_le_vec,x_l_te_vec,...
    x_r_le_vec,x_r_te_vec,alpha,beta,M)
```

The first step is to calculate the Prandtl-Glauert factor β_{PG}, not to be confused with the sideslip angle β, and the geometric coefficients a_0, a_1, b_0, and b_1. The supersonic Prandtl-Glauert factor is defined as:

$$\beta_{PG} = \sqrt{M_\infty^2 - 1}$$

Calculating the geometric coefficients requires a few algebraic manipulations. Using the definitions of the leading and trailing edges:

$$
\begin{aligned}
x_{l,le} &= a_0 + b_0 y_l \\
x_{r,le} &= a_0 + b_0 y_r \\
x_{l,te} &= a_1 + b_1 y_l \\
x_{r,te} &= a_1 + b_1 y_r
\end{aligned}
$$

Multiplying the first and third equations by (y_r/y_l) and eliminating b_0 and b_1 yields:

$$
\begin{aligned}
a_0 &= \frac{x_{r,le} - x_{l,le}(y_r/y_l)}{1 - (y_r/y_l)} \\
a_1 &= \frac{x_{r,te} - x_{l,te}(y_r/y_l)}{1 - (y_r/y_l)}
\end{aligned}
$$

If $y_l = 0$, a_0 and a_1 must instead be calculated multiplying the second and fourth equations by (y_l/y_r) and eliminating b_0 and b_1:

$$
\begin{aligned}
a_0 &= \frac{x_{r,le}(y_l/y_r) - x_{l,le}}{(y_l/y_r) - 1} \\
a_1 &= \frac{x_{r,te}(y_l/y_r) - x_{l,te}}{(y_l/y_r) - 1}
\end{aligned}
$$

b_0 and b_1 are calculated by subtracting the first equation from the second and the third equation from the fourth:

$$
\begin{aligned}
b_0 &= \frac{x_{r,le} - x_{l,le}}{y_r - y_l} \\
b_1 &= \frac{x_{r,te} - x_{l,te}}{y_r - y_l}
\end{aligned}
$$

```
%Supersonic Prandtl-Glauert factor
Beta = sqrt(M^2 - 1);

%Compute geometric coefficients
```

```
a0_vec = (x_r_le_vec - x_l_le_vec.*(y_r_vec./y_l_vec))...
    ./(1 - (y_r_vec./y_l_vec));
a0_vec(y_l_vec == 0) = (x_r_le_vec(y_l_vec == 0).*...
    (y_l_vec(y_l_vec == 0)./y_r_vec(y_l_vec == 0)) -...
    x_l_le_vec(y_l_vec == 0))./((y_l_vec(y_l_vec == 0)./...
    y_r_vec(y_l_vec == 0)) - 1);
a1_vec = (x_r_te_vec - x_l_te_vec.*(y_r_vec./y_l_vec))...
    ./(1 - (y_r_vec./y_l_vec));
a1_vec(y_l_vec == 0) = (x_r_te_vec(y_l_vec == 0).*...
    (y_l_vec(y_l_vec == 0)./y_r_vec(y_l_vec == 0)) -...
    x_l_te_vec(y_l_vec == 0))./((y_l_vec(y_l_vec == 0)./...
    y_r_vec(y_l_vec == 0)) - 1);
b0_vec = (x_r_le_vec - x_l_le_vec)./(y_r_vec - y_l_vec);
b1_vec = (x_r_te_vec - x_l_te_vec)./(y_r_vec - y_l_vec);
```

The most efficient way to compute the AIC matrix in MATLAB is to give the inputs to the function that computes the unit strength perturbation velocity component contributions $\hat{u}_{ij}$, $\hat{v}_{ij}$, and $\hat{w}_{ij}$ in matrix form. This can be easily accomplished by using MATLAB's [A,B] = meshgrid(a,b) function. This function takes input vectors a and b and outputs matrices A and B such that A contains b rows of vector a and B contains a columns of vector b. To be consistent with the AIC matrix layout, the a vector is any parameter relating to the panels and the b vector is any parameter relating to the control points.

```
%Construct matrices of all parameters needed for the AIC
   matrix where the
%(i,j) entry corresponds to the ith control point and the
   jth panel
[y_l_mat,x_cp_mat]    = meshgrid(y_l_vec,x_cp_vec);
[y_r_mat,y_cp_mat]    = meshgrid(y_r_vec,y_cp_vec);
[x_l_le_mat,z_cp_mat] = meshgrid(x_l_le_vec,z_cp_vec);
```

```
[x_l_te_mat,~]          = meshgrid(x_l_te_vec,x_cp_vec);
[x_r_le_mat,~]          = meshgrid(x_r_le_vec,x_cp_vec);
[x_r_te_mat,~]          = meshgrid(x_r_te_vec,x_cp_vec);
[a0_mat,~]              = meshgrid(a0_vec,x_cp_vec);
[a1_mat,~]              = meshgrid(a1_vec,x_cp_vec);
[b0_mat,~]              = meshgrid(b0_vec,x_cp_vec);
[b1_mat,~]              = meshgrid(b1_vec,x_cp_vec);
```

Before the perturbation velocity component contributions can be computed, the panels need to be trimmed such that the control points only see contributions from the portions of panels that lie upstream of their x locations. This must be done for each control point and panel combination. To accommodate the cases where a panel must be divided into two sections, shown in Figure 3-1 and as described in section 2.9, each panel and control point combination will have two separate contributions to the perturbation velocity component matrices. The first will be the contribution of the portion of the panel that lies entirely upstream of the control point (full panel). The second will be the contribution of the portion of the panel where only the leading edge lies upstream of the control point (truncated panel).

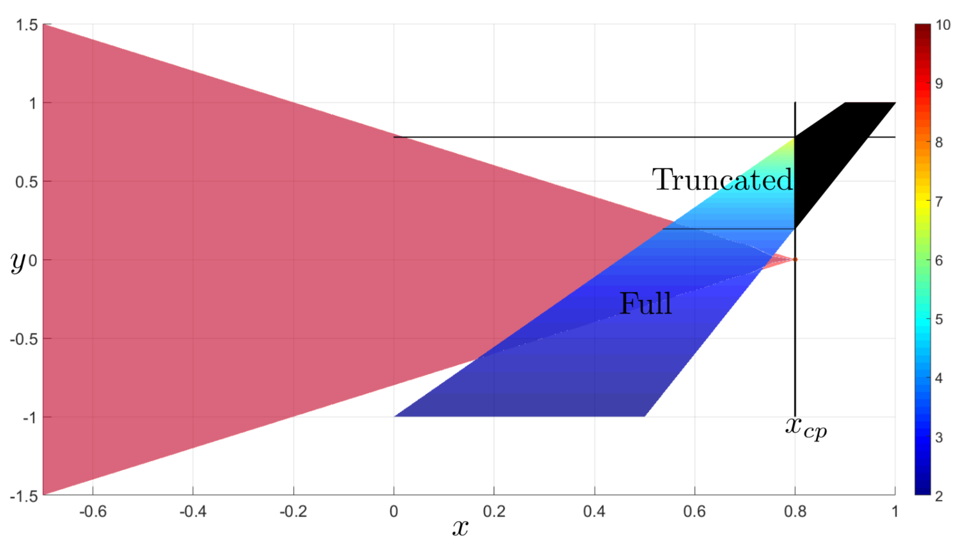

Figure 3-1: Panel Divided into Full and Truncated Portions

Initially, the singularity strength parameters of all full and truncated contributions are set to one and the left and right y locations of all full and truncated contributions are set to the left and right y locations of the panels.

```
%Trim panel upstream of control point
dm_mat_1   = ones(size(a0_mat));
dm_mat_2   = ones(size(a0_mat));
y_l_mat_1 = y_l_mat;
y_l_mat_2 = y_l_mat;
y_r_mat_1 = y_r_mat;
y_r_mat_2 = y_r_mat;
```

Each control point and panel combination is them examined and trimmed in two nested for loops.

```
for i = 1:length(x_cp_vec) %Loop over control points
    for j = 1:length(y_l_vec) %Loop over panels
```

The panel leading edge has a negative slope if b_0 is less than zero. If the slope is negative, the left edge of the panel is trimmed such that the leading edge of the trimmed panel lies entirely upstream of the control point. If the slope is positive, the right edge of the panel is trimmed. The y location where the leading edge has the same x location as the control point can be found using the equation for the leading edge:

$$\begin{aligned} x_{cp} &= a_0 + b_0 y \\ y &= \frac{x_{cp} - a_0}{b_0} \end{aligned}$$

The panel is only trimmed if the y value is within the y limits of the panel, as shown in Figure 3-2. If the left edge is being trimmed, the left y limit of both the full and truncated panels is set to the maximum of y_l and y. If the right edge is being trimmed, the right y limit of both the full and truncated panels is set to the minimum of y_r and y.

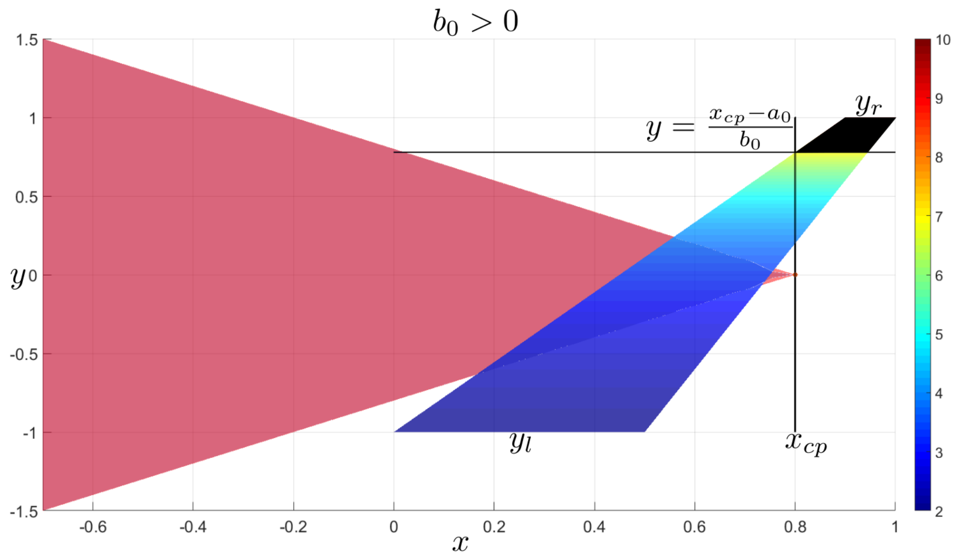

Figure 3-2: Control Point y Location Within Panel

```
if b0_mat(i,j) < 0 %If panel leading edge has a
   negative slope
    %Trim panel left edge
    y_l_mat(i,j) = max(y_l_mat(i,j),(x_cp_mat(i,j)
       -a0_mat(i,j))/...
        b0_mat(i,j));
    y_l_mat_1(i,j) = y_l_mat(i,j);
    y_l_mat_2(i,j) = y_l_mat(i,j);
else
    %Trim panel right edge
    y_r_mat(i,j) = min(y_r_mat(i,j),(x_cp_mat(i,j)
       -a0_mat(i,j))/...
        b0_mat(i,j));
    y_r_mat_1(i,j) = y_r_mat(i,j);
    y_r_mat_2(i,j) = y_r_mat(i,j);
end
```

If the control point's x location is less than the minimum of the panel leading edge corner points x locations, as shown in Figure 3-3, the control point's x location is upstream of the entire panel and the singularity strength parameters of both the full and truncated panels are set to zero.

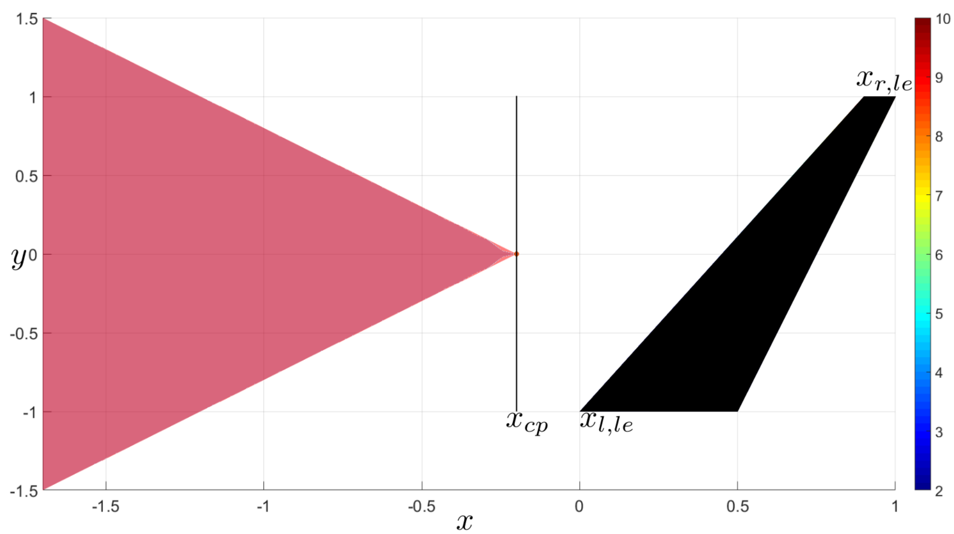

Figure 3-3: Control Point Upstream of Entire Panel

```
if x_cp_mat(i,j) < min(x_l_le_mat(i,j),x_r_le_mat(
   i,j)) %If control
    %point is upstream of entire panel
    %Set both truncated and full panel delta mus
       equal to zero
    dm_mat_1(i,j) = 0;
    dm_mat_2(i,j) = 0;
```

If the control point's x location is less than the maximum of the panel trailing edge corner points x locations and greater than or equal to the minimum of the panel leading edge corner points x locations, as shown in Figure 3-4, the control point's x location lies somewhere within the panel.

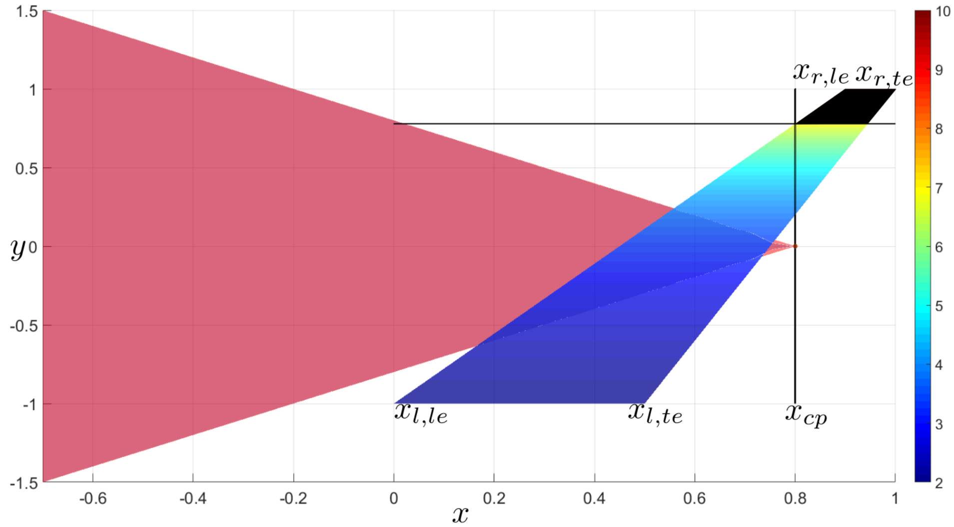

Figure 3-4: Control Point x Location Within Panel

```
elseif x_cp_mat(i,j) < max(x_l_te_mat(i,j),
   x_r_te_mat(i,j)) &&...
         x_cp_mat(i,j) >= min(x_l_le_mat(i,j),
            x_r_le_mat(i,j)) %If
    %the control point x location is within the
       panel
```

If the control point's x location is less than or equal to the minimum of the panel trailing edge corner points x locations, as shown in Figure 3-5, the control point's x location is upstream of the entire panel trailing edge and the singularity strength parameter of the full panel is set to zero.

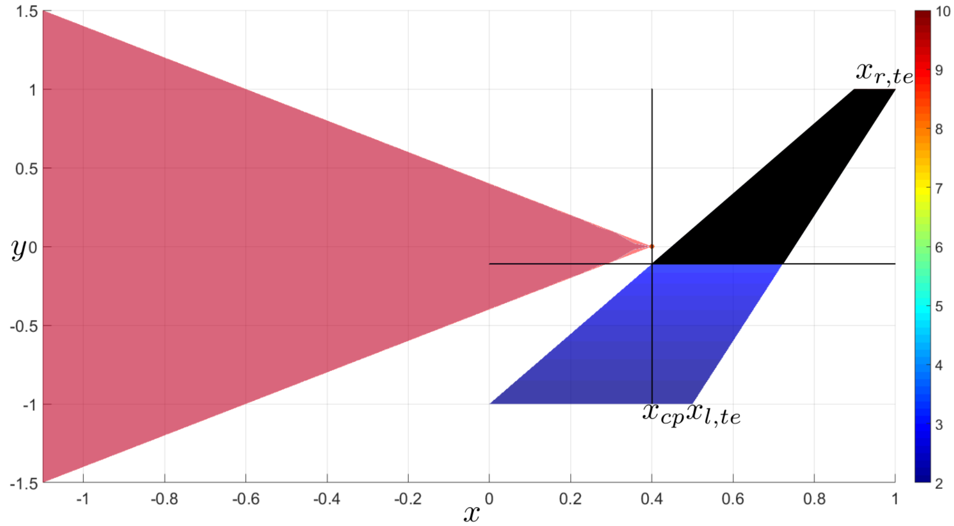

Figure 3-5: Control Point Upstream of Entire Panel Trailing Edge

```
if x_cp_mat(i,j) <= min(x_l_te_mat(i,j),
   x_r_te_mat(i,j)) %If
    %control point is upstream of entire
       trailing edge
    %Set full panel delta mu equal to zero
    dm_mat_1(i,j) = 0;
```

If the control point's x location lies somewhere within the panel and the control point's x location is not upstream of the entire panel trailing edge, as shown in Figure 3-6, the control point's x location lies somewhere within the panel trailing edge and the panel needs to be split into full and truncated portions. The y location where the trailing edge has the same x location as the control point can be found using the equation for the trailing edge:

$$\begin{aligned} x_{cp} &= a_1 + b_1 y \\ y &= \frac{x_{cp} - a_1}{b_1} \end{aligned}$$

The panel trailing edge has a positive slope if b_1 is greater than zero. If the slope is positive, the full panel portion is to the left of y and the truncated panel portion is to the right of y. The full panel right y limit is changed to y and the truncated panel left y limit is changed to y. If the slope is negative, the full panel portion is to the right of y and the truncated panel portion is to the left of y. The full panel left y limit is changed to y and the truncated panel right y limit is changed to y.

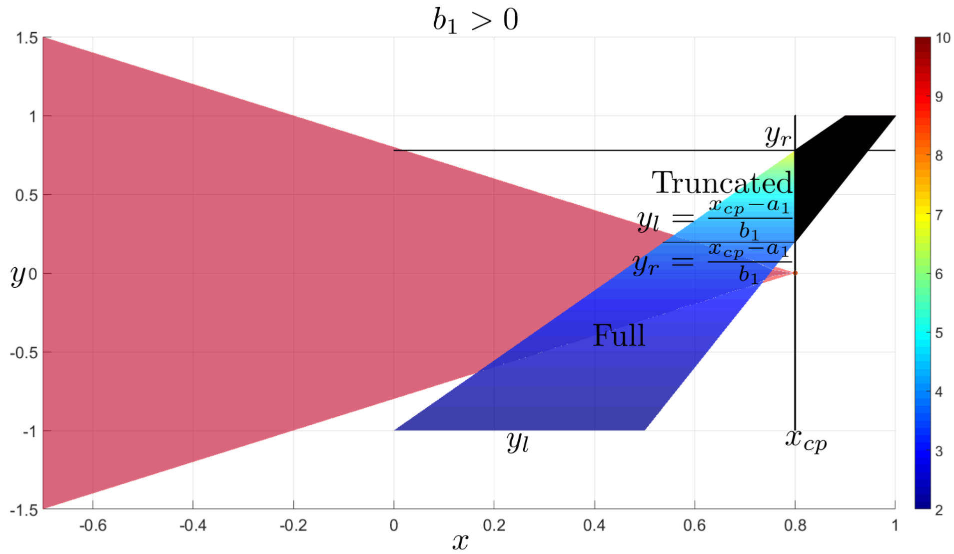

Figure 3-6: Control Point x Location Within Panel Trailing Edge

```
        else
            if b1_mat(i,j) > 0 %If panel trailing edge
                has a positive
                %slope
                %Divide panel at the y location where
                   the x location of
                %the control point intersects the
                   panel trailing edge
                %This y location becomes right y limit
                    of full panel
                %and left y limit of truncated panel
                y_r_mat_1(i,j) = (x_cp_mat(i,j) -
```

```
            a1_mat(i,j))/...
             b1_mat(i,j);
        y_l_mat_2(i,j) = (x_cp_mat(i,j) -
            a1_mat(i,j))/...
             b1_mat(i,j);
    else
        %Divide panel at the y location where
            the x location of
        %the control point intersects the
            panel trailing edge
        %This y location becomes left y limit
            of full panel
        %and right y limit of truncated panel
        y_l_mat_1(i,j) = (x_cp_mat(i,j) -
            a1_mat(i,j))/...
             b1_mat(i,j);
        y_r_mat_2(i,j) = (x_cp_mat(i,j) -
            a1_mat(i,j))/...
             b1_mat(i,j);
    end
end
```

If the control point's x location is not less than the maximum of the panel trailing edge corner points x locations, as shown in Figure 3-7, the control point's x location is downstream of the entire panel and the singularity strength parameter of the truncated panel is set to zero.

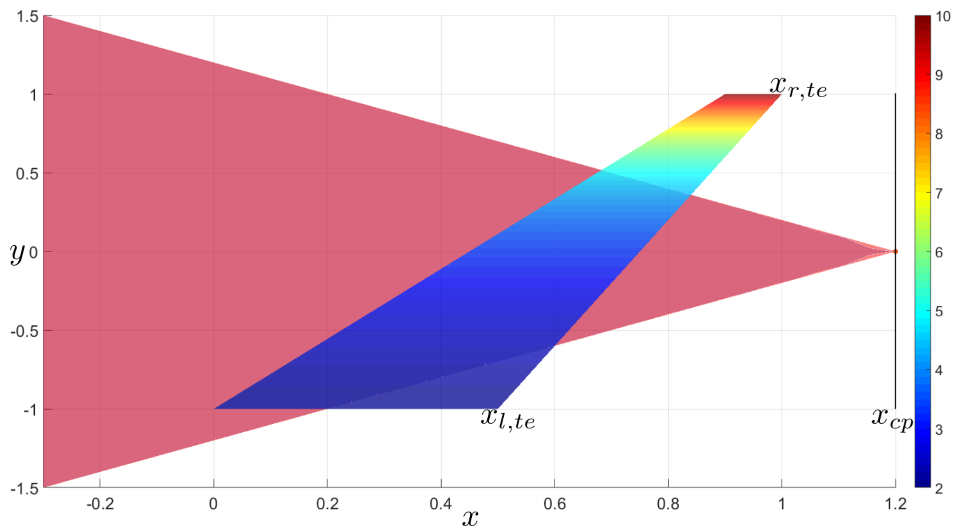

Figure 3-7: Control Point Downstream of Entire Panel

```
        else
            %Set truncated panel delta mu equal to zero
            dm_mat_2(i,j) = 0;
        end
    end
end
```

Once the panels have been trimmed, the full and truncated unit strength potential and perturbation velocity component contributions $\hat{\phi}$, $\hat{u}$, $\hat{v}$, and $\hat{w}$ can be evaluated for all panel and control point combinations. The following MATLAB functions evaluate the superposition integral expressions derived in Appendix A.

```
%Compute potential and perturbation velocity component
   contributions per
%unit delta mu at each control point from each panel
phi1 = phi_tapered_panel(x_cp_mat,y_cp_mat,z_cp_mat,a0_mat
   ,a1_mat,...
    b0_mat,b1_mat,y_l_mat_1,y_r_mat_1,dm_mat_1,Beta);
phi2 = phi_tapered_panel_truncated(x_cp_mat,y_cp_mat,
```

```
    z_cp_mat,a0_mat,...
      a1_mat,b0_mat,b1_mat,y_l_mat_2,y_r_mat_2,dm_mat_2,Beta
         );

[u_1,v_1,w_1] = V_tapered_panel(x_cp_mat,y_cp_mat,z_cp_mat
    ,a0_mat,...
      a1_mat,b0_mat,b1_mat,y_l_mat_1,y_r_mat_1,dm_mat_1,Beta
         );
[u_2,v_2,w_2] = V_tapered_panel_truncated(x_cp_mat,
    y_cp_mat,z_cp_mat,...
      a0_mat,a1_mat,b0_mat,b1_mat,y_l_mat_2,y_r_mat_2,
         dm_mat_2,Beta);
```

The full and truncated contributions are added to obtain the total contributions.

```
%Add full and truncated contributions
phi = phi1 + phi2;
u   = u_1 + u_2;
v   = v_1 + v_2;
w   = w_1 + w_2;
```

Certain entries in the matrices where the control point is outside the domain of dependence of a panel appear as NaNs. These entries are set to zero.

```
%Set NaN values to zero
phi(isnan(phi)) = 0;
u(isnan(u))     = 0;
v(isnan(v))     = 0;
w(isnan(w))     = 0;
```

The imaginary part of the matrices is discarded.

```
%Take real part
phi = real(phi);
```

```
u   = real(u);
v   = real(v);
w   = real(w);
```

The AIC matrix and right hand side vector are constructed.

```
%Construct AIC matrix
A = diag(n_x_cp)*u + diag(n_y_cp)*v + diag(n_z_cp)*w;

% Construct right hand side vector
b = -cosd(alpha)*cosd(beta)*n_x_cp + sind(beta)*n_y_cp -...
    sind(alpha)*cosd(beta)*n_z_cp;
```

The singularity strength parameters are solved for using MATLAB's backslash operator.

```
% Solve for doublet sheet delta mus
dm = A\b;
```

The potential and perturbation velocity components are calculated at the control points by multiplying the unit strength contribution matrices by the singularity strength parameter vector.

```
% Evaluate potential and perturbation velocity components
   at control points
phi = phi*dm;
u   = u*dm;
v   = v*dm;
w   = w*dm;
```

3.4 Calculating Lift, Induced Drag, and Sideforce

Once the perturbation velocity components are known at the control points, the lift, induced drag, and sideforce coefficients can be calculated. The pressure coefficient is defined as:

$$c_p \equiv \frac{p - p_\infty}{\frac{1}{2}\rho_\infty V_\infty^2}$$

which can be written in terms of the Mach number as:

$$c_p = \frac{2}{\gamma M_\infty^2}\left(\frac{p}{p_\infty} - 1\right)$$

The isentropic pressure equation can be written in terms of the perturbation velocity components:

$$\frac{p}{p_\infty} = \left\{1 - (\gamma - 1)M_\infty^2[u + \frac{1}{2}(u^2 + v^2 + w^2)]\right\}^{\frac{\gamma}{\gamma-1}}$$

Using the same Taylor expansion from section 2.3, taking the linear approximation of the isentropic pressure equation yields:

$$\frac{p}{p_\infty} \approx 1 - \frac{\gamma}{\gamma - 1}(\gamma - 1)M_\infty^2[u + \frac{1}{2}(u^2 + v^2 + w^2)] \approx 1 - \gamma M_\infty^2 u$$

Substituting this expression into the pressure coefficient gives the linearized pressure coefficient:

$$c_p = -2u$$

The normal force coefficient for each panel is calculated by integrating the difference between the lower and upper surface pressure coefficient over the panel area and dividing by the total panel area:

$$c_n = \frac{1}{A}\int\int (c_{p,l} - c_{p,u})\, da$$

It will be assumed that the pressure coefficient is constant within each panel and equal to the value at the control point, so the normal force coefficient is:

$$c_n = c_{p,l} - c_{p,u}$$

The u perturbation velocities are always equal and opposite on the upper and lower surfaces of a panel, so the normal force coefficient can be written as:

$$c_n = 4u$$

where u is assumed to be the upper surface perturbation velocity component.

The normal force coefficient is oriented in the direction of the normal vector of each panel's control point. This can be broken down into x, y, and z components:

$$\begin{aligned}
c_x &= \frac{n_x c_n}{\sqrt{n_x^2 + n_y^2 + n_z^2}} \\
c_y &= \frac{n_y c_n}{\sqrt{n_x^2 + n_y^2 + n_z^2}} \\
c_z &= \frac{n_z c_n}{\sqrt{n_x^2 + n_y^2 + n_z^2}}
\end{aligned}$$

The total force coefficients for the wing are calculated by taking an area weighted sum of the individual panel force coefficients and dividing by the reference area:

$$\begin{aligned}
C_x &= \frac{1}{A_{ref}} \sum_{i=1}^{N} c_{x,i} A_i \\
C_y &= \frac{1}{A_{ref}} \sum_{i=1}^{N} c_{y,i} A_i \\
C_z &= \frac{1}{A_{ref}} \sum_{i=1}^{N} c_{z,i} A_i
\end{aligned}$$

The lift, induced drag, and sideforce coefficients can be calculated by applying the appropriate transformation matrix to the total force coefficients:

$$\begin{bmatrix} C_D \\ C_Y \\ C_L \end{bmatrix} = \begin{bmatrix} \cos\beta & -\sin\beta & 0 \\ \sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix} \begin{bmatrix} C_x \\ C_y \\ C_z \end{bmatrix}$$

$$= \begin{bmatrix} \cos\beta\cos\alpha & -\sin\beta & \cos\beta\sin\alpha \\ \sin\beta\cos\alpha & \cos\beta & \sin\beta\sin\alpha \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix} \begin{bmatrix} C_x \\ C_y \\ C_z \end{bmatrix}$$

3.5 Analytical Solutions

The analytical solutions for some simple test cases are obtained using supersonic conical flow theory [5]. The u perturbation velocity component distribution on the upper surface of a lifting triangle with a supersonic leading edge is given by:

$$u = \mathrm{Re}\left\{ \frac{\alpha}{\beta_{PG}\pi} \frac{m}{\sqrt{m^2-1}} \left[\cos^{-1}\frac{1-m\tau}{m-\tau} + \cos^{-1}\frac{1+m\tau}{m+\tau} \right] \right\}$$

where m is the product of the half wingspan and the Prandtl-Glauert factor per unit root chord and:

$$\begin{aligned} \tau &= \frac{2\varepsilon}{1+\varepsilon^2} \\ \varepsilon &= \frac{y+iz}{\frac{x}{\beta_{PG}} + \sqrt{\left(\frac{x}{\beta_{PG}}\right)^2 - y^2 - z^2}} \end{aligned}$$

The u perturbation velocity component distribution on the upper surface of a lifting triangle with a root chord length of 1 and a half span of 2 in a flow where $\beta_{PG} = 1$ and $\alpha = 1^\circ$ is shown in Figure 3-8.

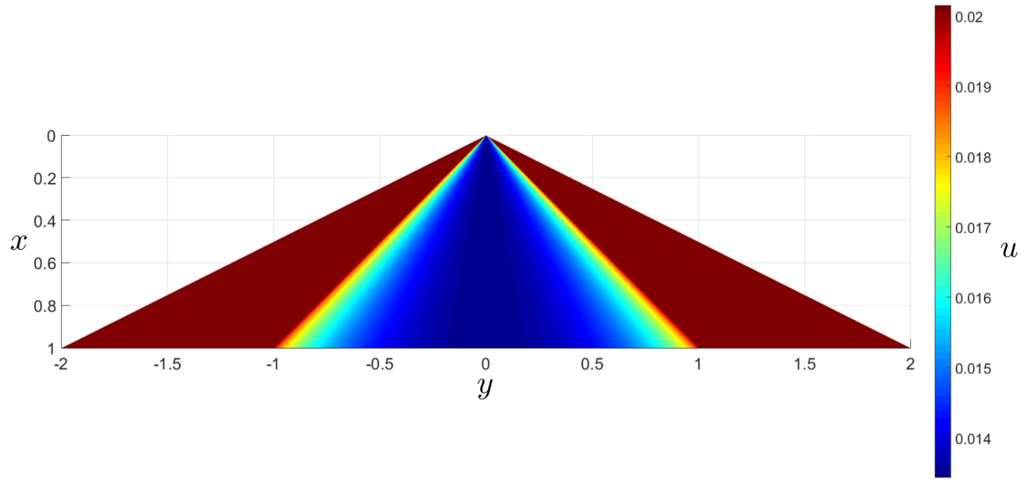

Figure 3-8: Supersonic Leading Edge Lifting Triangle Analytical u Distribution

The u perturbation velocity component distribution on the upper surface of a lifting triangle with a subsonic leading edge is given by:

$$u = \mathrm{Re}\left\{ \frac{m^2\alpha}{\beta_{PG} E(k)} \frac{1}{\sqrt{m^2 - \tau^2}} \right\}$$

where $k = \sqrt{1 - m^2}$ and $E(k)$ is the complete elliptic integral of the second kind evaluated for the parameter k.

The u perturbation velocity component distribution on the upper surface of a lifting triangle with a root chord length of 1 and a half span of 1/2 in a flow where $\beta_{PG} = 1$ and $\alpha = 1°$ is shown in Figure 3-9.

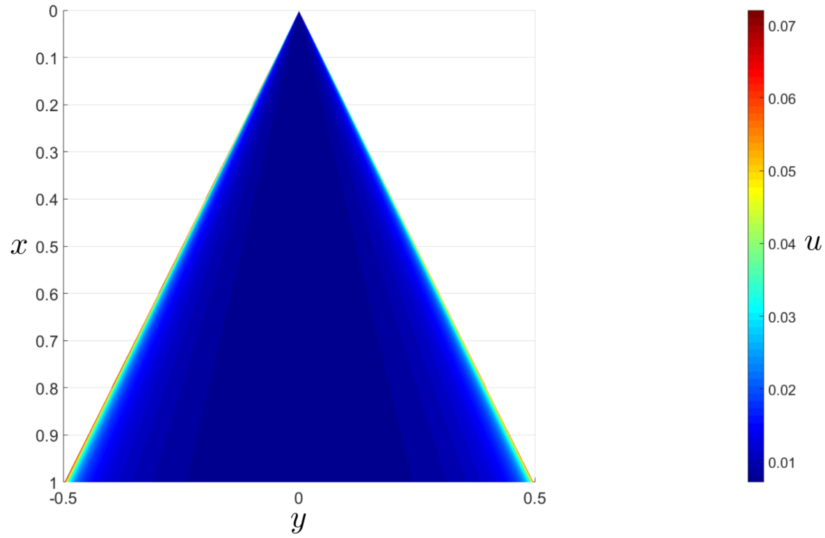

Figure 3-9: Subsonic Leading Edge Lifting Triangle Analytical u Distribution

The u perturbation velocity component distribution on the upper surface of a lifting rectangle is given by:

$$u = \mathrm{Re}\left\{\frac{\alpha}{\beta_{PG}\pi}\left[\cos^{-1}\sqrt{1-\tau_1} - \left(\frac{\pi}{2} - \cos^{-1}\sqrt{1+\tau_2}\right)\right]\right\}$$

where m is the half wingspan and:

$$\begin{aligned}
\tau_1 &= \frac{2\varepsilon_1}{1+\varepsilon_1^2} \\
\tau_2 &= \frac{2\varepsilon_2}{1+\varepsilon_2^2} \\
\varepsilon_1 &= \frac{(y+m)+iz}{\frac{x}{\beta_{PG}} + \sqrt{\left(\frac{x}{\beta_{PG}}\right)^2 - (y+m)^2 - z^2}} \\
\varepsilon_2 &= \frac{(y-m)+iz}{\frac{x}{\beta_{PG}} + \sqrt{\left(\frac{x}{\beta_{PG}}\right)^2 - (y-m)^2 - z^2}}
\end{aligned}$$

The u perturbation velocity component distribution on the upper surface of a lifting rectangle with a chord length of 1 and a half span of 2 in a flow where $\beta_{PG} = 1$ and $\alpha = 1^\circ$ is shown in Figure 3-10.

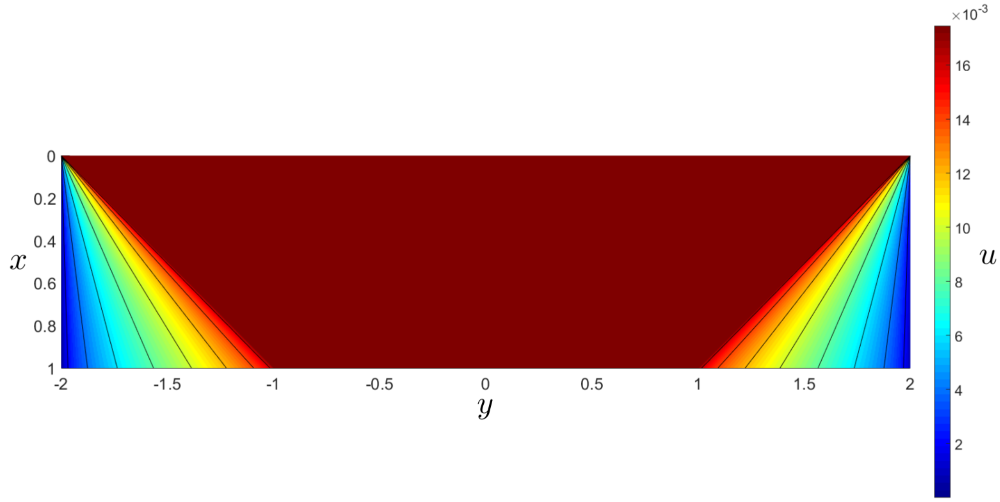

Figure 3-10: Lifting Rectangle Analytical u Distribution

The u perturbation velocity component distribution on the upper surface of a lifting square with a chord length of 1 and a half span of $1/2$ in a flow where $\beta_{PG} = 1$ and $\alpha = 1°$ is shown in Figure 3-11.

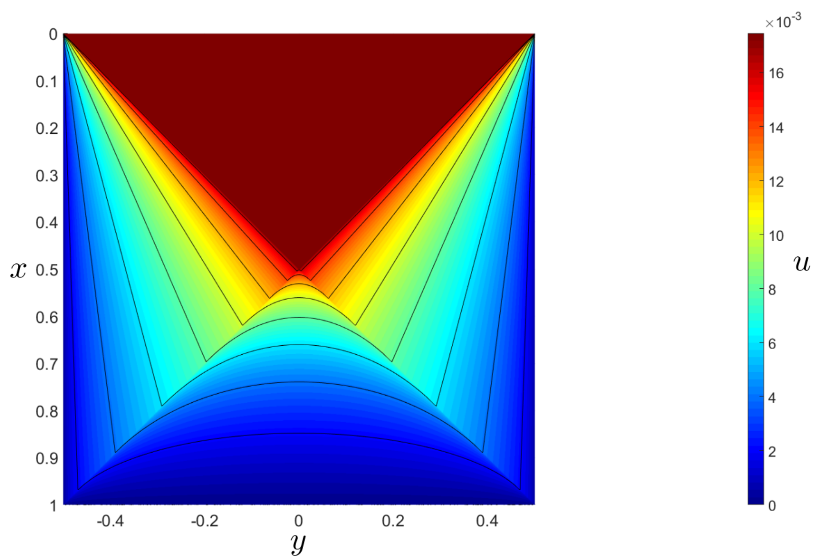

Figure 3-11: Lifting Square Analytical u Distribution

3.6 Choice of Control Point Location and Wing Discretization

In order to use the supersonic doublet panel code, the user must determine how to discretize the lifting surfaces into panels and where to put the control points. One way to ensure the panel side edges are parallel to the x-axis is to first discretize the lifting surfaces into chordwise strips as shown in Figure 3-12.

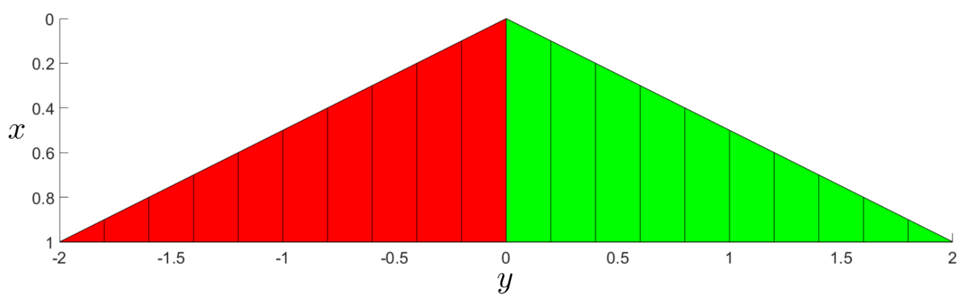

Figure 3-12: Lifting Surface Discretized into Chordwise Strips

Each strip can then be divided into trapezoidal panels with the control points placed intuitively at the panel centroids as shown in Figure 3-13.

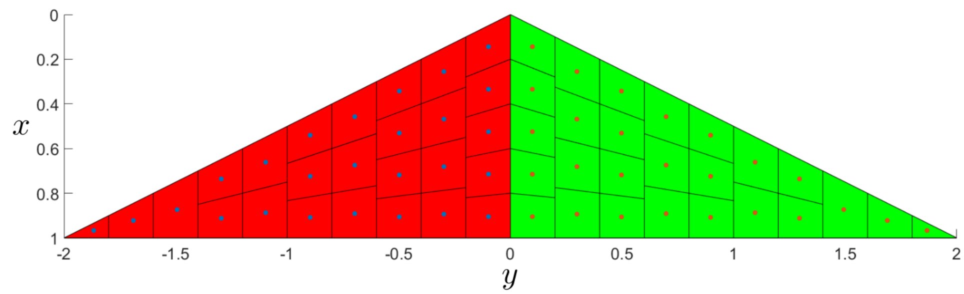

Figure 3-13: Lifting Surface Discretized into Trapezoidal Panels with Control Points at Centroids

The u perturbation velocity distribution of the lifting rectangle suggests that the ideal panel distribution is one that is more dense at the wing tips than at the root. This can be achieved by using a sine spacing for the panel strip edge y locations. The MATLAB functions y = sinspace(d1,d2,n,factor) and y = cosspace(d1,d2,n,factor) can be used to generate sine and cosine spaced points. These functions take input scalers d1, d2, n, and factor and output a vector y. d1 and d2 are the lower and upper limits of the output vector y, n is the number of points, and factor is used to reduce the degree of clustering. A factor value of 0.0 corresponds to the normal sine or cosine spacing and a factor value of 1.0 corresponds to a uniformly distributed set of points. For the lifting rectangle, good convergence is obtained for a factor value of 0.6 as shown in Figure 3-14.

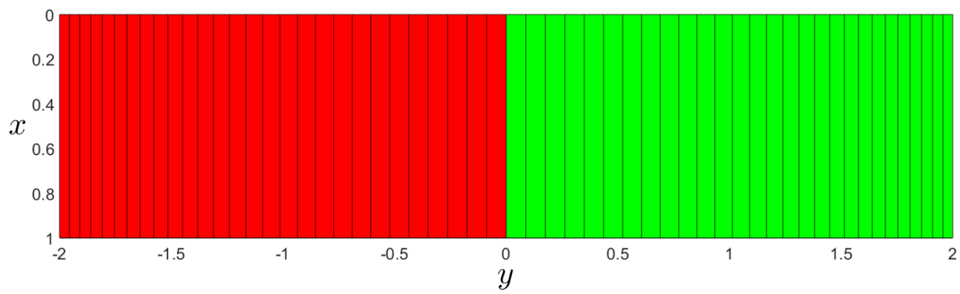

Figure 3-14: Lifting Rectangle Strips

The stability of this doublet panel method depends strongly on the aspect ratios of the panels. For most cases, an aspect ratio close to one gives the best result. Each strip is thus divided into panels such that the aspect ratio of each panel is as close to one as possible with control points at the panel centroids as shown in Figure 3-15. The narrower strips near the tip have more panels than the wider strips near the root.

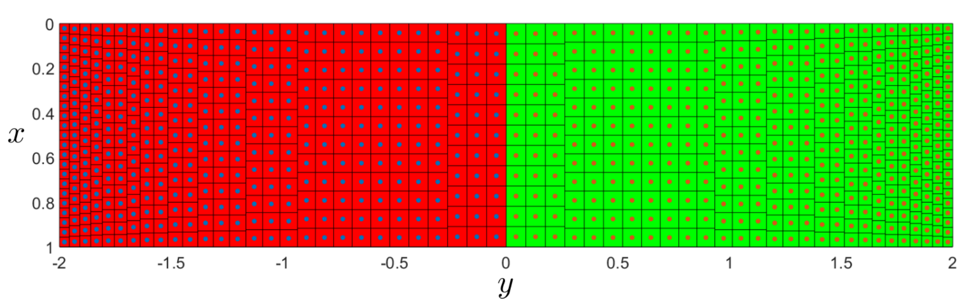

Figure 3-15: Lifting Rectangle Panels and Control Points

The numerical solution for the above discretization is shown in Figure 3-16 and the relative error for the u distribution in Figure 3-17.

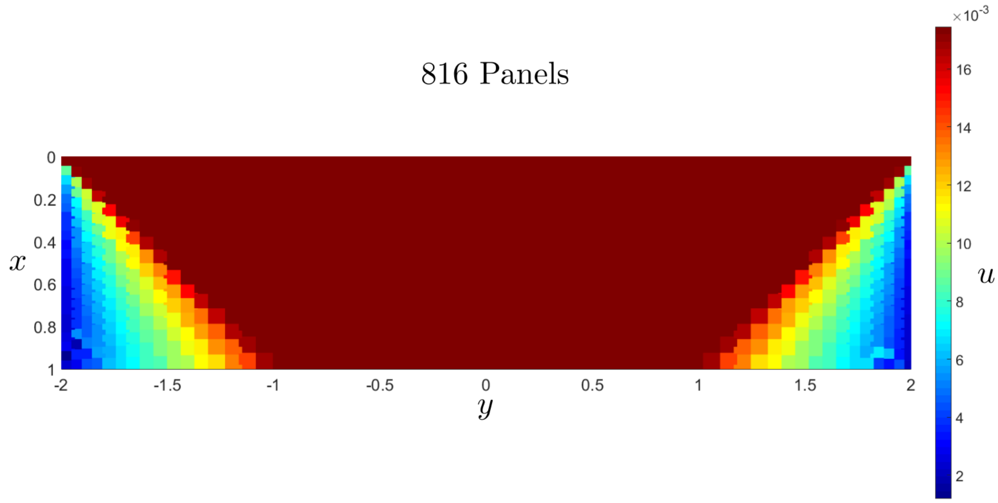

Figure 3-16: Lifting Rectangle Numerical u Distribution

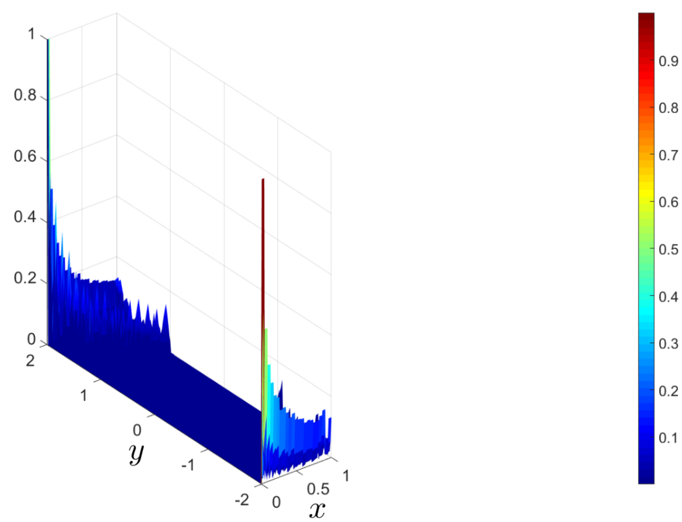

Figure 3-17: Lifting Rectangle u Distribution Relative Error

The numerical solution is a good approximation of the analytical u distribution, but the tip influence region is not completely resolved. This could be improved by increasing the number of panels; however, instability becomes a problem when the number of panels is increased further. Up until this point, the L^2 norm of the error in the u distribution e_u and the error in the lift coefficient e_{C_L} improve as the number of panels is increased as shown in Figures 3-18 and 3-19. where:

$$\begin{aligned} e_u &= \frac{\|u_{numerical} - u_{analytical}\|_{L^2}}{\|u_{analytical}\|_{L^2}} \\ e_{C_L} &= \frac{|C_{L_{numerical}} - C_{L_{analytical}}|}{|C_{L_{analytical}}|} \\ \|x\|_{L^2} &= \sqrt{x_1^2 + ... + x_n^2} \end{aligned}$$

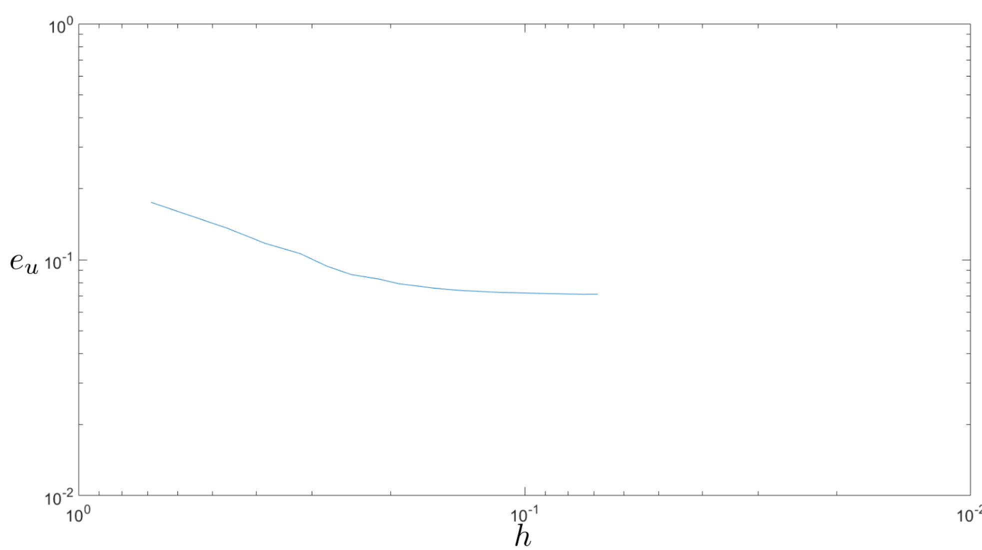

Figure 3-18: Lifting Rectangle u Error Convergence

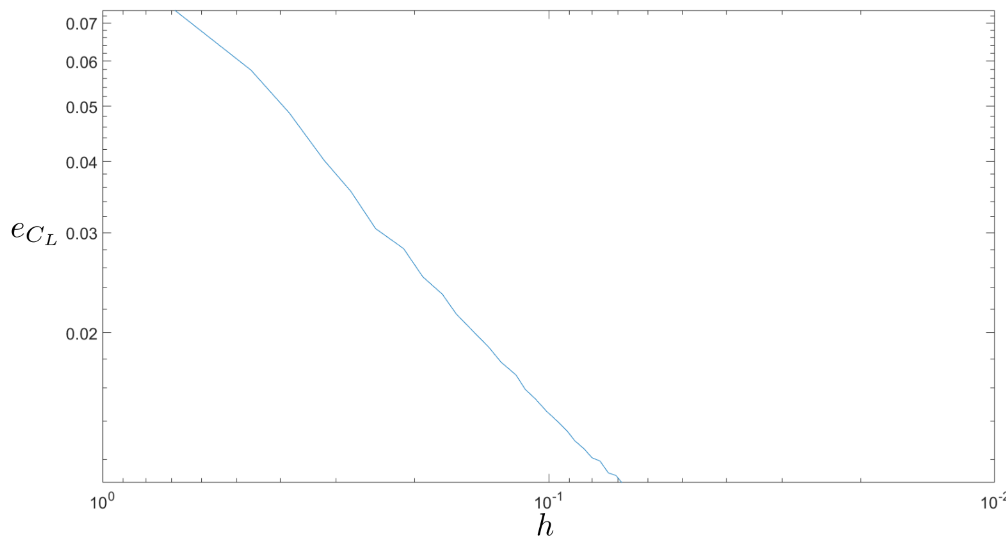

Figure 3-19: Lifting Rectangle C_L Error Convergence

For the lifting square, a good result is also obtained using a sine spacing for the panel strip edge y locations. Because the tips influence a large portion of the surface area, the panel density does not need to be as clustered at the tips. A larger factor value of 0.8 is used as shown in Figure 3-20.

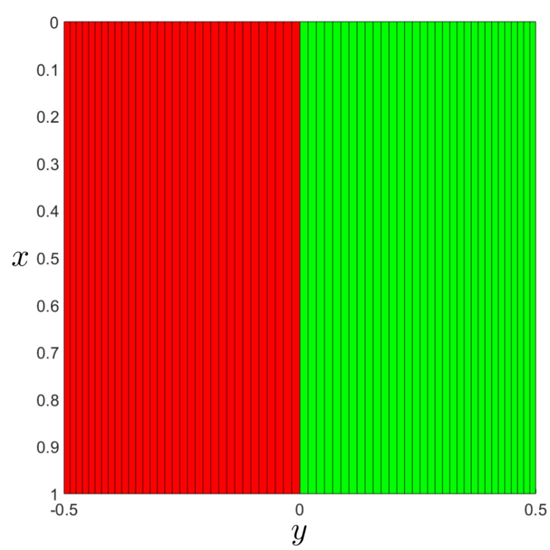

Figure 3-20: Lifting Square Strips

Again the strips are divided such that the panel aspect ratios are as close to one as possible and the control points are placed at the panel centroids as shown in Figure 3-21.

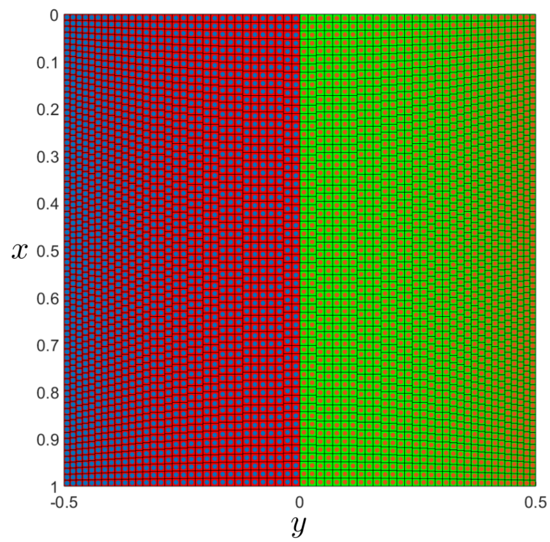

Figure 3-21: Lifting Square Panels and Control Points

The numerical solution for the above discretization is shown in Figure 3-22 and the relative error for the u distribution in Figure 3-23.

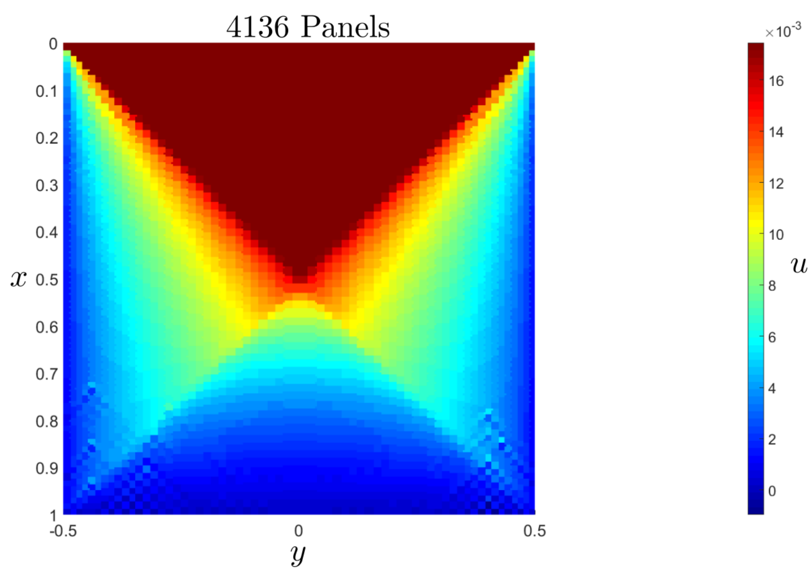

Figure 3-22: Lifting Square Numerical u Distribution

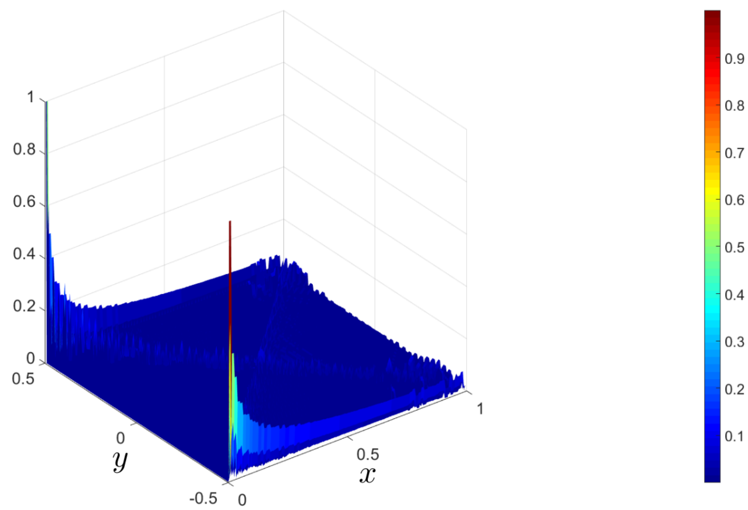

Figure 3-23: Lifting Square u Distribution Relative Error

This doublet panel method remains stable for a significantly higher number of panels with the lifting square. The error is very small everywhere except where the analytical solution has a large gradient. The convergence of the L^2 norm of the error in the u distribution and the error in the lift coefficient are shown in Figures 3-24 and 3-25.

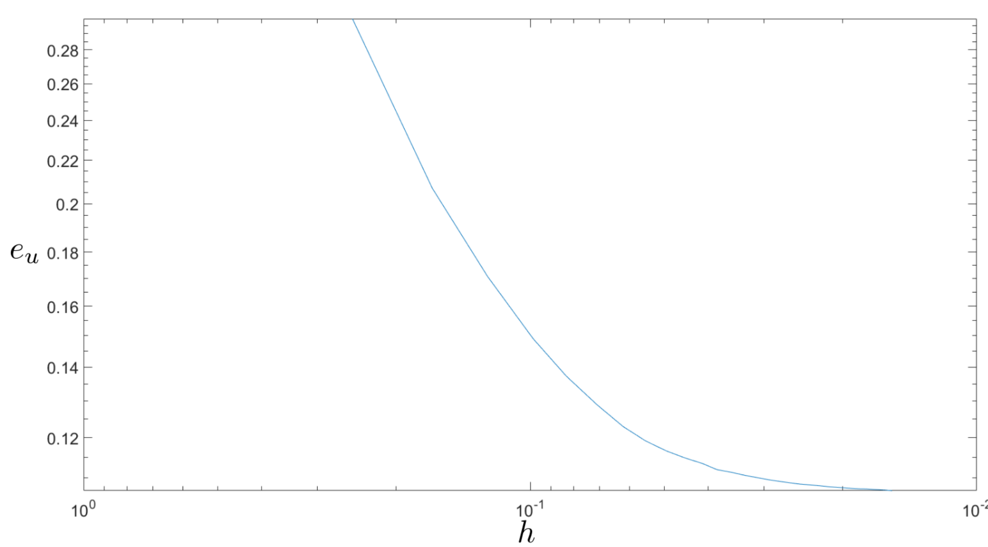

Figure 3-24: Lifting Square u Error Convergence

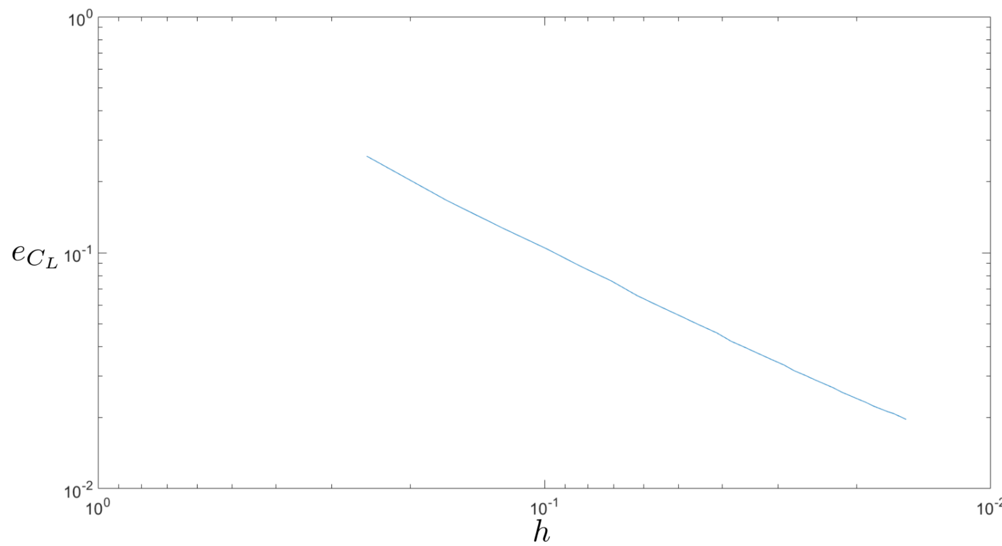

Figure 3-25: Lifting Square C_L Error Convergence

For the lifting triangle with a supersonic leading edge, a cosine distribution of the strip edge y locations with a factor value of 0.6 works well as shown in Figure 3-26.

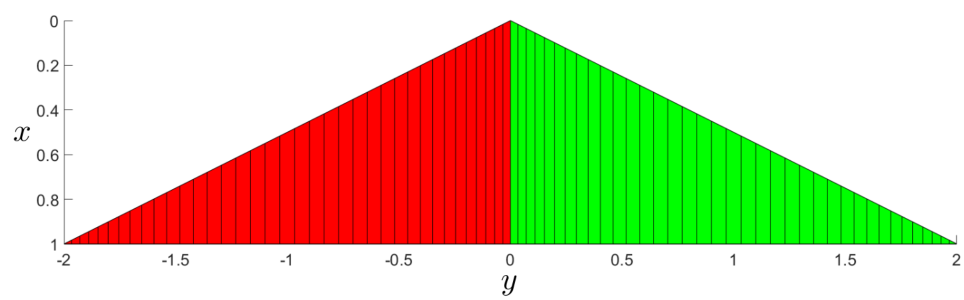

Figure 3-26: Supersonic Leading Edge Lifting Triangle Strips

Once again the strips are divided such that the panel aspect ratios are as close to one as possible and the control points are placed at the panel centroids as shown in Figure 3-27.

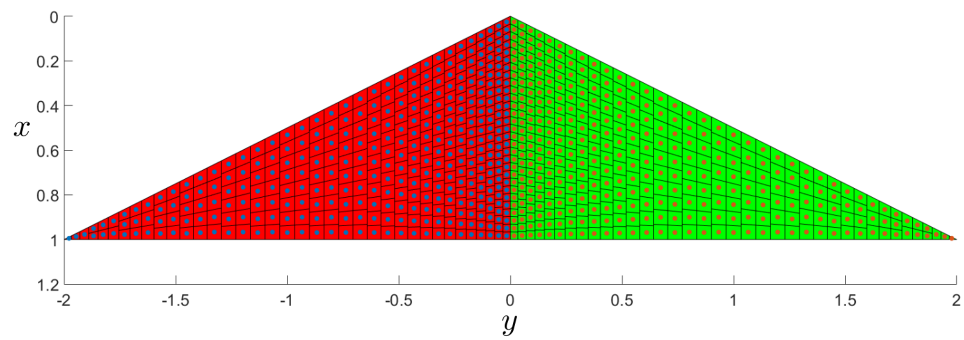

Figure 3-27: Supersonic Leading Edge Lifting Triangle Panels and Control Points

It is important to note that for lifting surfaces with a tip chord length of zero, the aspect ratio of the panels in the strips at the tips will be lower bounded by the aspect ratio of the wing. For this supersonic leading edge lifting triangle this does not cause a problem, but for other cases it might be advantageous to redefine the lifting surface so that the tip chord has a finite length.

The numerical solution for the above discretization is shown in Figure 3-28 and the relative error for the u distribution in Figure 3-29.

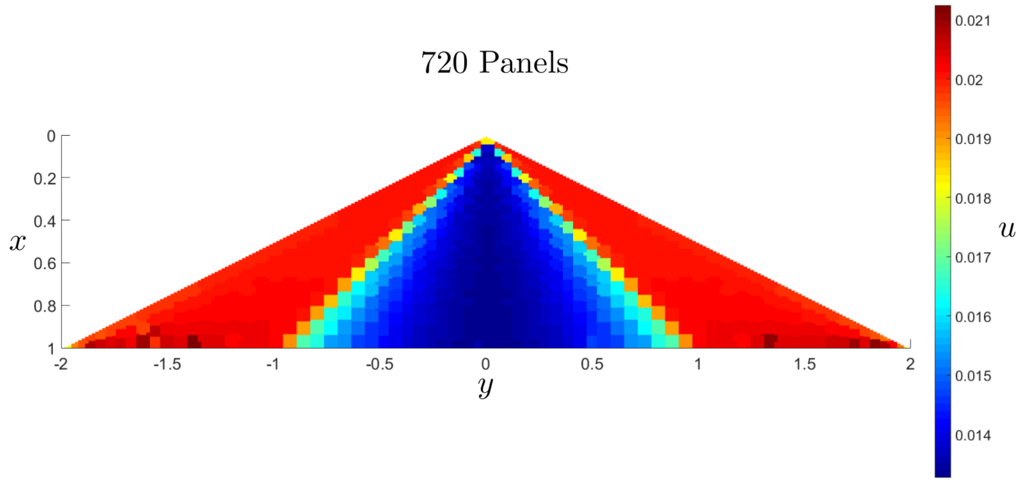

Figure 3-28: Supersonic Leading Edge Lifting Triangle Numerical u Distribution

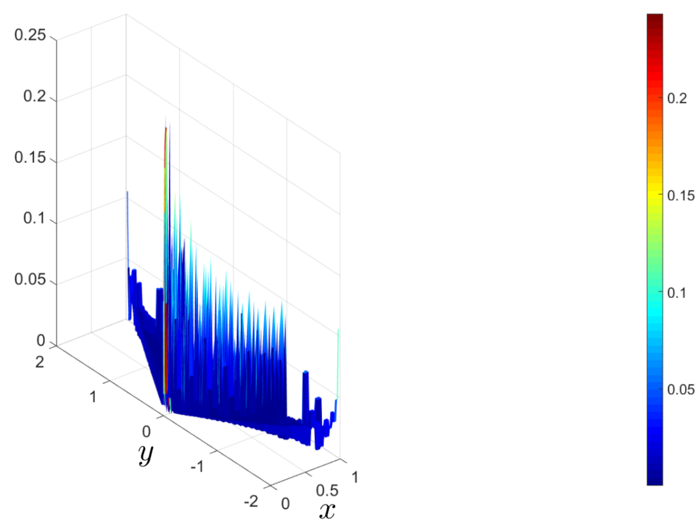

Figure 3-29: Supersonic Leading Edge Lifting Triangle u Distribution Relative Error

The numerical solution provides a good approximation of the analytical u distribution everywhere except near the edge of the Mach lines originating at the front tip. There is also an instability causing a slight overshoot near the trailing edge, which prevents increasing the number of panels beyond this point from being beneficial. The lift coefficient error does not improve as the number of panels is increased but it is very small even at course discretizations as shown in Figure 3-30. The L^2 norm of the error in the u distribution does improve slightly as shown in Figure 3-31.

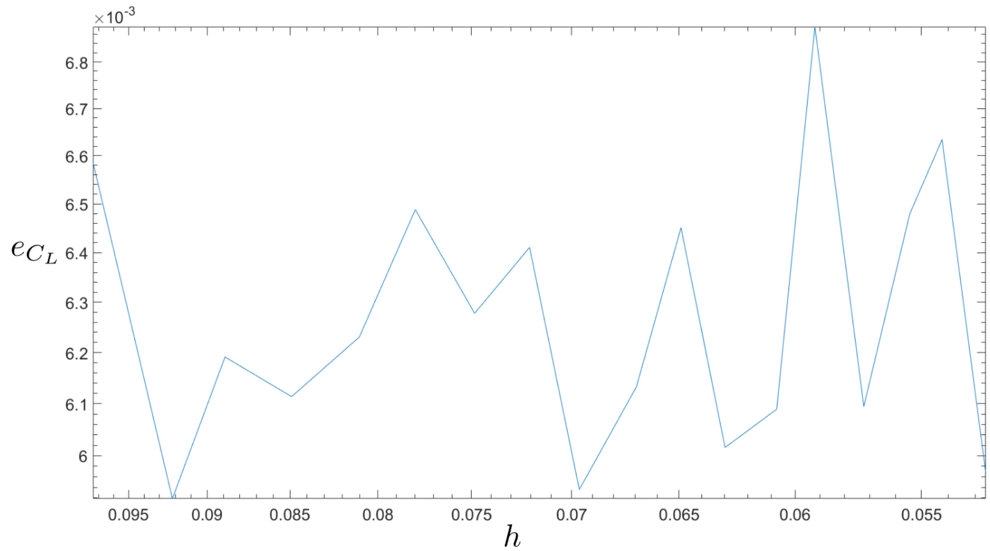

Figure 3-30: Supersonic Leading Edge Lifting Triangle C_L Error Convergence

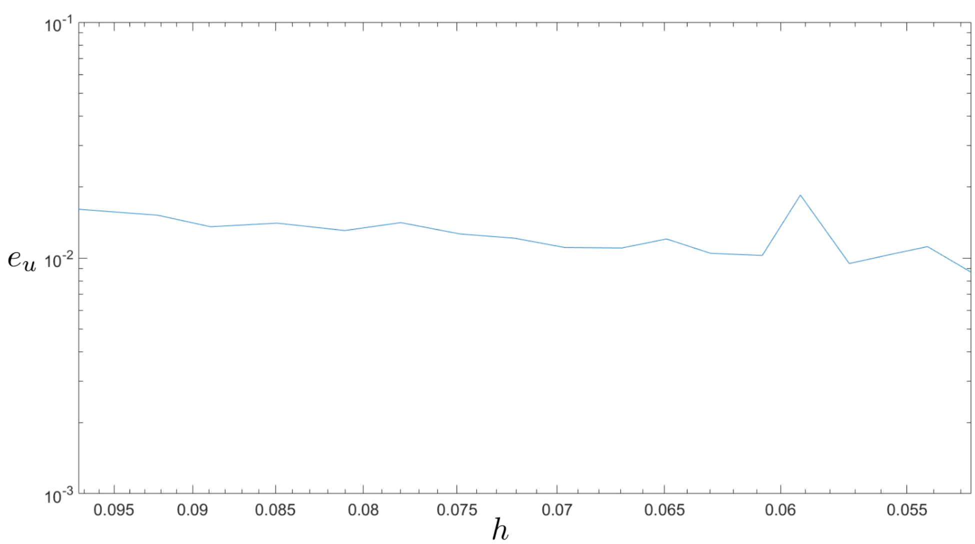

Figure 3-31: Supersonic Leading Edge Lifting Triangle u Error Convergence

The subsonic leading edge lifting triangle is by far the most difficult of the test case because of the leading edge singularity shown in Figure 3-32.

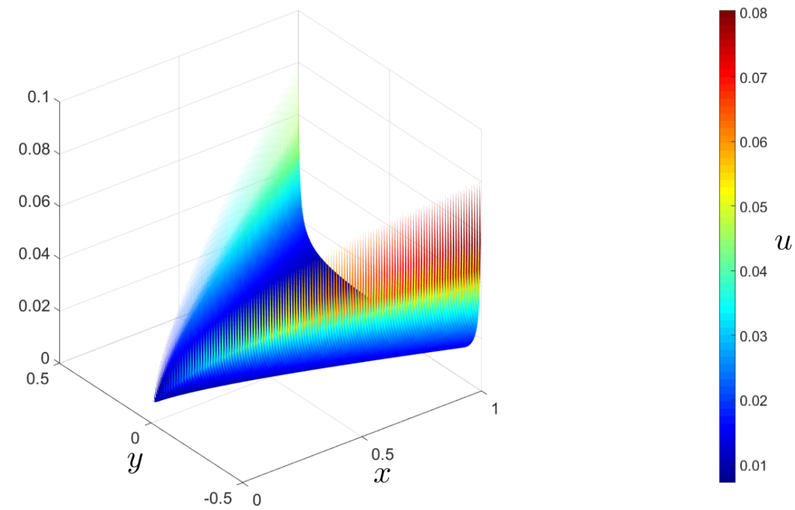

Figure 3-32: Subsonic Leading Edge Lifting Triangle Analytical u Distribution

The u perturbation velocity distribution of the subsonic leading edge lifting triangle suggests that the ideal panel distribution is one that is more dense at the leading edge than at the trailing edge. This can be achieved by using a sine spacing for the panel left and right leading and trailing edge x locations. The difference in panel size from the leading edge to the trailing edge should be made as big as possible without causing instability, so a factor value of 0.1 is used. The stability of this method for this case also requires a sine spacing in the strip edge y locations using the same factor value of 0.1. Like the previous cases the control points are placed at the panel centroids.

A consequence of using sine spacing in the streamwise direction is that the panels have different aspect ratios within each strip. To ensure that the panels near the leading edge do not have aspect ratios that are too high, the number of panels in each strip is selected so that if the panels were equally spaced, the aspect ratios would be as close to 0.1 as possible. This results in a discretization that is much more dense in the spanwise direction and much less dense in the chordwise direction than the discretizations used for the other test cases as shown in Figure 3-33.

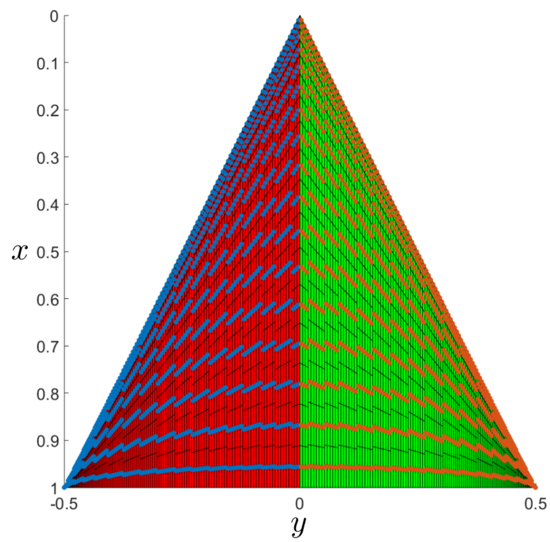

Figure 3-33: Subsonic Leading Edge Lifting Triangle Panels and Control Points

The numerical solution for the above discretization is very noisy, particularly near the leading edge as shown in Figure 3-34.

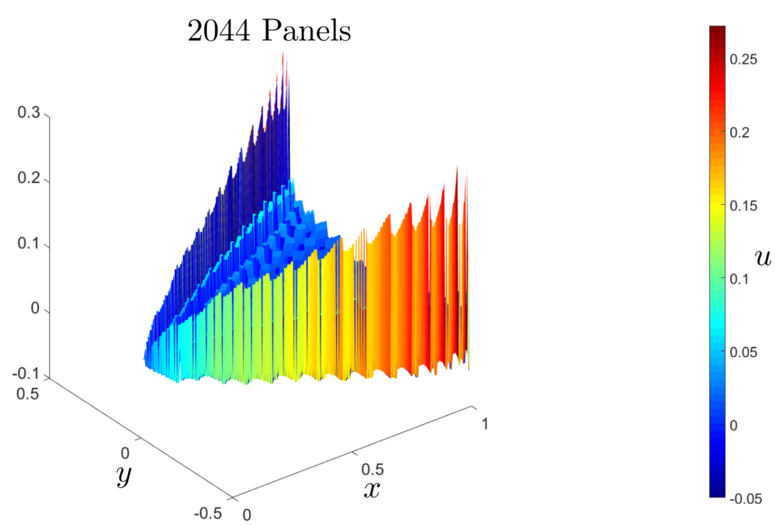

Figure 3-34: Subsonic Leading Edge Lifting Triangle Noisy Numerical u Distribution

Some of this noise can be mitigated by using a moving average filter for the u values within each strip. This can be done using the MATLAB z = smooth(y,span) function. This function takes in input vector y and scalar span and outputs vector

z. y is the original data, span is the number of points used to compute each entry in the output vector, and z is the output vector. For this case and this discretization, a span of 11 points works best. The resulting u distribution and error are shown in Figures 3-35 and 3-36. Note that the moving average is taken after the u values are interpolated using MATLAB's meshgrid function, which preserves the lift coefficient.

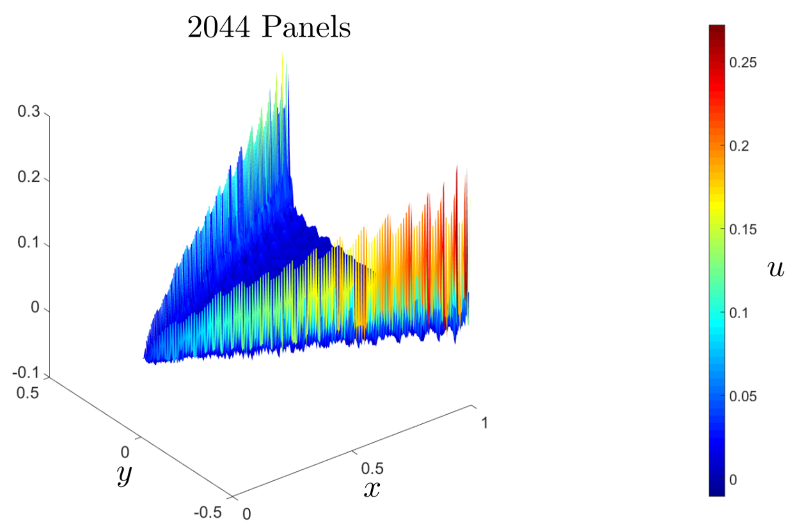

Figure 3-35: Subsonic Leading Edge Lifting Triangle Smooth Numerical u Distribution

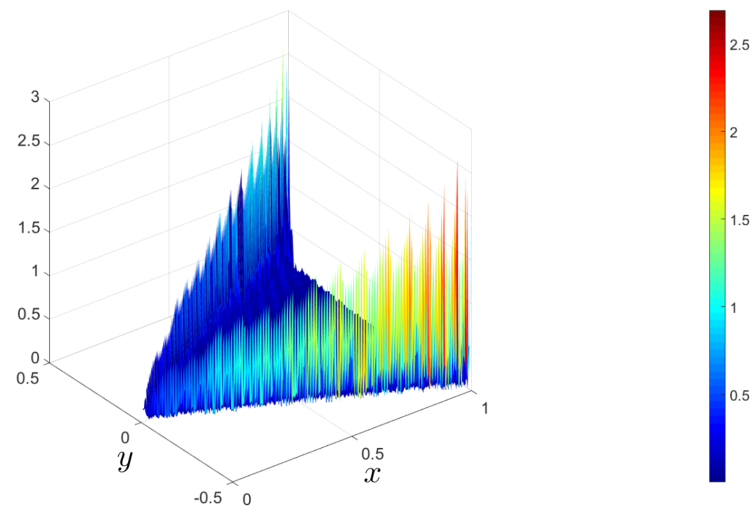

Figure 3-36: Subsonic Leading Edge Lifting Triangle u Distribution Relative Error

After much effort, both the L^2 norm of the u distribution error, shown in Figure 3-37, and the lift coefficient error, shown in Figure 3-38, converge monotonically.

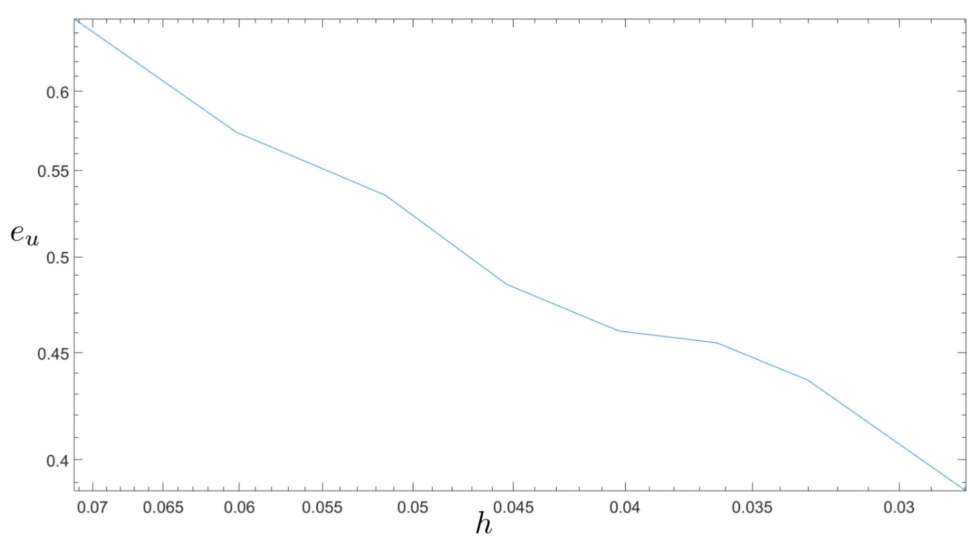

Figure 3-37: Subsonic Leading Edge Lifting Triangle u Error Convergence

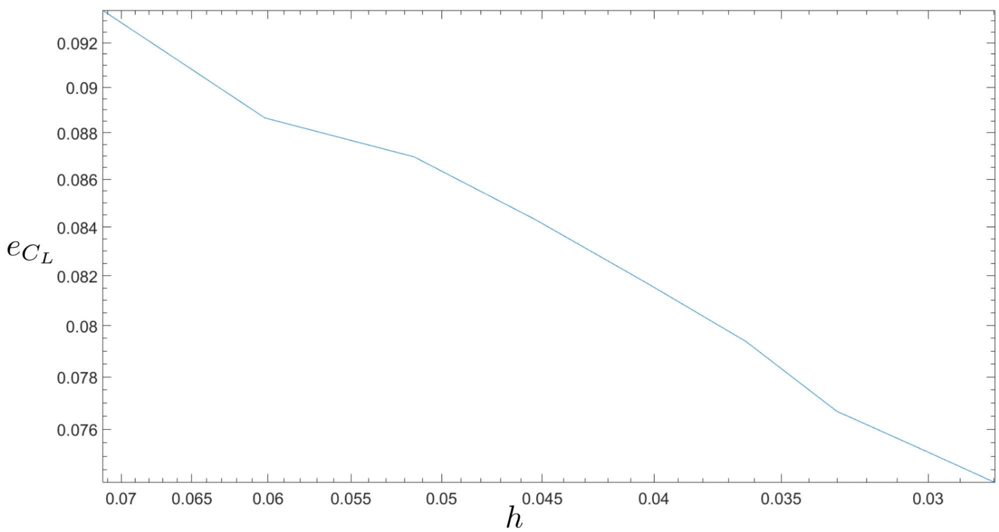

Figure 3-38: Subsonic Leading Edge Lifting Triangle C_L Error Convergence

Appendix A

Rubi-Mathematica Integration Steps

The superposition integrals for the tapered supersonic doublet panel potential and perturbation velocity components are evaluated using Wolfram Mathematica along with the Rubi rule-based mathematics symbolic integration rules [6]. The Rubi package contains over 6000 integration rules and allows for each step to be displayed.

A.1 Full Tapered Supersonic Doublet Panel Potential and Perturbation Velocity Components

```
(*Clear all variables*)
ClearAll["Global`*"]


(*Rubi integration setup*)
ShowSteps = True; (*False displays only the final answer,
True displays intermediate steps*)
RulesDirectory = NotebookDirectory[];
Get[RulesDirectory <> "Load Rubi.m"];
```

Modifying 7017 integration rules to display steps...

In[359]:=
```
(*Evaluate x indefinite integral*)
(1 / (2 * π)) * Int[((μ1 - μ0) / ((a1 - a0) + (b1 - b0) * y′)) * (z / ((y - y′) ^2 + z^2)) *
   (x - x′) / Sqrt[(x - x′) ^2 - (β^2) * ((y - y′) ^2 + z^2)], x′]
```

Rule 13: If !MatchQ[u, b_ v_ /; FreeQ[b, x]],

$$\int a\,u\,dx \longrightarrow \text{Dist}\left[a, \int u\,dx, x\right]$$

In[360]:=
$$\frac{1}{2\pi}\text{Dist}\left[\frac{z\,(\mu 0-\mu 1)}{(a0-a1+(b0-b1)\,y')\,((y-y')^2+z^2)}, \text{Int}\left[\frac{x-x'}{\sqrt{(x-x')^2-((y-y')^2+z^2)\,\beta^2}}, x'\right], x'\right]$$

Rule 36: If PolyQ[Pm, x] && PolyQ[Qm, x], let m = Expon[Pm, x], then if
Expon[Qm, x] == m - 1 && ZeroQ[Coeff[Qm, x, m - 1] ∂_x Pm - m Coeff[Pm, x, m] Qm],

$$\int (a+b\,Pm^n)^p\,Qm\,dx \longrightarrow \text{Dist}\left[\frac{\text{Coeff}[Qm, x, m-1]}{m\,\text{Coeff}[Pm, x, m]}, \text{Subst}\left[\int (a+b\,x^n)^p\,dx, x, Pm\right], x\right]$$

In[361]:=
$$-\frac{1}{2\pi}\text{Dist}\left[\frac{z\,(\mu 0-\mu 1)}{2\,(a0-a1+(b0-b1)\,y')\,((y-y')^2+z^2)}, \text{Subst}\left[\text{Int}\left[\frac{1}{\sqrt{x'-((y-y')^2+z^2)\,\beta^2}}, x'\right], x', (x-x')^2\right], x'\right]$$

Rule 43: If NonzeroQ[m + 1],

$$\int (a+b\,x)^m\,dx \longrightarrow \text{Simp}\left[\frac{(a+b\,x)^{m+1}}{b\,(m+1)}, x\right]$$

Out[361]=
$$-\frac{z\sqrt{(x-x')^2-((y-y')^2+z^2)\,\beta^2}\,(\mu 0-\mu 1)}{2\pi\,(a0-a1+(b0-b1)\,y')\,((y-y')^2+z^2)}$$

```
(*Evaluate at integration limits*)
```

In[362]:=
```
f[x′_] = -((z * Sqrt[(x - x′) ^2 - ((y - y′) ^2 + z^2) * β^2] * (μ0 - μ1)) /
   (2 * Pi * (a0 - a1 + (b0 - b1) * y′) * ((y - y′) ^2 + z^2)))
```

Out[362]=
$$-\frac{z\sqrt{(x-x')^2-((y-y')^2+z^2)\,\beta^2}\,(\mu 0-\mu 1)}{2\pi\,(a0-a1+(b0-b1)\,y')\,((y-y')^2+z^2)}$$

In[363]:=
```
Simplify[f[a1 + b1 * y′] - f[a0 + b0 * y′]]
```

Out[363]=
$$\left(z\left(\sqrt{(a0-x+b0\,y')^2-\left((y-y')^2+z^2\right)\beta^2}-\sqrt{(a1-x+b1\,y')^2-\left((y-y')^2+z^2\right)\beta^2}\right)(\mu0-\mu1)\right)\Big/\left(2\pi\,(a0-a1+(b0-b1)\,y')\left(y^2-2\,y\,y'+y'^2+z^2\right)\right)$$

In[364]:=
```
(*Evaluate y indefinite integral*)
Int[(z * (Sqrt[(a0 - x + b0 * y′)^2 - ((y - y′)^2 + z^2) * β^2] -
      Sqrt[(a1 - x + b1 * y′)^2 - ((y - y′)^2 + z^2) * β^2]) * (μ0 - μ1)) /
  (2 * Pi * (a0 - a1 + (b0 - b1) * y′) * (y^2 - 2 * y * y′ + y′^2 + z^2)), y′]
```

Rule 13: If !MatchQ[u, b_ v_ /; FreeQ[b, x]],

$$\int a\,u\,dx \longrightarrow \text{Dist}\left[a, \int u\,dx, x\right]$$

In[365]:=
$$\text{Dist}\Big[\frac{z(\mu0-\mu1)}{2\pi}, \text{Int}\Big[\left(\sqrt{(a0-x+b0\,y')^2-\left((y-y')^2+z^2\right)\beta^2}-\sqrt{(a1-x+b1\,y')^2-\left((y-y')^2+z^2\right)\beta^2}\right)\Big/\left((a0-a1+(b0-b1)\,y')\left(y^2-2\,y\,y'+y'^2+z^2\right)\right), y'\Big], y'\Big]$$

Rule 7000: If ZeroQ[n2 - 2 n] && PositiveIntegerQ[n],

let v = RationalFunctionExpand$\left[\frac{u}{a+b\,x^n+c\,x^{2n}}, x\right]$, then if SumQ[v],

$$\int \frac{u}{a+b\,x^n+c\,x^{n2}}\,dx \longrightarrow \int v\,dx$$

In[366]:=
$$\text{Dist}\Big[\frac{z(\mu0-\mu1)}{2\pi}, \text{Int}\Big[\left(\sqrt{\left(a0^2-2\,a0\,x+x^2-y^2\beta^2-z^2\beta^2+y'^2\left(b0^2-\beta^2\right)+y'\left(2\,a0\,b0-2\,b0\,x+2\,y\,\beta^2\right)\right)}\right)\Big/\left((a0-a1+(b0-b1)\,y')\left(y^2-2\,y\,y'+y'^2+z^2\right)\right)+\left(\sqrt{\left(a1^2-2\,a1\,x+x^2-y^2\beta^2-z^2\beta^2+y'^2\left(b1^2-\beta^2\right)+y'\left(2\,a1\,b1-2\,b1\,x+2\,y\,\beta^2\right)\right)}\right)\Big/\left((-a0+a1-(b0-b1)\,y')\left(y^2-2\,y\,y'+y'^2+z^2\right)\right), y'\Big], y'\Big]$$

Rule:

$$\text{Int}[a\,u+b\,v+\cdots, x] \longrightarrow a\int u\,dx+b\int v\,dx+\cdots$$

In[367]:=
$$\text{Dist}\Big[\frac{z(\mu0-\mu1)}{2\pi}, \text{Int}\Big[\left(\sqrt{\left(a0^2-2\,a0\,x+x^2-y^2\beta^2-z^2\beta^2+y'^2\left(b0^2-\beta^2\right)+y'\left(2\,a0\,b0-2\,b0\,x+2\,y\,\beta^2\right)\right)}\right)\Big/\left((a0-a1+(b0-b1)\,y')\left(y^2-2\,y\,y'+y'^2+z^2\right)\right), y'\Big], y'\Big]+\text{Dist}\Big[\frac{z(\mu0-\mu1)}{2\pi}, \text{Int}\Big[\left(\sqrt{\left(a1^2-2\,a1\,x+x^2-y^2\beta^2-z^2\beta^2+y'^2\left(b1^2-\beta^2\right)+y'\left(2\,a1\,b1-2\,b1\,x+2\,y\,\beta^2\right)\right)}\right)\Big/\left((-a0+a1-(b0-b1)\,y')\left(y^2-2\,y\,y'+y'^2+z^2\right)\right), y'\Big], y'\Big]$$

Rule 7000: If ZeroQ[n2 - 2 n] && PositiveIntegerQ[n],

let $v = \text{RationalFunctionExpand}\left[\frac{u}{a + b\, x^n + c\, x^{2\, n}}, x\right]$, then if SumQ[v],

$$\int \frac{u}{a + b\, x^n + c\, x^{n2}}\, dx \longrightarrow \int v\, dx$$

Rule 7000: If ZeroQ[n2 - 2 n] && PositiveIntegerQ[n],

let $v = \text{RationalFunctionExpand}\left[\frac{u}{a + b\, x^n + c\, x^{2\, n}}, x\right]$, then if SumQ[v],

$$\int \frac{u}{a + b\, x^n + c\, x^{n2}}\, dx \longrightarrow \int v\, dx$$

In[368]:=
$$\text{Dist}\Big[\frac{z\,(\mu0 - \mu1)}{2\,\pi}, \text{Int}\Big[\Big((b0 - b1)^2 \sqrt{\left(a0^2 - 2\, a0\, x + x^2 - y^2\, \beta^2 - z^2\, \beta^2 + y\prime^2\,(b0^2 - \beta^2) + y\prime\,(2\, a0\, b0 - 2\, b0\, x + 2\, y\, \beta^2)\right)}\Big) \Big/ \Big(\big(a0 - a1 + (b0 - b1)\, y\prime\big)\,\big(a0^2 - 2\, a0\, a1 + a1^2 + 2\, a0\, b0\, y - 2\, a1\, b0\, y - 2\, a0\, b1\, y + 2\, a1\, b1\, y + b0^2\, y^2 - 2\, b0\, b1\, y^2 + b1^2\, y^2 + b0^2\, z^2 - 2\, b0\, b1\, z^2 + b1^2\, z^2\big)\Big) + \Big(\big(a0 - a1 + 2\, b0\, y - 2\, b1\, y + (-b0 + b1)\, y\prime\big) \sqrt{\left(a0^2 - 2\, a0\, x + x^2 - y^2\, \beta^2 - z^2\, \beta^2 + y\prime^2\,(b0^2 - \beta^2) + y\prime\,(2\, a0\, b0 - 2\, b0\, x + 2\, y\, \beta^2)\right)}\Big) \Big/ \Big(\big(y^2 - 2\, y\, y\prime + y\prime^2 + z^2\big)\,\big(a0^2 - 2\, a0\, a1 + a1^2 + 2\, a0\, b0\, y - 2\, a1\, b0\, y - 2\, a0\, b1\, y + 2\, a1\, b1\, y + b0^2\, y^2 - 2\, b0\, b1\, y^2 + b1^2\, y^2 + b0^2\, z^2 - 2\, b0\, b1\, z^2 + b1^2\, z^2\big)\Big), y\prime\Big], y\prime\Big] + \text{Dist}\Big[\frac{z\,(\mu0 - \mu1)}{2\,\pi}, \text{Int}\Big[\Big((b0 - b1)^2 \sqrt{\left(a1^2 - 2\, a1\, x + x^2 - y^2\, \beta^2 - z^2\, \beta^2 + y\prime^2\,(b1^2 - \beta^2) + y\prime\,(2\, a1\, b1 - 2\, b1\, x + 2\, y\, \beta^2)\right)}\Big) \Big/ \Big(\big(-a0 + a1 - (b0 - b1)\, y\prime\big)\,\big(a0^2 - 2\, a0\, a1 + a1^2 + 2\, a0\, b0\, y - 2\, a1\, b0\, y - 2\, a0\, b1\, y + 2\, a1\, b1\, y + b0^2\, y^2 - 2\, b0\, b1\, y^2 + b1^2\, y^2 + b0^2\, z^2 - 2\, b0\, b1\, z^2 + b1^2\, z^2\big)\Big) + \Big(\big(-a0 + a1 - 2\, b0\, y + 2\, b1\, y + (b0 - b1)\, y\prime\big) \sqrt{\left(a1^2 - 2\, a1\, x + x^2 - y^2\, \beta^2 - z^2\, \beta^2 + y\prime^2\,(b1^2 - \beta^2) + y\prime\,(2\, a1\, b1 - 2\, b1\, x + 2\, y\, \beta^2)\right)}\Big) \Big/ \Big(\big(y^2 - 2\, y\, y\prime + y\prime^2 + z^2\big)\,\big(a0^2 - 2\, a0\, a1 + a1^2 + 2\, a0\, b0\, y - 2\, a1\, b0\, y - 2\, a0\, b1\, y + 2\, a1\, b1\, y + b0^2\, y^2 - 2\, b0\, b1\, y^2 + b1^2\, y^2 + b0^2\, z^2 - 2\, b0\, b1\, z^2 + b1^2\, z^2\big)\Big), y\prime\Big], y\prime\Big]$$

Rule:

$$\text{Int}[a\, u + b\, v + \cdots, x] \longrightarrow a \int u\, dx + b \int v\, dx + \cdots$$

Rule:

$$\text{Int}[a\, u + b\, v + \cdots, x] \longrightarrow a \int u\, dx + b \int v\, dx + \cdots$$

In[369]:=

```
Dist[(z (μ0 - μ1)) /
   (2 π (a0^2 + a1^2 - 2 a1 (b0 - b1) y - 2 a0 (a1 - (b0 - b1) y) + (b0 - b1)^2 (y^2 + z^2))),
  Int[1/(y^2 - 2 y y′ + y′^2 + z^2) (a0 - a1 + 2 b0 y - 2 b1 y + (-b0 + b1) y′)
     √(a0^2 - 2 a0 x + x^2 - y^2 β^2 - z^2 β^2 + y′^2 (b0^2 - β^2) + y′ (2 a0 b0 - 2 b0 x + 2 y β^2)),
   y′], y′] + Dist[(z (μ0 - μ1)) /
   (2 π (a0^2 + a1^2 - 2 a1 (b0 - b1) y - 2 a0 (a1 - (b0 - b1) y) + (b0 - b1)^2 (y^2 + z^2))),
  Int[1/(y^2 - 2 y y′ + y′^2 + z^2) (-a0 + a1 - 2 b0 y + 2 b1 y + (b0 - b1) y′)
     √(a1^2 - 2 a1 x + x^2 - y^2 β^2 - z^2 β^2 + y′^2 (b1^2 - β^2) + y′ (2 a1 b1 - 2 b1 x + 2 y β^2)),
   y′], y′] + Dist[((b0 - b1)^2 z (μ0 - μ1)) /
   (2 π (a0^2 + a1^2 - 2 a1 (b0 - b1) y - 2 a0 (a1 - (b0 - b1) y) + (b0 - b1)^2 (y^2 + z^2))),
  Int[(√(a0^2 - 2 a0 x + x^2 - y^2 β^2 - z^2 β^2 + y′^2 (b0^2 - β^2) + y′ (2 a0 b0 - 2 b0 x + 2 y β^2))) /
    (a0 - a1 + (b0 - b1) y′), y′], y′] + Dist[((b0 - b1)^2 z (μ0 - μ1)) /
   (2 π (a0^2 + a1^2 - 2 a1 (b0 - b1) y - 2 a0 (a1 - (b0 - b1) y) + (b0 - b1)^2 (y^2 + z^2))),
  Int[(√(a1^2 - 2 a1 x + x^2 - y^2 β^2 - z^2 β^2 + y′^2 (b1^2 - β^2) + y′ (2 a1 b1 - 2 b1 x + 2 y β^2))) /
    (-a0 + a1 - (b0 - b1) y′), y′], y′]
```

Rule 607:

If NonzeroQ[b^2 - 4 a c] && NonzeroQ[e^2 - 4 d f] && RationalQ[p] && p > 0 && NonzeroQ[p + q + 1],

$\int (g + h x) (a + b x + c x^2)^p (d + e x + f x^2)^q \, dx \longrightarrow$

$\text{Simp}\left[\frac{h (a + b x + c x^2)^p (d + e x + f x^2)^{q+1}}{2 f (p + q + 1)}, x\right] - \text{Dist}\left[\frac{1}{2 f (p + q + 1)},\right.$
$\int (a + b x + c x^2)^{p-1} (d + e x + f x^2)^q \text{Simp}[h p (b d - a e) + a (h e - 2 g f) (p + q + 1) + (2 h p (c d - a f) +$
$\left. b (h e - 2 g f) (p + q + 1)) x + (h p (c e - b f) + c (h e - 2 g f) (p + q + 1)) x^2, x] \, dx, x\right]$

Rule 607:

If NonzeroQ[b^2 - 4 a c] && NonzeroQ[e^2 - 4 d f] && RationalQ[p] && p > 0 && NonzeroQ[p + q + 1],

$\int (g + h x) (a + b x + c x^2)^p (d + e x + f x^2)^q \, dx \longrightarrow$

$\text{Simp}\left[\frac{h (a + b x + c x^2)^p (d + e x + f x^2)^{q+1}}{2 f (p + q + 1)}, x\right] - \text{Dist}\left[\frac{1}{2 f (p + q + 1)},\right.$
$\int (a + b x + c x^2)^{p-1} (d + e x + f x^2)^q \text{Simp}[h p (b d - a e) + a (h e - 2 g f) (p + q + 1) + (2 h p (c d - a f) +$
$\left. b (h e - 2 g f) (p + q + 1)) x + (h p (c e - b f) + c (h e - 2 g f) (p + q + 1)) x^2, x] \, dx, x\right]$

Rule 320: If

NonzeroQ[b^2 - 4 a c] && NonzeroQ[c d^2 - b d e + a e^2] && NonzeroQ[2 c d - b e] && RationalQ[p] && p > 0 && NonzeroQ[m + 2 p + 1] && (! RationalQ[m] || m < 1) && ! NegativeIntegerQ[m + 2 p] && IntQuadraticQ[a, b, c, d, e, m, p, x],

$\int (d + e x)^m (a + b x + c x^2)^p \, dx \longrightarrow \text{Simp}\left[\frac{(d + e x)^{m+1} (a + b x + c x^2)^p}{e (m + 2 p + 1)}, x\right] -$
$\text{Dist}\left[\frac{p}{e (m + 2 p + 1)}, \int (d + e x)^m \text{Simp}[b d - 2 a e + (2 c d - b e) x, x] (a + b x + c x^2)^{p-1} \, dx, x\right]$

Rule 320: If

NonzeroQ[b^2 - 4 a c] && NonzeroQ[c d^2 - b d e + a e^2] && NonzeroQ[2 c d - b e] && RationalQ[p] && p > 0 && NonzeroQ[m + 2 p + 1] && (! RationalQ[m] || m < 1) && ! NegativeIntegerQ[m + 2 p] && IntQuadraticQ[a, b, c, d, e, m, p, x],

$\int (d + e x)^m (a + b x + c x^2)^p \, dx \longrightarrow \text{Simp}\left[\frac{(d + e x)^{m+1} (a + b x + c x^2)^p}{e (m + 2 p + 1)}, x\right] -$
$\text{Dist}\left[\frac{p}{e (m + 2 p + 1)}, \int (d + e x)^m \text{Simp}[b d - 2 a e + (2 c d - b e) x, x] (a + b x + c x^2)^{p-1} \, dx, x\right]$

In[370]:=
```
-Dist[(z (μ0 - μ1)) /
    (2 π (a0^2 + a1^2 - 2 a1 (b0 - b1) y - 2 a0 (a1 - (b0 - b1) y) + (b0 - b1)^2 (y^2 + z^2))),
   Int[(-(b0 - b1) (a0 - x) (a0 y - x y + b0 (y^2 + z^2)) -
        (a0 - a1 + (b0 - b1) y) (a0^2 - 2 a0 x + x^2 - (y^2 + z^2) β^2) +
        y′^2 (a1 (b0^2 - β^2) - a0 (b0 b1 - β^2) - (b0 - b1) (b0 x - y β^2)) + y′ ((b0 - b1)
             (a0^2 - 2 a0 x + x^2 - b0^2 (y^2 + z^2)) - 2 (a0 - a1 + (b0 - b1) y) (a0 b0 - b0 x + y β^2))) /
      ((y^2 - 2 y y′ + y′^2 + z^2) √(a0^2 - 2 a0 x + x^2 - y^2 β^2 - z^2 β^2 + y′^2 (b0^2 - β^2) +
           y′ (2 a0 b0 - 2 b0 x + 2 y β^2))), y′], y′] - Dist[(z (μ0 - μ1)) /
   (2 π (a0^2 + a1^2 - 2 a1 (b0 - b1) y - 2 a0 (a1 - (b0 - b1) y) + (b0 - b1)^2 (y^2 + z^2))),
  Int[((b0 - b1) (a1 - x) (a1 y - x y + b1 (y^2 + z^2)) +
       (a0 - a1 + (b0 - b1) y) (a1^2 - 2 a1 x + x^2 - (y^2 + z^2) β^2) +
       y′^2 (-a1 (b0 b1 - β^2) + a0 (b1^2 - β^2) + (b0 - b1) (b1 x - y β^2)) +
       y′ (-(b0 - b1) (a1^2 - 2 a1 x + x^2 - b1^2 (y^2 + z^2)) +
          2 (a0 - a1 + (b0 - b1) y) (a1 b1 - b1 x + y β^2))) / ((y^2 - 2 y y′ + y′^2 + z^2)
       √(a1^2 - 2 a1 x + x^2 - y^2 β^2 - z^2 β^2 + y′^2 (b1^2 - β^2) + y′ (2 a1 b1 - 2 b1 x + 2 y β^2))),
   y′], y′] - Dist[((b0 - b1) z (μ0 - μ1)) /
   (4 π (a0^2 + a1^2 - 2 a1 (b0 - b1) y - 2 a0 (a1 - (b0 - b1) y) + (b0 - b1)^2 (y^2 + z^2))),
  Int[(2 y′ ((a0 - a1) (b0^2 - β^2) - (b0 - b1) (a0 b0 - b0 x + y β^2)) +
       2 ((a0 - a1) (a0 b0 - b0 x + y β^2) - (b0 - b1) (a0^2 - 2 a0 x + x^2 - (y^2 + z^2) β^2))) /
    ((a0 - a1 + (b0 - b1) y′) √(a0^2 - 2 a0 x + x^2 - y^2 β^2 - z^2 β^2 + y′^2 (b0^2 - β^2) +
          y′ (2 a0 b0 - 2 b0 x + 2 y β^2))), y′], y′] + Dist[((b0 - b1) z (μ0 - μ1)) /
   (4 π (a0^2 + a1^2 - 2 a1 (b0 - b1) y - 2 a0 (a1 - (b0 - b1) y) + (b0 - b1)^2 (y^2 + z^2))),
  Int[(-2 y′ ((a0 - a1) (b1^2 - β^2) - (b0 - b1) (a1 b1 - b1 x + y β^2)) -
       2 ((a0 - a1) (a1 b1 - b1 x + y β^2) - (b0 - b1) (a1^2 - 2 a1 x + x^2 - (y^2 + z^2) β^2))) /
    ((-a0 + a1 + (-b0 + b1) y′) √(a1^2 - 2 a1 x + x^2 - y^2 β^2 - z^2 β^2 +
          y′^2 (b1^2 - β^2) + y′ (2 a1 b1 - 2 b1 x + 2 y β^2))), y′], y′]
```

Rule 674: If NonzeroQ[$b^2 - 4\,a\,c$] && NonzeroQ[$e^2 - 4\,d\,f$],

$$\int \frac{A + B\,x + C\,x^2}{(a + b\,x + c\,x^2)\,\sqrt{d + e\,x + f\,x^2}}\,dx \longrightarrow$$

$$\text{Dist}\left[\frac{C}{c},\ \int \frac{1}{\sqrt{d + e\,x + f\,x^2}}\,dx,\ x\right] + \text{Dist}\left[\frac{1}{c},\ \int \frac{A\,c - a\,C + (B\,c - b\,C)\,x}{(a + b\,x + c\,x^2)\,\sqrt{d + e\,x + f\,x^2}}\,dx,\ x\right]$$

Rule 674: If NonzeroQ[$b^2 - 4\,a\,c$] && NonzeroQ[$e^2 - 4\,d\,f$],

$$\int \frac{A + B\,x + C\,x^2}{(a + b\,x + c\,x^2)\,\sqrt{d + e\,x + f\,x^2}}\,dx \longrightarrow$$

$$\text{Dist}\left[\frac{C}{c},\ \int \frac{1}{\sqrt{d + e\,x + f\,x^2}}\,dx,\ x\right] + \text{Dist}\left[\frac{1}{c},\ \int \frac{A\,c - a\,C + (B\,c - b\,C)\,x}{(a + b\,x + c\,x^2)\,\sqrt{d + e\,x + f\,x^2}}\,dx,\ x\right]$$

Rule 470: If NonzeroQ[$e\,f - d\,g$] && NonzeroQ[$b^2 - 4\,a\,c$] && NonzeroQ[$c\,d^2 - b\,d\,e + a\,e^2$],

$$\int (d + e\,x)^m\,(f + g\,x)\,(a + b\,x + c\,x^2)^p\,dx \longrightarrow$$

$$\text{Dist}\left[\frac{g}{e},\ \int (d + e\,x)^{m+1}\,(a + b\,x + c\,x^2)^p\,dx,\ x\right] + \text{Dist}\left[\frac{e\,f - d\,g}{e},\ \int (d + e\,x)^m\,(a + b\,x + c\,x^2)^p\,dx,\ x\right]$$

Rule 470: If NonzeroQ[$e\,f - d\,g$] && NonzeroQ[$b^2 - 4\,a\,c$] && NonzeroQ[$c\,d^2 - b\,d\,e + a\,e^2$],

$$\int (d + e\,x)^m\,(f + g\,x)\,(a + b\,x + c\,x^2)^p\,dx \longrightarrow$$

$$\text{Dist}\left[\frac{g}{e},\ \int (d + e\,x)^{m+1}\,(a + b\,x + c\,x^2)^p\,dx,\ x\right] + \text{Dist}\left[\frac{e\,f - d\,g}{e},\ \int (d + e\,x)^m\,(a + b\,x + c\,x^2)^p\,dx,\ x\right]$$

```
In[371]:= -Dist[(z (μ0 - μ1)) /
       (2 π (a0^2 + a1^2 - 2 a1 (b0 - b1) y - 2 a0 (a1 - (b0 - b1) y) + (b0 - b1)^2 (y^2 + z^2))),
      Int[((-b0 + b1) (a0 - x) (a0 y - x y + b0 (y^2 + z^2)) -
          (a0 - a1 + (b0 - b1) y) (a0^2 - 2 a0 x + x^2 - (y^2 + z^2) β^2) -
          (y^2 + z^2) (a1 (b0^2 - β^2) - a0 (b0 b1 - β^2) - (b0 - b1) (b0 x - y β^2)) +
          y′ ((b0 - b1) (a0^2 - 2 a0 x + x^2 - b0^2 (y^2 + z^2)) - 2 (a0 - a1 + (b0 - b1) y) (a0 b0 - b0 x +
                y β^2) + 2 y (a1 (b0^2 - β^2) - a0 (b0 b1 - β^2) - (b0 - b1) (b0 x - y β^2)))) /
        ((y^2 - 2 y y′ + y′^2 + z^2) √(a0^2 - 2 a0 x + x^2 - y^2 β^2 - z^2 β^2 + y′^2 (b0^2 - β^2) +
              y′ (2 a0 b0 - 2 b0 x + 2 y β^2))), y′], y′] - Dist[(z (μ0 - μ1)) /
      (2 π (a0^2 + a1^2 - 2 a1 (b0 - b1) y - 2 a0 (a1 - (b0 - b1) y) + (b0 - b1)^2 (y^2 + z^2))),
     Int[((b0 - b1) (a1 - x) (a1 y - x y + b1 (y^2 + z^2)) +
         (a0 - a1 + (b0 - b1) y) (a1^2 - 2 a1 x + x^2 - (y^2 + z^2) β^2) -
         (y^2 + z^2) (-a1 (b0 b1 - β^2) + a0 (b1^2 - β^2) + (b0 - b1) (b1 x - y β^2)) +
         y′ ((-b0 + b1) (a1^2 - 2 a1 x + x^2 - b1^2 (y^2 + z^2)) + 2 (a0 - a1 + (b0 - b1) y) (a1 b1 - b1 x +
               y β^2) + 2 y (-a1 (b0 b1 - β^2) + a0 (b1^2 - β^2) + (b0 - b1) (b1 x - y β^2)))) /
       ((y^2 - 2 y y′ + y′^2 + z^2) √(a1^2 - 2 a1 x + x^2 - y^2 β^2 - z^2 β^2 + y′^2 (b1^2 - β^2) +
             y′ (2 a1 b1 - 2 b1 x + 2 y β^2))), y′], y′] +
    Dist[(z (a1^2 (b0^2 - β^2) + a0^2 (b1^2 - β^2) + 2 a0 (b0 - b1) (b1 x - y β^2) + (b0 - b1)^2
             (x^2 - (y^2 + z^2) β^2) - 2 a1 (a0 (b0 b1 - β^2) + (b0 - b1) (b0 x - y β^2))) (μ0 - μ1)) /
      (2 π (a0^2 + a1^2 - 2 a1 (b0 - b1) y - 2 a0 (a1 - (b0 - b1) y) + (b0 - b1)^2 (y^2 + z^2))),
     Int[1 / ((a0 - a1 + (b0 - b1) y′)
          √(a0^2 - 2 a0 x + x^2 - y^2 β^2 - z^2 β^2 + y′^2 (b0^2 - β^2) + y′ (2 a0 b0 - 2 b0 x + 2 y β^2))), y′],
     y′] + Dist[(z (a1^2 (b0^2 - β^2) + a0^2 (b1^2 - β^2) + 2 a0 (b0 - b1) (b1 x - y β^2) + (b0 - b1)^2
             (x^2 - (y^2 + z^2) β^2) - 2 a1 (a0 (b0 b1 - β^2) + (b0 - b1) (b0 x - y β^2))) (μ0 - μ1)) /
      (2 π (a0^2 + a1^2 - 2 a1 (b0 - b1) y - 2 a0 (a1 - (b0 - b1) y) + (b0 - b1)^2 (y^2 + z^2))),
     Int[1 / ((-a0 + a1 + (-b0 + b1) y′)
          √(a1^2 - 2 a1 x + x^2 - y^2 β^2 - z^2 β^2 + y′^2 (b1^2 - β^2) + y′ (2 a1 b1 - 2 b1 x + 2 y β^2))), y′], y′]
```

Rule 623:

If NonzeroQ[b^2 - 4 a c] && NonzeroQ[e^2 - 4 d f] && NonzeroQ[b d - a e] && NegQ[b^2 - 4 a c], let

$q = \sqrt{(c\,d - a\,f)^2 - (b\,d - a\,e)\,(c\,e - b\,f)}$, then

$$\int \frac{g + h\,x}{(a + b\,x + c\,x^2)\,\sqrt{d + e\,x + f\,x^2}}\,dx \longrightarrow$$

$$\text{Dist}\Big[\frac{1}{2\,q}, \int \Big(\text{Simp}[h\,(b\,d - a\,e) - g\,(c\,d - a\,f - q) - (g\,(c\,e - b\,f) - h\,(c\,d - a\,f + q))\,x, x] \Big/ \Big((a + b\,x + c\,x^2)\,\sqrt{d + e\,x + f\,x^2}\Big)\Big)\,dx, x\Big] -$$
$$\text{Dist}\Big[\frac{1}{2\,q}, \int \Big(\text{Simp}[h\,(b\,d - a\,e) - g\,(c\,d - a\,f + q) - (g\,(c\,e - b\,f) - h\,(c\,d - a\,f - q))\,x, x] \Big/ \Big((a + b\,x + c\,x^2)\,\sqrt{d + e\,x + f\,x^2}\Big)\Big)\,dx, x\Big]$$

Rule 623:

If NonzeroQ[b^2 - 4 a c] && NonzeroQ[e^2 - 4 d f] && NonzeroQ[b d - a e] && NegQ[b^2 - 4 a c], let

$q = \sqrt{(c\,d - a\,f)^2 - (b\,d - a\,e)\,(c\,e - b\,f)}$, then

$$\int \frac{g + h\,x}{(a + b\,x + c\,x^2)\,\sqrt{d + e\,x + f\,x^2}}\,dx \longrightarrow$$

$$\text{Dist}\Big[\frac{1}{2\,q}, \int \Big(\text{Simp}[h\,(b\,d - a\,e) - g\,(c\,d - a\,f - q) - (g\,(c\,e - b\,f) - h\,(c\,d - a\,f + q))\,x, x] \Big/ \Big((a + b\,x + c\,x^2)\,\sqrt{d + e\,x + f\,x^2}\Big)\Big)\,dx, x\Big] -$$
$$\text{Dist}\Big[\frac{1}{2\,q}, \int \Big(\text{Simp}[h\,(b\,d - a\,e) - g\,(c\,d - a\,f + q) - (g\,(c\,e - b\,f) - h\,(c\,d - a\,f - q))\,x, x] \Big/ \Big((a + b\,x + c\,x^2)\,\sqrt{d + e\,x + f\,x^2}\Big)\Big)\,dx, x\Big]$$

Rule 310: If NonzeroQ[b^2 - 4 a c] && NonzeroQ[2 c d - b e],

$$\int \frac{1}{(d + e\,x)\,\sqrt{a + b\,x + c\,x^2}}\,dx \longrightarrow$$

$$\text{Dist}\Big[-2, \text{Subst}\Big[\int \frac{1}{4\,c\,d^2 - 4\,b\,d\,e + 4\,a\,e^2 - x^2}\,dx, x, \frac{2\,a\,e - b\,d - (2\,c\,d - b\,e)\,x}{\sqrt{a + b\,x + c\,x^2}}\Big], x\Big]$$

Rule 310: If NonzeroQ[b^2 - 4 a c] && NonzeroQ[2 c d - b e],

$$\int \frac{1}{(d + e\,x)\,\sqrt{a + b\,x + c\,x^2}}\,dx \longrightarrow$$

$$\text{Dist}\Big[-2, \text{Subst}\Big[\int \frac{1}{4\,c\,d^2 - 4\,b\,d\,e + 4\,a\,e^2 - x^2}\,dx, x, \frac{2\,a\,e - b\,d - (2\,c\,d - b\,e)\,x}{\sqrt{a + b\,x + c\,x^2}}\Big], x\Big]$$

In[372]:=

$$-\text{Dist}\Big[\big(z\,(\mu 0 - \mu 1)\big) \Big/ \Big(4\,\pi\,\big(a0^2 + x^2 - 2\,b0\,x\,y - 2\,a0\,(x - b0\,y) + b0^2\,(y^2 + z^2)\big)\,\big(a0^2 + a1^2 - 2\,a1\,(b0 - b1)\,y - 2\,a0\,(a1 - (b0 - b1)\,y) + (b0 - b1)^2\,(y^2 + z^2)\big)\Big),$$
$$\text{Int}\Big[\Big(-2\,(a1\,b0 - a0\,b1 - b0\,x + b1\,x)\,(a0\,y - x\,y + b0\,y^2 + b0\,z^2)\,(a0^2 - 2\,a0\,x + x^2 + 2\,a0\,b0\,y - 2\,b0\,x\,y + b0^2\,y^2 + b0^2\,z^2) + 2\,(a1\,b0 - a0\,b1 - (b0 - b1)\,x)\,(a0 - x + b0\,y)\,y'\,\big(a0^2 + x^2 - 2\,b0\,x\,y - 2\,a0\,(x - b0\,y) + b0^2\,(y^2 + z^2)\big)\Big) \Big/$$
$$\Big(\big(y^2 - 2\,y\,y' + y'^2 + z^2\big)\,\sqrt{\big(a0^2 - 2\,a0\,x + x^2 - y^2\,\beta^2 - z^2\,\beta^2 + y'^2\,(b0^2 - \beta^2) + y'\,(2\,a0\,b0 - 2\,b0\,x + 2\,y\,\beta^2)\big)}\Big), y'\Big], y'\Big] +$$
$$\text{Dist}\Big[\big(z\,(\mu 0 - \mu 1)\big) \Big/ \Big(4\,\pi\,\big(a0^2 + x^2 - 2\,b0\,x\,y - 2\,a0\,(x - b0\,y) + b0^2\,(y^2 + z^2)\big)$$

```
    (a0^2 + a1^2 - 2 a1 (b0 - b1) y - 2 a0 (a1 - (b0 - b1) y) + (b0 - b1)^2 (y^2 + z^2))),
  Int[(2 (a0 - x) (a0^2 + x^2 - 2 b0 x y - 2 a0 (x - b0 y) + b0^2 (y^2 + z^2))
        (a0^2 + a1 (x - b0 y) - a0 (a1 + x - 2 b0 y + b1 y) - (b0 - b1) (x y - b0 (y^2 + z^2))) +
      2 b0 y′ (a0^2 + x^2 - 2 b0 x y - 2 a0 (x - b0 y) + b0^2 (y^2 + z^2))
        (a0^2 + a1 (x - b0 y) - a0 (a1 + x - 2 b0 y + b1 y) - (b0 - b1) (x y - b0 (y^2 + z^2)))) /
    ((y^2 - 2 y y′ + y′^2 + z^2) √(a0^2 - 2 a0 x + x^2 - y^2 β^2 - z^2 β^2 + y′^2 (b0^2 - β^2) +
        y′ (2 a0 b0 - 2 b0 x + 2 y β^2))), y′], y′] - Dist[
  (z (μ0 - μ1)) / (4 π (a0^2 + a1^2 - 2 a1 (b0 - b1) y - 2 a0 (a1 - (b0 - b1) y) + (b0 - b1)^2 (y^2 + z^2))
    (a1^2 + x^2 - 2 b1 x y - 2 a1 (x - b1 y) + b1^2 (y^2 + z^2))),
  Int[(2 (a1 b0 - a0 b1 - b0 x + b1 x) (a1 y - x y + b1 y^2 + b1 z^2)
        (a1^2 - 2 a1 x + x^2 + 2 a1 b1 y - 2 b1 x y + b1^2 y^2 + b1^2 z^2) - 2 (a1 b0 - a0 b1 - (b0 - b1) x)
        (a1 - x + b1 y) y′ (a1^2 + x^2 - 2 b1 x y - 2 a1 (x - b1 y) + b1^2 (y^2 + z^2))) /
    ((y^2 - 2 y y′ + y′^2 + z^2) √(a1^2 - 2 a1 x + x^2 - y^2 β^2 - z^2 β^2 + y′^2 (b1^2 - β^2) +
        y′ (2 a1 b1 - 2 b1 x + 2 y β^2))), y′], y′] + Dist[
  (z (μ0 - μ1)) / (4 π (a0^2 + a1^2 - 2 a1 (b0 - b1) y - 2 a0 (a1 - (b0 - b1) y) + (b0 - b1)^2 (y^2 + z^2))
    (a1^2 + x^2 - 2 b1 x y - 2 a1 (x - b1 y) + b1^2 (y^2 + z^2))),
  Int[(2 (a1 - x) (a1^2 + x^2 - 2 b1 x y - 2 a1 (x - b1 y) + b1^2 (y^2 + z^2))
        (a1^2 - a1 (x + (b0 - 2 b1) y) - a0 (a1 - x + b1 y) + (b0 - b1) (x y - b1 (y^2 + z^2))) +
      2 b1 y′ (a1^2 + x^2 - 2 b1 x y - 2 a1 (x - b1 y) + b1^2 (y^2 + z^2))
        (a1^2 - a1 (x + (b0 - 2 b1) y) - a0 (a1 - x + b1 y) + (b0 - b1) (x y - b1 (y^2 + z^2)))) /
    ((y^2 - 2 y y′ + y′^2 + z^2) √(a1^2 - 2 a1 x + x^2 - y^2 β^2 - z^2 β^2 + y′^2 (b1^2 - β^2) +
        y′ (2 a1 b1 - 2 b1 x + 2 y β^2))), y′], y′] -
Dist[(z (a1^2 (b0^2 - β^2) + a0^2 (b1^2 - β^2) + 2 a0 (b0 - b1) (b1 x - y β^2) + (b0 - b1)^2
        (x^2 - (y^2 + z^2) β^2) - 2 a1 (a0 (b0 b1 - β^2) + (b0 - b1) (b0 x - y β^2))) (μ0 - μ1)) /
    (π (a0^2 + a1^2 - 2 a1 (b0 - b1) y - 2 a0 (a1 - (b0 - b1) y) + (b0 - b1)^2 (y^2 + z^2))),
  Subst[Int[1 / (-y′^2 + 4 (a0 - a1)^2 (b0^2 - β^2) - 4 (a0 - a1) (b0 - b1) (2 a0 b0 - 2 b0 x + 2 y β^2) +
        4 (b0 - b1)^2 (a0^2 - 2 a0 x + x^2 - y^2 β^2 - z^2 β^2)), y′], y′,
    (- (a0 - a1) (2 a0 b0 - 2 b0 x + 2 y β^2) + 2 (b0 - b1) (a0^2 - 2 a0 x + x^2 - y^2 β^2 - z^2 β^2) -
      y′ (2 (a0 - a1) (b0^2 - β^2) - (b0 - b1) (2 a0 b0 - 2 b0 x + 2 y β^2))) /
    (√(a0^2 - 2 a0 x + x^2 - y^2 β^2 - z^2 β^2 + y′^2 (b0^2 - β^2) + y′ (2 a0 b0 - 2 b0 x + 2 y β^2)))], y′] -
Dist[(z (a1^2 (b0^2 - β^2) + a0^2 (b1^2 - β^2) + 2 a0 (b0 - b1) (b1 x - y β^2) + (b0 - b1)^2
        (x^2 - (y^2 + z^2) β^2) - 2 a1 (a0 (b0 b1 - β^2) + (b0 - b1) (b0 x - y β^2))) (μ0 - μ1)) /
    (π (a0^2 + a1^2 - 2 a1 (b0 - b1) y - 2 a0 (a1 - (b0 - b1) y) + (b0 - b1)^2 (y^2 + z^2))),
  Subst[Int[1 / (-y′^2 + 4 (-a0 + a1)^2 (b1^2 - β^2) - 4 (-a0 + a1) (-b0 + b1)
        (2 a1 b1 - 2 b1 x + 2 y β^2) + 4 (-b0 + b1)^2 (a1^2 - 2 a1 x + x^2 - y^2 β^2 - z^2 β^2)), y′], y′,
    (- (-a0 + a1) (2 a1 b1 - 2 b1 x + 2 y β^2) + 2 (-b0 + b1) (a1^2 - 2 a1 x + x^2 - y^2 β^2 - z^2 β^2) -
      y′ (2 (-a0 + a1) (b1^2 - β^2) - (-b0 + b1) (2 a1 b1 - 2 b1 x + 2 y β^2))) /
    (√(a1^2 - 2 a1 x + x^2 - y^2 β^2 - z^2 β^2 + y′^2 (b1^2 - β^2) + y′ (2 a1 b1 - 2 b1 x + 2 y β^2)))], y′]
```

Rule 617: If
NonzeroQ[b^2 - 4 a c] && NonzeroQ[e^2 - 4 d f] && NonzeroQ[b d - a e] && ZeroQ[h^2 (b d - a e) - 2 g h (c d - a f) + g^2 (c e - b f)],

$$\int \frac{g + h x}{(a + b x + c x^2)\sqrt{d + e x + f x^2}}\, dx \longrightarrow$$

$$\text{Dist}\left[-2 g (g b - 2 a h), \text{Subst}\left[\int \frac{1}{\text{Simp}[g (g b - 2 a h) (b^2 - 4 a c) - (b d - a e) x^2, x]}\, dx, x, \frac{\text{Simp}[g b - 2 a h - (b h - 2 g c) x, x]}{\sqrt{d + e x + f x^2}}\right], x\right]$$

Rule 617: If
NonzeroQ[b^2 - 4 a c] && NonzeroQ[e^2 - 4 d f] && NonzeroQ[b d - a e] && ZeroQ[h^2 (b d - a e) - 2 g h (c d - a f) + g^2 (c e - b f)],

$$\int \frac{g + h x}{(a + b x + c x^2)\sqrt{d + e x + f x^2}}\, dx \longrightarrow$$

$$\text{Dist}\left[-2 g (g b - 2 a h), \text{Subst}\left[\int \frac{1}{\text{Simp}[g (g b - 2 a h) (b^2 - 4 a c) - (b d - a e) x^2, x]}\, dx, x, \frac{\text{Simp}[g b - 2 a h - (b h - 2 g c) x, x]}{\sqrt{d + e x + f x^2}}\right], x\right]$$

Rule 617: If
NonzeroQ[b^2 - 4 a c] && NonzeroQ[e^2 - 4 d f] && NonzeroQ[b d - a e] && ZeroQ[h^2 (b d - a e) - 2 g h (c d - a f) + g^2 (c e - b f)],

$$\int \frac{g + h x}{(a + b x + c x^2)\sqrt{d + e x + f x^2}}\, dx \longrightarrow$$

$$\text{Dist}\left[-2 g (g b - 2 a h), \text{Subst}\left[\int \frac{1}{\text{Simp}[g (g b - 2 a h) (b^2 - 4 a c) - (b d - a e) x^2, x]}\, dx, x, \frac{\text{Simp}[g b - 2 a h - (b h - 2 g c) x, x]}{\sqrt{d + e x + f x^2}}\right], x\right]$$

Rule 617: If
NonzeroQ[b^2 - 4 a c] && NonzeroQ[e^2 - 4 d f] && NonzeroQ[b d - a e] && ZeroQ[h^2 (b d - a e) - 2 g h (c d - a f) + g^2 (c e - b f)],

$$\int \frac{g + h x}{(a + b x + c x^2)\sqrt{d + e x + f x^2}}\, dx \longrightarrow$$

$$\text{Dist}\left[-2 g (g b - 2 a h), \text{Subst}\left[\int \frac{1}{\text{Simp}[g (g b - 2 a h) (b^2 - 4 a c) - (b d - a e) x^2, x]}\, dx, x, \frac{\text{Simp}[g b - 2 a h - (b h - 2 g c) x, x]}{\sqrt{d + e x + f x^2}}\right], x\right]$$

Rule 707: If NegQ$\left[\frac{a}{b}\right]$ && (PositiveQ[a] || NegativeQ[b]),

$$\int \frac{1}{a + b x^2}\, dx \longrightarrow \text{Simp}\left[\frac{\text{ArcTanh}\left[\frac{\sqrt{-b}\, x}{\sqrt{a}}\right]}{\sqrt{a}\,\sqrt{-b}}, x\right]$$

Rule 707: If NegQ$\left[\frac{a}{b}\right]$ && (PositiveQ[a] || NegativeQ[b]),

$$\int \frac{1}{a+b\,x^2}\,dx \longrightarrow \text{Simp}\left[\frac{\text{ArcTanh}\left[\frac{\sqrt{-b}\,x}{\sqrt{a}}\right]}{\sqrt{a}\,\sqrt{-b}},\ x\right]$$

```
In[373]:= (z √(a1² (b0² - β²) + a0² (b1² - β²) + 2 a0 (b0 - b1) (b1 x - y β²) +
              (b0 - b1)² (x² - (y² + z²) β²) - 2 a1 (a0 (b0 b1 - β²) + (b0 - b1) (b0 x - y β²))) (μ0 - μ1)
          ArcTanh[((a0 - a1) (a0 b0 - b0 x + y β²) - (b0 - b1) (a0² - 2 a0 x + x² - (y² + z²) β²) +
              y′ ((a0 - a1) (b0² - β²) - (b0 - b1) (a0 b0 - b0 x + y β²)))/
            (√(a0² - 2 a0 x + x² - (y² + z²) β² + y′² (b0² - β²) + 2 y′ (a0 b0 - b0 x + y β²))
              √(a1² (b0² - β²) + a0² (b1² - β²) + 2 a0 (b0 - b1) (b1 x - y β²) +
                  (b0 - b1)² (x² - (y² + z²) β²) - 2 a1 (a0 (b0 b1 - β²) + (b0 - b1) (b0 x - y β²))))])/
        (2 π (a0² + a1² - 2 a1 (b0 - b1) y - 2 a0 (a1 - (b0 - b1) y) + (b0 - b1)² (y² + z²))) -
       (z √(a1² (b0² - β²) + a0² (b1² - β²) + 2 a0 (b0 - b1) (b1 x - y β²) +
              (b0 - b1)² (x² - (y² + z²) β²) - 2 a1 (a0 (b0 b1 - β²) + (b0 - b1) (b0 x - y β²))) (μ0 - μ1)
          ArcTanh[((a0 - a1) (a1 b1 - b1 x + y β²) - (b0 - b1) (a1² - 2 a1 x + x² - (y² + z²) β²) +
              y′ ((a0 - a1) (b1² - β²) - (b0 - b1) (a1 b1 - b1 x + y β²)))/
            (√(a1² - 2 a1 x + x² - (y² + z²) β² + y′² (b1² - β²) + 2 y′ (a1 b1 - b1 x + y β²))
              √(a1² (b0² - β²) + a0² (b1² - β²) + 2 a0 (b0 - b1) (b1 x - y β²) +
                  (b0 - b1)² (x² - (y² + z²) β²) - 2 a1 (a0 (b0 b1 - β²) + (b0 - b1) (b0 x - y β²))))])/
        (2 π (a0² + a1² - 2 a1 (b0 - b1) y - 2 a0 (a1 - (b0 - b1) y) + (b0 - b1)² (y² + z²))) +
       Dist[(4 (a0 - x) (a1 b0 - a0 b1 - (b0 - b1) x)² z³ (a0 y - x y + b0 (y² + z²))
            (a0² + x² - 2 b0 x y - 2 a0 (x - b0 y) + b0² (y² + z²)) (μ0 - μ1))/
          (π (a0² + a1² - 2 a1 (b0 - b1) y - 2 a0 (a1 - (b0 - b1) y) + (b0 - b1)² (y² + z²))), Subst[
          Int[1/(2 (a0 - x) y′² (a0 y - x y + b0 (y² + z²)) - 32 (a0 - x) (a1 b0 - a0 b1 - (b0 - b1) x)²
                z⁴ (a0 y - x y + b0 (y² + z²)) (a0² + x² - 2 b0 x y - 2 a0 (x - b0 y) + b0² (y² + z²))²),
            y′], y′, (-4 (a0 - x) (a1 b0 - a0 b1 - (b0 - b1) x) z²
                (a0² + x² - 2 b0 x y - 2 a0 (x - b0 y) + b0² (y² + z²)) - 4 b0
                (a1 b0 - a0 b1 - (b0 - b1) x) y′ z² (a0² + x² - 2 b0 x y - 2 a0 (x - b0 y) + b0² (y² + z²)))/
           (√(a0² - 2 a0 x + x² - y² β² - z² β² + y′² (b0² - β²) + y′ (2 a0 b0 - 2 b0 x + 2 y β²)))], y′] +
       Dist[(4 (a1 - x) (a1 b0 - a0 b1 - (b0 - b1) x)² z³ (a1 y - x y + b1 (y² + z²))
            (a1² + x² - 2 b1 x y - 2 a1 (x - b1 y) + b1² (y² + z²)) (μ0 - μ1))/
          (π (a0² + a1² - 2 a1 (b0 - b1) y - 2 a0 (a1 - (b0 - b1) y) + (b0 - b1)² (y² + z²))), Subst[
          Int[1/(2 (a1 - x) y′² (a1 y - x y + b1 (y² + z²)) - 32 (a1 - x) (a1 b0 - a0 b1 - (b0 - b1) x)² z⁴
                (a1 y - x y + b1 (y² + z²)) (a1² + x² - 2 b1 x y - 2 a1 (x - b1 y) + b1² (y² + z²))²), y′],
          y′, (4 (a1 - x) (a1 b0 - a0 b1 - (b0 - b1) x) z² (a1² + x² - 2 b1 x y - 2 a1 (x - b1 y) +
                 b1² (y² + z²)) + 4 b1 (a1 b0 - a0 b1 - (b0 - b1) x)
               y′ z² (a1² + x² - 2 b1 x y - 2 a1 (x - b1 y) + b1² (y² + z²)))/
           (√(a1² - 2 a1 x + x² - y² β² - z² β² + y′² (b1² - β²) + y′ (2 a1 b1 - 2 b1 x + 2 y β²)))], y′] +
       Dist[(4 (a0 - x) z (a0 y - x y + b0 (y² + z²)) (a0² + x² - 2 b0 x y - 2 a0 (x - b0 y) + b0² (y² + z²))
```

```
    (a0^2 + a1 (x - b0 y) - a0 (a1 + x - 2 b0 y + b1 y) - (b0 - b1) (x y - b0 (y^2 + z^2)))^2
    (μ0 - μ1)) /
   (π (a0^2 + a1^2 - 2 a1 (b0 - b1) y - 2 a0 (a1 - (b0 - b1) y) + (b0 - b1)^2 (y^2 + z^2))),
  Subst[Int[1 / (2 (a0 - x) y′^2 (a0 y - x y + b0 (y^2 + z^2)) + 32 (a0 - x) z^2
        (a0 y - x y + b0 (y^2 + z^2)) (a0^2 + x^2 - 2 b0 x y - 2 a0 (x - b0 y) + b0^2 (y^2 + z^2))^2
        (a0^2 + a1 (x - b0 y) - a0 (a1 + x - 2 b0 y + b1 y) - (b0 - b1) (x y - b0 (y^2 + z^2)))^2), y′],
   y′, (4 (a0 - x + b0 y) y′ (a0^2 + x^2 - 2 b0 x y - 2 a0 (x - b0 y) + b0^2 (y^2 + z^2))
       (a0^2 + a1 (x - b0 y) - a0 (a1 + x - 2 b0 y + b1 y) - (b0 - b1) (x y - b0 (y^2 + z^2))) -
      4 (a0 y - x y + b0 (y^2 + z^2)) (a0^2 + x^2 - 2 b0 x y - 2 a0 (x - b0 y) + b0^2 (y^2 + z^2))
       (a0^2 + a1 (x - b0 y) - a0 (a1 + x - 2 b0 y + b1 y) - (b0 - b1) (x y - b0 (y^2 + z^2)))) /
     (√(a0^2 - 2 a0 x + x^2 - y^2 β^2 - z^2 β^2 + y′^2 (b0^2 - β^2) + y′ (2 a0 b0 - 2 b0 x + 2 y β^2)))], y′] +
 Dist[(4 (a1 - x) z (a1 y - x y + b1 (y^2 + z^2)) (a1^2 + x^2 - 2 b1 x y - 2 a1 (x - b1 y) + b1^2 (y^2 + z^2))
    (a1^2 - a1 (x + (b0 - 2 b1) y) - a0 (a1 - x + b1 y) + (b0 - b1) (x y - b1 (y^2 + z^2)))^2
    (μ0 - μ1)) /
   (π (a0^2 + a1^2 - 2 a1 (b0 - b1) y - 2 a0 (a1 - (b0 - b1) y) + (b0 - b1)^2 (y^2 + z^2))),
  Subst[Int[1 / (2 (a1 - x) y′^2 (a1 y - x y + b1 (y^2 + z^2)) + 32 (a1 - x) z^2
        (a1 y - x y + b1 (y^2 + z^2)) (a1^2 + x^2 - 2 b1 x y - 2 a1 (x - b1 y) + b1^2 (y^2 + z^2))^2
        (a1^2 - a1 (x + (b0 - 2 b1) y) - a0 (a1 - x + b1 y) + (b0 - b1) (x y - b1 (y^2 + z^2)))^2), y′],
   y′, (4 (a1 - x + b1 y) y′ (a1^2 + x^2 - 2 b1 x y - 2 a1 (x - b1 y) + b1^2 (y^2 + z^2))
       (a1^2 - a1 (x + (b0 - 2 b1) y) - a0 (a1 - x + b1 y) + (b0 - b1) (x y - b1 (y^2 + z^2))) -
      4 (a1 y - x y + b1 (y^2 + z^2)) (a1^2 + x^2 - 2 b1 x y - 2 a1 (x - b1 y) + b1^2 (y^2 + z^2))
       (a1^2 - a1 (x + (b0 - 2 b1) y) - a0 (a1 - x + b1 y) + (b0 - b1) (x y - b1 (y^2 + z^2)))) /
     (√(a1^2 - 2 a1 x + x^2 - y^2 β^2 - z^2 β^2 + y′^2 (b1^2 - β^2) + y′ (2 a1 b1 - 2 b1 x + 2 y β^2)))], y′]
```

Rule 709: If NegQ$\left[\frac{a}{b}\right]$,

$$\int \frac{1}{a + b\,x^2}\,dx \longrightarrow \text{Simp}\left[\frac{\sqrt{-\frac{a}{b}}\ \text{ArcTanh}\left[\frac{x}{\sqrt{-\frac{a}{b}}}\right]}{a},\ x\right]$$

Rule 709: If NegQ$\left[\frac{a}{b}\right]$,

$$\int \frac{1}{a + b\,x^2}\,dx \longrightarrow \text{Simp}\left[\frac{\sqrt{-\frac{a}{b}}\ \text{ArcTanh}\left[\frac{x}{\sqrt{-\frac{a}{b}}}\right]}{a},\ x\right]$$

Rule 706: If PosQ$\left[\frac{a}{b}\right]$,

$$\int \frac{1}{a + b\,x^2}\,dx \longrightarrow \text{Simp}\left[\frac{\sqrt{\frac{a}{b}}\ \text{ArcTan}\left[\frac{x}{\sqrt{\frac{a}{b}}}\right]}{a},\ x\right]$$

Rule 706: If PosQ$\left[\frac{a}{b}\right]$,

$$\int \frac{1}{a + b\,x^2}\,dx \longrightarrow \mathrm{Simp}\left[\frac{\sqrt{\frac{a}{b}}\ \mathrm{ArcTan}\left[\frac{x}{\sqrt{\frac{a}{b}}}\right]}{a},\ x\right]$$

Out[373]=
```
- (((a0^2 + a1 (x - b0 y) - a0 (a1 + x - 2 b0 y + b1 y) - (b0 - b1) (x y - b0 (y^2 + z^2)))
      (μ0 - μ1) ArcTan[(a0 y - x y - (a0 - x + b0 y) y′ + b0 (y^2 + z^2)) /
        (z √(a0^2 - 2 a0 x + x^2 - (y^2 + z^2) β^2 + y′^2 (b0^2 - β^2) + 2 y′ (a0 b0 - b0 x + y β^2)))]) /
    (2 π (a0^2 + a1^2 - 2 a1 (b0 - b1) y - 2 a0 (a1 - (b0 - b1) y) + (b0 - b1)^2 (y^2 + z^2)))) -
  ((a1^2 - a1 (x + (b0 - 2 b1) y) - a0 (a1 - x + b1 y) + (b0 - b1) (x y - b1 (y^2 + z^2)))
     (μ0 - μ1) ArcTan[(a1 y - x y - (a1 - x + b1 y) y′ + b1 (y^2 + z^2)) /
       (z √(a1^2 - 2 a1 x + x^2 - (y^2 + z^2) β^2 + y′^2 (b1^2 - β^2) + 2 y′ (a1 b1 - b1 x + y β^2)))]) /
   (2 π (a0^2 + a1^2 - 2 a1 (b0 - b1) y - 2 a0 (a1 - (b0 - b1) y) + (b0 - b1)^2 (y^2 + z^2))) +
  ((a1 b0 - a0 b1 - (b0 - b1) x) z (μ0 - μ1)
     ArcTanh[((a0 - x) (a1 b0 - a0 b1 - (b0 - b1) x) + b0 (a1 b0 - a0 b1 - (b0 - b1) x) y′) /
       ((a1 b0 - a0 b1 - (b0 - b1) x)
         √(a0^2 - 2 a0 x + x^2 - (y^2 + z^2) β^2 + y′^2 (b0^2 - β^2) + 2 y′ (a0 b0 - b0 x + y β^2)))]) /
   (2 π (a0^2 + a1^2 - 2 a1 (b0 - b1) y - 2 a0 (a1 - (b0 - b1) y) + (b0 - b1)^2 (y^2 + z^2))) -
  ((a1 b0 - a0 b1 - (b0 - b1) x) z (μ0 - μ1) ArcTanh[(a1 - x + b1 y′) /
       (√(a1^2 - 2 a1 x + x^2 - (y^2 + z^2) β^2 + y′^2 (b1^2 - β^2) + 2 y′ (a1 b1 - b1 x + y β^2)))]) /
   (2 π (a0^2 + a1^2 - 2 a1 (b0 - b1) y - 2 a0 (a1 - (b0 - b1) y) + (b0 - b1)^2 (y^2 + z^2))) +
  (z √(a1^2 (b0^2 - β^2) + a0^2 (b1^2 - β^2) + 2 a0 (b0 - b1) (b1 x - y β^2) +
         (b0 - b1)^2 (x^2 - (y^2 + z^2) β^2) - 2 a1 (a0 (b0 b1 - β^2) + (b0 - b1) (b0 x - y β^2))) (μ0 - μ1)
     ArcTanh[((a0 - a1) (a0 b0 - b0 x + y β^2) - (b0 - b1) (a0^2 - 2 a0 x + x^2 - (y^2 + z^2) β^2) +
         y′ ((a0 - a1) (b0^2 - β^2) - (b0 - b1) (a0 b0 - b0 x + y β^2))) /
       (√(a0^2 - 2 a0 x + x^2 - (y^2 + z^2) β^2 + y′^2 (b0^2 - β^2) + 2 y′ (a0 b0 - b0 x + y β^2))
         √(a1^2 (b0^2 - β^2) + a0^2 (b1^2 - β^2) + 2 a0 (b0 - b1) (b1 x - y β^2) +
             (b0 - b1)^2 (x^2 - (y^2 + z^2) β^2) - 2 a1 (a0 (b0 b1 - β^2) + (b0 - b1) (b0 x - y β^2))))]) /
   (2 π (a0^2 + a1^2 - 2 a1 (b0 - b1) y - 2 a0 (a1 - (b0 - b1) y) + (b0 - b1)^2 (y^2 + z^2))) -
  (z √(a1^2 (b0^2 - β^2) + a0^2 (b1^2 - β^2) + 2 a0 (b0 - b1) (b1 x - y β^2) +
         (b0 - b1)^2 (x^2 - (y^2 + z^2) β^2) - 2 a1 (a0 (b0 b1 - β^2) + (b0 - b1) (b0 x - y β^2))) (μ0 - μ1)
     ArcTanh[((a0 - a1) (a1 b1 - b1 x + y β^2) - (b0 - b1) (a1^2 - 2 a1 x + x^2 - (y^2 + z^2) β^2) +
         y′ ((a0 - a1) (b1^2 - β^2) - (b0 - b1) (a1 b1 - b1 x + y β^2))) /
       (√(a1^2 - 2 a1 x + x^2 - (y^2 + z^2) β^2 + y′^2 (b1^2 - β^2) + 2 y′ (a1 b1 - b1 x + y β^2))
         √(a1^2 (b0^2 - β^2) + a0^2 (b1^2 - β^2) + 2 a0 (b0 - b1) (b1 x - y β^2) +
             (b0 - b1)^2 (x^2 - (y^2 + z^2) β^2) - 2 a1 (a0 (b0 b1 - β^2) + (b0 - b1) (b0 x - y β^2))))]) /
   (2 π (a0^2 + a1^2 - 2 a1 (b0 - b1) y - 2 a0 (a1 - (b0 - b1) y) + (b0 - b1)^2 (y^2 + z^2)))
```

In[374]:=
```
(*Evaluate at integration limits*)
g[y′_] = -(((a0^2 + a1 * (x - b0 * y) - a0 * (a1 + x - 2 * b0 * y + b1 * y) -
          (b0 - b1) * (x * y - b0 * (y^2 + z^2))) * (μ0 - μ1) *
        ArcTan[(a0 * y - x * y - (a0 - x + b0 * y) * y′ + b0 * (y^2 + z^2)) /
    (z * Sqrt[a0^2 - 2 * a0 * x + x^2 - (y^2 + z^2) * β^2 +
              y′^2 * (b0^2 - β^2) + 2 * y′ * (a0 * b0 - b0 * x + y * β^2)]])]) /
     (2 * Pi * (a0^2 + a1^2 - 2 * a1 * (b0 - b1) * y - 2 * a0 * (a1 - (b0 - b1) * y) +
```

```
            (b0 - b1) ^2 * (y^2 + z^2)))) -
  ((a1^2 - a1 * (x + (b0 - 2 * b1) * y) - a0 * (a1 - x + b1 * y) +
        (b0 - b1) * (x * y - b1 * (y^2 + z^2))) * (μ0 - μ1) *
   ArcTan[(a1 * y - x * y - (a1 - x + b1 * y) * y′ + b1 * (y^2 + z^2)) /
        (z * Sqrt[a1^2 - 2 * a1 * x + x^2 - (y^2 + z^2) * β^2 +
              y′^2 * (b1^2 - β^2) + 2 * y′ * (a1 * b1 - b1 * x + y * β^2)])]) /
   (2 * Pi * (a0^2 + a1^2 - 2 * a1 * (b0 - b1) * y - 2 * a0 * (a1 - (b0 - b1) * y) +
        (b0 - b1) ^2 * (y^2 + z^2))) +
  ((a1 * b0 - a0 * b1 - (b0 - b1) * x) * z * (μ0 - μ1) *
      ArcTanh[((a0 - x) * (a1 * b0 - a0 * b1 - (b0 - b1) * x) +
           b0 * (a1 * b0 - a0 * b1 - (b0 - b1) * x) * y′) /
     ((a1 * b0 - a0 * b1 - (b0 - b1) * x) * Sqrt[a0^2 - 2 * a0 * x + x^2 - (y^2 + z^2) * β^2 +
              y′^2 * (b0^2 - β^2) + 2 * y′ * (a0 * b0 - b0 * x + y * β^2)])]) /
   (2 * Pi * (a0^2 + a1^2 - 2 * a1 * (b0 - b1) * y - 2 * a0 * (a1 - (b0 - b1) * y) +
        (b0 - b1) ^2 * (y^2 + z^2))) -
  ((a1 * b0 - a0 * b1 - (b0 - b1) * x) * z * (μ0 - μ1) *
      ArcTanh[(a1 - x + b1 * y′) / Sqrt[a1^2 - 2 * a1 * x + x^2 - (y^2 + z^2) * β^2 +
           y′^2 * (b1^2 - β^2) + 2 * y′ * (a1 * b1 - b1 * x + y * β^2)]]) /
   (2 * Pi * (a0^2 + a1^2 - 2 * a1 * (b0 - b1) * y - 2 * a0 * (a1 - (b0 - b1) * y) +
        (b0 - b1) ^2 * (y^2 + z^2))) +
  (z * Sqrt[a1^2 * (b0^2 - β^2) + a0^2 * (b1^2 - β^2) +
        2 * a0 * (b0 - b1) * (b1 * x - y * β^2) + (b0 - b1) ^2 * (x^2 - (y^2 + z^2) * β^2) -
        2 * a1 * (a0 * (b0 * b1 - β^2) + (b0 - b1) * (b0 * x - y * β^2))] * (μ0 - μ1) *
   ArcTanh[((a0 - a1) * (a0 * b0 - b0 * x + y * β^2) -
           (b0 - b1) * (a0^2 - 2 * a0 * x + x^2 - (y^2 + z^2) * β^2) +
           y′ * ((a0 - a1) * (b0^2 - β^2) - (b0 - b1) * (a0 * b0 - b0 * x + y * β^2))) /
     (Sqrt[a0^2 - 2 * a0 * x + x^2 - (y^2 + z^2) * β^2 + y′^2 * (b0^2 - β^2) +
              2 * y′ * (a0 * b0 - b0 * x + y * β^2)] * Sqrt[a1^2 * (b0^2 - β^2) +
              a0^2 * (b1^2 - β^2) + 2 * a0 * (b0 - b1) * (b1 * x - y * β^2) +
      (b0 - b1) ^2 * (x^2 - (y^2 + z^2) * β^2) - 2 * a1 *
               (a0 * (b0 * b1 - β^2) + (b0 - b1) * (b0 * x - y * β^2))])]) /
    (2 * Pi * (a0^2 + a1^2 - 2 * a1 * (b0 - b1) * y - 2 * a0 * (a1 - (b0 - b1) * y) +
        (b0 - b1) ^2 * (y^2 + z^2))) -
  (z * Sqrt[a1^2 * (b0^2 - β^2) + a0^2 * (b1^2 - β^2) +
        2 * a0 * (b0 - b1) * (b1 * x - y * β^2) + (b0 - b1) ^2 * (x^2 - (y^2 + z^2) * β^2) -
        2 * a1 * (a0 * (b0 * b1 - β^2) + (b0 - b1) * (b0 * x - y * β^2))] * (μ0 - μ1) *
   ArcTanh[((a0 - a1) * (a1 * b1 - b1 * x + y * β^2) -
           (b0 - b1) * (a1^2 - 2 * a1 * x + x^2 - (y^2 + z^2) * β^2) +
           y′ * ((a0 - a1) * (b1^2 - β^2) - (b0 - b1) * (a1 * b1 - b1 * x + y * β^2))) /
     (Sqrt[a1^2 - 2 * a1 * x + x^2 - (y^2 + z^2) * β^2 + y′^2 * (b1^2 - β^2) +
              2 * y′ * (a1 * b1 - b1 * x + y * β^2)] * Sqrt[a1^2 * (b0^2 - β^2) +
              a0^2 * (b1^2 - β^2) + 2 * a0 * (b0 - b1) * (b1 * x - y * β^2) +
      (b0 - b1) ^2 * (x^2 - (y^2 + z^2) * β^2) - 2 * a1 *
```

```
        (a0 * (b0 * b1 - β^2) + (b0 - b1) * (b0 * x - y * β^2))]])]) /
    (2 * Pi * (a0^2 + a1^2 - 2 * a1 * (b0 - b1) * y - 2 * a0 * (a1 - (b0 - b1) * y) +
        (b0 - b1)^2 * (y^2 + z^2)))
```

Out[374]=

$$-\Bigg(\bigg(\Big(\big(a0^2 + a1\,(x - b0\,y) - a0\,(a1 + x - 2\,b0\,y + b1\,y) - (b0 - b1)\,\big(x\,y - b0\,(y^2 + z^2)\big)\big)\,(\mu 0 - \mu 1)$$

$$\operatorname{ArcTan}\Big[\frac{a0\,y - x\,y - (a0 - x + b0\,y)\,y' + b0\,(y^2 + z^2)}{z\sqrt{a0^2 - 2\,a0\,x + x^2 - (y^2 + z^2)\,\beta^2 + y'^2\,(b0^2 - \beta^2) + 2\,y'\,(a0\,b0 - b0\,x + y\,\beta^2)}}\Big]\Big)\Big/$$

$$\big(2\,\pi\,\big(a0^2 + a1^2 - 2\,a1\,(b0 - b1)\,y - 2\,a0\,(a1 - (b0 - b1)\,y) + (b0 - b1)^2\,(y^2 + z^2)\big)\big)\bigg) -$$

$$\bigg(\Big(\big(a1^2 - a1\,(x + (b0 - 2\,b1)\,y) - a0\,(a1 - x + b1\,y) + (b0 - b1)\,\big(x\,y - b1\,(y^2 + z^2)\big)\big)\,(\mu 0 - \mu 1)$$

$$\operatorname{ArcTan}\Big[\frac{a1\,y - x\,y - (a1 - x + b1\,y)\,y' + b1\,(y^2 + z^2)}{z\sqrt{a1^2 - 2\,a1\,x + x^2 - (y^2 + z^2)\,\beta^2 + y'^2\,(b1^2 - \beta^2) + 2\,y'\,(a1\,b1 - b1\,x + y\,\beta^2)}}\Big]\Big)\Big/$$

$$\big(2\,\pi\,\big(a0^2 + a1^2 - 2\,a1\,(b0 - b1)\,y - 2\,a0\,(a1 - (b0 - b1)\,y) + (b0 - b1)^2\,(y^2 + z^2)\big)\big) +$$

$$\Big((a1\,b0 - a0\,b1 - (b0 - b1)\,x)\,z\,(\mu 0 - \mu 1)\,\operatorname{ArcTanh}\Big[\big((a0 - x)\,(a1\,b0 - a0\,b1 - (b0 - b1)\,x) +$$

$$b0\,(a1\,b0 - a0\,b1 - (b0 - b1)\,x)\,y'\big)\Big/\Big((a1\,b0 - a0\,b1 - (b0 - b1)\,x)$$

$$\sqrt{a0^2 - 2\,a0\,x + x^2 - (y^2 + z^2)\,\beta^2 + y'^2\,(b0^2 - \beta^2) + 2\,y'\,(a0\,b0 - b0\,x + y\,\beta^2)}\Big)\Big]\Big)\Big/$$

$$\big(2\,\pi\,\big(a0^2 + a1^2 - 2\,a1\,(b0 - b1)\,y - 2\,a0\,(a1 - (b0 - b1)\,y) + (b0 - b1)^2\,(y^2 + z^2)\big)\big) -$$

$$\Big((a1\,b0 - a0\,b1 - (b0 - b1)\,x)\,z\,(\mu 0 - \mu 1)$$

$$\operatorname{ArcTanh}\Big[\frac{a1 - x + b1\,y'}{\sqrt{a1^2 - 2\,a1\,x + x^2 - (y^2 + z^2)\,\beta^2 + y'^2\,(b1^2 - \beta^2) + 2\,y'\,(a1\,b1 - b1\,x + y\,\beta^2)}}\Big]\Big)\Big/$$

$$\big(2\,\pi\,\big(a0^2 + a1^2 - 2\,a1\,(b0 - b1)\,y - 2\,a0\,(a1 - (b0 - b1)\,y) + (b0 - b1)^2\,(y^2 + z^2)\big)\big) +$$

$$\Big(z\sqrt{\big(a1^2\,(b0^2 - \beta^2) + a0^2\,(b1^2 - \beta^2) + 2\,a0\,(b0 - b1)\,(b1\,x - y\,\beta^2) +}$$

$$\overline{(b0 - b1)^2\,\big(x^2 - (y^2 + z^2)\,\beta^2\big) - 2\,a1\,\big(a0\,(b0\,b1 - \beta^2) + (b0 - b1)\,(b0\,x - y\,\beta^2)\big)\big)}\,(\mu 0 - \mu 1)$$

$$\operatorname{ArcTanh}\Big[\big((a0 - a1)\,(a0\,b0 - b0\,x + y\,\beta^2) - (b0 - b1)\,\big(a0^2 - 2\,a0\,x + x^2 - (y^2 + z^2)\,\beta^2\big) +$$

$$y'\,\big((a0 - a1)\,(b0^2 - \beta^2) - (b0 - b1)\,(a0\,b0 - b0\,x + y\,\beta^2)\big)\big)\Big/$$

$$\Big(\sqrt{a0^2 - 2\,a0\,x + x^2 - (y^2 + z^2)\,\beta^2 + y'^2\,(b0^2 - \beta^2) + 2\,y'\,(a0\,b0 - b0\,x + y\,\beta^2)}$$

$$\sqrt{\big(a1^2\,(b0^2 - \beta^2) + a0^2\,(b1^2 - \beta^2) + 2\,a0\,(b0 - b1)\,(b1\,x - y\,\beta^2) +}$$

$$\overline{(b0 - b1)^2\,\big(x^2 - (y^2 + z^2)\,\beta^2\big) - 2\,a1\,\big(a0\,(b0\,b1 - \beta^2) + (b0 - b1)\,(b0\,x - y\,\beta^2)\big)\big)}\Big)\Big]\Big)\Big/$$

$$\big(2\,\pi\,\big(a0^2 + a1^2 - 2\,a1\,(b0 - b1)\,y - 2\,a0\,(a1 - (b0 - b1)\,y) + (b0 - b1)^2\,(y^2 + z^2)\big)\big) -$$

$$\Big(z \sqrt{\left(a1^2 \left(b0^2-\beta^2\right)+a0^2 \left(b1^2-\beta^2\right)+2\, a0\, (b0-b1) \left(b1\, x-y\, \beta^2\right)+ (b0-b1)^2 \left(x^2-\left(y^2+z^2\right) \beta^2\right)-2\, a1 \left(a0 \left(b0\, b1-\beta^2\right)+(b0-b1) \left(b0\, x-y\, \beta^2\right)\right)\right)} (\mu 0-\mu 1)$$

$$\mathrm{ArcTanh}\Big[\Big((a0-a1) \left(a1\, b1-b1\, x+y\, \beta^2\right)-(b0-b1) \left(a1^2-2\, a1\, x+x^2-\left(y^2+z^2\right) \beta^2\right)+ y\prime \left((a0-a1) \left(b1^2-\beta^2\right)-(b0-b1) \left(a1\, b1-b1\, x+y\, \beta^2\right)\right)\Big)\Big/$$

$$\Big(\sqrt{a1^2-2\, a1\, x+x^2-\left(y^2+z^2\right) \beta^2+y\prime^2 \left(b1^2-\beta^2\right)+2\, y\prime \left(a1\, b1-b1\, x+y\, \beta^2\right)} \sqrt{\left(a1^2 \left(b0^2-\beta^2\right)+a0^2 \left(b1^2-\beta^2\right)+2\, a0\, (b0-b1) \left(b1\, x-y\, \beta^2\right)+ (b0-b1)^2 \left(x^2-\left(y^2+z^2\right) \beta^2\right)-2\, a1 \left(a0 \left(b0\, b1-\beta^2\right)+(b0-b1) \left(b0\, x-y\, \beta^2\right)\right)\right)}\Big)\Big]\Big)\Big/$$

$$\left(2\, \pi \left(a0^2+a1^2-2\, a1\, (b0-b1)\, y-2\, a0\, (a1-(b0-b1)\, y)+(b0-b1)^2 \left(y^2+z^2\right)\right)\right)$$

(*Tapered Supersonic Doublet Panel Perturbation Velocity Potential*)

In[375]:= **ϕ[x_, y_, z_] = Simplify[g[y1] - g[y0]]**

Out[375]=

$$\Big((\mu 0-\mu 1) \Big(\left(a0^2+a1\, (x-b0\, y)-a0\, (a1+x-2\, b0\, y+b1\, y)+(b0-b1) \left(-x\, y+b0 \left(y^2+z^2\right)\right)\right)$$

$$\mathrm{ArcTan}\left[\frac{a0\, (y-y0)+x\, (-y+y0)+b0 \left(y^2-y\, y0+z^2\right)}{z \sqrt{a0^2-2\, a0\, x+x^2-\left(y^2+z^2\right) \beta^2+y0^2 \left(b0^2-\beta^2\right)+2\, y0 \left(a0\, b0-b0\, x+y\, \beta^2\right)}}\right]-$$

$$\left(a0^2+a1\, (x-b0\, y)-a0\, (a1+x-2\, b0\, y+b1\, y)+(b0-b1) \left(-x\, y+b0 \left(y^2+z^2\right)\right)\right)$$

$$\mathrm{ArcTan}\left[\frac{a0\, (y-y1)+x\, (-y+y1)+b0 \left(y^2-y\, y1+z^2\right)}{z \sqrt{a0^2-2\, a0\, x+x^2-\left(y^2+z^2\right) \beta^2+y1^2 \left(b0^2-\beta^2\right)+2\, y1 \left(a0\, b0-b0\, x+y\, \beta^2\right)}}\right]+$$

$$\left(a1^2-a1\, (x+(b0-2\, b1)\, y)-a0\, (a1-x+b1\, y)+(b0-b1) \left(x\, y-b1 \left(y^2+z^2\right)\right)\right)$$

$$\mathrm{ArcTan}\left[\frac{a1\, (y-y0)+x\, (-y+y0)+b1 \left(y^2-y\, y0+z^2\right)}{z \sqrt{a1^2-2\, a1\, x+x^2-\left(y^2+z^2\right) \beta^2+y0^2 \left(b1^2-\beta^2\right)+2\, y0 \left(a1\, b1-b1\, x+y\, \beta^2\right)}}\right]-$$

$$\left(a1^2-a1\, (x+(b0-2\, b1)\, y)-a0\, (a1-x+b1\, y)+(b0-b1) \left(x\, y-b1 \left(y^2+z^2\right)\right)\right)$$

$$\mathrm{ArcTan}\left[\frac{a1\, (y-y1)+x\, (-y+y1)+b1 \left(y^2-y\, y1+z^2\right)}{z \sqrt{a1^2-2\, a1\, x+x^2-\left(y^2+z^2\right) \beta^2+y1^2 \left(b1^2-\beta^2\right)+2\, y1 \left(a1\, b1-b1\, x+y\, \beta^2\right)}}\right]-$$

$$(a1\, b0-a0\, b1+(-b0+b1)\, x)\, z\, \mathrm{ArcTanh}\left[\frac{a0-x+b0\, y0}{\sqrt{a0^2-2\, a0\, x+x^2+2\, a0\, b0\, y0-2\, b0\, x\, y0+b0^2\, y0^2-y^2\, \beta^2+2\, y\, y0\, \beta^2-y0^2\, \beta^2-z^2\, \beta^2}}\right]+$$

$$(a1\, b0-a0\, b1+(-b0+b1)\, x)\, z\, \mathrm{ArcTanh}\left[\frac{a0-x+b0\, y1}{\sqrt{a0^2-2\, a0\, x+x^2+2\, a0\, b0\, y1-2\, b0\, x\, y1+b0^2\, y1^2-y^2\, \beta^2+2\, y\, y1\, \beta^2-y1^2\, \beta^2-z^2\, \beta^2}}\right]+$$

$$(a1\, b0-a0\, b1+(-b0+b1)\, x)\, z\, \mathrm{ArcTanh}\left[\frac{a1-x+b1\, y0}{\sqrt{a1^2-2\, a1\, x+x^2-\left(y^2+z^2\right) \beta^2+y0^2 \left(b1^2-\beta^2\right)+2\, y0 \left(a1\, b1-b1\, x+y\, \beta^2\right)}}\right]-$$

$$(a1\, b0-a0\, b1+(-b0+b1)\, x)\, z\, \mathrm{ArcTanh}\left[\frac{a1-x+b1\, y1}{\sqrt{a1^2-2\, a1\, x+x^2-\left(y^2+z^2\right) \beta^2+y1^2 \left(b1^2-\beta^2\right)+2\, y1 \left(a1\, b1-b1\, x+y\, \beta^2\right)}}\right]-$$

```
z √(a1^2 (b0^2 - β^2) + a0^2 (b1^2 - β^2) + 2 a0 (b0 - b1) (b1 x - y β^2) +
      (b0 - b1)^2 (x^2 - (y^2 + z^2) β^2) - 2 a1 (a0 (b0 b1 - β^2) + (b0 - b1) (b0 x - y β^2)))
  ArcTanh[(a0^2 b1 + a0 (-a1 b0 + b0 x - 2 b1 x + b0 b1 y0 + y β^2 - y0 β^2) +
       a1 (b0 x - b0^2 y0 + (-y + y0) β^2) + (b0 - b1) (-x^2 + b0 x y0 + (y^2 - y y0 + z^2) β^2)) /
    (√(a0^2 - 2 a0 x + x^2 - (y^2 + z^2) β^2 + y0^2 (b0^2 - β^2) + 2 y0 (a0 b0 - b0 x + y β^2))
      √(a1^2 (b0^2 - β^2) + a0^2 (b1^2 - β^2) + 2 a0 (b0 - b1) (b1 x - y β^2) + (b0 - b1)^2
         (x^2 - (y^2 + z^2) β^2) - 2 a1 (a0 (b0 b1 - β^2) + (b0 - b1) (b0 x - y β^2))))] +
z √(a1^2 (b0^2 - β^2) + a0^2 (b1^2 - β^2) + 2 a0 (b0 - b1) (b1 x - y β^2) +
      (b0 - b1)^2 (x^2 - (y^2 + z^2) β^2) - 2 a1 (a0 (b0 b1 - β^2) + (b0 - b1) (b0 x - y β^2)))
  ArcTanh[(-a1^2 b0 + a0 (-b1 x + b1^2 y0 + (y - y0) β^2) + a1 (a0 b1 + 2 b0 x - b1 x -
          b0 b1 y0 - y β^2 + y0 β^2) + (b0 - b1) (-x^2 + b1 x y0 + (y^2 - y y0 + z^2) β^2)) /
    (√(a1^2 - 2 a1 x + x^2 - (y^2 + z^2) β^2 + y0^2 (b1^2 - β^2) + 2 y0 (a1 b1 - b1 x + y β^2))
      √(a1^2 (b0^2 - β^2) + a0^2 (b1^2 - β^2) + 2 a0 (b0 - b1) (b1 x - y β^2) + (b0 - b1)^2
         (x^2 - (y^2 + z^2) β^2) - 2 a1 (a0 (b0 b1 - β^2) + (b0 - b1) (b0 x - y β^2))))] +
z √(a1^2 (b0^2 - β^2) + a0^2 (b1^2 - β^2) + 2 a0 (b0 - b1) (b1 x - y β^2) +
      (b0 - b1)^2 (x^2 - (y^2 + z^2) β^2) - 2 a1 (a0 (b0 b1 - β^2) + (b0 - b1) (b0 x - y β^2)))
  ArcTanh[(a0^2 b1 + a0 (-a1 b0 + b0 x - 2 b1 x + b0 b1 y1 + y β^2 - y1 β^2) +
       a1 (b0 x - b0^2 y1 + (-y + y1) β^2) + (b0 - b1) (-x^2 + b0 x y1 + (y^2 - y y1 + z^2) β^2)) /
    (√(a0^2 - 2 a0 x + x^2 - (y^2 + z^2) β^2 + y1^2 (b0^2 - β^2) + 2 y1 (a0 b0 - b0 x + y β^2))
      √(a1^2 (b0^2 - β^2) + a0^2 (b1^2 - β^2) + 2 a0 (b0 - b1) (b1 x - y β^2) + (b0 - b1)^2
         (x^2 - (y^2 + z^2) β^2) - 2 a1 (a0 (b0 b1 - β^2) + (b0 - b1) (b0 x - y β^2))))] -
z √(a1^2 (b0^2 - β^2) + a0^2 (b1^2 - β^2) + 2 a0 (b0 - b1) (b1 x - y β^2) +
      (b0 - b1)^2 (x^2 - (y^2 + z^2) β^2) - 2 a1 (a0 (b0 b1 - β^2) + (b0 - b1) (b0 x - y β^2)))
  ArcTanh[(-a1^2 b0 + a0 (-b1 x + b1^2 y1 + (y - y1) β^2) + a1 (a0 b1 + 2 b0 x - b1 x -
          b0 b1 y1 - y β^2 + y1 β^2) + (b0 - b1) (-x^2 + b1 x y1 + (y^2 - y y1 + z^2) β^2)) /
    (√(a1^2 - 2 a1 x + x^2 - (y^2 + z^2) β^2 + y1^2 (b1^2 - β^2) + 2 y1 (a1 b1 - b1 x + y β^2))
      √(a1^2 (b0^2 - β^2) + a0^2 (b1^2 - β^2) + 2 a0 (b0 - b1) (b1 x - y β^2) + (b0 - b1)^2
         (x^2 - (y^2 + z^2) β^2) - 2 a1 (a0 (b0 b1 - β^2) + (b0 - b1) (b0 x - y β^2))))])) /
(2 π (a0^2 + a1^2 + 2 a1 (-b0 + b1) y - 2 a0 (a1 + (-b0 + b1) y) + (b0 - b1)^2 (y^2 + z^2)))
```

(*Tapered Supersonic Doublet Panel Perturbation Velocity u Component*)

```
In[376]:= u = D[ϕ[x, y, z], x]
Out[376]= ((μ0 - μ1)
    (-((((a1 b0 - a0 b1 + (-b0 + b1) x) z (-(((-2 a0 + 2 x - 2 b0 y0) (a0 - x + b0 y0)) / (2 (a0^2 -
```

$$2\,a0\,x+x^2+2\,a0\,b0\,y0-2\,b0\,x\,y0+b0^2\,y0^2-y^2\,\beta^2+2\,y\,y0\,\beta^2-$$
$$y0^2\,\beta^2-z^2\,\beta^2)^{3/2})) - 1/(\surd(a0^2-2\,a0\,x+x^2+2\,a0\,b0\,y0-$$
$$2\,b0\,x\,y0+b0^2\,y0^2-y^2\,\beta^2+2\,y\,y0\,\beta^2-y0^2\,\beta^2-z^2\,\beta^2))))\,/$$
$$(1-(a0-x+b0\,y0)^2/(a0^2-2\,a0\,x+x^2+2\,a0\,b0\,y0-2\,b0\,x\,y0+b0^2\,y0^2-$$
$$y^2\,\beta^2+2\,y\,y0\,\beta^2-y0^2\,\beta^2-z^2\,\beta^2))) +$$
$$((a1\,b0-a0\,b1+(-b0+b1)\,x)\,z\,(-(((-2\,a0+2\,x-2\,b0\,y1)\,(a0-x+b0\,y1))\,/$$
$$(2\,(a0^2-2\,a0\,x+x^2+2\,a0\,b0\,y1-2\,b0\,x\,y1+b0^2\,y1^2-y^2\,\beta^2+$$
$$2\,y\,y1\,\beta^2-y1^2\,\beta^2-z^2\,\beta^2)^{3/2})) - 1/(\surd(a0^2-2\,a0\,x+x^2+2\,a0$$
$$b0\,y1-2\,b0\,x\,y1+b0^2\,y1^2-y^2\,\beta^2+2\,y\,y1\,\beta^2-y1^2\,\beta^2-z^2\,\beta^2))))\,/$$
$$(1-(a0-x+b0\,y1)^2/(a0^2-2\,a0\,x+x^2+2\,a0\,b0\,y1-2\,b0\,x\,y1+b0^2\,y1^2-$$
$$y^2\,\beta^2+2\,y\,y1\,\beta^2-y1^2\,\beta^2-z^2\,\beta^2)) +$$
$$((a0^2+a1\,(x-b0\,y)-a0\,(a1+x-2\,b0\,y+b1\,y)+(b0-b1)\,(-x\,y+b0\,(y^2+z^2)))$$
$$(-(((-2\,a0+2\,x-2\,b0\,y0)\,(a0\,(y-y0)+x\,(-y+y0)+b0\,(y^2-y\,y0+z^2)))\,/$$
$$(2\,z\,(a0^2-2\,a0\,x+x^2-(y^2+z^2)\,\beta^2+y0^2\,(b0^2-\beta^2)+$$
$$2\,y0\,(a0\,b0-b0\,x+y\,\beta^2))^{3/2})) + (-y+y0)\,/$$
$$(z\,\surd(a0^2-2\,a0\,x+x^2-(y^2+z^2)\,\beta^2+y0^2\,(b0^2-\beta^2)+2\,y0\,(a0\,b0-b0\,x+y\,\beta^2)))))\,/$$
$$(1+(a0\,(y-y0)+x\,(-y+y0)+b0\,(y^2-y\,y0+z^2))^2\,/$$
$$(z^2\,(a0^2-2\,a0\,x+x^2-(y^2+z^2)\,\beta^2+y0^2\,(b0^2-\beta^2)+2\,y0\,(a0\,b0-b0\,x+y\,\beta^2)))) -$$
$$((a0^2+a1\,(x-b0\,y)-a0\,(a1+x-2\,b0\,y+b1\,y)+(b0-b1)\,(-x\,y+b0\,(y^2+z^2)))$$
$$(-(((-2\,a0+2\,x-2\,b0\,y1)\,(a0\,(y-y1)+x\,(-y+y1)+b0\,(y^2-y\,y1+z^2)))\,/$$
$$(2\,z\,(a0^2-2\,a0\,x+x^2-(y^2+z^2)\,\beta^2+y1^2\,(b0^2-\beta^2)+$$
$$2\,y1\,(a0\,b0-b0\,x+y\,\beta^2))^{3/2})) + (-y+y1)\,/$$
$$(z\,\surd(a0^2-2\,a0\,x+x^2-(y^2+z^2)\,\beta^2+y1^2\,(b0^2-\beta^2)+2\,y1\,(a0\,b0-b0\,x+y\,\beta^2)))))\,/$$
$$(1+(a0\,(y-y1)+x\,(-y+y1)+b0\,(y^2-y\,y1+z^2))^2\,/$$
$$(z^2\,(a0^2-2\,a0\,x+x^2-(y^2+z^2)\,\beta^2+y1^2\,(b0^2-\beta^2)+2\,y1\,(a0\,b0-b0\,x+y\,\beta^2)))) +$$
$$((a1\,b0-a0\,b1+(-b0+b1)\,x)\,z\,(-(((-2\,a1+2\,x-2\,b1\,y0)\,(a1-x+b1\,y0))/(2\,(a1^2-$$
$$2\,a1\,x+x^2-(y^2+z^2)\,\beta^2+y0^2\,(b1^2-\beta^2)+2\,y0\,(a1\,b1-b1\,x+y\,\beta^2))^{3/2})) -$$
$$1/(\surd(a1^2-2\,a1\,x+x^2-(y^2+z^2)\,\beta^2+y0^2\,(b1^2-\beta^2)+2\,y0\,(a1\,b1-b1\,x+y\,\beta^2)))))\,/$$
$$(1-(a1-x+b1\,y0)^2/(a1^2-2\,a1\,x+x^2-(y^2+z^2)\,\beta^2+y0^2\,(b1^2-\beta^2)+$$
$$2\,y0\,(a1\,b1-b1\,x+y\,\beta^2))) +$$
$$((a1^2-a1\,(x+(b0-2\,b1)\,y)-a0\,(a1-x+b1\,y)+(b0-b1)\,(x\,y-b1\,(y^2+z^2)))$$
$$(-(((-2\,a1+2\,x-2\,b1\,y0)\,(a1\,(y-y0)+x\,(-y+y0)+b1\,(y^2-y\,y0+z^2)))\,/$$
$$(2\,z\,(a1^2-2\,a1\,x+x^2-(y^2+z^2)\,\beta^2+y0^2\,(b1^2-\beta^2)+$$
$$2\,y0\,(a1\,b1-b1\,x+y\,\beta^2))^{3/2})) + (-y+y0)\,/$$
$$(z\,\surd(a1^2-2\,a1\,x+x^2-(y^2+z^2)\,\beta^2+y0^2\,(b1^2-\beta^2)+2\,y0\,(a1\,b1-b1\,x+y\,\beta^2)))))\,/$$
$$(1+(a1\,(y-y0)+x\,(-y+y0)+b1\,(y^2-y\,y0+z^2))^2\,/$$
$$(z^2\,(a1^2-2\,a1\,x+x^2-(y^2+z^2)\,\beta^2+y0^2\,(b1^2-\beta^2)+2\,y0\,(a1\,b1-b1\,x+y\,\beta^2)))) -$$

$$((a1\,b0 - a0\,b1 + (-b0 + b1)\,x)\,z\,\Big(-\Big(((-2\,a1 + 2\,x - 2\,b1\,y1)\,(a1 - x + b1\,y1)) \Big/ \Big(2\,\big(a1^2 -$$
$$2\,a1\,x + x^2 - (y^2 + z^2)\,\beta^2 + y1^2\,(b1^2 - \beta^2) + 2\,y1\,(a1\,b1 - b1\,x + y\,\beta^2)\big)^{3/2}\Big)\Big) -$$
$$1 \Big/ \Big(\surd\big(a1^2 - 2\,a1\,x + x^2 - (y^2 + z^2)\,\beta^2 + y1^2\,(b1^2 - \beta^2) + 2\,y1\,(a1\,b1 - b1\,x + y\,\beta^2)\big)\Big)\Big)\Big) \Big/$$
$$\Big(1 - (a1 - x + b1\,y1)^2 \Big/ \big(a1^2 - 2\,a1\,x + x^2 - (y^2 + z^2)\,\beta^2 + y1^2\,(b1^2 - \beta^2) +$$
$$2\,y1\,(a1\,b1 - b1\,x + y\,\beta^2)\big)\Big) -$$
$$\Big(\big(a1^2 - a1\,(x + (b0 - 2\,b1)\,y) - a0\,(a1 - x + b1\,y) + (b0 - b1)\,(x\,y - b1\,(y^2 + z^2))\big)$$
$$\Big(-\Big(((-2\,a1 + 2\,x - 2\,b1\,y1)\,(a1\,(y - y1) + x\,(-y + y1) + b1\,(y^2 - y\,y1 + z^2))) \Big/$$
$$\Big(2\,z\,\big(a1^2 - 2\,a1\,x + x^2 - (y^2 + z^2)\,\beta^2 + y1^2\,(b1^2 - \beta^2) +$$
$$2\,y1\,(a1\,b1 - b1\,x + y\,\beta^2)\big)^{3/2}\Big)\Big) + (-y + y1) \Big/$$
$$\Big(z\,\surd\big(a1^2 - 2\,a1\,x + x^2 - (y^2 + z^2)\,\beta^2 + y1^2\,(b1^2 - \beta^2) + 2\,y1\,(a1\,b1 - b1\,x + y\,\beta^2)\big)\Big)\Big)\Big) \Big/$$
$$\Big(1 + \big(a1\,(y - y1) + x\,(-y + y1) + b1\,(y^2 - y\,y1 + z^2)\big)^2 \Big/$$
$$\Big(z^2\,\big(a1^2 - 2\,a1\,x + x^2 - (y^2 + z^2)\,\beta^2 + y1^2\,(b1^2 - \beta^2) + 2\,y1\,(a1\,b1 - b1\,x + y\,\beta^2)\big)\Big)\Big) -$$
$$\Big(z\,\surd\big(a1^2\,(b0^2 - \beta^2) + a0^2\,(b1^2 - \beta^2) + 2\,a0\,(b0 - b1)\,(b1\,x - y\,\beta^2) +$$
$$(b0 - b1)^2\,(x^2 - (y^2 + z^2)\,\beta^2) - 2\,a1\,(a0\,(b0\,b1 - \beta^2) + (b0 - b1)\,(b0\,x - y\,\beta^2))\big)$$
$$\Big(-\Big(\Big(\big(-2\,a1\,b0\,(b0 - b1) + 2\,a0\,(b0 - b1)\,b1 + 2\,(b0 - b1)^2\,x\big)\,\big(a0^2\,b1 +$$
$$a0\,(-a1\,b0 + b0\,x - 2\,b1\,x + b0\,b1\,y0 + y\,\beta^2 - y0\,\beta^2) + a1\,\big(b0\,x - b0^2\,y0 +$$
$$(-y + y0)\,\beta^2\big) + (b0 - b1)\,\big(-x^2 + b0\,x\,y0 + (y^2 - y\,y0 + z^2)\,\beta^2\big)\big)\Big) \Big/$$
$$\Big(2\,\surd\big(a0^2 - 2\,a0\,x + x^2 - (y^2 + z^2)\,\beta^2 + y0^2\,(b0^2 - \beta^2) + 2\,y0\,(a0\,b0 - b0\,x + y\,\beta^2)\big)$$
$$\big(a1^2\,(b0^2 - \beta^2) + a0^2\,(b1^2 - \beta^2) + 2\,a0\,(b0 - b1)\,(b1\,x - y\,\beta^2) + (b0 - b1)^2\,\big(x^2 -$$
$$(y^2 + z^2)\,\beta^2\big) - 2\,a1\,(a0\,(b0\,b1 - \beta^2) + (b0 - b1)\,(b0\,x - y\,\beta^2))\big)^{3/2}\Big)\Big) +$$
$$(a1\,b0 + a0\,(b0 - 2\,b1) + (b0 - b1)\,(-2\,x + b0\,y0)) \Big/ \Big(\surd\big(a0^2 - 2\,a0\,x + x^2 -$$
$$(y^2 + z^2)\,\beta^2 + y0^2\,(b0^2 - \beta^2) + 2\,y0\,(a0\,b0 - b0\,x + y\,\beta^2)\big)\,\surd\big(a1^2\,(b0^2 - \beta^2) +$$
$$a0^2\,(b1^2 - \beta^2) + 2\,a0\,(b0 - b1)\,(b1\,x - y\,\beta^2) + (b0 - b1)^2\,(x^2 - (y^2 + z^2)\,\beta^2) -$$
$$2\,a1\,(a0\,(b0\,b1 - \beta^2) + (b0 - b1)\,(b0\,x - y\,\beta^2))\big)\Big) - \Big((-2\,a0 + 2\,x - 2\,b0\,y0)$$
$$\big(a0^2\,b1 + a0\,(-a1\,b0 + b0\,x - 2\,b1\,x + b0\,b1\,y0 + y\,\beta^2 - y0\,\beta^2) + a1\,\big(b0\,x -$$
$$b0^2\,y0 + (-y + y0)\,\beta^2\big) + (b0 - b1)\,\big(-x^2 + b0\,x\,y0 + (y^2 - y\,y0 + z^2)\,\beta^2\big)\big)\Big) \Big/$$
$$\Big(2\,\big(a0^2 - 2\,a0\,x + x^2 - (y^2 + z^2)\,\beta^2 + y0^2\,(b0^2 - \beta^2) + 2\,y0\,(a0\,b0 - b0\,x + y\,\beta^2)\big)^{3/2}$$
$$\surd\big(a1^2\,(b0^2 - \beta^2) + a0^2\,(b1^2 - \beta^2) + 2\,a0\,(b0 - b1)\,(b1\,x - y\,\beta^2) + (b0 - b1)^2$$
$$(x^2 - (y^2 + z^2)\,\beta^2) - 2\,a1\,(a0\,(b0\,b1 - \beta^2) + (b0 - b1)\,(b0\,x - y\,\beta^2))\big)\Big)\Big)\Big) \Big/$$
$$\Big(1 - \big(a0^2\,b1 + a0\,(-a1\,b0 + b0\,x - 2\,b1\,x + b0\,b1\,y0 + y\,\beta^2 - y0\,\beta^2) + a1$$
$$\big(b0\,x - b0^2\,y0 + (-y + y0)\,\beta^2\big) + (b0 - b1)\,\big(-x^2 + b0\,x\,y0 + (y^2 - y\,y0 + z^2)\,\beta^2\big)\big)^2 \Big/$$
$$\Big(\big(a0^2 - 2\,a0\,x + x^2 - (y^2 + z^2)\,\beta^2 + y0^2\,(b0^2 - \beta^2) + 2\,y0\,(a0\,b0 - b0\,x + y\,\beta^2)\big)$$
$$\big(a1^2\,(b0^2 - \beta^2) + a0^2\,(b1^2 - \beta^2) + 2\,a0\,(b0 - b1)\,(b1\,x - y\,\beta^2) + (b0 - b1)^2$$
$$(x^2 - (y^2 + z^2)\,\beta^2) - 2\,a1\,(a0\,(b0\,b1 - \beta^2) + (b0 - b1)\,(b0\,x - y\,\beta^2))\big)\Big)\Big) +$$
$$\Big(z\,\surd\big(a1^2\,(b0^2 - \beta^2) + a0^2\,(b1^2 - \beta^2) + 2\,a0\,(b0 - b1)\,(b1\,x - y\,\beta^2) +$$
$$(b0 - b1)^2\,(x^2 - (y^2 + z^2)\,\beta^2) - 2\,a1\,(a0\,(b0\,b1 - \beta^2) + (b0 - b1)\,(b0\,x - y\,\beta^2))\big)$$
$$\Big(-\Big(\Big(\big(-2\,a1\,b0\,(b0 - b1) + 2\,a0\,(b0 - b1)\,b1 + 2\,(b0 - b1)^2\,x\big)\,\big(-a1^2\,b0 +$$
$$a0\,(-b1\,x + b1^2\,y0 + (y - y0)\,\beta^2) + a1\,\big(a0\,b1 + 2\,b0\,x - b1\,x - b0\,b1\,y0 -$$

$$y\,\beta^2 + y0\,\beta^2) + (b0 - b1)\,(-x^2 + b1\,x\,y0 + (y^2 - y\,y0 + z^2)\,\beta^2)))\,/$$
$$(2\,\surd(a1^2 - 2\,a1\,x + x^2 - (y^2 + z^2)\,\beta^2 + y0^2\,(b1^2 - \beta^2) + 2\,y0\,(a1\,b1 - b1\,x + y\,\beta^2))$$
$$(a1^2\,(b0^2 - \beta^2) + a0^2\,(b1^2 - \beta^2) + 2\,a0\,(b0 - b1)\,(b1\,x - y\,\beta^2) + (b0 - b1)^2\,(x^2 -$$
$$(y^2 + z^2)\,\beta^2) - 2\,a1\,(a0\,(b0\,b1 - \beta^2) + (b0 - b1)\,(b0\,x - y\,\beta^2)))^{3/2})) +$$
$$(a1\,(2\,b0 - b1) - a0\,b1 + (b0 - b1)\,(-2\,x + b1\,y0))\,/\,(\surd(a1^2 - 2\,a1\,x +$$
$$x^2 - (y^2 + z^2)\,\beta^2 + y0^2\,(b1^2 - \beta^2) + 2\,y0\,(a1\,b1 - b1\,x + y\,\beta^2))$$
$$\surd(a1^2\,(b0^2 - \beta^2) + a0^2\,(b1^2 - \beta^2) + 2\,a0\,(b0 - b1)\,(b1\,x - y\,\beta^2) + (b0 - b1)^2$$
$$(x^2 - (y^2 + z^2)\,\beta^2) - 2\,a1\,(a0\,(b0\,b1 - \beta^2) + (b0 - b1)\,(b0\,x - y\,\beta^2)))) -$$
$$((-2\,a1 + 2\,x - 2\,b1\,y0)\,(-a1^2\,b0 + a0\,(-b1\,x + b1^2\,y0 + (y - y0)\,\beta^2) + a1$$
$$(a0\,b1 + 2\,b0\,x - b1\,x - b0\,b1\,y0 - y\,\beta^2 + y0\,\beta^2) + (b0 - b1)$$
$$(-x^2 + b1\,x\,y0 + (y^2 - y\,y0 + z^2)\,\beta^2)))\,/$$
$$(2\,(a1^2 - 2\,a1\,x + x^2 - (y^2 + z^2)\,\beta^2 + y0^2\,(b1^2 - \beta^2) + 2\,y0\,(a1\,b1 - b1\,x + y\,\beta^2))^{3/2}$$
$$\surd(a1^2\,(b0^2 - \beta^2) + a0^2\,(b1^2 - \beta^2) + 2\,a0\,(b0 - b1)\,(b1\,x - y\,\beta^2) + (b0 - b1)^2$$
$$(x^2 - (y^2 + z^2)\,\beta^2) - 2\,a1\,(a0\,(b0\,b1 - \beta^2) + (b0 - b1)\,(b0\,x - y\,\beta^2)))))))\,/$$
$$(1 - (-a1^2\,b0 + a0\,(-b1\,x + b1^2\,y0 + (y - y0)\,\beta^2) + a1\,(a0\,b1 + 2\,b0\,x - b1\,x -$$
$$b0\,b1\,y0 - y\,\beta^2 + y0\,\beta^2) + (b0 - b1)\,(-x^2 + b1\,x\,y0 + (y^2 - y\,y0 + z^2)\,\beta^2))^2\,/$$
$$((a1^2 - 2\,a1\,x + x^2 - (y^2 + z^2)\,\beta^2 + y0^2\,(b1^2 - \beta^2) + 2\,y0\,(a1\,b1 - b1\,x + y\,\beta^2))$$
$$(a1^2\,(b0^2 - \beta^2) + a0^2\,(b1^2 - \beta^2) + 2\,a0\,(b0 - b1)\,(b1\,x - y\,\beta^2) + (b0 - b1)^2$$
$$(x^2 - (y^2 + z^2)\,\beta^2) - 2\,a1\,(a0\,(b0\,b1 - \beta^2) + (b0 - b1)\,(b0\,x - y\,\beta^2)))))) +$$
$$(z\,\surd(a1^2\,(b0^2 - \beta^2) + a0^2\,(b1^2 - \beta^2) + 2\,a0\,(b0 - b1)\,(b1\,x - y\,\beta^2) +$$
$$(b0 - b1)^2\,(x^2 - (y^2 + z^2)\,\beta^2) - 2\,a1\,(a0\,(b0\,b1 - \beta^2) + (b0 - b1)\,(b0\,x - y\,\beta^2)))$$
$$(-(((-2\,a1\,b0\,(b0 - b1) + 2\,a0\,(b0 - b1)\,b1 + 2\,(b0 - b1)^2\,x)\,(a0^2\,b1 +$$
$$a0\,(-a1\,b0 + b0\,x - 2\,b1\,x + b0\,b1\,y1 + y\,\beta^2 - y1\,\beta^2) + a1\,(b0\,x - b0^2\,y1 +$$
$$(-y + y1)\,\beta^2) + (b0 - b1)\,(-x^2 + b0\,x\,y1 + (y^2 - y\,y1 + z^2)\,\beta^2)))\,/$$
$$(2\,\surd(a0^2 - 2\,a0\,x + x^2 - (y^2 + z^2)\,\beta^2 + y1^2\,(b0^2 - \beta^2) + 2\,y1\,(a0\,b0 - b0\,x + y\,\beta^2))$$
$$(a1^2\,(b0^2 - \beta^2) + a0^2\,(b1^2 - \beta^2) + 2\,a0\,(b0 - b1)\,(b1\,x - y\,\beta^2) + (b0 - b1)^2\,(x^2 -$$
$$(y^2 + z^2)\,\beta^2) - 2\,a1\,(a0\,(b0\,b1 - \beta^2) + (b0 - b1)\,(b0\,x - y\,\beta^2)))^{3/2})) +$$
$$(a1\,b0 + a0\,(b0 - 2\,b1) + (b0 - b1)\,(-2\,x + b0\,y1))\,/\,(\surd(a0^2 - 2\,a0\,x + x^2 -$$
$$(y^2 + z^2)\,\beta^2 + y1^2\,(b0^2 - \beta^2) + 2\,y1\,(a0\,b0 - b0\,x + y\,\beta^2))\,\surd(a1^2\,(b0^2 - \beta^2) +$$
$$a0^2\,(b1^2 - \beta^2) + 2\,a0\,(b0 - b1)\,(b1\,x - y\,\beta^2) + (b0 - b1)^2\,(x^2 - (y^2 + z^2)\,\beta^2) -$$
$$2\,a1\,(a0\,(b0\,b1 - \beta^2) + (b0 - b1)\,(b0\,x - y\,\beta^2)))) - ((-2\,a0 + 2\,x - 2\,b0\,y1)$$
$$(a0^2\,b1 + a0\,(-a1\,b0 + b0\,x - 2\,b1\,x + b0\,b1\,y1 + y\,\beta^2 - y1\,\beta^2) + a1\,(b0\,x -$$
$$b0^2\,y1 + (-y + y1)\,\beta^2) + (b0 - b1)\,(-x^2 + b0\,x\,y1 + (y^2 - y\,y1 + z^2)\,\beta^2)))\,/$$
$$(2\,(a0^2 - 2\,a0\,x + x^2 - (y^2 + z^2)\,\beta^2 + y1^2\,(b0^2 - \beta^2) + 2\,y1\,(a0\,b0 - b0\,x + y\,\beta^2))^{3/2}$$
$$\surd(a1^2\,(b0^2 - \beta^2) + a0^2\,(b1^2 - \beta^2) + 2\,a0\,(b0 - b1)\,(b1\,x - y\,\beta^2) + (b0 - b1)^2$$
$$(x^2 - (y^2 + z^2)\,\beta^2) - 2\,a1\,(a0\,(b0\,b1 - \beta^2) + (b0 - b1)\,(b0\,x - y\,\beta^2)))))))\,/$$
$$(1 - (a0^2\,b1 + a0\,(-a1\,b0 + b0\,x - 2\,b1\,x + b0\,b1\,y1 + y\,\beta^2 - y1\,\beta^2) + a1$$
$$(b0\,x - b0^2\,y1 + (-y + y1)\,\beta^2) + (b0 - b1)\,(-x^2 + b0\,x\,y1 + (y^2 - y\,y1 + z^2)\,\beta^2))^2\,/$$
$$((a0^2 - 2\,a0\,x + x^2 - (y^2 + z^2)\,\beta^2 + y1^2\,(b0^2 - \beta^2) + 2\,y1\,(a0\,b0 - b0\,x + y\,\beta^2))$$

$$
\begin{array}{l}
(a1^2 (b0^2-\beta^2) + a0^2 (b1^2-\beta^2) + 2\, a0\, (b0-b1) (b1\, x - y\, \beta^2) + (b0-b1)^2 \\
\quad (x^2 - (y^2+z^2)\, \beta^2) - 2\, a1\, (a0\, (b0\, b1 - \beta^2) + (b0-b1)\, (b0\, x - y\, \beta^2))))) - \\
(z \surd(a1^2 (b0^2-\beta^2) + a0^2 (b1^2-\beta^2) + 2\, a0\, (b0-b1)\, (b1\, x - y\, \beta^2) + \\
\quad (b0-b1)^2 (x^2 - (y^2+z^2)\, \beta^2) - 2\, a1\, (a0\, (b0\, b1 - \beta^2) + (b0-b1)\, (b0\, x - y\, \beta^2))) \\
\quad (-(((-2\, a1\, b0\, (b0-b1) + 2\, a0\, (b0-b1)\, b1 + 2\, (b0-b1)^2\, x)\, (-a1^2\, b0 + \\
\qquad a0\, (-b1\, x + b1^2\, y1 + (y-y1)\, \beta^2) + a1\, (a0\, b1 + 2\, b0\, x - b1\, x - b0\, b1\, y1 - \\
\qquad y\, \beta^2 + y1\, \beta^2) + (b0-b1)\, (-x^2 + b1\, x\, y1 + (y^2 - y\, y1 + z^2)\, \beta^2))) / \\
\qquad (2 \surd(a1^2 - 2\, a1\, x + x^2 - (y^2+z^2)\, \beta^2 + y1^2\, (b1^2-\beta^2) + 2\, y1\, (a1\, b1 - b1\, x + y\, \beta^2)) \\
\qquad (a1^2 (b0^2-\beta^2) + a0^2 (b1^2-\beta^2) + 2\, a0\, (b0-b1)\, (b1\, x - y\, \beta^2) + (b0-b1)^2 (x^2 - \\
\qquad (y^2+z^2)\, \beta^2) - 2\, a1\, (a0\, (b0\, b1 - \beta^2) + (b0-b1)\, (b0\, x - y\, \beta^2)))^{3/2})) + \\
\quad (a1\, (2\, b0 - b1) - a0\, b1 + (b0-b1)\, (-2\, x + b1\, y1)) / (\surd(a1^2 - 2\, a1\, x + \\
\qquad x^2 - (y^2+z^2)\, \beta^2 + y1^2\, (b1^2-\beta^2) + 2\, y1\, (a1\, b1 - b1\, x + y\, \beta^2)) \\
\qquad \surd(a1^2 (b0^2-\beta^2) + a0^2 (b1^2-\beta^2) + 2\, a0\, (b0-b1)\, (b1\, x - y\, \beta^2) + (b0-b1)^2 \\
\qquad (x^2 - (y^2+z^2)\, \beta^2) - 2\, a1\, (a0\, (b0\, b1 - \beta^2) + (b0-b1)\, (b0\, x - y\, \beta^2)))) - \\
\quad ((-2\, a1 + 2\, x - 2\, b1\, y1)\, (-a1^2\, b0 + a0\, (-b1\, x + b1^2\, y1 + (y-y1)\, \beta^2) + a1 \\
\qquad (a0\, b1 + 2\, b0\, x - b1\, x - b0\, b1\, y1 - y\, \beta^2 + y1\, \beta^2) + (b0-b1) \\
\qquad (-x^2 + b1\, x\, y1 + (y^2 - y\, y1 + z^2)\, \beta^2))) / \\
\quad (2\, (a1^2 - 2\, a1\, x + x^2 - (y^2+z^2)\, \beta^2 + y1^2\, (b1^2-\beta^2) + 2\, y1\, (a1\, b1 - b1\, x + y\, \beta^2))^{3/2} \\
\qquad \surd(a1^2 (b0^2-\beta^2) + a0^2 (b1^2-\beta^2) + 2\, a0\, (b0-b1)\, (b1\, x - y\, \beta^2) + (b0-b1)^2 \\
\qquad (x^2 - (y^2+z^2)\, \beta^2) - 2\, a1\, (a0\, (b0\, b1 - \beta^2) + (b0-b1)\, (b0\, x - y\, \beta^2)))))) / \\
(1 - (-a1^2\, b0 + a0\, (-b1\, x + b1^2\, y1 + (y-y1)\, \beta^2) + a1\, (a0\, b1 + 2\, b0\, x - b1\, x - \\
\qquad b0\, b1\, y1 - y\, \beta^2 + y1\, \beta^2) + (b0-b1)\, (-x^2 + b1\, x\, y1 + (y^2 - y\, y1 + z^2)\, \beta^2))^2 / \\
\quad ((a1^2 - 2\, a1\, x + x^2 - (y^2+z^2)\, \beta^2 + y1^2\, (b1^2-\beta^2) + 2\, y1\, (a1\, b1 - b1\, x + y\, \beta^2)) \\
\quad (a1^2 (b0^2-\beta^2) + a0^2 (b1^2-\beta^2) + 2\, a0\, (b0-b1)\, (b1\, x - y\, \beta^2) + (b0-b1)^2 \\
\qquad (x^2 - (y^2+z^2)\, \beta^2) - 2\, a1\, (a0\, (b0\, b1 - \beta^2) + (b0-b1)\, (b0\, x - y\, \beta^2))))) + \\
(-a0 + a1 - (b0-b1)\, y)\, \text{ArcTan}[(a0\, (y-y0) + x\, (-y+y0) + b0\, (y^2 - y\, y0 + z^2)) / \\
\quad (z \surd(a0^2 - 2\, a0\, x + x^2 - (y^2+z^2)\, \beta^2 + y0^2\, (b0^2-\beta^2) + 2\, y0\, (a0\, b0 - b0\, x + y\, \beta^2)))] - \\
(-a0 + a1 - (b0-b1)\, y) \\
\text{ArcTan}[\\
\quad (a0\, (y-y1) + x\, (-y+y1) + b0\, (y^2 - y\, y1 + z^2)) / \\
\quad (z \surd(a0^2 - 2\, a0\, x + x^2 - (y^2+z^2)\, \beta^2 + y1^2\, (b0^2-\beta^2) + 2\, y1\, (a0\, b0 - b0\, x + y\, \beta^2)))] + \\
(a0 - a1 + (b0-b1)\, y)\, \text{ArcTan}[(a1\, (y-y0) + x\, (-y+y0) + b1\, (y^2 - y\, y0 + z^2)) / \\
\quad (z \surd(a1^2 - 2\, a1\, x + x^2 - (y^2+z^2)\, \beta^2 + y0^2\, (b1^2-\beta^2) + 2\, y0\, (a1\, b1 - b1\, x + y\, \beta^2)))] - \\
(a0 - a1 + (b0-b1)\, y)\, \text{ArcTan}[(a1\, (y-y1) + x\, (-y+y1) + b1\, (y^2 - y\, y1 + z^2)) / \\
\quad (z \surd(a1^2 - 2\, a1\, x + x^2 - (y^2+z^2)\, \beta^2 + y1^2\, (b1^2-\beta^2) + 2\, y1\, (a1\, b1 - b1\, x + y\, \beta^2)))] - \\
(-b0 + b1)\, z\, \text{ArcTanh}[(a0 - x + b0\, y0) / (\surd(a0^2 - 2\, a0\, x + x^2 + 2\, a0\, b0\, y0 - \\
\qquad 2\, b0\, x\, y0 + b0^2\, y0^2 - y^2\, \beta^2 + 2\, y\, y0\, \beta^2 - y0^2\, \beta^2 - z^2\, \beta^2))] + \\
(-b0 + b1)\, z\, \text{ArcTanh}[(a0 - x + b0\, y1) / (\surd(a0^2 - 2\, a0\, x + x^2 + 2\, a0\, b0\, y1 - \\
\qquad 2\, b0\, x\, y1 + b0^2\, y1^2 - y^2\, \beta^2 + 2\, y\, y1\, \beta^2 - y1^2\, \beta^2 - z^2\, \beta^2))] + \\
(-b0 + b1)\, z\, \text{ArcTanh}[(a1 - x + b1\, y0) / (\surd(a1^2 - 2\, a1\, x + x^2 - (y^2+z^2)\, \beta^2 + \\
\qquad y0^2\, (b1^2-\beta^2) + 2\, y0\, (a1\, b1 - b1\, x + y\, \beta^2)))] -
\end{array}
$$

```
(-b0 + b1) z ArcTanh[(a1 - x + b1 y1) / (√(a1^2 - 2 a1 x + x^2 - (y^2 + z^2) β^2 +
        y1^2 (b1^2 - β^2) + 2 y1 (a1 b1 - b1 x + y β^2)))] -
  ((-2 a1 b0 (b0 - b1) + 2 a0 (b0 - b1) b1 + 2 (b0 - b1)^2 x) z
     ArcTanh[(a0^2 b1 + a0 (-a1 b0 + b0 x - 2 b1 x + b0 b1 y0 + y β^2 - y0 β^2) +
          a1 (b0 x - b0^2 y0 + (-y + y0) β^2) + (b0 - b1) (-x^2 + b0 x y0 + (y^2 - y y0 + z^2) β^2)) /
        (√(a0^2 - 2 a0 x + x^2 - (y^2 + z^2) β^2 + y0^2 (b0^2 - β^2) + 2 y0 (a0 b0 - b0 x + y β^2))
          √(a1^2 (b0^2 - β^2) + a0^2 (b1^2 - β^2) + 2 a0 (b0 - b1) (b1 x - y β^2) + (b0 - b1)^2
              (x^2 - (y^2 + z^2) β^2) - 2 a1 (a0 (b0 b1 - β^2) + (b0 - b1) (b0 x - y β^2))))]) /
    (2 √(a1^2 (b0^2 - β^2) + a0^2 (b1^2 - β^2) + 2 a0 (b0 - b1) (b1 x - y β^2) +
          (b0 - b1)^2 (x^2 - (y^2 + z^2) β^2) - 2 a1 (a0 (b0 b1 - β^2) + (b0 - b1) (b0 x - y β^2)))) +
  ((-2 a1 b0 (b0 - b1) + 2 a0 (b0 - b1) b1 + 2 (b0 - b1)^2 x) z
     ArcTanh[(-a1^2 b0 + a0 (-b1 x + b1^2 y0 + (y - y0) β^2) +
          a1 (a0 b1 + 2 b0 x - b1 x - b0 b1 y0 - y β^2 + y0 β^2) +
          (b0 - b1) (-x^2 + b1 x y0 + (y^2 - y y0 + z^2) β^2)) /
        (√(a1^2 - 2 a1 x + x^2 - (y^2 + z^2) β^2 + y0^2 (b1^2 - β^2) + 2 y0 (a1 b1 - b1 x + y β^2))
          √(a1^2 (b0^2 - β^2) + a0^2 (b1^2 - β^2) + 2 a0 (b0 - b1) (b1 x - y β^2) + (b0 - b1)^2
              (x^2 - (y^2 + z^2) β^2) - 2 a1 (a0 (b0 b1 - β^2) + (b0 - b1) (b0 x - y β^2))))]) /
    (2 √(a1^2 (b0^2 - β^2) + a0^2 (b1^2 - β^2) + 2 a0 (b0 - b1) (b1 x - y β^2) +
          (b0 - b1)^2 (x^2 - (y^2 + z^2) β^2) - 2 a1 (a0 (b0 b1 - β^2) + (b0 - b1) (b0 x - y β^2)))) +
  ((-2 a1 b0 (b0 - b1) + 2 a0 (b0 - b1) b1 + 2 (b0 - b1)^2 x) z
     ArcTanh[(a0^2 b1 + a0 (-a1 b0 + b0 x - 2 b1 x + b0 b1 y1 + y β^2 - y1 β^2) +
          a1 (b0 x - b0^2 y1 + (-y + y1) β^2) + (b0 - b1) (-x^2 + b0 x y1 + (y^2 - y y1 + z^2) β^2)) /
        (√(a0^2 - 2 a0 x + x^2 - (y^2 + z^2) β^2 + y1^2 (b0^2 - β^2) + 2 y1 (a0 b0 - b0 x + y β^2))
          √(a1^2 (b0^2 - β^2) + a0^2 (b1^2 - β^2) + 2 a0 (b0 - b1) (b1 x - y β^2) + (b0 - b1)^2
              (x^2 - (y^2 + z^2) β^2) - 2 a1 (a0 (b0 b1 - β^2) + (b0 - b1) (b0 x - y β^2))))]) /
    (2 √(a1^2 (b0^2 - β^2) + a0^2 (b1^2 - β^2) + 2 a0 (b0 - b1) (b1 x - y β^2) +
          (b0 - b1)^2 (x^2 - (y^2 + z^2) β^2) - 2 a1 (a0 (b0 b1 - β^2) + (b0 - b1) (b0 x - y β^2)))) -
  ((-2 a1 b0 (b0 - b1) + 2 a0 (b0 - b1) b1 + 2 (b0 - b1)^2 x) z
     ArcTanh[(-a1^2 b0 + a0 (-b1 x + b1^2 y1 + (y - y1) β^2) +
          a1 (a0 b1 + 2 b0 x - b1 x - b0 b1 y1 - y β^2 + y1 β^2) +
          (b0 - b1) (-x^2 + b1 x y1 + (y^2 - y y1 + z^2) β^2)) /
        (√(a1^2 - 2 a1 x + x^2 - (y^2 + z^2) β^2 + y1^2 (b1^2 - β^2) + 2 y1 (a1 b1 - b1 x + y β^2))
          √(a1^2 (b0^2 - β^2) + a0^2 (b1^2 - β^2) + 2 a0 (b0 - b1) (b1 x - y β^2) + (b0 - b1)^2
              (x^2 - (y^2 + z^2) β^2) - 2 a1 (a0 (b0 b1 - β^2) + (b0 - b1) (b0 x - y β^2))))]) /
    (2 √(a1^2 (b0^2 - β^2) + a0^2 (b1^2 - β^2) + 2 a0 (b0 - b1) (b1 x - y β^2) +
          (b0 - b1)^2 (x^2 - (y^2 + z^2) β^2) - 2 a1 (a0 (b0 b1 - β^2) + (b0 - b1) (b0 x - y β^2))))) /
(2 π (a0^2 + a1^2 + 2 a1 (-b0 + b1) y - 2 a0 (a1 + (-b0 + b1) y) +
    (b0 - b1)^2
     (y^2 + z^2)))
```

(*Tapered Supersonic Doublet Panel Perturbation Velocity v Component*)

```
In[377]:= v = D[ϕ[x, y, z], y]
```

Out[377]=

$$((\mu 0-\mu 1)\ ((((a1\ b0-a0\ b1+(-b0+b1)\ x)\ (a0-x+b0\ y0)\ z\ (-2\ y\ \beta^2+2\ y0\ \beta^2))\ /$$
$$(2\ (a0^2-2\ a0\ x+x^2+2\ a0\ b0\ y0-2\ b0\ x\ y0+b0^2\ y0^2-y^2\ \beta^2+2\ y\ y0\ \beta^2-y0^2\ \beta^2-z^2\ \beta^2)^{3/2}$$
$$(1-(a0-x+b0\ y0)^2\ /\ (a0^2-2\ a0\ x+x^2+2\ a0\ b0\ y0-$$
$$2\ b0\ x\ y0+b0^2\ y0^2-y^2\ \beta^2+2\ y\ y0\ \beta^2-y0^2\ \beta^2-z^2\ \beta^2)))\ -$$
$$((a1\ b0-a0\ b1+(-b0+b1)\ x)\ (a0-x+b0\ y1)\ z\ (-2\ y\ \beta^2+2\ y1\ \beta^2))\ /$$
$$(2\ (a0^2-2\ a0\ x+x^2+2\ a0\ b0\ y1-2\ b0\ x\ y1+b0^2\ y1^2-y^2\ \beta^2+2\ y\ y1\ \beta^2-y1^2\ \beta^2-z^2\ \beta^2)^{3/2}$$
$$(1-(a0-x+b0\ y1)^2\ /\ (a0^2-2\ a0\ x+x^2+2\ a0\ b0\ y1-$$
$$2\ b0\ x\ y1+b0^2\ y1^2-y^2\ \beta^2+2\ y\ y1\ \beta^2-y1^2\ \beta^2-z^2\ \beta^2)))\ +$$
$$((a0^2+a1\ (x-b0\ y)-a0\ (a1+x-2\ b0\ y+b1\ y)+(b0-b1)\ (-x\ y+b0\ (y^2+z^2)))$$
$$(-(((a0\ (y-y0)+x\ (-y+y0)+b0\ (y^2-y\ y0+z^2))\ (-2\ y\ \beta^2+2\ y0\ \beta^2))\ /$$
$$(2\ z\ (a0^2-2\ a0\ x+x^2-(y^2+z^2)\ \beta^2+y0^2\ (b0^2-\beta^2)+$$
$$2\ y0\ (a0\ b0-b0\ x+y\ \beta^2))^{3/2}))+(a0-x+b0\ (2\ y-y0))\ /\ (z\ \sqrt{}(a0^2-$$
$$2\ a0\ x+x^2-(y^2+z^2)\ \beta^2+y0^2\ (b0^2-\beta^2)+2\ y0\ (a0\ b0-b0\ x+y\ \beta^2)))))\ /$$
$$(1+(a0\ (y-y0)+x\ (-y+y0)+b0\ (y^2-y\ y0+z^2))^2\ /$$
$$(z^2\ (a0^2-2\ a0\ x+x^2-(y^2+z^2)\ \beta^2+y0^2\ (b0^2-\beta^2)+2\ y0\ (a0\ b0-b0\ x+y\ \beta^2))))\ -$$
$$((a0^2+a1\ (x-b0\ y)-a0\ (a1+x-2\ b0\ y+b1\ y)+(b0-b1)\ (-x\ y+b0\ (y^2+z^2)))$$
$$(-(((a0\ (y-y1)+x\ (-y+y1)+b0\ (y^2-y\ y1+z^2))\ (-2\ y\ \beta^2+2\ y1\ \beta^2))\ /$$
$$(2\ z\ (a0^2-2\ a0\ x+x^2-(y^2+z^2)\ \beta^2+y1^2\ (b0^2-\beta^2)+$$
$$2\ y1\ (a0\ b0-b0\ x+y\ \beta^2))^{3/2}))+(a0-x+b0\ (2\ y-y1))\ /\ (z\ \sqrt{}(a0^2-$$
$$2\ a0\ x+x^2-(y^2+z^2)\ \beta^2+y1^2\ (b0^2-\beta^2)+2\ y1\ (a0\ b0-b0\ x+y\ \beta^2)))))\ /$$
$$(1+(a0\ (y-y1)+x\ (-y+y1)+b0\ (y^2-y\ y1+z^2))^2\ /$$
$$(z^2\ (a0^2-2\ a0\ x+x^2-(y^2+z^2)\ \beta^2+y1^2\ (b0^2-\beta^2)+2\ y1\ (a0\ b0-b0\ x+y\ \beta^2))))\ -$$
$$((a1\ b0-a0\ b1+(-b0+b1)\ x)\ (a1-x+b1\ y0)\ z\ (-2\ y\ \beta^2+2\ y0\ \beta^2))\ /$$
$$(2\ (a1^2-2\ a1\ x+x^2-(y^2+z^2)\ \beta^2+y0^2\ (b1^2-\beta^2)+2\ y0\ (a1\ b1-b1\ x+y\ \beta^2))^{3/2}$$
$$(1-(a1-x+b1\ y0)^2\ /$$
$$(a1^2-2\ a1\ x+x^2-(y^2+z^2)\ \beta^2+y0^2\ (b1^2-\beta^2)+2\ y0\ (a1\ b1-b1\ x+y\ \beta^2))))\ +$$
$$((a1^2-a1\ (x+(b0-2\ b1)\ y)-a0\ (a1-x+b1\ y)+(b0-b1)\ (x\ y-b1\ (y^2+z^2)))$$
$$(-(((a1\ (y-y0)+x\ (-y+y0)+b1\ (y^2-y\ y0+z^2))\ (-2\ y\ \beta^2+2\ y0\ \beta^2))\ /$$
$$(2\ z\ (a1^2-2\ a1\ x+x^2-(y^2+z^2)\ \beta^2+y0^2\ (b1^2-\beta^2)+$$
$$2\ y0\ (a1\ b1-b1\ x+y\ \beta^2))^{3/2}))+(a1-x+b1\ (2\ y-y0))\ /\ (z\ \sqrt{}(a1^2-$$
$$2\ a1\ x+x^2-(y^2+z^2)\ \beta^2+y0^2\ (b1^2-\beta^2)+2\ y0\ (a1\ b1-b1\ x+y\ \beta^2)))))\ /$$
$$(1+(a1\ (y-y0)+x\ (-y+y0)+b1\ (y^2-y\ y0+z^2))^2\ /$$
$$(z^2\ (a1^2-2\ a1\ x+x^2-(y^2+z^2)\ \beta^2+y0^2\ (b1^2-\beta^2)+2\ y0\ (a1\ b1-b1\ x+y\ \beta^2))))\ +$$
$$((a1\ b0-a0\ b1+(-b0+b1)\ x)\ (a1-x+b1\ y1)\ z\ (-2\ y\ \beta^2+2\ y1\ \beta^2))\ /$$
$$(2\ (a1^2-2\ a1\ x+x^2-(y^2+z^2)\ \beta^2+y1^2\ (b1^2-\beta^2)+2\ y1\ (a1\ b1-b1\ x+y\ \beta^2))^{3/2}$$
$$(1-(a1-x+b1\ y1)^2\ /$$

$$(a1^2 - 2\,a1\,x + x^2 - (y^2 + z^2)\,\beta^2 + y1^2\,(b1^2 - \beta^2) + 2\,y1\,(a1\,b1 - b1\,x + y\,\beta^2)))) -$$
$$((a1^2 - a1\,(x + (b0 - 2\,b1)\,y) - a0\,(a1 - x + b1\,y) + (b0 - b1)\,(x\,y - b1\,(y^2 + z^2)))$$
$$(-(((a1\,(y - y1) + x\,(-y + y1) + b1\,(y^2 - y\,y1 + z^2))\,(-2\,y\,\beta^2 + 2\,y1\,\beta^2))\,/$$
$$(2\,z\,(a1^2 - 2\,a1\,x + x^2 - (y^2 + z^2)\,\beta^2 + y1^2\,(b1^2 - \beta^2) +$$
$$2\,y1\,(a1\,b1 - b1\,x + y\,\beta^2))^{3/2})) + (a1 - x + b1\,(2\,y - y1))\,/\,(z\,\sqrt{}(a1^2 -$$
$$2\,a1\,x + x^2 - (y^2 + z^2)\,\beta^2 + y1^2\,(b1^2 - \beta^2) + 2\,y1\,(a1\,b1 - b1\,x + y\,\beta^2)))))\,/$$
$$(1 + (a1\,(y - y1) + x\,(-y + y1) + b1\,(y^2 - y\,y1 + z^2))^2\,/$$
$$(z^2\,(a1^2 - 2\,a1\,x + x^2 - (y^2 + z^2)\,\beta^2 + y1^2\,(b1^2 - \beta^2) + 2\,y1\,(a1\,b1 - b1\,x + y\,\beta^2)))) -$$
$$(z\,\sqrt{}(a1^2\,(b0^2 - \beta^2) + a0^2\,(b1^2 - \beta^2) + 2\,a0\,(b0 - b1)\,(b1\,x - y\,\beta^2) +$$
$$(b0 - b1)^2\,(x^2 - (y^2 + z^2)\,\beta^2) - 2\,a1\,(a0\,(b0\,b1 - \beta^2) + (b0 - b1)\,(b0\,x - y\,\beta^2)))$$
$$(-(((-2\,a0\,(b0 - b1)\,\beta^2 + 2\,a1\,(b0 - b1)\,\beta^2 - 2\,(b0 - b1)^2\,y\,\beta^2)$$
$$(a0^2\,b1 + a0\,(-a1\,b0 + b0\,x - 2\,b1\,x + b0\,b1\,y0 + y\,\beta^2 - y0\,\beta^2) + a1\,(b0\,x - b0^2$$
$$y0 + (-y + y0)\,\beta^2) + (b0 - b1)\,(-x^2 + b0\,x\,y0 + (y^2 - y\,y0 + z^2)\,\beta^2)))\,/$$
$$(2\,\sqrt{}(a0^2 - 2\,a0\,x + x^2 - (y^2 + z^2)\,\beta^2 + y0^2\,(b0^2 - \beta^2) + 2\,y0\,(a0\,b0 - b0\,x + y\,\beta^2))$$
$$(a1^2\,(b0^2 - \beta^2) + a0^2\,(b1^2 - \beta^2) + 2\,a0\,(b0 - b1)\,(b1\,x - y\,\beta^2) + (b0 - b1)^2\,(x^2 -$$
$$(y^2 + z^2)\,\beta^2) - 2\,a1\,(a0\,(b0\,b1 - \beta^2) + (b0 - b1)\,(b0\,x - y\,\beta^2)))^{3/2})) +$$
$$(a0\,\beta^2 - a1\,\beta^2 + (b0 - b1)\,(2\,y - y0)\,\beta^2)\,/\,(\sqrt{}(a0^2 - 2\,a0\,x + x^2 - (y^2 + z^2)\,\beta^2 +$$
$$y0^2\,(b0^2 - \beta^2) + 2\,y0\,(a0\,b0 - b0\,x + y\,\beta^2))\,\sqrt{}(a1^2\,(b0^2 - \beta^2) +$$
$$a0^2\,(b1^2 - \beta^2) + 2\,a0\,(b0 - b1)\,(b1\,x - y\,\beta^2) + (b0 - b1)^2\,(x^2 - (y^2 + z^2)\,\beta^2) -$$
$$2\,a1\,(a0\,(b0\,b1 - \beta^2) + (b0 - b1)\,(b0\,x - y\,\beta^2)))) - ((-2\,y\,\beta^2 + 2\,y0\,\beta^2)$$
$$(a0^2\,b1 + a0\,(-a1\,b0 + b0\,x - 2\,b1\,x + b0\,b1\,y0 + y\,\beta^2 - y0\,\beta^2) + a1\,(b0\,x -$$
$$b0^2\,y0 + (-y + y0)\,\beta^2) + (b0 - b1)\,(-x^2 + b0\,x\,y0 + (y^2 - y\,y0 + z^2)\,\beta^2)))\,/$$
$$(2\,(a0^2 - 2\,a0\,x + x^2 - (y^2 + z^2)\,\beta^2 + y0^2\,(b0^2 - \beta^2) + 2\,y0\,(a0\,b0 - b0\,x + y\,\beta^2))^{3/2}$$
$$\sqrt{}(a1^2\,(b0^2 - \beta^2) + a0^2\,(b1^2 - \beta^2) + 2\,a0\,(b0 - b1)\,(b1\,x - y\,\beta^2) + (b0 - b1)^2$$
$$(x^2 - (y^2 + z^2)\,\beta^2) - 2\,a1\,(a0\,(b0\,b1 - \beta^2) + (b0 - b1)\,(b0\,x - y\,\beta^2))))))\,/$$
$$(1 - (a0^2\,b1 + a0\,(-a1\,b0 + b0\,x - 2\,b1\,x + b0\,b1\,y0 + y\,\beta^2 - y0\,\beta^2) + a1\,(b0\,x -$$
$$b0^2\,y0 + (-y + y0)\,\beta^2) + (b0 - b1)\,(-x^2 + b0\,x\,y0 + (y^2 - y\,y0 + z^2)\,\beta^2))^2\,/$$
$$((a0^2 - 2\,a0\,x + x^2 - (y^2 + z^2)\,\beta^2 + y0^2\,(b0^2 - \beta^2) + 2\,y0\,(a0\,b0 - b0\,x + y\,\beta^2))$$
$$(a1^2\,(b0^2 - \beta^2) + a0^2\,(b1^2 - \beta^2) + 2\,a0\,(b0 - b1)\,(b1\,x - y\,\beta^2) + (b0 - b1)^2$$
$$(x^2 - (y^2 + z^2)\,\beta^2) - 2\,a1\,(a0\,(b0\,b1 - \beta^2) + (b0 - b1)\,(b0\,x - y\,\beta^2))))) +$$
$$(z\,\sqrt{}(a1^2\,(b0^2 - \beta^2) + a0^2\,(b1^2 - \beta^2) + 2\,a0\,(b0 - b1)\,(b1\,x - y\,\beta^2) +$$
$$(b0 - b1)^2\,(x^2 - (y^2 + z^2)\,\beta^2) - 2\,a1\,(a0\,(b0\,b1 - \beta^2) + (b0 - b1)\,(b0\,x - y\,\beta^2)))$$
$$(-(((-2\,a0\,(b0 - b1)\,\beta^2 + 2\,a1\,(b0 - b1)\,\beta^2 - 2\,(b0 - b1)^2\,y\,\beta^2)$$
$$(-a1^2\,b0 + a0\,(-b1\,x + b1^2\,y0 + (y - y0)\,\beta^2) + a1\,(a0\,b1 + 2\,b0\,x - b1\,x - b0\,b1$$
$$y0 - y\,\beta^2 + y0\,\beta^2) + (b0 - b1)\,(-x^2 + b1\,x\,y0 + (y^2 - y\,y0 + z^2)\,\beta^2)))\,/$$
$$(2\,\sqrt{}(a1^2 - 2\,a1\,x + x^2 - (y^2 + z^2)\,\beta^2 + y0^2\,(b1^2 - \beta^2) + 2\,y0\,(a1\,b1 - b1\,x + y\,\beta^2))$$
$$(a1^2\,(b0^2 - \beta^2) + a0^2\,(b1^2 - \beta^2) + 2\,a0\,(b0 - b1)\,(b1\,x - y\,\beta^2) + (b0 - b1)^2\,(x^2 -$$
$$(y^2 + z^2)\,\beta^2) - 2\,a1\,(a0\,(b0\,b1 - \beta^2) + (b0 - b1)\,(b0\,x - y\,\beta^2)))^{3/2})) +$$

$$\left(a0\,\beta^2 - a1\,\beta^2 + (b0 - b1)\,(2\,y - y0)\,\beta^2\right) \Big/ \Big(\sqrt{}\big(a1^2 - 2\,a1\,x + x^2 -$$
$$(y^2 + z^2)\,\beta^2 + y0^2\,(b1^2 - \beta^2) + 2\,y0\,(a1\,b1 - b1\,x + y\,\beta^2)\big)$$
$$\sqrt{}\big(a1^2\,(b0^2 - \beta^2) + a0^2\,(b1^2 - \beta^2) + 2\,a0\,(b0 - b1)\,(b1\,x - y\,\beta^2) + (b0 - b1)^2$$
$$(x^2 - (y^2 + z^2)\,\beta^2) - 2\,a1\,(a0\,(b0\,b1 - \beta^2) + (b0 - b1)\,(b0\,x - y\,\beta^2))\big)\Big)\Big) -$$
$$\Big(\big(-2\,y\,\beta^2 + 2\,y0\,\beta^2\big)\,\big(-a1^2\,b0 + a0\,(-b1\,x + b1^2\,y0 + (y - y0)\,\beta^2) +$$
$$a1\,(a0\,b1 + 2\,b0\,x - b1\,x - b0\,b1\,y0 - y\,\beta^2 + y0\,\beta^2) +$$
$$(b0 - b1)\,(-x^2 + b1\,x\,y0 + (y^2 - y\,y0 + z^2)\,\beta^2)\big)\Big) \Big/$$
$$\Big(2\,\big(a1^2 - 2\,a1\,x + x^2 - (y^2 + z^2)\,\beta^2 + y0^2\,(b1^2 - \beta^2) + 2\,y0\,(a1\,b1 - b1\,x + y\,\beta^2)\big)^{3/2}$$
$$\sqrt{}\big(a1^2\,(b0^2 - \beta^2) + a0^2\,(b1^2 - \beta^2) + 2\,a0\,(b0 - b1)\,(b1\,x - y\,\beta^2) + (b0 - b1)^2$$
$$(x^2 - (y^2 + z^2)\,\beta^2) - 2\,a1\,(a0\,(b0\,b1 - \beta^2) + (b0 - b1)\,(b0\,x - y\,\beta^2))\big)\Big)\Big)\Big) \Big/$$
$$\Big(1 - \big(-a1^2\,b0 + a0\,(-b1\,x + b1^2\,y0 + (y - y0)\,\beta^2) + a1\,(a0\,b1 + 2\,b0\,x - b1\,x -$$
$$b0\,b1\,y0 - y\,\beta^2 + y0\,\beta^2) + (b0 - b1)\,(-x^2 + b1\,x\,y0 + (y^2 - y\,y0 + z^2)\,\beta^2)\big)^2 \Big/$$
$$\Big(\big(a1^2 - 2\,a1\,x + x^2 - (y^2 + z^2)\,\beta^2 + y0^2\,(b1^2 - \beta^2) + 2\,y0\,(a1\,b1 - b1\,x + y\,\beta^2)\big)$$
$$\big(a1^2\,(b0^2 - \beta^2) + a0^2\,(b1^2 - \beta^2) + 2\,a0\,(b0 - b1)\,(b1\,x - y\,\beta^2) + (b0 - b1)^2$$
$$(x^2 - (y^2 + z^2)\,\beta^2) - 2\,a1\,(a0\,(b0\,b1 - \beta^2) + (b0 - b1)\,(b0\,x - y\,\beta^2))\big)\Big)\Big) +$$
$$\Big(z\,\sqrt{}\big(a1^2\,(b0^2 - \beta^2) + a0^2\,(b1^2 - \beta^2) + 2\,a0\,(b0 - b1)\,(b1\,x - y\,\beta^2) +$$
$$(b0 - b1)^2\,(x^2 - (y^2 + z^2)\,\beta^2) - 2\,a1\,(a0\,(b0\,b1 - \beta^2) + (b0 - b1)\,(b0\,x - y\,\beta^2))\big)\Big)$$
$$\Big(-\Big(\Big(\big(-2\,a0\,(b0 - b1)\,\beta^2 + 2\,a1\,(b0 - b1)\,\beta^2 - 2\,(b0 - b1)^2\,y\,\beta^2\big)$$
$$\big(a0^2\,b1 + a0\,(-a1\,b0 + b0\,x - 2\,b1\,x + b0\,b1\,y1 + y\,\beta^2 - y1\,\beta^2) + a1\,(b0\,x - b0^2$$
$$y1 + (-y + y1)\,\beta^2) + (b0 - b1)\,(-x^2 + b0\,x\,y1 + (y^2 - y\,y1 + z^2)\,\beta^2)\big)\Big) \Big/$$
$$\Big(2\,\sqrt{}\big(a0^2 - 2\,a0\,x + x^2 - (y^2 + z^2)\,\beta^2 + y1^2\,(b0^2 - \beta^2) + 2\,y1\,(a0\,b0 - b0\,x + y\,\beta^2)\big)$$
$$\big(a1^2\,(b0^2 - \beta^2) + a0^2\,(b1^2 - \beta^2) + 2\,a0\,(b0 - b1)\,(b1\,x - y\,\beta^2) + (b0 - b1)^2\,(x^2 -$$
$$(y^2 + z^2)\,\beta^2) - 2\,a1\,(a0\,(b0\,b1 - \beta^2) + (b0 - b1)\,(b0\,x - y\,\beta^2))\big)^{3/2}\Big)\Big) +$$
$$\left(a0\,\beta^2 - a1\,\beta^2 + (b0 - b1)\,(2\,y - y1)\,\beta^2\right) \Big/ \Big(\sqrt{}\big(a0^2 - 2\,a0\,x + x^2 -$$
$$(y^2 + z^2)\,\beta^2 + y1^2\,(b0^2 - \beta^2) + 2\,y1\,(a0\,b0 - b0\,x + y\,\beta^2)\big)$$
$$\sqrt{}\big(a1^2\,(b0^2 - \beta^2) + a0^2\,(b1^2 - \beta^2) + 2\,a0\,(b0 - b1)\,(b1\,x - y\,\beta^2) + (b0 - b1)^2$$
$$(x^2 - (y^2 + z^2)\,\beta^2) - 2\,a1\,(a0\,(b0\,b1 - \beta^2) + (b0 - b1)\,(b0\,x - y\,\beta^2))\big)\Big)\Big) -$$
$$\Big(\big(-2\,y\,\beta^2 + 2\,y1\,\beta^2\big)\,\big(a0^2\,b1 + a0\,(-a1\,b0 + b0\,x - 2\,b1\,x + b0\,b1\,y1 + y\,\beta^2 - y1\,\beta^2) +$$
$$a1\,(b0\,x - b0^2\,y1 + (-y + y1)\,\beta^2) +$$
$$(b0 - b1)\,(-x^2 + b0\,x\,y1 + (y^2 - y\,y1 + z^2)\,\beta^2)\big)\Big) \Big/$$
$$\Big(2\,\big(a0^2 - 2\,a0\,x + x^2 - (y^2 + z^2)\,\beta^2 + y1^2\,(b0^2 - \beta^2) + 2\,y1\,(a0\,b0 - b0\,x + y\,\beta^2)\big)^{3/2}$$
$$\sqrt{}\big(a1^2\,(b0^2 - \beta^2) + a0^2\,(b1^2 - \beta^2) + 2\,a0\,(b0 - b1)\,(b1\,x - y\,\beta^2) + (b0 - b1)^2$$
$$(x^2 - (y^2 + z^2)\,\beta^2) - 2\,a1\,(a0\,(b0\,b1 - \beta^2) + (b0 - b1)\,(b0\,x - y\,\beta^2))\big)\Big)\Big)\Big) \Big/$$
$$\Big(1 - \big(a0^2\,b1 + a0\,(-a1\,b0 + b0\,x - 2\,b1\,x + b0\,b1\,y1 + y\,\beta^2 - y1\,\beta^2) + a1\,(b0\,x -$$
$$b0^2\,y1 + (-y + y1)\,\beta^2) + (b0 - b1)\,(-x^2 + b0\,x\,y1 + (y^2 - y\,y1 + z^2)\,\beta^2)\big)^2 \Big/$$
$$\Big(\big(a0^2 - 2\,a0\,x + x^2 - (y^2 + z^2)\,\beta^2 + y1^2\,(b0^2 - \beta^2) + 2\,y1\,(a0\,b0 - b0\,x + y\,\beta^2)\big)$$
$$\big(a1^2\,(b0^2 - \beta^2) + a0^2\,(b1^2 - \beta^2) + 2\,a0\,(b0 - b1)\,(b1\,x - y\,\beta^2) + (b0 - b1)^2$$
$$(x^2 - (y^2 + z^2)\,\beta^2) - 2\,a1\,(a0\,(b0\,b1 - \beta^2) + (b0 - b1)\,(b0\,x - y\,\beta^2))\big)\Big)\Big) -$$
$$\Big(z\,\sqrt{}\big(a1^2\,(b0^2 - \beta^2) + a0^2\,(b1^2 - \beta^2) + 2\,a0\,(b0 - b1)\,(b1\,x - y\,\beta^2) +$$
$$(b0 - b1)^2\,(x^2 - (y^2 + z^2)\,\beta^2) - 2\,a1\,(a0\,(b0\,b1 - \beta^2) + (b0 - b1)\,(b0\,x - y\,\beta^2))\big)\Big)$$

$$\Big(-\Big(\Big(\Big(-2\,a0\,(b0-b1)\,\beta^2+2\,a1\,(b0-b1)\,\beta^2-2\,(b0-b1)^2\,y\,\beta^2\Big)$$
$$\Big(-a1^2\,b0+a0\,\big(-b1\,x+b1^2\,y1+(y-y1)\,\beta^2\big)+a1\,\big(a0\,b1+2\,b0\,x-b1\,x-b0\,b1$$
$$y1-y\,\beta^2+y1\,\beta^2\big)+(b0-b1)\,\big(-x^2+b1\,x\,y1+\big(y^2-y\,y1+z^2\big)\,\beta^2\big)\Big)\Big)\Big/$$
$$\Big(2\,\surd\Big(a1^2-2\,a1\,x+x^2-\big(y^2+z^2\big)\,\beta^2+y1^2\,\big(b1^2-\beta^2\big)+2\,y1\,\big(a1\,b1-b1\,x+y\,\beta^2\big)\Big)$$
$$\Big(a1^2\,\big(b0^2-\beta^2\big)+a0^2\,\big(b1^2-\beta^2\big)+2\,a0\,(b0-b1)\,\big(b1\,x-y\,\beta^2\big)+(b0-b1)^2\,\big(x^2-$$
$$\big(y^2+z^2\big)\,\beta^2\big)-2\,a1\,\big(a0\,\big(b0\,b1-\beta^2\big)+(b0-b1)\,\big(b0\,x-y\,\beta^2\big)\big)\Big)^{3/2}\Big)\Big)+$$
$$\big(a0\,\beta^2-a1\,\beta^2+(b0-b1)\,(2\,y-y1)\,\beta^2\big)\Big/\Big(\surd\Big(a1^2-2\,a1\,x+x^2-$$
$$\big(y^2+z^2\big)\,\beta^2+y1^2\,\big(b1^2-\beta^2\big)+2\,y1\,\big(a1\,b1-b1\,x+y\,\beta^2\big)\Big)$$
$$\surd\Big(a1^2\,\big(b0^2-\beta^2\big)+a0^2\,\big(b1^2-\beta^2\big)+2\,a0\,(b0-b1)\,\big(b1\,x-y\,\beta^2\big)+(b0-b1)^2$$
$$\big(x^2-\big(y^2+z^2\big)\,\beta^2\big)-2\,a1\,\big(a0\,\big(b0\,b1-\beta^2\big)+(b0-b1)\,\big(b0\,x-y\,\beta^2\big)\big)\Big)\Big)-$$
$$\Big(\big(-2\,y\,\beta^2+2\,y1\,\beta^2\big)\,\Big(-a1^2\,b0+a0\,\big(-b1\,x+b1^2\,y1+(y-y1)\,\beta^2\big)+$$
$$a1\,\big(a0\,b1+2\,b0\,x-b1\,x-b0\,b1\,y1-y\,\beta^2+y1\,\beta^2\big)+$$
$$(b0-b1)\,\big(-x^2+b1\,x\,y1+\big(y^2-y\,y1+z^2\big)\,\beta^2\big)\Big)\Big)\Big/$$
$$\Big(2\,\Big(a1^2-2\,a1\,x+x^2-\big(y^2+z^2\big)\,\beta^2+y1^2\,\big(b1^2-\beta^2\big)+2\,y1\,\big(a1\,b1-b1\,x+y\,\beta^2\big)\Big)^{3/2}$$
$$\surd\Big(a1^2\,\big(b0^2-\beta^2\big)+a0^2\,\big(b1^2-\beta^2\big)+2\,a0\,(b0-b1)\,\big(b1\,x-y\,\beta^2\big)+(b0-b1)^2$$
$$\big(x^2-\big(y^2+z^2\big)\,\beta^2\big)-2\,a1\,\big(a0\,\big(b0\,b1-\beta^2\big)+(b0-b1)\,\big(b0\,x-y\,\beta^2\big)\big)\Big)\Big)\Big)\Big)\Big/$$
$$\Big(1-\Big(-a1^2\,b0+a0\,\big(-b1\,x+b1^2\,y1+(y-y1)\,\beta^2\big)+a1\,\big(a0\,b1+2\,b0\,x-b1\,x-$$
$$b0\,b1\,y1-y\,\beta^2+y1\,\beta^2\big)+(b0-b1)\,\big(-x^2+b1\,x\,y1+\big(y^2-y\,y1+z^2\big)\,\beta^2\big)\Big)^2\Big/$$
$$\Big(\Big(a1^2-2\,a1\,x+x^2-\big(y^2+z^2\big)\,\beta^2+y1^2\,\big(b1^2-\beta^2\big)+2\,y1\,\big(a1\,b1-b1\,x+y\,\beta^2\big)\Big)$$
$$\Big(a1^2\,\big(b0^2-\beta^2\big)+a0^2\,\big(b1^2-\beta^2\big)+2\,a0\,(b0-b1)\,\big(b1\,x-y\,\beta^2\big)+(b0-b1)^2$$
$$\big(x^2-\big(y^2+z^2\big)\,\beta^2\big)-2\,a1\,\big(a0\,\big(b0\,b1-\beta^2\big)+(b0-b1)\,\big(b0\,x-y\,\beta^2\big)\big)\Big)\Big)\Big)+$$
$$(-a1\,b0-a0\,(-2\,b0+b1)+(b0-b1)\,(-x+2\,b0\,y))$$
$$\text{ArcTan}\Big[$$
$$\big(a0\,(y-y0)+x\,(-y+y0)+b0\,\big(y^2-y\,y0+z^2\big)\big)\Big/$$
$$\Big(z\,\surd\Big(a0^2-2\,a0\,x+x^2-\big(y^2+z^2\big)\,\beta^2+y0^2\,\big(b0^2-\beta^2\big)+2\,y0\,\big(a0\,b0-b0\,x+y\,\beta^2\big)\Big)\Big)\Big]-$$
$$(-a1\,b0-a0\,(-2\,b0+b1)+(b0-b1)\,(-x+2\,b0\,y))$$
$$\text{ArcTan}\Big[$$
$$\big(a0\,(y-y1)+x\,(-y+y1)+b0\,\big(y^2-y\,y1+z^2\big)\big)\Big/$$
$$\Big(z\,\surd\Big(a0^2-2\,a0\,x+x^2-\big(y^2+z^2\big)\,\beta^2+y1^2\,\big(b0^2-\beta^2\big)+2\,y1\,\big(a0\,b0-b0\,x+y\,\beta^2\big)\Big)\Big)\Big]+$$
$$(-a1\,(b0-2\,b1)-a0\,b1+(b0-b1)\,(x-2\,b1\,y))$$
$$\text{ArcTan}\Big[$$
$$\big(a1\,(y-y0)+x\,(-y+y0)+b1\,\big(y^2-y\,y0+z^2\big)\big)\Big/$$
$$\Big(z\,\surd\Big(a1^2-2\,a1\,x+x^2-\big(y^2+z^2\big)\,\beta^2+y0^2\,\big(b1^2-\beta^2\big)+2\,y0\,\big(a1\,b1-b1\,x+y\,\beta^2\big)\Big)\Big)\Big]-$$
$$(-a1\,(b0-2\,b1)-a0\,b1+(b0-b1)\,(x-2\,b1\,y))$$
$$\text{ArcTan}\Big[$$
$$\big(a1\,(y-y1)+x\,(-y+y1)+b1\,\big(y^2-y\,y1+z^2\big)\big)\Big/$$
$$\Big(z\,\surd\Big(a1^2-2\,a1\,x+x^2-\big(y^2+z^2\big)\,\beta^2+y1^2\,\big(b1^2-\beta^2\big)+2\,y1\,\big(a1\,b1-b1\,x+y\,\beta^2\big)\Big)\Big)\Big]-$$
$$\Big(z\,\big(-2\,a0\,(b0-b1)\,\beta^2+2\,a1\,(b0-b1)\,\beta^2-2\,(b0-b1)^2\,y\,\beta^2\big)$$
$$\text{ArcTanh}\Big[\Big(a0^2\,b1+a0\,\big(-a1\,b0+b0\,x-2\,b1\,x+b0\,b1\,y0+y\,\beta^2-y0\,\beta^2\big)+$$
$$a1\,\big(b0\,x-b0^2\,y0+(-y+y0)\,\beta^2\big)+(b0-b1)\,\big(-x^2+b0\,x\,y0+\big(y^2-y\,y0+z^2\big)\,\beta^2\big)\Big)\Big/$$
$$\Big(\surd\Big(a0^2-2\,a0\,x+x^2-\big(y^2+z^2\big)\,\beta^2+y0^2\,\big(b0^2-\beta^2\big)+2\,y0\,\big(a0\,b0-b0\,x+y\,\beta^2\big)\Big)$$

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          √(a1^2 (b0^2 - β^2) + a0^2 (b1^2 - β^2) + 2 a0 (b0 - b1) (b1 x - y β^2) + (b0 - b1)^2
                (x^2 - (y^2 + z^2) β^2) - 2 a1 (a0 (b0 b1 - β^2) + (b0 - b1) (b0 x - y β^2))))]) /
      (2 √(a1^2 (b0^2 - β^2) + a0^2 (b1^2 - β^2) + 2 a0 (b0 - b1) (b1 x - y β^2) +
            (b0 - b1)^2 (x^2 - (y^2 + z^2) β^2) - 2 a1 (a0 (b0 b1 - β^2) + (b0 - b1) (b0 x - y β^2)))) +
    (z (-2 a0 (b0 - b1) β^2 + 2 a1 (b0 - b1) β^2 - 2 (b0 - b1)^2 y β^2)
       ArcTanh[(-a1^2 b0 + a0 (-b1 x + b1^2 y0 + (y - y0) β^2) + a1 (a0 b1 + 2 b0 x - b1 x -
                b0 b1 y0 - y β^2 + y0 β^2) + (b0 - b1) (-x^2 + b1 x y0 + (y^2 - y y0 + z^2) β^2)) /
          (√(a1^2 - 2 a1 x + x^2 - (y^2 + z^2) β^2 + y0^2 (b1^2 - β^2) + 2 y0 (a1 b1 - b1 x + y β^2))
            √(a1^2 (b0^2 - β^2) + a0^2 (b1^2 - β^2) + 2 a0 (b0 - b1) (b1 x - y β^2) + (b0 - b1)^2
                (x^2 - (y^2 + z^2) β^2) - 2 a1 (a0 (b0 b1 - β^2) + (b0 - b1) (b0 x - y β^2))))]) /
      (2 √(a1^2 (b0^2 - β^2) + a0^2 (b1^2 - β^2) + 2 a0 (b0 - b1) (b1 x - y β^2) +
            (b0 - b1)^2 (x^2 - (y^2 + z^2) β^2) - 2 a1 (a0 (b0 b1 - β^2) + (b0 - b1) (b0 x - y β^2)))) +
    (z (-2 a0 (b0 - b1) β^2 + 2 a1 (b0 - b1) β^2 - 2 (b0 - b1)^2 y β^2)
       ArcTanh[(a0^2 b1 + a0 (-a1 b0 + b0 x - 2 b1 x + b0 b1 y1 + y β^2 - y1 β^2) +
             a1 (b0 x - b0^2 y1 + (-y + y1) β^2) + (b0 - b1) (-x^2 + b0 x y1 + (y^2 - y y1 + z^2) β^2)) /
          (√(a0^2 - 2 a0 x + x^2 - (y^2 + z^2) β^2 + y1^2 (b0^2 - β^2) + 2 y1 (a0 b0 - b0 x + y β^2))
            √(a1^2 (b0^2 - β^2) + a0^2 (b1^2 - β^2) + 2 a0 (b0 - b1) (b1 x - y β^2) + (b0 - b1)^2
                (x^2 - (y^2 + z^2) β^2) - 2 a1 (a0 (b0 b1 - β^2) + (b0 - b1) (b0 x - y β^2))))]) /
      (2 √(a1^2 (b0^2 - β^2) + a0^2 (b1^2 - β^2) + 2 a0 (b0 - b1) (b1 x - y β^2) +
            (b0 - b1)^2 (x^2 - (y^2 + z^2) β^2) - 2 a1 (a0 (b0 b1 - β^2) + (b0 - b1) (b0 x - y β^2)))) -
    (z (-2 a0 (b0 - b1) β^2 + 2 a1 (b0 - b1) β^2 - 2 (b0 - b1)^2 y β^2)
       ArcTanh[(-a1^2 b0 + a0 (-b1 x + b1^2 y1 + (y - y1) β^2) + a1 (a0 b1 + 2 b0 x - b1 x -
                b0 b1 y1 - y β^2 + y1 β^2) + (b0 - b1) (-x^2 + b1 x y1 + (y^2 - y y1 + z^2) β^2)) /
          (√(a1^2 - 2 a1 x + x^2 - (y^2 + z^2) β^2 + y1^2 (b1^2 - β^2) + 2 y1 (a1 b1 - b1 x + y β^2))
            √(a1^2 (b0^2 - β^2) + a0^2 (b1^2 - β^2) + 2 a0 (b0 - b1) (b1 x - y β^2) + (b0 - b1)^2
                (x^2 - (y^2 + z^2) β^2) - 2 a1 (a0 (b0 b1 - β^2) + (b0 - b1) (b0 x - y β^2))))]) /
      (2 √(a1^2 (b0^2 - β^2) + a0^2 (b1^2 - β^2) + 2 a0 (b0 - b1) (b1 x - y β^2) + (b0 - b1)^2
            (x^2 - (y^2 + z^2) β^2) - 2 a1 (a0 (b0 b1 - β^2) + (b0 - b1) (b0 x - y β^2)))))) /
(2 π (a0^2 + a1^2 + 2 a1 (-b0 + b1) y - 2 a0 (a1 + (-b0 + b1) y) +
    (b0 - b1)^2
     (y^2 + z^2))) - ((-2
     a0
     (-b0 + b1) + 2
     a1
     (-b0 + b1) + 2
     (b0 - b1)^2
     y) (μ0 - μ1)
  ((a0^2 + a1 (x - b0 y) - a0 (a1 + x - 2 b0 y + b1 y) + (b0 - b1) (-x y + b0 (y^2 + z^2)))
     ArcTan[
      (a0 (y - y0) + x (-y + y0) + b0 (y^2 - y y0 + z^2)) /
       (z √(a0^2 - 2 a0 x + x^2 - (y^2 + z^2) β^2 + y0^2 (b0^2 - β^2) + 2 y0 (a0 b0 - b0 x + y β^2)))] -
    (a0^2 + a1 (x - b0 y) - a0 (a1 + x - 2 b0 y + b1 y) + (b0 - b1) (-x y + b0 (y^2 + z^2)))
```

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 ArcTan[
   (a0 (y - y1) + x (-y + y1) + b0 (y^2 - y y1 + z^2)) /
    (z √(a0^2 - 2 a0 x + x^2 - (y^2 + z^2) β^2 + y1^2 (b0^2 - β^2) + 2 y1 (a0 b0 - b0 x + y β^2)))] +
(a1^2 - a1 (x + (b0 - 2 b1) y) - a0 (a1 - x + b1 y) + (b0 - b1) (x y - b1 (y^2 + z^2)))
 ArcTan[
   (a1 (y - y0) + x (-y + y0) + b1 (y^2 - y y0 + z^2)) /
    (z √(a1^2 - 2 a1 x + x^2 - (y^2 + z^2) β^2 + y0^2 (b1^2 - β^2) + 2 y0 (a1 b1 - b1 x + y β^2)))] -
(a1^2 - a1 (x + (b0 - 2 b1) y) - a0 (a1 - x + b1 y) + (b0 - b1) (x y - b1 (y^2 + z^2)))
 ArcTan[
   (a1 (y - y1) + x (-y + y1) + b1 (y^2 - y y1 + z^2)) /
    (z √(a1^2 - 2 a1 x + x^2 - (y^2 + z^2) β^2 + y1^2 (b1^2 - β^2) + 2 y1 (a1 b1 - b1 x + y β^2)))] -
(a1 b0 - a0 b1 + (-b0 + b1) x) z ArcTanh[(a0 - x + b0 y0) /
    (√(a0^2 - 2 a0 x + x^2 + 2 a0 b0 y0 - 2 b0 x y0 +
          b0^2 y0^2 - y^2 β^2 + 2 y y0 β^2 - y0^2 β^2 - z^2 β^2))] +
(a1 b0 - a0 b1 + (-b0 + b1) x) z ArcTanh[(a0 - x + b0 y1) / (√(a0^2 - 2 a0 x +
          x^2 + 2 a0 b0 y1 - 2 b0 x y1 + b0^2 y1^2 - y^2 β^2 + 2 y y1 β^2 - y1^2 β^2 - z^2 β^2))] +
(a1 b0 - a0 b1 + (-b0 + b1) x) z ArcTanh[(a1 - x + b1 y0) /
    (√(a1^2 - 2 a1 x + x^2 - (y^2 + z^2) β^2 + y0^2 (b1^2 - β^2) + 2 y0 (a1 b1 - b1 x + y β^2)))] -
(a1 b0 - a0 b1 + (-b0 + b1) x) z ArcTanh[(a1 - x + b1 y1) /
    (√(a1^2 - 2 a1 x + x^2 - (y^2 + z^2) β^2 + y1^2 (b1^2 - β^2) + 2 y1 (a1 b1 - b1 x + y β^2)))] -
z √(a1^2 (b0^2 - β^2) + a0^2 (b1^2 - β^2) + 2 a0 (b0 - b1) (b1 x - y β^2) +
      (b0 - b1)^2 (x^2 - (y^2 + z^2) β^2) - 2 a1 (a0 (b0 b1 - β^2) + (b0 - b1) (b0 x - y β^2)))
 ArcTanh[(a0^2 b1 + a0 (-a1 b0 + b0 x - 2 b1 x + b0 b1 y0 + y β^2 - y0 β^2) +
       a1 (b0 x - b0^2 y0 + (-y + y0) β^2) + (b0 - b1) (-x^2 + b0 x y0 + (y^2 - y y0 + z^2) β^2)) /
    (√(a0^2 - 2 a0 x + x^2 - (y^2 + z^2) β^2 + y0^2 (b0^2 - β^2) + 2 y0 (a0 b0 - b0 x + y β^2))
      √(a1^2 (b0^2 - β^2) + a0^2 (b1^2 - β^2) + 2 a0 (b0 - b1) (b1 x - y β^2) + (b0 - b1)^2
          (x^2 - (y^2 + z^2) β^2) - 2 a1 (a0 (b0 b1 - β^2) + (b0 - b1) (b0 x - y β^2))))] +
z √(a1^2 (b0^2 - β^2) + a0^2 (b1^2 - β^2) + 2 a0 (b0 - b1) (b1 x - y β^2) +
      (b0 - b1)^2 (x^2 - (y^2 + z^2) β^2) - 2 a1 (a0 (b0 b1 - β^2) + (b0 - b1) (b0 x - y β^2)))
 ArcTanh[(-a1^2 b0 + a0 (-b1 x + b1^2 y0 + (y - y0) β^2) +
       a1 (a0 b1 + 2 b0 x - b1 x - b0 b1 y0 - y β^2 + y0 β^2) +
       (b0 - b1) (-x^2 + b1 x y0 + (y^2 - y y0 + z^2) β^2)) /
    (√(a1^2 - 2 a1 x + x^2 - (y^2 + z^2) β^2 + y0^2 (b1^2 - β^2) + 2 y0 (a1 b1 - b1 x + y β^2))
      √(a1^2 (b0^2 - β^2) + a0^2 (b1^2 - β^2) + 2 a0 (b0 - b1) (b1 x - y β^2) + (b0 - b1)^2
          (x^2 - (y^2 + z^2) β^2) - 2 a1 (a0 (b0 b1 - β^2) + (b0 - b1) (b0 x - y β^2))))] +
z √(a1^2 (b0^2 - β^2) + a0^2 (b1^2 - β^2) + 2 a0 (b0 - b1) (b1 x - y β^2) +
      (b0 - b1)^2 (x^2 - (y^2 + z^2) β^2) - 2 a1 (a0 (b0 b1 - β^2) + (b0 - b1) (b0 x - y β^2)))
 ArcTanh[(a0^2 b1 + a0 (-a1 b0 + b0 x - 2 b1 x + b0 b1 y1 + y β^2 - y1 β^2) +
       a1 (b0 x - b0^2 y1 + (-y + y1) β^2) + (b0 - b1) (-x^2 + b0 x y1 + (y^2 - y y1 + z^2) β^2)) /
    (√(a0^2 - 2 a0 x + x^2 - (y^2 + z^2) β^2 + y1^2 (b0^2 - β^2) + 2 y1 (a0 b0 - b0 x + y β^2))
      √(a1^2 (b0^2 - β^2) + a0^2 (b1^2 - β^2) + 2 a0 (b0 - b1) (b1 x - y β^2) + (b0 - b1)^2
          (x^2 - (y^2 + z^2) β^2) - 2 a1 (a0 (b0 b1 - β^2) + (b0 - b1) (b0 x - y β^2))))] -
z √(a1^2 (b0^2 - β^2) + a0^2 (b1^2 - β^2) + 2 a0 (b0 - b1) (b1 x - y β^2) +
```

```
        (b0 - b1)^2 (x^2 - (y^2 + z^2) β^2) - 2 a1 (a0 (b0 b1 - β^2) + (b0 - b1) (b0 x - y β^2)))
    ArcTanh[(-a1^2 b0 + a0 (-b1 x + b1^2 y1 + (y - y1) β^2) +
          a1 (a0 b1 + 2 b0 x - b1 x - b0 b1 y1 - y β^2 + y1 β^2) +
          (b0 - b1) (-x^2 + b1 x y1 + (y^2 - y y1 + z^2) β^2)) /
        (√(a1^2 - 2 a1 x + x^2 - (y^2 + z^2) β^2 + y1^2 (b1^2 - β^2) + 2 y1 (a1 b1 - b1 x + y β^2))
          √(a1^2 (b0^2 - β^2) + a0^2 (b1^2 - β^2) + 2 a0 (b0 - b1) (b1 x - y β^2) + (b0 - b1)^2
              (x^2 - (y^2 + z^2) β^2) - 2 a1 (a0 (b0 b1 - β^2) + (b0 - b1) (b0 x - y β^2))))])) /
  (2 π (a0^2 + a1^2 + 2 a1 (-b0 + b1) y - 2 a0 (a1 + (-b0 + b1) y) + (b0 - b1)^2 (y^2 + z^2))^2)
```

(*Tapered Supersonic Doublet Panel Perturbation Velocity w Component*)

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In[378]:= w = D[ϕ[x, y, z], z]
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```
((μ0 - μ1) (-(((a1 b0 - a0 b1 + (-b0 + b1) x) (a0 - x + b0 y0) z^2 β^2) /
          ((a0^2 - 2 a0 x + x^2 + 2 a0 b0 y0 - 2 b0 x y0 + b0^2 y0^2 - y^2 β^2 + 2 y y0 β^2 - y0^2 β^2 - z^2 β^2)^(3/2)
            (1 - (a0 - x + b0 y0)^2 / (a0^2 - 2 a0 x + x^2 + 2 a0 b0 y0 -
                  2 b0 x y0 + b0^2 y0^2 - y^2 β^2 + 2 y y0 β^2 - y0^2 β^2 - z^2 β^2)))) +
      ((a1 b0 - a0 b1 + (-b0 + b1) x) (a0 - x + b0 y1) z^2 β^2) /
        ((a0^2 - 2 a0 x + x^2 + 2 a0 b0 y1 - 2 b0 x y1 + b0^2 y1^2 - y^2 β^2 + 2 y y1 β^2 - y1^2 β^2 - z^2 β^2)^(3/2)
          (1 - (a0 - x + b0 y1)^2 / (a0^2 - 2 a0 x + x^2 + 2 a0 b0 y1 -
                2 b0 x y1 + b0^2 y1^2 - y^2 β^2 + 2 y y1 β^2 - y1^2 β^2 - z^2 β^2))) +
      ((a0^2 + a1 (x - b0 y) - a0 (a1 + x - 2 b0 y + b1 y) + (b0 - b1) (-x y + b0 (y^2 + z^2)))
          (((a0 (y - y0) + x (-y + y0) + b0 (y^2 - y y0 + z^2)) β^2) / (a0^2 - 2 a0 x + x^2 -
                (y^2 + z^2) β^2 + y0^2 (b0^2 - β^2) + 2 y0 (a0 b0 - b0 x + y β^2))^(3/2) + (2 b0) /
              (√(a0^2 - 2 a0 x + x^2 - (y^2 + z^2) β^2 + y0^2 (b0^2 - β^2) + 2 y0 (a0 b0 - b0 x + y β^2))) -
            (a0 (y - y0) + x (-y + y0) + b0 (y^2 - y y0 + z^2)) / (z^2 √(a0^2 - 2 a0 x +
                  x^2 - (y^2 + z^2) β^2 + y0^2 (b0^2 - β^2) + 2 y0 (a0 b0 - b0 x + y β^2))))) /
        (1 + (a0 (y - y0) + x (-y + y0) + b0 (y^2 - y y0 + z^2))^2 /
            (z^2 (a0^2 - 2 a0 x + x^2 - (y^2 + z^2) β^2 + y0^2 (b0^2 - β^2) + 2 y0 (a0 b0 - b0 x + y β^2)))) -
      ((a0^2 + a1 (x - b0 y) - a0 (a1 + x - 2 b0 y + b1 y) + (b0 - b1) (-x y + b0 (y^2 + z^2)))
          (((a0 (y - y1) + x (-y + y1) + b0 (y^2 - y y1 + z^2)) β^2) / (a0^2 - 2 a0 x + x^2 -
                (y^2 + z^2) β^2 + y1^2 (b0^2 - β^2) + 2 y1 (a0 b0 - b0 x + y β^2))^(3/2) + (2 b0) /
              (√(a0^2 - 2 a0 x + x^2 - (y^2 + z^2) β^2 + y1^2 (b0^2 - β^2) + 2 y1 (a0 b0 - b0 x + y β^2))) -
            (a0 (y - y1) + x (-y + y1) + b0 (y^2 - y y1 + z^2)) / (z^2 √(a0^2 - 2 a0 x +
                  x^2 - (y^2 + z^2) β^2 + y1^2 (b0^2 - β^2) + 2 y1 (a0 b0 - b0 x + y β^2))))) /
        (1 + (a0 (y - y1) + x (-y + y1) + b0 (y^2 - y y1 + z^2))^2 /
            (z^2 (a0^2 - 2 a0 x + x^2 - (y^2 + z^2) β^2 + y1^2 (b0^2 - β^2) + 2 y1 (a0 b0 - b0 x + y β^2)))) +
      ((a1 b0 - a0 b1 + (-b0 + b1) x) (a1 - x + b1 y0) z^2 β^2) /
        ((a1^2 - 2 a1 x + x^2 - (y^2 + z^2) β^2 + y0^2 (b1^2 - β^2) + 2 y0 (a1 b1 - b1 x + y β^2))^(3/2)
          (1 - (a1 - x + b1 y0)^2 /
```

$$(a1^2 - 2\,a1\,x + x^2 - (y^2 + z^2)\,\beta^2 + y0^2\,(b1^2 - \beta^2) + 2\,y0\,(a1\,b1 - b1\,x + y\,\beta^2))))) +$$
$$((a1^2 - a1\,(x + (b0 - 2\,b1)\,y) - a0\,(a1 - x + b1\,y) + (b0 - b1)\,(x\,y - b1\,(y^2 + z^2)))$$
$$(((a1\,(y - y0) + x\,(-y + y0) + b1\,(y^2 - y\,y0 + z^2))\,\beta^2) / (a1^2 - 2\,a1\,x + x^2 -$$
$$(y^2 + z^2)\,\beta^2 + y0^2\,(b1^2 - \beta^2) + 2\,y0\,(a1\,b1 - b1\,x + y\,\beta^2))^{3/2} + (2\,b1) /$$
$$(\surd(a1^2 - 2\,a1\,x + x^2 - (y^2 + z^2)\,\beta^2 + y0^2\,(b1^2 - \beta^2) + 2\,y0\,(a1\,b1 - b1\,x + y\,\beta^2))) -$$
$$(a1\,(y - y0) + x\,(-y + y0) + b1\,(y^2 - y\,y0 + z^2)) / (z^2\,\surd(a1^2 - 2\,a1\,x +$$
$$x^2 - (y^2 + z^2)\,\beta^2 + y0^2\,(b1^2 - \beta^2) + 2\,y0\,(a1\,b1 - b1\,x + y\,\beta^2)))))) /$$
$$(1 + (a1\,(y - y0) + x\,(-y + y0) + b1\,(y^2 - y\,y0 + z^2))^2 /$$
$$(z^2\,(a1^2 - 2\,a1\,x + x^2 - (y^2 + z^2)\,\beta^2 + y0^2\,(b1^2 - \beta^2) + 2\,y0\,(a1\,b1 - b1\,x + y\,\beta^2))))) -$$
$$((a1\,b0 - a0\,b1 + (-b0 + b1)\,x)\,(a1 - x + b1\,y1)\,z^2\,\beta^2) /$$
$$((a1^2 - 2\,a1\,x + x^2 - (y^2 + z^2)\,\beta^2 + y1^2\,(b1^2 - \beta^2) + 2\,y1\,(a1\,b1 - b1\,x + y\,\beta^2))^{3/2}$$
$$(1 - (a1 - x + b1\,y1)^2 /$$
$$(a1^2 - 2\,a1\,x + x^2 - (y^2 + z^2)\,\beta^2 + y1^2\,(b1^2 - \beta^2) + 2\,y1\,(a1\,b1 - b1\,x + y\,\beta^2))))) -$$
$$((a1^2 - a1\,(x + (b0 - 2\,b1)\,y) - a0\,(a1 - x + b1\,y) + (b0 - b1)\,(x\,y - b1\,(y^2 + z^2)))$$
$$(((a1\,(y - y1) + x\,(-y + y1) + b1\,(y^2 - y\,y1 + z^2))\,\beta^2) / (a1^2 - 2\,a1\,x + x^2 -$$
$$(y^2 + z^2)\,\beta^2 + y1^2\,(b1^2 - \beta^2) + 2\,y1\,(a1\,b1 - b1\,x + y\,\beta^2))^{3/2} + (2\,b1) /$$
$$(\surd(a1^2 - 2\,a1\,x + x^2 - (y^2 + z^2)\,\beta^2 + y1^2\,(b1^2 - \beta^2) + 2\,y1\,(a1\,b1 - b1\,x + y\,\beta^2))) -$$
$$(a1\,(y - y1) + x\,(-y + y1) + b1\,(y^2 - y\,y1 + z^2)) / (z^2\,\surd(a1^2 - 2\,a1\,x +$$
$$x^2 - (y^2 + z^2)\,\beta^2 + y1^2\,(b1^2 - \beta^2) + 2\,y1\,(a1\,b1 - b1\,x + y\,\beta^2)))))) /$$
$$(1 + (a1\,(y - y1) + x\,(-y + y1) + b1\,(y^2 - y\,y1 + z^2))^2 /$$
$$(z^2\,(a1^2 - 2\,a1\,x + x^2 - (y^2 + z^2)\,\beta^2 + y1^2\,(b1^2 - \beta^2) + 2\,y1\,(a1\,b1 - b1\,x + y\,\beta^2))))) -$$
$$(z\,\surd(a1^2\,(b0^2 - \beta^2) + a0^2\,(b1^2 - \beta^2) + 2\,a0\,(b0 - b1)\,(b1\,x - y\,\beta^2) +$$
$$(b0 - b1)^2\,(x^2 - (y^2 + z^2)\,\beta^2) - 2\,a1\,(a0\,(b0\,b1 - \beta^2) + (b0 - b1)\,(b0\,x - y\,\beta^2)))$$
$$(((b0 - b1)^2\,z\,\beta^2\,(a0^2\,b1 + a0\,(-a1\,b0 + b0\,x - 2\,b1\,x + b0\,b1\,y0 + y\,\beta^2 - y0\,\beta^2) +$$
$$a1\,(b0\,x - b0^2\,y0 + (-y + y0)\,\beta^2) +$$
$$(b0 - b1)\,(-x^2 + b0\,x\,y0 + (y^2 - y\,y0 + z^2)\,\beta^2))) / (\surd(a0^2 - 2\,a0\,x + x^2 -$$
$$(y^2 + z^2)\,\beta^2 + y0^2\,(b0^2 - \beta^2) + 2\,y0\,(a0\,b0 - b0\,x + y\,\beta^2))\,(a1^2\,(b0^2 - \beta^2) +$$
$$a0^2\,(b1^2 - \beta^2) + 2\,a0\,(b0 - b1)\,(b1\,x - y\,\beta^2) + (b0 - b1)^2\,(x^2 - (y^2 + z^2)\,\beta^2) -$$
$$2\,a1\,(a0\,(b0\,b1 - \beta^2) + (b0 - b1)\,(b0\,x - y\,\beta^2)))^{3/2}) + (2\,(b0 - b1)\,z\,\beta^2) /$$
$$(\surd(a0^2 - 2\,a0\,x + x^2 - (y^2 + z^2)\,\beta^2 + y0^2\,(b0^2 - \beta^2) + 2\,y0\,(a0\,b0 - b0\,x + y\,\beta^2))$$
$$\surd(a1^2\,(b0^2 - \beta^2) + a0^2\,(b1^2 - \beta^2) + 2\,a0\,(b0 - b1)\,(b1\,x - y\,\beta^2) + (b0 - b1)^2$$
$$(x^2 - (y^2 + z^2)\,\beta^2) - 2\,a1\,(a0\,(b0\,b1 - \beta^2) + (b0 - b1)\,(b0\,x - y\,\beta^2)))) +$$
$$(z\,\beta^2\,(a0^2\,b1 + a0\,(-a1\,b0 + b0\,x - 2\,b1\,x + b0\,b1\,y0 + y\,\beta^2 - y0\,\beta^2) + a1\,(b0\,x -$$
$$b0^2\,y0 + (-y + y0)\,\beta^2) + (b0 - b1)\,(-x^2 + b0\,x\,y0 + (y^2 - y\,y0 + z^2)\,\beta^2))) /$$
$$((a0^2 - 2\,a0\,x + x^2 - (y^2 + z^2)\,\beta^2 + y0^2\,(b0^2 - \beta^2) + 2\,y0\,(a0\,b0 - b0\,x + y\,\beta^2))^{3/2}$$
$$\surd(a1^2\,(b0^2 - \beta^2) + a0^2\,(b1^2 - \beta^2) + 2\,a0\,(b0 - b1)\,(b1\,x - y\,\beta^2) + (b0 - b1)^2$$
$$(x^2 - (y^2 + z^2)\,\beta^2) - 2\,a1\,(a0\,(b0\,b1 - \beta^2) + (b0 - b1)\,(b0\,x - y\,\beta^2)))))) /$$
$$(1 - (a0^2\,b1 + a0\,(-a1\,b0 + b0\,x - 2\,b1\,x + b0\,b1\,y0 + y\,\beta^2 - y0\,\beta^2) + a1\,(b0\,x -$$
$$b0^2\,y0 + (-y + y0)\,\beta^2) + (b0 - b1)\,(-x^2 + b0\,x\,y0 + (y^2 - y\,y0 + z^2)\,\beta^2))^2 /$$

$$
\begin{aligned}
&((a0^2 - 2\,a0\,x + x^2 - (y^2 + z^2)\,\beta^2 + y0^2\,(b0^2 - \beta^2) + 2\,y0\,(a0\,b0 - b0\,x + y\,\beta^2)) \\
&\quad (a1^2\,(b0^2 - \beta^2) + a0^2\,(b1^2 - \beta^2) + 2\,a0\,(b0 - b1)\,(b1\,x - y\,\beta^2) + (b0 - b1)^2 \\
&\qquad (x^2 - (y^2 + z^2)\,\beta^2) - 2\,a1\,(a0\,(b0\,b1 - \beta^2) + (b0 - b1)\,(b0\,x - y\,\beta^2))))) + \\
&(z\,\surd(a1^2\,(b0^2 - \beta^2) + a0^2\,(b1^2 - \beta^2) + 2\,a0\,(b0 - b1)\,(b1\,x - y\,\beta^2) + \\
&\qquad (b0 - b1)^2\,(x^2 - (y^2 + z^2)\,\beta^2) - 2\,a1\,(a0\,(b0\,b1 - \beta^2) + (b0 - b1)\,(b0\,x - y\,\beta^2))) \\
&\quad (((b0 - b1)^2\,z\,\beta^2\,(-a1^2\,b0 + a0\,(-b1\,x + b1^2\,y0 + (y - y0)\,\beta^2) + \\
&\qquad a1\,(a0\,b1 + 2\,b0\,x - b1\,x - b0\,b1\,y0 - y\,\beta^2 + y0\,\beta^2) + \\
&\qquad (b0 - b1)\,(-x^2 + b1\,x\,y0 + (y^2 - y\,y0 + z^2)\,\beta^2))) \Big/ \\
&\quad (\surd(a1^2 - 2\,a1\,x + x^2 - (y^2 + z^2)\,\beta^2 + y0^2\,(b1^2 - \beta^2) + 2\,y0\,(a1\,b1 - b1\,x + y\,\beta^2)) \\
&\qquad (a1^2\,(b0^2 - \beta^2) + a0^2\,(b1^2 - \beta^2) + 2\,a0\,(b0 - b1)\,(b1\,x - y\,\beta^2) + (b0 - b1)^2 \\
&\qquad (x^2 - (y^2 + z^2)\,\beta^2) - 2\,a1\,(a0\,(b0\,b1 - \beta^2) + (b0 - b1)\,(b0\,x - y\,\beta^2)))^{3/2}) + \\
&\quad (2\,(b0 - b1)\,z\,\beta^2) \Big/ (\surd(a1^2 - 2\,a1\,x + x^2 - (y^2 + z^2)\,\beta^2 + y0^2\,(b1^2 - \beta^2) + \\
&\qquad 2\,y0\,(a1\,b1 - b1\,x + y\,\beta^2))\,\surd(a1^2\,(b0^2 - \beta^2) + a0^2\,(b1^2 - \beta^2) + \\
&\qquad 2\,a0\,(b0 - b1)\,(b1\,x - y\,\beta^2) + (b0 - b1)^2\,(x^2 - (y^2 + z^2)\,\beta^2) - \\
&\qquad 2\,a1\,(a0\,(b0\,b1 - \beta^2) + (b0 - b1)\,(b0\,x - y\,\beta^2)))) + \\
&\quad (z\,\beta^2\,(-a1^2\,b0 + a0\,(-b1\,x + b1^2\,y0 + (y - y0)\,\beta^2) + a1\,(a0\,b1 + 2\,b0\,x - b1\,x - \\
&\qquad b0\,b1\,y0 - y\,\beta^2 + y0\,\beta^2) + (b0 - b1)\,(-x^2 + b1\,x\,y0 + (y^2 - y\,y0 + z^2)\,\beta^2))) \Big/ \\
&\quad ((a1^2 - 2\,a1\,x + x^2 - (y^2 + z^2)\,\beta^2 + y0^2\,(b1^2 - \beta^2) + 2\,y0\,(a1\,b1 - b1\,x + y\,\beta^2))^{3/2} \\
&\qquad \surd(a1^2\,(b0^2 - \beta^2) + a0^2\,(b1^2 - \beta^2) + 2\,a0\,(b0 - b1)\,(b1\,x - y\,\beta^2) + (b0 - b1)^2 \\
&\qquad (x^2 - (y^2 + z^2)\,\beta^2) - 2\,a1\,(a0\,(b0\,b1 - \beta^2) + (b0 - b1)\,(b0\,x - y\,\beta^2)))))) \Big/ \\
&(1 - (-a1^2\,b0 + a0\,(-b1\,x + b1^2\,y0 + (y - y0)\,\beta^2) + a1\,(a0\,b1 + 2\,b0\,x - b1\,x - \\
&\qquad b0\,b1\,y0 - y\,\beta^2 + y0\,\beta^2) + (b0 - b1)\,(-x^2 + b1\,x\,y0 + (y^2 - y\,y0 + z^2)\,\beta^2))^2 \Big/ \\
&\quad ((a1^2 - 2\,a1\,x + x^2 - (y^2 + z^2)\,\beta^2 + y0^2\,(b1^2 - \beta^2) + 2\,y0\,(a1\,b1 - b1\,x + y\,\beta^2)) \\
&\qquad (a1^2\,(b0^2 - \beta^2) + a0^2\,(b1^2 - \beta^2) + 2\,a0\,(b0 - b1)\,(b1\,x - y\,\beta^2) + (b0 - b1)^2 \\
&\qquad (x^2 - (y^2 + z^2)\,\beta^2) - 2\,a1\,(a0\,(b0\,b1 - \beta^2) + (b0 - b1)\,(b0\,x - y\,\beta^2))))) + \\
&(z\,\surd(a1^2\,(b0^2 - \beta^2) + a0^2\,(b1^2 - \beta^2) + 2\,a0\,(b0 - b1)\,(b1\,x - y\,\beta^2) + \\
&\qquad (b0 - b1)^2\,(x^2 - (y^2 + z^2)\,\beta^2) - 2\,a1\,(a0\,(b0\,b1 - \beta^2) + (b0 - b1)\,(b0\,x - y\,\beta^2))) \\
&\quad (((b0 - b1)^2\,z\,\beta^2\,(a0^2\,b1 + a0\,(-a1\,b0 + b0\,x - 2\,b1\,x + b0\,b1\,y1 + y\,\beta^2 - y1\,\beta^2) + \\
&\qquad a1\,(b0\,x - b0^2\,y1 + (-y + y1)\,\beta^2) + \\
&\qquad (b0 - b1)\,(-x^2 + b0\,x\,y1 + (y^2 - y\,y1 + z^2)\,\beta^2))) \Big/ \\
&\quad (\surd(a0^2 - 2\,a0\,x + x^2 - (y^2 + z^2)\,\beta^2 + y1^2\,(b0^2 - \beta^2) + 2\,y1\,(a0\,b0 - b0\,x + y\,\beta^2)) \\
&\qquad (a1^2\,(b0^2 - \beta^2) + a0^2\,(b1^2 - \beta^2) + 2\,a0\,(b0 - b1)\,(b1\,x - y\,\beta^2) + (b0 - b1)^2 \\
&\qquad (x^2 - (y^2 + z^2)\,\beta^2) - 2\,a1\,(a0\,(b0\,b1 - \beta^2) + (b0 - b1)\,(b0\,x - y\,\beta^2)))^{3/2}) + \\
&\quad (2\,(b0 - b1)\,z\,\beta^2) \Big/ (\surd(a0^2 - 2\,a0\,x + x^2 - (y^2 + z^2)\,\beta^2 + y1^2\,(b0^2 - \beta^2) + \\
&\qquad 2\,y1\,(a0\,b0 - b0\,x + y\,\beta^2))\,\surd(a1^2\,(b0^2 - \beta^2) + a0^2\,(b1^2 - \beta^2) + \\
&\qquad 2\,a0\,(b0 - b1)\,(b1\,x - y\,\beta^2) + (b0 - b1)^2\,(x^2 - (y^2 + z^2)\,\beta^2) - \\
&\qquad 2\,a1\,(a0\,(b0\,b1 - \beta^2) + (b0 - b1)\,(b0\,x - y\,\beta^2)))) + \\
&\quad (z\,\beta^2\,(a0^2\,b1 + a0\,(-a1\,b0 + b0\,x - 2\,b1\,x + b0\,b1\,y1 + y\,\beta^2 - y1\,\beta^2) + a1\,(b0\,x - \\
&\qquad b0^2\,y1 + (-y + y1)\,\beta^2) + (b0 - b1)\,(-x^2 + b0\,x\,y1 + (y^2 - y\,y1 + z^2)\,\beta^2))) \Big/
\end{aligned}
$$

$$((a0^2 - 2\,a0\,x + x^2 - (y^2 + z^2)\,\beta^2 + y1^2\,(b0^2 - \beta^2) + 2\,y1\,(a0\,b0 - b0\,x + y\,\beta^2))^{3/2}$$
$$\surd(a1^2\,(b0^2 - \beta^2) + a0^2\,(b1^2 - \beta^2) + 2\,a0\,(b0 - b1)\,(b1\,x - y\,\beta^2) + (b0 - b1)^2$$
$$(x^2 - (y^2 + z^2)\,\beta^2) - 2\,a1\,(a0\,(b0\,b1 - \beta^2) + (b0 - b1)\,(b0\,x - y\,\beta^2)))))) \Big/$$
$$(1 - (a0^2\,b1 + a0\,(-a1\,b0 + b0\,x - 2\,b1\,x + b0\,b1\,y1 + y\,\beta^2 - y1\,\beta^2) + a1\,(b0\,x -$$
$$b0^2\,y1 + (-y + y1)\,\beta^2) + (b0 - b1)\,(-x^2 + b0\,x\,y1 + (y^2 - y\,y1 + z^2)\,\beta^2))^2 \Big/$$
$$((a0^2 - 2\,a0\,x + x^2 - (y^2 + z^2)\,\beta^2 + y1^2\,(b0^2 - \beta^2) + 2\,y1\,(a0\,b0 - b0\,x + y\,\beta^2))$$
$$(a1^2\,(b0^2 - \beta^2) + a0^2\,(b1^2 - \beta^2) + 2\,a0\,(b0 - b1)\,(b1\,x - y\,\beta^2) + (b0 - b1)^2$$
$$(x^2 - (y^2 + z^2)\,\beta^2) - 2\,a1\,(a0\,(b0\,b1 - \beta^2) + (b0 - b1)\,(b0\,x - y\,\beta^2))))) -$$
$$(z\,\surd(a1^2\,(b0^2 - \beta^2) + a0^2\,(b1^2 - \beta^2) + 2\,a0\,(b0 - b1)\,(b1\,x - y\,\beta^2) +$$
$$(b0 - b1)^2\,(x^2 - (y^2 + z^2)\,\beta^2) - 2\,a1\,(a0\,(b0\,b1 - \beta^2) + (b0 - b1)\,(b0\,x - y\,\beta^2)))$$
$$(((b0 - b1)^2\,z\,\beta^2\,(-a1^2\,b0 + a0\,(-b1\,x + b1^2\,y1 + (y - y1)\,\beta^2) +$$
$$a1\,(a0\,b1 + 2\,b0\,x - b1\,x - b0\,b1\,y1 - y\,\beta^2 + y1\,\beta^2) +$$
$$(b0 - b1)\,(-x^2 + b1\,x\,y1 + (y^2 - y\,y1 + z^2)\,\beta^2))) \Big/$$
$$(\surd(a1^2 - 2\,a1\,x + x^2 - (y^2 + z^2)\,\beta^2 + y1^2\,(b1^2 - \beta^2) + 2\,y1\,(a1\,b1 - b1\,x + y\,\beta^2))$$
$$(a1^2\,(b0^2 - \beta^2) + a0^2\,(b1^2 - \beta^2) + 2\,a0\,(b0 - b1)\,(b1\,x - y\,\beta^2) + (b0 - b1)^2$$
$$(x^2 - (y^2 + z^2)\,\beta^2) - 2\,a1\,(a0\,(b0\,b1 - \beta^2) + (b0 - b1)\,(b0\,x - y\,\beta^2)))^{3/2}) +$$
$$(2\,(b0 - b1)\,z\,\beta^2) \Big/ (\surd(a1^2 - 2\,a1\,x + x^2 - (y^2 + z^2)\,\beta^2 + y1^2\,(b1^2 - \beta^2) +$$
$$2\,y1\,(a1\,b1 - b1\,x + y\,\beta^2))\,\surd(a1^2\,(b0^2 - \beta^2) + a0^2\,(b1^2 - \beta^2) +$$
$$2\,a0\,(b0 - b1)\,(b1\,x - y\,\beta^2) + (b0 - b1)^2\,(x^2 - (y^2 + z^2)\,\beta^2) -$$
$$2\,a1\,(a0\,(b0\,b1 - \beta^2) + (b0 - b1)\,(b0\,x - y\,\beta^2)))) +$$
$$(z\,\beta^2\,(-a1^2\,b0 + a0\,(-b1\,x + b1^2\,y1 + (y - y1)\,\beta^2) + a1\,(a0\,b1 + 2\,b0\,x - b1\,x -$$
$$b0\,b1\,y1 - y\,\beta^2 + y1\,\beta^2) + (b0 - b1)\,(-x^2 + b1\,x\,y1 + (y^2 - y\,y1 + z^2)\,\beta^2))) \Big/$$
$$((a1^2 - 2\,a1\,x + x^2 - (y^2 + z^2)\,\beta^2 + y1^2\,(b1^2 - \beta^2) + 2\,y1\,(a1\,b1 - b1\,x + y\,\beta^2))^{3/2}$$
$$\surd(a1^2\,(b0^2 - \beta^2) + a0^2\,(b1^2 - \beta^2) + 2\,a0\,(b0 - b1)\,(b1\,x - y\,\beta^2) + (b0 - b1)^2$$
$$(x^2 - (y^2 + z^2)\,\beta^2) - 2\,a1\,(a0\,(b0\,b1 - \beta^2) + (b0 - b1)\,(b0\,x - y\,\beta^2)))))) \Big/$$
$$(1 - (-a1^2\,b0 + a0\,(-b1\,x + b1^2\,y1 + (y - y1)\,\beta^2) + a1\,(a0\,b1 + 2\,b0\,x - b1\,x -$$
$$b0\,b1\,y1 - y\,\beta^2 + y1\,\beta^2) + (b0 - b1)\,(-x^2 + b1\,x\,y1 + (y^2 - y\,y1 + z^2)\,\beta^2))^2 \Big/$$
$$((a1^2 - 2\,a1\,x + x^2 - (y^2 + z^2)\,\beta^2 + y1^2\,(b1^2 - \beta^2) + 2\,y1\,(a1\,b1 - b1\,x + y\,\beta^2))$$
$$(a1^2\,(b0^2 - \beta^2) + a0^2\,(b1^2 - \beta^2) + 2\,a0\,(b0 - b1)\,(b1\,x - y\,\beta^2) + (b0 - b1)^2$$
$$(x^2 - (y^2 + z^2)\,\beta^2) - 2\,a1\,(a0\,(b0\,b1 - \beta^2) + (b0 - b1)\,(b0\,x - y\,\beta^2))))) +$$
$$2\,b0\,(b0 - b1)\,z\,\text{ArcTan}[(a0\,(y - y0) + x\,(-y + y0) + b0\,(y^2 - y\,y0 + z^2)) \Big/$$
$$(z\,\surd(a0^2 - 2\,a0\,x + x^2 - (y^2 + z^2)\,\beta^2 + y0^2\,(b0^2 - \beta^2) + 2\,y0\,(a0\,b0 - b0\,x + y\,\beta^2)))] -$$
$$2\,b0\,(b0 - b1)\,z\,\text{ArcTan}[(a0\,(y - y1) + x\,(-y + y1) + b0\,(y^2 - y\,y1 + z^2)) \Big/$$
$$(z\,\surd(a0^2 - 2\,a0\,x + x^2 - (y^2 + z^2)\,\beta^2 + y1^2\,(b0^2 - \beta^2) + 2\,y1\,(a0\,b0 - b0\,x + y\,\beta^2)))] -$$
$$2\,(b0 - b1)\,b1\,z\,\text{ArcTan}[(a1\,(y - y0) + x\,(-y + y0) + b1\,(y^2 - y\,y0 + z^2)) \Big/$$
$$(z\,\surd(a1^2 - 2\,a1\,x + x^2 - (y^2 + z^2)\,\beta^2 + y0^2\,(b1^2 - \beta^2) + 2\,y0\,(a1\,b1 - b1\,x + y\,\beta^2)))] +$$
$$2\,(b0 - b1)\,b1\,z\,\text{ArcTan}[(a1\,(y - y1) + x\,(-y + y1) + b1\,(y^2 - y\,y1 + z^2)) \Big/$$
$$(z\,\surd(a1^2 - 2\,a1\,x + x^2 - (y^2 + z^2)\,\beta^2 + y1^2\,(b1^2 - \beta^2) + 2\,y1\,(a1\,b1 - b1\,x + y\,\beta^2)))] -$$
$$(a1\,b0 - a0\,b1 + (-b0 + b1)\,x)\,\text{ArcTanh}[(a0 - x + b0\,y0) \Big/$$
$$(\surd(a0^2 - 2\,a0\,x + x^2 + 2\,a0\,b0\,y0 - 2\,b0\,x\,y0 +$$

$$b0^2\, y0^2 - y^2\beta^2 + 2\, y\, y0\, \beta^2 - y0^2\beta^2 - z^2\beta^2)))] +$$
$$(a1\, b0 - a0\, b1 + (-b0 + b1)\, x)\, \text{ArcTanh}[(a0 - x + b0\, y1) / (\surd(a0^2 - 2\, a0\, x + x^2 +$$
$$2\, a0\, b0\, y1 - 2\, b0\, x\, y1 + b0^2\, y1^2 - y^2\beta^2 + 2\, y\, y1\, \beta^2 - y1^2\beta^2 - z^2\beta^2)))] +$$
$$(a1\, b0 - a0\, b1 + (-b0 + b1)\, x)\, \text{ArcTanh}[(a1 - x + b1\, y0) /$$
$$(\surd(a1^2 - 2\, a1\, x + x^2 - (y^2 + z^2)\,\beta^2 + y0^2\,(b1^2 - \beta^2) + 2\, y0\,(a1\, b1 - b1\, x + y\,\beta^2)))] -$$
$$(a1\, b0 - a0\, b1 + (-b0 + b1)\, x)\, \text{ArcTanh}[(a1 - x + b1\, y1) /$$
$$(\surd(a1^2 - 2\, a1\, x + x^2 - (y^2 + z^2)\,\beta^2 + y1^2\,(b1^2 - \beta^2) + 2\, y1\,(a1\, b1 - b1\, x + y\,\beta^2)))] +$$
$$((b0 - b1)^2\, z^2\beta^2\, \text{ArcTanh}[(a0^2\, b1 + a0\,(-a1\, b0 + b0\, x - 2\, b1\, x + b0\, b1\, y0 + y\,\beta^2 - y0\,\beta^2) +$$
$$a1\,(b0\, x - b0^2\, y0 + (-y + y0)\,\beta^2) + (b0 - b1)\,(-x^2 + b0\, x\, y0 + (y^2 - y\, y0 + z^2)\,\beta^2)) /$$
$$(\surd(a0^2 - 2\, a0\, x + x^2 - (y^2 + z^2)\,\beta^2 + y0^2\,(b0^2 - \beta^2) + 2\, y0\,(a0\, b0 - b0\, x + y\,\beta^2))$$
$$\surd(a1^2\,(b0^2 - \beta^2) + a0^2\,(b1^2 - \beta^2) + 2\, a0\,(b0 - b1)\,(b1\, x - y\,\beta^2) + (b0 - b1)^2$$
$$(x^2 - (y^2 + z^2)\,\beta^2) - 2\, a1\,(a0\,(b0\, b1 - \beta^2) + (b0 - b1)\,(b0\, x - y\,\beta^2)))))]) /$$
$$(\surd(a1^2\,(b0^2 - \beta^2) + a0^2\,(b1^2 - \beta^2) + 2\, a0\,(b0 - b1)\,(b1\, x - y\,\beta^2) +$$
$$(b0 - b1)^2\,(x^2 - (y^2 + z^2)\,\beta^2) - 2\, a1\,(a0\,(b0\, b1 - \beta^2) + (b0 - b1)\,(b0\, x - y\,\beta^2)))) -$$
$$\surd(a1^2\,(b0^2 - \beta^2) + a0^2\,(b1^2 - \beta^2) + 2\, a0\,(b0 - b1)\,(b1\, x - y\,\beta^2) +$$
$$(b0 - b1)^2\,(x^2 - (y^2 + z^2)\,\beta^2) - 2\, a1\,(a0\,(b0\, b1 - \beta^2) + (b0 - b1)\,(b0\, x - y\,\beta^2)))$$
$$\text{ArcTanh}[(a0^2\, b1 + a0\,(-a1\, b0 + b0\, x - 2\, b1\, x + b0\, b1\, y0 + y\,\beta^2 - y0\,\beta^2) +$$
$$a1\,(b0\, x - b0^2\, y0 + (-y + y0)\,\beta^2) + (b0 - b1)\,(-x^2 + b0\, x\, y0 + (y^2 - y\, y0 + z^2)\,\beta^2)) /$$
$$(\surd(a0^2 - 2\, a0\, x + x^2 - (y^2 + z^2)\,\beta^2 + y0^2\,(b0^2 - \beta^2) + 2\, y0\,(a0\, b0 - b0\, x + y\,\beta^2))$$
$$\surd(a1^2\,(b0^2 - \beta^2) + a0^2\,(b1^2 - \beta^2) + 2\, a0\,(b0 - b1)\,(b1\, x - y\,\beta^2) + (b0 - b1)^2$$
$$(x^2 - (y^2 + z^2)\,\beta^2) - 2\, a1\,(a0\,(b0\, b1 - \beta^2) + (b0 - b1)\,(b0\, x - y\,\beta^2)))))] -$$
$$((b0 - b1)^2\, z^2\beta^2\, \text{ArcTanh}[(-a1^2\, b0 + a0\,(-b1\, x + b1^2\, y0 + (y - y0)\,\beta^2) +$$
$$a1\,(a0\, b1 + 2\, b0\, x - b1\, x - b0\, b1\, y0 - y\,\beta^2 + y0\,\beta^2) +$$
$$(b0 - b1)\,(-x^2 + b1\, x\, y0 + (y^2 - y\, y0 + z^2)\,\beta^2)) /$$
$$(\surd(a1^2 - 2\, a1\, x + x^2 - (y^2 + z^2)\,\beta^2 + y0^2\,(b1^2 - \beta^2) + 2\, y0\,(a1\, b1 - b1\, x + y\,\beta^2))$$
$$\surd(a1^2\,(b0^2 - \beta^2) + a0^2\,(b1^2 - \beta^2) + 2\, a0\,(b0 - b1)\,(b1\, x - y\,\beta^2) + (b0 - b1)^2$$
$$(x^2 - (y^2 + z^2)\,\beta^2) - 2\, a1\,(a0\,(b0\, b1 - \beta^2) + (b0 - b1)\,(b0\, x - y\,\beta^2)))))]) /$$
$$(\surd(a1^2\,(b0^2 - \beta^2) + a0^2\,(b1^2 - \beta^2) + 2\, a0\,(b0 - b1)\,(b1\, x - y\,\beta^2) +$$
$$(b0 - b1)^2\,(x^2 - (y^2 + z^2)\,\beta^2) - 2\, a1\,(a0\,(b0\, b1 - \beta^2) + (b0 - b1)\,(b0\, x - y\,\beta^2)))) +$$
$$\surd(a1^2\,(b0^2 - \beta^2) + a0^2\,(b1^2 - \beta^2) + 2\, a0\,(b0 - b1)\,(b1\, x - y\,\beta^2) +$$
$$(b0 - b1)^2\,(x^2 - (y^2 + z^2)\,\beta^2) - 2\, a1\,(a0\,(b0\, b1 - \beta^2) + (b0 - b1)\,(b0\, x - y\,\beta^2)))$$
$$\text{ArcTanh}[(-a1^2\, b0 + a0\,(-b1\, x + b1^2\, y0 + (y - y0)\,\beta^2) +$$
$$a1\,(a0\, b1 + 2\, b0\, x - b1\, x - b0\, b1\, y0 - y\,\beta^2 + y0\,\beta^2) +$$
$$(b0 - b1)\,(-x^2 + b1\, x\, y0 + (y^2 - y\, y0 + z^2)\,\beta^2)) /$$
$$(\surd(a1^2 - 2\, a1\, x + x^2 - (y^2 + z^2)\,\beta^2 + y0^2\,(b1^2 - \beta^2) + 2\, y0\,(a1\, b1 - b1\, x + y\,\beta^2))$$
$$\surd(a1^2\,(b0^2 - \beta^2) + a0^2\,(b1^2 - \beta^2) + 2\, a0\,(b0 - b1)\,(b1\, x - y\,\beta^2) + (b0 - b1)^2$$
$$(x^2 - (y^2 + z^2)\,\beta^2) - 2\, a1\,(a0\,(b0\, b1 - \beta^2) + (b0 - b1)\,(b0\, x - y\,\beta^2)))))] -$$
$$((b0 - b1)^2\, z^2\beta^2\, \text{ArcTanh}[(a0^2\, b1 + a0\,(-a1\, b0 + b0\, x - 2\, b1\, x + b0\, b1\, y1 + y\,\beta^2 - y1\,\beta^2) +$$
$$a1\,(b0\, x - b0^2\, y1 + (-y + y1)\,\beta^2) + (b0 - b1)\,(-x^2 + b0\, x\, y1 + (y^2 - y\, y1 + z^2)\,\beta^2)) /$$
$$(\surd(a0^2 - 2\, a0\, x + x^2 - (y^2 + z^2)\,\beta^2 + y1^2\,(b0^2 - \beta^2) + 2\, y1\,(a0\, b0 - b0\, x + y\,\beta^2))$$
$$\surd(a1^2\,(b0^2 - \beta^2) + a0^2\,(b1^2 - \beta^2) + 2\, a0\,(b0 - b1)\,(b1\, x - y\,\beta^2) + (b0 - b1)^2$$
$$(x^2 - (y^2 + z^2)\,\beta^2) - 2\, a1\,(a0\,(b0\, b1 - \beta^2) + (b0 - b1)\,(b0\, x - y\,\beta^2)))))]) /$$
$$(\surd(a1^2\,(b0^2 - \beta^2) + a0^2\,(b1^2 - \beta^2) + 2\, a0\,(b0 - b1)\,(b1\, x - y\,\beta^2) +$$

```
        (b0 - b1)^2 (x^2 - (y^2 + z^2) β^2) - 2 a1 (a0 (b0 b1 - β^2) + (b0 - b1) (b0 x - y β^2)))) +
    √(a1^2 (b0^2 - β^2) + a0^2 (b1^2 - β^2) + 2 a0 (b0 - b1) (b1 x - y β^2) +
        (b0 - b1)^2 (x^2 - (y^2 + z^2) β^2) - 2 a1 (a0 (b0 b1 - β^2) + (b0 - b1) (b0 x - y β^2)))
     ArcTanh[(a0^2 b1 + a0 (-a1 b0 + b0 x - 2 b1 x + b0 b1 y1 + y β^2 - y1 β^2) +
          a1 (b0 x - b0^2 y1 + (-y + y1) β^2) + (b0 - b1) (-x^2 + b0 x y1 + (y^2 - y y1 + z^2) β^2)) /
        (√(a0^2 - 2 a0 x + x^2 - (y^2 + z^2) β^2 + y1^2 (b0^2 - β^2) + 2 y1 (a0 b0 - b0 x + y β^2))
          √(a1^2 (b0^2 - β^2) + a0^2 (b1^2 - β^2) + 2 a0 (b0 - b1) (b1 x - y β^2) + (b0 - b1)^2
              (x^2 - (y^2 + z^2) β^2) - 2 a1 (a0 (b0 b1 - β^2) + (b0 - b1) (b0 x - y β^2))))] +
    ((b0 - b1)^2 z^2 β^2 ArcTanh[(-a1^2 b0 + a0 (-b1 x + b1^2 y1 + (y - y1) β^2) +
           a1 (a0 b1 + 2 b0 x - b1 x - b0 b1 y1 - y β^2 + y1 β^2) +
           (b0 - b1) (-x^2 + b1 x y1 + (y^2 - y y1 + z^2) β^2)) /
         (√(a1^2 - 2 a1 x + x^2 - (y^2 + z^2) β^2 + y1^2 (b1^2 - β^2) + 2 y1 (a1 b1 - b1 x + y β^2))
           √(a1^2 (b0^2 - β^2) + a0^2 (b1^2 - β^2) + 2 a0 (b0 - b1) (b1 x - y β^2) + (b0 - b1)^2
               (x^2 - (y^2 + z^2) β^2) - 2 a1 (a0 (b0 b1 - β^2) + (b0 - b1) (b0 x - y β^2))))]) /
     (√(a1^2 (b0^2 - β^2) + a0^2 (b1^2 - β^2) + 2 a0 (b0 - b1) (b1 x - y β^2) +
          (b0 - b1)^2 (x^2 - (y^2 + z^2) β^2) - 2 a1 (a0 (b0 b1 - β^2) + (b0 - b1) (b0 x - y β^2)))) -
    √(a1^2 (b0^2 - β^2) + a0^2 (b1^2 - β^2) + 2 a0 (b0 - b1) (b1 x - y β^2) +
        (b0 - b1)^2 (x^2 - (y^2 + z^2) β^2) - 2 a1 (a0 (b0 b1 - β^2) + (b0 - b1) (b0 x - y β^2)))
     ArcTanh[(-a1^2 b0 + a0 (-b1 x + b1^2 y1 + (y - y1) β^2) +
          a1 (a0 b1 + 2 b0 x - b1 x - b0 b1 y1 - y β^2 + y1 β^2) +
          (b0 - b1) (-x^2 + b1 x y1 + (y^2 - y y1 + z^2) β^2)) /
        (√(a1^2 - 2 a1 x + x^2 - (y^2 + z^2) β^2 + y1^2 (b1^2 - β^2) + 2 y1 (a1 b1 - b1 x + y β^2))
          √(a1^2 (b0^2 - β^2) + a0^2 (b1^2 - β^2) + 2 a0 (b0 - b1) (b1 x - y β^2) + (b0 - b1)^2
              (x^2 - (y^2 + z^2) β^2) - 2 a1 (a0 (b0 b1 - β^2) + (b0 - b1) (b0 x - y β^2))))])) /
 (2 π (a0^2 + a1^2 + 2 a1 (-b0 + b1) y - 2 a0 (a1 + (-b0 + b1) y) +
      (b0 - b1)^2
       (y^2 + z^2))) -
((b0 - b1)^2 z (μ0 - μ1) ((a0^2 + a1 (x - b0 y) - a0 (a1 + x - 2 b0 y + b1 y) +
          (b0 - b1) (-x y + b0 (y^2 + z^2)))
     ArcTan[(a0 (y - y0) + x (-y + y0) + b0 (y^2 - y y0 + z^2)) /
        (z √(a0^2 - 2 a0 x + x^2 - (y^2 + z^2) β^2 + y0^2 (b0^2 - β^2) + 2 y0 (a0 b0 - b0 x + y β^2)))] -
    (a0^2 + a1 (x - b0 y) - a0 (a1 + x - 2 b0 y + b1 y) + (b0 - b1) (-x y + b0 (y^2 + z^2)))
     ArcTan[(a0 (y - y1) + x (-y + y1) + b0 (y^2 - y y1 + z^2)) /
        (z √(a0^2 - 2 a0 x + x^2 - (y^2 + z^2) β^2 + y1^2 (b0^2 - β^2) + 2 y1 (a0 b0 - b0 x + y β^2)))] +
    (a1^2 - a1 (x + (b0 - 2 b1) y) - a0 (a1 - x + b1 y) + (b0 - b1) (x y - b1 (y^2 + z^2)))
     ArcTan[(a1 (y - y0) + x (-y + y0) + b1 (y^2 - y y0 + z^2)) /
        (z √(a1^2 - 2 a1 x + x^2 - (y^2 + z^2) β^2 + y0^2 (b1^2 - β^2) + 2 y0 (a1 b1 - b1 x + y β^2)))] -
    (a1^2 - a1 (x + (b0 - 2 b1) y) - a0 (a1 - x + b1 y) + (b0 - b1) (x y - b1 (y^2 + z^2)))
     ArcTan[(a1 (y - y1) + x (-y + y1) + b1 (y^2 - y y1 + z^2)) /
        (z √(a1^2 - 2 a1 x + x^2 - (y^2 + z^2) β^2 + y1^2 (b1^2 - β^2) + 2 y1 (a1 b1 - b1 x + y β^2)))] -
    (a1 b0 - a0 b1 + (-b0 + b1) x) z ArcTanh[(a0 - x + b0 y0) /
        (√(a0^2 - 2 a0 x + x^2 + 2 a0 b0 y0 - 2 b0 x y0 +
```

$$
\begin{aligned}
&\qquad b0^2\, y0^2 - y^2 \beta^2 + 2\, y\, y0\, \beta^2 - y0^2 \beta^2 - z^2 \beta^2))] + \\
&(a1\, b0 - a0\, b1 + (-b0 + b1)\, x)\, z\, \mathrm{ArcTanh}[(a0 - x + b0\, y1) / (\surd(a0^2 - 2\, a0\, x + \\
&\qquad x^2 + 2\, a0\, b0\, y1 - 2\, b0\, x\, y1 + b0^2\, y1^2 - y^2 \beta^2 + 2\, y\, y1\, \beta^2 - y1^2 \beta^2 - z^2 \beta^2))] + \\
&(a1\, b0 - a0\, b1 + (-b0 + b1)\, x)\, z\, \mathrm{ArcTanh}[(a1 - x + b1\, y0) / \\
&\quad (\surd(a1^2 - 2\, a1\, x + x^2 - (y^2 + z^2)\, \beta^2 + y0^2\, (b1^2 - \beta^2) + 2\, y0\, (a1\, b1 - b1\, x + y\, \beta^2)))] - \\
&(a1\, b0 - a0\, b1 + (-b0 + b1)\, x)\, z\, \mathrm{ArcTanh}[(a1 - x + b1\, y1) / \\
&\quad (\surd(a1^2 - 2\, a1\, x + x^2 - (y^2 + z^2)\, \beta^2 + y1^2\, (b1^2 - \beta^2) + 2\, y1\, (a1\, b1 - b1\, x + y\, \beta^2)))] - \\
&z\, \surd(a1^2\, (b0^2 - \beta^2) + a0^2\, (b1^2 - \beta^2) + 2\, a0\, (b0 - b1)\, (b1\, x - y\, \beta^2) + \\
&\qquad (b0 - b1)^2\, (x^2 - (y^2 + z^2)\, \beta^2) - 2\, a1\, (a0\, (b0\, b1 - \beta^2) + (b0 - b1)\, (b0\, x - y\, \beta^2))) \\
&\ \mathrm{ArcTanh}[(a0^2\, b1 + a0\, (-a1\, b0 + b0\, x - 2\, b1\, x + b0\, b1\, y0 + y\, \beta^2 - y0\, \beta^2) + \\
&\qquad a1\, (b0\, x - b0^2\, y0 + (-y + y0)\, \beta^2) + (b0 - b1)\, (-x^2 + b0\, x\, y0 + (y^2 - y\, y0 + z^2)\, \beta^2)) / \\
&\quad (\surd(a0^2 - 2\, a0\, x + x^2 - (y^2 + z^2)\, \beta^2 + y0^2\, (b0^2 - \beta^2) + 2\, y0\, (a0\, b0 - b0\, x + y\, \beta^2)) \\
&\qquad \surd(a1^2\, (b0^2 - \beta^2) + a0^2\, (b1^2 - \beta^2) + 2\, a0\, (b0 - b1)\, (b1\, x - y\, \beta^2) + (b0 - b1)^2 \\
&\qquad\quad (x^2 - (y^2 + z^2)\, \beta^2) - 2\, a1\, (a0\, (b0\, b1 - \beta^2) + (b0 - b1)\, (b0\, x - y\, \beta^2))))] + \\
&z\, \surd(a1^2\, (b0^2 - \beta^2) + a0^2\, (b1^2 - \beta^2) + 2\, a0\, (b0 - b1)\, (b1\, x - y\, \beta^2) + \\
&\qquad (b0 - b1)^2\, (x^2 - (y^2 + z^2)\, \beta^2) - 2\, a1\, (a0\, (b0\, b1 - \beta^2) + (b0 - b1)\, (b0\, x - y\, \beta^2))) \\
&\ \mathrm{ArcTanh}[(-a1^2\, b0 + a0\, (-b1\, x + b1^2\, y0 + (y - y0)\, \beta^2) + \\
&\qquad a1\, (a0\, b1 + 2\, b0\, x - b1\, x - b0\, b1\, y0 - y\, \beta^2 + y0\, \beta^2) + \\
&\qquad (b0 - b1)\, (-x^2 + b1\, x\, y0 + (y^2 - y\, y0 + z^2)\, \beta^2)) / \\
&\quad (\surd(a1^2 - 2\, a1\, x + x^2 - (y^2 + z^2)\, \beta^2 + y0^2\, (b1^2 - \beta^2) + 2\, y0\, (a1\, b1 - b1\, x + y\, \beta^2)) \\
&\qquad \surd(a1^2\, (b0^2 - \beta^2) + a0^2\, (b1^2 - \beta^2) + 2\, a0\, (b0 - b1)\, (b1\, x - y\, \beta^2) + (b0 - b1)^2 \\
&\qquad\quad (x^2 - (y^2 + z^2)\, \beta^2) - 2\, a1\, (a0\, (b0\, b1 - \beta^2) + (b0 - b1)\, (b0\, x - y\, \beta^2))))] + \\
&z\, \surd(a1^2\, (b0^2 - \beta^2) + a0^2\, (b1^2 - \beta^2) + 2\, a0\, (b0 - b1)\, (b1\, x - y\, \beta^2) + \\
&\qquad (b0 - b1)^2\, (x^2 - (y^2 + z^2)\, \beta^2) - 2\, a1\, (a0\, (b0\, b1 - \beta^2) + (b0 - b1)\, (b0\, x - y\, \beta^2))) \\
&\ \mathrm{ArcTanh}[(a0^2\, b1 + a0\, (-a1\, b0 + b0\, x - 2\, b1\, x + b0\, b1\, y1 + y\, \beta^2 - y1\, \beta^2) + \\
&\qquad a1\, (b0\, x - b0^2\, y1 + (-y + y1)\, \beta^2) + (b0 - b1)\, (-x^2 + b0\, x\, y1 + (y^2 - y\, y1 + z^2)\, \beta^2)) / \\
&\quad (\surd(a0^2 - 2\, a0\, x + x^2 - (y^2 + z^2)\, \beta^2 + y1^2\, (b0^2 - \beta^2) + 2\, y1\, (a0\, b0 - b0\, x + y\, \beta^2)) \\
&\qquad \surd(a1^2\, (b0^2 - \beta^2) + a0^2\, (b1^2 - \beta^2) + 2\, a0\, (b0 - b1)\, (b1\, x - y\, \beta^2) + (b0 - b1)^2 \\
&\qquad\quad (x^2 - (y^2 + z^2)\, \beta^2) - 2\, a1\, (a0\, (b0\, b1 - \beta^2) + (b0 - b1)\, (b0\, x - y\, \beta^2))))] - \\
&z\, \surd(a1^2\, (b0^2 - \beta^2) + a0^2\, (b1^2 - \beta^2) + 2\, a0\, (b0 - b1)\, (b1\, x - y\, \beta^2) + \\
&\qquad (b0 - b1)^2\, (x^2 - (y^2 + z^2)\, \beta^2) - 2\, a1\, (a0\, (b0\, b1 - \beta^2) + (b0 - b1)\, (b0\, x - y\, \beta^2))) \\
&\ \mathrm{ArcTanh}[(-a1^2\, b0 + a0\, (-b1\, x + b1^2\, y1 + (y - y1)\, \beta^2) + \\
&\qquad a1\, (a0\, b1 + 2\, b0\, x - b1\, x - b0\, b1\, y1 - y\, \beta^2 + y1\, \beta^2) + \\
&\qquad (b0 - b1)\, (-x^2 + b1\, x\, y1 + (y^2 - y\, y1 + z^2)\, \beta^2)) / \\
&\quad (\surd(a1^2 - 2\, a1\, x + x^2 - (y^2 + z^2)\, \beta^2 + y1^2\, (b1^2 - \beta^2) + 2\, y1\, (a1\, b1 - b1\, x + y\, \beta^2)) \\
&\qquad \surd(a1^2\, (b0^2 - \beta^2) + a0^2\, (b1^2 - \beta^2) + 2\, a0\, (b0 - b1)\, (b1\, x - y\, \beta^2) + (b0 - b1)^2 \\
&\qquad\quad (x^2 - (y^2 + z^2)\, \beta^2) - 2\, a1\, (a0\, (b0\, b1 - \beta^2) + (b0 - b1)\, (b0\, x - y\, \beta^2))))])) / \\
&(\pi\, (a0^2 + a1^2 + 2\, a1\, (-b0 + b1)\, y - 2\, a0\, (a1 + (-b0 + b1)\, y) + (b0 - b1)^2\, (y^2 + z^2))^2)
\end{aligned}
$$

A.2 Truncated Tapered Supersonic Doublet Panel Potential and Perturbation Velocity Components

In[331]:=
```
(*Clear all variables*)
ClearAll["Global`*"]
```

In[332]:=
```
(*Rubi integration setup*)
ShowSteps = True; (*False displays only the final answer,
True displays intermediate steps*)
RulesDirectory = NotebookDirectory[];
Get[RulesDirectory <> "Load Rubi.m"];
```

Modifying 7017 integration rules to display steps...

In[380]:=
```
(*Evaluate x indefinite integral*)
(1/(2*π))*Int[((μ1 - μ0)/((a1 - a0) + (b1 - b0)*y′))*(z/((y - y′)^2 + z^2))*
    (x - x′)/Sqrt[(x - x′)^2 - (β^2)*((y - y′)^2 + z^2)], x′]
```

Rule 13: If !MatchQ[u, b_ v_ /; FreeQ[b, x]],

$$\int a\, u\, dx \longrightarrow \text{Dist}\left[a, \int u\, dx, x\right]$$

In[381]:=

$$\frac{\text{Dist}\left[\frac{z\,(\mu0-\mu1)}{(a0-a1+(b0-b1)\,y\prime)\,\left((y-y\prime)^2+z^2\right)}, \text{Int}\left[\frac{x-x\prime}{\sqrt{(x-x\prime)^2-\left((y-y\prime)^2+z^2\right)\beta^2}}, x\prime\right], x\prime\right]}{2\,\pi}$$

Rule 36: If PolyQ[Pm, x] && PolyQ[Qm, x], let m = Expon[Pm, x], then if
Expon[Qm, x] == m − 1 && ZeroQ[Coeff[Qm, x, m − 1] ∂_x Pm − m Coeff[Pm, x, m] Qm],

$$\int (a+b\,\text{Pm}^n)^p\,\text{Qm}\,dx \longrightarrow \text{Dist}\left[\frac{\text{Coeff}[\text{Qm}, x, m-1]}{m\,\text{Coeff}[\text{Pm}, x, m]}, \text{Subst}\left[\int (a+b\,x^n)^p\,dx, x, \text{Pm}\right], x\right]$$

In[382]:=

$$-\frac{\text{Dist}\left[\frac{z\,(\mu0-\mu1)}{2\,(a0-a1+(b0-b1)\,y\prime)\,\left((y-y\prime)^2+z^2\right)}, \text{Subst}\left[\text{Int}\left[\frac{1}{\sqrt{x\prime-\left((y-y\prime)^2+z^2\right)\beta^2}}, x\prime\right], x\prime, (x-x\prime)^2\right], x\prime\right]}{2\,\pi}$$

Rule 43: If NonzeroQ[m + 1],

$$\int (a+b\,x)^m\,dx \longrightarrow \text{Simp}\left[\frac{(a+b\,x)^{m+1}}{b\,(m+1)}, x\right]$$

Out[382]=

$$-\frac{z\,\sqrt{(x-x\prime)^2-\left((y-y\prime)^2+z^2\right)\beta^2}\,(\mu0-\mu1)}{2\,\pi\,(a0-a1+(b0-b1)\,y\prime)\,\left((y-y\prime)^2+z^2\right)}$$

In[383]:=
```
(*Evaluate at integration limits*)
f[x′_] = -((z*Sqrt[(x - x′)^2 - ((y - y′)^2 + z^2)*β^2]*(μ0 - μ1))/
    (2*Pi*(a0 - a1 + (b0 - b1)*y′)*((y - y′)^2 + z^2)))
```

Out[383]=

$$-\frac{z\,\sqrt{(x-x\prime)^2-\left((y-y\prime)^2+z^2\right)\beta^2}\,(\mu0-\mu1)}{2\,\pi\,(a0-a1+(b0-b1)\,y\prime)\,\left((y-y\prime)^2+z^2\right)}$$

In[384]:=
```
Simplify[f[x] - f[a0 + b0 * y′]]
```

Out[384]=

$$-\frac{z\left(\sqrt{-\left(y^2-2\,y\,y\prime+y\prime^2+z^2\right)\beta^2}-\sqrt{(a0-x+b0\,y\prime)^2-\left((y-y\prime)^2+z^2\right)\beta^2}\right)(\mu0-\mu1)}{2\,\pi\,(a0-a1+(b0-b1)\,y\prime)\,\left(y^2-2\,y\,y\prime+y\prime^2+z^2\right)}$$

In[385]:=
```
(*Evaluate y indefinite integral*)
Int[-((z * (Sqrt[(-(y^2 - 2 * y * y′ + y′^2 + z^2)) * β^2] -
    Sqrt[(a0 - x + b0 * y′)^2 - ((y - y′)^2 + z^2) * β^2]) * (μ0 - μ1)) /
  (2 * Pi * (a0 - a1 + (b0 - b1) * y′) * (y^2 - 2 * y * y′ + y′^2 + z^2))), y′]
```

Rule 13: If !MatchQ[u, b_ v_ /; FreeQ[b, x]],

$$\int a\,u\,dx \longrightarrow \mathrm{Dist}\left[a, \int u\,dx, x\right]$$

In[386]:=

$$-\mathrm{Dist}\left[\frac{z\,(\mu 0-\mu 1)}{2\,\pi},\ \mathrm{Int}\left[\frac{\sqrt{\left(-y^2+2\,y\,y\prime-y\prime^2-z^2\right)\beta^2}-\sqrt{\left(a0-x+b0\,y\prime\right)^2-\left((y-y\prime)^2+z^2\right)\beta^2}}{\left(a0-a1+(b0-b1)\,y\prime\right)\left(y^2-2\,y\,y\prime+y\prime^2+z^2\right)},\ y\prime\right],\ y\prime\right]$$

Rule 7000: If ZeroQ[n2 − 2 n] && PositiveIntegerQ[n],

let $v = \mathrm{RationalFunctionExpand}\left[\frac{u}{a+b\,x^n+c\,x^{2\,n}}, x\right]$, then if SumQ[v],

$$\int \frac{u}{a+b\,x^n+c\,x^{n2}}\,dx \longrightarrow \int v\,dx$$

In[387]:=

$$-\mathrm{Dist}\left[\frac{z\,(\mu 0-\mu 1)}{2\,\pi},\ \mathrm{Int}\left[\frac{\beta^2}{\left(-a0+a1-(b0-b1)\,y\prime\right)\sqrt{-\left(y^2-2\,y\,y\prime+y\prime^2+z^2\right)\beta^2}}+\left(\sqrt{\left(a0^2-2\,a0\,x+x^2-y^2\,\beta^2-z^2\,\beta^2+y\prime^2\left(b0^2-\beta^2\right)+y\prime\left(2\,a0\,b0-2\,b0\,x+2\,y\,\beta^2\right)\right)}\right)\Big/\left(\left(-a0+a1-(b0-b1)\,y\prime\right)\left(y^2-2\,y\,y\prime+y\prime^2+z^2\right)\right),\ y\prime\right],\ y\prime\right]$$

Rule:

$$\mathrm{Int}[a\,u+b\,v+\cdots, x] \longrightarrow a\int u\,dx+b\int v\,dx+\cdots$$

In[388]:=

$$-\mathrm{Dist}\left[\frac{z\,(\mu 0-\mu 1)}{2\,\pi},\ \mathrm{Int}\left[\left(\sqrt{\left(a0^2-2\,a0\,x+x^2-y^2\,\beta^2-z^2\,\beta^2+y\prime^2\left(b0^2-\beta^2\right)+y\prime\left(2\,a0\,b0-2\,b0\,x+2\,y\,\beta^2\right)\right)}\right)\Big/\left(\left(-a0+a1-(b0-b1)\,y\prime\right)\left(y^2-2\,y\,y\prime+y\prime^2+z^2\right)\right),\ y\prime\right],\ y\prime\right]-\mathrm{Dist}\left[\frac{z\,\beta^2\,(\mu 0-\mu 1)}{2\,\pi},\ \mathrm{Int}\left[\frac{1}{\left(-a0+a1-(b0-b1)\,y\prime\right)\sqrt{\left(-y^2+2\,y\,y\prime-y\prime^2-z^2\right)\beta^2}},\ y\prime\right],\ y\prime\right]$$

Rule 7000: If ZeroQ[n2 − 2 n] && PositiveIntegerQ[n],

let $v = \mathrm{RationalFunctionExpand}\left[\frac{u}{a+b\,x^n+c\,x^{2\,n}}, x\right]$, then if SumQ[v],

$$\int \frac{u}{a+b\,x^n+c\,x^{n2}}\,dx \longrightarrow \int v\,dx$$

Rule 1494:

If LinearQ[u, x] && QuadraticQ[v, x] && !(LinearMatchQ[u, x] && QuadraticMatchQ[v, x]),

$$\int u^m\,v^p\,dx \longrightarrow \int \mathrm{ExpandToSum}[u, x]^m\,\mathrm{ExpandToSum}[v, x]^p\,dx$$

In[389]:=
```
-Dist[z (μ0 - μ1)/(2 π), Int[
  ((b0 - b1)^2 √(a0^2 - 2 a0 x + x^2 - y^2 β^2 - z^2 β^2 + y′^2 (b0^2 - β^2) + y′ (2 a0 b0 - 2 b0 x + 2 y β^2))) /
   ((-a0 + a1 - (b0 - b1) y′) (a0^2 - 2 a0 a1 + a1^2 + 2 a0 b0 y - 2 a1 b0 y - 2 a0 b1 y +
        2 a1 b1 y + b0^2 y^2 - 2 b0 b1 y^2 + b1^2 y^2 + b0^2 z^2 - 2 b0 b1 z^2 + b1^2 z^2)) +
  ((-a0 + a1 - 2 b0 y + 2 b1 y + (b0 - b1) y′) √(a0^2 - 2 a0 x + x^2 - y^2 β^2 -
        z^2 β^2 + y′^2 (b0^2 - β^2) + y′ (2 a0 b0 - 2 b0 x + 2 y β^2))) /
   ((y^2 - 2 y y′ + y′^2 + z^2) (a0^2 - 2 a0 a1 + a1^2 + 2 a0 b0 y - 2 a1 b0 y - 2 a0 b1 y +
        2 a1 b1 y + b0^2 y^2 - 2 b0 b1 y^2 + b1^2 y^2 + b0^2 z^2 - 2 b0 b1 z^2 + b1^2 z^2)), y′], y′] -
 Dist[z β^2 (μ0 - μ1)/(2 π), Int[1/((-a0 + a1 + (-b0 + b1) y′) √(2 y y′ β^2 - y′^2 β^2 + (-y^2 - z^2) β^2)),
   y′], y′]
```

Rule:

```
Int[a u + b v + ⋯, x] ⟶ a ∫u ⅆx + b ∫v ⅆx + ⋯
```

Rule 310: If NonzeroQ[b^2 - 4 a c] && NonzeroQ[2 c d - b e],

```
∫ 1/((d + e x) √(a + b x + c x^2)) ⅆx ⟶
 Dist[-2, Subst[∫ 1/(4 c d^2 - 4 b d e + 4 a e^2 - x^2) ⅆx, x, (2 a e - b d - (2 c d - b e) x)/√(a + b x + c x^2)], x]
```

In[390]:=
```
-Dist[(z (μ0 - μ1)) /
   (2 π (a0^2 + a1^2 - 2 a1 (b0 - b1) y - 2 a0 (a1 - (b0 - b1) y) + (b0 - b1)^2 (y^2 + z^2))),
  Int[1/(y^2 - 2 y y′ + y′^2 + z^2) (-a0 + a1 - 2 b0 y + 2 b1 y + (b0 - b1) y′)
     √(a0^2 - 2 a0 x + x^2 - y^2 β^2 - z^2 β^2 + y′^2 (b0^2 - β^2) + y′ (2 a0 b0 - 2 b0 x + 2 y β^2)),
   y′], y′] - Dist[((b0 - b1)^2 z (μ0 - μ1)) /
  (2 π (a0^2 + a1^2 - 2 a1 (b0 - b1) y - 2 a0 (a1 - (b0 - b1) y) + (b0 - b1)^2 (y^2 + z^2))),
 Int[(√(a0^2 - 2 a0 x + x^2 - y^2 β^2 - z^2 β^2 + y′^2 (b0^2 - β^2) + y′ (2 a0 b0 - 2 b0 x + 2 y β^2))) /
   (-a0 + a1 - (b0 - b1) y′), y′], y′] + Dist[z β^2 (μ0 - μ1)/π, Subst[
  Int[1/(-y′^2 - 4 (-a0 + a1)^2 β^2 - 8 (-a0 + a1) (-b0 + b1) y β^2 + 4 (-b0 + b1)^2 (-y^2 - z^2) β^2),
   y′], y′, (-2 (-a0 + a1) y β^2 + 2 (-b0 + b1) (-y^2 - z^2) β^2 -
     y′ (-2 (-a0 + a1) β^2 - 2 (-b0 + b1) y β^2)) / (√(2 y y′ β^2 - y′^2 β^2 + (-y^2 - z^2) β^2))], y′]
```

Rule 607:

If NonzeroQ[$b^2 - 4\,a\,c$] && NonzeroQ[$e^2 - 4\,d\,f$] && RationalQ[p] && p > 0 && NonzeroQ[p + q + 1],

$$\int (g + h\,x)\,(a + b\,x + c\,x^2)^p\,(d + e\,x + f\,x^2)^q\,dx \longrightarrow$$

$$\text{Simp}\left[\frac{h\,(a + b\,x + c\,x^2)^p\,(d + e\,x + f\,x^2)^{q+1}}{2\,f\,(p + q + 1)},\,x\right] - \text{Dist}\left[\frac{1}{2\,f\,(p + q + 1)},\right.$$
$$\int (a + b\,x + c\,x^2)^{p-1}\,(d + e\,x + f\,x^2)^q\,\text{Simp}[h\,p\,(b\,d - a\,e) + a\,(h\,e - 2\,g\,f)\,(p + q + 1) + (2\,h\,p\,(c\,d - a\,f) +$$
$$\left. b\,(h\,e - 2\,g\,f)\,(p + q + 1))\,x + (h\,p\,(c\,e - b\,f) + c\,(h\,e - 2\,g\,f)\,(p + q + 1))\,x^2,\,x]\,dx,\,x\right]$$

Rule 320: If

NonzeroQ[$b^2 - 4\,a\,c$] && NonzeroQ[$c\,d^2 - b\,d\,e + a\,e^2$] && NonzeroQ[$2\,c\,d - b\,e$] && RationalQ[p] && p > 0 && NonzeroQ[m + 2 p + 1] && (! RationalQ[m] || m < 1) && ! NegativeIntegerQ[m + 2 p] && IntQuadraticQ[a, b, c, d, e, m, p, x],

$$\int (d + e\,x)^m\,(a + b\,x + c\,x^2)^p\,dx \longrightarrow \text{Simp}\left[\frac{(d + e\,x)^{m+1}\,(a + b\,x + c\,x^2)^p}{e\,(m + 2\,p + 1)},\,x\right] -$$
$$\text{Dist}\left[\frac{p}{e\,(m + 2\,p + 1)},\,\int (d + e\,x)^m\,\text{Simp}[b\,d - 2\,a\,e + (2\,c\,d - b\,e)\,x,\,x]\,(a + b\,x + c\,x^2)^{p-1}\,dx,\,x\right]$$

Rule 705: If PosQ$\left[\frac{a}{b}\right]$ && (NegativeQ[a] || NegativeQ[b]),

$$\int \frac{1}{a + b\,x^2}\,dx \longrightarrow -\text{Simp}\left[\frac{\text{ArcTan}\left[\frac{\sqrt{-b}\,x}{\sqrt{-a}}\right]}{\sqrt{-a}\,\sqrt{-b}},\,x\right]$$

In[391]:=

$$-\Big(\Big(z\,\beta\,(\mu0 - \mu1)\,\text{ArcTan}\Big[\Big(\big(a0\,y - a1\,y - (a0 - a1 + (b0 - b1)\,y)\,y\prime + (b0 - b1)\,(y^2 + z^2)\big)\,\beta\Big)\Big/$$
$$\Big(\sqrt{\left(a0^2 + a1^2 - 2\,a1\,(b0 - b1)\,y - 2\,a0\,(a1 - (b0 - b1)\,y) + (b0 - b1)^2\,(y^2 + z^2)\right)}$$
$$\sqrt{2\,y\,y\prime\,\beta^2 - y\prime^2\,\beta^2 - (y^2 + z^2)\,\beta^2}\Big)\Big]\Big)\Big/$$
$$\Big(2\,\pi\,\sqrt{\left(a0^2 + a1^2 - 2\,a1\,(b0 - b1)\,y - 2\,a0\,(a1 - (b0 - b1)\,y) + (b0 - b1)^2\,(y^2 + z^2)\right)}\Big)\Big) +$$
$$\text{Dist}\Big[(z\,(\mu0 - \mu1))\Big/\Big(2\,\pi\,\big(a0^2 + a1^2 - 2\,a1\,(b0 - b1)\,y -$$
$$2\,a0\,(a1 - (b0 - b1)\,y) + (b0 - b1)^2\,(y^2 + z^2)\big)\Big),$$
$$\text{Int}\Big[\Big((b0 - b1)\,(a0 - x)\,(a0\,y - x\,y + b0\,(y^2 + z^2)) +$$
$$(a0 - a1 + (b0 - b1)\,y)\,(a0^2 - 2\,a0\,x + x^2 - (y^2 + z^2)\,\beta^2) +$$
$$y\prime^2\,(-a1\,(b0^2 - \beta^2) + a0\,(b0\,b1 - \beta^2) + (b0 - b1)\,(b0\,x - y\,\beta^2)) +$$
$$y\prime\,(-(b0 - b1)\,(a0^2 - 2\,a0\,x + x^2 - b0^2\,(y^2 + z^2)) +$$
$$2\,(a0 - a1 + (b0 - b1)\,y)\,(a0\,b0 - b0\,x + y\,\beta^2))\Big)\Big/\Big((y^2 - 2\,y\,y\prime + y\prime^2 + z^2)$$
$$\sqrt{(a0^2 - 2\,a0\,x + x^2 - y^2\,\beta^2 - z^2\,\beta^2 + y\prime^2\,(b0^2 - \beta^2) + y\prime\,(2\,a0\,b0 - 2\,b0\,x + 2\,y\,\beta^2))}\Big),$$
$$y\prime\Big],\,y\prime\Big] - \text{Dist}\Big[\big((b0 - b1)\,z\,(\mu0 - \mu1)\big)\Big/$$
$$\Big(4\,\pi\,\big(a0^2 + a1^2 - 2\,a1\,(b0 - b1)\,y - 2\,a0\,(a1 - (b0 - b1)\,y) + (b0 - b1)^2\,(y^2 + z^2)\big)\Big),$$
$$\text{Int}\Big[\Big(-2\,y\prime\,\big((a0 - a1)\,(b0^2 - \beta^2) - (b0 - b1)\,(a0\,b0 - b0\,x + y\,\beta^2)\big) -$$
$$2\,\big((a0 - a1)\,(a0\,b0 - b0\,x + y\,\beta^2) - (b0 - b1)\,(a0^2 - 2\,a0\,x + x^2 - (y^2 + z^2)\,\beta^2)\big)\Big)\Big/$$
$$\Big((-a0 + a1 + (-b0 + b1)\,y\prime)\,\sqrt{(a0^2 - 2\,a0\,x + x^2 - y^2\,\beta^2 - z^2\,\beta^2 +}$$
$$\overline{y\prime^2\,(b0^2 - \beta^2) + y\prime\,(2\,a0\,b0 - 2\,b0\,x + 2\,y\,\beta^2))}\Big),\,y\prime\Big],\,y\prime\Big]$$

Rule 674: If NonzeroQ[b^2 - 4 a c] && NonzeroQ[e^2 - 4 d f],

$$\int \frac{A + B x + C x^2}{(a + b x + c x^2) \sqrt{d + e x + f x^2}} \, dx \longrightarrow$$

$$\text{Dist}\left[\frac{C}{c}, \int \frac{1}{\sqrt{d + e x + f x^2}} \, dx, x\right] + \text{Dist}\left[\frac{1}{c}, \int \frac{A c - a C + (B c - b C) x}{(a + b x + c x^2) \sqrt{d + e x + f x^2}} \, dx, x\right]$$

Rule 470: If NonzeroQ[e f - d g] && NonzeroQ[b^2 - 4 a c] && NonzeroQ[c d^2 - b d e + a e^2],

$$\int (d + e x)^m (f + g x) (a + b x + c x^2)^p \, dx \longrightarrow$$

$$\text{Dist}\left[\frac{g}{e}, \int (d + e x)^{m+1} (a + b x + c x^2)^p \, dx, x\right] + \text{Dist}\left[\frac{e f - d g}{e}, \int (d + e x)^m (a + b x + c x^2)^p \, dx, x\right]$$

In[392]:=

$$-\Bigg(\Bigg(z \beta (\mu 0 - \mu 1) \text{ArcTan}\Big[\Big((a0\, y - a1\, y - (a0 - a1 + (b0 - b1) y) y\prime + (b0 - b1) (y^2 + z^2)) \beta\Big) \Big/ \Big(\sqrt{(a0^2 + a1^2 - 2 a1 (b0 - b1) y - 2 a0 (a1 - (b0 - b1) y) + (b0 - b1)^2 (y^2 + z^2))} \sqrt{2 y y\prime \beta^2 - y\prime^2 \beta^2 - (y^2 + z^2) \beta^2}\Big)\Big]\Bigg) \Bigg/ \Bigg(2 \pi \sqrt{(a0^2 + a1^2 - 2 a1 (b0 - b1) y - 2 a0 (a1 - (b0 - b1) y) + (b0 - b1)^2 (y^2 + z^2))}\Bigg)\Bigg) +$$

$$\text{Dist}\Big[(z (\mu 0 - \mu 1)) \Big/ \Big(2 \pi \big(a0^2 + a1^2 - 2 a1 (b0 - b1) y - 2 a0 (a1 - (b0 - b1) y) + (b0 - b1)^2 (y^2 + z^2)\big)\Big),$$
$$\text{Int}\Big[\Big((b0 - b1) (a0 - x) (a0\, y - x\, y + b0 (y^2 + z^2)) + (a0 - a1 + (b0 - b1) y) (a0^2 - 2 a0\, x + x^2 - (y^2 + z^2) \beta^2) - (y^2 + z^2) (-a1 (b0^2 - \beta^2) + a0 (b0\, b1 - \beta^2) + (b0 - b1) (b0\, x - y \beta^2)) + y\prime \big((-b0 + b1) (a0^2 - 2 a0\, x + x^2 - b0^2 (y^2 + z^2)) + 2 (a0 - a1 + (b0 - b1) y) (a0\, b0 - b0\, x + y \beta^2) + 2 y (-a1 (b0^2 - \beta^2) + a0 (b0\, b1 - \beta^2) + (b0 - b1) (b0\, x - y \beta^2))\big)\Big) \Big/ \Big((y^2 - 2 y\, y\prime + y\prime^2 + z^2) \sqrt{(a0^2 - 2 a0\, x + x^2 - y^2 \beta^2 - z^2 \beta^2 + y\prime^2 (b0^2 - \beta^2) + y\prime (2 a0\, b0 - 2 b0\, x + 2 y \beta^2))}\Big), y\prime\Big], y\prime\Big] -$$

$$\text{Dist}\Big[\Big(z \big(a1^2 (b0^2 - \beta^2) + a0^2 (b1^2 - \beta^2) + 2 a0 (b0 - b1) (b1\, x - y \beta^2) + (b0 - b1)^2 (x^2 - (y^2 + z^2) \beta^2) - 2 a1 (a0 (b0\, b1 - \beta^2) + (b0 - b1) (b0\, x - y \beta^2))\big) (\mu 0 - \mu 1)\Big) \Big/ \Big(2 \pi \big(a0^2 + a1^2 - 2 a1 (b0 - b1) y - 2 a0 (a1 - (b0 - b1) y) + (b0 - b1)^2 (y^2 + z^2)\big)\Big),$$
$$\text{Int}\Big[1 \Big/ \Big((-a0 + a1 + (-b0 + b1) y\prime) \sqrt{(a0^2 - 2 a0\, x + x^2 - y^2 \beta^2 - z^2 \beta^2 + y\prime^2 (b0^2 - \beta^2) + y\prime (2 a0\, b0 - 2 b0\, x + 2 y \beta^2))}\Big), y\prime\Big], y\prime\Big]$$

Rule 623:

If NonzeroQ[$b^2 - 4\,a\,c$] && NonzeroQ[$e^2 - 4\,d\,f$] && NonzeroQ[$b\,d - a\,e$] && NegQ[$b^2 - 4\,a\,c$], let

$q = \sqrt{(c\,d - a\,f)^2 - (b\,d - a\,e)\,(c\,e - b\,f)}$, then

$$\int \frac{g + h\,x}{(a + b\,x + c\,x^2)\,\sqrt{d + e\,x + f\,x^2}}\,dx \longrightarrow$$

$$\text{Dist}\left[\frac{1}{2\,q},\ \int\left(\text{Simp}[h\,(b\,d - a\,e) - g\,(c\,d - a\,f - q) - (g\,(c\,e - b\,f) - h\,(c\,d - a\,f + q))\,x,\ x] \middle/ \left((a + b\,x + c\,x^2)\,\sqrt{d + e\,x + f\,x^2}\right)\right)dx,\ x\right] -$$

$$\text{Dist}\left[\frac{1}{2\,q},\ \int\left(\text{Simp}[h\,(b\,d - a\,e) - g\,(c\,d - a\,f + q) - (g\,(c\,e - b\,f) - h\,(c\,d - a\,f - q))\,x,\ x] \middle/ \left((a + b\,x + c\,x^2)\,\sqrt{d + e\,x + f\,x^2}\right)\right)dx,\ x\right]$$

Rule 310: If NonzeroQ[$b^2 - 4\,a\,c$] && NonzeroQ[$2\,c\,d - b\,e$],

$$\int \frac{1}{(d + e\,x)\,\sqrt{a + b\,x + c\,x^2}}\,dx \longrightarrow$$

$$\text{Dist}\left[-2,\ \text{Subst}\left[\int \frac{1}{4\,c\,d^2 - 4\,b\,d\,e + 4\,a\,e^2 - x^2}\,dx,\ x,\ \frac{2\,a\,e - b\,d - (2\,c\,d - b\,e)\,x}{\sqrt{a + b\,x + c\,x^2}}\right],\ x\right]$$

In[393]:=
```
-((z β (μ0 - μ1) ArcTan[((a0 y - a1 y - (a0 - a1 + (b0 - b1) y) y' + (b0 - b1) (y^2 + z^2)) β) /
      (√(a0^2 + a1^2 - 2 a1 (b0 - b1) y - 2 a0 (a1 - (b0 - b1) y) + (b0 - b1)^2 (y^2 + z^2))
        √(2 y y' β^2 - y'^2 β^2 - (y^2 + z^2) β^2))]) /
    (2 π √(a0^2 + a1^2 - 2 a1 (b0 - b1) y - 2 a0 (a1 - (b0 - b1) y) + (b0 - b1)^2 (y^2 + z^2)))) +
 Dist[(z (μ0 - μ1)) / (4 π (a0^2 + x^2 - 2 b0 x y - 2 a0 (x - b0 y) + b0^2 (y^2 + z^2))
     (a0^2 + a1^2 - 2 a1 (b0 - b1) y - 2 a0 (a1 - (b0 - b1) y) + (b0 - b1)^2 (y^2 + z^2))),
  Int[(2 (a1 b0 - a0 b1 - b0 x + b1 x) (a0 y - x y + b0 y^2 + b0 z^2) (a0^2 - 2 a0 x + x^2 +
          2 a0 b0 y - 2 b0 x y + b0^2 y^2 + b0^2 z^2) - 2 (a1 b0 - a0 b1 - (b0 - b1) x) (a0 - x + b0 y)
        y' (a0^2 + x^2 - 2 b0 x y - 2 a0 (x - b0 y) + b0^2 (y^2 + z^2))) / ((y^2 - 2 y y' + y'^2 + z^2)
       √(a0^2 - 2 a0 x + x^2 - y^2 β^2 - z^2 β^2 + y'^2 (b0^2 - β^2) + y' (2 a0 b0 - 2 b0 x + 2 y β^2))), y'],
  y'] - Dist[(z (μ0 - μ1)) / (4 π (a0^2 + x^2 - 2 b0 x y - 2 a0 (x - b0 y) + b0^2 (y^2 + z^2))
     (a0^2 + a1^2 - 2 a1 (b0 - b1) y - 2 a0 (a1 - (b0 - b1) y) + (b0 - b1)^2 (y^2 + z^2))),
  Int[(-2 (a0 - x) (a0^2 + x^2 - 2 b0 x y - 2 a0 (x - b0 y) + b0^2 (y^2 + z^2))
        (a0^2 + a1 (x - b0 y) - a0 (a1 + x - 2 b0 y + b1 y) - (b0 - b1) (x y - b0 (y^2 + z^2))) -
       2 b0 y' (a0^2 + x^2 - 2 b0 x y - 2 a0 (x - b0 y) + b0^2 (y^2 + z^2))
        (a0^2 + a1 (x - b0 y) - a0 (a1 + x - 2 b0 y + b1 y) - (b0 - b1) (x y - b0 (y^2 + z^2)))) /
     ((y^2 - 2 y y' + y'^2 + z^2) √(a0^2 - 2 a0 x + x^2 - y^2 β^2 - z^2 β^2 + y'^2 (b0^2 - β^2) +
          y' (2 a0 b0 - 2 b0 x + 2 y β^2))), y'], y'] + Dist[
  (z (a1^2 (b0^2 - β^2) + a0^2 (b1^2 - β^2) + 2 a0 (b0 - b1) (b1 x - y β^2) + (b0 - b1)^2 (x^2 - (y^2 + z^2) β^2) -
        2 a1 (a0 (b0 b1 - β^2) + (b0 - b1) (b0 x - y β^2))) (μ0 - μ1)) /
   (π (a0^2 + a1^2 - 2 a1 (b0 - b1) y - 2 a0 (a1 - (b0 - b1) y) + (b0 - b1)^2 (y^2 + z^2))),
  Subst[Int[1 / (-y'^2 + 4 (-a0 + a1)^2 (b0^2 - β^2) - 4 (-a0 + a1) (-b0 + b1)
          (2 a0 b0 - 2 b0 x + 2 y β^2) + 4 (-b0 + b1)^2 (a0^2 - 2 a0 x + x^2 - y^2 β^2 - z^2 β^2)), y'], y',
   (-(-a0 + a1) (2 a0 b0 - 2 b0 x + 2 y β^2) + 2 (-b0 + b1) (a0^2 - 2 a0 x + x^2 - y^2 β^2 - z^2 β^2) -
      y' (2 (-a0 + a1) (b0^2 - β^2) - (-b0 + b1) (2 a0 b0 - 2 b0 x + 2 y β^2))) /
    (√(a0^2 - 2 a0 x + x^2 - y^2 β^2 - z^2 β^2 + y'^2 (b0^2 - β^2) + y' (2 a0 b0 - 2 b0 x + 2 y β^2)))], y']
```

Rule 617: If
NonzeroQ[b^2 - 4 a c] && NonzeroQ[e^2 - 4 d f] && NonzeroQ[b d - a e] && ZeroQ[h^2 (b d - a e) - 2 g h (c d - a f) + g^2 (c e - b f)],

$$\int \frac{g + h x}{(a + b x + c x^2)\sqrt{d + e x + f x^2}} \, dx \longrightarrow$$

$$\text{Dist}\left[-2 g (g b - 2 a h), \text{Subst}\left[\int \frac{1}{\text{Simp}[g (g b - 2 a h) (b^2 - 4 a c) - (b d - a e) x^2, x]} \, dx, x, \frac{\text{Simp}[g b - 2 a h - (b h - 2 g c) x, x]}{\sqrt{d + e x + f x^2}}\right], x\right]$$

Rule 617: If
NonzeroQ[b^2 - 4 a c] && NonzeroQ[e^2 - 4 d f] && NonzeroQ[b d - a e] && ZeroQ[h^2 (b d - a e) - 2 g h (c d - a f) + g^2 (c e - b f)],

$$\int \frac{g + h x}{(a + b x + c x^2)\sqrt{d + e x + f x^2}} \, dx \longrightarrow$$

$$\text{Dist}\left[-2 g (g b - 2 a h), \text{Subst}\left[\int \frac{1}{\text{Simp}[g (g b - 2 a h) (b^2 - 4 a c) - (b d - a e) x^2, x]} \, dx, x, \frac{\text{Simp}[g b - 2 a h - (b h - 2 g c) x, x]}{\sqrt{d + e x + f x^2}}\right], x\right]$$

Rule 707: If NegQ$\left[\frac{a}{b}\right]$ && (PositiveQ[a] || NegativeQ[b]),

$$\int \frac{1}{a + b x^2} \, dx \longrightarrow \text{Simp}\left[\frac{\text{ArcTanh}\left[\frac{\sqrt{-b}\, x}{\sqrt{a}}\right]}{\sqrt{a}\,\sqrt{-b}}, x\right]$$

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In[394]:= -((z β (μ0 - μ1) ArcTan[((a0 y - a1 y - (a0 - a1 + (b0 - b1) y) y′ + (b0 - b1) (y^2 + z^2)) β) /
        (√(a0^2 + a1^2 - 2 a1 (b0 - b1) y - 2 a0 (a1 - (b0 - b1) y) + (b0 - b1)^2 (y^2 + z^2))
         √(2 y y′ β^2 - y′^2 β^2 - (y^2 + z^2) β^2))]) /
    (2 π √(a0^2 + a1^2 - 2 a1 (b0 - b1) y - 2 a0 (a1 - (b0 - b1) y) + (b0 - b1)^2 (y^2 + z^2)))) +
  (z √(a1^2 (b0^2 - β^2) + a0^2 (b1^2 - β^2) + 2 a0 (b0 - b1) (b1 x - y β^2) +
        (b0 - b1)^2 (x^2 - (y^2 + z^2) β^2) - 2 a1 (a0 (b0 b1 - β^2) + (b0 - b1) (b0 x - y β^2))) (μ0 - μ1)
     ArcTanh[((a0 - a1) (a0 b0 - b0 x + y β^2) - (b0 - b1) (a0^2 - 2 a0 x + x^2 - (y^2 + z^2) β^2) +
        y′ ((a0 - a1) (b0^2 - β^2) - (b0 - b1) (a0 b0 - b0 x + y β^2))) /
      (√(a0^2 - 2 a0 x + x^2 - (y^2 + z^2) β^2 + y′^2 (b0^2 - β^2) + 2 y′ (a0 b0 - b0 x + y β^2))
        √(a1^2 (b0^2 - β^2) + a0^2 (b1^2 - β^2) + 2 a0 (b0 - b1) (b1 x - y β^2) +
           (b0 - b1)^2 (x^2 - (y^2 + z^2) β^2) - 2 a1 (a0 (b0 b1 - β^2) + (b0 - b1) (b0 x - y β^2))))]) /
   (2 π (a0^2 + a1^2 - 2 a1 (b0 - b1) y - 2 a0 (a1 - (b0 - b1) y) + (b0 - b1)^2 (y^2 + z^2))) -
  Dist[(4 (a0 - x) (a1 b0 - a0 b1 - (b0 - b1) x)^2 z^3 (a0 y - x y + b0 (y^2 + z^2))
      (a0^2 + x^2 - 2 b0 x y - 2 a0 (x - b0 y) + b0^2 (y^2 + z^2)) (μ0 - μ1)) /
    (π (a0^2 + a1^2 - 2 a1 (b0 - b1) y - 2 a0 (a1 - (b0 - b1) y) + (b0 - b1)^2 (y^2 + z^2))), Subst[
    Int[1/(2 (a0 - x) y′^2 (a0 y - x y + b0 (y^2 + z^2)) - 32 (a0 - x) (a1 b0 - a0 b1 - (b0 - b1) x)^2 z^4
          (a0 y - x y + b0 (y^2 + z^2)) (a0^2 + x^2 - 2 b0 x y - 2 a0 (x - b0 y) + b0^2 (y^2 + z^2))^2), y′],
    y′, (4 (a0 - x) (a1 b0 - a0 b1 - (b0 - b1) x) z^2 (a0^2 + x^2 - 2 b0 x y - 2 a0 (x - b0 y) +
           b0^2 (y^2 + z^2)) + 4 b0 (a1 b0 - a0 b1 - (b0 - b1) x)
         y′ z^2 (a0^2 + x^2 - 2 b0 x y - 2 a0 (x - b0 y) + b0^2 (y^2 + z^2))) /
      (√(a0^2 - 2 a0 x + x^2 - y^2 β^2 - z^2 β^2 + y′^2 (b0^2 - β^2) + y′ (2 a0 b0 - 2 b0 x + 2 y β^2)))], y′] -
  Dist[(4 (a0 - x) z (a0 y - x y + b0 (y^2 + z^2)) (a0^2 + x^2 - 2 b0 x y - 2 a0 (x - b0 y) + b0^2 (y^2 + z^2))
      (a0^2 + a1 (x - b0 y) - a0 (a1 + x - 2 b0 y + b1 y) - (b0 - b1) (x y - b0 (y^2 + z^2)))^2
      (μ0 - μ1)) /
    (π (a0^2 + a1^2 - 2 a1 (b0 - b1) y - 2 a0 (a1 - (b0 - b1) y) + (b0 - b1)^2 (y^2 + z^2))),
   Subst[Int[1/(2 (a0 - x) y′^2 (a0 y - x y + b0 (y^2 + z^2)) + 32 (a0 - x) z^2
          (a0 y - x y + b0 (y^2 + z^2)) (a0^2 + x^2 - 2 b0 x y - 2 a0 (x - b0 y) + b0^2 (y^2 + z^2))^2
          (a0^2 + a1 (x - b0 y) - a0 (a1 + x - 2 b0 y + b1 y) - (b0 - b1) (x y - b0 (y^2 + z^2)))^2), y′],
    y′, (-4 (a0 - x + b0 y) y′ (a0^2 + x^2 - 2 b0 x y - 2 a0 (x - b0 y) + b0^2 (y^2 + z^2))
         (a0^2 + a1 (x - b0 y) - a0 (a1 + x - 2 b0 y + b1 y) - (b0 - b1) (x y - b0 (y^2 + z^2))) +
        4 (a0 y - x y + b0 (y^2 + z^2)) (a0^2 + x^2 - 2 b0 x y - 2 a0 (x - b0 y) + b0^2 (y^2 + z^2))
         (a0^2 + a1 (x - b0 y) - a0 (a1 + x - 2 b0 y + b1 y) - (b0 - b1) (x y - b0 (y^2 + z^2)))) /
      (√(a0^2 - 2 a0 x + x^2 - y^2 β^2 - z^2 β^2 + y′^2 (b0^2 - β^2) + y′ (2 a0 b0 - 2 b0 x + 2 y β^2)))], y′]
```

Rule 709: If $\text{NegQ}\left[\frac{a}{b}\right]$,

$$\int \frac{1}{a+b\,x^2}\,dx \longrightarrow \text{Simp}\left[\frac{\sqrt{-\frac{a}{b}}\ \text{ArcTanh}\left[\frac{x}{\sqrt{-\frac{a}{b}}}\right]}{a},\ x\right]$$

Rule 706: If $\text{PosQ}\left[\frac{a}{b}\right]$,

$$\int \frac{1}{a+b\,x^2}\,dx \longrightarrow \text{Simp}\left[\frac{\sqrt{\frac{a}{b}}\ \text{ArcTan}\left[\frac{x}{\sqrt{\frac{a}{b}}}\right]}{a},\ x\right]$$

Out[394]=

$$
\begin{aligned}
&-\Big(\Big(z\,\beta\,(\mu0-\mu1)\,\text{ArcTan}\Big[\Big(\big(a0\,y-a1\,y-(a0-a1+(b0-b1)\,y)\,y'+(b0-b1)\,\big(y^2+z^2\big)\big)\,\beta\Big)\Big/\\
&\qquad\Big(\sqrt{\big(a0^2+a1^2-2\,a1\,(b0-b1)\,y-2\,a0\,(a1-(b0-b1)\,y)+(b0-b1)^2\,\big(y^2+z^2\big)\big)}\\
&\qquad\ \sqrt{2\,y\,y'\,\beta^2-y'^2\,\beta^2-\big(y^2+z^2\big)\,\beta^2}\Big)\Big]\Big)\Big/\\
&\quad\Big(2\,\pi\,\sqrt{\big(a0^2+a1^2-2\,a1\,(b0-b1)\,y-2\,a0\,(a1-(b0-b1)\,y)+(b0-b1)^2\,\big(y^2+z^2\big)\big)}\Big)\Big)-\\
&\Big(\big(a0^2+a1\,(x-b0\,y)-a0\,(a1+x-2\,b0\,y+b1\,y)-(b0-b1)\,\big(x\,y-b0\,\big(y^2+z^2\big)\big)\big)\\
&\quad(\mu0-\mu1)\,\text{ArcTan}\Big[\big(a0\,y-x\,y-(a0-x+b0\,y)\,y'+b0\,\big(y^2+z^2\big)\big)\Big/\\
&\qquad\Big(z\,\sqrt{\big(a0^2-2\,a0\,x+x^2-\big(y^2+z^2\big)\,\beta^2+y'^2\,\big(b0^2-\beta^2\big)+2\,y'\,\big(a0\,b0-b0\,x+y\,\beta^2\big)\big)}\Big)\Big]\Big)\Big/\\
&\ \Big(2\,\pi\,\big(a0^2+a1^2-2\,a1\,(b0-b1)\,y-2\,a0\,(a1-(b0-b1)\,y)+(b0-b1)^2\,\big(y^2+z^2\big)\big)\Big)+\\
&\Big((a1\,b0-a0\,b1-(b0-b1)\,x)\,z\,(\mu0-\mu1)\,\text{ArcTanh}\Big[\big(a0-x+b0\,y'\big)\Big/\\
&\qquad\Big(\sqrt{\big(a0^2-2\,a0\,x+x^2-\big(y^2+z^2\big)\,\beta^2+y'^2\,\big(b0^2-\beta^2\big)+2\,y'\,\big(a0\,b0-b0\,x+y\,\beta^2\big)\big)}\Big)\Big]\Big)\Big/\\
&\ \Big(2\,\pi\,\big(a0^2+a1^2-2\,a1\,(b0-b1)\,y-2\,a0\,(a1-(b0-b1)\,y)+(b0-b1)^2\,\big(y^2+z^2\big)\big)\Big)+\\
&\Big(z\,\sqrt{\big(a1^2\,\big(b0^2-\beta^2\big)+a0^2\,\big(b1^2-\beta^2\big)+2\,a0\,(b0-b1)\,\big(b1\,x-y\,\beta^2\big)+}\\
&\qquad\qquad\overline{(b0-b1)^2\,\big(x^2-\big(y^2+z^2\big)\,\beta^2\big)-2\,a1\,\big(a0\,\big(b0\,b1-\beta^2\big)+(b0-b1)\,\big(b0\,x-y\,\beta^2\big)\big)\big)}\ (\mu0-\mu1)\\
&\quad\text{ArcTanh}\Big[\Big((a0-a1)\,\big(a0\,b0-b0\,x+y\,\beta^2\big)-(b0-b1)\,\big(a0^2-2\,a0\,x+x^2-\big(y^2+z^2\big)\,\beta^2\big)+\\
&\qquad\quad y'\,\big((a0-a1)\,\big(b0^2-\beta^2\big)-(b0-b1)\,\big(a0\,b0-b0\,x+y\,\beta^2\big)\big)\Big)\Big/\\
&\qquad\Big(\sqrt{\big(a0^2-2\,a0\,x+x^2-\big(y^2+z^2\big)\,\beta^2+y'^2\,\big(b0^2-\beta^2\big)+2\,y'\,\big(a0\,b0-b0\,x+y\,\beta^2\big)\big)}\\
&\qquad\quad\sqrt{\big(a1^2\,\big(b0^2-\beta^2\big)+a0^2\,\big(b1^2-\beta^2\big)+2\,a0\,(b0-b1)\,\big(b1\,x-y\,\beta^2\big)+}\\
&\qquad\qquad\quad\overline{(b0-b1)^2\,\big(x^2-\big(y^2+z^2\big)\,\beta^2\big)-2\,a1\,\big(a0\,\big(b0\,b1-\beta^2\big)+(b0-b1)\,\big(b0\,x-y\,\beta^2\big)\big)\big)}\Big)\Big]\Big)\Big/\\
&\ \Big(2\,\pi\,\big(a0^2+a1^2-2\,a1\,(b0-b1)\,y-2\,a0\,(a1-(b0-b1)\,y)+(b0-b1)^2\,\big(y^2+z^2\big)\big)\Big)
\end{aligned}
$$

In[395]:=
```
(*Evaluate at integration limits*)
g[y′_] =
 -((z*β*(μ0 - μ1)*ArcTan[((a0*y - a1*y - (a0 - a1 + (b0 - b1)*y)*y′ + (b0 - b1)*
              (y^2 + z^2))*β)/
     (Sqrt[a0^2 + a1^2 - 2*a1*(b0 - b1)*y - 2*a0*(a1 - (b0 - b1)*y) +
            (b0 - b1)^2*(y^2 + z^2)]*
      Sqrt[2*y*y′*β^2 - y′^2*β^2 - (y^2 + z^2)*β^2])])/
   (2*Pi*Sqrt[a0^2 + a1^2 - 2*a1*(b0 - b1)*y -
        2*a0*(a1 - (b0 - b1)*y) + (b0 - b1)^2*(y^2 + z^2)])) -
 ((a0^2 + a1*(x - b0*y) - a0*(a1 + x - 2*b0*y + b1*y) -
      (b0 - b1)*(x*y - b0*(y^2 + z^2)))*(μ0 - μ1)*
  ArcTan[(a0*y - x*y - (a0 - x + b0*y)*y′ + b0*(y^2 + z^2))/
    (z*Sqrt[a0^2 - 2*a0*x + x^2 - (y^2 + z^2)*β^2 + y′^2*(b0^2 - β^2) +
      2*y′*(a0*b0 - b0*x + y*β^2)])])/
   (2*Pi*(a0^2 + a1^2 - 2*a1*(b0 - b1)*y -
    2*a0*(a1 - (b0 - b1)*y) + (b0 - b1)^2*(y^2 + z^2))) +
 ((a1*b0 - a0*b1 - (b0 - b1)*x)*z*(μ0 - μ1)*
  ArcTanh[(a0 - x + b0*y′)/
      Sqrt[a0^2 - 2*a0*x + x^2 - (y^2 + z^2)*β^2 + y′^2*(b0^2 - β^2) +
     2*y′*(a0*b0 - b0*x + y*β^2)]])/(2*Pi*(a0^2 + a1^2 - 2*a1*(b0 - b1)*y -
    2*a0*(a1 - (b0 - b1)*y) + (b0 - b1)^2*(y^2 + z^2))) +
 (z*Sqrt[a1^2*(b0^2 - β^2) +
      a0^2*(b1^2 - β^2) + 2*a0*(b0 - b1)*(b1*x - y*β^2) +
   (b0 - b1)^2*(x^2 - (y^2 + z^2)*β^2) -
      2*a1*(a0*(b0*b1 - β^2) + (b0 - b1)*(b0*x - y*β^2))]*
  (μ0 - μ1)*ArcTanh[((a0 - a1)*(a0*b0 - b0*x + y*β^2) -
    (b0 - b1)*(a0^2 - 2*a0*x + x^2 - (y^2 + z^2)*β^2) +
    y′*((a0 - a1)*(b0^2 - β^2) - (b0 - b1)*(a0*b0 - b0*x + y*β^2)))/
   (Sqrt[a0^2 - 2*a0*x + x^2 - (y^2 + z^2)*β^2 + y′^2*(b0^2 - β^2) +
     2*y′*(a0*b0 - b0*x + y*β^2)]*
       Sqrt[a1^2*(b0^2 - β^2) + a0^2*(b1^2 - β^2) +
     2*a0*(b0 - b1)*(b1*x - y*β^2) + (b0 - b1)^2*(x^2 - (y^2 + z^2)*β^2) -
     2*a1*(a0*(b0*b1 - β^2) + (b0 - b1)*(b0*x - y*β^2))])])/
  (2*Pi*(a0^2 + a1^2 - 2*a1*(b0 - b1)*y -
      2*a0*(a1 - (b0 - b1)*y) + (b0 - b1)^2*(y^2 + z^2)))
```

```
Out[395]= -((z β (μ0 - μ1) ArcTan[((a0 y - a1 y - (a0 - a1 + (b0 - b1) y) y′ + (b0 - b1) (y^2 + z^2)) β) /
              (√(a0^2 + a1^2 - 2 a1 (b0 - b1) y - 2 a0 (a1 - (b0 - b1) y) + (b0 - b1)^2 (y^2 + z^2))
                √(2 y y′ β^2 - y′^2 β^2 - (y^2 + z^2) β^2))]) /
          (2 π √(a0^2 + a1^2 - 2 a1 (b0 - b1) y - 2 a0 (a1 - (b0 - b1) y) + (b0 - b1)^2 (y^2 + z^2)))) -
      ((a0^2 + a1 (x - b0 y) - a0 (a1 + x - 2 b0 y + b1 y) - (b0 - b1) (x y - b0 (y^2 + z^2)))
          (μ0 - μ1) ArcTan[(a0 y - x y - (a0 - x + b0 y) y′ + b0 (y^2 + z^2)) /
            (z √(a0^2 - 2 a0 x + x^2 - (y^2 + z^2) β^2 + y′^2 (b0^2 - β^2) + 2 y′ (a0 b0 - b0 x + y β^2)))]) /
        (2 π (a0^2 + a1^2 - 2 a1 (b0 - b1) y - 2 a0 (a1 - (b0 - b1) y) + (b0 - b1)^2 (y^2 + z^2))) +
      ((a1 b0 - a0 b1 - (b0 - b1) x) z (μ0 - μ1) ArcTanh[(a0 - x + b0 y′) /
            (√(a0^2 - 2 a0 x + x^2 - (y^2 + z^2) β^2 + y′^2 (b0^2 - β^2) + 2 y′ (a0 b0 - b0 x + y β^2)))]) /
        (2 π (a0^2 + a1^2 - 2 a1 (b0 - b1) y - 2 a0 (a1 - (b0 - b1) y) + (b0 - b1)^2 (y^2 + z^2))) +
      (z √(a1^2 (b0^2 - β^2) + a0^2 (b1^2 - β^2) + 2 a0 (b0 - b1) (b1 x - y β^2) +
              (b0 - b1)^2 (x^2 - (y^2 + z^2) β^2) - 2 a1 (a0 (b0 b1 - β^2) + (b0 - b1) (b0 x - y β^2))) (μ0 - μ1)
          ArcTanh[((a0 - a1) (a0 b0 - b0 x + y β^2) - (b0 - b1) (a0^2 - 2 a0 x + x^2 - (y^2 + z^2) β^2) +
              y′ ((a0 - a1) (b0^2 - β^2) - (b0 - b1) (a0 b0 - b0 x + y β^2))) /
            (√(a0^2 - 2 a0 x + x^2 - (y^2 + z^2) β^2 + y′^2 (b0^2 - β^2) + 2 y′ (a0 b0 - b0 x + y β^2))
              √(a1^2 (b0^2 - β^2) + a0^2 (b1^2 - β^2) + 2 a0 (b0 - b1) (b1 x - y β^2) +
                  (b0 - b1)^2 (x^2 - (y^2 + z^2) β^2) - 2 a1 (a0 (b0 b1 - β^2) + (b0 - b1) (b0 x - y β^2))))]) /
        (2 π (a0^2 + a1^2 - 2 a1 (b0 - b1) y - 2 a0 (a1 - (b0 - b1) y) + (b0 - b1)^2 (y^2 + z^2)))

In[396]:= (*Tapered Supersonic Doublet Panel Perturbation Velocity Potential*)
          ϕ[x_, y_, z_] = Simplify[g[y1] - g[y0]]
```

Out[396]=

$$
\begin{aligned}
&\frac{1}{2\pi}(\mu 0-\mu 1)\Bigg(\Big(z\,\beta\,\mathrm{ArcTan}\Big[\big(\big(a0\,(y-y0)+a1\,(-y+y0)+(b0-b1)\,\big(y^2-y\,y0+z^2\big)\big)\,\beta\big)\Big/\\
&\quad\Big(\sqrt{\big(a0^2+a1^2+2\,a1\,(-b0+b1)\,y-2\,a0\,(a1+(-b0+b1)\,y)+(b0-b1)^2\,\big(y^2+z^2\big)\big)}\\
&\quad\quad\sqrt{-\big(y^2-2\,y\,y0+y0^2+z^2\big)\,\beta^2}\Big)\Big]\Big)\Big/\\
&\Big(\sqrt{\big(a0^2+a1^2+2\,a1\,(-b0+b1)\,y-2\,a0\,(a1+(-b0+b1)\,y)+(b0-b1)^2\,\big(y^2+z^2\big)\big)}\Big)-\\
&\Big(z\,\beta\,\mathrm{ArcTan}\Big[\big(\big(a0\,(y-y1)+a1\,(-y+y1)+(b0-b1)\,\big(y^2-y\,y1+z^2\big)\big)\,\beta\big)\Big/\\
&\quad\Big(\sqrt{\big(a0^2+a1^2+2\,a1\,(-b0+b1)\,y-2\,a0\,(a1+(-b0+b1)\,y)+(b0-b1)^2\,\big(y^2+z^2\big)\big)}\\
&\quad\quad\sqrt{-\big(y^2-2\,y\,y1+y1^2+z^2\big)\,\beta^2}\Big)\Big]\Big)\Big/\\
&\Big(\sqrt{\big(a0^2+a1^2+2\,a1\,(-b0+b1)\,y-2\,a0\,(a1+(-b0+b1)\,y)+(b0-b1)^2\,\big(y^2+z^2\big)\big)}\Big)+\\
&\Big(\big(a0^2+a1\,(x-b0\,y)-a0\,(a1+x-2\,b0\,y+b1\,y)+(b0-b1)\,\big(-x\,y+b0\,\big(y^2+z^2\big)\big)\big)\\
&\quad\mathrm{ArcTan}\Big[\big(a0\,(y-y0)+x\,(-y+y0)+b0\,\big(y^2-y\,y0+z^2\big)\big)\Big/\\
&\quad\quad\Big(z\,\sqrt{\big(a0^2-2\,a0\,x+x^2-\big(y^2+z^2\big)\,\beta^2+y0^2\,\big(b0^2-\beta^2\big)+2\,y0\,\big(a0\,b0-b0\,x+y\,\beta^2\big)\big)}\Big)\Big]\Big)\Big/\\
&\big(a0^2+a1^2+2\,a1\,(-b0+b1)\,y-2\,a0\,(a1+(-b0+b1)\,y)+(b0-b1)^2\,\big(y^2+z^2\big)\big)-\\
&\Big(\big(a0^2+a1\,(x-b0\,y)-a0\,(a1+x-2\,b0\,y+b1\,y)+(b0-b1)\,\big(-x\,y+b0\,\big(y^2+z^2\big)\big)\big)\\
&\quad\mathrm{ArcTan}\Big[\big(a0\,(y-y1)+x\,(-y+y1)+b0\,\big(y^2-y\,y1+z^2\big)\big)\Big/\\
&\quad\quad\Big(z\,\sqrt{\big(a0^2-2\,a0\,x+x^2-\big(y^2+z^2\big)\,\beta^2+y1^2\,\big(b0^2-\beta^2\big)+2\,y1\,\big(a0\,b0-b0\,x+y\,\beta^2\big)\big)}\Big)\Big]\Big)\Big/\\
&\big(a0^2+a1^2+2\,a1\,(-b0+b1)\,y-2\,a0\,(a1+(-b0+b1)\,y)+(b0-b1)^2\,\big(y^2+z^2\big)\big)-\\
&\Big((a1\,b0-a0\,b1+(-b0+b1)\,x)\,z\,\mathrm{ArcTanh}\Big[(a0-x+b0\,y0)\Big/\\
&\quad\quad\Big(\sqrt{\big(a0^2-2\,a0\,x+x^2-\big(y^2+z^2\big)\,\beta^2+y0^2\,\big(b0^2-\beta^2\big)+2\,y0\,\big(a0\,b0-b0\,x+y\,\beta^2\big)\big)}\Big)\Big]\Big)\Big/\\
&\big(a0^2+a1^2+2\,a1\,(-b0+b1)\,y-2\,a0\,(a1+(-b0+b1)\,y)+(b0-b1)^2\,\big(y^2+z^2\big)\big)+\\
&\Big((a1\,b0-a0\,b1+(-b0+b1)\,x)\,z\,\mathrm{ArcTanh}\Big[(a0-x+b0\,y1)\Big/\\
&\quad\quad\Big(\sqrt{\big(a0^2-2\,a0\,x+x^2-\big(y^2+z^2\big)\,\beta^2+y1^2\,\big(b0^2-\beta^2\big)+2\,y1\,\big(a0\,b0-b0\,x+y\,\beta^2\big)\big)}\Big)\Big]\Big)\Big/\\
&\big(a0^2+a1^2+2\,a1\,(-b0+b1)\,y-2\,a0\,(a1+(-b0+b1)\,y)+(b0-b1)^2\,\big(y^2+z^2\big)\big)-\\
&\Big(z\,\sqrt{\Big(a1^2\,\big(b0^2-\beta^2\big)+a0^2\,\big(b1^2-\beta^2\big)+2\,a0\,(b0-b1)\,\big(b1\,x-y\,\beta^2\big)+}\\
&\quad\quad\overline{(b0-b1)^2\,\big(x^2-\big(y^2+z^2\big)\,\beta^2\big)-2\,a1\,\big(a0\,\big(b0\,b1-\beta^2\big)+(b0-b1)\,\big(b0\,x-y\,\beta^2\big)\big)\Big)}\\
&\quad\mathrm{ArcTanh}\Big[\big(a0^2\,b1+a0\,\big(-a1\,b0+b0\,x-2\,b1\,x+b0\,b1\,y0+y\,\beta^2-y0\,\beta^2\big)+\\
&\quad\quad a1\,\big(b0\,x-b0^2\,y0+(-y+y0)\,\beta^2\big)+(b0-b1)\,\big(-x^2+b0\,x\,y0+\big(y^2-y\,y0+z^2\big)\,\beta^2\big)\big)\Big/\\
&\quad\quad\Big(\sqrt{\big(a0^2-2\,a0\,x+x^2-\big(y^2+z^2\big)\,\beta^2+y0^2\,\big(b0^2-\beta^2\big)+2\,y0\,\big(a0\,b0-b0\,x+y\,\beta^2\big)\big)}\\
&\quad\quad\sqrt{\big(a1^2\,\big(b0^2-\beta^2\big)+a0^2\,\big(b1^2-\beta^2\big)+2\,a0\,(b0-b1)\,\big(b1\,x-y\,\beta^2\big)+(b0-b1)^2}\\
&\quad\quad\quad\overline{\big(x^2-\big(y^2+z^2\big)\,\beta^2\big)-2\,a1\,\big(a0\,\big(b0\,b1-\beta^2\big)+(b0-b1)\,\big(b0\,x-y\,\beta^2\big)\big)\big)}\Big)\Big]\Big)\Big/\\
&\big(a0^2+a1^2+2\,a1\,(-b0+b1)\,y-2\,a0\,(a1+(-b0+b1)\,y)+(b0-b1)^2\,\big(y^2+z^2\big)\big)+\\
&\Big(z\,\sqrt{\Big(a1^2\,\big(b0^2-\beta^2\big)+a0^2\,\big(b1^2-\beta^2\big)+2\,a0\,(b0-b1)\,\big(b1\,x-y\,\beta^2\big)+}\\
&\quad\quad\overline{(b0-b1)^2\,\big(x^2-\big(y^2+z^2\big)\,\beta^2\big)-2\,a1\,\big(a0\,\big(b0\,b1-\beta^2\big)+(b0-b1)\,\big(b0\,x-y\,\beta^2\big)\big)\Big)}\\
&\quad\mathrm{ArcTanh}\Big[\big(a0^2\,b1+a0\,\big(-a1\,b0+b0\,x-2\,b1\,x+b0\,b1\,y1+y\,\beta^2-y1\,\beta^2\big)+\\
&\quad\quad a1\,\big(b0\,x-b0^2\,y1+(-y+y1)\,\beta^2\big)+(b0-b1)\,\big(-x^2+b0\,x\,y1+\big(y^2-y\,y1+z^2\big)\,\beta^2\big)\big)\Big/\\
&\quad\quad\Big(\sqrt{\big(a0^2-2\,a0\,x+x^2-\big(y^2+z^2\big)\,\beta^2+y1^2\,\big(b0^2-\beta^2\big)+2\,y1\,\big(a0\,b0-b0\,x+y\,\beta^2\big)\big)}\\
&\quad\quad\sqrt{\big(a1^2\,\big(b0^2-\beta^2\big)+a0^2\,\big(b1^2-\beta^2\big)+2\,a0\,(b0-b1)\,\big(b1\,x-y\,\beta^2\big)+(b0-b1)^2}\\
&\quad\quad\quad\overline{\big(x^2-\big(y^2+z^2\big)\,\beta^2\big)-2\,a1\,\big(a0\,\big(b0\,b1-\beta^2\big)+(b0-b1)\,\big(b0\,x-y\,\beta^2\big)\big)\big)}\Big)\Big]\Big)\Big/\\
&\big(a0^2+a1^2+2\,a1\,(-b0+b1)\,y-2\,a0\,(a1+(-b0+b1)\,y)+(b0-b1)^2\,\big(y^2+z^2\big)\big)\Bigg)
\end{aligned}
$$

In[398]:=
```
(*Tapered Supersonic Doublet Panel Perturbation Velocity u Component*)
u = D[ϕ[x, y, z], x]
```

Out[398]=
```
 1
---- (μ0 - μ1)
2 π
 (-(((a1 b0 - a0 b1 + (-b0 + b1) x) z (-(((-2 a0 + 2 x - 2 b0 y0) (a0 - x + b0 y0)) / (2 (a0^2 -
              2 a0 x + x^2 - (y^2 + z^2) β^2 + y0^2 (b0^2 - β^2) + 2 y0 (a0 b0 - b0 x + y β^2))^(3/2))) -
        1 / (√(a0^2 - 2 a0 x + x^2 - (y^2 + z^2) β^2 + y0^2 (b0^2 - β^2) + 2 y0 (a0 b0 - b0 x + y β^2))))) /
     ((a0^2 + a1^2 + 2 a1 (-b0 + b1) y - 2 a0 (a1 + (-b0 + b1) y) + (b0 - b1)^2 (y^2 + z^2))
       (1 - (a0 - x + b0 y0)^2 /
          (a0^2 - 2 a0 x + x^2 - (y^2 + z^2) β^2 + y0^2 (b0^2 - β^2) + 2 y0 (a0 b0 - b0 x + y β^2))))) +
   ((a0^2 + a1 (x - b0 y) - a0 (a1 + x - 2 b0 y + b1 y) + (b0 - b1) (-x y + b0 (y^2 + z^2)))
      (-(((-2 a0 + 2 x - 2 b0 y0) (a0 (y - y0) + x (-y + y0) + b0 (y^2 - y y0 + z^2))) / (2 z
             (a0^2 - 2 a0 x + x^2 - (y^2 + z^2) β^2 + y0^2 (b0^2 - β^2) + 2 y0 (a0 b0 - b0 x + y β^2))^(3/2))) +
        (-y + y0) / (z √(a0^2 - 2 a0 x + x^2 - (y^2 + z^2) β^2 + y0^2 (b0^2 - β^2) +
              2 y0 (a0 b0 - b0 x + y β^2))))) /
    ((a0^2 + a1^2 + 2 a1 (-b0 + b1) y - 2 a0 (a1 + (-b0 + b1) y) + (b0 - b1)^2 (y^2 + z^2))
      (1 + (a0 (y - y0) + x (-y + y0) + b0 (y^2 - y y0 + z^2))^2 /
         (z^2 (a0^2 - 2 a0 x + x^2 - (y^2 + z^2) β^2 + y0^2 (b0^2 - β^2) + 2 y0 (a0 b0 - b0 x + y β^2))))) +
   ((a1 b0 - a0 b1 + (-b0 + b1) x) z (-(((-2 a0 + 2 x - 2 b0 y1) (a0 - x + b0 y1)) /
           (2 (a0^2 - 2 a0 x + x^2 - (y^2 + z^2) β^2 + y1^2 (b0^2 - β^2) + 2 y1 (a0 b0 - b0 x + y β^2))^(3/2))) -
        1 / (√(a0^2 - 2 a0 x + x^2 - (y^2 + z^2) β^2 + y1^2 (b0^2 - β^2) + 2 y1 (a0 b0 - b0 x + y β^2))))) /
    ((a0^2 + a1^2 + 2 a1 (-b0 + b1) y - 2 a0 (a1 + (-b0 + b1) y) + (b0 - b1)^2 (y^2 + z^2))
      (1 - (a0 - x + b0 y1)^2 /
         (a0^2 - 2 a0 x + x^2 - (y^2 + z^2) β^2 + y1^2 (b0^2 - β^2) + 2 y1 (a0 b0 - b0 x + y β^2)))) -
   ((a0^2 + a1 (x - b0 y) - a0 (a1 + x - 2 b0 y + b1 y) + (b0 - b1) (-x y + b0 (y^2 + z^2)))
      (-(((-2 a0 + 2 x - 2 b0 y1) (a0 (y - y1) + x (-y + y1) + b0 (y^2 - y y1 + z^2))) / (2 z
             (a0^2 - 2 a0 x + x^2 - (y^2 + z^2) β^2 + y1^2 (b0^2 - β^2) + 2 y1 (a0 b0 - b0 x + y β^2))^(3/2))) +
        (-y + y1) / (z √(a0^2 - 2 a0 x + x^2 - (y^2 + z^2) β^2 + y1^2 (b0^2 - β^2) +
              2 y1 (a0 b0 - b0 x + y β^2))))) /
    ((a0^2 + a1^2 + 2 a1 (-b0 + b1) y - 2 a0 (a1 + (-b0 + b1) y) + (b0 - b1)^2 (y^2 + z^2))
      (1 + (a0 (y - y1) + x (-y + y1) + b0 (y^2 - y y1 + z^2))^2 /
         (z^2 (a0^2 - 2 a0 x + x^2 - (y^2 + z^2) β^2 + y1^2 (b0^2 - β^2) + 2 y1 (a0 b0 - b0 x + y β^2))))) -
   (z √(a1^2 (b0^2 - β^2) + a0^2 (b1^2 - β^2) + 2 a0 (b0 - b1) (b1 x - y β^2) +
          (b0 - b1)^2 (x^2 - (y^2 + z^2) β^2) - 2 a1 (a0 (b0 b1 - β^2) + (b0 - b1) (b0 x - y β^2)))
      (-(((-2 a1 b0 (b0 - b1) + 2 a0 (b0 - b1) b1 + 2 (b0 - b1)^2 x)
              (a0^2 b1 + a0 (-a1 b0 + b0 x - 2 b1 x + b0 b1 y0 + y β^2 - y0 β^2) + a1 (b0 x - b0^2 y0 +
                    (-y + y0) β^2) + (b0 - b1) (-x^2 + b0 x y0 + (y^2 - y y0 + z^2) β^2))) /
            (2 √(a0^2 - 2 a0 x + x^2 - (y^2 + z^2) β^2 + y0^2 (b0^2 - β^2) + 2 y0 (a0 b0 - b0 x + y β^2))
```

$$(a1^2\ (b0^2-\beta^2)+a0^2\ (b1^2-\beta^2)+2\ a0\ (b0-b1)\ (b1\ x-y\ \beta^2)+(b0-b1)^2$$

$$(x^2-(y^2+z^2)\ \beta^2)-2\ a1\ (a0\ (b0\ b1-\beta^2)+(b0-b1)\ (b0\ x-y\ \beta^2)))^{3/2}))\ +$$

$$(a1\ b0+a0\ (b0-2\ b1)+(b0-b1)\ (-2\ x+b0\ y0))\ /$$

$$(\sqrt{(}a0^2-2\ a0\ x+x^2-(y^2+z^2)\ \beta^2+y0^2\ (b0^2-\beta^2)+2\ y0\ (a0\ b0-b0\ x+y\ \beta^2))$$

$$\sqrt{(}a1^2\ (b0^2-\beta^2)+a0^2\ (b1^2-\beta^2)+2\ a0\ (b0-b1)\ (b1\ x-y\ \beta^2)+(b0-b1)^2$$

$$(x^2-(y^2+z^2)\ \beta^2)-2\ a1\ (a0\ (b0\ b1-\beta^2)+(b0-b1)\ (b0\ x-y\ \beta^2))))\ -$$

$$((-2\ a0+2\ x-2\ b0\ y0)\ (a0^2\ b1+a0\ (-a1\ b0+b0\ x-2\ b1\ x+b0\ b1\ y0+y\ \beta^2-y0\ \beta^2)+a1$$

$$(b0\ x-b0^2\ y0+(-y+y0)\ \beta^2)+(b0-b1)\ (-x^2+b0\ x\ y0+(y^2-y\ y0+z^2)\ \beta^2)))\ /$$

$$(2\ (a0^2-2\ a0\ x+x^2-(y^2+z^2)\ \beta^2+y0^2\ (b0^2-\beta^2)+2\ y0\ (a0\ b0-b0\ x+y\ \beta^2))^{3/2}$$

$$\sqrt{(}a1^2\ (b0^2-\beta^2)+a0^2\ (b1^2-\beta^2)+2\ a0\ (b0-b1)\ (b1\ x-y\ \beta^2)+(b0-b1)^2$$

$$(x^2-(y^2+z^2)\ \beta^2)-2\ a1\ (a0\ (b0\ b1-\beta^2)+(b0-b1)\ (b0\ x-y\ \beta^2))))))\ /$$

$$((a0^2+a1^2+2\ a1\ (-b0+b1)\ y-2\ a0\ (a1+(-b0+b1)\ y)+(b0-b1)^2\ (y^2+z^2))$$

$$(1-(a0^2\ b1+a0\ (-a1\ b0+b0\ x-2\ b1\ x+b0\ b1\ y0+y\ \beta^2-y0\ \beta^2)+a1$$

$$(b0\ x-b0^2\ y0+(-y+y0)\ \beta^2)+(b0-b1)\ (-x^2+b0\ x\ y0+(y^2-y\ y0+z^2)\ \beta^2))^2\ /$$

$$((a0^2-2\ a0\ x+x^2-(y^2+z^2)\ \beta^2+y0^2\ (b0^2-\beta^2)+2\ y0\ (a0\ b0-b0\ x+y\ \beta^2))$$

$$(a1^2\ (b0^2-\beta^2)+a0^2\ (b1^2-\beta^2)+2\ a0\ (b0-b1)\ (b1\ x-y\ \beta^2)+(b0-b1)^2$$

$$(x^2-(y^2+z^2)\ \beta^2)-2\ a1\ (a0\ (b0\ b1-\beta^2)+(b0-b1)\ (b0\ x-y\ \beta^2))))))))\ +$$

$$(z\ \sqrt{(}a1^2\ (b0^2-\beta^2)+a0^2\ (b1^2-\beta^2)+2\ a0\ (b0-b1)\ (b1\ x-y\ \beta^2)\ +$$

$$(b0-b1)^2\ (x^2-(y^2+z^2)\ \beta^2)-2\ a1\ (a0\ (b0\ b1-\beta^2)+(b0-b1)\ (b0\ x-y\ \beta^2))))$$

$$(-(((-2\ a1\ b0\ (b0-b1)+2\ a0\ (b0-b1)\ b1+2\ (b0-b1)^2\ x)$$

$$(a0^2\ b1+a0\ (-a1\ b0+b0\ x-2\ b1\ x+b0\ b1\ y1+y\ \beta^2-y1\ \beta^2)+a1\ (b0\ x-b0^2\ y1\ +$$

$$(-y+y1)\ \beta^2)+(b0-b1)\ (-x^2+b0\ x\ y1+(y^2-y\ y1+z^2)\ \beta^2)))\ /$$

$$(2\ \sqrt{(}a0^2-2\ a0\ x+x^2-(y^2+z^2)\ \beta^2+y1^2\ (b0^2-\beta^2)+2\ y1\ (a0\ b0-b0\ x+y\ \beta^2))$$

$$(a1^2\ (b0^2-\beta^2)+a0^2\ (b1^2-\beta^2)+2\ a0\ (b0-b1)\ (b1\ x-y\ \beta^2)+(b0-b1)^2$$

$$(x^2-(y^2+z^2)\ \beta^2)-2\ a1\ (a0\ (b0\ b1-\beta^2)+(b0-b1)\ (b0\ x-y\ \beta^2)))^{3/2}))\ +$$

$$(a1\ b0+a0\ (b0-2\ b1)+(b0-b1)\ (-2\ x+b0\ y1))\ /$$

$$(\sqrt{(}a0^2-2\ a0\ x+x^2-(y^2+z^2)\ \beta^2+y1^2\ (b0^2-\beta^2)+2\ y1\ (a0\ b0-b0\ x+y\ \beta^2))$$

$$\sqrt{(}a1^2\ (b0^2-\beta^2)+a0^2\ (b1^2-\beta^2)+2\ a0\ (b0-b1)\ (b1\ x-y\ \beta^2)+(b0-b1)^2$$

$$(x^2-(y^2+z^2)\ \beta^2)-2\ a1\ (a0\ (b0\ b1-\beta^2)+(b0-b1)\ (b0\ x-y\ \beta^2))))\ -$$

$$((-2\ a0+2\ x-2\ b0\ y1)\ (a0^2\ b1+a0\ (-a1\ b0+b0\ x-2\ b1\ x+b0\ b1\ y1+y\ \beta^2-y1\ \beta^2)+a1$$

$$(b0\ x-b0^2\ y1+(-y+y1)\ \beta^2)+(b0-b1)\ (-x^2+b0\ x\ y1+(y^2-y\ y1+z^2)\ \beta^2)))\ /$$

$$(2\ (a0^2-2\ a0\ x+x^2-(y^2+z^2)\ \beta^2+y1^2\ (b0^2-\beta^2)+2\ y1\ (a0\ b0-b0\ x+y\ \beta^2))^{3/2}$$

$$\sqrt{(}a1^2\ (b0^2-\beta^2)+a0^2\ (b1^2-\beta^2)+2\ a0\ (b0-b1)\ (b1\ x-y\ \beta^2)+(b0-b1)^2$$

$$(x^2-(y^2+z^2)\ \beta^2)-2\ a1\ (a0\ (b0\ b1-\beta^2)+(b0-b1)\ (b0\ x-y\ \beta^2))))))\ /$$

$$((a0^2+a1^2+2\ a1\ (-b0+b1)\ y-2\ a0\ (a1+(-b0+b1)\ y)+(b0-b1)^2\ (y^2+z^2))$$

$$(1-(a0^2\ b1+a0\ (-a1\ b0+b0\ x-2\ b1\ x+b0\ b1\ y1+y\ \beta^2-y1\ \beta^2)+a1$$

$$(b0\ x-b0^2\ y1+(-y+y1)\ \beta^2)+(b0-b1)\ (-x^2+b0\ x\ y1+(y^2-y\ y1+z^2)\ \beta^2))^2\ /$$

$$((a0^2-2\ a0\ x+x^2-(y^2+z^2)\ \beta^2+y1^2\ (b0^2-\beta^2)+2\ y1\ (a0\ b0-b0\ x+y\ \beta^2))$$

$$(a1^2\ (b0^2-\beta^2)+a0^2\ (b1^2-\beta^2)+2\ a0\ (b0-b1)\ (b1\ x-y\ \beta^2)+(b0-b1)^2$$

$$\left(x^2-\left(y^2+z^2\right)\beta^2\right)-2\,a1\left(a0\left(b0\,b1-\beta^2\right)+(b0-b1)\left(b0\,x-y\,\beta^2\right)\right)))))+$$

$$((-a0+a1-(b0-b1)\,y)\,\mathrm{ArcTan}\Big[\left(a0\,(y-y0)+x\,(-y+y0)+b0\left(y^2-y\,y0+z^2\right)\right)/$$
$$\left(z\surd\left(a0^2-2\,a0\,x+x^2-\left(y^2+z^2\right)\beta^2+y0^2\left(b0^2-\beta^2\right)+2\,y0\left(a0\,b0-b0\,x+y\,\beta^2\right)\right)\right)\Big])/$$
$$\Big(a0^2+a1^2+2\,a1\,(-b0+b1)\,y-2\,a0\,(a1+(-b0+b1)\,y)+$$
$$(b0-b1)^2\left(y^2+z^2\right)\Big)-$$

$$((-a0+a1-(b0-b1)\,y)\,\mathrm{ArcTan}\Big[\left(a0\,(y-y1)+x\,(-y+y1)+b0\left(y^2-y\,y1+z^2\right)\right)/$$
$$\left(z\surd\left(a0^2-2\,a0\,x+x^2-\left(y^2+z^2\right)\beta^2+y1^2\left(b0^2-\beta^2\right)+2\,y1\left(a0\,b0-b0\,x+y\,\beta^2\right)\right)\right)\Big])/$$
$$\Big(a0^2+a1^2+2\,a1\,(-b0+b1)\,y-2\,a0\,(a1+(-b0+b1)\,y)+$$
$$(b0-b1)^2\left(y^2+z^2\right)\Big)-$$

$$((-b0+b1)\,z\,\mathrm{ArcTanh}\Big[(a0-x+b0\,y0)/$$
$$\left(\surd\left(a0^2-2\,a0\,x+x^2-\left(y^2+z^2\right)\beta^2+y0^2\left(b0^2-\beta^2\right)+2\,y0\left(a0\,b0-b0\,x+y\,\beta^2\right)\right)\right)\Big])/$$
$$\Big(a0^2+a1^2+2\,a1\,(-b0+b1)\,y-2\,a0\,(a1+(-b0+b1)\,y)+$$
$$(b0-b1)^2\left(y^2+z^2\right)\Big)+$$

$$((-b0+b1)\,z\,\mathrm{ArcTanh}\Big[(a0-x+b0\,y1)/$$
$$\left(\surd\left(a0^2-2\,a0\,x+x^2-\left(y^2+z^2\right)\beta^2+y1^2\left(b0^2-\beta^2\right)+2\,y1\left(a0\,b0-b0\,x+y\,\beta^2\right)\right)\right)\Big])/$$
$$\Big(a0^2+a1^2+2\,a1\,(-b0+b1)\,y-2\,a0\,(a1+(-b0+b1)\,y)+$$
$$(b0-b1)^2\left(y^2+z^2\right)\Big)-$$

$$\left(\left(-2\,a1\,b0\,(b0-b1)+2\,a0\,(b0-b1)\,b1+2\,(b0-b1)^2\,x\right)z\right.$$
$$\mathrm{ArcTanh}\Big[\Big(a0^2\,b1+a0\left(-a1\,b0+b0\,x-2\,b1\,x+b0\,b1\,y0+y\,\beta^2-y0\,\beta^2\right)+$$
$$a1\left(b0\,x-b0^2\,y0+(-y+y0)\,\beta^2\right)+(b0-b1)\left(-x^2+b0\,x\,y0+\left(y^2-y\,y0+z^2\right)\beta^2\right)\Big)/$$
$$\Big(\surd\left(a0^2-2\,a0\,x+x^2-\left(y^2+z^2\right)\beta^2+y0^2\left(b0^2-\beta^2\right)+2\,y0\left(a0\,b0-b0\,x+y\,\beta^2\right)\right)$$
$$\surd\Big(a1^2\left(b0^2-\beta^2\right)+a0^2\left(b1^2-\beta^2\right)+2\,a0\,(b0-b1)\left(b1\,x-y\,\beta^2\right)+(b0-b1)^2$$
$$\left(x^2-\left(y^2+z^2\right)\beta^2\right)-2\,a1\left(a0\left(b0\,b1-\beta^2\right)+(b0-b1)\left(b0\,x-y\,\beta^2\right)\right)\Big)\Big)\Big]\Big)/$$
$$\Big(2\left(a0^2+a1^2+2\,a1\,(-b0+b1)\,y-2\,a0\,(a1+(-b0+b1)\,y)+(b0-b1)^2\left(y^2+z^2\right)\right)$$
$$\surd\Big(a1^2\left(b0^2-\beta^2\right)+a0^2\left(b1^2-\beta^2\right)+2\,a0\,(b0-b1)\left(b1\,x-y\,\beta^2\right)+$$
$$(b0-b1)^2\left(x^2-\left(y^2+z^2\right)\beta^2\right)-2\,a1\left(a0\left(b0\,b1-\beta^2\right)+(b0-b1)\left(b0\,x-y\,\beta^2\right)\right)\Big)\Big)+$$

$$\left(\left(-2\,a1\,b0\,(b0-b1)+2\,a0\,(b0-b1)\,b1+2\,(b0-b1)^2\,x\right)z\right.$$
$$\mathrm{ArcTanh}\Big[\Big(a0^2\,b1+a0\left(-a1\,b0+b0\,x-2\,b1\,x+b0\,b1\,y1+y\,\beta^2-y1\,\beta^2\right)+$$
$$a1\left(b0\,x-b0^2\,y1+(-y+y1)\,\beta^2\right)+(b0-b1)\left(-x^2+b0\,x\,y1+\left(y^2-y\,y1+z^2\right)\beta^2\right)\Big)/$$
$$\Big(\surd\left(a0^2-2\,a0\,x+x^2-\left(y^2+z^2\right)\beta^2+y1^2\left(b0^2-\beta^2\right)+2\,y1\left(a0\,b0-b0\,x+y\,\beta^2\right)\right)$$
$$\surd\Big(a1^2\left(b0^2-\beta^2\right)+a0^2\left(b1^2-\beta^2\right)+2\,a0\,(b0-b1)\left(b1\,x-y\,\beta^2\right)+(b0-b1)^2$$
$$\left(x^2-\left(y^2+z^2\right)\beta^2\right)-2\,a1\left(a0\left(b0\,b1-\beta^2\right)+(b0-b1)\left(b0\,x-y\,\beta^2\right)\right)\Big)\Big)\Big]\Big)/$$
$$\Big(2\left(a0^2+a1^2+2\,a1\,(-b0+b1)\,y-2\,a0\,(a1+(-b0+b1)\,y)+(b0-b1)^2\left(y^2+z^2\right)\right)$$
$$\surd\Big(a1^2\left(b0^2-\beta^2\right)+a0^2\left(b1^2-\beta^2\right)+2\,a0\,(b0-b1)\left(b1\,x-y\,\beta^2\right)+$$
$$(b0-b1)^2\left(x^2-\left(y^2+z^2\right)\beta^2\right)-2\,a1\left(a0\left(b0\,b1-\beta^2\right)+(b0-b1)\left(b0\,x-y\,\beta^2\right)\right)\Big)\Big)\Big)$$

In[399]:=
```
(*Tapered Supersonic Doublet Panel Perturbation Velocity v Component*)
v = D[ϕ[x, y, z], y]
```

Out[399]=
$$\frac{1}{2\pi}(\mu0-\mu1)\,\Big(\Big(z\,\beta\,\Big(\left((2\,y-2\,y0)\left(a0\,(y-y0)+a1\,(-y+y0)+(b0-b1)\left(y^2-y\,y0+z^2\right)\right)\beta^3\right)/$$
$$\Big(2\surd\left(a0^2+a1^2+2\,a1\,(-b0+b1)\,y-2\,a0\,(a1+(-b0+b1)\,y)+(b0-b1)^2\left(y^2+z^2\right)\right)$$
$$\left(-\left(y^2-2\,y\,y0+y0^2+z^2\right)\beta^2\right)^{3/2}\Big)+((a0-a1+(b0-b1)\,(2\,y-y0))\,\beta)/$$

$$\Big(\sqrt{\left(a0^2+a1^2+2\,a1\,(-b0+b1)\,y-2\,a0\,(a1+(-b0+b1)\,y)+(b0-b1)^2\,\left(y^2+z^2\right)\right)}\,\sqrt{-\left(y^2-2\,y\,y0+y0^2+z^2\right)\beta^2}\Big)-$$

$$\Big(\Big(-2\,a0\,(-b0+b1)+2\,a1\,(-b0+b1)+2\,(b0-b1)^2\,y\Big)\,\Big(a0\,(y-y0)+a1\,(-y+y0)+(b0-b1)\,\left(y^2-y\,y0+z^2\right)\Big)\,\beta\Big)\Big/$$

$$\Big(2\,\left(a0^2+a1^2+2\,a1\,(-b0+b1)\,y-2\,a0\,(a1+(-b0+b1)\,y)+(b0-b1)^2\,\left(y^2+z^2\right)\right)^{3/2}\,\sqrt{-\left(y^2-2\,y\,y0+y0^2+z^2\right)\beta^2}\Big)\Big)\Big)\Big/$$

$$\Big(\sqrt{\left(a0^2+a1^2+2\,a1\,(-b0+b1)\,y-2\,a0\,(a1+(-b0+b1)\,y)+(b0-b1)^2\,\left(y^2+z^2\right)\right)}\,\Big(1-\Big(a0\,(y-y0)+a1\,(-y+y0)+(b0-b1)\,\left(y^2-y\,y0+z^2\right)\Big)^2\Big/\Big(\left(y^2-2\,y\,y0+y0^2+z^2\right)\left(a0^2+a1^2+2\,a1\,(-b0+b1)\,y-2\,a0\,(a1+(-b0+b1)\,y)+(b0-b1)^2\,\left(y^2+z^2\right)\right)\Big)\Big)\Big)-$$

$$\Big(z\,\beta\,\Big(\Big((2\,y-2\,y1)\,\Big(a0\,(y-y1)+a1\,(-y+y1)+(b0-b1)\,\left(y^2-y\,y1+z^2\right)\Big)\,\beta^3\Big)\Big/$$

$$\Big(2\,\sqrt{\left(a0^2+a1^2+2\,a1\,(-b0+b1)\,y-2\,a0\,(a1+(-b0+b1)\,y)+(b0-b1)^2\,\left(y^2+z^2\right)\right)}\,\left(-\left(y^2-2\,y\,y1+y1^2+z^2\right)\beta^2\right)^{3/2}\Big)+((a0-a1+(b0-b1)\,(2\,y-y1))\,\beta)\Big/$$

$$\Big(\sqrt{\left(a0^2+a1^2+2\,a1\,(-b0+b1)\,y-2\,a0\,(a1+(-b0+b1)\,y)+(b0-b1)^2\,\left(y^2+z^2\right)\right)}\,\sqrt{-\left(y^2-2\,y\,y1+y1^2+z^2\right)\beta^2}\Big)-$$

$$\Big(\Big(-2\,a0\,(-b0+b1)+2\,a1\,(-b0+b1)+2\,(b0-b1)^2\,y\Big)\,\Big(a0\,(y-y1)+a1\,(-y+y1)+(b0-b1)\,\left(y^2-y\,y1+z^2\right)\Big)\,\beta\Big)\Big/$$

$$\Big(2\,\left(a0^2+a1^2+2\,a1\,(-b0+b1)\,y-2\,a0\,(a1+(-b0+b1)\,y)+(b0-b1)^2\,\left(y^2+z^2\right)\right)^{3/2}\,\sqrt{-\left(y^2-2\,y\,y1+y1^2+z^2\right)\beta^2}\Big)\Big)\Big)\Big/$$

$$\Big(\sqrt{\left(a0^2+a1^2+2\,a1\,(-b0+b1)\,y-2\,a0\,(a1+(-b0+b1)\,y)+(b0-b1)^2\,\left(y^2+z^2\right)\right)}\,\Big(1-\Big(a0\,(y-y1)+a1\,(-y+y1)+(b0-b1)\,\left(y^2-y\,y1+z^2\right)\Big)^2\Big/\Big(\left(y^2-2\,y\,y1+y1^2+z^2\right)\left(a0^2+a1^2+2\,a1\,(-b0+b1)\,y-2\,a0\,(a1+(-b0+b1)\,y)+(b0-b1)^2\,\left(y^2+z^2\right)\right)\Big)\Big)\Big)+$$

$$\Big((a1\,b0-a0\,b1+(-b0+b1)\,x)\,(a0-x+b0\,y0)\,z\,\left(-2\,y\,\beta^2+2\,y0\,\beta^2\right)\Big)\Big/$$

$$\Big(2\,\left(a0^2+a1^2+2\,a1\,(-b0+b1)\,y-2\,a0\,(a1+(-b0+b1)\,y)+(b0-b1)^2\,\left(y^2+z^2\right)\right)\,\Big(a0^2-2\,a0\,x+x^2-\left(y^2+z^2\right)\beta^2+y0^2\,\left(b0^2-\beta^2\right)+2\,y0\,\left(a0\,b0-b0\,x+y\,\beta^2\right)\Big)^{3/2}$$

$$\Big(1-(a0-x+b0\,y0)^2\Big/\Big(a0^2-2\,a0\,x+x^2-\left(y^2+z^2\right)\beta^2+y0^2\,\left(b0^2-\beta^2\right)+2\,y0\,\left(a0\,b0-b0\,x+y\,\beta^2\right)\Big)\Big)\Big)+$$

$$\Big(\Big(a0^2+a1\,(x-b0\,y)-a0\,(a1+x-2\,b0\,y+b1\,y)+(b0-b1)\,\left(-x\,y+b0\,\left(y^2+z^2\right)\right)\Big)$$

$$\Big(-\Big(\Big(\Big(a0\,(y-y0)+x\,(-y+y0)+b0\,\left(y^2-y\,y0+z^2\right)\Big)\,\left(-2\,y\,\beta^2+2\,y0\,\beta^2\right)\Big)\Big/\Big(2\,z\,\Big(a0^2-2\,a0\,x+x^2-\left(y^2+z^2\right)\beta^2+y0^2\,\left(b0^2-\beta^2\right)+2\,y0\,\left(a0\,b0-b0\,x+y\,\beta^2\right)\Big)^{3/2}\Big)\Big)+$$

$$(a0-x+b0\,(2\,y-y0))\Big/\Big(z\,\sqrt{\left(a0^2-2\,a0\,x+x^2-\left(y^2+z^2\right)\beta^2+y0^2\,\left(b0^2-\beta^2\right)+2\,y0\,\left(a0\,b0-b0\,x+y\,\beta^2\right)\right)}\Big)\Big)\Big)\Big/$$

$$\Big(\left(a0^2+a1^2+2\,a1\,(-b0+b1)\,y-2\,a0\,(a1+(-b0+b1)\,y)+(b0-b1)^2\,\left(y^2+z^2\right)\right)$$

$$\Big(1+\Big(a0\,(y-y0)+x\,(-y+y0)+b0\,\left(y^2-y\,y0+z^2\right)\Big)^2\Big/$$

```
        (z^2 (a0^2 - 2 a0 x + x^2 - (y^2 + z^2) β^2 + y0^2 (b0^2 - β^2) + 2 y0 (a0 b0 - b0 x + y β^2))))) -
 (((a1 b0 - a0 b1 + (-b0 + b1) x) (a0 - x + b0 y1) z (-2 y β^2 + 2 y1 β^2)) /
  (2 (a0^2 + a1^2 + 2 a1 (-b0 + b1) y - 2 a0 (a1 + (-b0 + b1) y) + (b0 - b1)^2 (y^2 + z^2))
    (a0^2 - 2 a0 x + x^2 - (y^2 + z^2) β^2 + y1^2 (b0^2 - β^2) + 2 y1 (a0 b0 - b0 x + y β^2))^(3/2)
    (1 - (a0 - x + b0 y1)^2 /
        (a0^2 - 2 a0 x + x^2 - (y^2 + z^2) β^2 + y1^2 (b0^2 - β^2) + 2 y1 (a0 b0 - b0 x + y β^2))))) -
 ((a0^2 + a1 (x - b0 y) - a0 (a1 + x - 2 b0 y + b1 y) + (b0 - b1) (-x y + b0 (y^2 + z^2)))
    (-(((a0 (y - y1) + x (-y + y1) + b0 (y^2 - y y1 + z^2)) (-2 y β^2 + 2 y1 β^2)) / (2 z
             (a0^2 - 2 a0 x + x^2 - (y^2 + z^2) β^2 + y1^2 (b0^2 - β^2) + 2 y1 (a0 b0 - b0 x + y β^2))^(3/2))) +
      (a0 - x + b0 (2 y - y1)) / (z √(a0^2 - 2 a0 x + x^2 - (y^2 + z^2) β^2 +
             y1^2 (b0^2 - β^2) + 2 y1 (a0 b0 - b0 x + y β^2))))) /
  ((a0^2 + a1^2 + 2 a1 (-b0 + b1) y - 2 a0 (a1 + (-b0 + b1) y) + (b0 - b1)^2 (y^2 + z^2))
    (1 + (a0 (y - y1) + x (-y + y1) + b0 (y^2 - y y1 + z^2))^2 /
        (z^2 (a0^2 - 2 a0 x + x^2 - (y^2 + z^2) β^2 + y1^2 (b0^2 - β^2) + 2 y1 (a0 b0 - b0 x + y β^2))))) -
 (z √(a1^2 (b0^2 - β^2) + a0^2 (b1^2 - β^2) + 2 a0 (b0 - b1) (b1 x - y β^2) +
        (b0 - b1)^2 (x^2 - (y^2 + z^2) β^2) - 2 a1 (a0 (b0 b1 - β^2) + (b0 - b1) (b0 x - y β^2)))
    (-(((-2 a0 (b0 - b1) β^2 + 2 a1 (b0 - b1) β^2 - 2 (b0 - b1)^2 y β^2)
             (a0^2 b1 + a0 (-a1 b0 + b0 x - 2 b1 x + b0 b1 y0 + y β^2 - y0 β^2) + a1 (b0 x - b0^2 y0 +
                    (-y + y0) β^2) + (b0 - b1) (-x^2 + b0 x y0 + (y^2 - y y0 + z^2) β^2))) /
          (2 √(a0^2 - 2 a0 x + x^2 - (y^2 + z^2) β^2 + y0^2 (b0^2 - β^2) + 2 y0 (a0 b0 - b0 x + y β^2))
             (a1^2 (b0^2 - β^2) + a0^2 (b1^2 - β^2) + 2 a0 (b0 - b1) (b1 x - y β^2) + (b0 - b1)^2
                 (x^2 - (y^2 + z^2) β^2) - 2 a1 (a0 (b0 b1 - β^2) + (b0 - b1) (b0 x - y β^2)))^(3/2))) +
      (a0 β^2 - a1 β^2 + (b0 - b1) (2 y - y0) β^2) / (√(a0^2 - 2 a0 x + x^2 - (y^2 + z^2) β^2 +
             y0^2 (b0^2 - β^2) + 2 y0 (a0 b0 - b0 x + y β^2))
          √(a1^2 (b0^2 - β^2) + a0^2 (b1^2 - β^2) + 2 a0 (b0 - b1) (b1 x - y β^2) + (b0 - b1)^2
              (x^2 - (y^2 + z^2) β^2) - 2 a1 (a0 (b0 b1 - β^2) + (b0 - b1) (b0 x - y β^2)))) -
      ((-2 y β^2 + 2 y0 β^2) (a0^2 b1 + a0 (-a1 b0 + b0 x - 2 b1 x + b0 b1 y0 + y β^2 - y0 β^2) + a1
              (b0 x - b0^2 y0 + (-y + y0) β^2) + (b0 - b1) (-x^2 + b0 x y0 + (y^2 - y y0 + z^2) β^2))) /
        (2 (a0^2 - 2 a0 x + x^2 - (y^2 + z^2) β^2 + y0^2 (b0^2 - β^2) + 2 y0 (a0 b0 - b0 x + y β^2))^(3/2)
          √(a1^2 (b0^2 - β^2) + a0^2 (b1^2 - β^2) + 2 a0 (b0 - b1) (b1 x - y β^2) + (b0 - b1)^2
              (x^2 - (y^2 + z^2) β^2) - 2 a1 (a0 (b0 b1 - β^2) + (b0 - b1) (b0 x - y β^2)))))) /
  ((a0^2 + a1^2 + 2 a1 (-b0 + b1) y - 2 a0 (a1 + (-b0 + b1) y) + (b0 - b1)^2 (y^2 + z^2))
    (1 - (a0^2 b1 + a0 (-a1 b0 + b0 x - 2 b1 x + b0 b1 y0 + y β^2 - y0 β^2) + a1
             (b0 x - b0^2 y0 + (-y + y0) β^2) + (b0 - b1) (-x^2 + b0 x y0 + (y^2 - y y0 + z^2) β^2))^2 /
        ((a0^2 - 2 a0 x + x^2 - (y^2 + z^2) β^2 + y0^2 (b0^2 - β^2) + 2 y0 (a0 b0 - b0 x + y β^2))
          (a1^2 (b0^2 - β^2) + a0^2 (b1^2 - β^2) + 2 a0 (b0 - b1) (b1 x - y β^2) + (b0 - b1)^2
              (x^2 - (y^2 + z^2) β^2) - 2 a1 (a0 (b0 b1 - β^2) + (b0 - b1) (b0 x - y β^2)))))) +
 (z √(a1^2 (b0^2 - β^2) + a0^2 (b1^2 - β^2) + 2 a0 (b0 - b1) (b1 x - y β^2) +
```

$$(b0 - b1)^2 \; (x^2 - (y^2 + z^2) \beta^2) - 2 a1 \; (a0 \; (b0 \, b1 - \beta^2) + (b0 - b1) \; (b0 \, x - y \, \beta^2)))$$
$$(- (((-2 a0 \; (b0 - b1) \beta^2 + 2 a1 \; (b0 - b1) \beta^2 - 2 \; (b0 - b1)^2 y \, \beta^2)$$
$$(a0^2 b1 + a0 \; (-a1 \, b0 + b0 \, x - 2 b1 \, x + b0 \, b1 \, y1 + y \, \beta^2 - y1 \, \beta^2) + a1 \; (b0 \, x - b0^2 y1 +$$
$$(-y + y1) \beta^2) + (b0 - b1) \; (-x^2 + b0 \, x \, y1 + (y^2 - y \, y1 + z^2) \beta^2))) \Big/$$
$$(2 \surd (a0^2 - 2 a0 \, x + x^2 - (y^2 + z^2) \beta^2 + y1^2 \; (b0^2 - \beta^2) + 2 y1 \; (a0 \, b0 - b0 \, x + y \, \beta^2))$$
$$(a1^2 \; (b0^2 - \beta^2) + a0^2 \; (b1^2 - \beta^2) + 2 a0 \; (b0 - b1) \; (b1 \, x - y \, \beta^2) + (b0 - b1)^2$$
$$(x^2 - (y^2 + z^2) \beta^2) - 2 a1 \; (a0 \; (b0 \, b1 - \beta^2) + (b0 - b1) \; (b0 \, x - y \, \beta^2)))^{3/2})) +$$
$$(a0 \, \beta^2 - a1 \, \beta^2 + (b0 - b1) \; (2 y - y1) \beta^2) \Big/ (\surd (a0^2 - 2 a0 \, x + x^2 - (y^2 + z^2) \beta^2 +$$
$$y1^2 \; (b0^2 - \beta^2) + 2 y1 \; (a0 \, b0 - b0 \, x + y \, \beta^2))$$
$$\surd (a1^2 \; (b0^2 - \beta^2) + a0^2 \; (b1^2 - \beta^2) + 2 a0 \; (b0 - b1) \; (b1 \, x - y \, \beta^2) + (b0 - b1)^2$$
$$(x^2 - (y^2 + z^2) \beta^2) - 2 a1 \; (a0 \; (b0 \, b1 - \beta^2) + (b0 - b1) \; (b0 \, x - y \, \beta^2)))) -$$
$$((-2 y \, \beta^2 + 2 y1 \, \beta^2) \; (a0^2 b1 + a0 \; (-a1 \, b0 + b0 \, x - 2 b1 \, x + b0 \, b1 \, y1 + y \, \beta^2 - y1 \, \beta^2) + a1$$
$$(b0 \, x - b0^2 y1 + (-y + y1) \beta^2) + (b0 - b1) \; (-x^2 + b0 \, x \, y1 + (y^2 - y \, y1 + z^2) \beta^2))) \Big/$$
$$(2 \; (a0^2 - 2 a0 \, x + x^2 - (y^2 + z^2) \beta^2 + y1^2 \; (b0^2 - \beta^2) + 2 y1 \; (a0 \, b0 - b0 \, x + y \, \beta^2))^{3/2}$$
$$\surd (a1^2 \; (b0^2 - \beta^2) + a0^2 \; (b1^2 - \beta^2) + 2 a0 \; (b0 - b1) \; (b1 \, x - y \, \beta^2) + (b0 - b1)^2$$
$$(x^2 - (y^2 + z^2) \beta^2) - 2 a1 \; (a0 \; (b0 \, b1 - \beta^2) + (b0 - b1) \; (b0 \, x - y \, \beta^2)))))) \Big/$$
$$((a0^2 + a1^2 + 2 a1 \; (-b0 + b1) \; y - 2 a0 \; (a1 + (-b0 + b1) \; y) + (b0 - b1)^2 \; (y^2 + z^2))$$
$$(1 - (a0^2 b1 + a0 \; (-a1 \, b0 + b0 \, x - 2 b1 \, x + b0 \, b1 \, y1 + y \, \beta^2 - y1 \, \beta^2) + a1$$
$$(b0 \, x - b0^2 y1 + (-y + y1) \beta^2) + (b0 - b1) \; (-x^2 + b0 \, x \, y1 + (y^2 - y \, y1 + z^2) \beta^2))^2 \Big/$$
$$((a0^2 - 2 a0 \, x + x^2 - (y^2 + z^2) \beta^2 + y1^2 \; (b0^2 - \beta^2) + 2 y1 \; (a0 \, b0 - b0 \, x + y \, \beta^2))$$
$$(a1^2 \; (b0^2 - \beta^2) + a0^2 \; (b1^2 - \beta^2) + 2 a0 \; (b0 - b1) \; (b1 \, x - y \, \beta^2) + (b0 - b1)^2$$
$$(x^2 - (y^2 + z^2) \beta^2) - 2 a1 \; (a0 \; (b0 \, b1 - \beta^2) + (b0 - b1) \; (b0 \, x - y \, \beta^2)))))) -$$
$$((-2 a0 \; (-b0 + b1) + 2 a1 \; (-b0 + b1) + 2 \; (b0 - b1)^2 y) \; z \, \beta \, \mathrm{ArcTan}[$$
$$((a0 \; (y - y0) + a1 \; (-y + y0) + (b0 - b1) \; (y^2 - y \, y0 + z^2)) \; \beta) \Big/$$
$$(\surd (a0^2 + a1^2 + 2 a1 \; (-b0 + b1) \; y - 2 a0 \; (a1 + (-b0 + b1) \; y) + (b0 - b1)^2 \; (y^2 + z^2))$$
$$\sqrt{-(y^2 - 2 y \, y0 + y0^2 + z^2) \beta^2}\;)]) \Big/$$
$$(2 \; (a0^2 + a1^2 + 2 a1 \; (-b0 + b1) \; y - 2 a0 \; (a1 + (-b0 + b1) \; y) + (b0 - b1)^2 \; (y^2 + z^2))^{3/2}) +$$
$$((-2 a0 \; (-b0 + b1) + 2 a1 \; (-b0 + b1) + 2 \; (b0 - b1)^2 y) \; z \, \beta$$
$$\mathrm{ArcTan}[((a0 \; (y - y1) + a1 \; (-y + y1) + (b0 - b1) \; (y^2 - y \, y1 + z^2)) \; \beta) \Big/$$
$$(\surd (a0^2 + a1^2 + 2 a1 \; (-b0 + b1) \; y - 2 a0 \; (a1 + (-b0 + b1) \; y) + (b0 - b1)^2 \; (y^2 + z^2))$$
$$\sqrt{-(y^2 - 2 y \, y1 + y1^2 + z^2) \beta^2}\;)]) \Big/$$
$$(2 \; (a0^2 + a1^2 + 2 a1 \; (-b0 + b1) \; y - 2 a0 \; (a1 + (-b0 + b1) \; y) + (b0 - b1)^2 \; (y^2 + z^2))^{3/2}) +$$
$$((-a1 \, b0 - a0 \; (-2 b0 + b1) + (b0 - b1) \; (-x + 2 b0 \, y))$$
$$\mathrm{ArcTan}[(a0 \; (y - y0) + x \; (-y + y0) + b0 \; (y^2 - y \, y0 + z^2)) \; /$$
$$(z \surd (a0^2 - 2 a0 \, x + x^2 - (y^2 + z^2) \beta^2 + y0^2 \; (b0^2 - \beta^2) + 2 y0 \; (a0 \, b0 - b0 \, x + y \, \beta^2)))]) \Big/$$
$$(a0^2 + a1^2 + 2 a1 \; (-b0 + b1) \; y - 2 a0 \; (a1 + (-b0 + b1) \; y) +$$

```
    (b0 - b1)^2 (y^2 + z^2)) -
((-2 a0 (-b0 + b1) + 2 a1 (-b0 + b1) + 2 (b0 - b1)^2 y)
   (a0^2 + a1 (x - b0 y) - a0 (a1 + x - 2 b0 y + b1 y) + (b0 - b1) (-x y + b0 (y^2 + z^2)))
   ArcTan[(a0 (y - y0) + x (-y + y0) + b0 (y^2 - y y0 + z^2)) /
      (z √(a0^2 - 2 a0 x + x^2 - (y^2 + z^2) β^2 + y0^2 (b0^2 - β^2) + 2 y0 (a0 b0 - b0 x + y β^2)))]) /
 (a0^2 + a1^2 + 2 a1 (-b0 + b1) y - 2 a0 (a1 + (-b0 + b1) y) + (b0 - b1)^2 (y^2 + z^2))^2 -
((-a1 b0 - a0 (-2 b0 + b1) + (b0 - b1) (-x + 2 b0 y))
   ArcTan[(a0 (y - y1) + x (-y + y1) + b0 (y^2 - y y1 + z^2)) /
      (z √(a0^2 - 2 a0 x + x^2 - (y^2 + z^2) β^2 + y1^2 (b0^2 - β^2) + 2 y1 (a0 b0 - b0 x + y β^2)))]) /
 (a0^2 + a1^2 + 2 a1 (-b0 + b1) y - 2 a0 (a1 + (-b0 + b1) y) + (b0 - b1)^2 (y^2 + z^2)) +
((-2 a0 (-b0 + b1) + 2 a1 (-b0 + b1) + 2 (b0 - b1)^2 y)
   (a0^2 + a1 (x - b0 y) - a0 (a1 + x - 2 b0 y + b1 y) + (b0 - b1) (-x y + b0 (y^2 + z^2)))
   ArcTan[(a0 (y - y1) + x (-y + y1) + b0 (y^2 - y y1 + z^2)) /
      (z √(a0^2 - 2 a0 x + x^2 - (y^2 + z^2) β^2 + y1^2 (b0^2 - β^2) + 2 y1 (a0 b0 - b0 x + y β^2)))]) /
 (a0^2 + a1^2 + 2 a1 (-b0 + b1) y - 2 a0 (a1 + (-b0 + b1) y) + (b0 - b1)^2 (y^2 + z^2))^2 +
((a1 b0 - a0 b1 + (-b0 + b1) x)
   (-2 a0 (-b0 + b1) + 2 a1 (-b0 + b1) + 2 (b0 - b1)^2 y)
   z ArcTanh[(a0 - x + b0 y0) /
      (√(a0^2 - 2 a0 x + x^2 - (y^2 + z^2) β^2 + y0^2 (b0^2 - β^2) + 2 y0 (a0 b0 - b0 x + y β^2)))]) /
 (a0^2 + a1^2 + 2 a1 (-b0 + b1) y - 2 a0 (a1 + (-b0 + b1) y) + (b0 - b1)^2 (y^2 + z^2))^2 -
((a1 b0 - a0 b1 + (-b0 + b1) x) (-2 a0 (-b0 + b1) + 2 a1 (-b0 + b1) + 2 (b0 - b1)^2 y)
   z ArcTanh[(a0 - x + b0 y1) /
      (√(a0^2 - 2 a0 x + x^2 - (y^2 + z^2) β^2 + y1^2 (b0^2 - β^2) + 2 y1 (a0 b0 - b0 x + y β^2)))]) /
 (a0^2 + a1^2 + 2 a1 (-b0 + b1) y - 2 a0 (a1 + (-b0 + b1) y) + (b0 - b1)^2 (y^2 + z^2))^2 -
(z (-2 a0 (b0 - b1) β^2 + 2 a1 (b0 - b1) β^2 - 2 (b0 - b1)^2 y β^2)
   ArcTanh[(a0^2 b1 + a0 (-a1 b0 + b0 x - 2 b1 x + b0 b1 y0 + y β^2 - y0 β^2) +
         a1 (b0 x - b0^2 y0 + (-y + y0) β^2) + (b0 - b1) (-x^2 + b0 x y0 + (y^2 - y y0 + z^2) β^2)) /
      (√(a0^2 - 2 a0 x + x^2 - (y^2 + z^2) β^2 + y0^2 (b0^2 - β^2) + 2 y0 (a0 b0 - b0 x + y β^2))
         √(a1^2 (b0^2 - β^2) + a0^2 (b1^2 - β^2) + 2 a0 (b0 - b1) (b1 x - y β^2) + (b0 - b1)^2
            (x^2 - (y^2 + z^2) β^2) - 2 a1 (a0 (b0 b1 - β^2) + (b0 - b1) (b0 x - y β^2)))))]) /
 (2 (a0^2 + a1^2 + 2 a1 (-b0 + b1) y - 2 a0 (a1 + (-b0 + b1) y) + (b0 - b1)^2 (y^2 + z^2))
   √(a1^2 (b0^2 - β^2) + a0^2 (b1^2 - β^2) + 2 a0 (b0 - b1) (b1 x - y β^2) +
      (b0 - b1)^2 (x^2 - (y^2 + z^2) β^2) - 2 a1 (a0 (b0 b1 - β^2) + (b0 - b1) (b0 x - y β^2)))) +
((-2 a0 (-b0 + b1) + 2 a1 (-b0 + b1) + 2 (b0 - b1)^2 y) z
   √(a1^2 (b0^2 - β^2) + a0^2 (b1^2 - β^2) + 2 a0 (b0 - b1) (b1 x - y β^2) +
      (b0 - b1)^2 (x^2 - (y^2 + z^2) β^2) - 2 a1 (a0 (b0 b1 - β^2) + (b0 - b1) (b0 x - y β^2)))
   ArcTanh[(a0^2 b1 + a0 (-a1 b0 + b0 x - 2 b1 x + b0 b1 y0 + y β^2 - y0 β^2) +
         a1 (b0 x - b0^2 y0 + (-y + y0) β^2) + (b0 - b1) (-x^2 + b0 x y0 + (y^2 - y y0 + z^2) β^2)) /
      (√(a0^2 - 2 a0 x + x^2 - (y^2 + z^2) β^2 + y0^2 (b0^2 - β^2) + 2 y0 (a0 b0 - b0 x + y β^2))
         √(a1^2 (b0^2 - β^2) + a0^2 (b1^2 - β^2) + 2 a0 (b0 - b1) (b1 x - y β^2) + (b0 - b1)^2
            (x^2 - (y^2 + z^2) β^2) - 2 a1 (a0 (b0 b1 - β^2) + (b0 - b1) (b0 x - y β^2)))))]) /
 (a0^2 + a1^2 + 2 a1 (-b0 + b1) y - 2 a0 (a1 + (-b0 + b1) y) + (b0 - b1)^2 (y^2 + z^2))^2 +
(z (-2 a0 (b0 - b1) β^2 + 2 a1 (b0 - b1) β^2 - 2 (b0 - b1)^2 y β^2)
```

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$$\mathrm{ArcTanh}\Big[\left(a0^2\, b1 + a0\, (-a1\, b0 + b0\, x - 2\, b1\, x + b0\, b1\, y1 + y\, \beta^2 - y1\, \beta^2) +\right.$$
$$a1\, (b0\, x - b0^2\, y1 + (-y + y1)\, \beta^2) + (b0 - b1)\, (-x^2 + b0\, x\, y1 + (y^2 - y\, y1 + z^2)\, \beta^2)) \Big/$$
$$\Big(\sqrt{(a0^2 - 2\, a0\, x + x^2 - (y^2 + z^2)\, \beta^2 + y1^2\, (b0^2 - \beta^2) + 2\, y1\, (a0\, b0 - b0\, x + y\, \beta^2))}$$
$$\sqrt{(a1^2\, (b0^2 - \beta^2) + a0^2\, (b1^2 - \beta^2) + 2\, a0\, (b0 - b1)\, (b1\, x - y\, \beta^2) + (b0 - b1)^2}$$
$$(x^2 - (y^2 + z^2)\, \beta^2) - 2\, a1\, (a0\, (b0\, b1 - \beta^2) + (b0 - b1)\, (b0\, x - y\, \beta^2))))\Big]\Big) \Big/$$
$$\Big(2\, (a0^2 + a1^2 + 2\, a1\, (-b0 + b1)\, y - 2\, a0\, (a1 + (-b0 + b1)\, y) + (b0 - b1)^2\, (y^2 + z^2))$$
$$\sqrt{(a1^2\, (b0^2 - \beta^2) + a0^2\, (b1^2 - \beta^2) + 2\, a0\, (b0 - b1)\, (b1\, x - y\, \beta^2) +}$$
$$(b0 - b1)^2\, (x^2 - (y^2 + z^2)\, \beta^2) - 2\, a1\, (a0\, (b0\, b1 - \beta^2) + (b0 - b1)\, (b0\, x - y\, \beta^2)))\Big) -$$
$$\Big((-2\, a0\, (-b0 + b1) + 2\, a1\, (-b0 + b1) + 2\, (b0 - b1)^2\, y)\, z$$
$$\sqrt{(a1^2\, (b0^2 - \beta^2) + a0^2\, (b1^2 - \beta^2) + 2\, a0\, (b0 - b1)\, (b1\, x - y\, \beta^2) +}$$
$$(b0 - b1)^2\, (x^2 - (y^2 + z^2)\, \beta^2) - 2\, a1\, (a0\, (b0\, b1 - \beta^2) + (b0 - b1)\, (b0\, x - y\, \beta^2)))$$
$$\mathrm{ArcTanh}\Big[\left(a0^2\, b1 + a0\, (-a1\, b0 + b0\, x - 2\, b1\, x + b0\, b1\, y1 + y\, \beta^2 - y1\, \beta^2) +\right.$$
$$a1\, (b0\, x - b0^2\, y1 + (-y + y1)\, \beta^2) + (b0 - b1)\, (-x^2 + b0\, x\, y1 + (y^2 - y\, y1 + z^2)\, \beta^2)) \Big/$$
$$\Big(\sqrt{(a0^2 - 2\, a0\, x + x^2 - (y^2 + z^2)\, \beta^2 + y1^2\, (b0^2 - \beta^2) + 2\, y1\, (a0\, b0 - b0\, x + y\, \beta^2))}$$
$$\sqrt{(a1^2\, (b0^2 - \beta^2) + a0^2\, (b1^2 - \beta^2) + 2\, a0\, (b0 - b1)\, (b1\, x - y\, \beta^2) + (b0 - b1)^2}$$
$$(x^2 - (y^2 + z^2)\, \beta^2) - 2\, a1\, (a0\, (b0\, b1 - \beta^2) + (b0 - b1)\, (b0\, x - y\, \beta^2))))\Big]\Big) \Big/$$
$$(a0^2 + a1^2 + 2\, a1\, (-b0 + b1)\, y - 2\, a0\, (a1 + (-b0 + b1)\, y) + (b0 - b1)^2\, (y^2 + z^2))^2\Big)$$

In[400]:=

```
(*Tapered Supersonic Doublet Panel Perturbation Velocity w Component*)
w = D[ϕ[x, y, z], z]
```

Out[400]=

$$\frac{1}{2\,\pi}\,(\mu 0 - \mu 1)\, \Big(\Big(z\, \beta\, \Big((z\, (a0\, (y - y0) + a1\, (-y + y0) + (b0 - b1)\, (y^2 - y\, y0 + z^2))\, \beta^3) \Big/$$
$$\Big(\sqrt{(a0^2 + a1^2 + 2\, a1\, (-b0 + b1)\, y - 2\, a0\, (a1 + (-b0 + b1)\, y) + (b0 - b1)^2\, (y^2 + z^2))}$$
$$(-(y^2 - 2\, y\, y0 + y0^2 + z^2)\, \beta^2)^{3/2}\Big) + (2\, (b0 - b1)\, z\, \beta) \Big/$$
$$\Big(\sqrt{(a0^2 + a1^2 + 2\, a1\, (-b0 + b1)\, y - 2\, a0\, (a1 + (-b0 + b1)\, y) + (b0 - b1)^2\, (y^2 + z^2))}$$
$$\sqrt{-(y^2 - 2\, y\, y0 + y0^2 + z^2)\, \beta^2}\Big) -$$
$$((b0 - b1)^2\, z\, (a0\, (y - y0) + a1\, (-y + y0) + (b0 - b1)\, (y^2 - y\, y0 + z^2))\, \beta) \Big/$$
$$\Big((a0^2 + a1^2 + 2\, a1\, (-b0 + b1)\, y - 2\, a0\, (a1 + (-b0 + b1)\, y) + (b0 - b1)^2\, (y^2 + z^2))^{3/2}$$
$$\sqrt{-(y^2 - 2\, y\, y0 + y0^2 + z^2)\, \beta^2}\Big)\Big)\Big) \Big/$$
$$\Big(\sqrt{(a0^2 + a1^2 + 2\, a1\, (-b0 + b1)\, y - 2\, a0\, (a1 + (-b0 + b1)\, y) + (b0 - b1)^2\, (y^2 + z^2))}$$
$$\Big(1 - (a0\, (y - y0) + a1\, (-y + y0) + (b0 - b1)\, (y^2 - y\, y0 + z^2))^2 \Big/ \Big((y^2 - 2\, y\, y0 + y0^2 + z^2)$$
$$(a0^2 + a1^2 + 2\, a1\, (-b0 + b1)\, y - 2\, a0\, (a1 + (-b0 + b1)\, y) + (b0 - b1)^2\, (y^2 + z^2))\Big)\Big)\Big) -$$
$$\Big(z\, \beta\, \Big((z\, (a0\, (y - y1) + a1\, (-y + y1) + (b0 - b1)\, (y^2 - y\, y1 + z^2))\, \beta^3) \Big/$$
$$\Big(\sqrt{(a0^2 + a1^2 + 2\, a1\, (-b0 + b1)\, y - 2\, a0\, (a1 + (-b0 + b1)\, y) + (b0 - b1)^2\, (y^2 + z^2))}$$
$$(-(y^2 - 2\, y\, y1 + y1^2 + z^2)\, \beta^2)^{3/2}\Big) + (2\, (b0 - b1)\, z\, \beta) \Big/$$
$$\Big(\sqrt{(a0^2 + a1^2 + 2\, a1\, (-b0 + b1)\, y - 2\, a0\, (a1 + (-b0 + b1)\, y) + (b0 - b1)^2\, (y^2 + z^2))}$$

$$\sqrt{-\left(y^2-2\,y\,y1+y1^2+z^2\right)\beta^2}\,\Big)\,-$$

$$\Big(\left(b0-b1\right)^2 z\left(a0\,(y-y1)+a1\,(-y+y1)+(b0-b1)\left(y^2-y\,y1+z^2\right)\right)\beta\Big)\Big/$$

$$\Big(\Big(\left(a0^2+a1^2+2\,a1\,(-b0+b1)\,y-2\,a0\,(a1+(-b0+b1)\,y)+(b0-b1)^2\left(y^2+z^2\right)\right)^{3/2}$$

$$\sqrt{-\left(y^2-2\,y\,y1+y1^2+z^2\right)\beta^2}\,\Big)\Big)\Big)\Big/$$

$$\Big(\sqrt{\left(a0^2+a1^2+2\,a1\,(-b0+b1)\,y-2\,a0\,(a1+(-b0+b1)\,y)+(b0-b1)^2\left(y^2+z^2\right)\right)}$$

$$\Big(1-\left(a0\,(y-y1)+a1\,(-y+y1)+(b0-b1)\left(y^2-y\,y1+z^2\right)\right)^2\Big/\Big(\left(y^2-2\,y\,y1+y1^2+z^2\right)$$

$$\left(a0^2+a1^2+2\,a1\,(-b0+b1)\,y-2\,a0\,(a1+(-b0+b1)\,y)+(b0-b1)^2\left(y^2+z^2\right)\right)\Big)\Big)\Big)\,-$$

$$\Big((a1\,b0-a0\,b1+(-b0+b1)\,x)\,(a0-x+b0\,y0)\,z^2\beta^2\Big)\Big/$$

$$\Big(\left(a0^2+a1^2+2\,a1\,(-b0+b1)\,y-2\,a0\,(a1+(-b0+b1)\,y)+(b0-b1)^2\left(y^2+z^2\right)\right)$$

$$\left(a0^2-2\,a0\,x+x^2-\left(y^2+z^2\right)\beta^2+y0^2\left(b0^2-\beta^2\right)+2\,y0\left(a0\,b0-b0\,x+y\,\beta^2\right)\right)^{3/2}$$

$$\Big(1-(a0-x+b0\,y0)^2\Big/$$

$$\left(a0^2-2\,a0\,x+x^2-\left(y^2+z^2\right)\beta^2+y0^2\left(b0^2-\beta^2\right)+2\,y0\left(a0\,b0-b0\,x+y\,\beta^2\right)\right)\Big)\Big)\,+$$

$$\Big(\left(a0^2+a1\,(x-b0\,y)-a0\,(a1+x-2\,b0\,y+b1\,y)+(b0-b1)\left(-x\,y+b0\left(y^2+z^2\right)\right)\right)$$

$$\Big(\Big(\left(a0\,(y-y0)+x\,(-y+y0)+b0\left(y^2-y\,y0+z^2\right)\right)\beta^2\Big)\Big/$$

$$\left(a0^2-2\,a0\,x+x^2-\left(y^2+z^2\right)\beta^2+y0^2\left(b0^2-\beta^2\right)+2\,y0\left(a0\,b0-b0\,x+y\,\beta^2\right)\right)^{3/2}+$$

$$(2\,b0)\Big/\Big(\sqrt{\left(a0^2-2\,a0\,x+x^2-\left(y^2+z^2\right)\beta^2+y0^2\left(b0^2-\beta^2\right)+2\,y0\left(a0\,b0-b0\,x+y\,\beta^2\right)\right)}\Big)-$$

$$\left(a0\,(y-y0)+x\,(-y+y0)+b0\left(y^2-y\,y0+z^2\right)\right)\Big/$$

$$\Big(z^2\sqrt{\left(a0^2-2\,a0\,x+x^2-\left(y^2+z^2\right)\beta^2+y0^2\left(b0^2-\beta^2\right)+2\,y0\left(a0\,b0-b0\,x+y\,\beta^2\right)\right)}\Big)\Big)\Big)\Big/$$

$$\Big(\left(a0^2+a1^2+2\,a1\,(-b0+b1)\,y-2\,a0\,(a1+(-b0+b1)\,y)+(b0-b1)^2\left(y^2+z^2\right)\right)$$

$$\Big(1+\left(a0\,(y-y0)+x\,(-y+y0)+b0\left(y^2-y\,y0+z^2\right)\right)^2\Big/$$

$$\Big(z^2\left(a0^2-2\,a0\,x+x^2-\left(y^2+z^2\right)\beta^2+y0^2\left(b0^2-\beta^2\right)+2\,y0\left(a0\,b0-b0\,x+y\,\beta^2\right)\right)\Big)\Big)\Big)\,+$$

$$\Big((a1\,b0-a0\,b1+(-b0+b1)\,x)\,(a0-x+b0\,y1)\,z^2\beta^2\Big)\Big/$$

$$\Big(\left(a0^2+a1^2+2\,a1\,(-b0+b1)\,y-2\,a0\,(a1+(-b0+b1)\,y)+(b0-b1)^2\left(y^2+z^2\right)\right)$$

$$\left(a0^2-2\,a0\,x+x^2-\left(y^2+z^2\right)\beta^2+y1^2\left(b0^2-\beta^2\right)+2\,y1\left(a0\,b0-b0\,x+y\,\beta^2\right)\right)^{3/2}$$

$$\Big(1-(a0-x+b0\,y1)^2\Big/$$

$$\left(a0^2-2\,a0\,x+x^2-\left(y^2+z^2\right)\beta^2+y1^2\left(b0^2-\beta^2\right)+2\,y1\left(a0\,b0-b0\,x+y\,\beta^2\right)\right)\Big)\Big)\,-$$

$$\Big(\left(a0^2+a1\,(x-b0\,y)-a0\,(a1+x-2\,b0\,y+b1\,y)+(b0-b1)\left(-x\,y+b0\left(y^2+z^2\right)\right)\right)$$

$$\Big(\Big(\left(a0\,(y-y1)+x\,(-y+y1)+b0\left(y^2-y\,y1+z^2\right)\right)\beta^2\Big)\Big/$$

$$\left(a0^2-2\,a0\,x+x^2-\left(y^2+z^2\right)\beta^2+y1^2\left(b0^2-\beta^2\right)+2\,y1\left(a0\,b0-b0\,x+y\,\beta^2\right)\right)^{3/2}+$$

$$(2\,b0)\Big/\Big(\sqrt{\left(a0^2-2\,a0\,x+x^2-\left(y^2+z^2\right)\beta^2+y1^2\left(b0^2-\beta^2\right)+2\,y1\left(a0\,b0-b0\,x+y\,\beta^2\right)\right)}\Big)-$$

$$\left(a0\,(y-y1)+x\,(-y+y1)+b0\left(y^2-y\,y1+z^2\right)\right)\Big/$$

$$\Big(z^2\sqrt{\left(a0^2-2\,a0\,x+x^2-\left(y^2+z^2\right)\beta^2+y1^2\left(b0^2-\beta^2\right)+2\,y1\left(a0\,b0-b0\,x+y\,\beta^2\right)\right)}\Big)\Big)\Big)\Big/$$

$$\Big(\left(a0^2+a1^2+2\,a1\,(-b0+b1)\,y-2\,a0\,(a1+(-b0+b1)\,y)+(b0-b1)^2\left(y^2+z^2\right)\right)$$

$$\Big(1+\left(a0\,(y-y1)+x\,(-y+y1)+b0\left(y^2-y\,y1+z^2\right)\right)^2\Big/$$

$$\Big(z^2\left(a0^2-2\,a0\,x+x^2-\left(y^2+z^2\right)\beta^2+y1^2\left(b0^2-\beta^2\right)+2\,y1\left(a0\,b0-b0\,x+y\,\beta^2\right)\right)\Big)\Big)\Big)\,-$$

$$\Big(z\sqrt{\Big(a1^2\left(b0^2-\beta^2\right)+a0^2\left(b1^2-\beta^2\right)+2\,a0\,(b0-b1)\left(b1\,x-y\,\beta^2\right)+}$$

$$(b0 - b1)^2 \; (x^2 - (y^2 + z^2)\; \beta^2) - 2\; a1\; (a0\; (b0\; b1 - \beta^2) + (b0 - b1)\; (b0\; x - y\; \beta^2)))$$

$$(((b0 - b1)^2\; z\; \beta^2\; (a0^2\; b1 + a0\; (-a1\; b0 + b0\; x - 2\; b1\; x + b0\; b1\; y0 + y\; \beta^2 - y0\; \beta^2) + a1$$

$$(b0\; x - b0^2\; y0 + (-y + y0)\; \beta^2) + (b0 - b1)\; (-x^2 + b0\; x\; y0 + (y^2 - y\; y0 + z^2)\; \beta^2))) \;/$$

$$(\sqrt{}(a0^2 - 2\; a0\; x + x^2 - (y^2 + z^2)\; \beta^2 + y0^2\; (b0^2 - \beta^2) + 2\; y0\; (a0\; b0 - b0\; x + y\; \beta^2))$$

$$(a1^2\; (b0^2 - \beta^2) + a0^2\; (b1^2 - \beta^2) + 2\; a0\; (b0 - b1)\; (b1\; x - y\; \beta^2) + (b0 - b1)^2$$

$$(x^2 - (y^2 + z^2)\; \beta^2) - 2\; a1\; (a0\; (b0\; b1 - \beta^2) + (b0 - b1)\; (b0\; x - y\; \beta^2)))^{3/2}) +$$

$$(2\; (b0 - b1)\; z\; \beta^2) \;/\; (\sqrt{}(a0^2 - 2\; a0\; x + x^2 - (y^2 + z^2)\; \beta^2 + y0^2\; (b0^2 - \beta^2) +$$

$$2\; y0\; (a0\; b0 - b0\; x + y\; \beta^2))$$

$$\sqrt{}(a1^2\; (b0^2 - \beta^2) + a0^2\; (b1^2 - \beta^2) + 2\; a0\; (b0 - b1)\; (b1\; x - y\; \beta^2) + (b0 - b1)^2$$

$$(x^2 - (y^2 + z^2)\; \beta^2) - 2\; a1\; (a0\; (b0\; b1 - \beta^2) + (b0 - b1)\; (b0\; x - y\; \beta^2)))) +$$

$$(z\; \beta^2\; (a0^2\; b1 + a0\; (-a1\; b0 + b0\; x - 2\; b1\; x + b0\; b1\; y0 + y\; \beta^2 - y0\; \beta^2) + a1$$

$$(b0\; x - b0^2\; y0 + (-y + y0)\; \beta^2) + (b0 - b1)\; (-x^2 + b0\; x\; y0 + (y^2 - y\; y0 + z^2)\; \beta^2))) \;/$$

$$((a0^2 - 2\; a0\; x + x^2 - (y^2 + z^2)\; \beta^2 + y0^2\; (b0^2 - \beta^2) + 2\; y0\; (a0\; b0 - b0\; x + y\; \beta^2))^{3/2}$$

$$\sqrt{}(a1^2\; (b0^2 - \beta^2) + a0^2\; (b1^2 - \beta^2) + 2\; a0\; (b0 - b1)\; (b1\; x - y\; \beta^2) + (b0 - b1)^2$$

$$(x^2 - (y^2 + z^2)\; \beta^2) - 2\; a1\; (a0\; (b0\; b1 - \beta^2) + (b0 - b1)\; (b0\; x - y\; \beta^2)))))) \;/$$

$$((a0^2 + a1^2 + 2\; a1\; (-b0 + b1)\; y - 2\; a0\; (a1 + (-b0 + b1)\; y) + (b0 - b1)^2\; (y^2 + z^2))$$

$$(1 - (a0^2\; b1 + a0\; (-a1\; b0 + b0\; x - 2\; b1\; x + b0\; b1\; y0 + y\; \beta^2 - y0\; \beta^2) + a1$$

$$(b0\; x - b0^2\; y0 + (-y + y0)\; \beta^2) + (b0 - b1)\; (-x^2 + b0\; x\; y0 + (y^2 - y\; y0 + z^2)\; \beta^2))^2 \;/$$

$$((a0^2 - 2\; a0\; x + x^2 - (y^2 + z^2)\; \beta^2 + y0^2\; (b0^2 - \beta^2) + 2\; y0\; (a0\; b0 - b0\; x + y\; \beta^2))$$

$$(a1^2\; (b0^2 - \beta^2) + a0^2\; (b1^2 - \beta^2) + 2\; a0\; (b0 - b1)\; (b1\; x - y\; \beta^2) + (b0 - b1)^2$$

$$(x^2 - (y^2 + z^2)\; \beta^2) - 2\; a1\; (a0\; (b0\; b1 - \beta^2) + (b0 - b1)\; (b0\; x - y\; \beta^2)))))) +$$

$$(z\; \sqrt{}(a1^2\; (b0^2 - \beta^2) + a0^2\; (b1^2 - \beta^2) + 2\; a0\; (b0 - b1)\; (b1\; x - y\; \beta^2) +$$

$$(b0 - b1)^2\; (x^2 - (y^2 + z^2)\; \beta^2) - 2\; a1\; (a0\; (b0\; b1 - \beta^2) + (b0 - b1)\; (b0\; x - y\; \beta^2)))$$

$$(((b0 - b1)^2\; z\; \beta^2\; (a0^2\; b1 + a0\; (-a1\; b0 + b0\; x - 2\; b1\; x + b0\; b1\; y1 + y\; \beta^2 - y1\; \beta^2) + a1$$

$$(b0\; x - b0^2\; y1 + (-y + y1)\; \beta^2) + (b0 - b1)\; (-x^2 + b0\; x\; y1 + (y^2 - y\; y1 + z^2)\; \beta^2))) \;/$$

$$(\sqrt{}(a0^2 - 2\; a0\; x + x^2 - (y^2 + z^2)\; \beta^2 + y1^2\; (b0^2 - \beta^2) + 2\; y1\; (a0\; b0 - b0\; x + y\; \beta^2))$$

$$(a1^2\; (b0^2 - \beta^2) + a0^2\; (b1^2 - \beta^2) + 2\; a0\; (b0 - b1)\; (b1\; x - y\; \beta^2) + (b0 - b1)^2$$

$$(x^2 - (y^2 + z^2)\; \beta^2) - 2\; a1\; (a0\; (b0\; b1 - \beta^2) + (b0 - b1)\; (b0\; x - y\; \beta^2)))^{3/2}) +$$

$$(2\; (b0 - b1)\; z\; \beta^2) \;/\; (\sqrt{}(a0^2 - 2\; a0\; x + x^2 - (y^2 + z^2)\; \beta^2 + y1^2\; (b0^2 - \beta^2) +$$

$$2\; y1\; (a0\; b0 - b0\; x + y\; \beta^2))$$

$$\sqrt{}(a1^2\; (b0^2 - \beta^2) + a0^2\; (b1^2 - \beta^2) + 2\; a0\; (b0 - b1)\; (b1\; x - y\; \beta^2) + (b0 - b1)^2$$

$$(x^2 - (y^2 + z^2)\; \beta^2) - 2\; a1\; (a0\; (b0\; b1 - \beta^2) + (b0 - b1)\; (b0\; x - y\; \beta^2)))) +$$

$$(z\; \beta^2\; (a0^2\; b1 + a0\; (-a1\; b0 + b0\; x - 2\; b1\; x + b0\; b1\; y1 + y\; \beta^2 - y1\; \beta^2) + a1$$

$$(b0\; x - b0^2\; y1 + (-y + y1)\; \beta^2) + (b0 - b1)\; (-x^2 + b0\; x\; y1 + (y^2 - y\; y1 + z^2)\; \beta^2))) \;/$$

$$((a0^2 - 2\; a0\; x + x^2 - (y^2 + z^2)\; \beta^2 + y1^2\; (b0^2 - \beta^2) + 2\; y1\; (a0\; b0 - b0\; x + y\; \beta^2))^{3/2}$$

$$\sqrt{}(a1^2\; (b0^2 - \beta^2) + a0^2\; (b1^2 - \beta^2) + 2\; a0\; (b0 - b1)\; (b1\; x - y\; \beta^2) + (b0 - b1)^2$$

$$(x^2 - (y^2 + z^2)\; \beta^2) - 2\; a1\; (a0\; (b0\; b1 - \beta^2) + (b0 - b1)\; (b0\; x - y\; \beta^2)))))) \;/$$

$$((a0^2 + a1^2 + 2\; a1\; (-b0 + b1)\; y - 2\; a0\; (a1 + (-b0 + b1)\; y) + (b0 - b1)^2\; (y^2 + z^2))$$

$$(1 - (a0^2\; b1 + a0\; (-a1\; b0 + b0\; x - 2\; b1\; x + b0\; b1\; y1 + y\; \beta^2 - y1\; \beta^2) + a1$$

$$\left(b0\,x-b0^2\,y1+(-y+y1)\,\beta^2\right)+(b0-b1)\,\left(-x^2+b0\,x\,y1+\left(y^2-y\,y1+z^2\right)\beta^2\right)\right)^2\Big/$$

$$\left(\left(a0^2-2\,a0\,x+x^2-\left(y^2+z^2\right)\beta^2+y1^2\,\left(b0^2-\beta^2\right)+2\,y1\,\left(a0\,b0-b0\,x+y\,\beta^2\right)\right)\right.$$

$$\left(a1^2\,\left(b0^2-\beta^2\right)+a0^2\,\left(b1^2-\beta^2\right)+2\,a0\,(b0-b1)\,\left(b1\,x-y\,\beta^2\right)+(b0-b1)^2\right.$$

$$\left.\left.\left(x^2-\left(y^2+z^2\right)\beta^2\right)-2\,a1\,\left(a0\,\left(b0\,b1-\beta^2\right)+(b0-b1)\,\left(b0\,x-y\,\beta^2\right)\right)\right)\right)\right)\right)-$$

$$\left((b0-b1)^2\,z^2\,\beta\,\mathrm{ArcTan}\left[\left(\left(a0\,(y-y0)+a1\,(-y+y0)+(b0-b1)\,\left(y^2-y\,y0+z^2\right)\right)\beta\right)\Big/\right.\right.$$

$$\left(\sqrt{\left(a0^2+a1^2+2\,a1\,(-b0+b1)\,y-2\,a0\,(a1+(-b0+b1)\,y)+(b0-b1)^2\,\left(y^2+z^2\right)\right)}\right.$$

$$\left.\left.\left.\sqrt{-\left(y^2-2\,y\,y0+y0^2+z^2\right)\beta^2}\right)\right]\right)\Big/$$

$$\left(a0^2+a1^2+2\,a1\,(-b0+b1)\,y-2\,a0\,(a1+(-b0+b1)\,y)+(b0-b1)^2\,\left(y^2+z^2\right)\right)^{3/2}+$$

$$\left(\beta\,\mathrm{ArcTan}\left[\left(\left(a0\,(y-y0)+a1\,(-y+y0)+(b0-b1)\,\left(y^2-y\,y0+z^2\right)\right)\beta\right)\Big/\right.\right.$$

$$\left(\sqrt{\left(a0^2+a1^2+2\,a1\,(-b0+b1)\,y-2\,a0\,(a1+(-b0+b1)\,y)+(b0-b1)^2\,\left(y^2+z^2\right)\right)}\right.$$

$$\left.\left.\left.\sqrt{-\left(y^2-2\,y\,y0+y0^2+z^2\right)\beta^2}\right)\right]\right)\Big/$$

$$\left(\sqrt{\left(a0^2+a1^2+2\,a1\,(-b0+b1)\,y-2\,a0\,(a1+(-b0+b1)\,y)+(b0-b1)^2\,\left(y^2+z^2\right)\right)}\right)+$$

$$\left((b0-b1)^2\,z^2\,\beta\,\mathrm{ArcTan}\left[\left(\left(a0\,(y-y1)+a1\,(-y+y1)+(b0-b1)\,\left(y^2-y\,y1+z^2\right)\right)\beta\right)\Big/\right.\right.$$

$$\left(\sqrt{\left(a0^2+a1^2+2\,a1\,(-b0+b1)\,y-2\,a0\,(a1+(-b0+b1)\,y)+(b0-b1)^2\,\left(y^2+z^2\right)\right)}\right.$$

$$\left.\left.\left.\sqrt{-\left(y^2-2\,y\,y1+y1^2+z^2\right)\beta^2}\right)\right]\right)\Big/$$

$$\left(a0^2+a1^2+2\,a1\,(-b0+b1)\,y-2\,a0\,(a1+(-b0+b1)\,y)+(b0-b1)^2\,\left(y^2+z^2\right)\right)^{3/2}-$$

$$\left(\beta\,\mathrm{ArcTan}\left[\left(\left(a0\,(y-y1)+a1\,(-y+y1)+(b0-b1)\,\left(y^2-y\,y1+z^2\right)\right)\beta\right)\Big/\right.\right.$$

$$\left(\sqrt{\left(a0^2+a1^2+2\,a1\,(-b0+b1)\,y-2\,a0\,(a1+(-b0+b1)\,y)+(b0-b1)^2\,\left(y^2+z^2\right)\right)}\right.$$

$$\left.\left.\left.\sqrt{-\left(y^2-2\,y\,y1+y1^2+z^2\right)\beta^2}\right)\right]\right)\Big/$$

$$\left(\sqrt{\left(a0^2+a1^2+2\,a1\,(-b0+b1)\,y-2\,a0\,(a1+(-b0+b1)\,y)+(b0-b1)^2\,\left(y^2+z^2\right)\right)}\right)+$$

$$\left(2\,b0\,(b0-b1)\,z\,\mathrm{ArcTan}\left[\left(a0\,(y-y0)+x\,(-y+y0)+b0\,\left(y^2-y\,y0+z^2\right)\right)\Big/\right.\right.$$

$$\left.\left.\left(z\,\sqrt{\left(a0^2-2\,a0\,x+x^2-\left(y^2+z^2\right)\beta^2+y0^2\,\left(b0^2-\beta^2\right)+2\,y0\,\left(a0\,b0-b0\,x+y\,\beta^2\right)\right)}\right)\right]\right)\Big/$$

$$\left(a0^2+a1^2+2\,a1\,(-b0+b1)\,y-2\,a0\,(a1+(-b0+b1)\,y)+(b0-b1)^2\,\left(y^2+z^2\right)\right)-$$

$$\left(2\,(b0-b1)^2\,z\,\left(a0^2+a1\,(x-b0\,y)-a0\,(a1+x-2\,b0\,y+b1\,y)+\right.\right.$$

$$\left.(b0-b1)\,\left(-x\,y+b0\,\left(y^2+z^2\right)\right)\right)$$

$$\mathrm{ArcTan}\left[\left(a0\,(y-y0)+x\,(-y+y0)+b0\,\left(y^2-y\,y0+z^2\right)\right)\Big/\right.$$

$$\left.\left.\left(z\,\sqrt{\left(a0^2-2\,a0\,x+x^2-\left(y^2+z^2\right)\beta^2+y0^2\,\left(b0^2-\beta^2\right)+2\,y0\,\left(a0\,b0-b0\,x+y\,\beta^2\right)\right)}\right)\right]\right)\Big/$$

$$\left(a0^2+a1^2+2\,a1\,(-b0+b1)\,y-2\,a0\,(a1+(-b0+b1)\,y)+(b0-b1)^2\,\left(y^2+z^2\right)\right)^2-$$

$$\left(2\,b0\,(b0-b1)\,z\,\mathrm{ArcTan}\left[\left(a0\,(y-y1)+x\,(-y+y1)+b0\,\left(y^2-y\,y1+z^2\right)\right)\Big/\right.\right.$$

$$\left.\left.\left(z\,\sqrt{\left(a0^2-2\,a0\,x+x^2-\left(y^2+z^2\right)\beta^2+y1^2\,\left(b0^2-\beta^2\right)+2\,y1\,\left(a0\,b0-b0\,x+y\,\beta^2\right)\right)}\right)\right]\right)\Big/$$

$$\left(a0^2+a1^2+2\,a1\,(-b0+b1)\,y-2\,a0\,(a1+(-b0+b1)\,y)+(b0-b1)^2\,\left(y^2+z^2\right)\right)+$$

$$\left(2\,(b0-b1)^2\,z\,\left(a0^2+a1\,(x-b0\,y)-a0\,(a1+x-2\,b0\,y+b1\,y)+\right.\right.$$

$$\left.(b0-b1)\,\left(-x\,y+b0\,\left(y^2+z^2\right)\right)\right)$$

$$\mathrm{ArcTan}\left[\left(a0\,(y-y1)+x\,(-y+y1)+b0\,\left(y^2-y\,y1+z^2\right)\right)\Big/\right.$$

$$\left.\left.\left(z\,\sqrt{\left(a0^2-2\,a0\,x+x^2-\left(y^2+z^2\right)\beta^2+y1^2\,\left(b0^2-\beta^2\right)+2\,y1\,\left(a0\,b0-b0\,x+y\,\beta^2\right)\right)}\right)\right]\right)\Big/$$

$$
\begin{aligned}
&(a0^2 + a1^2 + 2\,a1\,(-b0 + b1)\,y - 2\,a0\,(a1 + (-b0 + b1)\,y) + (b0 - b1)^2\,(y^2 + z^2))^2 + \\
&(2\,(b0 - b1)^2\,(a1\,b0 - a0\,b1 + (-b0 + b1)\,x)\,z^2\,\text{ArcTanh}[(a0 - x + b0\,y0)\,/ \\
&\quad (\surd(a0^2 - 2\,a0\,x + x^2 - (y^2 + z^2)\,\beta^2 + y0^2\,(b0^2 - \beta^2) + 2\,y0\,(a0\,b0 - b0\,x + y\,\beta^2))))])\,/ \\
&(a0^2 + a1^2 + 2\,a1\,(-b0 + b1)\,y - 2\,a0\,(a1 + (-b0 + b1)\,y) + (b0 - b1)^2\,(y^2 + z^2))^2 - \\
&((a1\,b0 - a0\,b1 + (-b0 + b1)\,x)\,\text{ArcTanh}[(a0 - x + b0\,y0)\,/ \\
&\quad (\surd(a0^2 - 2\,a0\,x + x^2 - (y^2 + z^2)\,\beta^2 + y0^2\,(b0^2 - \beta^2) + 2\,y0\,(a0\,b0 - b0\,x + y\,\beta^2))))])\,/ \\
&(a0^2 + a1^2 + 2\,a1\,(-b0 + b1)\,y - 2\,a0\,(a1 + (-b0 + b1)\,y) + (b0 - b1)^2\,(y^2 + z^2)) - \\
&(2\,(b0 - b1)^2\,(a1\,b0 - a0\,b1 + (-b0 + b1)\,x)\,z^2\,\text{ArcTanh}[(a0 - x + b0\,y1)\,/ \\
&\quad (\surd(a0^2 - 2\,a0\,x + x^2 - (y^2 + z^2)\,\beta^2 + y1^2\,(b0^2 - \beta^2) + 2\,y1\,(a0\,b0 - b0\,x + y\,\beta^2))))])\,/ \\
&(a0^2 + a1^2 + 2\,a1\,(-b0 + b1)\,y - 2\,a0\,(a1 + (-b0 + b1)\,y) + (b0 - b1)^2\,(y^2 + z^2))^2 + \\
&((a1\,b0 - a0\,b1 + (-b0 + b1)\,x)\,\text{ArcTanh}[(a0 - x + b0\,y1)\,/ \\
&\quad (\surd(a0^2 - 2\,a0\,x + x^2 - (y^2 + z^2)\,\beta^2 + y1^2\,(b0^2 - \beta^2) + 2\,y1\,(a0\,b0 - b0\,x + y\,\beta^2))))])\,/ \\
&(a0^2 + a1^2 + 2\,a1\,(-b0 + b1)\,y - 2\,a0\,(a1 + (-b0 + b1)\,y) + (b0 - b1)^2\,(y^2 + z^2)) + \\
&((b0 - b1)^2\,z^2\,\beta^2\,\text{ArcTanh}[(a0^2\,b1 + a0\,(-a1\,b0 + b0\,x - 2\,b1\,x + b0\,b1\,y0 + y\,\beta^2 - y0\,\beta^2) + \\
&\qquad a1\,(b0\,x - b0^2\,y0 + (-y + y0)\,\beta^2) + (b0 - b1)\,(-x^2 + b0\,x\,y0 + (y^2 - y\,y0 + z^2)\,\beta^2))\,/ \\
&\quad (\surd(a0^2 - 2\,a0\,x + x^2 - (y^2 + z^2)\,\beta^2 + y0^2\,(b0^2 - \beta^2) + 2\,y0\,(a0\,b0 - b0\,x + y\,\beta^2)) \\
&\qquad \surd(a1^2\,(b0^2 - \beta^2) + a0^2\,(b1^2 - \beta^2) + 2\,a0\,(b0 - b1)\,(b1\,x - y\,\beta^2) + (b0 - b1)^2 \\
&\qquad\quad (x^2 - (y^2 + z^2)\,\beta^2) - 2\,a1\,(a0\,(b0\,b1 - \beta^2) + (b0 - b1)\,(b0\,x - y\,\beta^2)))))])\,/ \\
&((a0^2 + a1^2 + 2\,a1\,(-b0 + b1)\,y - 2\,a0\,(a1 + (-b0 + b1)\,y) + (b0 - b1)^2\,(y^2 + z^2)) \\
&\quad \surd(a1^2\,(b0^2 - \beta^2) + a0^2\,(b1^2 - \beta^2) + 2\,a0\,(b0 - b1)\,(b1\,x - y\,\beta^2) + \\
&\qquad (b0 - b1)^2\,(x^2 - (y^2 + z^2)\,\beta^2) - 2\,a1\,(a0\,(b0\,b1 - \beta^2) + (b0 - b1)\,(b0\,x - y\,\beta^2)))) + \\
&(2\,(b0 - b1)^2\,z^2\,\surd(a1^2\,(b0^2 - \beta^2) + a0^2\,(b1^2 - \beta^2) + 2\,a0\,(b0 - b1)\,(b1\,x - y\,\beta^2) + \\
&\qquad (b0 - b1)^2\,(x^2 - (y^2 + z^2)\,\beta^2) - 2\,a1\,(a0\,(b0\,b1 - \beta^2) + (b0 - b1)\,(b0\,x - y\,\beta^2))) \\
&\quad \text{ArcTanh}[(a0^2\,b1 + a0\,(-a1\,b0 + b0\,x - 2\,b1\,x + b0\,b1\,y0 + y\,\beta^2 - y0\,\beta^2) + \\
&\qquad a1\,(b0\,x - b0^2\,y0 + (-y + y0)\,\beta^2) + (b0 - b1)\,(-x^2 + b0\,x\,y0 + (y^2 - y\,y0 + z^2)\,\beta^2))\,/ \\
&\quad (\surd(a0^2 - 2\,a0\,x + x^2 - (y^2 + z^2)\,\beta^2 + y0^2\,(b0^2 - \beta^2) + 2\,y0\,(a0\,b0 - b0\,x + y\,\beta^2)) \\
&\qquad \surd(a1^2\,(b0^2 - \beta^2) + a0^2\,(b1^2 - \beta^2) + 2\,a0\,(b0 - b1)\,(b1\,x - y\,\beta^2) + (b0 - b1)^2 \\
&\qquad\quad (x^2 - (y^2 + z^2)\,\beta^2) - 2\,a1\,(a0\,(b0\,b1 - \beta^2) + (b0 - b1)\,(b0\,x - y\,\beta^2)))))])\,/ \\
&(a0^2 + a1^2 + 2\,a1\,(-b0 + b1)\,y - 2\,a0\,(a1 + (-b0 + b1)\,y) + (b0 - b1)^2\,(y^2 + z^2))^2 - \\
&(\surd(a1^2\,(b0^2 - \beta^2) + a0^2\,(b1^2 - \beta^2) + 2\,a0\,(b0 - b1)\,(b1\,x - y\,\beta^2) + \\
&\qquad (b0 - b1)^2\,(x^2 - (y^2 + z^2)\,\beta^2) - 2\,a1\,(a0\,(b0\,b1 - \beta^2) + (b0 - b1)\,(b0\,x - y\,\beta^2))) \\
&\quad \text{ArcTanh}[(a0^2\,b1 + a0\,(-a1\,b0 + b0\,x - 2\,b1\,x + b0\,b1\,y0 + y\,\beta^2 - y0\,\beta^2) + \\
&\qquad a1\,(b0\,x - b0^2\,y0 + (-y + y0)\,\beta^2) + (b0 - b1)\,(-x^2 + b0\,x\,y0 + (y^2 - y\,y0 + z^2)\,\beta^2))\,/ \\
&\quad (\surd(a0^2 - 2\,a0\,x + x^2 - (y^2 + z^2)\,\beta^2 + y0^2\,(b0^2 - \beta^2) + 2\,y0\,(a0\,b0 - b0\,x + y\,\beta^2)) \\
&\qquad \surd(a1^2\,(b0^2 - \beta^2) + a0^2\,(b1^2 - \beta^2) + 2\,a0\,(b0 - b1)\,(b1\,x - y\,\beta^2) + (b0 - b1)^2 \\
&\qquad\quad (x^2 - (y^2 + z^2)\,\beta^2) - 2\,a1\,(a0\,(b0\,b1 - \beta^2) + (b0 - b1)\,(b0\,x - y\,\beta^2)))))])\,/ \\
&(a0^2 + a1^2 + 2\,a1\,(-b0 + b1)\,y - 2\,a0\,(a1 + (-b0 + b1)\,y) + (b0 - b1)^2\,(y^2 + z^2)) - \\
&((b0 - b1)^2\,z^2\,\beta^2\,\text{ArcTanh}[(a0^2\,b1 + a0\,(-a1\,b0 + b0\,x - 2\,b1\,x + b0\,b1\,y1 + y\,\beta^2 - y1\,\beta^2) + \\
&\qquad a1\,(b0\,x - b0^2\,y1 + (-y + y1)\,\beta^2) + (b0 - b1)\,(-x^2 + b0\,x\,y1 + (y^2 - y\,y1 + z^2)\,\beta^2))\,/ \\
&\quad (\surd(a0^2 - 2\,a0\,x + x^2 - (y^2 + z^2)\,\beta^2 + y1^2\,(b0^2 - \beta^2) + 2\,y1\,(a0\,b0 - b0\,x + y\,\beta^2)) \\
&\qquad \surd(a1^2\,(b0^2 - \beta^2) + a0^2\,(b1^2 - \beta^2) + 2\,a0\,(b0 - b1)\,(b1\,x - y\,\beta^2) + (b0 - b1)^2 \\
&\qquad\quad (x^2 - (y^2 + z^2)\,\beta^2) - 2\,a1\,(a0\,(b0\,b1 - \beta^2) + (b0 - b1)\,(b0\,x - y\,\beta^2)))))])\,/ \\
&((a0^2 + a1^2 + 2\,a1\,(-b0 + b1)\,y - 2\,a0\,(a1 + (-b0 + b1)\,y) + (b0 - b1)^2\,(y^2 + z^2))
\end{aligned}
$$

$$\sqrt{\left(a1^2\left(b0^2-\beta^2\right)+a0^2\left(b1^2-\beta^2\right)+2\,a0\,(b0-b1)\left(b1\,x-y\,\beta^2\right)+(b0-b1)^2\left(x^2-\left(y^2+z^2\right)\beta^2\right)-2\,a1\left(a0\left(b0\,b1-\beta^2\right)+(b0-b1)\left(b0\,x-y\,\beta^2\right)\right)\right)}\Bigg)\Bigg)-$$

$$\Bigg(2\,(b0-b1)^2 z^2\sqrt{\left(a1^2\left(b0^2-\beta^2\right)+a0^2\left(b1^2-\beta^2\right)+2\,a0\,(b0-b1)\left(b1\,x-y\,\beta^2\right)+(b0-b1)^2\left(x^2-\left(y^2+z^2\right)\beta^2\right)-2\,a1\left(a0\left(b0\,b1-\beta^2\right)+(b0-b1)\left(b0\,x-y\,\beta^2\right)\right)\right)}$$

$$\mathrm{ArcTanh}\Big[\Big(a0^2\,b1+a0\left(-a1\,b0+b0\,x-2\,b1\,x+b0\,b1\,y1+y\,\beta^2-y1\,\beta^2\right)+a1\left(b0\,x-b0^2\,y1+(-y+y1)\,\beta^2\right)+(b0-b1)\left(-x^2+b0\,x\,y1+\left(y^2-y\,y1+z^2\right)\beta^2\right)\Big)\Big/$$

$$\Big(\sqrt{\left(a0^2-2\,a0\,x+x^2-\left(y^2+z^2\right)\beta^2+y1^2\left(b0^2-\beta^2\right)+2\,y1\left(a0\,b0-b0\,x+y\,\beta^2\right)\right)}\sqrt{\left(a1^2\left(b0^2-\beta^2\right)+a0^2\left(b1^2-\beta^2\right)+2\,a0\,(b0-b1)\left(b1\,x-y\,\beta^2\right)+(b0-b1)^2\left(x^2-\left(y^2+z^2\right)\beta^2\right)-2\,a1\left(a0\left(b0\,b1-\beta^2\right)+(b0-b1)\left(b0\,x-y\,\beta^2\right)\right)\right)}\Big)\Big]\Bigg)\Big/$$

$$\left(a0^2+a1^2+2\,a1\,(-b0+b1)\,y-2\,a0\,(a1+(-b0+b1)\,y)+(b0-b1)^2\left(y^2+z^2\right)\right)^2+$$

$$\Bigg(\sqrt{\left(a1^2\left(b0^2-\beta^2\right)+a0^2\left(b1^2-\beta^2\right)+2\,a0\,(b0-b1)\left(b1\,x-y\,\beta^2\right)+(b0-b1)^2\left(x^2-\left(y^2+z^2\right)\beta^2\right)-2\,a1\left(a0\left(b0\,b1-\beta^2\right)+(b0-b1)\left(b0\,x-y\,\beta^2\right)\right)\right)}$$

$$\mathrm{ArcTanh}\Big[\Big(a0^2\,b1+a0\left(-a1\,b0+b0\,x-2\,b1\,x+b0\,b1\,y1+y\,\beta^2-y1\,\beta^2\right)+a1\left(b0\,x-b0^2\,y1+(-y+y1)\,\beta^2\right)+(b0-b1)\left(-x^2+b0\,x\,y1+\left(y^2-y\,y1+z^2\right)\beta^2\right)\Big)\Big/$$

$$\Big(\sqrt{\left(a0^2-2\,a0\,x+x^2-\left(y^2+z^2\right)\beta^2+y1^2\left(b0^2-\beta^2\right)+2\,y1\left(a0\,b0-b0\,x+y\,\beta^2\right)\right)}\sqrt{\left(a1^2\left(b0^2-\beta^2\right)+a0^2\left(b1^2-\beta^2\right)+2\,a0\,(b0-b1)\left(b1\,x-y\,\beta^2\right)+(b0-b1)^2\left(x^2-\left(y^2+z^2\right)\beta^2\right)-2\,a1\left(a0\left(b0\,b1-\beta^2\right)+(b0-b1)\left(b0\,x-y\,\beta^2\right)\right)\right)}\Big)\Big]\Bigg)\Big/$$

$$\left(a0^2+a1^2+2\,a1\,(-b0+b1)\,y-2\,a0\,(a1+(-b0+b1)\,y)+(b0-b1)^2\left(y^2+z^2\right)\right)\Bigg)$$

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