SYMPLECTIC SINGULARITIES, PERIODIC ORBITS OF THE BILLIARD BALL MAP, AND THE OBSTACLE PROBLEM

by

ALAN WILLIAM MAGNUSON

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Signature of Author Signature redacted
Department of Mathematics

Certified by Signature redacted
Richard B. Melrose
Thesis Supervisor

Accepted by Signature redacted
Nesmith C. Ankeny
Chairman, Departmental Graduate Committee

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ABSTRACT

In Part I, given $\Omega \subset R^{n+1}$, a strictly convex, smoothly bounded region, it is shown here that a non-degenerate closed geodesic $\mathcal{O}$ in the boundary $\partial\Omega$ induces a sequence of closed rays in Ω converging uniformly to $\mathcal{O}$.

In Part II, the normal forms for nested hypersurfaces in a symplectic manifold are used to analyze the obstacle bypassing problem around a biasymptotic ray.

Thesis Supervisor: Dr. Richard B. Melrose

Title: Professor of Mathematics

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INTRODUCTION

We consider two problems in the geometry of Riemannian manifolds with boundary.

1. Let Ω be a smoothly bounded strictly convex region in R^{m+1}, its boundary. A reflected ray in Ω is a path consisting of connected line segments inscribed in $\partial\Omega$ and satisfying Snell's reflection law there.

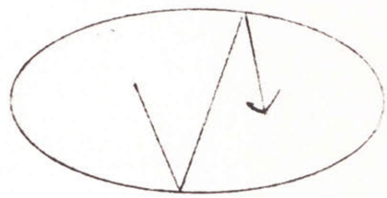


Figure I.1

We are interested in the existence of closed rays in Ω. Poincaré proved the existence of a closed ray with n vertices for each $n > 1$. Using minimax methods, Birkhoff [B1] extended the argument of Poincaré to establish the existence of a second closed ray with n vertices for each integer n, when $\Omega \subset R^2$.

Melrose [M1] has observed that this problem may be analyzed by consideration of a pair of transversally intersecting hypersurfaces in the symplectic manifold T^*R^{m+1}, the cotangent bundle over R^{m+1}. These hypersurfaces are the space of covectors of unit length, S^*R^{m+1}, and the hypersurface, $T^*_{\partial\Omega}R^{m+1}$ formed by the restriction of T^*R^{m+1} to the boundary $\partial\Omega$. This observation transfers a significant part of the analysis of reflected rays to the general problem of studying transversally intersecting hypersurfaces in a symplectic manifold.

Recall that a symplectic manifold is an even dimensional manifold X together with a closed non-degenerate 2-form ω. By Darboux's theorem, there always exist local coordinates (p_i, q_i) (called Darboux coordinates) such that ω has the normal form:

$$\omega = \sum_{i=0}^{m} dp_i \wedge dq_i \qquad \dim X = 2m + 2$$

In other words, the only local invariant of a symplectic manifold is its dimension.

Let $F \to X$ be a hypersurface in X. As a corollary to the proof of Darboux's theorem, any hypersurface in X may be brought to normal form. There will exist local Darboux coordinates (p_i, q_i) such that

$$F = \{q_0 = 0\}$$

The restriction of ω to F, $i^*\omega$, has rank 2m - 2, so its kernel defines a characteristic direction at each point of F. The integral curves of this distribution are called the bicharacteristics of F. Let $G \to X$ be another hypersurface in X - transversal to F. Let J be the manifold $F \cap G$, and let $x \in J$. The obvious local invariants at x are the tangency of the bicharacteristic of F through x, ℓx, to the hypersurface G, and the tangency of Kx, the corresponding G bicharacteristic to F. If ℓx is transversal to G, then Kx is transversal to F and there are no additional invariants. As a corollary of the proof of Darboux's theorem, one may always introduce Darboux coordinates (q_0, p_0) in a neighborhood of x with

$$F = \{q_0 = 0\}$$

$$G = \{p_0 = 0\}$$

If ℓx is tangent to G, then one can show (Section 2) that Kx is tangent to F. Such a point is called a glancing point of (F, G). If ℓx is tangent to G to first order and Kx is tangent to F to first order, x is called a non-degenerate glancing point. Melrose [M1] has shown that in a neighborhood of such a point, one may introduce Darboux coordinates (q_i, p_i) with

$$F = \{q_0 = 0\}$$

$$G = \{1/2\, p_0^2 - q_0 - p_1 = 0\}$$

In Section 2, it will be shown that reflected rays in Ω may be described by a symplectic map $B : B^*\partial\Omega \to B^*\partial\Omega$,

$$B^*\partial\Omega = \{(x,\xi) \in T^*\partial\Omega : |\xi| \leq 1\}$$

called the billiard ball map. If Ω is strictly convex, the glancing points of the pair $(S^*R^3, T^*_{\partial\Omega}R^3)$ will be non-degenerate. The normal form for non-degenerate glancing points will imply that B may be locally interpolated by the flow of a Hamiltonian vector field. Through analysis of this flow, it will be proved here that for each closed geodesic, $\mathcal{O} \subset \partial\Omega$, in general position, there exists an infinite family of closed rays with vertices lying in a neighborhood of $\mathcal{O}$. Indeed, it will be shown that the closed rays in this family approximate $\mathcal{O}$ uniformly as the number of vertices tends to infinity.

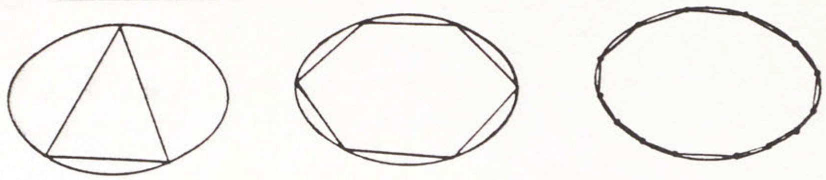


Figure I.2

2. Consider the obstacle bypassing problem for a smoothly bounded compact region $\Omega \subset R^3$. The objects of interest are the paths of minimal length that bypass the obstacle Ω. These paths will be made up of geodesic segments on the obstacle surface $\partial\Omega$ connected by line segments in R^3 tangent to $\partial\Omega$.

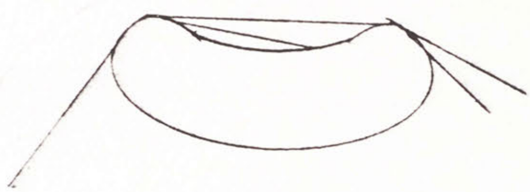


Figure I.3

As in the case of the closed ray problem, this problem may also be analyzed in terms of the transversal hypersurfaces S^*R^3 and $T^*_{\partial\Omega}R^3$. In the obstacle bypassing problem, however, this approach leads to difficulties due to the fact that Ω is no longer assumed to be strictly convex. Because of this the glancing points of $(S^*R^3, T^*_{\partial\Omega}R^3)$ will not necessarily be non-degenerate.

Melrose [M2] has shown that hypersurfaces (F, G) in a symplectic manifold will have formal moduli at degenerate glancing points.

There will be many normal forms for (F,G). However, Arnol'd has observed [A1] that much of the analysis of the obstacle bypassing problem may be described in terms of the nested hypersurfaces $(S^*R^3, S^*_{\partial\Omega}R^3)$.

For a general symplectic manifold X, classification of the nested pair (F, J) represents a first step in any classification of normal forms for (F, G). For the nested pair, (F, J), the obvious local invariant at $x \in J$ is the tangency of the F bicharacteristic, ℓx, to J. Around points where ℓx is transversal or simply tangent to J, there will be Darboux coordinates (p_i, q_i) such that

Transversal case

$$F = \{q_0 = 0\}$$

$$J = \{q_0 = p_0 = 0\}$$

Simply tangent case

$$F = \{q_0 = 0\}$$

$$J = \{1/2\ p_0^2 - p_1 = q_0 = 0\}$$

When ℓx is tangent to J to second or third order, then provided J is in general position, there will exist Darboux coordinates, (p_i, q_i) such that

$$F = \{q_0 = 0\}$$

Second order contact (inflectional point)

$$J = \{q_0 = 0;\ p_0^3 + p_0\, p_1 + q_1 = 0\}$$

Third order contact

$$J = \{q_0 = 0;\ \ p_0^4 + p_0^2\, p_2 + p_0\, p_1 + q_1 = 0\}$$

The normal form for second order contact was established by Melrose in [M2] while the third order normal form was established in the category of formal power series by Arnol'd in [A1] and extended to the category of smooth diffeomorphisms by Melrose and the author in [MMag].

These equivalence theorems permit introduction of coordinate systems that simplify the study of the obstacle bypassing problem. In Part 2, they will be used to discuss the behavior of length minimizing paths near a line of third order contact to $\partial\Omega$.

PART I

EXISTENCE OF CLOSED RAYS

In Part I, we show that reflected rays in Ω may be described by the billiard map B: $B^*\partial\Omega \to B^*\partial\Omega$ which will be a symplectomorphism of the co-ball bundle over $\partial\Omega$. As a direct consequence of the classification theorems, described in the introduction, it is shown that B may be locally interpolated by a hamiltonian flow. Analysis of this flow, in conjunction with an extremal condition of Poincaré will demonstrate the existence of periodic points of B which will correspond to closed rays of Ω.

Section 1: Poincaré's Proof of the Existence of Closed Rays

Let Ω be a strictly convex, smoothly bounded region of R^{m+1} with boundary $\partial\Omega$. Let P_n be the space of n-polygons inscribed in the region Ω. Note that this space contains all closed rays with n vertices. In $(\partial\Omega)^n = \partial\Omega x \ldots x\partial\Omega$, let R_n be the open dense set of ordered n-tuples $(x_0, \ldots, x_{n-1})$ such that $x_0 \neq x_{n-1}$; $x_i \neq x_{i+1}$ $(i = 0, \ldots, n-2)$. Then P_n is the image of R_n under the map sending a point in $(\partial\Omega)^n$ to an inscribed polygon in Ω.

(x_0, x_1, x_2, x_3) $\longrightarrow$

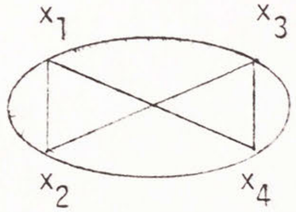


Figure 1.1

Note that this map is a local diffeomorphism in a neighborhood of any point in R_n.

Geometrically, a reflected ray is distinguished among inscribed polygons by the fact that Snell's reflection law is satisfied at each vertex. Poincaré observed that this geometric condition is equivalent to an extremal condition derived from Hamilton's variational principle.

Let $Q : \partial\Omega \to R^{m+1}$ be the vector valued function defined by the inclusion $\partial\Omega \to R^{m+1}$ and set

$$Q_i(x_0, \ldots, x_{n-1}) = Q(x_i) \quad i = 0, \ldots, n-1$$

Define $L_n: (\partial\Omega)^n \to R$ by

$$L_n = |Q_0 - Q_1| + |Q_1 - Q_2| + \ldots + |Q_{n-1} - Q_0|$$

Theorem 1.1 (Poincaré): The inscribed n polygon in Ω defined by linking consecutive points $x_0, \ldots, x_{n-1}$ is a closed reflected ray iff $(x_0, \ldots, x_{n-1})$ is a critical point of L_n in $R_n \subset (\partial\Omega)^n$.

Proof: Note that R_n is precisely the set of points in $(\partial\Omega)^n$ where L_n is smooth. At $(x_0, \ldots, x_{n-1})$,

$$\begin{aligned} dL_n = &\left\langle \frac{Q_0 - Q_1}{|Q_0 - Q_1|} - \frac{Q_{n-1} - Q_0}{|Q_{n-1} - Q_0|}, dQ_0 \right\rangle \\ &+ \left\langle \frac{Q_1 - Q_2}{|Q_1 - Q_2|} - \frac{Q_0 - Q_1}{|Q_0 - Q_1|}, dQ_1 \right\rangle \\ &+ \ldots + \left\langle \frac{Q_{n-1} - Q_0}{|Q_{n-1} - Q_0|} - \frac{Q_{n-2} - Q_{n-1}}{|Q_{n-2} - Q_{n-1}|}, dQ_{n-1} \right\rangle \end{aligned} \tag{1.1}$$

dQ_i is a vector valued 1-form taking values in $T_{x_i} R^{m+1}$ for k = 0, ..., n-1 and < , > is the canonical inner product on R^{m+1}.

Since the dQ_i are independent, $dL_n(x_0, \ldots, x_{n-1}) = 0$ iff

$$\left\langle \frac{Q_i - Q_{i+1}}{|Q_i - Q_{i+1}|} - \frac{Q_{i-1} - Q_i}{|Q_{i-1} - Q_i|} \, , \; dQ_i \right\rangle = 0 \qquad (1.2)$$

for i = 0, ..., n-1

but this equation is satisfied iff the vector

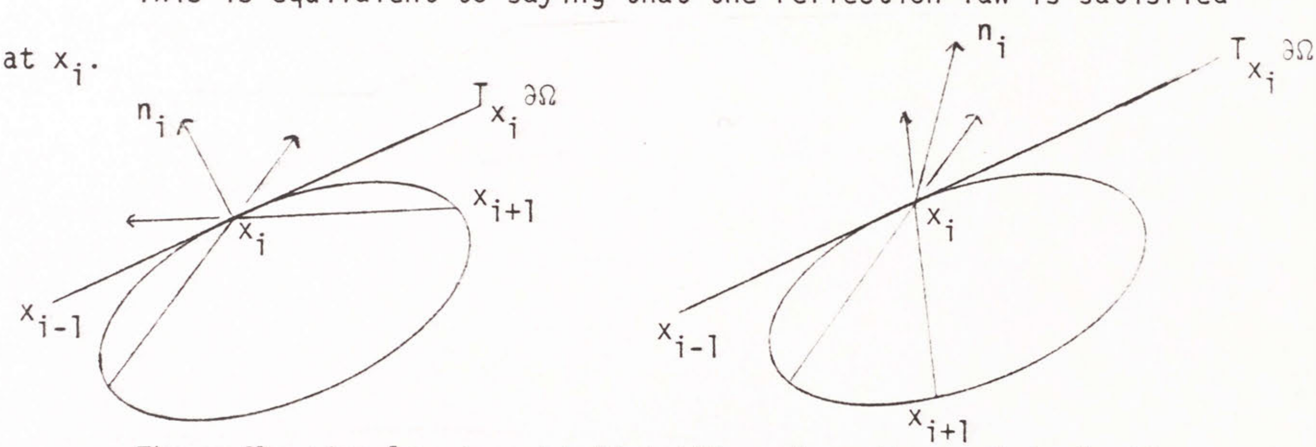
$$n_i = \frac{Q_i - Q_{i+1}}{|Q_i - Q_{i+1}|} - \frac{Q_{i-1} - Q_i}{|Q_{i-1} - Q_i|} \qquad (1.3)$$

is orthogonal to the image of the vector valued 1 form (dQ_i). But

$$\text{Image } (dQ_i) = T_{x_i} \partial\Omega \subset R^{m+1}$$

so (1.2) is satisfied precisely when n_i is orthogonal to $T_{x_i} \partial\Omega$.

This is equivalent to saying that the reflection law is satisfied at x_i.



The reflection law is satisfied iff n_i is orthogonal to $T_{x_i} \partial\Omega$.

Figure 1.2

Consequently, $dL_n = 0$ at $(x_0, \ldots, x_{n-1})$ iff the reflection law is satisfied at each x_i.

The extremal criterion of Poincaré reduces the problem of finding closed reflected rays to the problem of finding critical points of L_n. L_n is a continuous function defined on the compact set $(\partial\Omega)^n$ hence it is maximized and minimized over that set. The minimum corresponds to n-tuples $X = (x_0, \ldots, x_{n-1}) \subset (\partial\Omega)^n$ with

$$x_0 = x_1 = \ldots = x_{n-1}$$

however,

Proposition 1.2: The maximum of L_n is attained in R_n and corresponds to a closed ray of Ω.

Proof: Note that if $X = (x_0, \ldots, x_{n-1}) \notin R_n$ then $x_i = x_{i+1}$ for some i. In this case, the triangle inequality and strict convexity of Ω imply that there are polygons $X' \in R_n$ close to X with

$$L_n(X') > L_n(X)$$

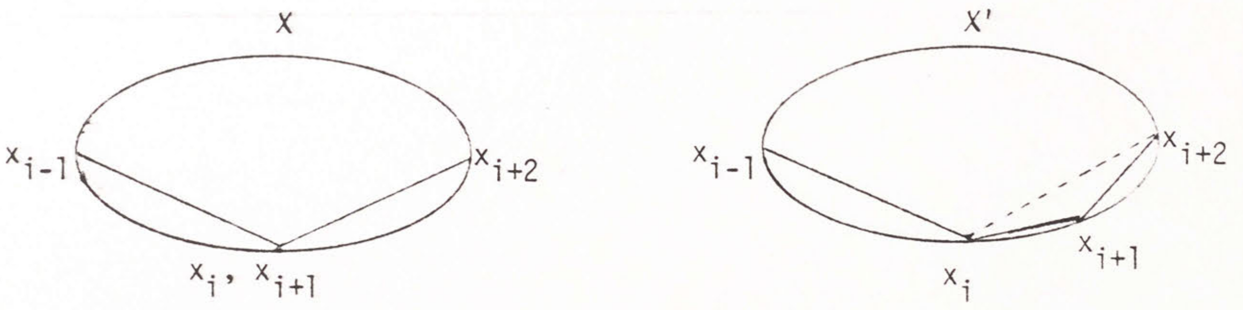
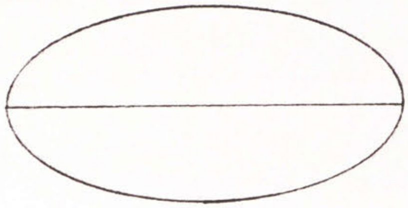


Figure 1.3

Hence L_n is maximized at some point $(x_0, \ldots, x_{n-1}) \in R_n$. Since L_n is smooth on R_n, $(x_0, \ldots, x_{n-1})$ is a critical point of L_n and hence defines a closed ray.



When n = 2, the closed ray corresponds to the longest chord in the interior of the region.

Figure 1.4

While this argument is valid for $\Omega \subset R^{m+1}$ for all $m > 0$, closed rays having kn vertices, $(k,n) > 1$, might not be prime. They could be iterates of closed rays with fewer vertices. For example the closed ray with 2n vertices maximizing L_{2n} must be the n^{th} iterate of the closed ray pictured in Figure 1.4. So, in reality, the extremal condition by itself only guarantees the existence of prime closed rays for each prime number p.

Poincaré's original argument was written for $\Omega \subset R^2$. In this case an additional topological constraint can be used to address the problem of iterated closed rays. Assign to each element of R_n a positive integer called the winding number, N. The winding number resolves each R_n into a finite number of components. L_n may be maximized on each component. In particular, maximizing L_n over the component with winding number, $N = 1$, will establish the existence of a prime closed ray with n vertices for each $n > 1$.

N = 1 N = 2

Trace the sequence of vertices $x_0, \ldots, x_{n-1}$ clockwise around $\partial\Omega$. N is the number of times one passes x_0.

Figure 1.5

The case $\Omega \subset R^2$ allows one more extension of these arguments due to Birkhoff. Birkhoff observed that given a vertex number $n > 1$, and winding number N, the extremal criterion of Poincaré would yield a second closed ray with n vertices corresponding to the minimax of L_n. Let $X = (x_0, \ldots, x_{n-1})$ be a point that maximizes L_n among the n-tuples with winding number N, and set $X' = (x_1, \ldots, x_{n-1}, x_0)$. Let a be a real number and define the set L_a to be

$$L_a = \{X \in (\partial\Omega)^n : L_n(X) > a\}$$

The minimax of L_n associated to X and X' is

$$\text{minimax } L_n = \max\{a : X, X' \text{ lie in the same connected component of } L_a\}$$

When $\Omega \subset R^2$, L_n is a C^1 function on $(\partial\Omega)^n$, thus the set

$$M = \{X \in (\partial\Omega)^n : L_n(X) = \text{minimax } L_n\}$$

must contain a critical point of L_n. (For an exposition of this argument, see, for example, [Math1].) This critical point represents a second closed ray with n vertices and winding number, N.

In summary, when $\Omega \subset R^2$, these arguments guarantee the existence of two closed rays for each winding number N and sufficiently large vertex number n. Unfortunately, in higher dimensions, neither the winding number argument or the minimax argument apply in any straightforward way. The problem with the winding number is obvious. In higher dimensions, there is no easy way to define it.

The problem with generalizing the minimax argument to higher dimensions is that when $\Omega \in R^{m+1}$, $m > 1$, the function L_n is no longer C^1 on $(\partial\Omega)^n$. This is not necessarily a fatal problem. The minimax argument will apply in certain situations. For example, when $n = 2$, there exists $a \in R$, $0 < a <$ minimax L_n such that $L_a \subset R_2$. This means that the singular set of L_2 and the minimax set, M, lie in disjoint open neighborhoods of $(\partial\Omega)^2$. Under these circumstances, the hypotheses of the minimax argument are satisfied and the set M will contain a critical point of L_2. (The corresponding closed ray is pictured in Figure 1.6.)

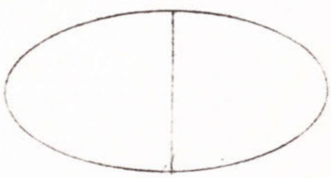


Figure 1.6

Unfortunately, $a > 0$ such that $a <$ minimax L_n and $L_a \subset R_n$ does not exist in general and this condition is not easy to check when $n > 2$.

As one further drawback in higher dimensions, none of these arguments give very much information about the position of the closed rays in Ω. Correcting this deficiency is the key to obtaining better existence theorems for the higher dimensional cases. Observe that as the angle of incidence of its segments approaches zero, the limiting behavior of a reflected ray is that of a geodesic in the boundary $\partial\Omega$. In the following sections, it will be shown that this limiting behavior may be used to reduce the higher dimensional problem to one very similar to the case $\Omega \subset R^2$.

Section 2

THE BILLIARD BALL MAP AND THE INTERPOLATING HAMILTONIAN

To understand the relationship between reflected rays in Ω and geodesics in $\partial\Omega$, it is convenient to introduce an alternative description of reflected rays given by the billiard ball map on the co-ball bundle of $\partial\Omega$.

$$B: B^*\partial\Omega \to B^*\partial\Omega$$

$$B^*\partial\Omega = \{(x,\xi) \in T^*\partial\Omega : |\xi| \leq 1\}$$

The billiard ball map is defined as follows (refer to figure 2.1.) Given $x \in \partial\Omega$ let $n(x)$ be the outward unit normal vector to at x. If $\xi \in T^*_x\partial\Omega$ with $|\xi| < 1$ there is a unique unit vector $n \in (R^{m+1})^*$ such that

$$n(x) \lrcorner\, n < 0$$

$\lrcorner$ denotes interior product

and

$$v \lrcorner\, \xi = v \lrcorner\, n$$

for v tangent to $\partial\Omega$ at x. Via the canonical inner product on R^{m+1} identify n with a unit vector $e(x,\xi) \in R^{m+1}$. Convexity of $\partial\Omega$ guarantees the existence of x', the unique point of intersection of $\partial\Omega$ with the positive line segment, $\{x + te, t > 0\}$. Let ξ' be the unique element of $T^*_{x'}\partial\Omega$ for which

$$\langle e(x,\xi), v\rangle = v \lrcorner \xi'$$

for v tangent to $\partial\Omega$ at x'. For (x,ξ) in the interior of $B^*\partial\Omega$, i.e. $\{(x,\xi) : |\xi| < 1\}$, B is the map sending (x,ξ) to (x',ξ'). Extend B to the rest of $B^*\partial\Omega$ by requiring

$$B(x,\xi) = (x,\xi) \text{ when } |\xi| = 1$$

As a corollary to the following discussion, it will be shown that $B: B^*\partial\Omega \to B^*\partial\Omega$ preserves the canonical symplectic structure of $B^*\partial\Omega$.

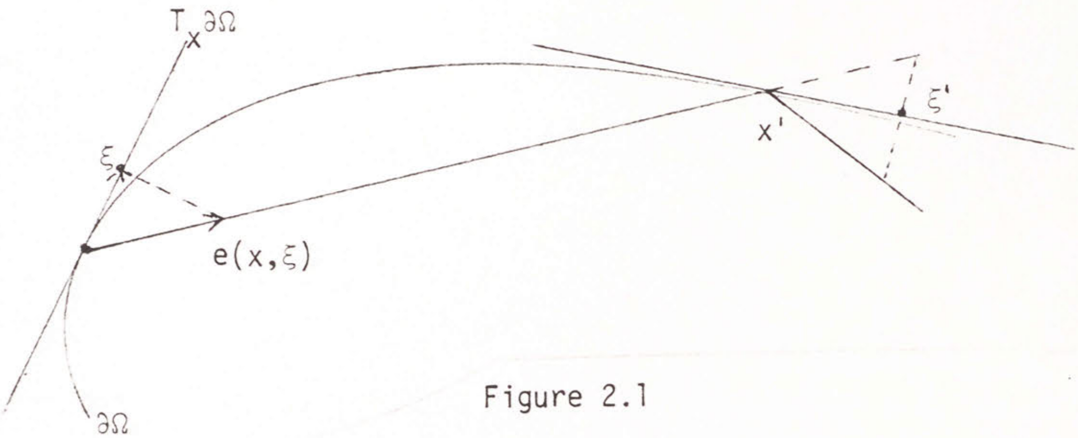


Figure 2.1

Figure 2.1 shows that the projection of an orbit of B onto determines the sequence of vertices of a reflected ray and conversely, a reflected ray determines an orbit of B. Closed rays then correspond to periodic points of B.

The billiard ball map arises naturally as an invariant defined by the intersection of two hypersurfaces in T^*R^{m+1}. Consider the following general situation. Let (X,ω) be a symplectic manifold (Dim X = 2m+2). Recall that the symplectic structure is given by ω, a closed non-degenerate two-form. If $F \to X$ is a hypersurface, then the pull back of ω to F defines a foliation of F by bicharacteristic lines. Let $f \in C^\infty(X)$ be a defining function of the hypersurface F, i.e.

$$F = \{x \in X : f(x) = 0\} \; ; \quad df \neq 0 \text{ on } F\}$$

Recall that the symplectic two-form defines a map from T_x^*X to T_xX by

$$V \in T_xX \rightarrow V \lrcorner\, \omega_x \in T_x^*X$$

The fact that ω is non-degenerate implies that any function $f \in C^\infty(X)$ defines a hamiltonian vector field, H_f, on X, where

$$df = H_f \lrcorner\, \omega$$

Since

$$H_f \lrcorner\, df = \omega(H_f, H_f) = 0$$

H_f is tangent to the level sets of f. Next suppose that $x \in F$ and $V \in T_xF$.

$$\begin{aligned} \omega(H_f, V) &= V \lrcorner\, df \\ &= 0 \end{aligned}$$

Thus H_f lies in the one-dimensional kernel of the pull back of ω to F. In particular, this means that H_f is tangent to the bicharacteristic lines of F.

Let $x \in F$, locally about x, the quotient space M_F of the bicharacteristic foliation will be a smooth 2m-2 dimensional manifold. M_F has a canonical symplectic structure ω_F defined by

$$\pi_F^* \,\omega_F = i^*\omega$$

where π_F, i

$$\begin{array}{ccc} F & \xrightarrow{\ i\ } & X \\ {\scriptstyle \pi_F}\downarrow & & \\ M_F & & \end{array}$$

are the canonical projection and inclusion of F into M_F and X respectively.

ω also defines a Lie bracket on $C^\infty(X)$ where the bracket is given by

$$\{f,g\} = H_F \lrcorner \, dg$$

$\{\ ,\ \}$ is called the Poisson bracket. Anti symmetry of $\{\ ,\ \}$ follows from the anti symmetry of ω and the fact that ω is closed will ensure that $\{\ ,\ \}$ satisfies the Jacobi identity. (See [AM1].)

Let F and G be transversally intersecting hypersurfaces in X with defining functions $f,g \in C^\infty(X)$. Form the manifold $J = F \cap G$ and set

$$K = \{x \in J : \{f,g\}(x) = 0\}$$

K is called the set of glancing points. Let $x \in K$, then

$$H_f \lrcorner \, dg = \{f,g\}(x) = 0$$

and

$$H_g \lrcorner \, df = \{g,f\}(x) = 0$$

so $x \in K$ iff the bicharacteristic of F through x, ℓ_x, is tangent to G and the bicharacteristic of G through x, K_x, is tangent to F. Let $x \in K$.

Locally, the spaces of bicharacteristics of F and G, M_F, and M_G respectively are defined as smooth manifolds for a neightborhood of x. Letting π_F and π_G denote the canonical projection of points in F and G to the bicharacteristic lines containing them, consider the following commutative diagram.

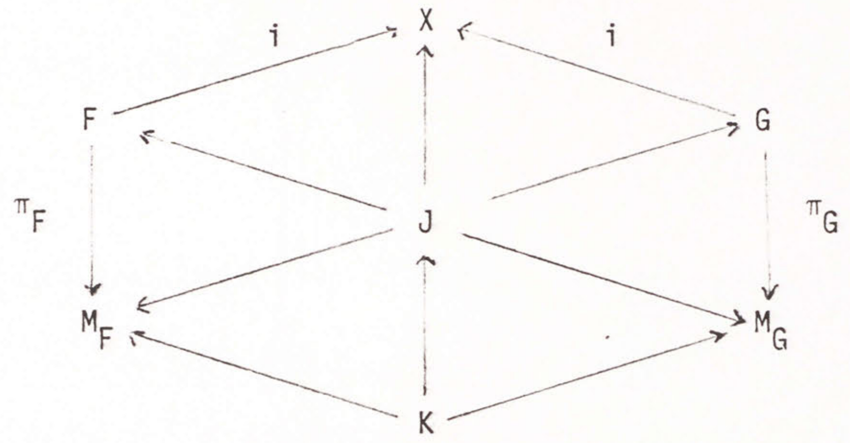


Figure 2.2

In J\K, the restrictions

$$\pi_F : J \to M_F$$

$$\pi_G : J \to M_G$$

are local diffeomorphisms. If $x \in K$ then x is said to be a non-degenerate glancing point if

$$\{f,\{f,g\}\}(x) \neq 0$$

$$\{g,\{f,g\}\}(x) \neq 0$$

Geometrically, this is the situation where ℓx is simply tangent to G and

K_x is simply tangent to F. At a non-degenerate glancing point $x \in K$, the restrictions of π_F and π_G to J have simple fold singularities.

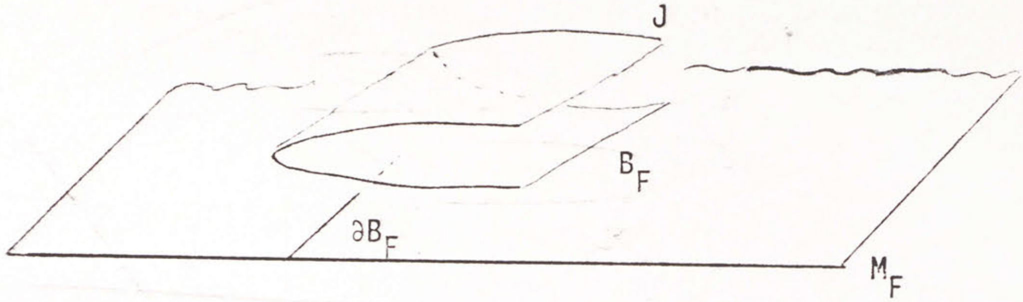


Figure 2.3

Thus, in a neighborhood of such a point, $B_F = \pi_F(J) \subset M_F$ has the structure of a manifold with boundary. Locally, ∂B_F may be identified with K and π_F has a pair of inverses, $\alpha_\pm : B_F \to J$ such that

$$\pi_F \circ \alpha_\pm = \text{id} \quad \text{on } B_F$$

The analogous considerations apply to G and define a space $B_G \subset M_G$ and maps $\beta_\pm : B_G \to J$. Finally, the maps

$$\delta_\pm : B_F \to B_F$$

$$\delta_\pm = \pi_F \circ \beta_\pm \circ \pi_G \circ \alpha_\mp$$

are the generalized Billiard maps of the pair (F,G). These maps will be symplectic with respect to the symplectic structure on M_F.

The major result of Melrose [M1] is that there exist Darboux coordinates $(p_0, \ldots, p_m, q_0, \ldots, q_m)$ for X about x so that

$$F = \{q_0 = 0\}$$

$$G = \{4p_0^2 - q_0 - p_m = 0\}$$

These coordinates induce canonical coordinates $(q_1, \ldots, q_m, p_1, \ldots, p_m)$ on $B_F \subset M_F$. With respect to these coordinates

$$\partial B_F = \{p_m = 0\}$$

and

$$\delta_\pm : B_F \to B_F$$

takes the form

$$\delta_\pm : (q_1, \ldots, q_m, p_1, \ldots, p_m) \to (q_1, \ldots, q_m \pm p_m^{1/2}, p_1, \ldots, p_m)$$

Equivalently, we can say the billiard maps $\delta_\pm$ are locally interpolated by the hamiltonian flow generated by the function p_m.

$$\delta_\pm = \exp(\pm p_m^{\frac{1}{2}} H_{p_m})$$

Recall that the cotangent bundle of any manifold has a canonical symplectic structure. In the symplectic manifold, T^*R^{m+1}, define the following hypersurfaces

$$S^*R^3 = \{(x,\xi) \in T^*R^3 : |\xi| = 1\}$$

$$T^*_{\partial\Omega}R^3 = \{(x,\xi) \in T^*R^3 : x \in \partial\Omega\}$$

S^*R^{m+1} and $T^*_{\partial\Omega}R^{m+1}$ play the role of transversally intersecting hypersurfaces in T^*R^{m+1}. The intersection manifold, J, may be identified as

$$S^*_{\partial\Omega}R^{m+1} = \{(x,\xi) \in T^*R^{m+1} : |\xi| = 1;\ x \in \partial\Omega\}$$

The spaces of bicharacteristics of S^*R^{m+1} and $T^*_{\partial\Omega}R^{m+1}$, M_E and M_B are both defined globally over their respective hypersurfaces. M_E is the space of oriented lines in R^{m+1}. The points in B_E are those lines passing through Ω. There is a natural identification

$$M_E \approx T^*S^m$$

given by mapping $(e,\xi) \in T^*S^m$ to the oriented line

$$\ell = \{\ell + et : t \in R\}$$

Remark: To regard ξ as an element of R^{m+1}, use the canonical inner product of R^{m+1} to identify T^*S^m with a subspace of R^{m+1}.

M_B may be identified with $T^*\partial\Omega$ via the projection

$$T^*_{\partial\Omega}R^{m+1} \to T^*\partial\Omega$$

given by the restriction map $T^*_x R^{m+1} \to T^*_x\partial\Omega$. With this identification

$$B_B = B^*\partial\Omega$$

These identifications are symplectic in that the canonical symplectic structure on M_B agrees with that of $T^*\partial\Omega$ and the canonical symplectic structure of M_E agrees with that of T^*S^m up to a change of sign. Finally, using the canonical inner product on R^{m+1}, (or the Riemannian structure for a general Riemannian manifold) there is a natural inclusion

$$T^*\partial\Omega \to T^*_{\partial\Omega}R^{m+1}$$

preserving symplectic structures. (The fact that the inclusion preserves symplectic structures means that the pull back of the symplectic form of T^*R^{m+1} to $T^*\partial\Omega$ is the symplectic form of $T^*\partial\Omega$. This fact will be discussed at length in Section 9.) Via this inclusion,

$$S^*\partial\Omega \to S^*_{\partial\Omega}R^{m+1}$$

is the manifold of glancing points K. Figure 2.2 specializes to

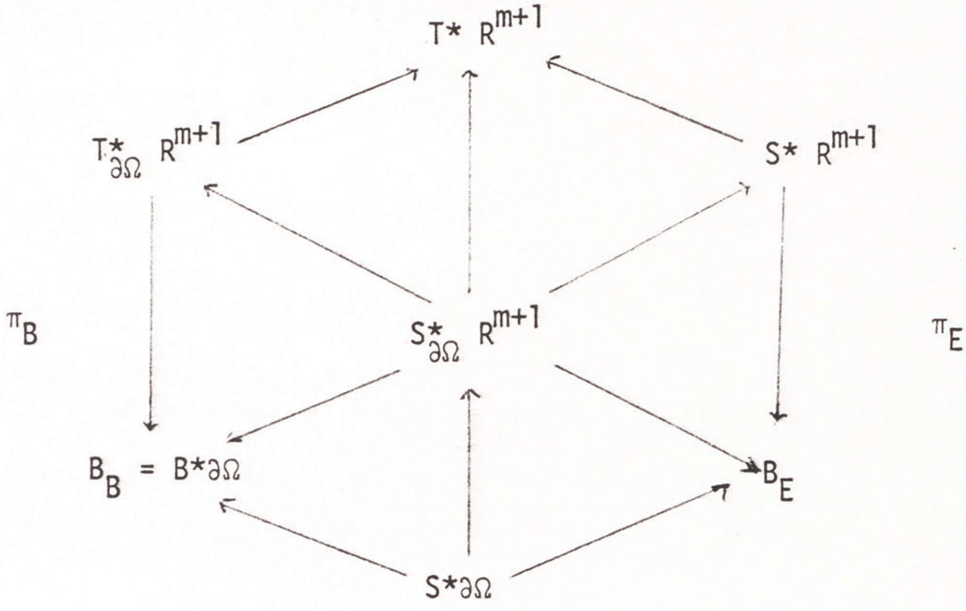
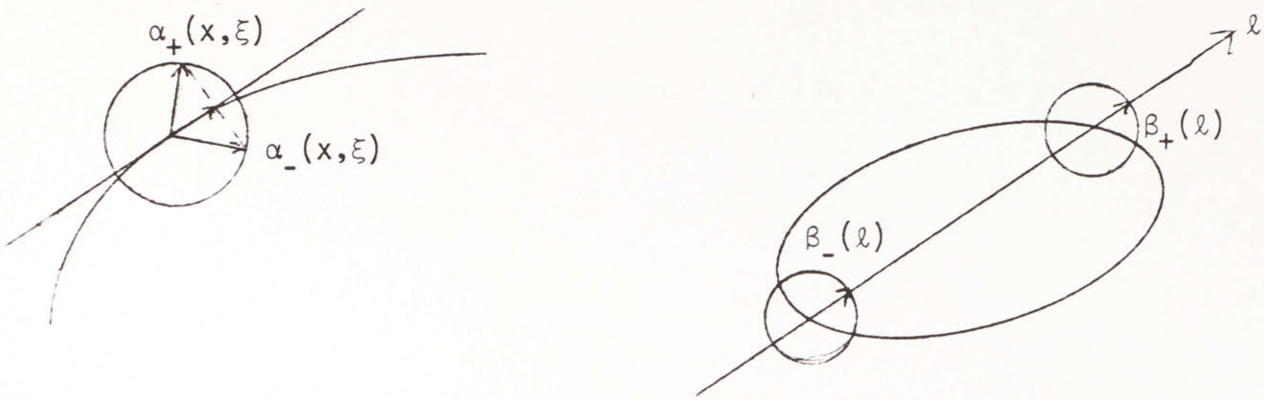


Figure 2.4

Given that Ω is strictly convex, the points in $K \cong S^*\partial\Omega$ consist entirely of non-degenerate glancing points. (The proof of this fact will be deferred to Section 9.) Inverses $\alpha_\pm$, $\beta_\pm$ for π_B, π_E are defined globally on B_B, B_E respectively.



Pictorial representations of $\alpha_\pm$, $\beta_\pm$

Figure 2.5

Finally, the billiard map may be identified with the composition

$$B = \pi_B \circ \beta_+ \circ \pi_E \circ \alpha_-$$

The equivalence theorem for non-degenerate glancing points implies that for $(x,\xi) \in S^*\partial\Omega \subset B^*\partial\Omega$, there exists a neighborhood $U \subset B^*\partial\Omega$, $(x,\xi) \in U$ and a function $\zeta \in C^\infty(U)$ such that

$$B = \exp(-\zeta^{\frac{1}{2}} H_\zeta)$$

ζ is called a local interpolating hamiltonian for B. ζ is a defining function for the hypersurface $U \cap S^*\partial\Omega$. As a corollary to the proof of Melrose [M1], ζ is unique to the extent that if ζ' is another interpolating hamiltonian on U, $\zeta - \zeta'$ vanishes to all orders at $S^*\partial\Omega$.

Section 3

BILLIARD HEURISTICS

In general, the interpolating hamiltonian cannot be defined globally on $B^*\partial\Omega$, but for the sake of illustration, assume that ζ exists globally on such a $B^*\partial\Omega$. Consider the following situation. Let (X,ω) be a symplectic manifold, dim $X = 2m + 2$. Let $F \to X$ be a hypersurface and suppose that F contains a closed bicharacteristic $\bar{\mathcal{O}} \subset F$. Let $p \in \bar{\mathcal{O}}$ and let $W \to F$ be a local transversal section of $\bar{\mathcal{O}}$ (dim $W = 2m$) at p. The F bicharacteristics near $\bar{\mathcal{O}}$ induce a local diffeomorphism

$$\Theta : W \to W$$
$$\Theta(p) = p$$

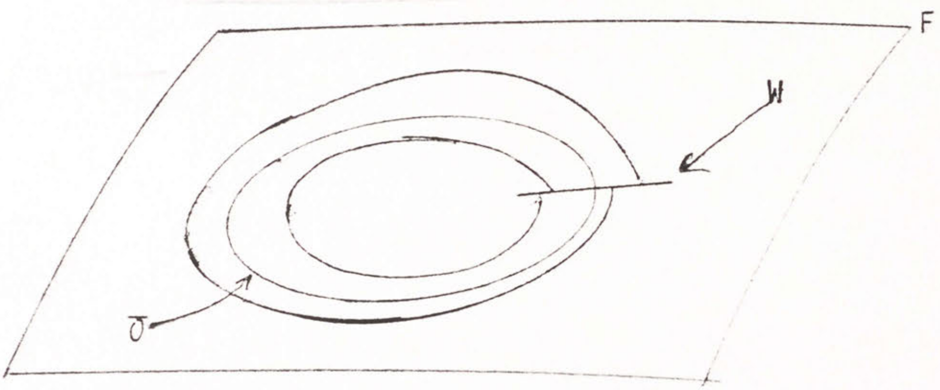


Figure 3.1

called a Poincaré map.

Remark: Technically, Θ is a map $\Theta : U_0 \to U_1$ where U_0, U_1 are neighborhoods of W each containing p. The author has overlooked this point to avoid yet another layer of notation.

The derivative of Θ induces an isomorphism

$$D\Theta : T_pW \to T_pW$$

The bicharacteristic $\overline{\mathcal{O}}$ is called non-degenerate if $D\Theta$ does not have 1 as an eigenvalue. This notion is well defined. Given a different $p' \in \overline{\mathcal{O}}$ and transversal section W', the bicharacteristics of F determine a local diffeomorphism

$$\psi : W \to W'$$

$$\psi(p) = p'$$

In particular, the map $D\psi : T_pW \to T_pW'$ conjugates $D\Theta$ to $D\Theta'$, consequently, $D\Theta$ and $D\Theta'$ have the same spectrum.

Next, let $f \in C^\infty(X)$ be a defining function of F, let H_f be the Hamiltonian vector field associated to f. The bicharacteristics of F will be integral curves of H_f. Thus $\overline{\mathcal{O}}$ will be a closed orbit of H_f. Let $\overline{W} \to X$ be a transversal section of $\overline{\mathcal{O}}$ in X at p. $W = \overline{W} \cap F$ will be a transversal section of $\overline{\mathcal{O}}$ in F. Then the integral curves of H_f define a Poincaré map $\Theta_f : \overline{W} \to \overline{W}$ extending $\Theta : W \to W$.

Lemma 3.1: There exists a path $\gamma : (-\varepsilon, \varepsilon) \to \overline{W}$, transversal to F with $\gamma(0) = p$ such that

$$\Theta_f \circ \gamma(r) = \gamma(r) \text{ for all } r \in (-\varepsilon, \varepsilon)$$

i.e. the non-degenerate bicharacteristic $\overline{\mathcal{O}}$ determines a cylinder of closed orbits of H_f.

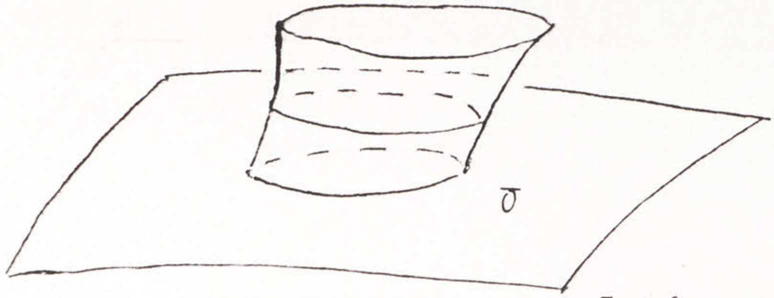


Figure 3.2

Proof: The result follows from the implicit function theorem. Introduce coordinates $(q_1, \ldots, q_{2m})$ on $\overline{W}$ and extend them to coordinates $(q_1, \ldots, q_{2m}, f)$ on $\overline{W}$. Then $\Theta_f : \overline{W} \to \overline{W}$ is defined in coordinates by

$$\Theta_f : (q_i, f) \to (q_i', f)$$

Keep in mind that when $f = 0$, $\Theta_f = \Theta$. Form the map $\Phi : R^{2m+1} \to R^{2m}$ defined by

$$\Phi(q_i, \ldots, q_{2m}, f) = (q_1 - q_1', \ldots, q_{2m} - q_{2m}')$$

For $f = 0$, $q = 0$, the derivatives of $D\Phi$ in the q variables may be written as

$$D\Phi = 1 - D\Theta$$

Since $\mathcal{O}$ is non-degenerate, $1 - D\Theta$ is invertible and the implicit function theorem implies the existence of functions $q_i : (-\varepsilon, \varepsilon) \to R$ such that

$$\Theta_f(q_i(f), f) = (q_i(f), f)$$

f is automatically preserved since any Hamiltonian is preserved by its Hamiltonian flow.

Remark: The points on the path $\gamma(t)$ are periodic points of the Hamiltonian flow generated by H_f. They define a smooth one parameter family of closed orbits of H_f. Since f is constant on these orbits and γ is transversal to F, this family may be parameterized (locally) by f. The orbit cylinder associated to H_f and $\overline{\mathcal{O}}$ is the map

$$\mathcal{T}_f : S^1 \times [-\varepsilon,\varepsilon] \to X$$

which maps $S^1 \times \{r\}$ into the closed orbit defined by $f = r$.

The cosphere bundle $S^*\partial\Omega$ is a hypersurface in the symplectic manifold $B^*\partial\Omega$. A standard result from symplectic geometry is that the bicharacteristics of the cosphere bundle of a Riemannian manifold, M, project to geodesics of M. Indeed, there will be a bijective correspondence between oriented geodesics of M and bicharacteristics of S^*M.

Let $\mathcal{O}$ be an oriented closed geodesic in $\partial\Omega$ and $\overline{\mathcal{O}}$ be the corresponding bicharacteristic in $S^*\partial\Omega$. $\mathcal{O}$ is said to be non-degenerate if $\overline{\mathcal{O}}$ is non-degenerate. One defining function of $S^*\partial\Omega$ is the kinetic energy function.

$$E(x,\xi) = 1/2\ |\xi|^2$$

In this case, the orbit cylinder, $\mathcal{T}_E$, is simply the family of closed H_E orbits generated by the homogeneous structure of the fibers in $T^*\partial\Omega$. The interpolating Hamiltonian, ζ, is also a defining function of $S^*\partial\Omega$. Thus a transversal section $\overline{W} \subset B^*\partial\Omega$ of $\overline{\mathcal{O}}$ at $p \in \overline{\mathcal{O}}$ defines a curve

$$\gamma : (1 - \varepsilon, 1] \to B^*\partial\Omega \qquad\qquad \gamma \pitchfork S^*\partial\Omega$$

$$\gamma(1) = p$$

of periodic points of the flow generated by H_ζ. Let $\tau(r)$ be the period of $\gamma(r)$. Periodic points of the billiard ball map will correspond to solutions of the equation

$$\tau(r) = n(\zeta(\gamma(r)))^{1/2} \qquad n = 2, 3, \ldots \tag{3.2}$$

Since $\tau(r)$ is smooth in r, $\tau(1) \neq 0$, and

$$(\zeta(\gamma(r)))^{1/2} \approx O(1 - r)^{1/2}$$

(3.2) will have a unique solution r_n for each sufficiently large n.

Finally, to incorporate the winding number, N, into this construction, it suffices to consider the Poincaré maps $(\Theta)^N$ and $(\Theta_f)^N$ in Lemma 3.1 and extend the notion of a non-degenerate closed bicharacteristic to that of a N-non-degenerate closed bicharacteristic.

Remark: Since the path γ defined in (3.1) may be constructed for each point in $\overline{\mathcal{O}}$ and Equation (3.2) may be solved for each such γ, existence of an interpolating Hamiltonian defined on a neighborhood of $\overline{\mathcal{O}}$ would imply the existence of a 1-parameter family of periodic orbits of B for each winding number N and sufficiently large vertex number n. This is unlikely to be true in general.

Section 4

EXTENSION OF THE INTERPOLATING HAMILTONIAN

In general, while the interpolating Hamiltonian, ζ, is not well defined on a neighborhood of $S^*\partial\Omega \subset B^*\partial\Omega$, it may be extended to a neighborhood of any compact segment of a bicharacteristic in $S^*\partial\Omega$. This fact will be proved in this section. Unfortunately, there is a global obstruction to extending ζ to a neighborhood of a closed bicharacteristic $\bar{\mathcal{O}}$. This problem may be overcome by considering a neighborhood $V \subset B^*\partial\Omega$, $\bar{\mathcal{O}} \subset V$ satisfying

$$V \text{ is diffeomorphic to } U \times S^1$$

$$U \subset \bar{R}^+ \times R^{2m} \tag{4.1}$$

First, note that a winding number may be defined in this setting.

Lemma 4.1: For sufficiently small $V \subset B^*\partial\Omega$, V - a neighborhood of $\bar{\mathcal{O}}$ of the form (4.1), it is possible to assign a canonical winding number to any sequence of points $(x_0, \xi_0), \ldots, (x_{n-1}, \xi_{n-1})$ in V.

Proof: The Riemannian structure of $\partial\Omega$ will define a canonical local normal fibration of $B^*\partial\Omega$ over the closed bicharacteristic which induces a canonical retraction of V to $\bar{\mathcal{O}}$. So a sequence of points $(x_0, \xi_0), \ldots,$ (x_{n-1}, ξ_{n-1}) in V defines a sequence of points in $\bar{\mathcal{O}}$ and any sequence of points in $\bar{\mathcal{O}}$ has a canonical winding number, N, defined, e.g., as in Figure 1.5. This is well defined so long as V lies in the domain of the normal fibration.

Next, let $\tilde{V}$ be the universal cover of V. Let $\mathcal{O}$ be the pull back of $\overline{\mathcal{O}}$ to $\tilde{V}$. Given any finite segment I of $\tilde{\mathcal{O}}$, the billiard map may be lifted to $\tilde{B}$, a map defined on an open neighborhood, V_I, of I. By restricting V_I, this ploy permits construction of an interpolating Hamiltonian $\tilde{\zeta}$ for $\tilde{B}$ on V_I. The interpolating flow may be analyzed using Poincaré maps as in Section 3, but there will be a loss of control in one degree of freedom due to the fact that $\tilde{\zeta}$ is not periodic on V_I, hence will not be globally invariant under the interpolating Hamiltonian flow. Nevertheless, a winding number, N, will define Poincaré maps which will yield, for each sufficiently large n, a family of nearly periodic points

$$\rho_n : S^1 \to V$$

uniquely characterized by the properties

$$B(x(s), \xi(s)) = (x(s), \xi'(s)) \qquad s \in S^1$$
$$\xi'(s) = \alpha\xi(s) \quad \text{for some } \alpha > 0 \tag{4.2}$$

and

The sequence $\{(x_0, \xi_0), \ldots, (x_{n-1}, \xi_{n-1})\}$ defined by $(x_i, \xi_i) = B^i(x(s), \xi(s)) \in V$ will have winding number N. (4.3)

The extremal criterion of Poincaré applied to this family will yield a pair of closed rays with n vertices.

With $\overline{\mathcal{O}}$ a closed bicharacteristic of $S^*\partial\Omega$. Let $\overline{W} \subset B^*\partial\Omega$ be a transversal section of $\overline{\mathcal{O}}$, $W = \overline{W} \cap S^*\partial\Omega$. Form the space $\overline{W} \times R$ and use the geodesic flow generated by the Hamiltonian

$$r(x,\xi) = |\xi|$$

to define a map $\lambda : \overline{W} \times R \to B^*\partial\Omega$

$$\lambda(p, t) = \exp(L_\infty \, tH_r)(p)$$

$$L_\infty = \text{arclength } \mathcal{O} \tag{4.4}$$

Defining a symplectic structure on $\overline{W} \times R$ by $\tilde{\omega} = \lambda^*\omega$, λ projects the bicharacteristics of the hypersurface $W \times R \subset \overline{W} \times R$ onto the bicharacteristics of $S^*\partial\Omega$. In particular, λ projects $\tilde{\mathcal{O}} = \{0\} \times R$ onto the closed bicharacteristic $\overline{\mathcal{O}} \subset S^*\partial\Omega$, and

$$\lambda(0,0) = p_0 \subset W$$
$$\lambda(0,t + k) = \lambda(0,t)$$

Let I be a compact segment of $\tilde{\mathcal{O}}$, $I = \{0\} \times [a,b]$. By restricting V_I to be a suitable, small neighborhood of I, $V_I \subset W \times R$

$$\lambda : V_I \to V \subset B^*\partial\Omega$$

will be a covering map from V_I to a neighborhood V of $B^*\partial\Omega$ of the type described in (4.1).

Moreover, since B fixes $S^*\partial\Omega$,

Lemma 4.2: V_I may be chosen small enough so that the billiard ball map lifts to a map

$$\tilde{B} : V_I \to \overline{W} \times R$$

Proof: Given $p \in \overline{\mathcal{O}} \subset B^*\partial\Omega$, since λ is a covering map, there exists $U \subset V$, $p \in U$ such that $\lambda^{-1}(U)$ is a finite union of disjoint connected open sets.

$$\lambda^{-1}(U) = \bigcup_{i=1}^{n} \tilde{U}_i$$

$$\lambda^{-1}(p) = \{p_i\}_{i=1,\ \dots,\ n}$$

with $\lambda : \tilde{U}_i \to U$ a diffeomorphism. Since $p \in S^*\partial\Omega$, $B(p) = p$, so there exists $U' \subset U$ such that $B(U') \subset U$. Define $\tilde{B}$ on $\tilde{U}_i'$, $\tilde{U}_i' = \lambda^{-1}(U') \cap \tilde{U}_i$ by

$$\lambda \circ \tilde{B} = B \circ \lambda \tag{4.5}$$

Since I is covered by a finite collection of $\tilde{U}_i'$, it is clear that B may be defined on a neighborhood of I.

Theorem 4.3: By shrinking V_I further, it is possible to define a function $\tilde{\zeta} \in C^\infty(V_I)$ such that

$$\tilde{B} = \exp(-\tilde{\zeta}^{1/2} H_{\tilde{\zeta}}) \tag{4.6}$$

and $\tilde{\zeta}$ is a defining function for $\partial V_I = \lambda^{-1}(S^*\partial\Omega) \cap V_I$.

Proof: Let $\tilde{p}_0 = (0,0) \in W \times R$. $p = \lambda(\tilde{p}_0)$. There exists $U \subset B^*\partial\Omega$, $p \in U$ with an interpolating Hamiltonian defined on U. Define ζ on the $\tilde{p}_0$ component of $\lambda^{-1}(U)$, $\tilde{U}_0$ by

$$\tilde{\zeta} = \zeta \circ \lambda \tag{4.7}$$

$\tilde{\zeta}$ is extended from $\tilde{U}_0$ by requiring

$$\tilde{\zeta} = \tilde{\zeta} \circ \tilde{B} \tag{4.7a}$$

By construction, $\tilde{\zeta}$ is a local interpolating Hamiltonian of the lifted billiard map $\tilde{B}$ on $\tilde{U}_0$, i.e., (4.6) holds on $\tilde{U}_0$. It will be shown that Equation (4.6) implies that $\tilde{B}$ acts on V_I as a singular shift. Thus Equation (4.7a) defines a smooth extension of ζ on $V_I \cap \lambda^{-1}\text{int}(B^*\partial\Omega)$. It is not clear that this extension will be smooth at ∂V_I.

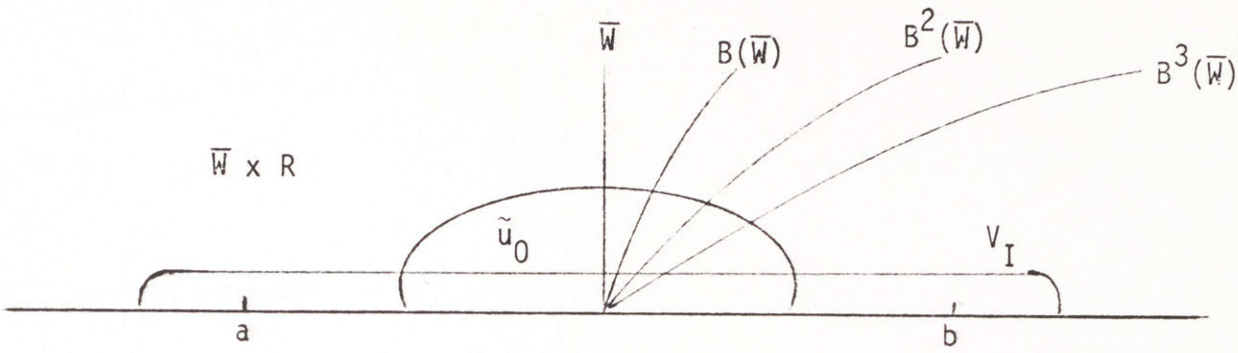


Figure 4.1

Local existence, (Eqn. (4.7)), implies that $\tilde{\zeta}$ is defined and smooth on a neighborhood of $0 \times (-t_1, t_1) \subset \overline{W} \times R$. It suffices to show that (4.7a) defines a smooth extension of $\tilde{\zeta}$ to a neighborhood of $0 \times (-t_1, t_1 + \varepsilon)$ for some $\varepsilon > 0$ (extension at the left endpoint being entirely analogous.)

Let $\tilde{p}_1 = (0, t_1) \in \overline{W} \times R$. Then by existence of a local interpolating Hamiltonian, there exists a neighborhood $N \subset V_I$ of p and coordinates $(q_1, \ldots, q_{2m}, z, \zeta')$ on N so that ζ' is an interpolating Hamiltonian of $\tilde{B}$ and in these coordinates

$$\tilde{B}(q,z,\zeta') = (q, z - \zeta'^{1/2}, \zeta')$$

$$\tilde{\mathcal{O}} = \{q_i = 0, \zeta' = 0\} \tag{4.8}$$

$$\tilde{p}_1 = 0$$

$$\frac{\partial}{\partial t} \lrcorner\, dz > C > 0$$

In these coordinates, there exists $\delta > 0$ such that

$$N = \{0 < |q|,\ \zeta' < \delta;\ -\delta^2/2 < z < \delta^2/2\}$$

and that the set

$$M = \{0 < |q|,\ \zeta' < \delta;\ -\delta^2/2 < z < -\delta^2/4\}$$

lies in $\tilde{U}_0 \cap N$. On M, $\tilde{\zeta}$ and ζ' are both well defined, smooth interpolating Hamiltonians of B. The derivatives of $\tilde{\zeta}$ are bounded on M, and $\tilde{\zeta} - \tilde{\zeta}'$ vanishes to infinite order uniformly on M as $\zeta' \to 0$.

Set

$$n_1(z,\zeta') = \left[\left[\frac{z + \delta^2/2}{(\zeta')^{1/2}}\right]\right] \tag{4.9}$$

[[ ]] = greatest integer function

$$n_2(z,\zeta') = \left[\left[\frac{z + {}^2/2}{(\zeta')^{1/2}} - 1/2\right]\right] \tag{4.10}$$

For either function,

$$-\delta^2/2 < z - n_i(z,\zeta')(\zeta')^{1/2} < -\delta^2/4 \qquad i = 1,2$$

So that $\tilde{B}^{n_i}(\tilde{p}) \in M$ for all $\tilde{p} \in N$. Formulas (4.7a), (4.8) then imply that on N,

$$\tilde{\zeta}(q,z,\zeta') = \tilde{\zeta}(q,z - n_i(z,\zeta')(\zeta')^{1/2},\ \zeta') \quad i = 1,2 \tag{4.11}$$

for points $\tilde{p} \in N$ where n_i is locally constant. Since $\tilde{\zeta} \circ \tilde{B} = \tilde{\zeta}$ on M, these two equations agree where their domains overlap. Together, they may be used to study the smoothness of the extension of $\tilde{\zeta}$ to N.

Lemma 4.4: On N, extend $\tilde{\zeta}$ to $\zeta' = 0$ by

$$\partial^{\alpha}\tilde{\zeta} = \partial^{\alpha}\zeta' \text{ where } \zeta' = 0$$

$$\partial^{\alpha} = \frac{\partial^{\alpha_1}}{\partial q^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial z^{\alpha_2}} \frac{\partial^{\alpha_3}}{\partial \zeta'^{\alpha_3}}$$

Then $\tilde{\zeta}$ is smooth on N.

Proof: It is sufficient to show that $|\partial^{\alpha}\tilde{\zeta} - \partial^{\alpha}\zeta|$ vanishes to infinite order as $\zeta' \to 0$.

$$\left|\partial^{\alpha}(\tilde{\zeta} - \zeta')_{(q,z,\zeta')}\right| < \sum_{\substack{k+j<\alpha_3 \\ k,j>0}} C_{k,j}\, n_i(z,\zeta')^k\, \zeta'^{k/2-j} |\partial^{\alpha_{k,j}}(\tilde{\zeta} - \zeta')|$$

$$\alpha_{k,j} = (\alpha_1, \alpha_2 + k, \alpha_3 - k - j)$$

Recall $n_i(z,\zeta') < \dfrac{\delta^2}{(\zeta')^{1/2}}$ so that

$$|\partial^{\alpha}(\tilde{\zeta} - \zeta')| < \sum_{\substack{k+j<\alpha_3 \\ k,j>0}} C'_{k,j}(\zeta')^{-j} |\partial^{\alpha_{f,j}}(\zeta - \zeta')_{(q,z - n_i \cdot (\zeta')^{1/2},\zeta')}|$$

Finally, recall that $\zeta - \zeta'$ vanishes to all orders of ζ' uniformly on M. So for $n > 0$, there exists $K > 0$ such that on M

$$|\partial^{\alpha_{k,j}}(\tilde{\zeta} - \zeta')| < K(\zeta')^n$$

Hence, for $n' > 0$, there exists $K' > 0$ such that on N

$$|\partial^{\alpha_{k,j}}(\tilde{\zeta} - \zeta')_{(q,z,\zeta')}| < K'(\zeta')^n$$

Since $\tilde{\zeta}$ can be extended smoothly to a neighborhood of $0 \times (-t, t + \varepsilon) \subset \overline{W} \times R$, $\tilde{\zeta}$ can be extended to a neighborhood of $I = \{0\} \times [a,b] \subset \overline{W} \times R$.

Section 5

EXISTENCE OF ALMOST PERIODIC POINTS

In this section, given a winding number, N, and a non-degenerate closed geodesic, $\mathcal{O}$, it will be shown that for sufficiently large n, there will exist a family of nearly periodic points.

$$\rho_n : S^1 \to V$$

satisfying and determined by (4.2), (4.3) where V is a neighborhood $\mathcal{O} \subset V \subset B^*\partial\Omega$ satisfying the hypotheses of Lemma 4.1.

Let I be the segment

$$I = \{0\} \times [-i, N+i] \to \overline{W} \times R$$

From the previous section, there exists a neighborhood $V_I \subset \overline{W} \times R$, $I \subset V_I$ with

$$\lambda : V_I \to V$$

where V, described above, is a neighborhood of $\overline{\mathcal{O}}$, diffeomorphic to $U \times S'$ where $U \subset \bar{R}^+ \times R^{2m}$. The billiard map may be lifted to a map $\tilde{B}$: $V_I \to \overline{W} \times R$ satisfying

$$\lambda \circ \tilde{B} = B \circ \lambda$$

and there exists a function $\tilde{\zeta} \in C^\infty(V_I)$ such that

$$\tilde{B} = \exp(-\tilde{\zeta}^{1/2} H_{\tilde{\zeta}})$$

We use the canonical local normal fibration of $B^*\partial\Omega$ over $\overline{\mathcal{O}}$ to define an an S^1 parameterized family of transversal local sections $\overline{W}_s \subset B^*\partial\Omega$ of $\overline{\mathcal{O}}$ in V.

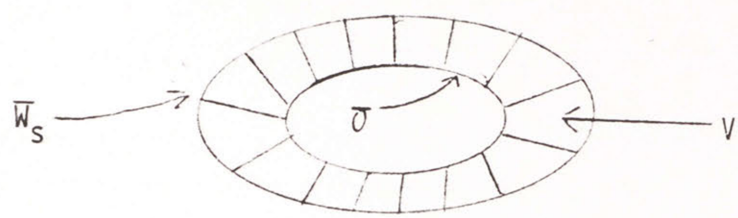


Figure 5.1

Set $W_\zeta = \partial W_s = S^*\partial\Omega \cap \overline{W}_s$. Then, via λ, these transversal sections define local transversal sections $\overline{W}_s$ of $\mathcal{O}$ in V_I.

Note that λ restricts to diffeomorphisms

$$\lambda : \tilde{W}_{s+i} \to W_s \quad i = 0, \ldots, N \qquad s \in [-1/2, 3/2]$$

By these correspondences, the integral curves of $H_{\tilde{\zeta}}$ define a map

$$\tilde{\Theta}_s : \tilde{W}_s \to \tilde{W}_{s+N} \qquad s \in [-1/2, 3/2]$$

which induces a Poincaré map

$$\overline{\Theta}_s : \overline{W}_s \to \overline{W}_s$$

which extends the N^{th} iterate of the Poincaré map

$$\Theta_s : W_s \to W_s$$

defined by the bicharacteristics of $S^*\partial\Omega$ in Section 3.

Definition 5.1: The bicharacteristic $\mathcal{O}$ is N-non-degenerate iff $(D\Theta)^N$ does not have 1 as an eigenvalue.

For the next lemma, let $\Xi : B^*\partial\Omega\ 0_{\partial\Omega} \to S^*\partial\Omega$ be the map $\Xi : (x,\xi) \to (x,\xi/|\xi|)$.

Lemma 5.2: Assume that Θ is N-non-degenerate. There is a unique smooth path $\gamma_s : (1 - \varepsilon, 1] \to \overline{W}_s$; s (-1/2, 3/2) such that $\gamma_s \pitchfork S^*\partial\Omega$ and

$$\Xi \circ \overline{\Theta}_s \circ \gamma_s(r) = \Xi \circ \gamma_s(r) \tag{5.1}$$

Moreover, γ_s is smooth in s.

Proof: Introduce local coordinates $(q_1, \ldots, q_{2m})$ on W_s and extend them to be constant on the radii of the fibers of $B^*\partial\Omega$. Then with $r(x,\xi) = |\xi|$, (q_i, r) form a coordinate system on $\overline{W}_s$. In coordinates, let $\overline{\Theta}_s : \overline{W}_s \to \overline{W}_s$ be given by

$$\overline{\Theta}_s : (q_i, r) \to (q_i', r')$$

As in Lemma 3.1, form the map

$$\Phi : (q_i, r) \to (q_i - q_i')$$

The non-degeneracy condition on $\mathcal{O}$ implies that the derivative of Φ in the q variables is non-degenerate at (0,0) so the implicit function theorem implies the existence of unique functions $q_i(r)$ such that

$$\overline{\Theta}_s(q_i(r), r) = (q_i(r), r')$$

Note that, in general, r does not equal r'. This represents the loss of control mentioned in Section 4.

$\gamma_s(r)$ is the path defined by the $q_i(r)$. In coordinates

$$\gamma_s(r) = (q_i(r), r)$$

It is clear that $\gamma_s(r)$ varies smoothly in s for s (-1/2, 3/2).

Let $\tau_s(r)$ be the return time of the point $\gamma_s(r)$ defined as follows. Let $\tilde{\gamma}_s(r)$ be the point over $\gamma_s(r)$ in $\tilde{W}_s$. $\tau_s(r)$ is the time required to flow along $H_{\tilde{\zeta}}$ from $\tilde{\gamma}_s(r)$ to $\tilde{\Theta}_s(\tilde{\gamma}_s(r))$. In Section 3, Equation (3.2) gave the condition for distinguishing periodic points of B within the path $\gamma(r)$. The analogous condition here is

$$n(\tilde{\zeta}(\tilde{\gamma}_s(r)))^{1/2} = \tau_s(r) \tag{5.2}$$

Lemma 5.3: There exists $N_0 > 0$ such that for $n > N_0$ there exists a unique function $r_n(s)$, $s \in (-1/2, 3/2)$ such that

$$n(\tilde{\zeta}(\gamma_s(r_n(s)))) = \tau_s(r_n(s)) \tag{5.3}$$

Proof: $\tau_s(r)$ is a smooth function in r and s with

$$\tau_s(1) = \tau_0 > 0$$

where τ_0 is the period of $\overline{\mathcal{O}}$ as an orbit of H_ζ. (Note that $H_{\tilde{\zeta}}$ is a well defined vector field on $S^*\partial\Omega$.)

$\tilde{\zeta} \circ \tilde{\gamma}_s(r)$ is a smooth function in r and s with

$$M'(1 - r) < |\tilde{\zeta} \circ \tilde{\gamma}_s(r)| < M(1 - r) \qquad M, M' > 0$$
$$0 < (1-r) < \delta$$

Consequently, for $s \in (-1/2, 3/2)$, Equation (5.3) has a unique solution $r_n(s)$, for n sufficiently large (see Figure 5.2). This solution will vary smoothly in s.

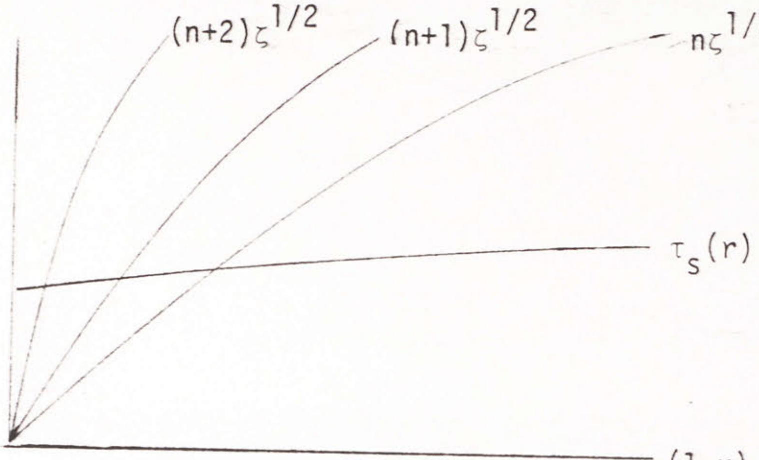


Figure 5.2

Set $\rho_n(s) = \sigma_s(r_n(s))$. From Lemmas 5.2, 5.3, $\rho_n(s)$ is the unique point in $\overline{W}_s$ such that

$$B^n(\rho_n(s)) \subset \overline{W}_s \tag{5.4}$$

$$q_i[B^n(\rho_n(s))] = q_i(\rho_n(s)) \qquad i = 1, \ldots, 2m \tag{5.5}$$

and

the sequence $(x_0, \xi_0), \ldots, (x_{n-1}, \xi_{n-1})$
has winding number N in V, where $(x_i, \xi_i) = B^i(\rho_n(s))$ (5.6)

Letting $(x_0, \xi_0) = \rho_n(s)$, $(x_n, \xi_n) = B^n(x_0, \xi_0)$, conditions (5.4), (5.5) are equivalent to

$$x_n = x_0$$
$$\xi_n = C\xi_0 \qquad C > 0 \tag{5.7}$$

Theorem 5.4: ρ_n is a smooth map $\rho_n : S^1 \to V$

Proof: $\overline{W}_s$ is periodic in s. $\rho_n(s)$ is locally smooth in s and defined uniquely in $\overline{W}_s$ by (5.6), (5.7). Consequently, $\rho_n(s)$ is periodic in s.

A point (x,ξ) in $B^*\partial\Omega$ defines an element of $(\partial\Omega)^n$ by

$$(x_i, \xi_i) = B^i(x,\xi)$$

In the next section, we use this fact to define L_n on ρ_n and look for periodic points of B among the critical points of $L_n|\rho_n$.

For Section 6, the following is proved here.

Proposition 5.5: There exists N > 0, C, C' > 0 such that

$$C' n^{-2} < (1 - r) \circ \rho_n(s) = 1 - r_n(s) < C n^{-2} \qquad n > N_0$$

Proof: Since $\tau_s(r)$ is continuous and bounded on V_I, with $\tau_s(1) = \tau_0$, there exists $\delta_1 > 0$ and constants K_1, K_1' such that

$$K_1 < \tau_s(r) < K_1' \qquad 0 < (1 - r) < \delta_1$$

Since $(1 - r)$, and $\tilde{\zeta}$ are defining functions of ∂V_I, there exist K_2, K_2', $\delta_2 > 0$ such that

$$K_2\tilde{\zeta} < (1 - r) < K_2'\tilde{\zeta} \qquad 0 < (1 - r) < \delta_2$$

Since $\lim_{n\to\infty} r_n(s) \to 1$ uniformly in s, there exists $N_0 > 0$ such that for $n > N_0$

$$(1 - r_n(s)) < \min(\delta_1, \delta_2)$$

So by Lemma 5.3, when $n > N_0$,

$$K_1^2\, n^{-2} < \tilde{\zeta} \circ \rho_n < K_1'^2\, n^{-2}$$

and hence

$$(K_1)^2\, K_2 n^{-2} < (1 - r) \circ \rho_n = (1 - r_n) < (K_1')^2\, K_2'\, n^{-2}$$

Proposition 5.6: ρ_n converges uniformly to the characteristic $\overline{\mathcal{O}}$ in the C^1 topology.

Proof: $\gamma : (-1/2, 3/2) \times (1 - \varepsilon, 1] \to B^*\partial\Omega$ is a smooth map with

$$\gamma_s(r_n(s)) = \rho_n(s)$$

and the path $\gamma_s(1)$ a smooth parameterization of the closed bicharacteristic $\overline{\mathcal{O}}$. By Proposition 5.5, $r_n - 1$ converges uniformly to zero as $n \to \infty$ and so the result follows.

Section 6

PERIODIC POINTS OF B

In the previous section, a winding number N and a N-non-degenerate closed geodesic $\mathcal{O}$, determined for each sufficiently large n a path

$$\rho_n : S^1 \to V$$

The extremal criterion of Poincare will detect periodic points of B within ρ_n.

Pull back $Q\colon \partial\Omega \to R^{m+1}$ to a function $Q\colon\ B^*\partial\Omega \to R^{m+1}$ and set $H_n : B^*\partial\Omega \to R$ equal to

$$H = |Q - Q\circ B| + |Q\circ B - Q\circ B^2| + \ldots + |Q\circ B^{n-1} - Q|$$

Let $(x,\xi) \in B^*\partial\Omega$ and define $(x_0, \xi_0), \ldots, (x_{n-1}, \xi_{n-1})$ by

$$(x_i, \xi_i) = B^i(x,\xi)$$

Theorem 6.1: If $(x_0, \xi_0), \ldots, (x_{n-1}, \xi_{n-1})$ is a closed orbit of B, then (x_0, ξ_0) is a critical point of H_n.

Proof: Define $h_n : B^*\partial\Omega \to (\partial\Omega)^n$ by

$$(x_i, \xi_i) \to (x_0, \ldots, x_{n-1})$$

where

$$(x_i, \xi_i) = B^i(x,\xi)$$

Note that $H_n = h_n^* L_n$. So if $(x_0, \xi_0), \ldots, (x_{n-1}, \xi_{n-1})$ is a periodic orbit of B, $(x_0, \ldots, x_{n-1})$ is a critical point of L_n and thus (x,ξ) is a critical point of $H_n = h_n^* L_n$.

Lemma 6.2: A formula for dH_n

If the n-tuple $(x_0, \ldots, x_{n-1})$ is defined by

$$(x_i, \xi_i) = B^i(x,\xi) \text{ for some } (x,\xi) \in B^*\partial\Omega$$

then the reflection law will be satisfied at the points $x_1, \ldots, x_{n-1}$. Consequently, all but the first and last terms of Equation (1.1) vanish. So

$$dH_n = h_n^* \, dL_n =$$

$$\left\langle \frac{Q \circ B^{n-1} - Q}{|Q \circ B^{n-1} - Q|} - \frac{Q \circ B^{n-2} - Q \circ B^{n-1}}{|Q \circ B^{n-2} - Q \circ B^{n-1}|}, \; d(Q \circ B^{n-1}) \right\rangle$$

$$+ \left\langle \frac{Q - Q \circ B}{|Q - Q \circ B|} - \frac{Q \circ B^{n-1} - Q}{|Q \circ B^{n-1} - Q|}, \; dQ \right\rangle \qquad (6.1)$$

Set $(x', \xi') = B(x, \xi)$. Then in rough terms, the first term of $dH_n(x,\xi)$ measures the separation of x and x'. It will vanish iff $x = x'$. The second term measures the difference between ξ and ξ'. When $x = x'$, it will vanish when $\xi = \xi'$. Unfortunately, in this situation, dH_n can sometimes vanish when $\xi \neq \xi'$, but this will not be a significant problem.

Consider the composition $H_n \circ \rho_n : S^1 \to R$. We need to calculate

$$\left.\frac{d(H_n \circ \rho_n)}{ds}\right|_{s = s_0}$$

So let

$$(x,\xi) = \rho_n(x)$$
$$(x', \xi') = B^n(x,\xi) \tag{6.2}$$

Then by construction

$$x' = x$$
$$\xi' = C\xi \qquad C > 0$$

So by Lemma 6.2.

$$dH_n(x,\xi) = \left\langle \frac{Q - Q \circ B}{|Q - Q \circ B|} - \frac{Q \circ B^{n-1} - Q}{|Q \circ B^{n-1} - Q|} , dQ_{(x,\xi)} \right\rangle \tag{6.3}$$

Regarding ξ, ξ' as elements of $T_x\partial\Omega$ via the Riemannian structure (6.3) may be rewritten as

$$dH_n(x,\xi) = <\xi - \xi', dQ_{(x,\xi)}>$$
$$= (1 - C)<\xi, dQ_{(x,\xi)}> \tag{6.4}$$

From (6.4),

$$\left.\frac{d(H_n \circ \rho_n)}{ds}\right|_{s=s_0} = (1 - C)\left\langle \xi, \rho_{n_*}(\frac{\partial}{\partial s}) \lrcorner dQ_{(x,\xi)} \right\rangle \tag{6.5}$$

Let $\pi : T^*\partial\Omega \to \partial\Omega$ be the canonical projection, then the vector

$$K_n(s) = \rho_{n_*}(\frac{\partial}{\partial s}) \lrcorner dQ_{(x,\xi)} \in T_x\partial\Omega \tag{6.6}$$

may be identified as the tangent vector to the path $\pi \circ \rho_n$ at x in $\partial\Omega$.

Lemma 6.3: There exists $N_0 > 0$ such that $n > N_0$ implies

$$<\xi(s),\ K_n(s)> \neq 0 \qquad s \in S^1 \tag{6.7}$$

Proof: According to Proposition 5.6, the path $\rho_n : S^1 \to B^*\partial\Omega$ converges, in the C^1 topology, to the bicharacteristic $\bar{\mathcal{O}}$ as $n\to\infty$. Thus the vectors, $\xi(s)$ and $K_n(s)$, defined by Equations (6.2), (6.7), respectively, converge to tangent vectors to the geodesic $\mathcal{O}$ - uniformly in s. Thus for sufficiently large n,

$$<\xi(s),\ K_n(s)> \ \neq\ 0 \text{ for all } s \in S^1$$

Let $n > N_0$, as in Lemma 6.3. The function $H_n \circ \rho_n : S^1 \to R$ must have two critical points, s_1, s_2 corresponding to its maximum and minimum.

Theorem 6.4: $(x,\xi) = \rho_n(s_i)$ (i = 1 or 2) is a periodic point of B with period n.

Proof: From previous results, $(x',\xi') = B(x,\xi)$ satisfies

$$x = x'$$

$$\xi' = C\xi \qquad C > 0$$

Equations (6.5), (6.6) imply

$$0 = (1 - C)<\xi,\ K_n>$$

So by Lemma 6.3, C = 1 and so $(x,\xi) = (x',\ \xi')$.

This completes the proof of the existence of closed rays in Ω. In the following sections, we examine the length spectrum of the closed rays, winding number $N = 1$, associated to the closed geodesic $\mathcal{O}$.

Section 7

THE LENGTH SPECTRUM

The length spectrum associated to the region Ω is the set of lengths of the closed reflected rays in Ω. Guillemin and Melrose [GM1] have shown that the length spectrum of Ω is a symplectic invariant of the symplectic manifold $B^*\partial\Omega$ and the symplectic map B. For background, their argument is included here.

Let ω be the symplectic form on $B^*\partial\Omega(\partial\Omega \subset R^{m+1}, m > 1)$. Let β be a one form on $B^*\partial\Omega$ such that $\omega = -d\beta$. Then since $B: B^*\partial\Omega \to B^*\partial\Omega$ is a symplectic map and $B^*\partial\Omega$ is simply connected, $\beta - B^*\beta$ is an exact form. Define F on $B^*\partial\Omega$ by

$$F\big|_{S^*\partial\Omega} = 0 \tag{7.1}$$

$$dF = \beta - B^*\beta \tag{7.2}$$

F will be a smooth function on the interior of $B^*\partial\Omega$.

Theorem 7.1 (Guillemin, Melrose): Given a closed orbit of B

$$\mathcal{J}_n = (x_0, \xi_0), \ldots, (x_{n-1}, \xi_{n-1})$$

$$L_n(\mathcal{J}_n) = \sum_{i=0}^{n-1} F(x_i, \xi_i). \tag{7.3}$$

This means that if $B^*\partial\Omega$ is considered as an abstract manifold with symplectic structure and the map B given, then $L_n(\mathcal{J}_n)$ may be recovered.

Theorem 7.1 is a corollary of the following lemmas.

Lemma 7.2: Given β' satisfying $\omega = -d\beta'$, define F' according to Equations (7.1), (7.2). Then

$$\sum_{i=0}^{n-1} F(x_i, \xi_i) = \sum_{i=0}^{n-1} F'(x_i, \xi_i) \qquad (7.4)$$

In other words the summation in Equation (7.3) is independent of the choice of the one form β that defines F.

Proof: Since $d(\beta - \beta') = \omega - \omega = 0$, $\beta - \beta'$ is an exact form. Let $G \quad C^{\infty}(B^*\partial\Omega)$ be a function such that

$$dG = \beta - \beta' .$$

Then

$$F - F' = G \circ B - G.$$

So since (x_0, ξ_0) is a periodic point of B with period n,

$$\begin{aligned}\sum_{i=0}^{n-1} F(x_i, \xi_i) &= \sum_{i=0}^{n-1} F'(x_i, \xi_i) + (G \circ B - G)(x_i, \xi_i) \\ &= \sum_{i=0}^{n-1} F'(x_i, \xi_i) + G \circ B(x_{n-1}, \xi_{n-1}) - G(x_0, \xi_0) \\ &= \sum_{i=0}^{n-1} F'(x_i, \xi_i) . \qquad (7.6)\end{aligned}$$

Define the function H on $B^*\partial\Omega$ by $H(x,\xi) = |x - x'|$ where $(x', \xi') = B(x,\xi)$. Equivalently, H may be written in terms of the map $Q : B^*\partial\Omega \to R^{m+1}$ as $H = |Q - B \circ Q|$. Note that

$$L_n(\mathcal{J}_n) = \sum_{i=0}^{n-1} H(x_i, \xi_i) . \qquad (7.7)$$

Let α be the canonical 1-form on $T^*\partial\Omega$. The symplectic form is given by $\omega = -d\alpha$. Given $V \in T_{(x,\xi)}(B^*\partial\Omega)$,

$$V \lrcorner\, \alpha(x,\xi) = \pi_* V \lrcorner\, \xi \quad ,$$

where $\pi\colon T^*\partial\Omega \to \partial\Omega$ is the canonical projection. Alternatively, using the Riemannian metric, $<\ ,\ >$, to identify $T^*_x\partial\Omega$ with $T_x\partial\Omega \subset R^{m+1}$, α may be written in terms of the R^{m+1} valued 1-form, dQ, as

$$\alpha = <\xi,\ dQ_{(x,\xi)}> \tag{7.8}$$

Lemma 7.3:

$$dH = \alpha - B^*\alpha$$

Proof:

$$dH_{(x,\xi)} = \left\langle \frac{Q - Q \circ B}{|Q - Q \circ B|}(x,\xi),\ dQ_{(x,\xi)} \right\rangle - \left\langle \frac{Q - Q \circ B}{|Q - Q \circ B|}(x,\xi),\ (B^*\, dQ)_{(x,\xi)} \right\rangle \tag{7.10}$$

As in Section 2, set

$$e(x,\xi) = \frac{Q - Q \circ B}{|Q - Q \circ B|}(x,\xi) \tag{7.11}$$

If $(x', \xi') = B(x,\xi)$, the one forms $(dQ)_{(x,\xi)}$ and $(B^*\, dQ)_{(x,\xi)}$ take values in $T_x\partial\Omega$ and $T_{x'}\partial\Omega$ respectively. Thus, by the construction of B, outlined in Section 2,

$$\left\langle e(x,\xi),\ dQ_{(x,\xi)}\right\rangle = <\xi,\ dQ_{(x,\xi)}> = \alpha(x,\xi) \tag{7.12}$$

$$\left\langle e(x,\xi),\ (B^*dQ)_{(x,\xi)}\right\rangle = \left\langle \xi',\ (B^*dQ)_{(x,\xi)}\right\rangle \tag{7.13}$$

But

$$\begin{aligned}(B^*\alpha)(x,\xi) &= B^*(\alpha(x',\ \xi')) \\ &= <B^*(dQ_{(x',\xi')}),\ \xi'> \\ &= <(B^*dQ)_{(x,\xi)},\ \xi'> \quad .\end{aligned}$$

So that $dH = \alpha - B^*\alpha$.

Lemmas 7.2, 7.3 along with Equation (7.7) prove Theorem 7.1.

In the case where $\Omega \subset R^*$, Marvizi and Melrose [MMar1] extended the preceding argument to derive an asymptotic formula giving the lengths of closed rays (of fixed winding number) in terms of the number of their vertices. Let $\mathcal{J}_n = \{(x_0,\ _0),\ \ldots,\ (x_{n-1},\ _{n-1})\}$ be a closed billiard orbit with winding number $N = 1$. Then, taking L_∞ to be the arclength of $\partial\Omega$, $L_n(\mathcal{J}_n)$ may be written as a formal power series in n^{-2}.

$$L_n(\mathcal{J}_n) \sim L_\infty + \sum_{k=1}^{\infty} C_k n^{-2k} \tag{7.14}$$

modulo errors of order n^{-2k}
each $k > 0$.

This formula and its analogs for higher winding numbers holds for all strictly convex $\Omega \subset R^2$. In higher dimensions, it will be generalized here to describe the lengths of the family of closed rays associated to a closed geodesic $\mathcal{O} \subset \partial\Omega$. To derive this higher dimensional version, it is convenient to introduce a globally defined, approximate interpolating Hamiltonian, Γ, on $B^*\partial\Omega$.

Lemma 7.4: There exists a function $\Gamma \in C^\infty(B^*\partial\Omega)$ such that $\Gamma - \zeta$ vanishes to infinite order at $S^*\partial\Omega$ for any local interpolating Hamiltonian ζ. Γ is called an approximate interpolating Hamiltonian.

Proof: Since $S^*\partial\Omega$ is compact, local existence of interpolating Hamiltonians implies that there exists a partition of unity $\{\emptyset_i\}$ on $B^*\partial\Omega$ and interpolating Hamiltonians $\{\zeta_i\}$ with ζ_i defined on a neighborhood of $\mathrm{supp}\emptyset_i$. Define Γ by

$$\Gamma = \sum_i \emptyset_i \, \zeta_i \, .$$

Let ζ be a local interpolating Hamiltonian defined on a open set U. By local uniqueness, $\zeta_i - \xi$ vanishes to infinite order at $S^*\partial\Omega$ on $U \cap \mathrm{supp}\phi_i$. Hence $\zeta - \Gamma$ will vanish to infinite order at $S^*\partial\Omega \cap U$.

As a corollary to Lemma 7.4, Γ will be a defining function for $S^*\partial\Omega$ and so by the construction in Section 3, there will be a 1 parameter family, $\mathcal{T}_\Gamma$, of closed H_Γ orbits associated to the non-degenerate closed bicharacteristic $\overline{\mathcal{O}}$. In general terms, the following calculations will approximate the relevant features of the billiard system by an approximating billiard system on the two-dimensional image of $\mathcal{T}_\Gamma$.

Define an approximate billiard map A: $B^*\partial\Omega \to B^*\partial\Omega$ by

$$A = \exp(-\Gamma^{1/2} H_\Gamma)$$

A is a symplectic map which is smooth on the interior of $B^*\partial\Omega$ and fixes $S^*\partial\Omega$. Define H' on $B^*\partial\Omega$ by

$$H'\Big|_{S^*\partial\Omega} = 0 \qquad\qquad (7.16)$$

$$dH' = \alpha - A^*\alpha \ .$$

The orbit cylinder for Γ is a map

$$\mathcal{T}_\Gamma : S^1 \times [0, \varepsilon) \to V \subset B^*\partial\Omega \qquad\qquad (7.17)$$

where $S^1 \times \{t\}$ is mapped to the closed H_Γ orbit defined by $\Gamma = t$. Using the path ρ_n defined in Section 5, define a path σ_n by

$$\sigma_n(s) = \mathcal{T}_\Gamma(s, \Gamma\circ\rho_n(s)) \quad . \qquad\qquad (7.18)$$

Heuristically, σ_n is the projection of ρ_n onto the orbit cylinder, $\mathcal{T}_\Gamma$.

Given $(x_0, \xi_0) \in \mathcal{J}_n$, then $(x_0, \xi_0) = \rho_n(s_0)$ for some s_0. Set $(y_0, \eta_0) = \sigma_n(s_0)$. As the first step in the reduction to the two-dimensional case, we wish to show that given k, there exists $N > 0$ such that when $n > N$,

$$L_n(\mathcal{J}_n) = \sum_{i=0}^{n} (H\circ B^i)(x_0, \xi_0)$$

$$= \sum_{i=0}^{n} (H'\circ A^i)(y_0, \eta_0)$$

modulo errors of order Γ^k . (7.19)

Let $I = \{0\} \times [-1/2, 5] \subset \overline{W} \times R$. Then by Theorem 4.3, there exists $V_I \subset \overline{W} \times R$ with a function $\zeta \subset C^\infty(V_I)$ such that $\tilde{B} = \exp(-\zeta^{1/2} H_\zeta)$, is the lift of the billiard map to V_I. Next, pull Γ back to $\overline{W} \times R$. $\zeta - \Gamma$ will vanish to infinite order at ∂V_I. Set $\tilde{A} = \exp(-\Gamma^{1/2} H\ )$, $\tilde{A}$ is the lift of the approximate billiard map A to V_I.

By restricting V_I if necessary, there will exist functions ψ_i, ϕ_i, z defined on V_I such that $(\phi_i, \psi_i, z, \Gamma)$ are Darboux coordinates on V_I. In these coordinates,

$$\omega = dz \wedge d\Gamma + \sum_{i=1}^{m} d\psi_i \wedge d\phi_i \qquad (7.21)$$

so that $H_\Gamma = \frac{\partial}{\partial z}$.

Let $I' = \{0\} \times [-1/4, 5/4] \subset \overline{W} \times R$ and let V' be a neighborhood of I' in V_I. Let $\delta > 0$ satisfy $\exp(tH_\zeta)(V') \subset V_I$ for all $t < \delta$.

Lemma 7.5: Given $k > 0$, there exists $C > 0$ such that

$$\left.\begin{array}{l} |z \circ B^i - z \circ A^i|,\ |\Gamma \circ B^i - \Gamma \circ A^i| \\ |\psi \circ B^i - \psi \circ A^i|,\ |\phi \circ B^i - \phi \circ A^i| \end{array}\right\} < C'\ \Gamma^K$$

$$\text{for } i < \delta\zeta^{1/2} \quad \text{and } p \in V' .$$

Proof: Since Γ and ζ are both defining functions of ∂V_I, there exist constants M, M' > 0 such that

$$M\zeta < \Gamma < M'\zeta .$$

Since $H_\zeta - H_\Gamma$ vanishes to infinite order at ∂V_I, there exists M" > 0 such that, for example

$$|\frac{d}{dt}(z_0 \exp(tH_\zeta)) - 1| < M''\zeta^k \tag{7.22}$$

Since ζ is constant on the flow lines of H_ζ, (7.22) implies that

$$|z \circ \exp(tH_\zeta) - t - z| < tM'' \zeta^k$$

for all $p \in V'$ where $\exp(tH_\zeta)$ is defined.

This means that for $p \in V'$, $i < \delta\zeta^{-1/2}$,

$$\begin{aligned} |z \circ B^i - z \circ A^i| &= |z \circ \exp(i\zeta^{1/2}H_\zeta) - z + i^{1/2}| \\ &< i\zeta^{1/2}M''\zeta^k + |i\zeta^{1/2} - i^{1/2}| \\ &< C\Gamma^k . \end{aligned}$$

The other coordinates are treated in exactly the same way.

Lemma 7.6: $H - H'$ vanishes to infinite order at $S^*\partial\Omega$.

Proof: It suffices to work locally. Using the coordinates of Lemma 7.5, given $k > 0$, there exists a function $f \in C^\infty(R \times V_I)$ and $\delta > 0$ such that

$$z \circ \exp(tH_\zeta) - z \circ \exp(tH_\Gamma) = \Gamma^k f(t,p)$$

$$t < \delta .$$

Hence

$$z \circ B - z \circ A = \Gamma^{k+1} f(-\zeta^{1/2}, p) + \zeta^{1/2} - \Gamma^{1/2}$$

so that

$$d(z \circ B - z \circ A) = (k + 1)\, \Gamma^k f\, d\Gamma + \Gamma^{k+1} d_p f$$

$$- \frac{1}{2} \zeta^{-1/2} \frac{\partial f}{\partial t} \Gamma^{k+1} + 1/2(\zeta^{-1/2} d\zeta - \Gamma^{1/2} d\Gamma)$$

Each term on the right can be bounded by Γ^k. Hence $B^*dz - A^*dz$ vanishes to infinite order on $S^*\partial\Omega$. Analogous results hold for the functions Γ, ψ_i, ϕ_i. Together with Lemma 7.5, this implies that

$$d(H - H') = A^*\alpha - B^*\alpha$$

vanishes to infinite order at $S^*\partial\Omega$. This implies that $H - H'$ is a C^1 function vanishing to infinite order at $S^*\partial\Omega$.

Lemma 7.7: Given a function $f \in C^\infty(B^*\partial\Omega)$ and $k > 0$, there exists $M, N > 0$ such that when $n > N$,

$$|f \circ B^i \circ \rho_n - f \circ A^i \circ \rho_n| < M\Gamma^k \quad i < n$$

where this inequality is valid for all points in $\lambda(V')$.

Proof: Pull f back to V_I. Since f has bounded derivatives on V_I, there exist constants $K, K' > 0$ such that when $p, p' \in V_I$ and

$$D(p, p') = |z(p) - z(p')| + |\Gamma(p) - \Gamma(p')|$$
$$+ \sum_{i=1}^{m} |\psi_i(p) - \psi_i(p')| + |\phi_i(p) - \phi_i(p')| \qquad (7.23)$$

satisfies $D(p, p') < K'$, then

$$|f(p) - f(p')| < K\, D(p, p') .$$

Since $\lambda(V')$ is a neighborhood of $\overline{O}$, it contains the image of ρ_n for n sufficiently large. Hence ρ_n may be lifted to a path segment in V'. Next, the winding number of the sequence defined by ρ_n, $\rho_n \circ B$, ..., $\rho_n \circ B^{n-1}$ is $N = 1$. Thus, in Lemma 7.5, δ may be chosen so that $n < \delta\ (\Gamma \circ \rho_n)^{-1/2}$ for all n. Hence

$$D(B^i \circ \rho_n, A^i \circ \rho_n) < M\,\Gamma^k \qquad i < n$$

and the estimate follows.

Lemma 7.8: Given k, there exists C such that

$$|H \circ B^i \circ \rho_n - H' \circ A^i \circ \rho_n| < C\,\Gamma^k, \qquad i < n$$

valid for n sufficiently large.

Proof:

$$|H \circ B^i - H' \circ A^i| < |H \circ B^i - H \circ A^i| + |H \circ A^i - H'\,A^i| .$$

By the triangle inequality,

$$|H \circ B^i - H \circ A^i| < |Q \circ B^i - Q \circ A^i| + |Q \circ B^{i+1} - Q \circ BA^i|$$

and Lemma 7.7 implies that both terms on the right may be bounded by Γ^k. Next, by Lemma 7.6,

$$|H - H'| < M\,\Gamma^k .$$

Thus, since Γ is invariant under A,

$$|H \circ A^i - H' \circ A^i| < M\Gamma^k.$$

Lemma 7.9: Given $f \in C^\infty(B^*\partial\Omega)$, $k > 0$, there exists M, $N > 0$ such that when $n > N$,

$$|f \circ A^i \circ \rho_n - f \circ A^i \circ \sigma_n| < M(\Gamma \circ \rho_n)^k \, . \qquad (7.24)$$

As a corollary, this estimate will hold when $f = H'$.

Proof: Using the analog of Prop. 7.5 for the interpolating flows $\exp(tH_\zeta)$ and $\exp(tH_\Gamma)$, it may be shown that the Poincaré map $\overline{\Theta} : \overline{W}_s \to \overline{W}_s$ defined in Section 5 agrees with the Poincaré map $\Theta_\Gamma : \overline{W}_s \to \overline{W}_s$ to infinite order at $\partial\overline{W}_s$. Thus the path $\gamma_s(r)$, defined in Section 5 contacts the orbit cylinder $\mathcal{T}_\Gamma$, to infinite order at $\gamma_s(1)$. Since $\rho_n(s)$ lies on $\gamma_s(s)$, (7.24) holds for $i = 0$. Lifting ρ_n and σ_n to V', this implies that the quantity $D(p, p')$ from equation (7.23) satisfies

$$D(\sigma_n, \rho_n) < M(\Gamma \circ \rho_n)^k \, .$$

But from (7.23) and the fact that $\Gamma \circ \sigma_n = \Gamma \circ \rho_n$,

$$D(A^i \circ \rho_n, A^i \circ \sigma_n) = D(\rho_n, \sigma_n) \, .$$

Thus the estimate for f follows by pulling f back to V_I and applying the argument of Lemma 7.7. The extension to H' follows from the triangle inequality and Lemma 7.6 as in Lemma 7.9.

Together, Lemmas 7.8 and 7.9 imply

$$\sum_{i=1}^{n-1} H \circ B^i \circ \rho_n = \sum_{i=0}^{n-1} H' \circ A^i \circ \rho_n$$

$$= \sum_{i=0}^{n-1} H' \circ A^i \circ \sigma_n \qquad (7.25)$$

modulo errors of order $(\Gamma)^k$.

From this point, the analysis may be restricted to $S^1 \times [0, \varepsilon)$. s is defined so as to restrict to the arclength coordinate on $S^1 \times \{0\}$. So on $S^1 \times [0, \varepsilon)$, (s, Γ) will form a system of coordinates. For $(s, \Gamma) \in S^1 \times [0, \varepsilon)$, define $z(s,\Gamma)$ to be the flow time along H_Γ from $(0, \Gamma)$ to (s,Γ). Where z is well defined, z and Γ form a set of Darboux coordinates, with $\omega = dz \wedge d\Gamma$. Unfortunately, the form dz is only well defined on $0 \leq s < 1$. Set

$$I(\Gamma) = z(1,\Gamma).$$

Then $d(z/I)$ is a globally defined one form on $S^1 \times [0, \varepsilon)$. Set

$$J(\Gamma) = \int_0^\Gamma I(t)\, dt$$

and define the 1 form β' by

$$\beta' = J(\Gamma)\, d(z/I) + L_\infty\, ds . \qquad (7.25)$$

Since $\omega = dz \wedge d\Gamma$, $d\beta' = -\omega$. So let F' be defined by the conditions

$$dF' = \beta' - A^*\beta'$$

$$F' \Big|_{\Gamma = 0} = 0$$

From (7.21)

$$F' = 2/3\ \Gamma^{3/2} + \frac{J(\Gamma)}{I(\Gamma)}\,\Gamma^{1/2} + s \circ A - s\ .$$

Next since

$$\int_{S^1 x\{0\}} \alpha = \int_{\bar{\mathcal{O}}} \alpha = L_\infty \quad .$$

$$\int_{S^1 x\{0\}} \beta' = \int_{S^1} L_\infty ds = L_\infty \quad .$$

$\alpha - \beta'$ is exact. So by the argument of Lemma 7.2, there exists $G \in C^\infty(S^1 \times [0,\varepsilon))$ such that

$$H' = F' + G \circ A - G\ .$$

So

$$\sum_{i=0}^{n-1} H' \circ A^i(s,\Gamma) = n[(2/3)\Gamma^{3/2} - \frac{J(\Gamma)}{I(\Gamma)}\,\Gamma^{1/2}] + s \circ A^n - s + G \circ A^n - G\ .$$

Now, let $(x_0, \xi_0) \in \mathcal{J}_n$ and let (y_0, η_0) be defined by (7.18), then

Lemma 7.10: Given $k > 0$, there exist C, $N > 0$ such that for $n > N$,

$$|(G \circ A^n - G)(y_0, \eta_0)| < C\,\Gamma^k.$$

Proof: Extend G to a smooth function on B, then the result follows from Lemmas 7.7 and 7.9, and the fact that (x_0, ξ_0) is a periodic point of B with period n.

Similarly, given the hypotheses of Lemma 7.10

$$|(s \circ A^n - s)(y_0, \eta_0) - L_\infty| < C\,\Gamma^k$$

so combining (7.22) with these last results,

$$L_n(\mathcal{J}_n) = n[(2/3)\Gamma^{3/2} - \frac{J(\Gamma)}{I(\Gamma)}\,\Gamma^{1/2}] + L_\infty \quad \text{modulo } \Gamma^k \tag{7.27}$$

where Γ is evaluated on any point in $\mathcal{J}_n$.

While the functions I and J depend on the choice of Γ, their Taylor series at $\Gamma = 0$ will be well defined. Let $\overline{W}_0$ be the transversal section s = 0 defined in Section 5. For $p \in \overline{W}_0$, let $\overline{I}$ be the return time of p, i.e., the time required to flow along H_Γ from $p \in \overline{W}_0$ to $\Theta(p) \in \overline{W}_0$. The function I in Equation (7.21) is precisely the pull back of $\overline{I}$ to $S^1 \times [0, \varepsilon)$. In fact, $I(\Gamma)$ is the period of the H_Γ orbit defined by $\Gamma = t$.

The estimates in Lemma 7.7 may be used to show that a different choice of Γ, say Γ', will change $\overline{I}$ by terms vanishing to infinite order at $\partial\overline{W}_0$. Also, one may show, along the lines of Lemma 7.9, that $\mathcal{T}_\Gamma$ and $\mathcal{T}_{\Gamma'}$ are tangent to infinite order at $\overline{\mathcal{O}}$. Consequently, any ambiguity in the function I will vanish to infinite order at $\Gamma = 0$. In particular, the Taylor coefficients

$$I_{k+1} = \frac{d^k\, I}{d\,\Gamma^k}$$

are defined independently of the choice of Γ.

By examining I in more detail, one may show that

Lemma 7.11: Given (x_0, ξ_0),

$$I(\Gamma(x_0, \xi_0)) = n(\Gamma(x_0, \xi_0))^{1/2} \quad (7.26)$$

$$\text{modulo } \Gamma^k \qquad k > 0 .$$

Proof: By lifting to V_I, define the return time $\tilde{I}$: $\overline{W}_0 \to R$ for the interpolating flow $\exp(tH_\zeta)$. By Equation 5.

$$\tilde{I}(x_0, \xi_0) = n(\zeta(x_0, \xi_0)) \quad (7.27)$$

$\overline{I} - \tilde{I}$ will vanish to infinite order at $\partial\overline{W}_0$ so (7.27) may be approximated by (7.26).

Lemma 7.11 implies that $I(\Gamma(x_0, \xi_0))$, $(x_0, \xi_0) \in \mathcal{J}_n$ has a well defined asymptotic expansion in powers of n^{-2}. Substituting this expansion into (7.27) will prove (7.14) for the case $\Omega \subset R^{m+1}$.

APPENDIX 1

CALCULATION OF THE TAYLOR SERIES OF ζ

Let $p \subset S^*\partial\Omega$. Then there exists $U \subset B^*\partial\Omega$, $P \subset U$ and an interpolating Hamiltonian $\zeta \in C^\infty(U)$. Since $r(x,\xi) = |\xi|$ is a defining function for the hypersurface $S'\partial\Omega$, there exist functions $\zeta_1, \zeta_2, \ldots$ $C^\infty(U)$, constant on the radii of $B^*\partial\Omega$, such that

$$\zeta = (1 - r)\zeta_1 + \ldots + (1 - r)^k\zeta_k + R_{k+1}$$

where R_{k+1} is a function vanishing to order k+1 at $S^*\partial\Omega$. The uniqueness properties of interpolating Hamiltonians imply that the ζ_k can be identified with globally defined smooth functions on $S^*\partial\Omega$. In this section, it will be shown that for $(x,\xi) \in S^*\partial\Omega$,

$$\zeta_1 = 2(\pi_x(\xi_0, \xi_0))^{-2/3} ,$$

where π denotes the second fundamental form of the hypersurface $\partial\Omega$.

Let $(x_0, \xi_0) \in S^*\partial\Omega$. This point defines an oriented geodesic $\mathcal{O}$ in $\partial\Omega$ and a bicharacteristic, $\overline{\mathcal{O}}$, in $S^*\partial\Omega$. Using the Riemannian structure on $\partial\Omega$, define an orthogonal section, $\overline{W}_0 \subset \partial\Omega$, to $\mathcal{O}$ at x_0 (use the exponential map). W_0 induces a transversal section, $\overline{W}_0$, to $\overline{\mathcal{O}}$ in $B^*\partial\Omega$.

Let $q_i, \ldots, q_{2m}$ be coordinates on the submanifold $\partial\overline{W}_0 = \overline{W}_0 \cap S^*\partial\Omega$. Extend these to a neighborhood of (x_0, ξ_0) in $B^*\partial\Omega$ by requiring them to be constant on the bicharacteristics of $S^*\partial\Omega$, and constant on the radii of the fibers in $B^*\partial\Omega$. Take s to be the arclength coordinate along the bicharacteristics of $S^*\partial\Omega$ extended to be constant in the radial direction and normalized by the condition:

$$s\big|_{\overline{W}_0} = 0$$

Then (q_i, s, r) form a local coordinate system of $B^*\partial\Omega$ in a neighborhood of (x_0, ξ_0).

Let $\alpha(\sigma) = (x_0, \sigma \circ \xi_0)$, $\sigma < 1$. In coordinates α is given by

$$r(\sigma) = \sigma \; ; \quad (s, q_i) = 0 \quad .$$

Let $\beta(\sigma) = B \circ \alpha(\sigma)$. Calculate $s \circ \beta(\sigma)$ using the fact that $B = \exp(-\zeta^{1/2}H_\zeta)$.

$$s \circ \beta(\sigma) = s - \zeta^{1/2}H_\zeta s + 0(1 - \sigma)$$

where the terms on the right are evaluated at $\alpha(\sigma)$.

In terms of ζ_1,

$$(\zeta)^{1/2} = (1 - r)^{1/2}(\zeta_1)^{1/2} + 0(1 - r) \; .$$

Next since

$$d\zeta = - \zeta_1 dr + 0(1 - r) \; ,$$

$$H_\zeta = -\zeta_1 H_r + 0(1 - r)$$

$$= - \zeta_1 \frac{\partial}{\partial s} + 0(1 - r) \; .$$

Hence

$$H_\zeta s = -\zeta_1 + 0(1 - r) \; .$$

Combining these formulas gives

$$s \circ \beta(\sigma) = (1 - \sigma)^{1/2}(\zeta_1)^{3/2} + 0(1 - \sigma) \; .$$

Alternatively, $s \circ \beta(\sigma)$ may be calculated in terms of the curvature of $\partial\Omega$. Use the Riemannian metric to introduce a normal fibration of $\partial\Omega$ over $\mathcal{O}$ in a neighborhood of x_0. Let S measure the arclength along $\mathcal{O}$ with $S(x_0) = 0$. Extend S to a neighborhood of x_0 by requiring that S be constant on the fibers of the normal fibration.

Let $\delta(\sigma)$ be the projection of $\beta(\sigma)$ onto $\partial\Omega$. Let $T(\sigma)$ be the arclength along δ from x_0 to $\delta(\sigma)$. δ is a planar curve, and $T(\sigma)$ has an expansion in powers of $(1 - \sigma)^{1/2}$ with coefficients determined by the curvature of . It will be shown here that:

$$T(\sigma) - S \circ \delta(\sigma) = 0(1 - \sigma),$$

$$S \circ \delta(\sigma) - s \circ \beta(\sigma) = 0(1 - \sigma)$$

so that ζ_1 can be determined by the expansion of $T(\sigma)$.

Proposition A1.1: $T(\sigma) - S \circ \delta(\sigma) = 0(1 - \sigma)$

Proof: The curve δ is tangent to the geodesic $\mathcal{O}$ up to second order at x_0. Hence S may be approximated on δ by the arclength T, measured along δ with errors that are third order in T.

$$S \circ \delta(\sigma) = T(\sigma) + 0(T^3)$$

$$= T(\sigma) + 0(1 - \sigma) .$$

Lemma A1.2: The curve β may be smoothly parameterized by the coordinate s. Moreover, if the bicharacteristic $\mathcal{O}$ is parameterized by s, then the two curves have second order contact at s = 0.

Proof: There exist coordinates $\phi_1, \ldots, \phi_{2m}$, z, such that B is given by $(\phi_i, z, \zeta) \to (\phi_i, z + \zeta^{1/2}, \zeta)$. The path α is defined by

$$z = 0;\ \phi = \phi_i(\zeta) \text{ with } \phi_i(\zeta) = O(\zeta)\ .$$

And $\overline{\mathcal{O}}$ is defined by

$$\phi_i,\ \zeta = 0\ .$$

In these coordinates, β is defined by

$$z = \zeta^{1/2};\quad \phi_i = \phi_i(\zeta)$$

or equivalently,

$$\phi_i = \phi_i(z^2)$$

$$\zeta = z$$

Proposition A1.3: $S \circ \delta(\sigma) - s \circ \beta(\sigma) = O(1 - \sigma)$.

Proof: A1.3 follows from A1.2 and the following:

Regard S as a function on $B^*\partial\Omega$. The level sets of S and s define 1 parameter families of transversal sections of $\overline{\mathcal{O}}$ in $S^*\partial\Omega$, W_s and W_s' respectively. Let $p(s) \in \overline{\mathcal{O}}$ denote the point of intersection of $\mathcal{O}$ and W_s.

Claim: $T_pW_s = T_pW_s'$ (A1.1)

Proof: The Riemannian metric may be used to identify $T^*\partial\Omega$ with $T\partial\Omega$. In this identification, the symplectic form may be written in terms of the

the metric as

$$\omega(X, Y) = \langle X_n, Y_v \rangle - \langle X_v, Y_n \rangle \quad \text{(A1.2)}$$

where X_V and X_h denote the vertical and horizontal components of X.

Equation (A1.2) may be used to identify $T_{p(s)} W_s$ and $T_{p(s)} W'$ as the symplectic orthocomplement of the space

$$V = \{\frac{\partial}{\partial s} , \frac{\partial}{\partial r}\}$$

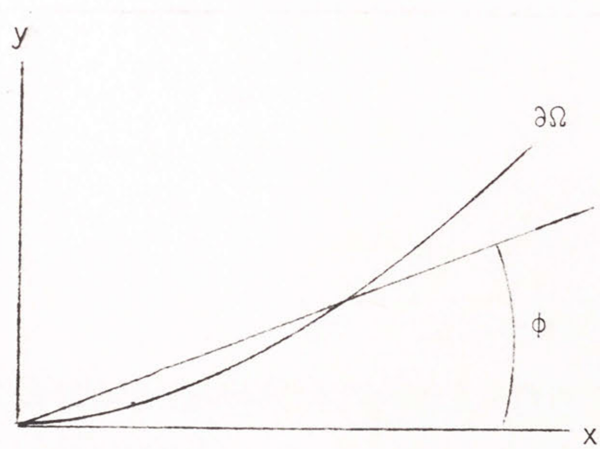
Equation (A1.1) implies that

$$S(q_i, s) = s + \sum O(q_i^2) \quad \text{(A1.3)}$$

the result now follows from Lemma A1.2.

Combining equations, conclude that

$$T(\sigma) = (1 - \sigma)^{1/2}(\zeta_1)^{3/2} + O(1 - \sigma)$$



$\cos\phi = \sigma$

Figure A1.3

Parameterize the planar curve δ by its arclength T. Let $K(T)$ denote the curvature of δ as a function of arclength. Set

$$\theta(T) = \int_0^T K(t)\, dt\ .$$

Then

$$X(T) = \int_0^T \cos(\theta(t))\, dt$$

$$Y(T) = \int_0^T \sin(\theta(t))\, dt\ .$$

Compute the low order terms of X, Y:

$$\cos(\theta(T)) = 1 - \frac{K^2T^2}{2} + 0(T^3)$$

$$X(T) = T = \frac{K^2T^3}{6} + 0(T^4)$$

$$\sin(\theta(T)) = KT + 0(T^2)$$

$$Y(T) = \frac{KT^2}{2} + 0(T^3)$$

$$\tan\phi = \frac{Y(T)}{X(T)} = (1/2)\ KT + 0(T^2)$$

$$= (1/2)(1 - \sigma)^{1/2}\ K(\zeta_1)^{3/2} + 0(1 - \sigma)$$

while

$$\tan\phi = \frac{(1 - \sigma^2)^{1/2}}{\sigma} = (2)^{1/2}(1 - \sigma)^{1/2} + 0(1 - \sigma)$$

Hence

$$\zeta_1 = 2(K)^{-2/3}$$

Recall that if π denotes the fundamental form of $\partial\Omega$ at x_0. Then K is precisely $K = \pi_{x_0}(\xi_0, \xi_0)$. Thus we have shown

$$\zeta_1(x_0, \xi_0) = 2(\pi_{x_0}(\xi_0, \xi_0))^{-2/3}$$

APPENDIX 2

THE TANGENT SPACE TO THE ORBIT CYLINDER

Consider the orbit cylinder

$$\mathcal{T}_\Gamma : S^1 \times [0, \varepsilon) \to B^*\partial\Omega$$

defined by the approximate interpolating Hamiltonian Γ. As an application of the computation of Appendix 1, in this section, we will compute the tangent space of this orbit cylinder for points p on the closed bicharacteristic $\overline{\mathcal{O}}$.

In particular, if $p \in \overline{\mathcal{O}}$, then $T_p(\mathcal{T}_\Gamma)$ will be spanned by S, the tangent vector to $\overline{\mathcal{O}}$ and a second vector transversal to $S^*\partial\Omega$. This second vector may be taken to be R + v, where R is the radial direction of the fibers, and $v \in T_p\, S^*\partial\Omega$.

Let $r(x,\xi) = |\xi|$. Then the Hamiltonian vector field H_r generates the geodesic flow on $B^*\partial\Omega$. Let τ_G and τ_B denote the periods of the closed bicharacteristic $\overline{\mathcal{O}}$ considered as a closed orbit of $\exp(tH_r)$ and $\exp(tH_\Gamma)$ respectively. Let G and F denote the fixed time maps

$$G = \exp(\tau_G\, H_r)$$

$$F = \exp(\tau_B\, H_\Gamma) \quad .$$

The vector, v, will be determined, modulo S by the equation

$$(1 - DG)v = w \qquad \text{(A2.1)}$$

where $w \in T_p S^*\partial\Omega$ is a tangent vector determined, modulo S, by the equations

$$\omega(w, u) = u \lrcorner \beta \qquad \text{for all } u \in T_p(S^*\partial\Omega)$$

and

$$\beta = \zeta_1^{-1} \int_{-\tau_G}^{0} [\exp(tH_r)^* \, d\zeta_1] dt$$

If $p \in \overline{\mathcal{O}}$, then p is a fixed point of both G and F. Consider the symplectic linear map

$$DG : T_p(B^*\partial\Omega) \to T_p^*(B^*\partial\Omega)$$

and let V be the generalized eigenspace associated to 1. The non-degeneracy condition on $\overline{\mathcal{O}}$ implies that V is precisely two dimensional (see [AM1], pages 524, 573-576). Indeed, V is readily identified as

$$V = \text{span}\{R, S\}$$

As in Proposition A1.3, let $V^\perp$ denote the symplectic orthocomplement of V. $V^\perp$ is invariant under DG and since the restriction of ω to V is non-degenerate (R is transversal to $S^*\partial\Omega$), $V \cap V^\perp = 0$ and $T_p(B^*\partial\Omega) = V + V^\perp$. In addition, the non-degeneracy assumption implies that

$$(Id - DG) : V^\perp \to V^\perp$$

is invertible.

Analogous considerations apply to the map $DF : T_p(B^*\partial\Omega) \to T^*(B^*\partial\Omega)$. The corresponding eigenspace V' is two dimensional and is spanned by ζ and some vector transversal to $S^*\partial\Omega$. Most importantly, V' is $T_p(\mathcal{T}_\Gamma)$.

Proposition A2.1: $V' = \text{span}\{S, R + v\}$ where $v \in V^\perp$ is defined by the equations

$$DF(R) = w \bmod V, \quad w \in V^\perp$$

$$v = (1 - DG)^{-1} w .$$

Proof: Since r,Γ are defining functions of $S^*\partial\Omega$, both DF and DG restrict to maps $T_p(S^*\partial\Omega) \to T_p(S^*\partial\Omega)$. Moreover, because both H_r and H_Γ flow along the characteristics of $S^*\partial\Omega$,

$$DF = DG \mod S \qquad (\text{on } T_p(S^*\partial\Omega)) .$$

Claim: $DF(R) = R + w \bmod S$, where $w \in V^\perp$

Proof: $$0 \neq \omega(S, R) = (DF(S), DF(R))$$

$$= (S, DF(R)) .$$

Hence $R - DF(R)$ lies in the symplectic orthocomplement of S, i.e. $T_p(S^*\partial\Omega)$.

Since $T_p(B^*\partial\Omega) = V + V^\perp$, there exists a unique $v \in V^\perp$ such that $V' = \{R + v, S\}$, v satisfies

$$DF(R + v) = R + w + DG(v) \qquad \text{mod } S$$

and

$$DF(R + v) = R + v \qquad \text{mod } S$$

Solving these two equations for v gives

$$v = (1 - DG)^{-1} w$$

Proposition A2.3: Identification of w.

Let $u \in V^{\perp}$, then

$$0 = \omega(R, u) = \omega(DF(R), DF(u))$$

$$= \omega(R, DF(u)) + \omega(w, DG(u))$$

Since ω is non-degenerate on $V^{\perp}$, w is defined by

$$\omega(w, DG(u)) = -\omega(R, DF(u))$$

where $-\omega(R, DF(u))$ is the S component of $DF(u)$. Given that $H_{\Gamma} = -\zeta_1 H_r$ on $S^*\partial\Omega$, this quantity is given by the following equation:

$$-\omega(R, DF(u)) = \zeta_1^{-1} \int_0^{\tau_G} (\exp(tH_r)_* u \lrcorner d\zeta_1) \, dt$$

PART II

THE OBSTACLE PROBLEM

In the obstacle problem, consider $\Omega \subset R^3$, a smoothly bounded, compact region in R^3. The objects of interest are the length minimizing paths in R^3 which bypass the obstacle Ω. It is clear that these paths will consist of geodesic segments in the boundary surface $\partial\Omega$ joined by line segments in R^3 tangent to the geodesic segments at their endpoints.

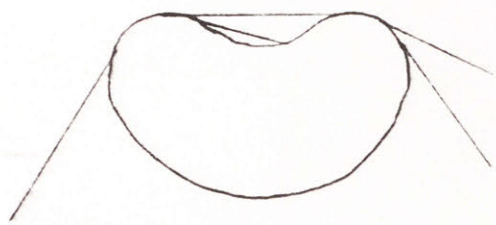


Figure II.1

This is a variational problem with constraints that may be analyzed by analogy with the variational problem of finding length minimizing paths in a Riemannian manifold. Let (M, < >) be a Riemannian manifold. The paths in M that are length extremal are geodesics. In symplectic geometry, the space of oriented geodesic segments with initial point x_0 may be identified with a Lagrangian submanifold $\Lambda_0 \subset T^*M$.

Analogously, the corresponding space of extremal bypassing paths in $R^3 \backslash \Omega$ may be identified with a Lagrangian submanifold in T^*R^3. In the obstacle problem, however, this Lagrangian submanifold will have singularities. These singularities may be analyzed using the normal forms for nested hypersurfaces in a symplectic manifold.

Section 8: Characteristics and the Space of Tangent Rays

Let (M, < , >) be a Riemannian manifold. The metric defines a hypersurface $S^*M \subset T^*M$ whose bicharacteristics project to geodesics on M. In the obstacle problem, this correspondence between bicharacteristics and geodesics may be used to describe extremal paths in $R^3\backslash\Omega$.

Let $\mathcal{O} \subset \partial\Omega$ be an oriented geodesic segment. $\mathcal{O}$ will determine a unique bicharacteristic $\overline{\mathcal{O}} \subset S^*\partial\Omega \subset T^*\partial\Omega$. By using the Riemannian metric to identify

$$T^*R^3 \cong TR^3$$

$$T^*\partial\Omega \cong T\partial\Omega$$

there is a natural inclusion $T^*\partial\Omega \to T^*R^3$.

Use this inclusion to map $S^*\partial\Omega$ into S^*R^3. Continue $\overline{\mathcal{O}}$ into S^*R^3 along the backward and forward rays of the S^*R^3 bicharacteristics passing through the endpoints of $\overline{\mathcal{O}}$. The projection of this extended bicharacteristic is a length extremal path in $R^3\backslash\Omega$.

The S^*R^3 bicharacteristics passing through $S^*\partial\Omega$ are called tangent rays (Arnold [A1]). The space of tangent rays, Δ, will be a hypersurface with singularities in the manifold of S^*R^3 bicharacteristics, M_E. Given the identification, of M_E with oriented lines in R^3, Δ will be shown to be the space of lines tangent to $\partial\Omega$ while the singular points will be the lines having higher order contact.

Remark: The inclusion $T^*\partial\Omega \to T^*R^3$ is not only well defined for any Riemannian manifold M with submanifold N, but

Proposition 8.1: Let ω, $\omega_{\partial\Omega}$ denote the canonical symplectic forms on T^*R^3 and $T^*\partial\Omega$ respectively. Then

$$i^*\omega = \omega_{\partial\Omega}$$

where

$$i : T^*\partial\Omega \to T^*R^3$$

Proof: Given any Riemannian manifold (M, < , >), the symplectic structure defined on TR^3 by the identification $T^*R^3 = TR^3$ is given in terms of < , > by

$$\omega(X, Y) = \langle X_h, Y_v\rangle - \langle X_v, Y_h\rangle \qquad (8.1)$$

where X_v, X_h denote the vertical and horizontal components of $X \in T_p(TR^3)$. The proposition now follows from the fact that $\partial\Omega$ carries the metric induced by R^3.

Corollary 8.2: If ω_E is the canonical symplectic structure on M_E then,

$$\pi_E^*\omega_E = i^*\omega_{\partial\Omega}$$

where $\pi_E : S^*\partial\Omega \to M_E$ is the restriction of the map $\pi_E : S^*R^3 \to M_E$ and $i : S^*\partial\Omega \to T^*\partial\Omega$ is the inclusion.

Proof: This follows from the fact that if $j : S^*R^3 \to T^*R^3$ is the inclusion then ω_E is defined by

$$\pi_E^*\omega_E = j^*\omega$$

where ω is the symplectic structure of T^*R^3.

Note that these results also hold for general Riemannian manifolds.

Section 9

SINGULARITIES OF THE SPACE OF TANGENT RAYS

From Section 2, let (X,ω) be a symplectic manifold, (F,G) transversally intersecting hypersurfaces in X with defining functions f, g. Setting $J = F \cap G$, the set K, of glancing points was defined as

$$K = \{x \in J : \{f,g\}(x) = 0\}$$

The projections

$$\pi_F : J \to M_F$$
$$\pi_G : J \to M_G .$$

were analyzed around non-degenerate glancing points. These points were defined by the conditions

$$x \in K$$
$$\{f, \{f, g\}\}(x) \neq 0 \qquad (9.5)$$
$$\{g, \{f, g\}\}(x) \neq 0 \qquad (9.6)$$

In the obstacle problem, the region Ω is not assumed to be strictly convex. Hence, the intersecting hypersurfaces $(S^*R^3, T^*_{\partial\Omega} R^3)$ will have degenerate glancing points. Generically, for $\Omega \subset R^3$, the obstacle problem will require consideration of the cases where (9.5) is relaxed to

$$\{f, \{f,g\}\}(x) = 0$$
$$\{f, \{f, \{f,g\}\}\}(x) \neq 0 \qquad (9.5')$$

and

$$\{f, \{f,g\}\}(x) = \{f, \{f, \{f,g\}\}\}(x) = 0$$

$$i_K^* (d\{f, \{f,g\}\}) \neq 0$$

$$\{f, \{f, \{f, \{f,g\}\}\}\}(x) \neq 0 \qquad (9.5'')$$

Geometrically, these conditions correspond to the cases where the bicharacteristic of F through x, ℓx, contacts G to second and third order at x. In each case, condition (9.6) will continue to imply that $\pi_G : J \to M_G$ has a simple fold singularity at $x \in K$ and that K is a smooth hypersurface of J in a neighborhood of x. For the projection π_F, condition (9.5') implies that $\pi_F : J \to M_F$ has a cusp $(S_{1,1,0})$ singularity at x, while (9.5") implies that π_F has a swallowtail singularity $(S_{1,1,1,0})$ at x.

Definition: Points satisfying (9.4), (9.5'), (9.6) are called inflection points while points satisfying (9.4), (9.5"), (9.6) are called swallowtail points.

Melrose has shown [M2] that there are formal obstructions to the existence of a normal form of the pair (F, G) around a cusp or swallowtail point. Fortunately, in the obstacle problem, one may examine the space of tangent rays, Δ, by studying the nested hypersurfaces $(S^*R^3, S^*_{\partial\Omega}R^3)$. In the general setting, this corresponds to studying the nested pair (F, J). For a pair of transversally intersecting hypersurfaces, (F, G), the nested pair (F, J) admits the following normal forms:

Theorem 9.2: If $x \in K$, then there exist Darboux coordinates (p_i, q_i) such that

if x is a non-degenerate glancing point of (F, G),

$$F = \{q_0, 0\}$$

$$J = \{p_0^2 + p_1 = 0; q_0 = 0\}$$

if x is an inflection point of (F, G):

$$F = \{q_0 = 0\}$$

$$J = \{q_0 = 0; p_0^3 + p_0 p_1 + q_1 = 0\}$$

finally, if x is a swallowtain point of (F, G) and

$$F = \{q_0 = 0\}$$

$$J = \{q_0 = 0; p_0^4 + p_0^2 q_2 + p_0 q_1 + p_1 = 0\}$$

For non-degenerate glancing points, Theorem 9.2 was the first step in classifying the pair (F, G) in [M1]. Inflection points were analyzed in the contact category in [M2]. Finally, the swallowtail case was first considered by Arnol'd in [A1], who proved Theorem 9.2 over the category of formal power series. This result was extended to the category of smooth diffeomorphisms by Melrose and the author in [MMag].

These normal forms are used to analyze the singularities of Δ, the space of tangent rays in M_E. Recall that Δ is the image of $S^*\partial\Omega$ under the projection

$$\pi_E : S^*R^3 \to M_E$$

In Theorem 9.3 below, the space of tangent rays, Δ, is identified as the space of lines tangent to $\partial\Omega$. A tangent ray, ℓ, is said to be an asymptotic or biasymptotic tangent ray if ℓ is tangent to $\partial\Omega$ to second or third order respectively.

Theorem 9.3: $(x, \xi) \in S^*_{\partial\Omega}R^3$ is a glancing point of $(S^*R^3, T^*_{\partial\Omega}R^3)$ iff $(x, \xi) \in S^*\partial\Omega$. Hence, (x,ξ) is a glancing point iff the oriented line $\ell = \pi_E(x,\xi)$ is tangent to $\partial\Omega$ at x. (x, ξ) is a non-degenerate glancing point iff ℓ is simply tangent to $\partial\Omega$, an inflectional point if ℓ is an asymptotic ray, and a swallowtail point if ℓ is a biasymptotic ray and $\partial\Omega$ is in general position.

Proof: Let $(x, \xi) \in S^*R^3$. Locally, about x, $\partial\Omega$ may be considered as a graph over its tangent plane. There will exist a function $\phi \in C^\infty(R^2)$ with $\phi(0) = 0$ and $d\phi(0) = 0$ such that $\partial\Omega$ is defined in R^3 by

$$q_3 = \phi(q_1, q_2)$$

for some choice of orthogonal linear coordinates on R^3. The q_i's induce canonical Darboux coordinates (q_i, p_i) on T^*R^3. In these coordinates,

$$G = q_3 - \phi(q_1, q_2)$$

$$E(x,\xi) = 1/2|\xi| = 1/2 \sum p_i^2$$

are defining functions for $T^*_{\partial\Omega}R^3$ and S^*R^3 respectively. Calculate Hamiltonian vector fields for E and G

$$H_E = \sum_{i=1}^{3} p_i \frac{\partial}{\partial q_i}$$

$$H_G = - \frac{\partial}{\partial p_3} + \sum_{i=1}^{2} \frac{\partial \phi}{\partial q_i} \frac{\partial}{\partial p_i}$$

and consider the iterated Poisson bracket:

$$\begin{aligned} \{E, \{E, \ldots, \{E,G\}\}\ldots\} (x, \xi) &= (H_E{}^K G)(x, \xi) \\ &= \frac{d^F}{dt^F} G(x + t\ ) \Big|_{t=0} \end{aligned} \tag{9.9}$$

Since G is a defining function of $\partial\Omega$ in R^3, from (9.9), k = 1, it follows that ℓ is tangent to $\partial\Omega$ iff $(x,\xi) \in S^*\partial\Omega$ and that $S^*\partial\Omega$ is precisely the set of glancing points of $\{S^*R^3, T^*_{\partial\Omega}R^3\}$. Note also, that Equations (7.5) and (7.5') are satisfied when ℓ is a simply tangent or asymptotic ray respectively. If ℓ is a biasymptotic ray, then

$$\{E, \{E,G\}\} = \{E, \{E, \{E,G\}\}\} = 0$$

$$\{E, \{E, \{E, \{E,G\}\}\}\} = 0$$

but $i^*_{S^*\partial\Omega}(d\{E, \{E,G\}\}) \neq 0$ may or may not hold. In general, it must be assumed as an additional hypothesis. Hence, the assumption of general position.

Next, let $(x,\xi) \in S^*\partial\Omega$. ξ may be assumed to be $\xi = dq_1$. Then,

$$\{G, \{E,G\}\}(x,\xi) = -1$$

Consequently, (9.6) is satisfied at every glancing point of $\{S^*R^3, T^*_{\partial\Omega}R^3\}$.

Remark: A path, ρ, in $R^3\backslash\Omega$ is said to be locally length minimizing if every finite segment of ρ is length minimizing.

Corollary 9.3: A locally length minimizing path cannot contain an asymptotic ray.

Proof: From Theorem 9.2, an asymptotic ray is the image of an inflectional point $(y, \xi) \in S^*\partial\Omega$. Equations (9.5) and (9.9) imply that the curvature of $\partial\Omega$ in the direction ξ is concave near the point y. Thus, if the curve ρ leaves $\partial\Omega$ along an asymptotic ray, then it will be possible to define comparison curves having shorter length.

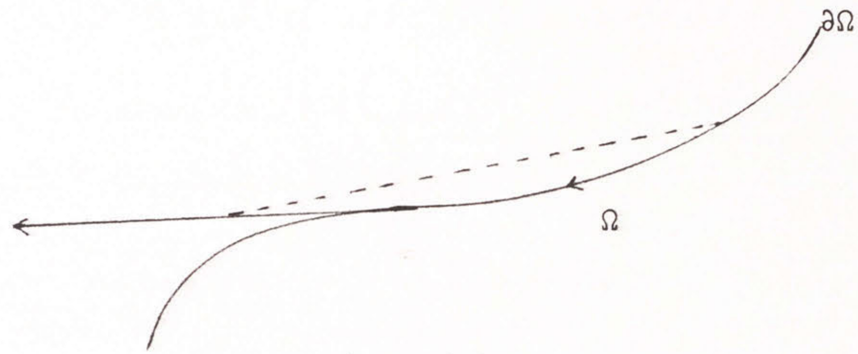


Figure 9.1

The normal forms in Theorem 9.2 imply that

Theorem 9.4: Let $(x, \xi) \in S^*\partial\Omega$ and set $\ell = \pi_E(x,\xi) \in M_E$. If (x,ξ) is a non-degenerate glancing point of the pair $(S^*R^3, T^*_{\partial\Omega}R^3)$, then $\Delta \subset M_E$ is a smooth hypersurface in a neighborhood of ℓ and there exist Darboux coordinates (q_1, q_2, p_1, p_2) defined in a neighborhood of $\ell \in M_E$ such that

$$\Delta = \{q_1 = 0\}$$

Similarly, if $(x, \xi) \in S^*\partial\Omega$ is a cusp point, then π_E has a cusp singularity at ℓ and there exists Darboux coordinates (q_1, q_2, p_1, p_2) such that

$$\Delta = \{p_1, q_1 ; p_0^3 + p_1 p_0 + q_1 \text{ has a double root for some } p_0.\}$$

Finally, if (x,ξ) $S^*\partial\Omega$ is a swallowtail point, then π_E has a swallowtail singularity. Provided that the pair $\{S^*R^3, T^*_{\partial\Omega}R^3\}$ is in general position, there exist Darboux coordinates (q_1, q_2, p_1, p_2) such that

$$\Delta = \{q_1, q_2, p_1, p_2 : p_0^4 + p_0^2 p_1 + q_2 p_0 + p_2 \text{ has a real double root.}\}$$

Proof: These normal forms follow directly from the normal forms of Theorem 9.9. In each case, the projection $\pi_E : S^*R^3 \to M_E$ is given by

$$(q_1, q_2, p_0, p_1, p_2) \to (q_1, q_2, p_1, p_2)$$

Since q_1, q_2, p_1, p_2 are constant on the bicharacteristics of S^*R^3 and the pull back of ω to S^*R^3 is given locally as

$$dq_1 \wedge dp_1 + dq_2 \wedge dp_2$$

q_1, q_2, p_1, p_2 form Darboux coordinates in M_E.

Section 10

THE LAGRANGIAN SUBMANIFOLD OF FALLING RAYS

Definition: Let (X,ω) be a symplectic manifold. A submanifold $i : L \to X$ is Lagrangian if $\dim L = 1/2 \dim X$ and $i^*\omega = 0$.

Let $x_0 \in R^3\backslash\Omega$. Define $\Gamma x_0 \subset S^*R^3$ as

$$\Gamma x_0 = \{(x\,,\ \xi) \in T^*R^3 : |\xi| = 1;\ x = x_0 + t\xi,\quad t \in R\}$$

The tangent rays passing through x_0 may be identified with the image under π_E of $\Gamma x_0 \cap S^*\partial\Omega$.

Proposition 3.1: Let $\ell \in \Delta \subset M_E$ be a tangent ray touching $\partial\Omega$ at $y_0 \in \partial\Omega$ and passing through x_0. Then there exists $\xi_0 \in S^*_{y_0}\partial\Omega$ with $\ell = \pi_E(y_0, \xi_0)$. If ℓ is a regular point of Δ, then Γx_0 is transversal to $S^*\partial\Omega$ at (y_0, ξ_0). That is, $\Gamma x_0 \cap S^*\partial\Omega$ is a smooth 1-dimensional submanifold of $S^*\partial\Omega$ in a neighborhood of (y_0, n_0).

Proof: Let π ; $T^*R^3 \to R^3$ be the canonical projection. Note that for $(x,\xi) \in \Gamma x_0$, $x \neq x_0$, the projection $\pi : \Gamma x_0 \to R^3$ is regular. This implies that $S^*_{\partial\Omega}R^3 \pitchfork \Gamma x_0$ at (y_0, ξ_0).

Next, since $\ell \in \Delta$ is a regular point, (y_0, ξ_0) is a non-degenerate glancing point of $(S^*R^3, T^*_{\partial\Omega}R^3)$. Let $\overline{\ell}$ be the S^*R^3 bicharacteristic passing through (y_0, ξ_0). Then $\overline{\ell}$ is contained in Γx_0, $\overline{\ell}$ is tangent to $S^*_{\partial\Omega}R^3$, but, by the non-degeneracy assumption, $\overline{\ell}$ is not tangent to $S^*\partial\Omega$. Thus

$$T_{(y_0,\ \xi_0)}(S^*_{\partial\Omega}R^3) \subset T_{(y_0,\ \xi_0)}(\Gamma x_0) + T_{(y_0,\ \xi_0)}(S^*\partial\Omega)$$

so that $\Gamma x_0 \pitchfork S^*\partial\Omega$ at (y_0, n_0) in S^*R^3.

Corollary 3.2: Since $\pi : \Gamma x_0 \to R^3$ is regular at (y_0, ξ_0), the tangent rays through x_0 trace out a smooth curve S in a neighborhood of $y_0 \in \partial\Omega$ S is $\pi(S^*\partial\Omega \cap \Gamma x_0)$.

At each point $y \in S$, the tangent ray, ℓy, passing through x_0 and y determines a unique oriented geodesic in $\partial\Omega$. This geodesic is just the natural continuation of ℓy into the obstacle surface, $\partial\Omega$. Using the correspondence between oriented geodesics and the bicharacteristics of $S^*\partial\Omega$, the one parameter family of oriented geodesics defined in this way may be identified with $\overline{\Lambda x_0}$, the 2-dimensional manifold formed by the union of the $S^*\partial\Omega$ bicharacteristics passing through $S^*\partial\Omega \cap \Gamma x_0$.

Let $U \subset \partial\Omega$ be the domain covered by the geodesics. As long as U is sufficiently small, each point $x \in U$ lies on a unique geodesic in the 1-parameter family. The tangent space at x to this geodesic determines a tangent ray. This tangent ray is called a falling ray associated to x_0. Let $\Lambda x_0 \subset \Delta$ denote the space of such falling rays.

Proposition 3.3: Λx_0 is a Lagrangian submanifold of M_E.

Proof: Λx_0 is the image under π_E of the two-dimensional manifold $\overline{\Lambda x_0}$. Any two-dimensional submanifold that is the union of the bicharacteristics of a hypersurface in a four-dimensional symplectic manifold will be Lagrangian. Hence $\overline{\Lambda x_0}$ is a Lagrangian submanifold of $\Gamma^*\partial\Omega$. Thus the result follows from Corollary 8.2.

Note also that Corollary 8.2 implies that π_E maps bicharacteristics of $S^*\partial\Omega$ to bicharacteristics of $\Delta \subset M_E$.

Section 11

SINGULARITIES OF Λx_0 AROUND A BIASYMPTOTIC RAY

The singularities of Λx_0 will occur at the same kind of points as the hypersurface Δ, at points corresponding to asymptotic and biasymptotic rays. From Corollary 9.3a, asymptotic rays cannot be incorporated within length minimizing paths. Biasymptotic rays, however, may appear in length minimizing paths. For this reason, it is important to examine the singularity of Λx_0 about a biasymptotic ray.

The major consequence of Theorem 9.3 by Arnol'd in [A1] is that when the obstacle Ω and $x_0 \in R^3 \setminus \Omega$ are in general position, Λx_0 has, close to a biasymptotic ray, a singularity diffeomorphic with the singularity of the open swallowtail at zero. (This result was at the level of formal power series.) The extension of Theorem 9.2, due to the author and Melrose, extends this result to the category of smooth diffeomorphisms. The open swallowtail is the set $\Lambda_0 \subset R^4$

$$\Lambda_0 = \{(p_1, p_2, q_1, q_2) \in R^4 ;$$

$$\Phi(p_0) = \frac{p_0^5}{5} + \frac{p_0^3 p_i}{3} + \frac{q_2 p_0^2}{2} + p_2 p_0 + \frac{q_1}{2}$$

$$\text{has a root of multiplicity } \geqslant 3\}$$

Let $\ell \in \Delta$ be a biasymptotic ray. By Theorem 2.3, in a neighborhood of ℓ, M_E is symplectomorphic to R^4 by a map carrying Δ to the surface:

$$\sum = \{(p_1, p_2, q_1, q_2) \in R^4 : p_0^4 + p_0^2 p_1 + q_0 p_0 + p_2 \text{ has a double root}\}$$

$\sum$ is the product of the ordinary swallowtail in R^3 with R. Its projection into (p_1, p_2, q_2) space is drawn in Figure 11.1. The lower edges J_+, J_- correspond to asymptotic rays in Δ while the swallowtail tip corresponds to biasymptotic rays.

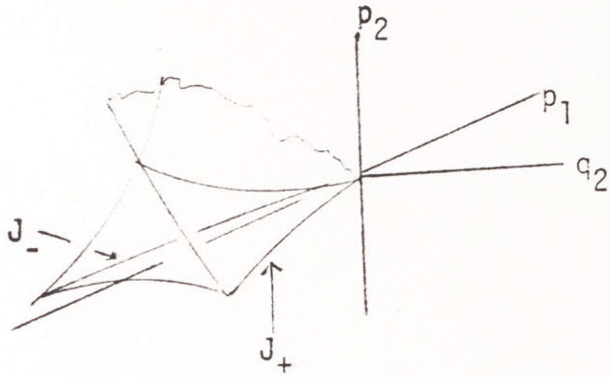


Figure 11.1

In particular, $\sum$ is self-intersecting along the two-dimensional surface I, defined by

$$I = \{q_2 = 0 \ , \quad p_2 = (\frac{p_1}{2})^2\}$$

Points in I correspond to rays that contact $\partial\Omega$ at two points. The reduction of Λx_0 to normal form begins with a description of the bicharacteristics of $\sum$. In [Al], Arnol'd showed that the bicharacteristics of were given by translations along the q_1 axis of curves defined by the conditions:

1. $\Phi(p_0) = \frac{p_0^5}{5} + \frac{p_1 p_0^3}{3} + \frac{q_2 p_0^2}{2} + p_2 p_0 + \frac{q_1}{2}$ has a root of multiplicity ≥ 3. (11.1)

2. $p_1 = c$ - an arbitrary constant.

Alternatively, the bicharacteristics are given in parametric form by

$$p_1 = c; \; q_2 = -4\xi^2 - 2c \; ; \; p_2 = 3\xi^4 + c\xi^2$$

$$q_1 = -12/5 \; \xi^5 - 2/3 \; c\xi^3 + K \qquad (11.2)$$

where c and K are constants.

Lemma 4.1 (Arnol'd): A Lagrangian submanifold, Λ, in general position lying on $\sum$ can be locally reduced to normal form

$$\Lambda_0 = \{(p_1, q_1, p_2, q_2) \; ; \; \Phi \text{ has a root of multiplicity } \geq 3\}$$

Proof: Since Λ is composed of characteristics of $\sum$, it suffices to reduce to normal form a 1 parameter sub-family in general position in the two parameter family of Equation (11.2).

For such a family, K may be taken to be a smooth function K(c). Then the symplectomorphism defined by

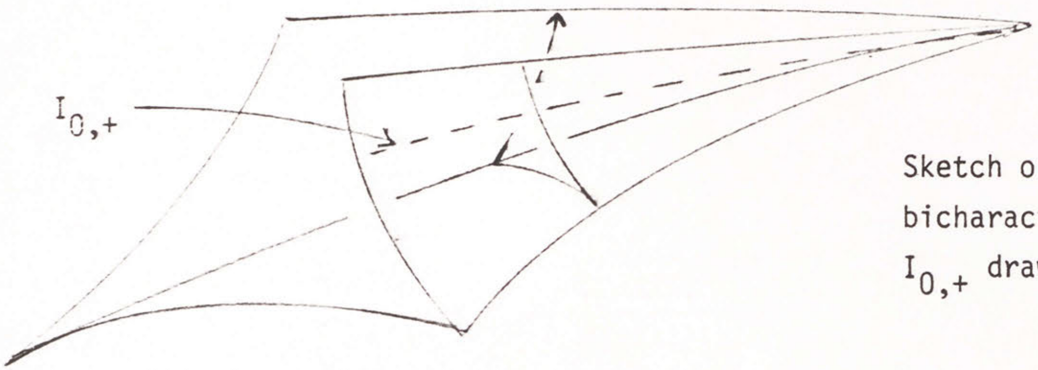
$$(p_1, p_2, q_1, q_2) \to (p_1, p_2, q_1 - K(p_1), p_2, q_2)$$

carries Λ to normal form.

While the swallowtail has a self-intersection set I = $\{q_2 = 0;$

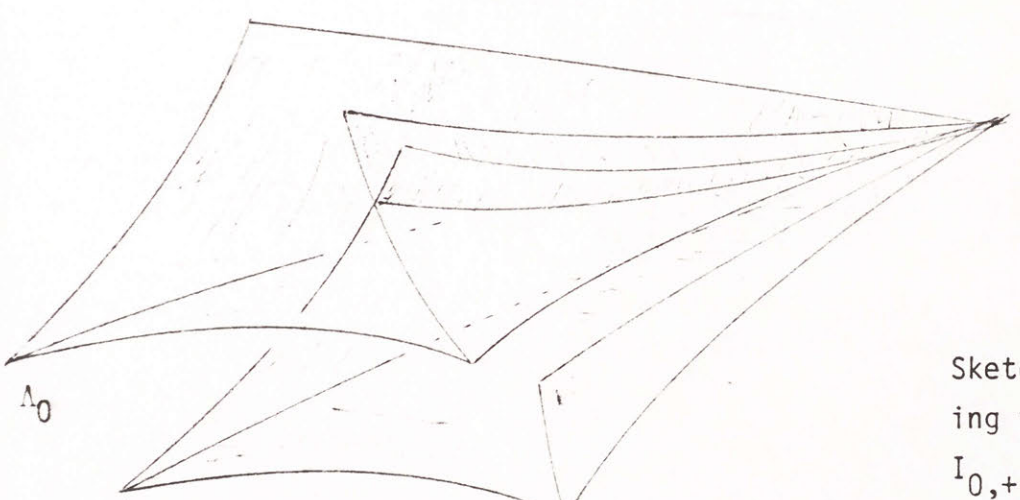
$p_2 = p_1^2/4\}$, Λ_0 is not self-intersecting. However, Λ_0 does intersect I in two curves, $I_{0,+}$, $I_{0,-}$ defined by the condition $\xi = \pm(-c/2)^{1/2}$ in Equation (11.2).

$\sum$ has two families of bicharacteristics passing through $I_{0,+}$. One family corresponds to Λ_0. The other defines a second Lagrangian submanifold Λ_1. Λ_1 intersects Λ_0 along $I_{0,+}$.



Sketch of Λ_0 showing bicharacteristics and $I_{0,+}$ drawn in.

Figure 11.2



Sketch of Λ_0, Λ_1 showing intersection line $I_{0,+}$.

Figure 11.3

In particular, $I_{0,+}$ and $I_{0,-}$ describe families of falling rays in Λx_0 that are tangent to Λx_0 at two points.

Remark: Note that Λ_1 will also intersect I in two curves $I_{1,+}$, $I_{1,-}$. Note also that $I_{1,-} = I_{0,+}$.

Section 12

EXAMPLE: ANALYSIS OF LENGTH MINIMIZING PATHS AROUND A BIASYMPTOTIC RAY

Consider the situation illustrated in Figure 12.1.

The geodesics emanating from the contact line, S, establish a map from $U \subset \partial\Omega$ to the Lagrangian submanifold $\Lambda x_0 \subset M_E$. We assume that $y_0 \in U$ is mapped to a biasymptotic ray in Λx_0 and that the region lying between the curves J_- and J_+ in Λx_0 corresponds to a region of concavity as shown in Figure 12.1.

Following the geodesic from $y \in S$ to $y' \in I_{0,+}$, the length minimizing path will follow one of two routes, it can either continue on along the geodesic into the floor of the valley, ρ_v, or it can jump across the valley along the fallong ray tangent to the right hand ridge at y' and the left hand ridge at y", ρ_r. At y", the falling ray defines a geodesic in $\partial\Omega$ continuing the path into $\partial\Omega$.

By simple comparison arguments, a length minimizing path may not travel through the region of concavity bounded by J_+ and J_-. Thus the interesting path is the one crossing the valley via the falling ray, ρ_r.

This path is described in terms of the Lagrangian submanifolds Λ_0, Λ_1. The path of tangent rays defined by ρ_r follows the Δ bicharacteristic contained in Λ_0 up to the intersection line $I_{0,+}$. Then it leaves $I_{0,+}$ by the Δ bicharacteristic contained in Λ_1. Since the bicharacteristics of Δ determine bicharacteristics of $S^*\partial\Omega$, this description actually specifies ρ_r.

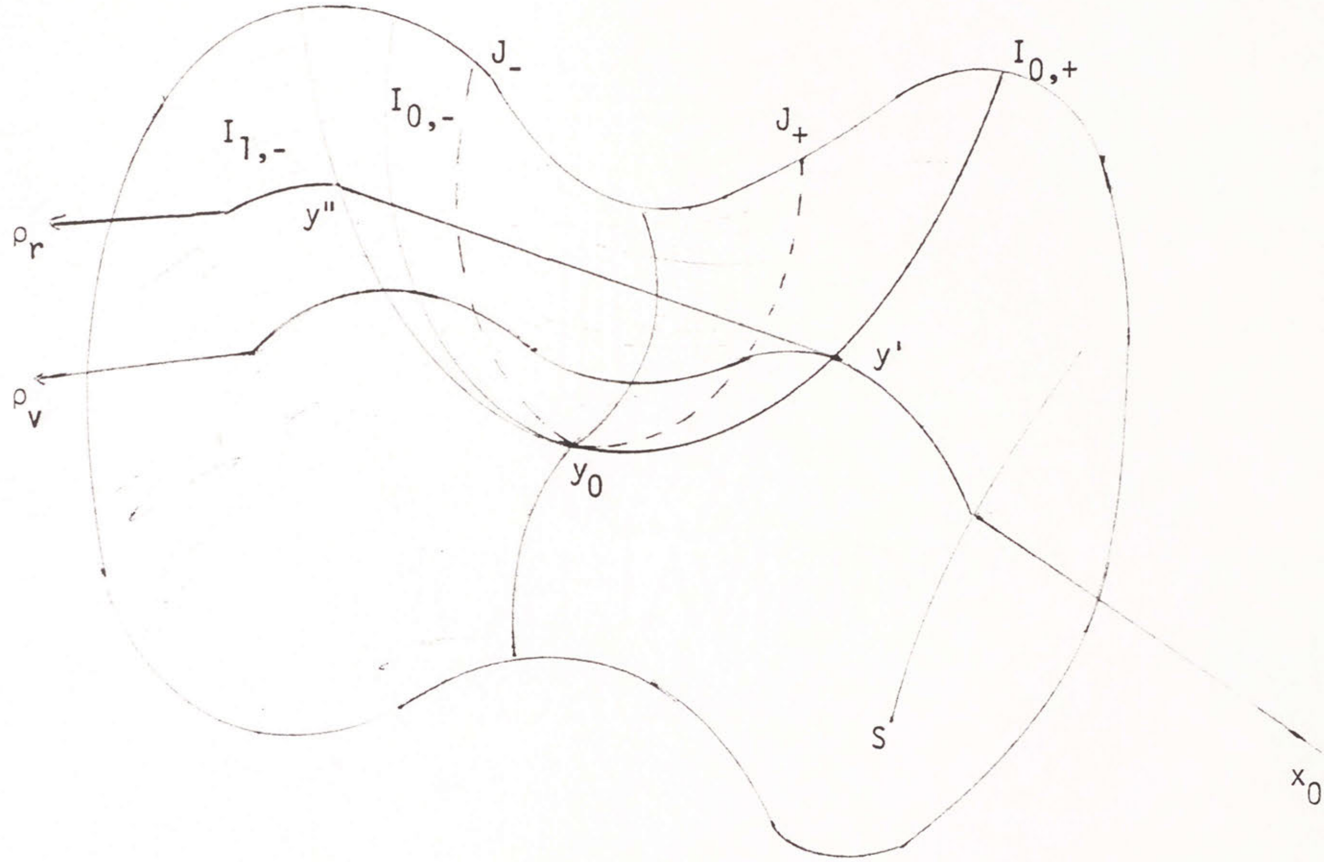
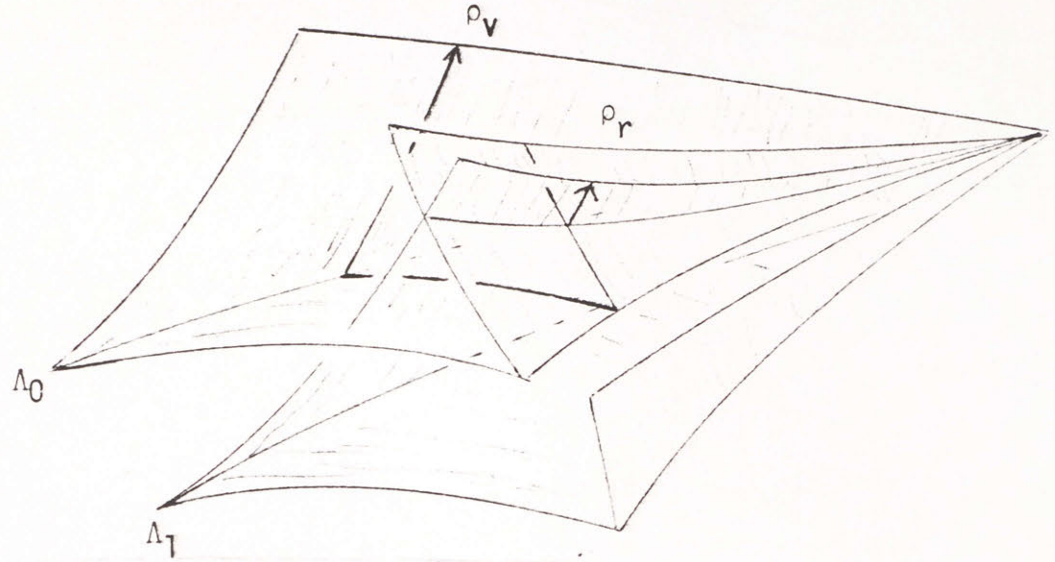


Figure 12.1



Sketch of Λ_0, Λ_1, showing intersection line $I_{0,+}$ and the bicharacteristics corresponding to ρ_v and ρ_r.

Figure 12.2

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