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We consider infinite-horizon economies populated by a continuum of agents subject to idiosyncratic shocks. This framework contains models of saving and capital accumulation with incomplete markets in the spirit of works by Bewley, Aiyagari, and Huggett; models of entry, exit, and industry dynamics in the spirit of Hopenhayn's work; and dynamic models of occupational choice and search models as special cases. Robust and easy-to-apply comparative statics results are established with respect to exogenous parameters as well as various kinds of changes in the Markov processes governing the law of motion of the idiosyncratic shocks.

I. Introduction

In several settings, heterogeneous agents make dynamic choices with rewards determined by market prices or aggregate externalities, which are in turn given as the aggregates of the decisions of all agents in the market. Because there are sufficiently many agents (i.e., the economy is "large" or "competitive"), all ignore their impact on these aggregate variables. The equilibrium in general takes the form of a stationary distribution of decisions (or state variables such as assets), which remains invariant while

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agents each experience changes in their decisions over time as a result of stochastic shocks and their dynamic responses to them. Examples include the following: (1) Bewley-Aiyagari style models (e.g., Bewley [1986], Aiyagari [1994], or the closely related line developed by Huggett [1993]): In these models, each household is subject to idiosyncratic labor income shocks and makes saving and consumption decisions taking future prices as given. Prices are then determined as functions of the aggregate capital stock of the economy, resulting from all households' saving decisions. (2) Models of industry equilibrium in the spirit of Hopenhayn (1992), where each firm has access to a stochastically evolving production technology and decides how much to produce and whether to exit given market prices, which are determined as a function of total production in the economy. (3) Models of dynamic occupational choice with or without credit constraints and with stochastic income and savings (e.g., Mookherjee and Ray 2003; Buera, Kaboski, and Shin 2011; Moll 2012). (4) Models with aggregate learning-by-doing externalities in the spirit of Arrow (1962) and Romer (1986), where potentially heterogeneous firms make production decisions, taking their future productivity as given, and aggregate productivity is determined as a function of total current or past production. (5) Search models in the spirit of Diamond (1982) and Mortensen and Pissarides (1994), where current production and search effort decisions depend on future thickness of the market. (6) Models of capital accumulation and international trade with factor price equalization (e.g., Ventura 1997).[1]

Despite the common structure across these and several other models, little is known in terms of how the stationary equilibria respond to a range of shocks including changes in preference and production parameters and changes in the distribution of (idiosyncratic) shocks influencing each agent's decisions. For example, even though the Bewley-Aiyagari model has become a workhorse in modern dynamic macroeconomics, most studies rely on numerical analysis to characterize its implications.

In this paper, we provide a general framework for the study of *large dynamic economies*, nesting the above-mentioned models (or their generalizations), and show how "robust" comparative statics of stationary equilibria of these economies can be derived in a simple and tractable manner. Here "robust" comparative statics refers to results, similar to those in, for example, Milgrom and Roberts (1994) and Milgrom and Shannon (1994), that hold under minimal conditions and without necessitating knowledge of specific functional forms and parameter values.

Our first substantive theorem, presented in Section III, builds on Smithson's (1971) set-valued fixed-point theorem and establishes monotonicity

[1] Models in categories 4–6 are typically set up without individual-level heterogeneity and with only limited stochastic shocks, making stationary equilibria symmetric. Our analysis covers significant generalizations of these models in which agents can be of different types and subject to idiosyncratic shocks represented by arbitrary Markov processes.

properties of fixed points of a class of mappings defined over general (nonlattice) spaces. In particular, it establishes that the set of fixed points of an upper hemicontinuous correspondence inherits the various monotonicity properties of the correspondence in question. This result is critical for deriving comparative statics of stationary equilibria in this class of models since strategies correspond to random variables and are thus not defined over spaces that are lattices in any natural order.[2]

Our second set of results, presented in Section IV, utilizes these monotonicity properties to show how the stationary equilibria of large dynamic economies respond to a range of exogenous shocks affecting a subset (or all) of the agents. In particular, we show that when a subset of agents is affected by positive shocks, defined as shocks that increase individual strategies for a given (market) aggregate, the greatest and least stationary equilibrium aggregates always increase. The economic intuition of this result stems from the fact that we are studying a market equilibrium aggregating the behavior of all the agents in the economy. A positive shock to a subset of agents increases their strategies. When the strategies of all other agents are held constant, the aggregate must increase. This increase in aggregate can induce countervailing indirect effects since we are not imposing any assumptions on how the aggregate affects individual strategies. However, in the greatest and least equilibria, these indirect effects can never overturn the direct effects; if they did, there would be no increase in aggregate to start with. Consequently, the greatest and the least stationary equilibrium aggregates must increase.

To illustrate these results, let us return to the Bewley-Aiyagari model mentioned above. In a version of this model in which agents have different utility functions, labor income processes, and borrowing limits, we derive robust comparative static results with respect to changes in the discount factor, borrowing limits, the parameters of the utility function (e.g., the level of risk aversion), and the parameters of the production function. In each case, we show that, under minimal and natural assumptions, changes that increase the action of individual agents for a given sequence of market aggregates translate into an increase in the least and greatest stationary equilibrium aggregates (capital-labor ratios). The response of all other macroeconomic variables (in the greatest and least stationary equilibria) can then be derived from the behavior of the capital-labor ratio. Importantly, as we discuss below, our results provide consider-

[2] The difficulty arises in the analysis of how an individual's stationary strategy changes in response to changes in parameters. This analysis always involves a fixed-point comparative statics problem since stationary strategies are fixed points of the adjoint Markov operator in stochastic dynamic programming problems (see Stokey and Lucas [1989, 317] or App. B for the more general case of Markov correspondences). The adjoint Markov operator maps a probability distribution into a probability distribution, so its domain or range is not a lattice in any natural order (Hopenhayn and Prescott 1992).

able information on aggregate behavior even though in the Bewley-Aiyagari model nothing can be said about how individual behavior changes in general.

Our third set of results, developed in Section V, turns to an analysis of the implications of changes in the Markov processes governing the behavior of stochastic shocks. In Bewley-Aiyagari models, this corresponds to changes in the distribution of productivity or labor income ranked in terms of first-order stochastic dominance or, more interestingly, in terms of mean-preserving spreads. The latter type of result allows us to address questions related to the impact on market aggregates of greater uncertainty in individual earnings.

In each case, our results are intuitive, easy to apply, and robust. A noteworthy feature of our results is that in most cases, though how aggregate behavior can be determined robustly, very little or nothing can be said about individual behavior: regularity of (market) aggregates is accompanied with irregularity of individual behavior. This highlights that our results are not a consequence of some implicit strong assumptions; in particular, large dynamic economies are not implicitly assumed to be supermodular or monotone (e.g., Mirman, Morand, and Reffett 2008).

Our paper builds on two literatures. The first is the study of large dynamic economies, which includes, among others, Jovanovic (1982), Bewley (1986), Jovanovic and Rosenthal (1988), Hopenhayn (1992), Huggett (1993), Aiyagari (1994), Ericson and Pakes (1995), and Miao (2002). Though some of these papers contain certain specific results on how equilibria change with parameters (e.g., the effect of relaxing borrowing limits in Aiyagari [1994], which we discuss further below, and that of productivity on entry in Hopenhayn [1992]), they do not present the general approach or the robust comparative static results provided here. To the best of our knowledge, none of these papers contains comparative statics either with respect to general changes in preferences and technology or with respect to changes in distributions of shocks, in particular, with respect to mean-preserving spreads.

Second, our work is related to the robust comparative statics literature (e.g., Milgrom and Roberts 1994; Milgrom and Shannon 1994). Selten (1970) and Corchón (1994) introduced and provided comparative statics for aggregative games in which payoffs to individual agents depend on their own strategies and an aggregate of others' strategies. In Acemoglu and Jensen (2013), we provided more general comparative statics results for static aggregative games, thus extending the approach of Milgrom and Roberts (1994) to aggregative games (the earlier literature on aggregative games, including Corchón [1994], exclusively relied on the implicit function theorem). In Acemoglu and Jensen (2010), we considered large static environments in which payoffs depend on aggregates (and individuals ignored their impact on aggregates). To the best of our knowledge, the current paper is the first to provide general comparative statics results for dynamic economies.

Only a few works have obtained comparative statics results in related dynamic economies. Most notably, Aiyagari's original work and Miao (2002) study certain properties of stationary equilibria in the Bewley-Aiyagari model. Their approach can be applied only in more restrictive environments (in particular, without ex ante heterogeneity) and for more limited parameter changes than the one developed in this paper. In addition, their approach faces some additional challenges and necessitates strong assumptions that, as explained in the working paper of our work (Acemoglu and Jensen 2012), are unnecessary.[3] Also related to our results is the study by Huggett (2004), who studies the impact of earning risk for an individual's savings decisions. Our results on increased earning risk mentioned above not only generalize but also extend Huggett's study from a partial to a general equilibrium setting.

We believe that the results provided here are significant for several reasons. First, as discussed at length in Milgrom and Roberts (1994), standard comparative statics methods such as those based on the implicit function theorem often run into difficulty unless there are strong parametric restrictions, and in the presence of such restrictions, the economic role of different ingredients of the model may be blurred. The existence of multiple equilibria, a common occurrence in dynamic equilibrium models, is also a challenge to these standard approaches. Second, the dynamic general equilibrium nature of such economies makes implicit function theorem type results difficult or impossible to apply, motivating the reliance of most of the literature in this area on numerical analysis (see, e.g., Sargent and Ljungqvist's [2004] textbook analysis of Bewley-

[3] Briefly, their approach proceeds as follows: First, using firms' profit maximization conditions, the wage rate is expressed in terms of the interest rate $w = w(r)$. Second, households' savings (capital supply) can be derived as a function of the sequence of interest rates after substituting $w_t = w(r_t)$ for the wage at each date in the budget constraint. Third, focusing on an individual and keeping $r_t = r$ all t, the effect of parameter changes on the capital supply can now be determined. As noted in n. 2, this part implicitly involves comparing fixed points on nonlattice spaces. Both Aiyagari (1994) and Miao (2002) achieve this by placing strong assumptions on the problem, which ensure that individual strategies as a function of the interest rate are *unique* and *stable* (in the sense that individual strategies are myopically stable with the aggregate held fixed). In particular, this requires (typically difficult-to-verify) cross-restrictions on preferences, technology, and the Markov process governing the labor productivity shocks. These cross-restrictions also imply that the borrowing constraint binds for all levels of the interest rate at the worst realization of the shock (see, e.g., Aiyagari 1993, 39; Miao 2002, assumption 1.b). Finally, given unique and stable stationary demand for capital represented by a schedule $D(r)$, this approach derives equilibrium comparative static results combining this demand with a schedule for the supply of capital, $S(r)$. However, even with the strong assumptions imposed in Aiyagari (1994) and Miao (2002), as we explain in Acemoglu and Jensen (2012), $S(r)$ can be easily downward sloping depending on income and substitution effects—even with unique stationary strategies for agents. This creates additional challenges that have been ignored in previous work (how they can be tackled is outlined in Acemoglu and Jensen [2012]).

Aiyagari and the related Huggett models). The results from numerical analysis may be sensitive to parameter values and the existence of multiple equilibria. They are also silent about the role of different assumptions of the model on the results. Our approach overcomes these difficulties by providing robust comparative static results for the entire set of equilibria. We believe that these problems increase the utility of our results and techniques, at the very least as a complement to existing, largely numerical methods of analysis, by clarifying the economic role of existing assumptions.

The paper is structured as follows: Section II defines large dynamic economies and (stationary) equilibria and establishes their existence under general conditions. In Sections III–V, we present our main comparative statics results. Section VI contains several applications of our results, and Section VII briefly sketches how they can be extended to models with multiple aggregates and provides some applications. Proofs are placed in Appendix A, and Appendix B contains results from stochastic dynamic programming used throughout the paper.

II. Large Dynamic Economies

We begin by describing the general class of large dynamic economies and prove the existence of equilibrium and stationary equilibrium. As we will see in Section VI, a number of important macroeconomic models fit into this general framework including Hopenhayn's (1992) model of firm dynamics and the Bewley-Aiyagari model. In this section, we will use the Bewley-Aiyagari model to illustrate and motivate our assumptions.

A. *Preferences and Technology*

The basic setting is an infinite-horizon, discrete-time economy populated by a continuum of agents $\mathcal{I} = [0, 1]$.[4] Each agent $i \in [0, 1]$ is subject to (uninsurable) idiosyncratic shocks $z_{i,t} \in Z_i \subseteq \mathbb{R}^M$ that follow a Markov process with transition function P_i. We assume throughout that $(z_{i,t})_{t=0}^{\infty}$ has a unique invariant distribution μ_{z_i}. A special case of this is when the $z_{i,t}$'s are independent and identically distributed, in which case $z_{i,t}$ has the distribution μ_{z_i} for all t.

Agent i's action set is $X_i \subseteq \mathbb{R}^n$, and given initial conditions $(x_{i,0}, z_{i,0}) \in X_i \times Z_i$, she solves

[4] Throughout, all sets are equipped with the Lebesgue measure and Borel algebra (and products of sets with the product measure and product algebra). For a set Z, the Borel algebra is denoted by $\mathcal{B}(Z)$, and the set of probability measures on $(Z, \mathcal{B}(Z))$ is denoted by $\mathcal{P}(Z)$. For simplicity, we consider only the case in which $\mathcal{I} = [0, 1]$, but our results hold for any nonatomic measure space of agents. This includes a setting such as that of Al-Najjar (2004), where the set of agents is countable and the measure is finitely additive.

$$\sup \mathbb{E}_0\left[\sum_{t=0}^{\infty} \beta^t u_i(x_{i,t}, x_{i,t+1}, z_{i,t}, Q_t, a_i)\right]$$
$$\text{subject to } x_{i,t+1} \in \Gamma_i(x_{i,t}, z_{i,t}, Q_t, a_i),\ t = 0, 1, 2, \ldots. \quad (1)$$

Here $\beta \in (0, 1)$ is the *discount factor*; $a_i \in A_i \subseteq \mathbb{R}^P$ is a *vector of parameters* with respect to which we wish to do comparative statics; and $Q_t \in \mathcal{Q} \subseteq \mathbb{R}$ is the *market aggregate* (or simply *aggregate*) at time t discussed below. Aside from these variables, (1) is seen to be a standard dynamic programming problem as treated at length in, for example, Stokey and Lucas (1989), with $u_i : X_i^2 \times Z_i \times \mathcal{Q} \times A_i \to \mathbb{R}$ the *instantaneous payoff* (*utility*) *function* and $\Gamma_i : X_i \times Z_i \times \mathcal{Q} \times A_i \to 2^{X_i}$ *the constraint correspondence.* A *strategy* $\mathbf{x}_i = (x_{i,1}, x_{i,2}, \ldots)$ is a sequence of random variables defined on the histories of shocks, that is, a sequence of measurable maps $x_{i,t} : Z_i^{t-1} \to X_i$, where $Z_i^{t-1} \equiv \prod_{\tau=0}^{t-1} Z_i$. For each t, $x_{i,t}$ gives rise to a (probability) distribution on X_i referred to as the *distribution* of $x_{i,t}$. A *feasible strategy* is one that satisfies the constraints in (1), and an *optimal strategy* is a solution to (1). When a strategy is optimal, it is denoted by $\mathbf{x}_i^*$. The following standard assumption ensures the existence of optimal strategies (Stokey and Lucas 1989, chap. 9). A transition function has the Feller property if the associated Markov operator maps the set of bounded continuous functions into itself (220).

Assumption 1. For each $i \in \mathcal{I}$, P_i has the Feller property, X_i and Z_i are compact, u_i is bounded and continuous, and Γ_i is continuous with nonempty and compact values.

Example 1 (Bewley-Aiyagari). In Bewley-Aiyagari economies, households maximize their discounted utility defined over their sequence of consumption $(c_{i,t})_{t=0}^{\infty}$,

$$\mathbb{E}_0\left[\sum_{t=0}^{\infty} \beta^t \tilde{u}_i(c_{i,t})\right] \quad (2)$$

subject to the constraint

$$\tilde{\Gamma}_i(x_{i,t}, c_{i,t}, z_{i,t}, Q_t) = \{(x_{i,t+1}, c_{i,t+1}) \in [-\underline{b}_i, \bar{b}_i] \times [0, \bar{c}_i]: x_{i,t+1} \leq r(Q_t)x_{i,t} + w(Q_t)z_{i,t} - c_{i,t}\}, \quad (3)$$

where $x_{i,t}$ is the asset holdings of household i; $z_{i,t} \in Z_i \subseteq \mathbb{R}$ denotes its labor productivity/endowment, which follows a Markov process; and $\underline{b}_i$ is a lower bound on assets capturing both natural debt limits and other borrowing constraints (e.g., the natural debt limit in the stationary equilibrium with rate of return r on the assets would be $\underline{b}_i = -z_i^{\min}/[r-1] < 0$, where $z_i^{\min}$ is the worst realization of $z_{i,t}$). The upper bound on assets, $\bar{b}_i$, ensures compactness of actions and can be chosen so that it does not bind in equilibrium. It is worth noting that the borrowing (credit) constraint $\underline{b}_i$ need not bind for a household even when the worst realization of the shock, $z_i^{\min}$, occurs. Therefore, our setting nests the complete markets case as well as "mixed" cases in which borrowing constraints bind on or off the equilibrium path for some but not all households. Crucially, utility functions, the distribution of labor endowments, and the lower bound on assets can vary across households. All households face the same prices, in particular, the wage rate $w_t = w(Q_t)$ and the interest rate $r_t = r(Q_t)$ at date t, which, in turn, depend on the capital-labor ratio in the economy, Q_t. Specifically, with competitive markets, $r_t = r(Q_t) = f'(Q_t)$ and $w_t = w(Q_t) = f(Q_t) - f'(Q_t)Q_t$, where f denotes the aggregate per capita production function (which is naturally taken to be continuous, differentiable, and concave).

Assuming that $\tilde{u}_i$ is increasing, we can solve for $c_{i,t}$ in terms of $x_{i,t+1}$ and write the decision problem in the form (1) where

$$u_i(x_{i,t}, x_{i,t+1}, z_{i,t}, Q_t, a_i) = \tilde{u}_i(r(Q_t)x_{i,t} + w(Q_t)z_{i,t} - x_{i,t+1}) \tag{4}$$

and

$$\Gamma_i(x_{i,t}, z_{i,t}, Q_t, a_i) = \{y_i \in [\underline{b}_i, \bar{b}_i] : y_{i,t} \le r(Q_t)x_{i,t} + w(Q_t)z_{i,t}\}. \tag{5}$$

It is easy to see that assumption 1 will hold under standard continuity conditions on $\tilde{u}_i$ and f.

This example also illustrates the "reduction" in the dimension of the problem due to the aggregate variable—in this case the capital-labor ratio Q_t—through which all market interactions take place. Since Q_t is deterministic, this entails a *no aggregate uncertainty* assumption (see, e.g., Lucas 1980; Bewley 1986; Aiyagari 1994). The details of how individual uncertainty is removed at the aggregate level follow.

B. *Markets and Aggregates*

The aggregate Q_t is determined by a so-called *aggregator*, which is a function that "cancels out" individual-level uncertainty by mapping random variables into real numbers. Our *baseline aggregator* is the simple integral ("average") of the strategies of players, with the integral being the *Pettis integral* (Uhlig 1996):[5]

[5] Briefly, the Pettis integral defines $\int_{[0,1]} x_{i,t}\,di$ as the limit in L^2-norm of the sequence of "Riemann sums," $\sum_{i=1}^{n} x_{\tau_{i,t}}(\tau_i - \tau_{i-1})$, $n = 1, 2, 3, \ldots$, for a narrowing sequence of subdivisions $0 = \tau_1 < \tau_2 < \cdots < \tau_n = 1$, $n = 1, 2, 3, \ldots$. It is clear that if any countable subsequence of $(x_{i,t})_{i\in\mathcal{I}}$ satisfies a law of large numbers, this limit will be a degenerate random variable with its unit mass at $\int_{[0,1]} \mathbb{E}(x_{i,t})\,di$. Intuitively, this implies that the integral is evaluated as the "mean of the means," which is essentially the approach adopted in both Bewley (1986) and Aiyagari (1994). Bewley explicitly defines the aggregate over random variables as the mean of the means (1986, 81). Aiyagari integrates over the distributional strategies, which, as

$$Q_t = H((x_{i,t})_{i\in\mathcal{I}}) = \int_{[0,1]} x_{i,t}\,di. \tag{6}$$

There are of course many other ways to define integrals of random variables, and the Pettis integral is subject to the valid criticism that its definition obscures the connection with the underlying sample space (Al-Najjar 2004; Sun 2006). Nevertheless, the issue of which one of several different approaches to the law of large numbers issue is chosen has little relevance for our results, which all remain valid under any of these choices.

Example 1 (continued). The baseline aggregator in (6) is natural in the Bewley-Aiyagari model, where $x_{i,t}$ is savings (capital holdings) of household i at date t. When the population is normalized to one, Q_t as given in (6) is the capital-labor ratio. Note a feature of an aggregator clearly illustrated in this case: the definition of the aggregator is closely related to market clearing. In particular, in the Bewley-Aiyagari model, (6) is the capital market–clearing condition.

Our general definition of an aggregator is an extension of that in (6). Let $\tilde{x}_i$ and x_i be random variables on a set X_i with distributions $\tilde{\mu}_{x_i}$ and μ_{x_i}. Then we say that $\tilde{x}_i$ *first-order stochastically dominates* x_i, written $\tilde{x}_i \succeq_{\text{st}} x_i$, if

$$\int_{X_i} f(x_i)\tilde{\mu}_{x_i}(dx_i) \geq \int_{X_i} f(x_i)\mu_{x_i}(dx_i)$$

for any increasing function $f: X_i \to \mathbb{R}$. A function H that maps a vector of random variables $(\tilde{x}_i)_{i\in\mathcal{I}}$ into a real number is said to be *increasing* if it is increasing in the first-order stochastic dominance order $\succeq_{\text{st}}$, that is, if $H((\tilde{x}_i)_{i\in\mathcal{I}}) \geq H((x_i)_{i\in\mathcal{I}})$ whenever $\tilde{x}_i \succeq_{\text{st}} x_i$ for all $i \in \mathcal{I}$. The function H is *continuous* if it is continuous in the weak $*$-topology on its domain (see, e.g., Stokey and Lucas 1989; Hopenhayn and Prescott 1992).

Definition 1 (Aggregator). An *aggregator* is a continuous and increasing function H that maps the agents' strategies at date t into a real number $Q_t \in \mathcal{Q}$ (with $\mathcal{Q} \subseteq \mathbb{R}$ denoting the range of H). The value

$$Q_t = H((x_{i,t})_{i\in\mathcal{I}}) \tag{7}$$

is referred to as the (market) aggregate at date t.[6]

long as aggregate distributions are deterministic, leads to the same outcome. Note also that since agents maximize expected payoffs, there is no difference between a degenerate random variable and a real number, and we may therefore simply set $H((x_{i,t})_{i\in\mathcal{I}}) = \int_{[0,1]} \mathbb{E}(x_{i,t})\,di$. See Uhlig (1996) and the appendices in Acemoglu and Jensen (2010, 2012) for further details.

[6] Note that if H is an aggregator, then so is any continuous and increasing transformation of H. Thus (6) represents, up to a monotone transformation, the class of separable functions, which are therefore a special case of our definition of an aggregator (see, e.g., Acemoglu and Jensen [2013] on separable aggregators).

It is straightforward to see that both properties in definition 1 are satisfied for our baseline aggregator (6). In fact, the conditions in definition 1 will naturally be satisfied for any reasonable aggregation procedure (including, in particular, those of Al-Najjar [2004] and Sun [2006]).

C. *Equilibrium*

We are now ready to define an equilibrium in large dynamic economies.

Definition 2 (Equilibrium). Fix initial conditions $(z_{i,0}, x_{i,0})_{i\in\mathcal{I}}$. Then an equilibrium $\{\mathbf{Q}^*, (\mathbf{x}_i^*)_{i\in\mathcal{I}}\}$ is a sequence of market aggregates and a strategy for each of the agents such that the following conditions hold:

1. Optimality: For each agent $i \in \mathcal{I}$, $\mathbf{x}_i^* = (x_{i,1}^*, x_{i,2}^*, x_{i,3}^*, \ldots)$ solves (1) given $\mathbf{Q}^* = (Q_0^*, Q_1^*, Q_2^*, \ldots)$ and the initial conditions $(z_{i,0}, x_{i,0})$.
2. Market clearing: $Q_t^* = H((x_{i,t}^*)_{i\in\mathcal{I}})$ for each $t = 0, 1, 2, \ldots$.

With the baseline aggregator (6), assumption 1 is sufficient to guarantee the existence of an equilibrium due to the "convexifying" effect of set-valued integration (Aumann 1965; see also assumption 2). In particular, payoff functions need not be concave, and constraint correspondences need not have convex graphs. With our general class of aggregators (not necessarily taking the simple form [6]), either we have to assume this convexifying feature directly or, alternatively, we must impose concavity and convex graph conditions on the agents. This is the content of the next assumption. To simplify notation, from now on we write $u_i(x_i, y_i, z_i, Q, a_i)$ in place of $u_i(x_{i,t}, x_{i,t+1}, z_{i,t}, Q_t, a_i)$, and similarly, we write $\Gamma_i(x_i, z_i, Q, a_i)$ for the constraint correspondence.

Assumption 2. At least one of the following two conditions holds:

- For each agent, X_i is convex, and given any choice of z_i, Q, and a_i, $u_i(x_i, y_i, z_i, Q, a_i)$ is concave in (x_i, y_i) and $\Gamma_i(\cdot, z_i, Q, a_i)$ has a convex graph.
- The aggregator H is convexifying; that is, for any subset B of the set of joint strategies such that $H(b)$ is well defined for all $b \in B$, the image $H(B) = \{H(b) \in \mathbb{R} : b \in B\} \subseteq \mathbb{R}$ is convex.[7]

We now have the following theorem.

Theorem 1 (Existence of equilibrium). Under assumptions 1 and 2, there exists an equilibrium for any choice of initial conditions $(z_{i,0}, x_{i,0})_{i\in\mathcal{I}}$.

[7] A convexifying aggregator is defined quite generally here. In most situations, the statement that $H(b)$ must be well defined has a more specific meaning, namely, that b is a sequence of joint strategies that is measurable across agents or across agent types (see Acemoglu and Jensen [2012, app. III] for further details).

As with all other results, the proof of theorem 1 is presented in Appendix A.

D. *Stationary Equilibria*

Our focus in this paper is on stationary equilibria. At the individual level, stationarity of $\mathbf{x}_i^*$ means that at any two dates $t, t' \in \mathbb{N}$, $x_{i,t}^*$ and $x_{i,t'}^*$ have the same distribution $\mu_{\mathbf{x}_i^*} \in \mathcal{P}(X_i)$. At the aggregate level, stationarity simply means that $\mathbf{Q}^*$ is a constant sequence. The simplest way to define a stationary equilibrium in stochastic dynamic settings involves assuming that the initial conditions $(x_{i,0}, z_{i,0})$ are random variables.

Definition 3 (Stationary equilibrium). A stationary equilibrium $\{Q^*, (\mathbf{x}_i^*)_{i\in\mathcal{I}}\}$ is a (market) aggregate and a stationary strategy for each of the agents such that the following conditions hold:

1. Optimality: For each agent $i \in \mathcal{I}$, the stationary strategy $\mathbf{x}_i^* = (x_{i,1}^*, x_{i,2}^*, x_{i,3}^*, \ldots)$ with distribution $\mu_{\mathbf{x}_i^*}$ solves (1) given $\mathbf{Q}^* = (Q^*, Q^*, Q^*, \ldots)$, and the randomly drawn initial conditions $(x_{i,0}, z_{i,0}) \sim \mu_{\mathbf{x}_i^*} \times \mu_{z_i}$.
2. Market clearing: $Q^* = H((x_{i,t}^*)_{i\in\mathcal{I}})$ for $t = 0, 1, 2, \ldots$.[8]

With a slight abuse of terminology, we will refer to a (market) aggregate Q^* of a stationary equilibrium as an equilibrium aggregate. The set of equilibrium aggregates given $a = (a_i)_{i\in\mathcal{I}}$ is denoted by $\mathcal{E}(a)$, and we refer to the least and greatest elements in $\mathcal{E}(a)$ as *the least and greatest equilibrium aggregates*, respectively.

In a stationary equilibrium, agent i faces a stationary sequence of aggregates $(Q^*, Q^*, \ldots)$ and solves a stationary dynamic programming problem whose value function v_i is determined by the following functional equation:

$$v_i(x_i, z_i, Q^*, a_i) = \sup_{y_i \in \Gamma_i(x_i, z_i, Q^*, a_i)} \left[u_i(x_i, y_i, z_i, Q^*, a_i) + \beta \int v_i(y_i, z_i', Q^*, a_i) P_i(z_i, dz_i') \right]. \tag{8}$$

As is well known, this functional equation has a unique solution v_i under assumption 1 (see, e.g., chap. 9 in Stokey and Lucas [1989]). Given v_i, the (*stationary*) *policy correspondence* is determined by

[8] Note that with stationary strategies, the market clears at all dates if it clears at just a single date. So condition 2 is equivalent to $Q^* = H((x_{i,0}^*)_{i\in\mathcal{I}})$.

$$G_i(x_i, z_i, Q^*, a_i) = \arg \sup_{y_i \in \Gamma_i(x_i, z_i, Q^*, a_i)} \left[u_i(x_i, y_i, z_i, Q^*, a_i) + \beta \int v_i(y_i, z_i', Q^*, a_i) P_i(z_i, dz_i') \right]. \quad (9)$$

When the idiosyncratic shock process $z_{i,t}$ is stationary, the stationary distribution $\mu_{\mathbf{x}_i^*}$ of definition 3 is simply an invariant distribution for this decision problem (see App. B for further details). To ensure the existence of such invariant distributions/stationary strategies, we impose the following assumption (for easy reference, the mathematical concepts used in the definition are defined in a remark immediately after the definition).

Assumption 3. X_i is a lattice, and given any choice of z_i, Q, and a_i, $u_i(x_i, y_i, z_i, Q, a_i)$ is supermodular in (x_i, y_i) and the graph of $\Gamma_i(\cdot, z_i, Q, a_i)$ is a sublattice of $X_i \times X_i$.

Remark 1. The set X_i is a *lattice* if for any two elements $x_i^1, x_i^2 \in X_i$, the supremum $x_i^1 \vee x_i^2$ as well as the infimum $x_i^1 \wedge x_i^2$ both lie in X_i. When $X_i \subseteq \mathbb{R}$ (one-dimensional action sets), this holds trivially. Fixing and suppressing (z_i, Q, a_i), Γ_i's graph is a *sublattice* of $X_i \times X_i$ if for all $x_i^1, x_i^2 \in X$, $y_i^1 \in \Gamma_i(x_i^1)$ and $y_i^2 \in \Gamma_i(x_i^2)$ imply that $y_i^1 \wedge y_i^2 \in \Gamma_i(x_i^1 \wedge x_i^2)$ and $y_i^1 \vee y_i^2 \in \Gamma_i(x_i^1 \vee x_i^2)$. When $X_i \subseteq \mathbb{R}$, this will hold if and only if the correspondence is *ascending* (or increasing in the strong set order) in x_i, meaning that for all $x_i^2 \geq x_i^1$ in X_i, $y_i^1 \in \Gamma_i(x_i^1)$ and $y_i^2 \in \Gamma_i(x_i^2)$ imply that $y_i^1 \wedge y_i^2 \in \Gamma_i(x_i^1)$ and $y_i^1 \vee y_i^2 \in \Gamma_i(x_i^2)$. Finally, u_i is *supermodular* in (x_i, y_i) if

$$u_i(x_i^1 \vee x_i^2, y_i^1 \vee y_i^2) + u_i(x_i^1 \wedge x_i^2, y_i^1 \wedge y_i^2) \geq u_i(x_i^1, y_i^1) + u_i(x_i^2, y_i^2)$$

for all $x_i^1, x_i^2 \in X_i$ and $y_i^1, y_i^2 \in X_i$. See, for example, Topkis (1998) for further details.

As proved in theorem B2 in Appendix B, the policy correspondence G_i will be ascending in x_i under assumption 3. So for all $x_i^2 \geq x_i^1$ and $y_i^j \in G_i(x_i^j, z_i, Q, a_i)$, $j = 1, 2$, we have $y_i^1 \wedge y_i^2 \in G_i(x_i^1, z_i, Q, a_i)$ and $y_i^1 \vee y_i^2 \in G_i(x_i^2, z_i, Q, a_i)$. Economically, this means that the current decision is increasing in the last period's decision (e.g., higher past savings will increase current savings). In large dynamic economies, this is typically a rather weak requirement as opposed to assuming that G_i is ascending in Q_t, which is highly restrictive (and which we do not assume).

Example 1 (continued). In the Bewley-Aiyagari model,

$$u_i(x_i, y_i, z_i, Q, a_i) = \tilde{u}_i(r(Q)x_i + w(Q)z_i - y_i);$$

hence u_i will be supermodular in (x_i, y_i) if and only if the individual instantaneous utility function $\tilde{u}_i$ is concave. This is true in general, but it is easiest to see in the twice-differentiable case: since $D^2_{x_i y_i} u_i = -r(Q)\tilde{u}_i''$,

$D^2_{x_i y_i} u_i \geq 0$ (supermodularity) holds if and only if $\tilde{u}_i'' \leq 0$ (concavity). As for the sublattice property, as noted in remark 1, $\Gamma_i(\cdot, z_i, Q)$ will be a sublattice of $X_i \times X_i$ if and only if $\Gamma_i(x_i, z_i, Q)$ is ascending in x_i (this is true in general when X_i is one-dimensional). It is straightforward to verify that this is indeed the case.

When the previous three assumptions are combined, large dynamic economies always have a stationary equilibrium, and least and greatest equilibrium aggregates are well defined.

Theorem 2 (Existence of stationary equilibrium). Suppose assumptions 1–3 hold. Then there exists a stationary equilibrium and the set of equilibrium aggregates is compact. In particular, there always exist least and greatest equilibrium aggregates.

Existence of a stationary equilibrium can also be established without assumption 3 under convexity and concavity assumptions (see the first alternative in assumption 2). But, as the previous example also indicates, assumption 3 is usually more natural in large dynamic economies, and moreover, it plays an important role for our comparative statics analysis in later sections, so we impose it now for simplicity.[9]

III. Monotonicity of Fixed Points

At the heart of our substantive results is a theorem that enables us to establish monotonicity of fixed points defined over general (nonlattice) spaces. Comparative statics of equilibria boils down to studying the behavior of the fixed points of some mapping $F : X \times \Theta \to 2^X$, where $x \in X$ is the variable of interest—in most of our applications, a probability distribution—and $\theta \in \Theta$ are exogenous parameters. Defining the set of fixed points,

$$\Lambda(\theta) \equiv \{x \in X : x \in F(x, \theta)\},$$

the question is thus how $\Lambda(\theta)$ varies with $\theta \in \Theta$. The technical problem associated with large economies is that when agents' strategies are random variables (probability measures), their strategy sets will generally not be lattices in any natural order (Hopenhayn and Prescott 1992, 1389). Furthermore, for general equilibrium analysis, one cannot work with increasing selections from optimal strategies, making it necessary to

[9] Note that without concavity and convexity assumptions, a stationary equilibrium as we have defined it here (with individual strategies also stationary) may not exist even if the aggregator is convexifying (the second alternative in assumption 2). In contrast, an equilibrium in which the distribution of states and actions is invariant will exist under assumption 1 if the aggregator is convexifying. This can be proved by essentially the same argument as that used to prove theorem 2 in Jovanovic and Rosenthal (1988). (Note, however, that in the anonymous sequential games setting of Jovanovic and Rosenthal's paper, individual strategies are not required to be stationary in a stationary equilibrium.)

study the set-valued case in general.[10] In large dynamic economies, F is an adjoint Markov correspondence that maps probability measures into sets of probability measures. The adjoint Markov correspondence is defined formally in Appendix B, where we also prove (theorems B1 and B2) that it will satisfy the following monotonicity properties under this paper's main assumptions.

Definition 4 (Type I and type II monotonicity [Smithson 1971]). Let X and Y be ordered sets with order $\preceq$. A correspondence $F : X \to 2^Y$ is

1. type I monotone if for all $x_1 \preceq x_2$ and $y_1 \in F(x_1)$, there exists $y_2 \in F(x_2)$ such that $y_1 \preceq y_2$;
2. type II monotone if for all $x_1 \preceq x_2$ and $y_2 \in F(x_2)$, there exists $y_1 \in F(x_1)$ such that $y_1 \preceq y_2$.

When a correspondence F is defined on a product set, $F : X \times \Theta \to 2^Y$, where Θ is also a partially ordered set, we say that F is *type I* (*type II*) *monotone* in θ if $F : \{x\} \times \Theta \to 2^Y$ is type I (type II) monotone for each $x \in X$. Type I/II monotonicity in x is defined similarly by keeping Θ fixed. If $F : X \times \Theta \to 2^Y$ is type I (type II) monotone in x as well as in θ, we simply say that F is type I (type II) monotone. Note that for a correspondence F to be type I or type II monotone, unlike the cases of monotonicity with respect to the weak or strong set orders, no specific order structure for the values or domain of F is required (Shannon 1995). As mentioned, this is critical for the study of large dynamic economies, where F is an adjoint Markov correspondence.

The main result, on which all the rest of our results build, is as follows.

Theorem 3 (Comparing equilibria). Let X be a compact topological space equipped with a closed order $\succeq$, Θ a partially ordered set, and let $F : X \times \{\theta\} \to 2^X$ be upper hemicontinuous for each $\theta \in \Theta$. Define the (possibly empty-valued) fixed-point correspondence $\Lambda(\theta) = \{x \in X : x \in F(x, \theta)\}$, $\Lambda : \Theta \to 2^X \cup \varnothing$. Then if F is type I monotone, so is Λ; and if F is type II monotone, so is Λ.

Mathematically, the idea of theorem 3 is to use the fixed-point theorem of Smithson (1971) instead of Tarski's fixed-point theorem as used by, among others, Topkis (1998) or the Knaster-Tarski theorem used by Hopenhayn and Prescott (1992). Note that theorem 3 is a natural gen-

[10] In general, increasing selections may not exist in the setting of the present paper; but more importantly, even when they exist, general equilibrium analysis requires all invariant distributions to be taken into account (the reason is that when market variables change, a property of a specific selection, such as this being the greatest selection, may be lost). This makes it impossible to use a result along the lines of corollary 3 in Hopenhayn and Prescott (1992), which concerns (single-valued) increasing functions.

eralization of corollary 3 in Hopenhayn and Prescott (1992).[11] Also useful for our focus is the next result providing an analogue of the standard approach of selecting the least and greatest equilibria from the fixed-point correspondence (those exist in the lattice case but will generally not exist in the setting that is relevant for us).

Theorem 4. Let $\Lambda(\theta) \subseteq X$ be the fixed-point set of theorem 3 (for given $\theta \in \Theta$), and suppose that it is nonempty, that is, $\Lambda(\theta) \neq \emptyset$ for $\theta \in \Theta$. Consider a continuous and increasing function $H : X \to \mathbb{R}$, and define the least and greatest selections from $H \circ \Lambda(\theta) : \bar{h}(\theta) = \sup_{x \in \Lambda(\theta)} H(x)$ and $\underline{h}(\theta) = \inf_{x \in \Lambda(\theta)} H(x)$. Then if Λ is type I monotone, $\bar{h}$ will be increasing; and if Λ is type II monotone, $\underline{h}$ will be increasing.

The proof of theorem 4 simply uses upper hemicontinuity and standard results on existence of a maximum. As always, proofs are in Appendix A.

IV. Changes in Exogenous Variables

In this section, we use the results from the previous section to derive two general comparative statics results. First we define changes in the exogenous parameters $a = (a_i)_{i \in \mathcal{I}}$ that are *positive shocks* as changes that increase individual strategies given market aggregates. We then establish (theorem 5) that the least and greatest equilibrium aggregates increase in response to positive shocks. Interestingly, for this result we do not need to assume anything about how the sequence of market variables $(Q_0, Q_1, Q_2, \ldots)$ enters into the payoff functions and constraint correspondences (aside from continuity, see assumptions 1 and 2).[12] So our assumptions do not restrict us to supermodular games or monotone economies (e.g., Mirman et al. 2008). This is key for many applications, including all of those we discuss in Section VI. The result is truly about the market level: Without additional supermodularity or monotonicity assumptions, individual strategies' response will in general be highly irregular as we illustrate through several examples in Section VI, but at the market level, the irregularity of individual behavior is nonetheless restricted so as to lead to considerable aggregate regularity.[13] In this sec-

[11] Hopenhayn and Prescott (1992) consider the case of a function $f : X \times \Theta \to X$, where $\Lambda(\theta) = \{x \in X : x = f(x, \theta)\}$. Their corollary 3 can be recast in our language as saying that Λ will be type I and type II monotone if f is increasing in (x, θ). See also n. 10.

[12] Our results are valid for a finite number of agents as long as these all take the market aggregates as given. This reiterates that our results are not "aggregation" results that depend on the continuum assumption.

[13] A natural first approach to comparative statics in general equilibrium economies would be, first, to pin down individual responses and then aggregate over them. The previous discussion highlights that this is not the strategy we adopt; in fact, this strategy would not work because, as we discuss further below, individual responses to the shocks we

tion's second main result (theorem 6), we trace the effect of the previous parameter changes on individual strategies.

A. *Comparative Statics with Respect to Changes in Exogenous Parameters*

Recall the stationary policy correspondence G_i defined in (9), which, for a (stationary) equilibrium aggregate Q, gives the current action as a function of the past action x_i, the idiosyncratic shock z_i, and the exogenous parameter vector a_i. A positive shock is simply defined as a parameter change that makes the set of current actions increase given Q, x_i, and z_i.

Definition 5 (Positive shocks). Consider a change in the exogenous parameters of agent $i \in \mathcal{I}$ from a_i' to a_i'', say, where $a_i'' \neq a_i'$. Such a parameter change is called a positive shock if $G_i(x_i, z_i, Q, a_i)$ is ascending in a_i from a_i' to a_i'', that is, if $y_i'' \vee y_i' \in G_i(x_i, z_i, Q, a_i'')$ and $y_i'' \wedge y_i' \in G_i(x_i, z_i, Q, a_i')$ for all $y_i' \in G_i(x_i, z_i, Q, a_i')$ and $y_i'' \in G_i(x_i, z_i, Q, a_i'')$.

To clarify the definition, consider the case in which G_i is single valued, $G_i = \{g_i\}$. In this case, definition 5 simply says that g_i must increase with the parameter change (for given Q, x_i, and z_i): $g_i(x_i, z_i, Q, a_i'') \geq g_i(x_i, z_i, Q, a_i')$. The statement in definition 5 is just the natural set-valued version of this statement. In most cases, we will have $a_i'' > a_i'$, but the definition does not require this (e.g., we will see in lemma 2 below that, under certain conditions, a decrease in the discount factor/level of patience may be a positive shock in large dynamic economies).

The obvious problem with definition 5 is that it refers directly to the stationary policy correspondence G_i. We show below how one can establish that a given parameter change is a positive shock from fundamentals.

Theorem 5 (Comparative statics of positive shocks). Under assumptions 1–3, a positive shock to the exogenous parameters of any subset $\tilde{\mathcal{I}} \subseteq \mathcal{I}$ of the agents (with no shock to the exogenous parameters of the remaining agents in $\mathcal{I} \setminus \tilde{\mathcal{I}}$) will lead to an increase in the least and greatest equilibrium aggregates.

Note that by definition, a positive shock to the a_i's of a subset of agents will lead to increases in those agents' current actions for fixed market aggregates, past actions, and realizations of the idiosyncratic shocks (the rest of the agents, which are not shocked, do not change their actions for fixed market aggregates). So the first-order/partial equilibrium effect of a positive shock is always positive. But in general equilibrium, the market aggregates will also change; in particular, the initial change in strategies will affect the equilibrium aggregate, which will lead to additional

consider are typically "irregular." Rather, the strong (and "regular") comparative statics results here are a consequence of our focus on market aggregates and of the equilibrium forces affecting aggregate variables.

changes in everyone's strategies, further changing equilibrium aggregates, and so on until a new equilibrium is reached. As discussed above, we have assumed essentially nothing about how the market aggregates enter into the agents' decision problems. The proof of theorem 5 in Appendix A shows that this result nevertheless obtains by combining theorems 3 and 4.[14]

Under additional assumptions, we can also specify what happens to individual behavior when agents are subjected to positive shocks. We know from theorem 5 that the market aggregate Q will increase in extremal stationary equilibria. Hence we can simply treat Q as an exogenous variable for an individual i alongside the truly exogenous parameters a_i. In keeping with definition 5, we say that Q is a *positive shock* for agent i if $G_i(x_i, z_i, Q, a_i)$ is ascending in Q. If $G_i(x_i, z_i, Q, a_i)$ is descending in Q (ascending in $-Q$), $-Q$ is a positive shock for the agent or, more straightforwardly, *Q is a negative shock*. Note that given theorem 5 and these definitions, the individual comparative statics question becomes a completely standard comparative statics problem (where we can use the results of, among others, Topkis [1978], Milgrom and Shannon [1994], and Quah [2007]). The following result is just one instance of this based primarily on Topkis (1978).

THEOREM 6 (Individual comparative statics). Suppose that the conditions in theorem 5 are satisfied. Then a positive shock to a subset $\tilde{\mathcal{I}} \subseteq \mathcal{I}$ of the agents will lead to

- a first-order stochastic dominance increase in the distribution of the least and greatest stationary equilibrium strategies of any agent $i \in \mathcal{I}$ for whom increases in Q are positive shocks,
- a first-order stochastic dominance decrease in the distribution of the least and greatest stationary equilibrium strategies of any agent $i \in \mathcal{I} \setminus \tilde{\mathcal{I}}$ for whom increases in Q are negative shocks.

Note that in the special case in which Q is a positive shock for all agents, the economy will be monotone/supermodular. In this case, theorem 6 implies that a positive shock will lead to a first-order stochastic dominance increase in the distribution of all agents' least and greatest stationary equilibrium strategies. As we have previously discussed, theorem 5 requires that increases in Q are neither positive nor negative shocks for all or even a single agent, so it is clear that individual-level predictions re-

[14] The intuition is related to that of the famous *correspondence principle*, which states that with sufficient regularity of the equilibrium mapping, a lot can be said about an economy's comparative statics properties. But whereas the correspondence principle requires one to select stable equilibria, our formulation selects the extremal equilibria (the least and greatest equilibrium aggregates), and furthermore, regularity, in our setting, is exclusively a market-level phenomenon.

quire much more stringent assumptions than predictions at the market level.

B. *Identifying Positive Shocks*

We now provide easy-to-verify sufficient conditions for positive shocks. The instantaneous utility function $u_i = u_i(x_i, y_i, z_i, Q, a_i)$ of an agent $i \in \mathcal{I}$ exhibits increasing differences in y_i and a_i if $u_i(x_i, y_i^2, z_i, Q, a_i) - u_i(x_i, y_i^1, z_i, Q, a_i)$ is nondecreasing in a_i whenever $y_i^2 \geq y_i^1$. If X_i, $A_i \subseteq \mathbb{R}$ and u_i is differentiable, increasing differences in y_i and a_i is equivalent to having $D^2_{y_i a_i} u_i \geq 0$ (Topkis 1998). Agent i's constraint correspondence $\Gamma_i = \Gamma_i(x_i, z_i, Q, a_i)$ is said to have strict complementarities in (x_i, a_i) if for any fixed choice of (z_i, Q) it holds for all $x_i^2 \geq x_i^1$ and $a_i^2 \geq a_i^1$, that $y \in \Gamma(x_i^1, z_i, Q, a_i^2)$ and $\tilde{y} \in \Gamma(x_i^2, z_i, Q, a_i^1)$ implies $y \wedge \tilde{y} \in \Gamma(x_i^1, z_i, Q, a_i^1)$ and $y \vee \tilde{y} \in \Gamma(x_i^2, z_i, Q, a_i^2)$. The concept of strict complementarities is due to Hopenhayn and Prescott (1992). It is weaker than assuming that the graph of Γ_i is a sublattice of $X_i \times X_i \times A_i$ for given (z_i, Q).

The proof of the next lemma is omitted (and is essentially identical to but slightly more straightforward than the proof of lemma 2).

Lemma 1. Suppose that assumptions 1 and 3 are satisfied for agent $i \in \mathcal{I}$. If $u_i = u_i(x_i, y_i, z_i, Q, a_i)$ exhibits increasing differences in y_i and a_i and $\Gamma_i = \Gamma_i(x_i, z_i, Q, a_i)$ exhibits strict complementarities in x_i and a_i, then any increase in a_i is a positive shock for agent i.

Example 1 (continued). Consider again the Bewley-Aiyagari economy, where, as established previously, the constraint correspondence takes the form

$$\Gamma_i(x_i, z_i, Q, a_i) = \{y_i \in [a_i, \bar{b}_i] : y_i \leq r(Q)x_i + w(Q)z_i\},$$

with the borrowing limit treated as an exogenous parameter, that is, $a_i = -\underline{b}_i$. Clearly, for $x_i^2 \geq x_i^1$, $a_i^2 \geq a_i^1$, $y \in [a_i^2, r(Q)x_i^1 + w(Q)z_i]$, and $\tilde{y} \in [a_i^1, r(Q)x_i^2 + w(Q)z_i]$, we have

$$y \wedge \tilde{y} = \min\{y, \tilde{y}\} \in [a_i^1, r(Q)x_i^1 + w(Q)z_i]$$

and

$$y \vee \tilde{y} = \max\{y, \tilde{y}\} \in [a_i^2, r(Q)x_i^2 + w(Q)z_i].$$

So Γ_i has strict complementarities in (x_i, a_i). Consequently, a "tightening" of the borrowing limits in a Bewley-Aiyagari economy will be a positive shock (note that since a_i does not affect the utility function in this case, the increasing differences part of the previous lemma is trivially satisfied).

The next result deals with changes in the discount factor/level of patience.

Lemma 2. Suppose that assumptions 1 and 3 are satisfied for agent $i \in \mathcal{I}$. Then if $u_i = u_i(x_i, y_i, z_i, Q)$ is increasing in x_i and $\Gamma_i = \Gamma_i(x_i, z_i, Q)$ is expansive in x_i (i.e., $x_i \leq \tilde{x}_i \Rightarrow \Gamma_i(x_i, z_i, Q) \subseteq \Gamma_i(\tilde{x}_i, z_i, Q)$), an increase in the discount factor β is a positive shock for agent i. If, instead, u_i is decreasing in x_i and Γ_i is contractive in $x_i(x_i \leq \tilde{x}_i \Rightarrow \Gamma_i(x_i, z_i, Q) \supseteq \Gamma_i(\tilde{x}_i, z_i, Q))$, a decrease in the discount factor β is a positive shock for agent i.

Finally, the next lemma provides another set of sufficient conditions for positive shocks that turn out to be useful in several settings. It applies directly to so-called *homogeneous programming problems* (see, e.g., Alvarez and Stokey 1998), and as the example below shows, it covers certain types of productivity shocks in the Bewley-Aiyagari model when combined with lemma 1.

Lemma 3. Assume that $u_i(x_i, y_i, z_i, Q, a_i)$ is homogeneous in strategies and exogenous variables (i.e., $u_i(\lambda x_i, \lambda y_i, z_i, Q, \lambda a_i) = \lambda^k u_i(x_i, y_i, z_i, Q, a_i)$ for all $\lambda > 0$ and some $k \in \mathbb{R}$)[15] and that the constraint is a cone (i.e., $y_i \in \Gamma_i(x_i, z_i, Q, a_i) \Leftrightarrow \lambda y_i \in \Gamma_i(\lambda x_i, z_i, Q, \lambda a_i)$ for all $\lambda > 0$). Then any increase in a_i is a positive shock for player i.

Example 1 (continued). In the Bewley-Aiyagari model, $u_i(x_i, y_i, z_i, Q, a_i) = \tilde{u}_i(r(Q)x_i - y_i + w(Q)a_i z_i)$ will be homogeneous in (x_i, y_i, a_i) if and only if the household has homothetic preferences (because we have homothetic preferences whenever $\tilde{u}_i$ is homogeneous; see Jensen 2012c, 811). Furthermore, when we ignore upper and lower bounds on assets, the constraint correspondence takes the form $\Gamma_i(x_i, z_i, Q, a_i) = \{y_i \in \mathbb{R} : y_i \leq r(Q)x_i + w(Q)a_i z_i\}$ and is clearly a cone. It follows from lemma 3 that if a household's borrowing constraint is nonbinding, then an increase in a_i is a positive shock. But as discussed above, a tightening of the borrowing constraint—possibly to a level where it binds—is also a positive shock.

V. Changes in Distributions

In this section, we present our comparative statics results in response to changes in the distribution of the idiosyncratic shock processes. Our first result (theorem 7) deals with first-order stochastic dominant changes in the shock processes. Loosely speaking, first-order stochastic changes will lead to higher equilibrium aggregates if at the individual level (i) a higher shock in a period increases the strategy in that period (assumption 4), and (ii) given constant aggregates, a first-order stochastic increase makes the individuals increase their strategies (assumption 5). As we explain immediately after theorem 7, condition ii is somewhat stringent; for instance, it does not hold in the setting of the Bewley-Aiyagari

[15] In the case $k = 0$, we follow the usual convention that the function must be equal to the logarithm of a homogeneous of degree 1 function (cf. assumption 1 in Jensen [2012c]).

model. However, condition ii plays no role for our the next theorem (theorem 8), which is this section's main result. This theorem shows that condition i together with certain easy-to-verify "third-order conditions" on the instantaneous utility function (and standard convexity and concavity conditions on the constraint correspondences and payoff functions) implies that any mean-preserving spread of the stochastic processes of idiosyncratic shocks will increase the equilibrium aggregate.

For the results in this section, the exogenous parameters $(a_i)_{i\in\mathcal{I}}$ play no role, and we suppress them to simplify notation.

Assumption 4. The function $u_i(x_i, y_i, z_i, Q)$ exhibits increasing differences in y_i and z_i, and $\Gamma_i(x_i, z_i, Q)$ is ascending in z_i.

When coupled with assumption 3, assumption 4 implies that the policy correspondence $G_i(x_i, z_i, Q, a_i)$ is ascending in z_i (Hopenhayn and Prescott 1992). Intuitively, it means that a larger value of z_i will lead to an increase in actions. For example, in the Bewley-Aiyagari model, when savings is a normal good, a higher z_i will increase income and savings.

A. *First-Order Stochastic Dominant Changes*

We begin by looking at first-order stochastic dominance increases in the distribution of $z_{i,t}$ for all or a subset of the agents. We now impose an additional assumption involving once again Hopenhayn and Prescott's (1992) notion of strict complementarities. Recall that, according to this notion, Γ_i has strict complementarities in (x_i, z_i) if the following is true: for all $x_i^2 \geq x_i^1$ and $z_i^2 \geq z_i^1$, $y \in \Gamma_i(x_i^1, z_i^2, Q)$ and $\tilde{y} \in \Gamma_i(x_i^2, z_i^1, Q)$ (for any fixed value of Q) imply that $y \wedge \tilde{y} \in \Gamma_i(x_i^1, z_i^1, Q)$ and $y \vee \tilde{y} \in \Gamma_i(x_i^2, z_i^2, Q)$.

Assumption 5. The function $u_i(x_i, y_i, z_i, Q)$ exhibits increasing differences in x_i and z_i, and $\Gamma_i(x_i, z_i, Q)$ has strict complementarities in (x_i, z_i).

Suppose that the stationary distribution of z_i, μ_{z_i}, is ordered by first-order stochastic dominance. Then assumptions 3–5 together ensure that $G_i(x_{i,t}, z_{i,t}, \mu_{z_i})$, the policy correspondence of agent i, when parameterized by μ_{z_i}, is ascending in μ_{z_i} (Hopenhayn and Prescott 1992). It is intuitively clear that when this is so, a first-order stochastic dominant increase in μ_i will lead to an increase in the optimal strategy of agent i. Then, as with our previous results, the main contribution of the next theorem is to show that this will translate into an increase in equilibrium aggregates.

Theorem 7 (Comparative statics of first-order stochastic dominance changes). Under assumptions 1–5, a first-order stochastic dominance increase in the stationary distribution of $z_{i,t}$ for all i (or any subset hereof) will lead to an increase in the least and greatest equilibrium aggregates.

It is also straightforward to see that theorem 6 carries over to this case to obtain individual comparative statics results once the change in the aggregate is determined. We omit this result to economize on space.

B. Mean-Preserving Spreads

We now investigate how mean-preserving spreads of the stationary distributions of the individual-level stochastic processes affect equilibrium outcomes. Recall that μ_{z_i} is a *mean-preserving spread* of μ'_{z_i} if and only if $\mu_{z_i} \succeq_{\text{cx}} \mu'_{z_i}$, where $\succeq_{\text{cx}}$ is the convex order ($\mu_{z_i} \succeq_{\text{cx}} \mu'_{z_i}$ if and only if $\int f(\tau)\mu(\tau) \geq \int f(\tau)\mu'(\tau)$ for all convex functions f).

EXAMPLE 1 (continued). In the Bewley-Aiyagari setting, the focus would be on a mean-preserving spread of the labor endowments/earnings process. The economic question would be whether more uncertain earning prospects will lead to higher capital-labor and output-labor ratios in equilibrium. Note that this question can be thought of as a natural extension to a general equilibrium setting of the partial equilibrium analysis of the impact of a mean-preserving spread of labor income risk on precautionary saving (e.g., Huggett 2004).

For the result to follow, we need additional structure on the individuals' decision problems. Recall also that a correspondence $\Gamma : X \to 2^X$ has a *convex graph* if for all x, $\tilde{x} \in X$ and $y \in \Gamma(x)$ and

$$\tilde{y} \in \Gamma(\tilde{x}) : \lambda y + (1-\lambda)\tilde{y} \in \Gamma(\lambda x + (1-\lambda)\tilde{x})$$

for all $\lambda \in [0,1]$.

ASSUMPTION 6.

1. $X_i \subseteq \mathbb{R}$ for all i.
2. $\Gamma_i(\cdot, z_i, Q) : X_i \to 2^{X_i}$ and $\Gamma_i(x_i, \cdot, Q) : Z_i \to 2^{X_i}$ have convex graphs and $u_i(x_i, y_i, z_i, Q)$ is concave in (x_i, y_i), strictly concave in y_i, and increasing in x_i.

Assumption 6 is standard (see, e.g., Stokey and Lucas 1989) and is easily satisfied in all the applications we consider in this paper. Moreover, part 1 of this assumption can be dispensed with (it is adopted for notational convenience).

DEFINITION 6. Let $k \geq 0$. A function $f : X \to \mathbb{R}_+$ is k-convex (k-concave) if the following conditions hold:

- When $k \neq 1$, the function $[1/(1-k)][f(x)]^{1-k}$ is convex (concave).
- When $k = 1$, the function $\log f(x)$ is convex (concave); that is, f is log-convex (log-concave).

A detailed treatment of the concepts of k-convexity and k-concavity can be found in Jensen (2012a). The essence of the concepts is that k-convexity is a strengthening of (conventional) convexity, while k-concavity is a weakening of concavity. So in terms of the conditions on the derivatives in this section's main result, which follows next, the requirement is loosely that some derivatives must be "a little more than convex" while others must be "a little less than concave." In light of the literature on precautionary savings (again see, e.g., Huggett [2004] and references therein), it should not be surprising that we need to place some conditions on the curvature of the partial derivatives (third derivatives). The economic intuition of these conditions is also straightforward: under the theorem's conditions, mean-preserving spreads will amount to "positive shocks" in the sense that, given equilibrium aggregates, they will make the affected individuals increase their strategies (in the convex order defined by mean-preserving spread). In the Bewley-Aiyagari model, this effect is driven by the precautionary savings motive.

THEOREM 8 (Comparative statics of mean-preserving spreads). Suppose that assumptions 1–4 and 6 hold for all agents, and in addition, assume that each u_i is differentiable and satisfies the following upper boundary condition,

$$\lim_{y_i^n \uparrow \sup \Gamma_i(x_i, z_i, Q)} D_{y_i} u_i(x_i, y_i^n, z_i, Q) = -\infty ,$$

which ensures that $\sup \Gamma_i(x_i, z_i, Q)$ will never be optimal given (x_i, z_i, Q). Then a mean-preserving spread to the invariant distribution μ_{z_i} of any subset of agents $\mathcal{I}' \subseteq \mathcal{I}$ will lead to an increase in the least and greatest equilibrium aggregates if, for each $i \in \mathcal{I}$, there exists a $k_i \geq 0$ such that $-D_{y_i} u_i(x_i, y_i, z_i, Q)$ is k_i-concave in (x_i, y_i) and (y_i, z_i); and $D_{x_i} u_i(x_i, y_i, z_i, Q)$ is k_i-convex in (x_i, y_i) and (y_i, z_i).

Theorem 8 provides a fairly easy-to-apply result showing how changes in the individual-level noise affect market aggregates. Mathematically, mean-preserving spreads increase individual-level actions whenever the policy correspondence defined in (9) is convex in x_i (note that the policy correspondence will be single-valued/a function under assumption 6, so this statement is unambiguous). The assumptions imposed in theorem 8 ensure such convexity of policy functions.[16]

VI. Applications

In this section we apply our comparative statics results to a number of canonical large dynamic economies. We emphasize how the requisite assumptions can be easily verified.

A. *The Bewley-Aiyagari Model*

We have already presented the basics of the Bewley-Aiyagari model in example 1. Here let us slightly generalize our treatment by defining Q_t

[16] See Jensen (2012b) for a detailed treatment of this issue. See also Carroll and Kimball (1996) and Huggett (2004) for the special case of income allocation problems.

as the aggregate capital to "effective" labor ratio at date t. This will again be the relevant market aggregate. In particular, suppose that the aggregate production function of the economy is given by $F(K_t, AL_t)$, where A is labor-augmenting productivity. Then

$$Q_t = \frac{K_t}{A\bar{L}_t},$$

where $\bar{L}_t$ is the total labor endowment of the economy. This market aggregate uniquely determines the wage as $w_t = Aw(Q_t)$ and the interest rate $r_t = r(Q_t)$ at date t via the usual marginal product conditions. Clearly, an improvement in labor-augmenting productivity leaves the interest rate unchanged at a fixed effective capital-labor ratio but increases the wage rate. In what follows, with some abuse of terminology, we continue to refer to Q_t as the capital-labor ratio of the economy, dropping the qualifier "effective" when this causes no confusion.

Household i chooses assets $x_{i,t}$ and consumption $c_{i,t}$ at each date in order to maximize discounted utility as given in (2) subject to the constraint correspondence in (3).

As outlined above, when the instantaneous utility function of agent i, $\tilde{u}_i$, is increasing, we can substitute for $c_{i,t}$ to write (2) as

$$\mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t u_i(x_i, y_i, z_i, Q, a_i)\right],$$

where

$$u_i(x_i, y_i, z_i, Q, a_i) \equiv v_i(r(Q)x_i + Aw(Q)z_i - y_i).$$

The associated constraint correspondence then becomes

$$\Gamma_i(x_i, z_i, Q) = \{y_i \in [-\underline{b}_i, \bar{b}_i] : y_i \leq r(Q)x_i + Aw(Q)z_i\}. \tag{10}$$

It is clear then that this (generalized) Bewley-Aiyagari model is a large dynamic economy. Note also that the specification chosen here generalizes the original model considered by Bewley and Aiyagari by allowing rich heterogeneity across agents. Denote the total labor endowment $\int_{[0,1]} z_{i,t}\,di$ by $\bar{L}_t$. Then the aggregate can be written simply as $Q_t = \int_{[0,1]} x_{i,t}\,di/A\bar{L}_t$. In stationary equilibrium, $\bar{L}_t$ is constant, so when A is also constant, $A\bar{L}_t$ can be normalized to unity, and the aggregator can be taken as $\int_{[0,1]} x_{i,t}\,di$ exactly as in our baseline aggregator, (6) (below we will also consider changes in A).

We next verify assumptions 1–3. Assumption 1 is trivially satisfied under the general conditions (continuity, compactness), and assumption 2 holds because the baseline aggregator is convexifying. Assumption 3 was verified for the Bewley-Aiyagari model in Section II.D, and as noted there, the supermodularity requirement will hold if and only if the instantaneous utility function $\tilde{u}_i$ is concave.

We also note that u_i is increasing in x_i and that Γ_i is expansive in x_i (these additional properties are used in lemma 2, where an expansive correspondence is also defined). Then using lemmas 1–3 (which in particular imply that an increase in the discount factor β, a tightening of the borrowing limits, changes, and preferences that reduce the marginal utility of consumption, and improvements in labor-augmenting technology A are positive shocks) and applying theorem 5, we obtain the following comparative statics results.[17]

Proposition 1. Consider the generalized Bewley-Aiyagari model as described above. Then:

- An increase in the discount rate β will lead to an increase in the least and greatest capital-labor ratios in equilibrium, as well as an increase in the associated least and greatest equilibrium output per capita.
- Any tightening of the borrowing limits (a decrease in $\underline{b}_i$ for all or a subset of households) is a positive shock and consequently leads to an increase in the least and greatest capital-labor ratios in equilibrium, as well as an increase in the associated least and greatest equilibrium output per capita. This statement remains valid when borrowing limits are endogenous ($\underline{b}_i$ is a function of Q), where a tightening means that $\underline{b}_i$ decreases for any fixed value of Q.
- Let a_i parameterize the instantaneous utility function $v_i = v_i(c_i, a_i)$, where c_i denotes consumption at a point in time, and consider the effect of a decrease in marginal utility; that is, assume that $D^2_{c_i a_i} v_i \leq 0$. Then an increase in a_i (for any subset of the agents not of measure zero) will lead to an increase in the least and greatest capital-labor ratios in equilibrium, as well as an increase in the associated least and greatest equilibrium output per capita.
- Suppose in addition that u_i's are homothetic. Then an increase in A will lead to an increase in the least and greatest (effective) capital-labor ratios in equilibrium, as well as an increase in the associated least and greatest equilibrium output per capita.

One of the implications of proposition 1 is that tighter borrowing constraints increase output per capita under fairly general conditions (including endogenous borrowing constraints). Thus we significantly generalize the results of Aiyagari (1994) and Miao (2002). Proposition 1 also

[17] Since, given our results so far, the proofs of all of the propositions in this section are straightforward, we omit them to save space.

implies that a “more credit-rationed” economy (where a larger fraction of households have binding borrowing constraints) will have higher equilibrium capital-labor and output-per-labor ratios. These conclusions follow from the fact that tighter borrowing constraints force agents to increase their precautionary savings levels when they face the prospect of being borrowing constrained at “bad” realizations of shocks.

Finally, the last part of the proposition shows that improvements in the labor-augmenting productivity also increase (effective) capital-labor ratios and equilibrium output per capita.

We can further use the results in proposition 1 to briefly discuss why in general very little can be said about individual behavior even though we can obtain quite strong results on aggregates. Consider, for example, an increase in β. At given Q, this is a positive shock and thus will increase the savings (asset holdings) of all households, raising the aggregate capital-labor ratio. As the aggregate capital-labor ratio increases, however, the wage rate increases and the interest rate falls, potentially discouraging savings. In fact, even a small increase in Q may have a significant impact on the savings of some households depending on income and substitution effects. Thus in equilibrium, a subset of households will typically reduce their savings while some others increase theirs. In fact, it is in general very difficult to say which households will reduce and which will increase their savings, because this will depend on the exact changes in the wage and interest rates. Nevertheless, the essence of the results here is that in the aggregate, savings and thus Q must go up.

A second case that illustrates the previous point even more sharply is that of a population of households for all of whom an increase in Q is a negative shock as defined in Section IV.A. When this holds, any household will lower its savings when Q increases. Now imagine that a subset of the households (with positive measure) have their borrowing constraints tightened. Then from proposition 1, the equilibrium aggregate, Q, will increase. But any household whose borrowing constraint remains the same must then lower its savings (and some of the households who do experience tightened borrowing constraints may also lower their savings as well). In the aggregate, all such falls in savings are more than counteracted by households who save more, however.

We next turn to distributional comparative statics. Assumption 4 requires that $u_i(x_i, y_i, z_i, Q)$ is supermodular in y_i and z_i and that $\Gamma_i(x_i, z_i, Q)$ is ascending in z_i, both of which follow from the same argument as that used above to verify that u_i is supermodular in x_i and y_i and that Γ_i is ascending in x_i (this is simply because x_i and z_i enter in an entirely “symmetric” way in u_i and Γ_i). Next turning to assumption 6, it is straightforward to verify that Γ_i has a convex graph as required. The concavity parts of assumption 6 hold if we take v_i to be strictly concave (note that this corresponds to assuming that households are risk averse). Next let us turn to the required k-concavity and k-convexity conditions of theorem 8. Specifically, for each household i, there must exist a $k_i \geq 0$ such that $-D_{y_i} u_i(x_i, y_i, z_i, Q)$ is k_i-concave in (x_i, y_i) as well as (y_i, z_i) and $D_{x_i} u_i(x_i, y_i, z_i, Q)$ is k_i-convex in (x_i, y_i) and in (z_i, y_i). Because the argument of v_i is linear in y_i, x_i, and z_i, all of these conditions will be satisfied simultaneously if and only if $Dv_i(c_i)$ is k_i-concave as well as k_i-convex. In other words, $[1/(1 - k_i)][Dv_i(c_i)]^{1-k_i}$ must be linear in c_i. Clearly, strict concavity in addition requires that $k_i > 0$. Differentiating twice, setting it equal to zero, and rearranging this yields the condition

$$\frac{D^3 v_i(c_i) Dv_i(c_i)}{[D^2 v_i(c_i)]^2} = k_i > 0. \tag{11}$$

This is exactly the condition that v_i belongs to the hyperbolic absolute risk aversion (HARA) class (Carroll and Kimball 1996). Most commonly used utility functions are in fact in the HARA class, including those that exhibit either constant absolute risk aversion or constant relative risk aversion (see, e.g., Carroll and Kimball 1996). Conveniently, such functions will also satisfy the boundary condition of theorem 8. So picking v_i in the HARA class is sufficient for all of the conditions of theorem 8 to hold, and so we get the following proposition.

PROPOSITION 2. Consider the generalized Bewley-Aiyagari model, and assume that v_i belongs to the HARA class for all i. Then a mean-preserving spread to (any subset of) the households' noise environments will lead to an increase in the greatest and least equilibrium capital-labor ratios and an increase in the associated least and greatest equilibrium per capita outputs.

Proposition 2 shows that an observation made by Aiyagari (1994, 671) in the context of an example is in fact true in general: an economy with idiosyncratic shocks will induce higher savings and output per capita than an otherwise-identical economy without any uncertainty. Proposition 2 is also closely related to a result in Huggett (2004), which shows that an individual agent's accumulation of wealth will increase if she is subjected to higher earnings risk (in particular, this result is valid for preferences that are a subset of the HARA class; cf. Huggett 2004, 776). Proposition 2 can thus be seen as extending Huggett's individual-level result to the market/general equilibrium level.

It is also useful to note several generalizations of the model we have discussed here in which our results can be applied without modification.

1. We can endogenize labor supply by assuming that households derive utility from consumption c and leisure h (see, e.g., Marcet, Obiols-Homs, and Weil 2007). Assume that household i is endowed with $\bar{l}_i$ units of labor, so labor supplied is $l_{i,t} = \bar{l}_i - h_{i,t}$, where $h_{i,t}$ is leisure consumed at time t by household i, and we interpret $z_{i,t}$ as the productivity of the la-

bor supply to the market by this household at time t. Define the indirect utility function

$$v_i(\tilde{c}, wz_i) = \max_{c,h}\{u_i(c, h) : c + hwz_i = \tilde{c}\},$$

where u_i is the instantaneous utility function defined over consumption and leisure. The individual household's decision problem can then be written as the maximization of

$$\mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t v_i(\tilde{c}_{i,t}, w(Q_t)z_{i,t})\right]$$

subject to the constraint

$$\tilde{\Gamma}_i(x_{i,t}, \tilde{c}_{i,t}, z_{i,t}, Q_t) = \{(x_{i,t+1}, \tilde{c}_{i,t+1}) \in [-\underline{b}_i, \bar{b}_i] \times [0, \bar{c}_i] : \\ x_{i,t+1} \leq r(Q_t)x_{i,t} + w(Q_t)\bar{l}_i z_{i,t} - \tilde{c}_{i,t}\}.$$

This is clearly a large dynamic economy, the aggregate (the capital-labor ratio) being now $Q_t = \int_i x_{i,t}di/A\int_i z_{i,t}(\bar{l}_i - h_{i,t})di$. When consumption and leisure are complements ($D^2_{ch}u \geq 0$), the optimal choice of leisure $h_{i,t}$ will be increasing in $x_{i,t}$; hence this reduces to our standard formulation of an aggregator, and, in particular, this aggregator will be monotone in $(x_{i,t})_{i\in\mathcal{I}}$.

2. We can generalize the households' payoff functions to allow for relative comparisons. For instance, (2), can be replaced with

$$\mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t v_i\left(c_{i,t}, \frac{x_{i,t}}{Q_t}\right)\right], \tag{12}$$

so that households derive utility consumption and their relative standing in the society (in terms of mean wealth). The constraint correspondence is the same as above, and this is clearly still a large dynamic economy. This specification could result, for example, from a simplified version of Cole, Mailath, and Postlewaite (1992), where the authors study a model of status. In their model, the marriage market allocates spouses on the basis of status determined as a function of wealth, generating an additional incentive for wealth accumulation. Even though in their model it is the full distribution of wealth that matters (because it is the rank of an individual that determines his or her marriage prospects), the formulation here is closely related and more tractable and readily allows our results to be applied.

B. Hopenhayn's Model of Entry, Exit, and Firm Dynamics

Another prominent example in which our results can be applied straightforwardly is Hopenhayn (1992). A continuum of price-taking firms $\mathcal{I}$ is subject to idiosyncratic productivity shocks with $z_{i,t} \in Z = [0, 1]$ denoting firm i's shock at date t. Firms endogenously enter and exit the market. Upon entry, a firm's productivity is drawn from a fixed probability distribution ν, and from then on (as long as the firm remains active), its productivity follows a monotone Markov process with transition function $\Gamma(z, A)$.[18] Let us restrict attention to stationary equilibria in which the sequence of market prices is constant and equal to $p > 0$. Then at any point in time, the value of an active firm with productivity $z \in Z$ is determined by the value function V, which is the solution to the following functional equation:

$$V(p, z) = \max_{d\in\{0,1\}, x\in\mathbb{R}_+} \left\{[px - C(x, z) - c] + d\beta \int V(p, z')\Gamma(z, dz')\right\}. \quad (13)$$

Here C is the cost function for producing x given productivity shock z, and $c > 0$ is a fixed cost paid each period by incumbent firms; β is the discount rate; d is a variable that captures active firms' option to exit ($d =$ 1 means that the firm remains active; $d = 0$ that it exits); C is continuous, strictly decreasing in z, and strictly convex and increasing in x with $\lim_{x\to\infty} C'(x, z) = \infty$ for all z. This ensures that there exists a unique function V that satisfies this equation. Let $d^*(z, p)$ and $x^*(z, p)$ denote the optimal exit and output strategies for a firm with productivity z facing the (stationary) price p. Clearly, the firm will exit if $\int V(p, z')\Gamma(z, dz') < 0$. Since V will be strictly decreasing in z, this determines a unique (price-dependent) exit cutoff $\bar{z}_p \in Z$ such that $d^*(z, p) = 0$ if and only if $z < \bar{z}_p$.

Any firm that is inactive at date t may enter after paying an entry cost $\gamma(M) > 0$, where M is the measure of firms entering at that date, and γ is a strictly increasing function.[19] Given p and the value function V determined from p as described above, new firms will consequently keep entering until their expected profits equal the entry cost:

$$\int V(p, z')\nu(dz') - \gamma(M) = 0, \quad (14)$$

where ν is the distribution of productivity for new entrants. Given p (and from there V), this determines a unique measure of entrants M_p. Given M_p and the above-determined exit threshold $\bar{z}_p$, the stationary distribution of the productivities of active firms must satisfy

[18] Given $z_{i,t}$, the probability that the shock at time $t + 1$, $z_{i,t+1}$, will be in the set $A \subseteq Z$ is $\Gamma(z_{i,t}, A)$. Monotonicity means that higher productivity at date t makes higher productivity at date $t + 1$ more likely (mathematically, $\Gamma(z', \cdot)$ first-order stochastically dominates $\Gamma(z, \cdot)$ whenever $z' \geq z$).

[19] This increasing cost of entry would result, e.g., because there is a scarce factor necessary for entry (e.g., land or managerial talent). Hopenhayn (1992) assumes that $\gamma(M)$ is independent of M. Our assumption simplifies the exposition, but it is not critical for our results.

$$\mu_p(A) = \int_{z_i \geq \bar{z}_p} \Gamma(z_i, A)\mu_p(dz_i) + M\nu(A) \quad \text{all } A \in \mathcal{B}(Z), \tag{15}$$

where $\mathcal{B}(Z)$ denotes the set of Borel subsets of Z.[20]

The stationary equilibrium price level p^* can now be determined as

$$p^* = D\left[\int x^*(z_i, p^*)\mu_{p^*}(dz_i)\right], \tag{16}$$

where D is the inverse demand function for the product of this industry, which is assumed to be continuous and strictly decreasing. This equation makes it clear that the key aggregate (market) variable in this economy, the price level p, is determined as an aggregate of the stochastic outputs of a large set of firms.

In consequence, it is intuitive that the Hopenhayn model is a special case of our framework. To bring this out more clearly, let the distribution of productivities across the active firms $\mathcal{N} \subseteq \mathcal{I}$ at some date t be denoted by $\eta_p : \mathcal{N} \to Z$ (note that this mapping depends on p). Then μ_p is precisely the image measure; that is, $\mu_p(A) = \eta\{i \in N : \eta_p(i) \in A\}$, where η is the Lebesgue measure and A is any Borel subset of Z. Hence

$$\int_{\mathcal{N}} x^*(\eta_p(i), p)di = \int_Z x^*(z, p)\mu_p(dz).$$

In words, the expected output of the "average" active firm equals the integral of $x^*(\cdot, p)$ under the measure μ_p. Defining $\tilde{x}_i(p) \equiv x^*(\eta_p(i), p)$, (16) can be equivalently written as

$$p = H((\tilde{x}_i(p))_{i\in\mathcal{I}}) \equiv D\left[\int_{\mathcal{N}} \tilde{x}_i(p)di\right].$$

Clearly, this defines an aggregator as in definition 1 (note that for H to be an increasing function, we must reverse the order on individual strategies).[21] Here $\tilde{x}_i(p)$ is the strategy of a firm given the stationary price level p. Note that this is a random variable $x^*(\cdot, p)$ defined on the probability space $(Z, \mathcal{B}(Z), \mu_p)$, where μ_p (the frequency distribution of the active firms' productivities) in general will depend not only on p but also on any exogenous parameters of the model. Therefore, shocks will affect $\tilde{x}_i(p)$ through two channels: directly through x^* and indirectly through the change in the distribution μ_p.

[20] Hopenhayn (1992) refers to the measure μ_p as the *state of the industry*.

[21] Alternatively, one can use as aggregate the inverse of the price level, p^{-1}. Hopenhayn (1992, 1131, n. 5) briefly discusses the difficulties associated with integrals across random variables and the law of large numbers. Hopenhayn's favored solution—which involves dependency across firms—will not pose any difficulties for our analysis.

It is straightforward to verify that assumptions 1 and 2 hold for active firms (i.e., conditioned on $d = 1$). Assumption 3 is also satisfied since for a given productivity level z, a firm will choose output to maximize $px - C(x, z, a) - c$ (here a is an exogenous parameter affecting costs), and thus the payoff function depends only on x and thus trivially satisfies the supermodularity assumption. Since there is no constraint other than $x \geq 0$ on this problem, the assumption that the graph of the constraint correspondence is a sublattice of $X_i \times X_i$ is also immediately satisfied. From this observation, it also follows that, for active firms, an increase in a will be a positive shock if and only if $D^2_{xa}C(x, z, a) \leq 0$. In other words, any shock that lowers the marginal cost (given p and z) is a positive shock. We also impose the natural restriction that $D_aC(x, z, a) \leq 0$, which implies that $V(z, p, a)$ is increasing in a. Finally, note also that such a shock also makes firms more likely to be active.

These observations enable us to apply theorem 5 to the Hopenhayn model. In addition, note that the right-hand side of (15) is type I and type II monotone in μ_p as well as in $-\bar{z}_p$ and M.[22] Therefore, theorem 3 implies that an increase in M or a decrease in $\bar{z}_p$ will lead to a (first-order stochastic dominance) increase in the distribution μ_p. Hence the aggregate p will decrease not only with positive shocks (recall that we have reversed the order on individual strategies) but also with other changes in parameters that lower $\bar{z}_p$ or raise M.[23]

Proposition 3. In the Hopenhayn model as described here,

1. a decrease in the fixed cost of operation c or a (first-order) increase in the transition function Γ lowers the equilibrium price and increases aggregate output;
2. a first-order stochastic increase in the entrants' productivity distribution ν lowers the equilibrium price and increases aggregate output;

[22] In this statement, μ_p is ordered by first-order stochastic dominance. The right-hand side of (15),

$$F(\mu(\cdot), \bar{z}_p, M) = \int_{z_i \geq \bar{z}_p} \Gamma(z_i, \cdot)\mu(dz_i) + M\nu(\cdot),$$

is single valued, so type I and type II monotonicity coincide with monotonicity in the usual sense. Note that $\int_{z_i \geq \bar{z}_p} \Gamma(z_i, \cdot)\mu(dz_i)$ is simply the adjoint of Γ imputed at $\bar{z}_p$. From this follows immediately that F will be monotone in μ_p since Γ is monotone (and it also easily follows that a decrease in $\bar{z}_p$ will lead to a first-order stochastic increase in F). That F is monotone in M (as well as in ν ordered by first-order stochastic dominance) is straightforward to verify.

[23] When $V(z, p, a)$ is increasing in a—which our assumption that $D_aC(x, z, a) \leq 0$ guarantees—an increase in a will lead to an increase in M (which can be directly seen from eq. [14]) and thus to an increase in μ_p. The fact that $D(\int_Z x^*(z, p)\mu_p(dz))$ decreases when $\mu_p(z)$ undergoes a type I and/or type II increase is a consequence of theorem 4.

3. a positive shock to the firms' profit functions, that is, an increase in a with $D_a C \leq 0$ and $D^2_{xa} C \leq 0$, lowers the equilibrium price and increases aggregate output.

It is also useful to note that, as in the Bewley-Aiyagari model, the effects on individual firms are uncertain and may easily go in opposing directions. Take a decline in the fixed costs of operation c to illustrate this for the first part of the proposition. Such a decline leaves the profit-maximizing choice of output for incumbents, $x(p, z)$, unchanged for any given price and level of productivity, but it will affect the state of the industry μ_p. The reason is that as c declines, the value of a firm with any given productivity $V(p, z)$ increases and the exit cutoff $\bar{z}_p$ also decreases, making it less likely that any active firm will exit in any period. The increase in $V(p, z)$ leads to greater entry, which together with the decline in $\bar{z}_p$ leads to an increase in μ_p, thus raising aggregate output. But as aggregate output increases, the equilibrium price will fall, which leads to counteracting effects on $V(p, z)$ as well as $\bar{z}_p$ (a decrease and an increase, respectively). The combined consequence for any firm with a given productivity level z is uncertain: for many types of firms the indirect effects may dominate, reducing their output, and some types of firms might choose to exit. Nevertheless, aggregate output necessarily increases and the equilibrium price necessarily declines. Similarly in part 2, the result is again driven by the impact of the shift in ν on μ_p; the resulting decline in p is a counteracting effect, reducing firm-level output at given productivity level z. Finally, in part 3, a positive shock directly raises $x(p, z, a)$ for all p, z and also raises the value function V, increasing μ_p and thus also increasing aggregate output and lowering the equilibrium price. Because the resulting decrease in p counteracts this effect, the overall impact on a firm of a given productivity level z is again uncertain. This discussion therefore illustrates that the types of results contained in proposition 3 would not have been possible by studying comparative statics at the individual firm level; indeed, similarly with some of the results discussed in proposition 1, there will generally be no regularity at the individual level.

Several natural extensions of the Hopenhayn model are also covered by our results. These include models that incorporate learning by doing at the firm level (so that current productivity depends on past production) and models in which firms undertake costly investments to improve their productivity.

C. Other Applications

To economize on space, we will sketch the other applications briefly, without providing formal results.

Occupational choice models.—Our framework can be applied to models in which households accumulate wealth and choose their occupations subject to shocks and potentially subject to credit constraints (e.g., between production work and entrepreneurship). Such models have been analyzed by, among others, Banerjee and Newman (1993), Mookherjee and Ray (2003), Buera (2009), Buera et al. (2011), Moll (2012), and Caselli and Gennaioli (2013). We again let Q_t denote the aggregate capital-labor ratio at date t, and suppose that household i chooses its assets $x_{i,t}$ and consumption $c_{i,t}$ to maximize its discounted utility as given by (2). To become an entrepreneur, each household needs to invest at least $\underline{k} > 0$. Credit constraints are modeled by assuming that a household can borrow at most a fraction $\phi \geq 0$ of its current asset holding $x_{i,t}$. Note that, as in our generalized Bewley-Aiagari model, these credit constraints need not bind for all or even a single consumer (in particular, what follows includes as a special case the setting with complete markets). Entrepreneurs also face an idiosyncratic risk denoted by $\eta_{i,t}$, which we assume is realized after the decisions for time t (and is serially uncorrelated). In particular, this implies that the earnings of household i at time t are either

$$y^W_{i,t} = r(Q_t)x_{i,t} + w(Q_t)z_{i,t}$$

or

$$y^E_{i,t} = r(Q_t)(x_{i,t} - k_{i,t}) + \eta_{i,t} f(Q_t)k_{i,t} - w(Q_t)\frac{Q_t}{k_{i,t}}$$

$$\text{if } k_{i,t} \geq \underline{k} \text{ and } x_{i,t} \geq \frac{k_{i,t}}{1+\phi}.$$

Intuitively, the entrepreneur has net savings (after borrowing) $x_{i,t} - k_{i,t}$ and earns the market interest rate on net savings. To become an entrepreneur, investment, $k_{i,t}$, needs to exceed the minimum investment, that is, $k_{i,t} \geq \underline{k}$, and to finance this, asset holdings plus borrowing need to cover his investment, so $x_{i,t} \geq k_{i,t}/(1+\phi)$. In addition, since all entrepreneurs are ex ante identical, they will hold to the same capital-labor ratio (which will also be equal to the aggregate, Q_t), and the second and the third terms in the above expression simply correspond to the return to entrepreneurs. Then we can write the constraint correspondence of a household as

$$\begin{aligned}&\tilde{\Gamma}_i(x_{i,t}, c_{i,t}, k_{i,t}, z_{i,t}, Q_t)\\ &\quad = \{(x_{i,t+1}, c_{i,t+1}, k_{i,t}) \in [-\underline{b}_i, \bar{b}_i] \times [0, \bar{c}_i] \times [\underline{k}, (1+\phi)x_{i,t}] :\\ &\qquad x_{i,t+1} \leq \max\{y^W_{i,t}, y^E_{i,t}\} - c_{i,t}\}.\end{aligned}$$

In this case, the aggregate becomes

$$Q_t = \frac{\int_{i \in [0,1]} x_{i,t}\, di}{\int_{i \in \Lambda_t} z_{i,t}\, di},$$

where $\Lambda_t \subset [0, 1]$ denotes the set of households who choose to become workers.

Models with aggregate externalities.—A variety of models in which a large number of firms or economic actors create an aggregate externality on others would also be a special case of our framework. Well-known examples include Arrow (1962) and Romer (1986) (though Romer's paper is one of endogenous growth and thus is not formally covered by the results presented so far). Generalizations of this class of models with heterogeneity across firms and stochastic shocks are straightforwardly covered by our results. For example, we can consider a continuum $\mathcal{I}$ of firms each with production function for a homogeneous final good given by $y_{i,t} = f(k_{i,t}, A_{i,t}Q_t)$, where f exhibits diminishing returns to scale and is increasing in both of its arguments, and $A_{i,t}$ is independent across producers and follows a Markov process (which again can vary across firms). Each firm faces an exogenous cost of capital R. The aggregate in this case would be $Q_t = \int k_{i,t}\, di$, summarizing the externalities across firms. One could also consider "learning by doing" type externalities that are a function of past cumulative output, that is, $Q_t = \sum_{\tau=t-T-1}^{t-1} \int y_{i,\tau}\, di$ for some $T < \infty$. Under these assumptions, all the results derived below can be applied to this model.

Search models.—Search models in the spirit of Diamond (1982), Mortensen (1982), and Mortensen and Pissarides (1994), in which members of a single population match pairwise or with firms on the other side of the market to form productive relationships, also constitute a special case of this framework. In Diamond's model, for example, individuals first make costly investments in order to produce ("collect a coconut") and then search for others who have also done so to form trading relationships. The aggregate variable, taken as given by each agent, is the fraction of agents that are searching for partners. This determines matching probabilities and thus the optimal strategies of each agent. Various generalizations of Diamond's model, or for that matter other search models, can also be studied using the framework presented below.

One relevant example in this context is Acemoglu and Shimer (2000), which combines elements from directed search models of Moen (1997) and Acemoglu and Shimer (1999) together with Bewley-Aiyagari style models. In this environment, each individual decides whether to apply to high-wage or low-wage jobs, recognizing that high-wage jobs will have more applicants and thus lower offer rates (these offer rates and exact wages are determined in equilibrium as a function of applications decisions). Individuals have concave preferences and do not have access to outside insurance opportunities, so they use their own savings to smooth consumption. Unemployed workers with limited assets then prefer to apply to low-wage jobs. Acemoglu and Shimer (2000) assumed a fixed interest rate and used numerical methods to give a glimpse of the structure of equilibrium and to argue that high unemployment benefits can increase output by encouraging more workers to apply to high-wage jobs. This model—and in fact a version with an endogenous interest rate—can also be cast as a special case of our framework, and thus, in addition to basic existence results, a range of comparative static results can be obtained readily.

International trade with capital accumulation.—We can also apply our results to various models of trade such as the dynamic Heckscher-Ohlin model (with factor price equalization) of Ventura (1997) or the version in Acemoglu (2009, chap. 19.3). There are $M \in \mathbb{N}$ countries indexed by $m \in \{1, \ldots, M\}$ and two goods that can be traded internationally without any costs or barriers. One good is produced with capital only and has the same technology in all countries. The other good is produced with labor only and has technology $A^m \bar{L}_t^m$ in country m. The utility function is the same in all countries and is homothetic, and the objective takes the form

$$\mathbb{E}_0\left[\sum_{t=0}^{\infty} \beta^t v_i(c_{i,t}^1, c_{i,t}^2)\right]. \tag{17}$$

The constraint correspondence can be written as

$$\tilde{\Gamma}_i(x_{i,t}, c_{i,t}^1, c_{i,t}^2, z_{i,t}, Q_t) = \{(x_{i,t+1}, c_{i,t+1}^1, c_{i,t+1}^2) \in [-\underline{b}_i, \bar{b}_i] \times [0, \bar{c}]^2 : x_{i,t+1} \leq r(Q_t)x_{i,t} + w(Q_t)z_{i,t} - c_{i,t}^1 - c_{i,t}^2\}.$$

Here we explicitly use that, because each sector uses only one factor, factor price equalization applies for any level of the capital stock in each country. This implies that the wage and interest rate faced by households are the same throughout the world and will be determined from the world capital-labor ratio (the aggregate):

$$Q_t = \sum_m Q_t^m \Big/ \sum_m A^m \bar{L}_t^m,$$

where $\bar{L}_t^m = \int z_{i,t}^m di$ (the labor endowment at date t in country m) and $Q_t^m = \int x_{i,t}^m di$ (savings in country m at date t).

VII. Extensions

In this section we briefly explain how our results can be extended to a broader set of economies, including, most importantly, models with multiple aggregates. We begin with the theory and then illustrate this by means of three applications drawn from different areas of economics (a model of status and savings, a model of political competition, and a model of Ricardian international trade).

Consider the class of large dynamic economies as defined in Section II but with multidimensional market aggregates, $Q_t \in \mathbb{R}^M$ for $t = 0, 1, 2, \ldots$, where $M > 1$. All of the definitions and assumptions in Section II extend naturally to this case, as do the results on existence of equilibrium and stationary equilibrium.[24] With multidimensional aggregates, the aggregator H will be a vector-valued function, $H = (H^m)_{m=1}^M$ (and all conditions in Sec. II are then naturally required to hold for each coordinate function H^m, $m = 1, \ldots, M$). Definition 3 then simply requires that in a stationary equilibrium, (1) agents still choose optimal strategies given equilibrium aggregates now given by $Q^* = (Q^{*,1}, \ldots, Q^{*,M})$, and (2) all M markets must now clear at all dates: $Q^{m*} = H^m((x^*_{i,t})_{i\in\mathcal{I}})$ for $t = 0, 1, 2, \ldots$ and all $m = 1, 2, \ldots, M$. Note also that we maintain the definition of positive shocks (definition 5).

In the following theorem we require that the mth aggregate is determined by agents in a prespecified group of agents $\mathcal{I}^m \subseteq \mathcal{I}$, $m = 1, \ldots, M$. Note that groups may overlap, though in many applications, including those we consider below, they may be given by disjoint sets (e.g., citizens residing in different neighborhoods or countries, or agents working in different sectors).

Theorem 9 (Comparative statics with multidimensional aggregates). Consider a large dynamic economy with $M \in \mathbb{N}$ aggregates determined by M groups of agents $\mathcal{I}^1, \mathcal{I}^2, \ldots, \mathcal{I}^M \subseteq \mathcal{I}$:

$$Q_t^m = H^m((x_{i,t})_{i\in\mathcal{I}^m}) \quad \text{for } m = 1, 2, \ldots, M. \tag{18}$$

Assume that each H^m is an aggregator in accordance with definition 1 and that assumptions 1–3 are satisfied. We then have the following conditions:

- If for each group $m \in \{1, \ldots, M\}$, $Q^{m'}$ is a positive shock for each agent $i \in \mathcal{I}^m$ for all $m' \neq m$, then a positive shock (to any subset of the agents) increases the least and greatest equilibrium aggregates ($Q^{*,m}$ for all $m \in \{1, \ldots, M\}$).

[24] One result that is not preserved in the multidimensional setting is the existence of a least and greatest equilibrium aggregate. So this is one thing our result below must explicitly deal with.

- If for a group $m \in \{1, \ldots, M\}$ the remaining equilibrium aggregates $Q^{m'}$, $m' \neq m$ are uniquely determined for any fixed value of Q^m, a positive shock to any subset of agents who are only in group m (i.e., a positive shock to any subset of $\mathcal{I}^m \backslash (\cup_{m' \neq m} \mathcal{I}^{m'})$) increases the least and greatest equilibrium aggregates in market m, $Q^{*,m}$.

The first part of theorem 9 covers as a special case multidimensional supermodular economies, that is, economies in which each Q^m is a positive shock for every agent (which can be obtained by sending $\mathcal{I}^1 = \cdots = \mathcal{I}^M = \mathcal{I}$); but of course, the first part of the theorem is more general and does not depend on having a supermodular economy. Note also that the conclusion in the second part of the theorem is more restrictive: only agents who are in group m—and not in any other group—can be affected by the shocks, and the conclusion concerns only the mth equilibrium aggregate.

Finally, though we have focused here on theorem 5, under similar conditions, theorems 6–8 can also be similarly generalized (omitted once again to economize on space). In the remainder of this section, we sketch three applications to show that theorem 9 covers some very interesting classes of economies with multidimensional aggregates.

Savings, wealth, and status motives.—Our first application is a generalization of the neoclassical growth model with status discussed at the end of Section VI.A. When agents' saving decisions are, in part, motivated by status motives, the relevant status is often related to own wealth relative to mean wealth in agents' local neighborhoods (rather than relative to the whole society). Such a model would correspond to a large dynamic economy with multiple aggregates. For simplicity, consider the case with two neighborhoods corresponding to two groups, that is, $M = 2$, and naturally these groups are nonoverlapping, that is, $\mathcal{I} = \mathcal{I}^1 \cup \mathcal{I}^2, \mathcal{I}^1 \cap \mathcal{I}^2 = \emptyset$, where $\mathcal{I}^m$ is the set of agents living in neighborhood $m = 1, 2$. The problem of a household $i \in \mathcal{I}^m$ is again given by (12), but the second argument is now $x_{i,t}/Q_t^m$, where Q_t^m is average wealth in neighborhood m, $Q_t^m = \int_{i \in \mathcal{I}^m} x_{i,t} di$. The constraint correspondence is generalized in an obvious way by conditioning on (Q^1, Q^2) and writing the relevant market prices as $r(\alpha^1 Q_t^1 + \alpha^2 Q_t^2)$ and $w(\alpha^1 Q_t^1 + \alpha^2 Q_t^2)$, where α^1 is the fraction of households in neighborhood 1 and $\alpha^2 = 1 - \alpha^1$. This economy satisfies the conditions of theorem 9 under natural assumptions. The conditions featured in the first part of theorem 9 are particularly intuitive: savings of agents in each of the neighborhoods must be—all else equal—(weakly) increasing in the level of average wealth in the other neighborhood ($Q^{m'}$ a positive shock for $m' \neq m$). When this holds, theorem 9 implies that positive shocks—defined in exactly the same way as in Section VI.A—increase the least and the greatest equilibrium aggregates and output per capita. When this is not met, one can alternatively use the second part of theorem 9, which requires that a neighborhood, when considered in isolation, has a unique equilibrium aggregate.[25]

Dynamic political competition.—A second application is dynamic political competition between two groups. Consider a society consisting of two such groups, each with total measure normalized to one for simplicity. Suppose that one of these groups will be in power at any point in time. The one in power can tax the other one. Which one is in power is determined by a contest function depending on the aggregate wealth of two groups. Define the two aggregates, Q_t^1 and Q_t^2, as the total wealth levels of the two groups. Suppose that the contest function is such that group 1 will be in power with probability $(Q_t^1)^\theta/[(Q_t^1)^\theta + (Q_t^2)^\theta]$, and of course the other group is in power with the complementary probability. Households in each group again maximize their discounted utility as given by (2). The constraint correspondence is now given by

$$\begin{aligned}\tilde{\Gamma}_i(x_{i,t}, c_{i,t}, z_{i,t}, Q_t^1, Q_t^2, s_t) \\ = \{(l_{i,t}, x_{i,t+1}, c_{i,t+1}) \in \mathbb{R}_+ \times [-\underline{b}_i, \bar{b}_i] \times [0, \bar{c}_i] : \\ x_{i,t+1} \leq [s_t + (1 - s_t)\eta] r(Q_t^1 + Q_t^2) x_{i,t} + w(Q_t^1 + Q_t^2) z_{i,t} l_{i,t} \\ + (1 - \eta) r(Q_t^1 + Q_t^2) Q_t^2 - c_{i,t}\}.\end{aligned}$$

Here, s_t is an indicator for group 1 being in power. If group 2 is in power, it takes away a fraction $1 - \eta$ of group 1's capital. Likewise, when it is in power, group 1 takes a fraction $1 - \eta$ of the other group's capital and distributes it among its members equally. The probability that $s_t = 1$ is $(Q_t^1)^\theta/[(Q_t^1)^\theta + (Q_t^2)^\theta]$, and this is realized before decisions at time t are made. This model has the exact same structure as the previous application (two aggregates influencing decisions of all agents, but the aggregates being determined independently by nonoverlapping groups). Hence the same line of argument applies in this case.

Ricardian international trade.—The final example is a model of Ricardian international trade (similar to that in Acemoglu and Ventura [2002] but without endogenous growth). Suppose that the world economy consists of two countries and two goods. There is neither migration nor capital flow across countries, but the two countries can trade. Country 1 has a comparative advantage in good 1 and country 2 in good 2. Suppose to simplify things that only country 1 can produce good 1 and vice versa. The production functions are denoted by f^1 and f^2. All households maximize discounted utility given by (17). There are no costs of trade, so

[25] Such uniqueness can be ensured by an argument similar to that found in Aiyagari (1993), though with the caveats we raised in n. 3 that this requires additional conditions to ensure that the supply of capital is upward sloping, motivating our somewhat greater emphasis on the first part of the theorem.

the relative price of the two goods will be the same in both countries. Again we can reduce this to an economy with two aggregates, (Q^1, Q^2), defined as the capital-labor ratios in the two countries, and the results in this section can again be applied readily.

VIII. Conclusion

There are relatively few known comparative static results on the structure of equilibria in dynamic economies. Many existing analytic results, such as those in growth models (overviewed in Acemoglu [2009]), are obtained using closed-form characterizations and rely heavily on functional forms. Many other works study the structure of such models using numerical analysis. This paper developed a general and fairly easy-to-apply framework for robust comparative statics about the structure of stationary equilibria in such dynamic economies. Our results are "robust" in the sense defined by Milgrom and Roberts (1994) in that they do not rely on parametric assumptions but on qualitative economic properties, such as utility functions exhibiting increasing differences in choice variables and certain parameters. Nevertheless, and importantly from the viewpoint of placing the contribution within the broader literature, none of our main results follow because of the supermodularity of the game or the economy. In fact, our key technical result, which underlies all our substantive results, is introduced to enable us to work with spaces that are not lattices. From an economic viewpoint, the fact that our results concern market aggregates, and contain little information about individual behavior, is a reflection of lack of supermodularity or monotonicity in the environments we consider.

Well-known models that are special cases of our framework include models of saving and capital accumulation with incomplete markets along the lines of work by Bewley, Aiyagari, and Huggett and models of industry equilibrium along the lines of work by Hopenhayn as well as search models and models of occupational choice with saving decisions and credit constraints. In all cases, our results enable us to establish—to the best of our knowledge—much stronger and more general results than those available in the literature, while at the same time clarifying why such results obtain this class of models. They also lead to a new set of comparative static results in response to first-order and second-order stochastic dominance shifts in distributions representing uncertainty in these models. All the major comparative static results provided in the paper are truly about the structure of equilibrium, not about individual behavior. This is highlighted by the fact that in most cases, while robust and general results can be obtained about how market outcomes behave, little can be said about individual behavior, which is in fact often quite irregular.

We believe that our framework and methods are useful both because they clarify the underlying economic forces, for example, in demonstrating that robust comparative statics applies to aggregate market variables, and because they can be applied readily in a range of problems.

Appendix A

Proofs

We now present the proofs of the main results from the text. Some of these proofs rely on technical results presented in Appendix B.

Proof of Theorem 1

Only a brief sketch is provided since this can be shown by essentially the same argument as that of theorem 1 in Jovanovic and Rosenthal (1988): For agent i, let $\mathcal{X}_i$ denote the set of strategies (these are infinite sequences of random variables as described in the main text), and let $\gamma_i(\mathbf{Q}) \subseteq \mathcal{X}_i$ denote the set of optimal strategies for agent i given the sequence of aggregates $\mathbf{Q} \in \prod_{t=0}^{\infty} \mathcal{Q}$, where $\prod_{t=0}^{\infty} \mathcal{Q}$ with the supremum norm $\| \mathbf{Q} \| = \sup_t | Q_t |$ is a compact and convex topological space; $\mathcal{X}_i$ is equipped with the topology of pointwise convergence, where each coordinate converges if and only if the random variable converges in the weak *-topology. Under assumption 1, $\gamma_i : \prod_{t=0}^{\infty} \mathcal{Q} \to 2^{\mathcal{X}_i}$ will be non–empty valued and upper hemicontinuous. Consider the upper hemicontinuous correspondence $\mathcal{H}(\mathbf{Q}) = \{H((\mathbf{x}_i)_{i\in\mathcal{I}}) : \mathbf{x}_i \in \gamma_i(\mathbf{Q}) \text{ for } i \in \mathcal{I}\}$. Since $\mathcal{H}$ will be convex valued under assumption 2, a fixed point $\mathbf{Q}^* \in \mathcal{H}(\mathbf{Q}^*)$ exists by the Kakutani-Glicksberg-Fan theorem; $\mathbf{Q}^*$ is a sequence of equilibrium aggregates with associated equilibrium strategies $\mathbf{x}_i^* \in \gamma_i(\mathbf{Q}^*)$, $i \in \mathcal{I}$. Q.E.D.

Proof of Theorem 2

Rather than proving this theorem directly, we refer to the proof of theorem 5 from which existence of a stationary equilibrium follows quite easily. Indeed, in that proof it is shown that Q is an equilibrium aggregate given a if and only if $Q \in \hat{H}(Q, a)$, where $\hat{H}$ is an upper hemicontinuous and convex-valued correspondence that maps a compact and convex subset of the reals into itself. Existence therefore follows from Kakutani's fixed-point theorem. The set of equilibrium aggregates will be compact as a direct consequence of the boundedness of the set of feasible equilibrium aggregates (a consequence of continuity of H and assumption 1) and the upper hemicontinuity of $\hat{H}$. Consequently, a least and a greatest equilibrium aggregate will always exist. Q.E.D.

For the proof of theorem 3 we need the following result from Smithson (1971). A subset C of an ordered set X is a *chain* in X if it is totally ordered, that is, if $c \succeq c'$ or $c' \succeq c$ for all $c, c' \in C$. If any chain in X has its supremum in X, then X is said to be *chain complete*.

Smithson's theorem. Let X be a chain-complete partially ordered set and $F: X \to 2^X$ a type I monotone correspondence. Assume that for any chain C in X

and any monotone selection from the restriction of F to C, $f: C \rightarrow X$, there exists $y_0 \in F(\sup C)$ such that $f(x) \leq y_0$ for all $x \in C$. Then, if there exists a point $e \in X$ and a point $y \in F(e)$ such that $e \preceq y$, F has a fixed point.

REMARK 2. Smithson's theorem has a parallel statement for type II monotone correspondences. In particular (see Smithson 1971, 306, remark), the conclusion (existence of a fixed point) remains valid for type II monotone correspondences if the hypotheses are altered as follows: (i) The set X is assumed to be lower chain complete rather than chain complete (a partially ordered set is lower chain complete if each nonempty chain has an infimum in the set). (ii) The condition on monotone selections on chains is altered as follows: For any chain C in X and any monotone selection from the restriction of F to C, $f: C \rightarrow X$, there exists $y_0 \in F(\inf C)$ such that $f(x) \geq y_0$ for all $x \in C$. (iii) Instead of elements $e \in X$ and $y \in F(e)$ with $e \preceq y$, there must exist $e \in X$ and $y \in F(e)$ with $e \succeq y$.

Proof of Theorem 3

We prove only the type I monotone case (the type II monotone case is similar). Compactness of X together with the fact that the order $\succeq$ is assumed to be closed ensures the chain completeness as well as lower chain completeness of $(X, \succeq)$.[26] The condition in Smithson's theorem on the supremum (and infimum in the type II case) of chains is satisfied because F is upper hemicontinuous. Indeed, let C be a chain with supremum $\sup C \in X$, and let $f: C \rightarrow X$ be a monotone selection from $F: C \rightarrow 2^X$. Let $\bar{f} \equiv \sup\{f(c) : c \in C\}$. Now choose a sequence $(c_n)_{n=1}^{\infty}$ from C with $c_{n+1} \succeq c_n$ for all n, $\lim_{n\to\infty} c_n = \sup C$, and $\lim_{n\to\infty} f(c_n) = \bar{f}$. It follows then from upper hemicontinuity of F that $\bar{f} \in F(\sup C)$. In addition, $\bar{f} \geq f(c)$ for all $c \in C$. This proves the claim. Now pick $\theta_2 \succeq \theta_1$ and a fixed point $x_1 \in \Lambda(\theta_1)$. We must show that there will exist an $x_2 \in \Lambda(\theta_2)$ with $x_2 \succeq x_1$. The proof of Smithson's theorem reveals that in fact there will always exist a fixed point x^* with $x^* \succeq e$ where e is an element as described in the theorem. When this observation is applied to the correspondence $F(\cdot, \theta_2)$, the conclusion of theorem 3 will follow if there exists $y \in F(x_1, \theta_2)$ with $y \succeq x_1$. But since $x_1 \in F(x_1, \theta_1)$, this follows directly from F's type I monotonicity in θ. Q.E.D.

Proof of Theorem 4

We prove only that $\bar{h}(\theta)$ is increasing (the other case is similar). The function $\bar{h}(\theta)$ is well defined because H is continuous and $\Lambda(\theta)$ is compact (the fixed-point set of an upper hemicontinuous correspondence on a compact set is always compact). Pick $\theta_1 \leq \theta_2$, and let $x_1 \in \Lambda(\theta_1)$ be an element such that $\bar{h}(\theta) = H(x_1)$. Since $\Lambda(\theta)$ is type I monotone, there will exist $x_2 \in \Lambda(\theta_2)$ such that $x_2 \succeq x_1$. Since H is monotone,

[26] A partially ordered set in which all chains have an infimum as well as a supremum is usually simply said to be *complete* (e.g., Ward 1954, 148). In the present setting in which X is topological and the order $\succeq$ is closed, the claim that compactness implies completeness follows from Ward's theorem 3 because any closed chain will be compact (any closed subset of a compact set is compact).

$$\bar{h}(\theta_2) = \sup_{x \in \Lambda(\theta_2)} H(x) \geq H(x_2) \geq H(x_1) = \bar{h}(\theta_1).$$

Q.E.D.

Proof of Theorem 5

We first provide a brief road map. The proof has three steps: In the first step, theorem 3 is used to show that for any fixed equilibrium aggregate Q, the set of stationary distributions for each individual will be type I and type II increasing in the exogenous variables $a = (a_i)_{i \in \mathcal{I}}$. In step 2, a map $\hat{H}$ that for each Q and a gives a set of aggregates is constructed. The fixed points of this map are precisely the set of equilibrium aggregates given a. Crucially, the least and greatest selections from $\hat{H}$ will be increasing in a by theorem 4. Using this, the third and final step uses an argument from Milgrom and Roberts (1994) and Acemoglu and Jensen (2013) to show that the equilibrium aggregates must also be increasing in a.

For each agent i, let a_i' and a_i'' be the parameter vectors associated with the positive shock. Throughout the proof, a_i is restricted to the set $A_i \equiv \{a_i', a_i''\}$ ordered by $a_i'' \succeq_i^* a_i'$ (that is to say, an order is placed on $\{a_i'', a_i'\}$ such that a_i'' is larger than a_i' in this order). Hence $G_i(x_i, z_i, Q, a_i)$ will be ascending in a_i when a_i is a positive shock. Note that if the shock $a = (a_i)_{i \in \mathcal{I}}$ does not affect an agent i', we may use the same construction now restricting $A_{i'}$ to a singleton (in which case G_i trivially is ascending in $a_{i'}$). This allows us to speak of a positive shock $a = (a_i)_{i \in \mathcal{I}}$ without having to specify the subset of agents affected by the shock.

Step 1: Fix $Q \in \mathcal{Q}$. Consider the agents' stationary policy correspondences $G_i : X_i \times Z_i \times \{Q\} \times A_i \to 2^{X_i}$, $i \in \mathcal{I}$ defined in equation (9), and for given Q and a_i, let $T^*_{Q,a_i} : \mathcal{P}(X_i) \to 2^{\mathcal{P}(X_i)}$ denote the adjoint Markov correspondence induced by G_i. By theorem B2 in Appendix B, each G_i will have a least and a greatest selection, and both of these selections will be increasing in x_i. Therefore, by theorem B1, T^*_{Q,a_i} will be type I and type II monotone when $\mathcal{P}(X_i)$ is equipped with the first-order stochastic dominance order $\succeq_{\text{st}}$. Since $(\mathcal{P}(X_i), \succeq_{\text{st}})$ has an infimum (namely, the degenerate distribution placing probability one on $\inf X_i$), this implies that the invariant distribution correspondence $F_i : \mathcal{Q} \times A_i \to 2^{\mathcal{P}(X_i)}$, given by $F_i(Q, a_i) = \{\mu \in \mathcal{P}(X_i) : \mu \in T^*_{Q,a_i}\mu\}$ is non–empty valued and upper hemicontinuous (theorem B3). Next we use our results from Section III. Since, again by theorem B1, T^*_{Q,a_i} is also type I and type II monotone in a_i, we can use theorem 3 to conclude that the invariant distribution correspondence F_i will be type I and type II monotone in a_i (F_i has nonempty values by theorem B3). This is true for every $i \in \mathcal{I}$; hence the joint correspondence

$$F = (F_i)_{i \in \mathcal{I}} : \mathcal{Q} \times A \to 2^{\Pi_{i \in \mathcal{I}} \mathcal{P}(X_i)}$$

is type I and type II monotone in $a = (a_i)_{i \in \mathcal{I}} \in A = \prod_{i \in \mathcal{I}} A_i$.

Step 2: For a distribution $x \in \prod_{i \in \mathcal{I}} \mathcal{P}(X_i)$, denote the random variable $id : X \to X$ on the probability space $(X, \prod_i \mathcal{B}(X_i), x)$ by $\hat{x}$. Given the aggregator H, define a mapping $\tilde{H}$ from distributions into the reals by the convention that $\tilde{H}(x) H(\hat{x})$ for all $x \in \prod_{i \in \mathcal{I}} \mathcal{P}(X_i)$. Next, consider

$$\hat{H}(Q, a) = \{\tilde{H}(x) \in \mathbb{R} : x \in F(Q, a) \text{ for all } i\}.$$

It is clear from the definition of a stationary equilibrium that Q^* is a (stationary) equilibrium aggregate given $a \in A$ if and only if $Q^* \in \hat{H}(Q^*, a)$. Under assumption 2, either (i) G_i will be convex valued for all i and therefore F will be convex valued or (ii) H will be convexifying. In either case, $\hat{H}$ will have convex values. Since H (and therefore $\tilde{H}$) is continuous and each $F_i(Q, a_i)$ is upper hemicontinuous (theorem B3), $\hat{H}$ will in addition be upper hemicontinuous (in particular, it has a least and a greatest selection). Now fix Q. Since $F(Q, \cdot)$ is type I and type II monotone and H is increasing, we can use theorem 4 to conclude that $\hat{H}(Q, \cdot)$'s least and greatest selections will be increasing. Note that since F is generally not type I or type II monotone in Q, the previous conclusion refers only to changes in a holding Q fixed.

Step 3: Let $Q_{\min} \equiv \tilde{H}((\delta_{\inf X_i})_{i \in \mathcal{I}})$ and $Q_{\min} \equiv \tilde{H}((\delta_{\sup X_i})_{i \in \mathcal{I}})$, where δ_{x_i} denotes the degenerate measure on X_i with its mass at x_i. It is then clear that $Q \geq Q_{\min}$ for all $Q \in \hat{H}(Q_{\min})$ and $Q \leq Q_{\max}$ for all $Q \in \hat{H}(Q_{\max})$. It follows that, for every $a \in A$, $\hat{H}(\cdot, a) : [Q_{\min}, Q_{\max}] \to 2^{[Q_{\min}, Q_{\max}]}$. That the least and greatest solutions to the fixed-point problem $Q^* \in \hat{H}(Q^*, a)$ are increasing in a now follows from the argument used in the proof of lemma 2 in Acemoglu and Jensen (2013). There it was shown that any correspondence $\hat{H}(\cdot, a) : [Q_{\min}, Q_{\max}] \to 2^{[Q_{\min}, Q_{\max}]}$ that is upper hemicontinuous and convex valued and for each fixed value of $Q \in [Q_{\min}, Q_{\max}]$ has least and greatest selections that are increasing in a will satisfy the conditions of corollary 2 in Milgrom and Roberts (1994). Milgrom and Roberts's result in turn says that the least and greatest fixed points $Q \in \hat{H}(Q, a)$ will be increasing in a. This completes the proof of the theorem in view of the fact that the least and greatest fixed points of $\hat{H}$ are the least and greatest equilibrium aggregates. Q.E.D.

Proof of Lemma 2

The value function of agent i as given by (8) can always be determined by value function iteration. Fix Q and suppress it for notational simplicity. The term v_i equals the pointwise limit of the sequence $(v_i^n)_{n=0}^{\infty}$ determined by

$$v_i^{n+1}(x_i, z_i, \beta) = \sup_{y_i \in \Gamma_i(x_i, z_i)} \left[u_i(x_i, y_i, z_i) + \beta \int v_i^n(y_i, z_i', \beta) P_i(z_i, dz_i') \right], \tag{A1}$$

where v^0 may be picked arbitrarily. Choose $v^0(x_i, z_i, \beta)$ that is increasing and supermodular in x_i and exhibits increasing differences in x_i and β. Since integration preserves supermodularity and increasing differences, $\int v_i^0(y_i, z_i', \beta) P_i(z_i, dz_i')$ will be supermodular in y_i and exhibit increasing differences in y_i and β. It immediately follows from Topkis's theorem on preservation of supermodularity under maximization (Topkis 1998, theorem 2.7.6) that v_i^1 will be supermodular in x_i. By recursion then, $v_i^2, v_i^3, \ldots$ are all supermodular in x_i, and so is the pointwise limit v_i (Topkis 1998, lemma 2.6.1). It is then straightforward to show that when v_i^n is increasing in y_i, u_i is increasing in x_i, and Γ_i is expansive in x_i, v_i^{n+1} will be increasing in x_i; hence the pointwise limit v_i will also be increasing in x_i. Since $\int v_i^0(y_i, z_i', \beta) P_i(z_i, dz_i')$ exhibits increasing differences in y_i and β and is increasing in y_i, $\beta\int v_i^0(y_i, z_i', \beta)P_i(z_i, dz_i')$ will exhibit increasing differences in y_i and β.[27] It follows from Hopenhayn and Prescott (1992, lemma 1) that v_i^1 will exhibit increasing differences in x_i and β, and again this property recursively carries over to the pointwise limit v_i. By Topkis's monotonicity theorem, we conclude that the policy correspondence $G_i(x_i, z_i, Q, \beta)$ will be ascending in β (for fixed x_i, z_i, and Q).

For the second part, simply substitute $\tilde{y}_i = -y_i$ and $\tilde{x}_i = -x_i$ and follow the previous proof using the increasing value function $v^n(\tilde{x}_i, \beta)$ in order to conclude that $y_i = -G_i(-x_i, \beta)$ will be descending in β and hence that a decrease in β is a positive shock. Q.E.D.

Proof of Lemma 3

The conclusion of this lemma follows directly since under homogeneity, the economy can be recast in the transformed strategies $\tilde{x}_{i,t} = x_{i,t}/a_i$ (all i and t), which yields an economy that is independent of $a = (a_i)_{i\in\mathcal{I}}$. Thus, when a is changed, the effect on the individual strategies is given by $x_{i,t} = a_i\tilde{x}_{i,t}$, where $\tilde{x}_{i,t}$ is fixed. It is clear then that an increase in a_i is a positive shock. Q.E.D.

Proof of Theorem 6

The conclusions are trivial consequences of the comparative statics results of Topkis (1978) and the first part of the proof of theorem 5. The reason is that Q can now be treated as an exogenous variable (alongside a) so that we in effect are dealing with just the question of how an individual's set of stationary strategies varies with Q and a. Q.E.D.

Proof of Theorem 7

This proof is essentially identical to the proof of theorem 5. As mentioned after assumption 5, $G_i(x_{i,t}, z_{i,t}, \mu_{z_i})$ will be ascending in μ_{z_i} when stationary distributions are ordered by first-order stochastic dominance (Hopenhayn and Prescott 1992). Therefore, first-order stochastic increases in μ_{z_i} for (a subset of) agents will correspond to "positive shocks" in the same way as increases in exogenous parameters in the proof of theorem 5. Theorem 7 then follows from the same argument that was used to prove theorem 5. Q.E.D.

Proof of Theorem 8

The basic idea is to show that a mean-preserving spread to the distributions of the agents' environment constitutes a "positive shock" in the sense that it leads to an increase in individuals' stationary strategies for any fixed equilibrium aggre-

[27] Let $f(y, \beta)$ exhibit increasing differences and be increasing in y. Then $\beta f(\tilde{y}, \beta) - \beta f(y, \beta)$ is clearly increasing in β for $\tilde{y} \geq y$, showing that $\beta f(y, \beta)$ increasing differences.

gate Q. Specifically, what we show is that the set of stationary strategies will be type I and II monotone in mean-preserving spreads when the set of stationary strategies is equipped with the convex-increasing order. Once again theorem 3 plays a critical role because the spaces we work with have no lattice structure. Once it has been established that mean-preserving spreads are in this sense "positive shocks," the proof follows the proof of theorem 5.

We begin by noting that under assumption 6, the policy correspondence (9) will be single valued; that is, $G_i(x_i, z_i, Q) = \{g_i(x_i, z_i, Q)\}$, where g_i is the (unique) policy function. For a given stationary market aggregate $Q \in \mathcal{Q}$, an agent's optimal strategy is therefore described by the following stochastic difference equation:

$$x_{i,t+1} = g_i(x_{i,t}, z_{i,t}, Q, \mu_{z_i}). \tag{A2}$$

Note that here we have made g_i's dependence on the distribution of $z_{i,t}$ explicit. We already know that g_i will be increasing in x_i and z_i (assumptions 3 and 4). By theorem 8 of Jensen (2012b), g_i will in addition be convex in x_i as well as in z_i under the conditions of the theorem. We now turn to proving that g_i will be $\succeq_{\text{cx}}$-increasing in μ_{z_i} (precisely, this means that $g_i(x_{i,t}, z_{i,t}, Q, \tilde{\mu}_{z_i}) \geq g_i(x_{i,t}, z_{i,t}, Q, \mu_{z_i})$ whenever $\tilde{\mu}_{z_i} \succeq_{\text{cx}} \mu_{z_i}$). From Jensen (2012b; corollary in the proof of theorem 8 applied with $k = 0$), $D_{x_i} v_i(x_i, z_i, Q)$ will (in the sense of agreeing with a function with these properties almost everywhere) be convex in z_i because $D_{x_i} u_i(x_i, y_i, z_i, Q)$ is nondecreasing in y_i and convex in (z_i, y_i) (the latter is true because k_i-convexity is stronger than convexity). This verifies one of the conditions of the following lemma (the other is supermodularity, already used). The lemma is stated in some generality because it is of independent interest (note that Q is suppressed in the lemma's statement).

Lemma 4. Assume that $u_i(x_i, y_i, z_i)$ is supermodular in (x_i, y_i) and denote the value function by $v_i(x_i, z_i, \mu_{z_i})$, where μ_{z_i} is the stationary distribution of z_i. Let x_i be ordered by the usual Euclidean order and μ_{z_i} be ordered by $\succeq_{\text{cx}}$. Then the value function exhibits increasing differences in x_i and μ_{z_i} if for all $\tilde{x}_i \geq x_i$ the following function is convex in z_i (for all fixed μ_{z_i}):

$$v_i(\tilde{x}_i, z_i, \mu_{z_i}) - v_i(x_i, z_i, \mu_{z_i}).$$

If the value function $v_i(x_i, z_i, \mu_{z_i})$ exhibits increasing differences in x_i and μ_{z_i}, then $\int v_i(y_i, z'_i, \mu_{z_i})\mu_{z_i}(dz'_i)$ exhibits increasing differences in y_i and μ_{z_i}. If, in addition, v_i is supermodular in y_i, the policy function $g_i(x_i, z_i, \mu_{z_i})$ is increasing in μ_{z_i}.

Proof. Let v_i^n denote the nth iterate of the value function and consider the $n+1$th iterate

$$v_i^{n+1}(x, z, \mu_{z_i}) = \sup_{y \in \Gamma_i(x,z)} \left\{ u_i(x, y, z) + \beta \int v_i^n(y, z', \mu_{z_i})\mu_{z_i}(dz') \right\}.$$

Assume by induction that v_i^n exhibits increasing differences in (y, μ_{z_i}) and that the hypothesis of the theorem holds for v_i^n. When $\tilde{y} \geq y$ and $\mu_{z_i} \succeq_{\text{cx}} \mu'_{z_i}$, we then have

$$\int v_i^n(\tilde{y}, z', \mu_{z_i}) - v_i^n(y, z', \mu_{z_i})\mu_{z_i}(dz')$$
$$\geq \int v_i^n(\tilde{y}, z', \mu_{z_i}) - v_i^n(y, z', \mu_{z_i})\mu'_z(dz')$$
$$\geq \int v_i^n(\tilde{y}, z', \mu'_{z_i}) - v_i^n(y, z', \mu'_{z_i})\mu'_z(dz').$$

Here the first inequality follows from the definition of the convex order, and the second inequality follows from increasing differences of v_i^n in (y, μ_{z_i}). Note that this evaluation implies the second conclusion of the lemma once the first has been established. Since $u_i(x, y, z) + \beta\int v_i^n(y, z', \mu_{z_i})\mu_{z_i}(dz')$ is supermodular in (x, y) by assumption and trivially exhibits increasing differences in (x, μ_{z_i}), it follows from the preservation of increasing differences under maximization that $v^{n+1}(x, z, \mu_{z_i})$ exhibits increasing differences in (x, μ_{z_i}). The first conclusion of the lemma now follows from a standard argument (increasing differences is a property that is pointwise closed and the value function is the pointwise limit of the sequence v^n, $n = 0, 1, 2, \ldots$). Q.E.D.

To prove theorem 8, we begin with some notation. For a set Z, let $\mathcal{P}(Z)$ denote the set of probability distributions on Z with the Borel algebra. A distribution $\lambda \in \mathcal{P}(Z)$ is greater than another probability distribution $\tilde{\lambda} \in \mathcal{P}(Z)$ in the monotone convex order (written $\lambda \succeq_{\text{cxi}} \tilde{\lambda}$) if $\int_Z f(\tau)\lambda(d\tau) \geq \int_Z f(\tau)\tilde{\lambda}(d\tau)$ for all convex and increasing functions $f: Z \to \mathbb{R}$ for which the integrals exist (Shaked and Shanthikumar 2007, chap. 4.A). The stochastic difference equation (A2) gives rise to a transition function $P_{Q,\mu_{z_i}}$ in the usual way (here $x_i \in X_i$ and A_i is a Borel subset of X_i):

$$P_{Q,\mu_{z_i}}(x_i, A) \equiv \mu_{z_i}(\{z_i \in Z_i : g_i(x_i, z_i, Q, \mu_{z_i}) \in A\}). \tag{A3}$$

This in turn determines the adjoint Markov operator:

$$T^*_{Q,\mu_{z_i}}\mu_{x_i} = \int P_{Q,\mu_{z_i}}(x_i, \cdot)\mu_{x_i}(dx_i); \tag{A4}$$

$\mu^*_{x_i}$ is an invariant distribution for (A2) if and only if it is a fixed point for $T^*_{Q,\mu_{z_i}}$, that is, $\mu^*_{x_i} = T^*_{Q,\mu_{z_i}}\mu^*_{x_i}$. We are first going to use the fact that g_i is convex and increasing in x_i to show that $T^*_{Q,\mu_{z_i}}$ will be a $\succeq_{\text{cxi}}$-monotone operator. In other words, we will show that $\tilde{\mu}_{x_i} \succeq_{\text{cxi}} \mu_{x_i} \Rightarrow T^*_{Q,\mu_{z_i}}\tilde{\mu}_{x_i} \succeq_{\text{cxi}} T^*_{Q,\mu_{z_i}}\mu_{x_i}$. The statement that $T^*_{Q,\mu_{z_i}}\tilde{\mu}_{x_i} \succeq_{\text{cxi}} T^*_{Q,\mu_{z_i}}\mu_{x_i}$ by definition means that for all convex and increasing functions $f: Z \to \mathbb{R}$,

$$\int f(\tau)T^*_{Q,\mu_{z_i}}\tilde{\mu}_{x_i}(d\tau) \geq \int f(\tau)T^*_{Q,\mu_{z_i}}\mu_{x_i}(d\tau).$$

But since this is equivalent to

$$\int_{Z_i}\left[\int_{X_i} f(g_i(x_i, z_i, Q, \mu_{z_i}))\tilde{\mu}_{x_i}(dx_i)\right]\mu_{z_i}(dz_i)$$
$$\geq \int_{Z_i}\left[\int_{X_i} f(g_i(x_i, z_i, Q, \mu_{z_i}))\mu_{x_i}(dx_i)\right]\mu_{z_i}(dz_i),$$

we immediately see that this inequality will hold whenever $\tilde{\mu}_{x_i} \succeq_{\text{cxi}} \mu_{x_i}$ (the composition of two convex and increasing functions is convex and increasing). This proves that $T^*_{Q,\mu_{z_i}}$ is a $\succeq_{\text{cxi}}$-monotone operator.

Our next objective is to prove that $\tilde{\mu}_{z_i} \succeq_{\text{cx}} \mu_{z_i} \Rightarrow T^*_{Q,\tilde{\mu}_{z_i}}\mu_{x_i} \succeq_{\text{cxi}} T^*_{Q,\mu_{z_i}}\mu_{x_i}$ for all $\mu_{x_i} \in \mathcal{P}(X_i)$. As above, we can rewrite the statement that $T^*_{Q,\tilde{\mu}_{z_i}}\mu_{x_i} \succeq_{\text{cxi}} T^*_{Q,\mu_{z_i}}\mu_{x_i}$:

$$\begin{aligned}&\int_{Z_i}\left[\int_{X_i} f(g_i(x_i, z_i, Q, \tilde{\mu}_{z_i}))\mu_{x_i}(dx_i)\right]\tilde{\mu}_{z_i}(dz_i)\\ &\geq \int_{Z_i}\left[\int_{X_i} f(g_i(x_i, z_i, Q, \mu_{z_i}))\mu_{x_i}(dx_i)\right]\mu_{z_i}(dz_i).\end{aligned} \tag{A5}$$

Since f is increasing and g_i is $\succeq_{\text{cx}}$-increasing in μ_{z_i}, it is obvious that for all $z_i \in Z_i$,

$$\int_{X_i} f(g_i(x_i, z_i, Q, \tilde{\mu}_{z_i}))\mu_{x_i}(dx_i) \geq \int_{X_i} f(g_i(x_i, z_i, Q, \mu_{z_i}))\mu_{x_i}(dx_i).$$

Hence

$$\begin{aligned}&\int_{Z_i}\left[\int_{X_i} f(g_i(x_i, z_i, Q, \tilde{\mu}_{z_i}))\mu_{x_i}(dx_i)\right]\tilde{\mu}_{z_i}(dz_i)\\ &\geq \int_{Z_i}\left[\int_{X_i} f(g_i(x_i, z_i, Q, \mu_{z_i}))\mu_{x_i}(dx_i)\right]\tilde{\mu}_{z_i}(dz_i).\end{aligned} \tag{A6}$$

But we also have[28]

$$\begin{aligned}&\int_{Z_i}\left[\int_{X_i} f(g_i(x_i, z_i, Q, \mu_{z_i}))\mu_{x_i}(dx_i)\right]\tilde{\mu}_{z_i}(dz_i)\\ &\geq \int_{Z_i}\left[\int_{X_i} f(g_i(x_i, z_i, Q, \mu_{z_i}))\mu_{x_i}(dx_i)\right]\mu_{z_i}(dz_i).\end{aligned} \tag{A7}$$

Combining (A6) and (A7), we get (A5) under the condition that $\tilde{\mu}_{z_i} \succeq_{\text{cx}} \mu_{z_i}$ as desired.

We are now ready to use theorem 3 to conclude that

$$F_i(Q, \mu_{z_i}) \equiv \{\mu_{x_i} \in \mathcal{P}(X_i) : \mu_{x_i} = T^*_{Q,\mu_{z_i}}\mu_{x_i}\}$$

[28] To verify (A7), reverse the order of integration and use the convexity of $f(g_i(x_i, \cdot, Q, \tilde{\mu}_{z_i}))$ and the definition of $\succeq_{\text{cx}}$.

will be type I and type II monotone in μ_{z_i} when $\mathcal{P}(Z_i)$ is equipped with the order $\succeq_{cx}$ and $\mathcal{P}(X_i)$ is equipped with $\succeq_{cxi}$.[29] Note that in the language of theorem 3, $F = \{T^*_{Q,\mu_{z_i}}\}$ and t corresponds to μ_{z_i}.

The rest of the proof proceeds as in the proof of theorem 5 with $(\mu_{z_i})_{i\in\mathcal{I}}$ replacing the exogenous variables $(a_i)_{i\in\mathcal{I}}$ in that proof. In particular, let $F(Q, \mu_z) = F_i(Q, \mu_{z_i})_{i\in\mathcal{I}}$, where $\mu_z = (\mu_{z_i})_{i\in\mathcal{I}}$, and consider precisely as in step 2 of the proof of theorem 5 (see that proof for the definition of $\tilde{H}$):

$$\hat{H}(Q, a) = \{\tilde{H}(x) \in \mathbb{R} : x \in F(Q, \mu_z) \text{ for all } i\}.$$

This establishes that a mean-preserving spread to (any subset of) the agents leads to an increase in the least and greatest equilibrium aggregates. Q.E.D.

Proof of Theorem 9

As in step 1 of the proof of theorem 5, let $F_i(Q^1, \ldots, Q^M, a_i)$ denote the set of stationary strategies for agent i given stationary sequences of the M aggregates and the exogenous parameters a_i. Again, F_i is non–empty valued, jointly upper hemicontinuous, and type I and II monotone in a_i. Now fix a group $m \in \{1, \ldots, M\}$, set $a^m \equiv (a_i)_{i\in\mathcal{I}^m}$, and follow step 2 in the proof of theorem 5 in order to define

$$\hat{H}^m(Q^1, \ldots, Q^M, a^m) = \{\tilde{H}^m(x^m) \in \mathbb{R} : x^m \in F^m(Q^1, \ldots, Q^M, a^m)\}.$$

Then $(Q^1, \ldots, Q^m)$ is a vector of equilibrium aggregates if and only if $Q^m \in \hat{H}^m(Q^1, \ldots, Q^M, a_m)$ for all $m = 1, \ldots, M$.

Then the steps of the proof from theorem 5 apply identically and imply that each $\hat{H}^m : [Q^m_{\min}, Q^m_{\max}] \rightarrow 2^{[Q^m_{\min}, Q^m_{\max}]}$ is convex valued, upper hemicontinuous in $(Q^1, \ldots, Q^M)$, and ascending in a^m (here $Q^m_{\min}$ and $Q^m_{\max}$ are defined by using $\hat{H}^m$ instead of $\hat{H}$ in step 3 in the proof of theorem 5).

Next, for the first statement in theorem 9, replace the one-dimensional version of the result in Milgrom and Roberts (1994) with their theorem 4. Note that this theorem in particular tells us that least and greatest equilibrium aggregates exist.

For the second statement in theorem 9, we can then apply corollary 3 in Milgrom and Roberts (1994) (here the statement concerns the mth equilibrium aggregate, and thus smallest and largest aggregates exist by the same argument).

Finally, note that the results from Milgrom and Roberts (1994) applied here are, in fact, formulated for the case in which each $\hat{H}^m$ is single valued. However, the proofs immediately extend to the set-valued case. Q.E.D.

Appendix B

Dynamic Programming with Transition Correspondences

Consider a standard recursive stochastic programming problem with functional equation

[29] The order $\succeq_{cxi}$ is a closed order on $\mathcal{P}(X_i)$.

$$v(x, z) = \sup_{y\in\Gamma(x,z)} \left[u(y, x, z) + \beta \int v(y, z')P(dz', z) \right]. \tag{B1}$$

As is well known, (B1) has a unique solution $v^* : X \times Z \to \mathbb{R}$ (and this will be a continuous function) when (B1) satisfies assumption 1 (Stokey and Lucas 1989). From v^*, the *policy correspondence* $G : X \times Z \to 2^X$ is then defined by

$$G(x, z) = \arg \sup_{y\in\Gamma(x,z)} u(y, x, z) + \beta \int v^*(y, z')P(dz', z). \tag{B2}$$

Clearly, G will be upper hemicontinuous under the above assumptions. A *policy function* is a measurable selection from G, that is, a measurable function $g : X \times Z \to X$ such that $g(x, z) \in G(x, z)$ in $X \times Z$. Throughout it is understood that $X \times Z$ is equipped with the product σ-algebra, $\mathcal{B}(X) \otimes \mathcal{B}(Z)$. Recall that a correspondence such as G is (upper) measurable if the inverse image of every open set is measurable, that is, if

$$G^{-1}(O) \equiv \{(x, z) \in X \times Z : G(x, z) \cap O \neq \varnothing\} \in \mathcal{B}(X) \otimes \mathcal{B}(Z),$$

whenever $O \subseteq X$ is open. An upper hemicontinuous correspondence is measurable (Aubin and Frankowska 1990, proposition 8.2.1).[30] Since a measurable correspondence has a measurable selection (their theorem 8.1.3), any upper hemicontinuous policy correspondence admits a policy function g. Let $\mathcal{G}$ denote the set of measurable selections from G, which was just shown to be nonempty.

Given a policy function $g \in \mathcal{G}$, an $x \in X$, and a measurable set $A \times B \in \mathcal{B}(X \times Z)$, let

$$P_g((x, z), A \times B) \equiv P(z, B)\chi_A(g(x, z)). \tag{B3}$$

For fixed $(x, z) \in X \times Z$, $P_g((x, z), \cdot)$ is a measure, and for fixed $A \times B \in \mathcal{B}(X \times Z)$, $P_g(\cdot, A)$ is measurable (the last statement is a consequence of Fubini's theorem). So P_g is a transition function.

The set of measurable policy functions $\mathcal{G}$ then gives rise to the *transition correspondence*:

$$P((x, z), \cdot) = \{P_g((x, z), \cdot) : g \in \mathcal{G}\}.$$

Intuitively, given a state (x_t, z_t) at date t, there is a set of probability measures $P((x_t, z_t), \cdot)$ each of which may describe the probability of being in a set $A \times B \in \mathcal{B}(X \times Z)$ at $t + 1$.

LEMMA 1 (The transition correspondence is upper hemicontinuous). Consider a sequence $(y_n)_{n=0}^{\infty}$ in $X \times Z$ that converges to a limit point $y \in X \times Z$. Let $P_{g_n}(y_n, \cdot) \in P(y_n, \cdot)$ be an associated sequence of transition functions from the transition correspondence P. Then for any weakly convergent subsequence $P_{g_{n_m}}(y_{n_m}, \cdot)$, there exists a $P_g(y, \cdot) \in P(y, \cdot)$ such that $P_{g_{n_m}}(y_{n_m}, \cdot) \to {}_w P_g(y, \cdot)$.

[30] Specifically, this is true when $X \times Z$ is a metric space with the Borel algebra and a complete σ-finite measure (see Aubin and Frankowska [1990] for details and a proof).

The proof is omitted.

REMARK 3. Since an upper hemicontinuous correspondence is measurable, we get what Blume (1982) calls a multivalued stochastic kernel $K : X \times Z \to 2^{\mathcal{P}(X\times Z)}$ by taking $P((x, z), \cdot) = K(x, z)$ for all $(x, z) \in X \times Z$.

Given $g \in \mathcal{G}$, define the adjoint Markov operator in the usual way from the transition function P_g:

$$T_g^*\lambda(A \times B) = \int Q(z, B)\chi_A(g(x, z))\lambda(d(x, z)). \tag{B4}$$

Next define the *adjoint Markov correspondence*:

$$T^*\lambda = \{T_g^*\lambda\}_{g\in\mathcal{G}}. \tag{B5}$$

To clarify, T^* maps a probability measure $\lambda \in \mathcal{P}(X \times Z)$ into a set of probability measures, namely, the set $\{T_g^*\lambda : g \in \mathcal{G}\} \subseteq \mathcal{P}(X \times Z)$. A probability measure λ^* is *invariant* if

$$\lambda^* \in T^*\lambda^*. \tag{B6}$$

Of course, this is the same as saying that there exists $g \in \mathcal{G}$ such that $\lambda^* = T_g^*\lambda^*$.

LEMMA B2 (The adjoint Markov correspondence is upper hemicontinuous). Let $\lambda_n \to_w \lambda$, and consider a sequence (μ_n) with $\mu_n \in T^*\lambda_n$. Then for any convergent subsequence $\mu^{n_m} \to_w \mu$, it holds that $\mu \in T^*\lambda$.

Proof. This is a direct consequence of proposition 2.3 in Blume (1982) (see remark 3). Q.E.D.

One way to prove existence of an invariant distribution with transition correspondences is based on convexity, upper hemicontinuity, and the Kakutani-Glicksberg-Fan theorem (Blume 1982). Alternatively, one can look at suitable increasing selections and prove existence along the lines of Hopenhayn and Prescott (1992), who study monotone Markov processes and use the Knaster-Tarski theorem (for an early study on monotone Markov processes, see Kalmykov [1962]). However, for this paper's focus, we need a set-valued existence result that integrates with the results of Section III. In particular, the results in Section III do not require monotonicity in the noise process z_t. Mathematically, this can be accomplished by using the disintegration theorem and the set-valued fixed-point theorem of Smithson (1971), and this is what we will do below.

We begin by proving a new result stating that if the policy correspondence $G(x, z)$ has an increasing and measurable greatest (respectively, least) selection in x (for fixed z), then the adjoint Markov correspondence will be type I (respectively, type II) monotone in the following order:[31]

$$\lambda \succeq_{X-\text{FOD}} \tilde{\lambda} \Leftrightarrow [\lambda(\cdot, B) \succeq_{\text{FOD}} \tilde{\lambda}(\cdot, B) \text{ for all } B \in \mathcal{B}(Z)].$$

Note that if $\lambda \succeq_{X-\text{FOD}} \tilde{\lambda}$, then $\lambda_x = \lambda(\cdot, Z) \succeq_{\text{FOD}} \tilde{\lambda}(\cdot, Z) = \tilde{\lambda}_x$; that is, x's marginal distribution given λ first-order dominates the marginal distribution given $\tilde{\lambda}$. For any invariant distribution μ_z for z_t, define $\Omega(\mu_z) = \{\lambda \in \mathcal{P}(X \times Z) : \lambda(X, \cdot) =$

[31] This is clearly reflexive and transitive since so is $\succeq_{\text{FOD}}$. It is also antisymmetric since if $\lambda(\cdot, B) \succeq_{\text{FOD}} \tilde{\lambda}(\cdot, B)$ and $\lambda(\cdot, B) \preceq_{\text{FOD}} \tilde{\lambda}(\cdot, B)$ for all $B \in \mathcal{B}(Z)$, then $\lambda(\cdot, B) = \tilde{\lambda}(\cdot, B)$ in distribution for all $B \in \mathcal{B}(Z)$.

$\mu_z(\cdot)\}$. Clearly, $T^*:\Omega(\mu_z)\to 2^{\Omega(\mu_z)}$, and if $\lambda, \tilde{\lambda}\in\Omega(\mu_z)$ and $\lambda \succeq_{X-\mathrm{FOD}} \tilde{\lambda}$, then $\lambda(\cdot \mid z) \succeq_{\mathrm{FOD}} \tilde{\lambda}(\cdot \mid z)$ for almost every (a.e.) $z\in Z$, where $(\lambda(\cdot|z))_{z\in Z}$ is the disintegrated family of measures with respect to Z.[32]

THEOREM B1. Assume that the policy correspondence $G: X\times\{z\}\to 2^X$ has an increasing greatest (least) selection for each fixed $z\in Z$. Then the adjoint Markov correspondence T^* is type I (type II) monotone on any subset $\Omega(\mu_z)\subseteq \mathcal{P}(X\times Z)$ with respect to the order $\succeq_{X-\mathrm{FOD}}$. If G depends on an exogenous variable $a\in A$ so that $G: X\times\{z\}\times A\to 2^X$ and the greatest (least) selection from G is increasing in a, then T_a^* will in addition be type I (type II) monotone in any subset $\Omega(\mu_z)\subseteq\mathcal{P}(X\times Z)$ with respect to the order $\succeq_{X-\mathrm{FOD}}$.

Proof. We prove the greatest/type I case only (the second case is similar). Consider probability measures $\mu_2 \succeq_{X-\mathrm{FOD}} \mu_1$. We wish to show that for any $\lambda_1\in T^*\mu_1$, there exists $\lambda_2\in T^*\mu_2$ such that $\lambda_2 \succeq_{X-\mathrm{FOD}} \lambda_1$. The term $\lambda_1\in T^*\mu_1$ if and only if there exists a measurable selection $g_1\in\mathcal{G}$ such that, for all $A\times B\in\mathcal{B}(X\times Z)$,

$$\lambda_1(A, B) = \int_{X\times Z} Q(z, B)\chi_A(g_1(x, z))\mu_1(d(x, z));$$

similarly for $\lambda_2\in T^*\mu_2$, where we denote the (not yet determined) measurable selection by $g_2\in\mathcal{G}$. Given these measurable selections, we have $\lambda_2 \succeq_{X-\mathrm{FOD}} \lambda_1$ if and only if for every increasing function f, the following holds for all $B\in\mathcal{B}(Z)$:

[32] For a measure λ on $X\times Z$, let $\pi: X\times Z\to Z$ denote the natural projection onto Z (so the fibers are given by $\pi^{-1}(z) = X\times\{z\}$). By the disintegration theorem, there exists a Borel family of probability measures $(\lambda(\cdot|z))_{z\in Z}$ such that for any (integrable) function,

$$\int_{X\times Z} f(x, z)\lambda(dx, dz) = \int_Z\left[\int_X f(x, z)\lambda(dx \mid z)\right]\lambda(\pi^{-1}(dz)),$$

where $\lambda(\pi^{-1}(B)) = \lambda(X\times B) = \mu_z(B)$ (the marginal measure on Z). This family of measures is referred to as the Z disintegrated family of measures. Since

$$\int_X g(x)\lambda(dx, B) = \int_{X\times Z} g(x)\chi_B(z)\lambda(dx, dz),$$

we get that $\tilde{\lambda}(\cdot, B)\succeq_{\mathrm{FOD}}\lambda(\cdot, B)$ if and only if for any increasing function $g: X\to\mathbb{R}$ it holds that

$$\int_B\left[\int_X g(x)\tilde{\lambda}(dx \mid z)\right]\mu(dz) \geq \int_B\left[\int_X g(x)\lambda(dx \mid z)\right]\mu(dz).$$

Hence if $\tilde{\lambda}(\cdot, B)\succeq_{\mathrm{FOD}}\lambda(\cdot, B)$ for all $B\in\mathcal{B}(Z)$, it must hold that for all g,

$$\int_X g(x)\tilde{\lambda}(dx \mid z) \geq \int_X g(x)\lambda(dx \mid z)$$

for μ-a.e. $z\in Z$ (if this were not the case, we could choose g and $B\in\mathcal{B}(Z)$ with

$$\int_B\left[\int_X g(x)\tilde{\lambda}(dx \mid z)\right]\mu(dz) < \int_B\left[\int_X g(x)\lambda(dx \mid z)\right]\mu(dz),$$

contradicting that $\tilde{\lambda}(\cdot, B)\succeq_{\mathrm{FOD}}\lambda(\cdot, B)$). Thus if $\tilde{\lambda}\succeq_{X-\mathrm{FOD}}\lambda$ with $\lambda, \tilde{\lambda}\in\Omega(\mu_z)$, $\tilde{\lambda}(\cdot \mid z)\succeq\lambda(\cdot \mid z)$ for a.e. z.

$$\int_X f(x)\lambda_2(dx, B) \geq \int_X f(x)\lambda_1(dx, B) \Leftrightarrow$$
$$\int_{X\times Z} Q(z, B)f \circ g_2(x, z)\mu_2(d(x, z)) \geq \int_{X\times Z} Q(z, B)f \circ g_1(x, z)\mu_1(d(x, z)). \tag{B7}$$

But taking g_2 to be the greatest selection from G (which is measurable), it is clear that for all $B \in \mathcal{B}(Z)$,

$$\int_{X\times Z} Q(z, B)f \circ g_2(x, z)\mu_1(d(x, z)) \geq \int_{X\times Z} Q(z, B)f \circ g_1(x, z)\mu_1(d(x, z)). \tag{B8}$$

In addition, since g_2 is increasing in x, the function $x \mapsto Q(z, B)f \circ g_2(x, z)$ is increasing in x. Since $\mu_2 \succeq_{X-\text{FOD}} \mu_1$ and $\mu_1, \mu_2 \in \Omega(\mu_z)$, $\mu_2(\cdot \mid z) \succeq_{\text{FOD}} \mu_1(\cdot \mid z)$ for a.e. $z \in Z$, and so for any $B \in \mathcal{B}(Z)$,

$$\begin{aligned}
&\int_{X\times Z} Q(z, B)f \circ g_2(x, z)\mu_2(d(x, z)) \\
&\quad = \int_Z \left[\int_X Q(z, B)f \circ g_2(x, z)\mu_2(dx \mid z)\right]\mu_z(dz) \\
&\quad \geq \int_Z \left[\int_X Q(z, B)f \circ g_2(x, z)\mu_1(dx \mid z)\right]\mu_z(dz) \\
&\quad = \int_{X\times Z} Q(z, B)f \circ g_2(x, z)\mu_1(d(x, z)).
\end{aligned}$$

Now simply combine the previous inequality with (B8) to get (B7). Thus we have proved that if G has an increasing maximal selection, T^* will be type I monotone. The statements concerning the variable $a \in A$ are proved by the same argument and are omitted. Q.E.D.

By a straightforward modification of proposition 2 in Hopenhayn and Prescott (1992), it can be shown that G will have least and greatest selections that are increasing in x under this paper's main conditions.

THEOREM B2. Let u and Γ satisfy assumptions 1 and 3. Then the policy correspondence $G : X \times Z \to 2^X$ will, for each fixed $z \in Z$, be ascending in x; in particular, it will have least and greatest selections, and these will be increasing in x.

Proof. Fix $z \in Z$. By iteration on the value function, we can use theorem 2.7.6 in Topkis (1998) to conclude that $v^*(x, z)$ will be supermodular in x when $u(x, y, z)$ is supermodular in (x, y) and the graph of $\Gamma(\cdot, z)$ is a sublattice of $X \times X$. Since supermodularity is preserved by integration, $\int v^*(y, z')P(dz', z)$ is supermodular in y. Considering that

$$G(x, z) = \arg \sup_{y\in\Gamma(x,z)} u(y, x, z) + \beta \int v^*(y, z')P(dz', z),$$

the statement of the theorem now follows directly from Topkis's theorem. Q.E.D.

We now get the following existence result. Note that unless T^* is also convex valued (which is not assumed here), the set of invariant distributions will generally not be convex.

Theorem B3 (Existence in the type I/II monotone case). Assume that the adjoint Markov correspondence is either type I or type II order preserving. In addition, assume that the state space (strategy set) has an infimum. Then T^* has a fixed point (there exists an invariant measure). In addition, the fixed-point correspondence will be upper hemicontinuous if T^* is upper hemicontinuous in (μ, θ), where θ is a parameter.

Proof. The idea is to apply Smithson's theorem to the mapping T^* on $(\Omega(\mu_z), \succeq_{\mathrm{FOD}-X})$, where μ_z is a fixed invariant measure for z_t. By proposition 1 in Hopenhayn and Prescott (1992), $(\Omega(\mu_z), \succeq_{\mathrm{FOD}-X})$ is chain complete. As shown in the proof of theorem 3, the condition on the supremum of chains in Smithson's theorem follows directly from upper hemicontinuity of T^*. It remains, therefore, only to establish the existence of some $\mu \in \Omega(\mu_z)$ such that there exists a $\lambda \in T^*\mu$ with $\mu \preceq_{\mathrm{FOD}-X} \lambda$. To this end, we follow Hopenhayn and Prescott's (1992) proof of corollary 2 and pick a measure δ_a from $\mathcal{P}(X)$ that places probability one on the infimum $\{a\} \equiv \inf X \in X$. Then $\lambda \succeq_{\mathrm{FOD}-X} \delta_a \times \mu_z$ (where $\times$ denotes product measure) for all $\lambda \in \Omega(\mu_z)$. It is then clear that if we take $\mu = \delta_a \times \mu_z$, we have $\lambda \succeq_{\mathrm{FOD}-X} \mu$ for (in fact, every) $\lambda \in T^*\mu$. We conclude that T^* has a fixed point. The upper hemicontinuity claim is trivial under the stated assumptions. Q.E.D.

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